AH-Subspaces in Neutrosophic Vector Spaces

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Abstract

In this paper, we introduce the concept of AH-subspace of a neutrosophic vector space and AHS-linear transformations. We study elementary properties of these concepts such as Kernel, AH-Quotient, and dimension.

Keywords: Neutrosophic vector space , AH-supspace , AHS-subspace , AH-Quotient.

1. Introduction

Neutrosophy, as a new branch of philosophy and of logic, founded by Smarandache, got its way in algebraic structure studies. Many neutrosophic algebraic structures were defined and studied such as neutrosophic groups, neutrosophic rings, neutrosophic refined rings, and neutrosophic vector spaces. See [3,4,5,6,9]. In 2019 and 2020 Smarandache [12,13,14] generalized the classical Algebraic Structures to NeutroAlgebraic Structures (or NeutroAlgebra) {whose operations and axioms are partially true, partially indeterminate, and partially false} as an extension of Partial Algebra, and to AntiAlgebraic Structures (or AntiAlgebra) {whose operations and axioms are totally false}.

AH-substructures were firstly defined in neutrosophic rings in [1], and then they have been studied in refined neutrosophic rings in [2]. These structures have had many symmetric properties that illustrate a bridge between classical algebra and neutrosophic algebra. In this paper, we try to define AH-subspace and AHS-subspace of a neutrosophic vector space and introduce some of its elementary properties. Also, some interesting concepts were defined and used in this study, such as neutrosophic AH-linear transformations, and AH-Quotient.

Motivation

This work is a continuation of works done in [1,2], that established the theory of neutrosophic AH-substructures in neutrosophic algebraic structures.

2. Preliminaries

Definition 2.3:[5] Let (V, +, ·) be a vector space over the field K then (V(I), +, ·) is called a weak neutrosophic vector space over the field K, and it is called a strong neutrosophic vector space if it is a vector space over the neutrosophic field K(I).

DOI: 10.5281/zenodo.3841628

Received: February29, 2020 Revised: April 28, 2020 Accepted: May 16, 2020

A neutrosophic field K(I) is a triple (K(I),+, ·), where K is a classical field. A neutrosophic field is not a field by classical meaning, but it is a ring.

Elements of V(I) have the form $\boldsymbol{x + yI; x , y \in V}$, i.e V(I) can be written as $\boldsymbol{V(I) = V + VI}$.

Definition 2.4: [5] Let V(I) be a strong neutrosophic vector space over the neutrosophic field K(I) and W(I) be a non empty set of V(I), then W(I) is called a strong neutrosophic subspace if W(I) itself is a strong neutrosophic vector space.

Definition 2.5:[5] Let U(I) and W(I) be two strong neutrosophic subspaces of V(I) then we say that V(I) is a direct sum of U(I) and W(I) if and only if for each element x ∈ $\boldsymbol{V(I)}$ then x can be written uniquely as x = y + z such y ∈ $\boldsymbol{U(I)}$ and z ∈ $\boldsymbol{W(I)}$

Definition 2.6:[5]

Let U(I) and W(I) be twostrong neutrosophic subspaces of V(I) and let $f: V(I) \to W(I)$, we say that f is a neutrosophic vector space homomorphism if

(a) $f(I) = I$.

(b) f is a vector space homomorphism.

We define the kernel of f by Ker f = { x∈ $V(I)$; $f(x)$ = 0 }.

Definition 2.7:[5]

Let $v_1, v_2 .. v_s \in V(I)$ and $x \in V(I)$ we say that x is a linear combination of $\{ v_i; i = 1, .., s\}$ if

$x = a_1v_1 + \cdots + a_sv_s$ such that $a_i \in K(I)$.

The set $\{ v_i; i = 1, .., s\}$ is called linearly independent if $a_1v_1 + \cdots + a_sv_s = 0$ implies $a_i = 0$ for all i.

Theorem 2.10: [5]

If $\{ v_1 , \dots , v_s \}$ is a bases of V(I) and $f: V(I) \to W(I)$ is a neutrosophic vector space homomorphism then $\{f(v_1) , \dots , f(v_s)\}$ is a bases of W(I).

Definition 2.8: [1]

Let R(I) be a neutrosophic ring and $P = P_0 + P_1 I = \{a_0 + a_1 I;\ a_0 \in P_0, a_1 \in P_1\}$.

(a) We say that P is an AH-ideal if P_0 and P_1 are ideals in the ring R.

(b) We say that P is an AHS-ideal if $P_0 = P_1$.

Definition 2.9: [2]

Let (R(I_1, I_2), $+$, $\times$) be a refined neutrosophic ring, and P_0, P_1, P_2 be three ideals in the ring R then the set

$P = (P_0, P_1 I_1, P_2 I_2) = \{(a, bI_1, cI_2); a \in P_0, b \in P_1, c \in P_2\}$ is called a refined neutrosophic AH-ideal.

If $P_0 = P_1 = P_2$ then P is called a refined neutrosophic AHS-ideal.

DOI: 10.5281/zenodo.3841628

3. Main concepts and discussion

Definition 3.1:

Let V(I) = V+VI be a strong/weak neutrosophic vector space, the set

$S = P + QI = \{x + yI; x \in P, y \in Q\}$, where P and Q are subspaces of V is called an AH-subspace of V(I).

If $P = Q$ then S is called an AHS-subspace of V(I).

Example 3.2:

We have V = R^2 is a vector space, $P =< (0,1) >,\ Q =< (1,0) >$, are two subspaces of V. The set

$S = P + QI = \{(0, a) + (b, 0)I; a, b \in R\}$ is an AH-subspace of V(I).

The set $L = P + PI = \{(0, a) + (0, b)I\}; a, b \in R$ is an AHS-subspace of V(I).

Theorem 3.3:

Let $V(I) = V + VI$ be a neutrosophic weak vector space, and let $S = P + QI$ be an AH-subspace of $V(I)$, i.e Q, P are supspaces of V, then S is a subspace by the classical meaning.

Proof:

Suppose that $x = a + bI, y = c + dI \in S; a, c \in P,\ b, d \in Q$, we have

$x + y = (a + c) + (b + d)I \in S$. For each scalar $m \in K$ we obtain $m.x = m.a + (m.b)I \in S$, since P and Q are subspaces; thus $S = P + QI$ is a subspace of V(I) over the field K.

Remark 3.4:

An AH-subspace S is not necessary a subspace of neutrosophic strong vector space V(I) over a neutrosophic field K(I), see Example 3.5.

Example 3.5:

Let S be the AH-subspace defined in Example 3.2, and $m = 1 + I \in R(I)$ be a neutrosophic scalar, and $x = (0,1) + (2,0)I \in S$, we have:

$m.x = (1 + I) \cdot x = (0,1) + [(2,0) + (0,1) + (2,0)]I = (0,1) + (4,1)I$, since (4,1) does not belong to Q, we find that $m \cdot x$ is not in *S*, thus *S* is not a subspace of V(I).

The following theorem shows that any AHS-subspace is a subspace of a neutrosophic strong vector space.

Theorem 3.6:

Let V(I) be a neutrosophic strong vector space over a neutrosophic field K(I), let $S = P + PI$ be an AHS-subspace. S is a subspace of V(I).

Proof:

Suppose that $x = a + bI, y = c + dI \in S; a, c, b,$ and $c \in P$, we have

$x+y=(a+c)+(b+d)I \in S$. Let $m=x+yI \in K(I)$ be a neutrosophic scalar, we find

$m \cdot x=(x.a)+(y.a+y.b+x.b)I \in S$, since $y.a+y.b+x.b \in P$, thus we get the desired result.

Definition 3.7:

(a) Let V and W be two vector spaces, $L_V: V \to W$ be a linear transformation. The AHS-linear transformation can be defined as follows:

$L: V(I) \to W(I); L(a+bI)=L_V(a)+L_V(b)I$.

(b) If $S = P+QI$ is an AH-subspace of V(I), $L(S)=L_V(P)+L_V(Q)I$.

(c) If $S = P+QI$ is an AH-subspace of W(I), $L^{-1}(S)=L_W^{-1}(P)+L_W^{-1}(Q)I$.

(d) $AH-Ker\ L = KerL_V+KerL_V\ I=\{x+yI; x,y \in KerL_V\}$.

Theorem 3.8:

Let W(I) and V(I) be two neutrosophic strong/weak vector spaces, and $L: V(I) \to W(I)$ be an AHS-linear transformation, we have:

(a) $AH-Ker\ L$ is an AHS-subspace of V(I).

(b) If $S = P+QI$ is an AH-subspace of V(I), $L(S)$ is an AH-subspace of W(I).

(c) If $S = P+QI$ is an AH-subspace of W(I), $L^{-1}(S)$ is an AH-subspace of V(I).

Proof:

(a) Since $KerL_V$ is a subspace of V, we find that $AH-Ker\ L = KerL_V+KerL_V\ I$ is an AHS-subspace of V(I).

(b) We have $L(S)=L_V(P)+L_V(Q)I$; thus L(S) is an AH-subspace of W(I), since $L_V(P), L_V(Q)$are subspaces of W.

(c) By regarding $L^{-1}(S)=L_W^{-1}(P)+L_W^{-1}(Q)I$, $L_W^{-1}(P)$ and $L_W^{-1}(Q)$ are subspaces of V, we obtain that $L^{-1}(S)$ is an AH-subspace of V(I).

Theorem 3.9:

Let W(I) and V(I) be two neutrosophic strong vector spaces over a neutrosophic field K(I), and $L: V(I) \to W(I)$ be an AHS-linear transformation, we have:

$L(x+y)=L(x)+L(y), L(m.x)=m.L(x)$, for all $x,y \in V(I), m \in K(I)$.

Proof:

Suppose $x=a+bI, y=c+dI; a,b,c,d \in V, and\ m=s+tI \in K(I)$, we have

$$L(x+y)=L([a+c]+[b+d]I)=L_V(a+c)+L_V(b+d)I=$$

$$[L_V(a)+L_V(b)I]+[L_V(c)+L_V(d)I]=L(x)+L(y).$$

$m.x=(s.a)+(s.b+t.a+t.b)I$, $L(m.x)=L_V(s.a)+L_V(s.b+t.a+t.b)I$

DOI: 10.5281/zenodo.3841628

$= s.L_V(a) + [s.L_V(b) + t.L_V(a) + t.L_V(b)]I = (s + tI).(L_V(a) + L_V(b)I) = m.L(x)$.

Theorem 3.10:

Let $S = P + QI$ be an AH-subspace of a neutrosophic weak vector space V(I) over a field K, suppose that

$X = \{x_i; 1 \leq i \leq n\}$ is a bases of P and $Y = \{y_j; 1 \leq j \leq m\}$ is a bases of Q then $X \cup YI$ is a bases of S.

Proof:

Let $z = x + yI$ be an arbitrary element in S; $x \in P, y \in Q$. Since P and Q are subspaces of V we can write

$x = a_1x_1 + a_2x_2 + \cdots + a_nx_n;\ a_i \in K\ and\ x_i \in X, y = b_1y_1 + b_2y_2 + \cdots + b_my_m;\ b_i \in K, y_i \in Y.$

Now we obtain $z = (a_1x_1 + \cdots + a_nx_n) + (b_1y_1I + \cdots + b_my_mI)$; thus $X \cup YI$ generates the subspace S.

$X \cup YI$ is linearly independent set. Assume that $(a_1x_1 + \cdots + a_nx_n) + (b_1y_1I + \cdots + b_my_mI) = 0$, this implies

$a_1x_1 + a_2x_2 + \cdots + a_nx_n = 0$ and $(b_1y_1 + b_2y_2 + \cdots + b_my_m)I = 0$. Since X and Y are linearly independent sets over K, we get $a_i = b_j = 0\ for\ all\ i,j$ and $X \cup YI$ is linearly independent then it is a bases of S.

Result 3.11:

Let $S = P + QI$ be an AH-subspace of a neutrosophic weak vector space V(I) with finite dimension over a field K, from Theorem 3.10 and the fact that $X \cap YI = \emptyset$, we find $\dim(S) = \dim(P) + \dim(Q)$.

Example 3.12:

Let V $=R^3$, P=<(0,0,1)>, Q=<(0,1,0)> be two subspaces of V,

(a) $S = P + QI = \{(0,0,m) + (0,n,0)I;\ m,n \in R\ \}$is an AH-subspace of V(I).

(b) The set $\{(0,0,1), (0,1,0)I\}$ is a basis of S, $\dim(S) = \dim(P) + \dim(Q) = 1 + 1 = 2$.

(c) $L_V: V \to V;\ L_V(x,y,z) = (x + y, y, z) for\ all\ x, y, z \in R$ is a linear transformation, the corresponding AHS-linear transformation is $L: V(I) \to V(I); L[(x,y,z) + (a,b,c)I] = L_V(x,y,z) + L_V(a,b,c)I =$

$(x + y, y, z) + (a + b, b, c)I.$

(d) $L(S) = L_V(P) + L_V(Q) = L_V\{(0,0,m)\} + L_V\{(0,n,0)\}I = \{(0,0,m) + (n,n,0)I\};\ m,n \in R$, which is an AH-subspace of V(I).

Example 3.13 :

Let V=R^2, W=R^3, $L_V: V \to W;\ L_V(x,y) = (x + y, x + y, x + y)$ is a linear transformation. The corresponding AHS-linear transformation is

$L: V(I) \to W(I); L[(x,y) + (a,b)I] = (x + y, x + y, x + y) + (a + b, a + b, a + b)I.$

$KerL_V =< (1,-1) >, AH - KerL = KerL_V + KerL_V\ I =< (1,-1) > +< (1,-1) > I =$

$\{(a,-a) + (b,-b)I; a,b \in R\}$ which is an AHS-subspace of V(I).

It is clear that $\dim(KerL) = 1 + 1 = 2$ according to Theorem 3.10.

DOI: 10.5281/zenodo.3841628

Definition 3.14:

Let V(I) be a neutrosophic strong/weak vector space, $S = P + QI$ be an AH-subspace of V(I), we define

AH-Quotient as:

$V(I)/S = V/P + (V/Q)\,I = (x + P) + (y + Q)I; x, y \in V$.

Theorem 3.15:

Let V(I) be a neutrosophic weak vector space over a field K, and $S = P + QI$ be an AH-subspace of V(I). The AH-Quotient $V(I)/S$ is a vector space over the field K with respect to the following operations:

Addition: $[(x + P) + (y + Q)I] + [(a + P) + (b + Q)]I = (x + a + P) + (y + b + Q)I; x, y, a, b \in V$.

Multiplication by a scalar: $(m).[(x + P) + (y + Q)I] = (m.x + P) + (m.y + Q)I$;

$x, y \in V$ and $m \in K$.

Proof:

It is easy to check the operations completely are well defined, and $(V(I)/S\,, +)$ is abelian group.

Let $z = [(x + P) + (y + Q)I] \in V(I)/S$, we have $1.z = z$.

Assume that $m, n \in K$, we have $m.(n.z) = m.[(n.x + P) + (n.y + Q)I] = (m.n.x + P) + (m.n.y + Q)I = (m.n).z$.

$(m + n).z = [(m + n).x + P] + [(m + n).y + Q]I = m.z + n.z$.

Let $h = [(a + P) + (b + Q)I] \in V(I)/S$, $z + h = (x + a + P) + (y + b + Q)I$,

$m.(z + h) = (m.x + m.a + P) + (m.y + m.b + Q)m.z + m.h$.

Example 3.16:

We have V = R^2 is a vector space over the field R, $P =< (0,1) >, Q =< (1,0) >$ are two subspaces of V,

$S = P + QI = \{(0, a) + (b, 0)I; a, b \in R\}$ is an AH-subspace of V(I).

The AH-Quotient is $V(I)/S = \{[(x, y) + P] + [(a, b) + Q]I; x, y, a, b \in V\}$.

We clarify operations on $V(I)/S$ as follows:

$x = [(2,1) + P] + [(1,3) + Q]I, y = [(2,5) + P] + [(1,1) + Q]I$ are two elements in $V(I)/S\,, m = 3\ is\ a\ scalar\ in\ R$.

$x + y = [(4,6) + P] + [(2,4) + Q]I$, $3.x = [(6,3) + P] + [(3,9) + Q]I$.

Remark 3.17:

If $S = P + PI$ is an AHS-subspace of a neutrosophic weak vector space V(I) over the field K, then AH-Quotient $V(I)/S = V/P + V/P\,I$ is a weak neutrosophic vector space, since V/P is a vector space.

DOI: 10.5281/zenodo.3841628

5. Conclusion

In this article, we have defined the concepts of AH-subspace, AHS-subspace, and AHS-linear transformation in neutrosophic vector spaces. Also, we have studied some basic properties of these concepts.

References:

[1] Abobala, M., "On Some Special Substructures of Neutrosophic Rings and Their Properties", International Journal of Neutrosophic Science", Vol 4, pp.72-81, 2020.

[2] Abobala, M., "On Some Special Substructures of Refined Neutrosophic Rings", International Journal of Neutrosophic Science, Vol 5, pp.59-66, 2020.

[3] Abobala, M,. "Classical Homomorphisms Between Refined Neutrosophic Rings and Neutrosophic Rings", International Journal of Neutrosophic Science, Vol 5, pp.72-75, 2020.

[4] Adeleke, E.O., Agboola, A.A.A., and Smarandache, F., "Refined Neutrosophic Rings I", International Journal of Neutrosophic Science, Vol 2, pp. 77-81, 2020.

[5] Agboola, A.A.A,. and Akinleye, S.A,. "Neutrosophic Vector Spaces", Neutrosophic Sets and Systems, Vol 4 , pp 9-17, 2014.

[6] Agboola, A.A.A,. Akwu, A.D,. and Oyebo, Y.T,. "Neutrosophic Groups and Subgroups", International .J .Math.Combin, Vol 3, pp. 1-9, 2012.

[7] Agboola, A.A.A., Akinola, A.D., and Oyebola, O.Y.," NeutrosophicRings I ", International J.Mathcombin, vol(4), pp 1-14, 2011.

[8] Bera, T and Mahapatra, N. K., "On Neutrosophic Soft Field", IJMTT, 56(7), pp.472-494, 2018.

[9] Bera, T and Mahapatra, N. K., "On Neutrosophic Soft Linear Spaces", Fuzzy Information and Engineering., 9, pp.299-324, 2017.

[10] Injrou, S., "Linear Algebra 2 ", Tishreen University Press, pp.180-230, 2015.

[11] Kandasamy, V.W.B,. and Smarandache, F., "Some Neutrosophic Algebraic Structures and Neutrosophic N-Algebraic Structures", Hexis, Phonex, Arizona 2006.

[12] Smarandache, F.,"NeutroAlgebra is a Generalization of Partial Algebra", International Journal of Neutrosophic Science (IJNS), Vol. 2, No. 1, pp. 08-17, 2020.

[13] Smarandache, F., "Introduction to NeutroAlgebraic Structures and AntiAlgebraic Structures, in Advances of Standard and Nonstandard Neutrosophic Theories," Pons Publishing House Brussels, Belgium, Ch. 6, pp. 240-265, 2019.

[14] Smarandache, F.,"Introduction to NeutroAlgebraic Structures and AntiAlgebraic Structures (revisited)", Neutrosophic Sets and Systems, vol. 31, pp. 1-16, 2020. DOI: 10.5281/zenodo.3638232.

DOI: 10.5281/zenodo.3841628