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ORIGINAL RESEARCH

A new decision-making method based on bipolar neutrosophic directed hypergraphs

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Abstract Directed hypergraphs are widely used as a tool to solve and model the problems appearing in computer science and operations research. Bipolar neutrosophic models are more flexible and applicable because these models study neutrosophic behavior positively as well as negatively. In this research study, we present a new frame work for handling bipolar neutrosophic information by combining the bipolar neutrosophic sets with directed hypergraphs. We introduce certain new concepts, including bipolar neutrosophic directed hypergraphs, regular bipolar neutrosophic directed hypergraphs, homomorphism and isomorphism on bipolar neutrosophic directed hypergraphs. Further, we study some isomorphic properties of strong bipolar neutrosophic directed hypergraphs. In particular, we consider interesting applications of bipolar neutrosophic directed hypergraphs in decision-making, and we develop efficient algorithm to solve decision-making problems.

Keywords Bipolar neutrosophic directed hypergraphs · Regular bipolar neutrosophic directed hypergraphs · Isomorphism · Decision-making · Algorithm

Mathematics Subject Classification 03E72 · 68R10 · 68R05

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1 Introduction

Classical set theory is the most fundamental concept of mathematics, which inherits the collection of distinct individuals or items that share some common property. If an element belongs to some set, membership value 1 is assigned to that element, otherwise 0 will be its membership value. That is, there are only two possible membership values for the elements of a classical set, 0 and 1. This crisp membership function deals suitably for mathematics and binary operations, but it does not describe nicely the real World phenomena which cannot be characterized in completely false or completely true manner. A fuzzy set, defined by Zadeh [33], permits its elements to possess variable values of membership. The membership value 1 is given to those elements that completely belong to the set and 0 is to those objects that are not in the set. An element that partially belongs to the set will own a membership value between 0 and 1. Intuitionistic fuzzy sets (IFSs), generalization of fuzzy sets, were introduced by Atanassov [11] in 1986. In 1994, Zhang [32] introduced bipolar fuzzy sets (BFSs) as an extention of fuzzy sets. BFSs study the positive as well as the negative behavior of real World problems. Bipolar valued fuzzy sets were introduced by Lee [18]. A variety of decision making problems, which are based on two-sided information are solved by using BF models. Neutrosophic sets (NSs), a generalized framework of fuzzy sets and intuitionistic fuzzy sets, were introduced by Smarandache [27] in 1998. Mainly, the concept of NS is to distinguish every assertion in three dimensional neutrosophic space, where the individual dimension describes the truth (t), the indeterminacy (I) and the falsehood (f) of the assertion, respectively, that is under consideration, where t, I, f are independent real subsets of $]-0, 1+[$ with no any restriction on their sum. Wang et al. [29] introduced the single-valued neutrosophic sets (SVNSs) as a subclass of the NSs to apply it more conveniently to engineering problems. Multicriteria decision-making method and single-valued neutrosophic minimum spanning tree and its clustering were studied by Ye [30,31]. Recently, as an extension of BFSs and SVNSs, Deli et al. [15] defined bipolar neutrosophic sets (BNSs).

In 1975, fuzzy graphs were introduced by Rosenfeld [24]. Bipolar fuzzy graphs were introduced by Akram [1]. Akram et al. [6] applied the bipolar fuzzy digraphs in decision support systems. Akram and Shahzadi [10] introduced single-valued neutrosophic graphs. When there is indeterminacy in relations between vertices, the use of fuzzy graphs [24] and its extended forms, including intuitionistic fuzzy graphs [22], bipolar fuzzy graphs [1] are not appropriate. Broumi et al. [13] introduced bipolar single-valued neutrosophic graphs. Akram and Sarwar [8] redefined the definition of bipolar neutrosophic graphs and discussed applications of neutrosophic graphs. Hypergraphs, a generalization of graphs, have been widely and deeply studied in Berge [12]. A hypergraph is an extension of a classical graph in this way that a hyperedge can combine two or more than two vertices. Hypergraphs have many applications in various fields, including biological sciences, computer science and natural sciences. Just as hypergraphs are the generalization of ordinary graphs, directed hypergraphs [16] are the extension of ordinary directed graphs. The crisp hypergraphs are insufficient to explain all real World problems. To study the degree of dependence of an object to the other, Kaufamnn [17] applied the concept of fuzzy sets to hypergraphs. Mordeson and Nair [20] presented fuzzy graphs and fuzzy hypergraphs. The concept of interval-valued

fuzzy sets was applied to hypergraphs by Chen [14]. Generalization and redefinition of fuzzy hypergraphs were discussed by Lee-Kwang and Lee [19]. Samanta and Pal [28] introduced bipolar fuzzy hypergraphs. Later on, certain properties of bipolar fuzzy hypergraphs were discussed by Akram et al. [7]. Novel applications of m-polar fuzzy hypergraphs were studied by Akram and Sarwar [9]. Isomorphism properties on fuzzy hypergraphs and strong fuzzy hypergraphs were discussed by Radhamani and Radhika [25], Radhika et al. [26], respectively. Parvathi and Thilagavathi [23] proposed intuitionistic fuzzy directed hypergraphs. Myithili et al. [21] introduced certain types of intuitionistic fuzzy directed hypergraphs. On the other hand, Akram and Luqman [2] discussed certain concepts of bipolar fuzzy directed hypergraphs. Akram and Luqman [4] studied single-valued neutrosophic directed hypergraphs. They also discussed intuitionistic single-valued neutrosophic hypergraphs [5]. Akram and Luqman [3] introduced bipolar neutrosophic hypergraphs. In this research study, we present a new frame work for handling bipolar neutrosophic information by combining the bipolar neutrosophic sets with directed hypergraphs. We introduce certain new concepts, including bipolar neutrosophic directed hypergraphs, homomorphism and isomorphism on bipolar neutrosophic directed hypergraphs. Further, we study some isomorphic properties of strong bipolar neutrosophic directed hypergraphs. In particular, we consider interesting applications of bipolar neutrosophic directed hypergraphs in decision-making, and we develop efficient algorithm to solve decision-making problems.
The layout of this paper is as follows: In Sect. 2, novel concepts of bipolar neutrosophic directed hypergraphs, simple, elementary, support simple and sectionally elementary bipolar neutrosophic directed hypergraphs are introduced. Further, concepts of homomorphism, weak isomorphism, co-weak isomorphism and isomorphism on bipolar neutrosophic directed hypergraphs are discussed. We introduce strong bipolar neutrosophic directed hypergraphs and study their certain properties. Section 3 deals with the discussion that how the concept of bipolar neutrosophic directed hypergraphs can be used to understand and analyze the real World applications. In the last section, we conclude our results.

2 Bipolar neutrosophic directed hypergraphs

Definition 2.1 [15] A bipolar neutrosophic set (BNS) N in V is defined as

$$N = \{(e, t_N^+(e), I_N^+(e), f_N^+(e), t_N^-(e), I_N^-(e), f_N^-(e))|e \in V\},$$

where $t_N^+, I_N^+, f_N^+ : V \to [0, 1]$ and $t_N^-, I_N^-, f_N^- : V \to [-1, 0]$. The positive membership values $t_N^+(e), I_N^+(e), f_N^+(e)$ denote the truth, indeterminacy and falsity degrees of a certain element $e \in V$, which indicate that the element captivates the property of bipolar neutrosophic set N. The negative membership values $t_N^-(e), I_N^-(e), f_N^-(e)$ denote the truth, indeterminacy and falsity membership of $e \in V$, which indicate the satisfaction of an element to some counter property corresponding to a bipolar neutrosophic set N.

Definition 2.2 [15] Let $N_1 = \{(e, t_{N_1}^+(e), I_{N_1}^+(e), f_{N_1}^+(e), t_{N_1}^-(e), I_{N_1}^-(e), f_{N_1}^-(e))|e \in V\}$ and $N_2 = \{(e, t_{N_2}^+(e), I_{N_2}^+(e), f_{N_2}^+(e), t_{N_2}^-(e), I_{N_2}^-(e), f_{N_2}^-(e))|e \in V\}$ be two bipolar neutrosophic sets. Then their union is defined as $(N_1 \cup N_2)(e) = \{\max\{t^+_{N_1}(e), t^+_{N_2}(e)\}$, $\frac{I^+_{N_1}(e)+I^+_{N_2}(e)}{2}$, $\min\{f^+_{N_1}(e), f^+_{N_2}(e)\}$, $\min\{t^-_{N_1}(e), t^-_{N_2}(e)\}$, $\frac{I^-_{N_1}(e)+I^-_{N_2}(e)}{2}$, $\max\{f^-_{N_1}(e)$, $f^-_{N_2}(e)\}\}$, for all $e \in V$.

Definition 2.3 [8] A bipolar neutrosophic relation on a non-empty set V is defined as a bipolar neutrosophic subset of $V \times V$ of the form $R = \{(mn), t^+_R(mn), I^+_R(mn)$, $f^+_R(mn), t^-_R(mn), I^-_R(mn), f^-_R(mn) | mn \in V \times V\}$, where $t^+_R, I^+_R, f^+_R, t^-_R, I^-_R, f^-_R$ are defined by the mappings $t^+_R, I^+_R, f^+_R : V \times V \to [0, 1]$ and $t^-_R, I^-_R, f^-_R : V \times V \to [-1, 0]$.

Definition 2.4 The *support* of a BNS $N = \{(e, t^+_N(e), I^+_N(e), f^+_N(e), t^-_N(e), I^-_N(e)$, $f^-_N(e)) | e \in V\}$ is defined as $supp(N) = supp^+(N) \cup supp^-(N)$, where $supp^+(N) = \{e \in V | t^+_N(e) \neq 0, I^+_N(e) \neq 0, f^+_N(e) \neq 0\}$ and $supp^-(N) = \{e \in V | t^-_N(e) \neq 0$, $I^-_N(e) \neq 0, f^-_N(e) \neq 0\}$. $supp^+(N)$ is called the *positive support* and $supp^-(N)$ is called the *negative support* of N.

Definition 2.5 [8] A bipolar neutrosophic graph on a non-empty set V is an ordered pair $G = (A, B)$, where A is a bipolar neutrosophic set on V and B is a bipolar neutrosophic relation in V such that

$$t^+_B(vy) \leq t^+_A(v) \wedge t^+_A(y), I^+_B(vy) \leq I^+_A(v) \wedge I^+_A(y), f^+_B(vy) \leq f^+_A(v) \vee f^+_A(y),$$
$$t^-_B(vy) \geq t^-_A(v) \vee t^-_A(y), I^-_B(vy) \geq I^-_A(v) \vee I^-_A(y), f^-_B(vy) \geq f^-_A(v) \wedge f^-_A(y),$$

for all $v, y \in V$. Note that $D(vy) = (0, 0, 0, 0, 0, 0) = \mathbf{0}$, for all $vy \in V \times V \setminus E$.

Akram and Luqman [3] introduced the concept of bipolar neutrosophic hypergraphs.

Definition 2.6 [3] Let V be a non-empty set. A *bipolar neutrosophic hypergraph H* on V is defined as an ordered pair $H = (\mu, \rho)$, where $\mu = \{\mu_1, \mu_2, \mu_3, \ldots, \mu_n\}$ is a finite collection of bipolar neutrosophic subsets on V and ρ is a bipolar neutrosophic relation on bipolar neutrosophic subsets μ_i such that

1. (i)

$$t^+_\rho(E_k) = t^+_\rho(\{x_1, x_2, x_3, \ldots, x_m\}) \leq \min\{t^+_{\mu_i}(x_1), t^+_{\mu_i}(x_2), \ldots, t^+_{\mu_i}(x_m)\},$$
$$I^+_\rho(E_k) = I^+_\rho(\{x_1, x_2, x_3, \ldots, x_m\}) \leq \min\{I^+_{\mu_i}(x_1), I^+_{\mu_i}(x_2), \ldots, I^+_{\mu_i}(x_m)\},$$
$$f^+_\rho(E_k) = f^+_\rho(\{x_1, x_2, x_3, \ldots, x_m\}) \leq \max\{f^+_{\mu_i}(x_1), f^+_{\mu_i}(x_2), \ldots, f^+_{\mu_i}(x_m)\}.$$

(ii)

$$t^-_\rho(E_k) = t^-_\rho(\{x_1, x_2, x_3, \ldots, x_m\}) \geq \max\{t^-_{\mu_i}(x_1), t^-_{\mu_i}(x_2), \ldots, t^-_{\mu_i}(x_m)\},$$
$$I^-_\rho(E_k) = I^-_\rho(\{x_1, x_2, x_3, \ldots, x_m\}) \geq \max\{I^-_{\mu_i}(x_1), I^-_{\mu_i}(x_2), \ldots, I^-_{\mu_i}(x_m)\},$$
$$f^-_\rho(E_k) = f^-_\rho(\{x_1, x_2, x_3, \ldots, x_m\}) \geq \min\{f^-_{\mu_i}(x_1), f^-_{\mu_i}(x_2), \ldots, f^-_{\mu_i}(x_m)\},$$

for all $x_1, x_2, x_3, \ldots, x_m \in V$.

2. $\bigcup_i supp(\mu_i(x)) = V$, for all $\mu_i \in \mu$.

We now present the concept of bipolar neutrosophic directed hypergraphs.

Definition 2.7 A *bipolar neutrosophic directed hypergraph* (BNDHG) with underlying set V is an ordered pair $G = (\sigma, \varepsilon)$, where σ is non-empty set of vertices and ε is a set of bipolar neutrosophic directed hyperarcs (or hyperedges).

A *bipolar neutrosophic directed hyperarc (or hyperedge)* $\varepsilon_i \in \varepsilon$ is an ordered pair $(T(\varepsilon_i), H(\varepsilon_i))$, such that $T(\varepsilon_i) \subset V$, $T(\varepsilon_i) \neq \emptyset$, is called its *tail* and $H(\varepsilon_i) \neq T(\varepsilon_i)$ is its *head*.

Definition 2.8 Let $G = (\sigma, \varepsilon)$ be a BNDHG. The order of G, denoted by $O(G)$, is defined as $O(G) = (O^+(G), O^-(G))$, where $O^+(G) = \sum_{x\in V} \wedge\sigma_i^+(x)$ and $O^-(G) = \sum_{x\in V} \vee\sigma_i^-(x)$.

The *size* of G, denoted by $S(G)$, is defined as $S(G) = (S^+(G), S^-(G))$, where $S^+(G) = \sum_{E_k\subset V} \varepsilon^+(E_k)$, $S^-(G) = \sum_{E_k\subset V} \varepsilon^-(E_k)$.

In a bipolar neutrosophic directed hypergraph, the vertices u_i and u_j are *adjacent* vertices if they both belong to the same bipolar neutrosophic directed hyperedge. Two bipolar neutrosophic directed hyperedges ε_i and ε_j are called *adjacent* if they have non-empty intersection, i.e., $supp(\varepsilon_i) \cap supp(\varepsilon_j) \neq \emptyset$, $i \neq j$.

Definition 2.9 A bipolar neutrosophic directed hypergraph $G = (\sigma, \varepsilon)$ is *simple* if it contains no repeated directed hyperedges, i.e., if $\varepsilon_j, \varepsilon_k \in \varepsilon$ and $\varepsilon_j \subseteq \varepsilon_k$, then $\varepsilon_j = \varepsilon_k$.

A bipolar neutrosophic directed hypergraph $G = (\sigma, \varepsilon)$ is said to be *support simple* if $\varepsilon_j, \varepsilon_k \in \varepsilon$, $supp(\varepsilon_j)$ =$supp(\varepsilon_k)$ and $\varepsilon_j \subseteq \varepsilon_k$, then $\varepsilon_j = \varepsilon_k$.

A bipolar neutrosophic directed hypergraph $G = (\sigma, \varepsilon)$ is *strongly support simple* if $\varepsilon_j, \varepsilon_k \in \varepsilon$ and $supp(\varepsilon_j) = supp(\varepsilon_k)$, then $\varepsilon_j = \varepsilon_k$.

Example 2.1 Consider a bipolar neutrosophic directed hypergraph $G = (\sigma, \varepsilon)$, where $\sigma = \{\sigma_1, \sigma_2, \sigma_3, \sigma_4\}$ be the family of bipolar neutrosophic subsets on $V = \{v_1, v_2, v_3, v_4, v_5, v_6\}$, as shown in Fig. 1, such that

$$\begin{aligned}
\sigma_1 = \big\{&(v_1, 0.1, 0.2, 0.3, -0.1, -0.2, -0.3), (v_2, 0.2, 0.2, 0.3, -0.2,\\
&-0.2, -0.3), (v_5, 0.3, 0.2, 0.3, -0.3, -0.2, -0.3)\big\},\\
\sigma_2 = \big\{&(v_1, 0.1, 0.2, 0.3, -0.1, -0.2, -0.3), (v_3, 0.4, 0.2, 0.3, -0.4,\\
&-0.2, -0.3), (v_6, 0.3, 0.2, 0.3, -0.3, -0.2, -0.3)\big\},\\
\sigma_3 = \big\{&(v_6, 0.1, 0.2, 0.3, -0.1, -0.2, -0.3), (v_4, 0.2, 0.2, 0.3, -0.2,\\
&-0.2, -0.3), (v_5, 0.3, 0.2, 0.3, -0.3, -0.2, -0.3)\big\},\\
\sigma_4 = \big\{&(v_6, 0.1, 0.2, 0.3, -0.1, -0.2, -0.3), (v_2, 0.2, 0.2, 0.3, -0.2,\\
&-0.2, -0.3), (v_5, 0.3, 0.2, 0.3, -0.3, -0.2, -0.3)\big\}.
\end{aligned}$$

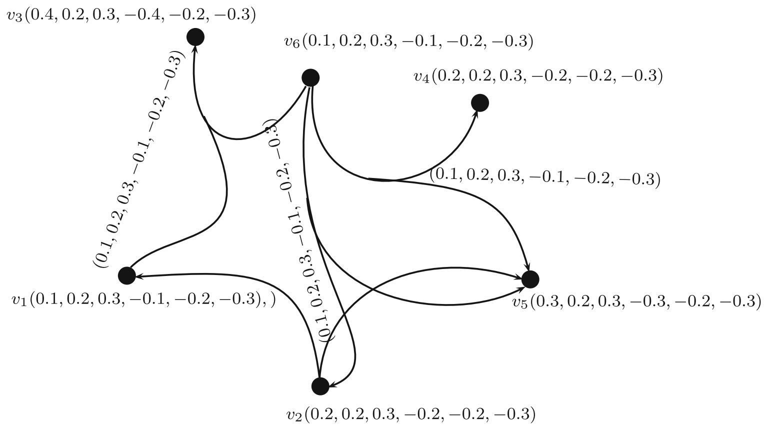

Fig. 1 Bipolar neutrosophic directed hypergraph

Bipolar neutrosophic relation ε is defined as

$$\begin{aligned}
\varepsilon(v_1, v_2, v_5) &= (0.1, 0.2, 0.3, -0.1, -0.2, -0.3),\\
\varepsilon(v_1, v_3, v_6) &= (0.1, 0.2, 0.3, -0.1, -0.2, -0.3),\\
\varepsilon(v_6, v_4, v_5) &= (0.1, 0.2, 0.3, -0.1, -0.2, -0.3),\\
\varepsilon(v_6, v_2, v_5) &= (0.1, 0.2, 0.3, -0.1, -0.2, -0.3).
\end{aligned}$$

Note that, G is simple, strongly support simple and support simple, that is, it contains no repeated directed hyperedges and if whenever $\varepsilon_j, \varepsilon_k \in \varepsilon$ and $supp(\varepsilon_j) = supp(\varepsilon_k)$, then $\varepsilon_j = \varepsilon_k$. Further, $o(G) = (1.1, 1.2, 1.8, -1.1, -1.2, -1.8)$ and $s(G) = (0.4, 0.6, 1.4, -0.4, -0.6, -1.4)$.

Definition 2.10 (i) The *height* of a bipolar neutrosophic directed hypergraph $G = (\sigma, \varepsilon)$, denoted by $h(G)$, is defined as $h(G) = \{\max(\varepsilon_l), \max(\varepsilon_m), \min(\varepsilon_n) | \varepsilon_l, \varepsilon_m, \varepsilon_n \in \varepsilon\}$, where $\varepsilon_l = \max t^{+}_{\varepsilon_j}(x_i)$, $\varepsilon_m = \max I^{+}_{\varepsilon_j}(x_i)$, $\varepsilon_n = \min f^{+}_{\varepsilon_j}(x_i)$.

(ii) The *depth* of a bipolar neutrosophic directed hypergraph $G = (\sigma, \varepsilon)$, denoted by d(G), is defined as $d(G) = \{\min(\varepsilon_l), \min(\varepsilon_m), \max(\varepsilon_n) | \varepsilon_l, \varepsilon_m, \varepsilon_n \in \varepsilon\}$, where $\varepsilon_l = \min t^{-}_{\varepsilon_j}(x_i)$, $\varepsilon_m = \min I^{-}_{\varepsilon_j}(x_i)$, $\varepsilon_n = \max f^{-}_{\varepsilon_j}(x_i)$.
The functions $t^{+}_{\varepsilon_j}(x_i)$, $I^{+}_{\varepsilon_j}(x_i)$ and $f^{+}_{\varepsilon_j}(x_i)$ denote the positive truth, indeterminacy and falsity membership values of vertex x_i to the hyperedge ε_j, respectively, $t^{-}_{\varepsilon_j}(x_i)$, $I^{-}_{\varepsilon_j}(x_i)$ and $f^{-}_{\varepsilon_j}(x_i)$ denote the negative truth, indeterminacy and falsity membership values of vertex x_i to the hyperedge ε_j, respectively.

Definition 2.11 Let $\varepsilon = (\varepsilon^{-}, \varepsilon^{+})$ be a directed hyperedge in a BNDHG. Then the vertex set ε^{-} is called the *in-set* and the vertex set ε^{+} is called the *out-set* of the directed hyperedge ε. It is not necessary that the sets ε^{-}, ε^{+} will be disjoint. The hyperedge ε is called the join of the vertices of ε^{-} and ε^{+}.

Definition 2.12 The *in-degree* $D_G^-(v)$ of a vertex v is defined as the sum of membership degrees of all those directed hyperedges such that v is contained in their out-set, i.e.,

$$D_G^-(v) = \left(\sum_{v \in H(E_k)} \varepsilon^+(E_k), \sum_{v \in H(E_k)} \varepsilon^-(E_k) \right).$$

The *out-degree* $D_G^+(v)$ of a vertex v is defined as the sum of membership degrees of all those directed hyperedges such that v is contained in their in-set, i.e.,

$$D_G^+(v) = \left(\sum_{v \in T(E_k)} \varepsilon^+(E_k), \sum_{v \in T(E_k)} \varepsilon^-(E_k) \right).$$

Definition 2.13 A bipolar neutrosophic directed hypergraph $G = (\sigma, \varepsilon)$ is called *k-regular* if in-degrees and out-degrees of all the vertices in G are same.

Example 2.2 Consider a bipolar neutrosophic directed hypergraph $G = (\sigma, \varepsilon)$ as shown in Fig. 2, where $\sigma = \{\sigma_1, \sigma_2, \sigma_3, \sigma_4\}$ is the family of bipolar neutrosophic subsets on $V = \{v_1, v_2, v_3, v_4, v_5, v_6\}$ and

$$\begin{aligned}
\sigma_1 = \big\{&(v_1, 0.1, 0.1, 0.1, -0.2, -0.2, -0.2), (v_2, 0.1, 0.1, 0.1, -0.2, -0.2, -0.2),\\
&(v_5, 0.1, 0.1, 0.1, -0.2, -0.2, -0.2)\big\},\\
\sigma_2 = \big\{&(v_1, 0.1, 0.1, 0.1, -0.2, -0.2, -0.2), (v_3, 0.1, 0.1, 0.1, -0.2, -0.2, -0.2),\\
&(v_6, 0.1, 0.1, 0.1, -0.2, -0.2, -0.2)\big\},\\
\sigma_3 = \big\{&(v_6, 0.1, 0.1, 0.1, -0.2, -0.2, -0.2), (v_4, 0.1, 0.1, 0.1, -0.2, -0.2, -0.2),\\
&(v_3, 0.1, 0.1, 0.1, -0.2, -0.2, -0.2)\big\},\\
\sigma_4 = \big\{&(v_2, 0.1, 0.1, 0.1, -0.2, -0.2, -0.2), (v_4, 0.1, 0.1, 0.1, -0.2, -0.2, -0.2),\\
&(v_5, 0.1, 0.1, 0.1, -0.2, -0.2, -0.2)\big\}.
\end{aligned}$$

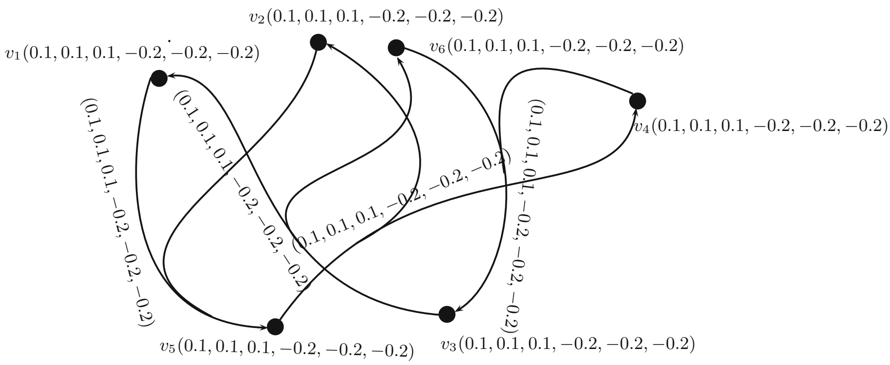

Fig. 2 Regular BNDHG

The bipolar neutrosophic relation ε is defined as

$$\varepsilon(v_1, v_2, v_5) = (0.1, 0.1, 0.1, -0.2, -0.2, -0.2),$$
$$\varepsilon(v_1, v_3, v_6) = (0.1, 0.1, 0.1, -0.2, -0.2, -0.2),$$
$$\varepsilon(v_6, v_4, v_3) = (0.1, 0.1, 0.1, -0.2, -0.2, -0.2),$$
$$\varepsilon(v_2, v_4, v_5) = (0.1, 0.1, 0.1, -0.2, -0.2, -0.2).$$

By routine calculations, we see that the bipolar neutrosophic directed hypergraph is regular.

Note that, $D_G^-(v_1) = (0.1, 0.1, 0.1, -0.2, -0.2, -0.2) = D_G^+(v_1)$ and $D_G^-(v_2) = (0.1, 0.1, 0.1, -0.2, -0.2, -0.2) = D_G^+(v_2)$. Similarly, $D_G^-(v_3) = D_G^+(v_3)$, $D_G^-(v_4) = D_G^+(v_4)$, $D_G^-(v_5) = D_G^+(v_5)$ and $D_G^-(v_6) = D_G^+(v_6)$. Hence G is regular bipolar neutrosophic directed hypergraph.

We now discuss the basic properties of isomorphism on bipolar neutrosophic directed hypergraphs.

Definition 2.14 Let $G = (\sigma, \varepsilon)$ and $G' = (\sigma', \varepsilon')$ be two bipolar neutrosophic directed hypergraphs, where $\sigma = \{\sigma_1, \sigma_2, \sigma_3, \ldots, \sigma_k\}$ and $\sigma' = \{\sigma'_1, \sigma'_2, \sigma'_3, \ldots, \sigma'_k\}$. A *homomorphism* of BNDHGs $\chi : G \to G'$ is a mapping $\chi : V \to V'$ which satisfies

1.
$$\wedge t^+_{\sigma_i}(u) \le \wedge t^+_{\sigma'_i}(\chi(u)), \wedge I^+_{\sigma_i}(u) \le \wedge I^+_{\sigma'_i}(\chi(u)), \vee f^+_{\sigma_i}(u) \ge \vee f^+_{\sigma'_i}(\chi(u)),$$
$$\vee t^-_{\sigma_i}(u) \ge \vee t^-_{\sigma'_i}(\chi(u)), \vee I^-_{\sigma_i}(u) \ge \vee I^-_{\sigma'_i}(\chi(u)), \wedge f^-_{\sigma_i}(u) \le \wedge f^-_{\sigma'_i}(\chi(u)),$$
for all $u \in V$.

2.
$$t^+_\varepsilon(u_1, u_2, u_3, \ldots, u_k) \le t^+_{\varepsilon'}(\chi(u_1), \chi(u_2), \chi(u_3), \ldots, \chi(u_k)),$$
$$I^+_\varepsilon(u_1, u_2, u_3, \ldots, u_k) \le I^+_{\varepsilon'}(\chi(u_1), \chi(u_2), \chi(u_3), \ldots, \chi(u_k)),$$
$$f^+_\varepsilon(u_1, u_2, u_3, \ldots, u_k) \ge f^+_{\varepsilon'}(\chi(u_1), \chi(u_2), \chi(u_3), \ldots, \chi(u_k)),$$
$$t^-_\varepsilon(u_1, u_2, u_3, \ldots, u_k) \ge t^-_{\varepsilon'}(\chi(u_1), \chi(u_2), \chi(u_3), \ldots, \chi(u_k)),$$
$$I^-_\varepsilon(u_1, u_2, u_3, \ldots, u_k) \ge I^-_{\varepsilon'}(\chi(u_1), \chi(u_2), \chi(u_3), \ldots, \chi(u_k)),$$
$$f^-_\varepsilon(u_1, u_2, u_3, \ldots, u_k) \le f^-_{\varepsilon'}(\chi(u_1), \chi(u_2), \chi(u_3), \ldots, \chi(u_k)),$$
for all $\{u_1, u_2, u_3, \ldots, u_k\} = E_i \subset V$.

Note that, for a homomorphism $\chi : G \to G'$, $\chi(\varepsilon) = (T(\chi(\varepsilon)), H(\chi(\varepsilon)))$ is an hyperarc in G' if $\varepsilon = (T(\varepsilon), H(\varepsilon))$ is an hyperarc in G.

Definition 2.15 A *weak isomorphism* $\chi : G \to G'$ is a mapping $\chi : V \to V'$ which is a bijective homomorphism and satisfies

$$\begin{aligned}
&\wedge t^+_{\sigma_i}(v) = \wedge t^+_{\sigma'_i}(\chi(v)), \wedge I^+_{\sigma_i}(v) = \wedge I^+_{\sigma'_i}(\chi(v)), \vee f^+_{\sigma_i}(v) = \vee f^+_{\sigma'_i}(\chi(v)),\\
&\vee t^-_{\sigma_i}(v) = \vee t^-_{\sigma'_i}(\chi(v)), \vee I^-_{\sigma_i}(v) = \vee I^-_{\sigma'_i}(\chi(v)), \wedge f^-_{\sigma_i}(v) = \wedge f^-_{\sigma'_i}(\chi(v)),
\end{aligned}$$

for all $v \in V$.

Definition 2.16 A *co-weak isomorphism* $\chi : G \to G'$ is a mapping $\chi : V \to V'$ which is a bijective homomorphism and satisfies

$$\begin{aligned}
t^+_{\varepsilon}(x_1, x_2, x_3, \ldots, x_k) &= t^+_{\varepsilon'}(\chi(x_1), \chi(x_2), \chi(x_3), \ldots, \chi(x_k)),\\
I^+_{\varepsilon}(x_1, x_2, x_3, \ldots, x_k) &= I^+_{\varepsilon'}(\chi(x_1), \chi(x_2), \chi(x_3), \ldots, \chi(x_k)),\\
f^+_{\varepsilon}(x_1, x_2, x_3, \ldots, x_k) &= f^+_{\varepsilon'}(\chi(x_1), \chi(x_2), \chi(x_3), \ldots, \chi(x_k)),\\
t^-_{\varepsilon}(x_1, x_2, x_3, \ldots, x_k) &= t^-_{\varepsilon'}(\chi(x_1), \chi(x_2), \chi(x_3), \ldots, \chi(x_k)),\\
I^-_{\varepsilon}(x_1, x_2, x_3, \ldots, x_k) &= I^-_{\varepsilon'}(\chi(x_1), \chi(x_2), \chi(x_3), \ldots, \chi(x_k)),\\
f^-_{\varepsilon}(x_1, x_2, x_3, \ldots, x_k) &= f^-_{\varepsilon'}(\chi(x_1), \chi(x_2), \chi(x_3), \ldots, \chi(x_k)),
\end{aligned}$$

for all $\{x_1, x_2, x_3, \ldots, x_k\} = E_i \subset V$.

Definition 2.17 An *isomorphism* of BNDHGs $\chi : G \to G'$ is a mapping $\chi : V \to V'$ which is bijective homomorphism and satisfies

1.

$$\begin{aligned}
&\wedge t^+_{\sigma_i}(u) = \wedge t^+_{\sigma'_i}(\chi(u)), \wedge I^+_{\sigma_i}(u) = \wedge I^+_{\sigma'_i}(\chi(u)), \vee f^+_{\sigma_i}(u) = \vee f^+_{\sigma'_i}(\chi(u)),\\
&\vee t^-_{\sigma_i}(u) = \vee t^-_{\sigma'_i}(\chi(u)), \vee I^-_{\sigma_i}(u) = \vee I^-_{\sigma'_i}(\chi(u)), \wedge f^-_{\sigma_i}(u) = \wedge f^-_{\sigma'_i}(\chi(u)),
\end{aligned}$$

   for all $u \in V$.

2.

$$\begin{aligned}
t^+_{\varepsilon}(u_1, u_2, u_3, \ldots, u_k) &= t^+_{\varepsilon'}(\chi(u_1), \chi(u_2), \chi(u_3), \ldots, \chi(u_k)),\\
I^+_{\varepsilon}(u_1, u_2, u_3, \ldots, u_k) &= I^+_{\varepsilon'}(\chi(u_1), \chi(u_2), \chi(u_3), \ldots, \chi(u_k)),\\
f^+_{\varepsilon}(u_1, u_2, u_3, \ldots, u_k) &= f^+_{\varepsilon'}(\chi(u_1), \chi(u_2), \chi(u_3), \ldots, \chi(u_k)),\\
t^-_{\varepsilon}(u_1, u_2, u_3, \ldots, u_k) &= t^-_{\varepsilon'}(\chi(u_1), \chi(u_2), \chi(u_3), \ldots, \chi(u_k)),\\
I^-_{\varepsilon}(u_1, u_2, u_3, \ldots, u_k) &= I^-_{\varepsilon'}(\chi(u_1), \chi(u_2), \chi(u_3), \ldots, \chi(u_k)),\\
f^-_{\varepsilon}(u_1, u_2, u_3, \ldots, u_k) &= f^-_{\varepsilon'}(\chi(u_1), \chi(u_2), \chi(u_3), \ldots, \chi(u_k)),
\end{aligned}$$

   for all $\{u_1, u_2, u_3, \ldots, u_k\} = E_i \subset V$.

If two bipolar neutrosophic directed hypergraphs G and G' are isomorphic, we denote it as $G \cong G'$.

Example 2.3 Let $\sigma = \{\sigma_1, \sigma_2, \sigma_3\}$ and $\sigma' = \{\sigma_1', \sigma_2', \sigma_3'\}$ be the families of bipolar neutrosophic subsets on $V = \{v_1, v_2, v_3, v_4, v_5, v_6\}$ and $V' = \{v_1', v_2', v_3', v_4', v_5', v_6'\}$, respectively, as:

$$\begin{aligned}
\sigma_1 = \{&(v_1, 0.1, 0.2, 0.3, -0.1, -0.2, -0.3), (v_2, 0.2, 0.2, 0.3, -0.2,\\
&-0.2, -0.3), (v_5, 0.3, 0.2, 0.3, -0.3, -0.2, -0.3),\\
&(v_6, 0.4, 0.2, 0.5, -0.4, -0.2, -0.5)\},\\
\sigma_2 = \{&(v_1, 0.1, 0.2, 0.3, -0.1, -0.2, -0.3), (v_3, 0.4, 0.2, 0.3, -0.4,\\
&-0.2, -0.3), (v_6, 0.3, 0.2, 0.3, -0.3, -0.2, -0.3)\},\\
\sigma_3 = \{&(v_6, 0.1, 0.2, 0.3, -0.1, -0.2, -0.3), (v_4, 0.2, 0.2, 0.3, -0.2,\\
&-0.2, -0.3), (v_5, 0.3, 0.2, 0.3, -0.3, -0.2, -0.3)\},
\end{aligned}$$

and

$$\begin{aligned}
\sigma_1' = \{&(v_4', 0.1, 0.2, 0.3, -0.1, -0.2, -0.3), (v_3', 0.2, 0.2, 0.3,\\
&-0.2, -0.2, -0.3), (v_6', 0.3, 0.2, 0.3, -0.3,\\
&-0.2, -0.3), (v_5', 0.4, 0.2, 0.5, -0.4, -0.2, -0.5)\},\\
\sigma_2' = \{&(v_4', 0.1, 0.2, 0.3, -0.1, -0.2, -0.3), (v_2', 0.4, 0.2, 0.3, -0.4,\\
&-0.2, -0.3), (v_5', 0.3, 0.2, 0.3, -0.3, -0.2, -0.3)\},\\
\sigma_3' = \{&(v_5', 0.1, 0.2, 0.3, -0.1, -0.2, -0.3), (v_1', 0.2, 0.2, 0.3, -0.2,\\
&-0.2, -0.3), (v_6', 0.3, 0.2, 0.3, -0.3, -0.2, -0.3)\}.
\end{aligned}$$

The bipolar neutrosophic relations ε and ε' are defined as

$$\begin{aligned}
\varepsilon(v_1, v_2, v_5, v_6) &= (0.1, 0.2, 0.5, -0.1, -0.2, -0.5),\\
\varepsilon(v_1, v_3, v_6) &= (0.1, 0.2, 0.3, -0.1, -0.2, -0.3),\\
\varepsilon(v_6, v_4, v_5) &= (0.1, 0.2, 0.3, -0.1, -0.2, -0.3),\\
\varepsilon'(v_4', v_3', v_6', v_5') &= (0.1, 0.2, 0.5, -0.1, -0.2, -0.5),\\
\varepsilon'(v_4', v_2', v_5') &= (0.1, 0.2, 0.3, -0.1, -0.2, -0.3),\\
\varepsilon'(v_5', v_1', v_6') &= (0.1, 0.2, 0.3, -0.1, -0.2, -0.3).
\end{aligned}$$

Define a mapping $\chi : V \to V'$ as $\chi(v_1) = v_4'$, $\chi(v_2) = v_3'$, $\chi(v_3) = v_2'$, $\chi(v_4) = v_1'$, $\chi(v_5) = v_6'$ and $\chi(v_6) = v_5'$. Note that,

$$\begin{aligned}
\sigma_1(v_1) &= (0.1, 0.2, 0.3, -0.1, -0.2, -0.3) = \sigma_1'(v_4') = \sigma_1'(\chi(v_1)),\\
\sigma_1(v_2) &= (0.2, 0.2, 0.3, -0.2, -0.2, -0.3) = \sigma_1'(v_3') = \sigma_1'(\chi(v_2)),\\
\sigma_2(v_3) &= (0.4, 0.2, 0.3, -0.4, -0.2, -0.3) = \sigma_2'(v_2') = \sigma_2'(\chi(v_3)),\\
\sigma_3(v_4) &= (0.2, 0.2, 0.3, -0.2, -0.2, -0.3) = \sigma_3'(v_1') = \sigma_3'(\chi(v_4)),\\
\sigma_3(v_5) &= (0.3, 0.2, 0.3, -0.3, -0.2, -0.3) = \sigma_3'(v_6') = \sigma_3'(\chi(v_5)).
\end{aligned}$$

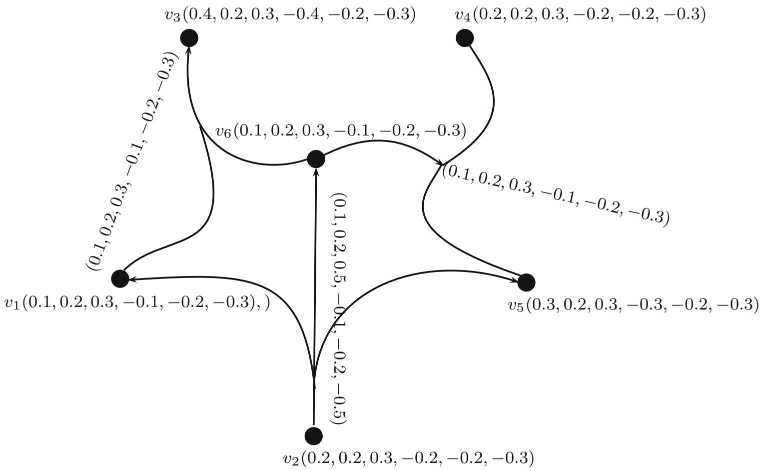

Fig. 3 G

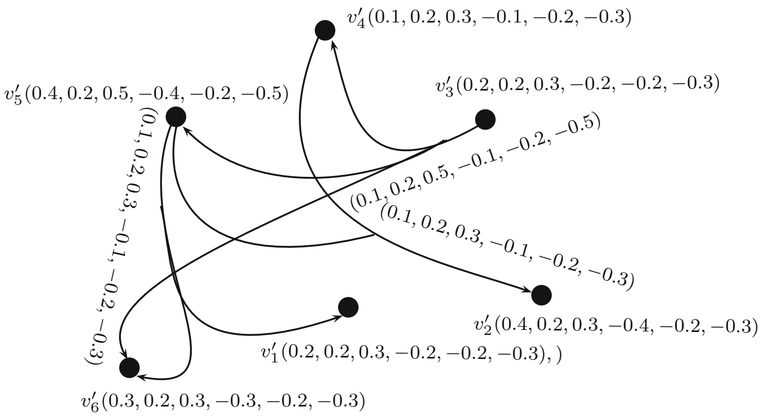

Fig. 4 G'

Similarly, $\sigma(v) = \sigma'(\chi(v))$, for all $v \in V$ and $\varepsilon(\{v_1, v_2, v_3, \dots, v_k\})= \varepsilon'(\{\chi(v_1), \chi(v_2), \chi(v_3), \dots, \chi(v_k)\})$, for all $v_k \in V$. Hence G and G' are isomorphic and the corresponding BNDHGs are shown in Figs. 3 and 4, respectively.

Note that, $\chi(\varepsilon) = (T(\chi(\varepsilon), H(\chi(\varepsilon)))$ is an hyperarc in G' if $\varepsilon = (T(\varepsilon), H(\varepsilon))$ is an hyperarc in G.

Remark 2.1 A weak isomorphism of bipolar neutrosophic directed hypergraphs preserves the membership degrees of vertices but not necessarily the membership degrees of directed hyperedges. A co-weak isomorphism of bipolar neutrosophic directed hypergraphs preserves the membership degrees of directed hyperedges but not necessarily the membership degrees of vertices.

In isomorphism of crisp hypergraphs, isomorphic hypergraphs have same degree as well as the order. The same also holds in bipolar neutrosophic directed hypergraphs.

Theorem 2.1 *Let G and G' be two isomorphic BNDHGs. Then they both have the same order and size.*

Proof Let $G = (\sigma, \varepsilon)$ and $G' = (\sigma', \varepsilon')$ be two bipolar neutrosophic directed hypergraphs, where $\sigma = \{\sigma_1, \sigma_2, \sigma_3, \cdots, \sigma_k\}$ and $\sigma' = \{\sigma_1', \sigma_2', \sigma_3', \ldots, \sigma_k'\}$ be the family of bipolar neutrosophic sets defined on V and V',respectively. Let $\chi : V \to V'$ be an isomorphism between G and G' then

$\wedge\sigma_i^+(z) = \wedge\sigma_i'^+(\chi(z))$, $\vee\sigma_i^-(z) = \vee\sigma_i'^-(\chi(z))$, for all $z \in V$ and

$\varepsilon(z_1, z_2, z_3, \ldots, z_k) = \varepsilon'(\chi(z_1), \chi(z_2), \chi(z_3), \ldots, \chi(z_k))$, for all $\{z_1, z_2, z_3, \ldots, z_k\} = E_i \subset V$.

$$\mathcal{O}^+(G) = \sum_{z \in V} \wedge\sigma_i^+(z) = \sum_{z \in V} \wedge\sigma_i'^+(\chi(z)) = \sum_{z' \in V'} \wedge\sigma_i'^+(z') = \mathcal{O}^+(G'),$$
$$\mathcal{O}^-(G) = \sum_{z \in V} \vee\sigma_i^-(z) = \sum_{z \in V} \vee\sigma_i'^-(\chi(z)) = \sum_{z' \in V'} \vee\sigma_i'^+(z') = \mathcal{O}^-(G'),$$
$$\mathcal{S}^+(G) = \sum_{E_k \subset V} \varepsilon^+(E_k) = \sum_{E_k \subset V} \varepsilon'^+(\chi(E_k)) = \sum_{E_k' \subset V'} \varepsilon'^+(E_k') = \mathcal{S}^+(G'),$$
$$\mathcal{S}^-(G) = \sum_{E_k \subset V} \varepsilon^-(E_k) = \sum_{E_k \subset V} \varepsilon'^-(\chi(E_k)) = \sum_{E_k' \subset V'} \varepsilon'^-(E_k') = \mathcal{S}^-(G').$$

This completes the proof.

Theorem 2.2 *Isomorphism between bipolar neutrosophic hypergraphs is an equivalence relation.*

Proof Let $G = (V, \sigma, \varepsilon)$, $G' = (V', \sigma', \varepsilon')$ and $G'' = (V'', \sigma'', \varepsilon'')$ be bipolar neutrosophic directed hypergraphs having underlying sets V, V' and V'', respectively.

i Reflexive: Consider the identity mapping $\chi : V \to V$, such that $\chi(z) = z$, for all $z \in V$. Then χ is bijective homomorphism and satisfies
$\wedge\sigma_i^+(z) = \wedge\sigma_i'^+(\chi(z))$, $\vee\sigma_i^-(z) = \vee\sigma_i'^-(\chi(z))$, for all $z \in V$ and

$$\varepsilon^+(z_1, z_2, z_3, \ldots, z_k) = \varepsilon'^+(\chi(z_1), \chi(z_2), \chi(z_3), \ldots, \chi(z_k)),$$
$$\varepsilon^-(z_1, z_2, z_3, \ldots, z_k) = \varepsilon'^-(\chi(z_1), \chi(z_2), \chi(z_3), \ldots, \chi(z_k)),$$

for all $\{z_1, z_2, z_3, \ldots, z_k\} = E_k \subset V$. Hence χ is an isomorphism of bipolar neutrosophic directed hypergraphs to itself. Thus reflexive relation is satisfied.

ii Symmetric: Let $\chi : V \to V'$ be an isomorphism between G and G', then χ is bijective mapping and $\chi(z) = z'$, for all $z \in V$. From the isomorphism of χ, we have $\wedge\sigma_i^+(z) = \wedge\sigma_i'^+(\chi(z))$, $\vee\sigma_i^-(z) = \vee\sigma_i'^-(\chi(z))$, for all $z \in V$ and

$$\varepsilon^+(z_1, z_2, z_3, \ldots, z_k) = \varepsilon'^+(\chi(z_1), \chi(z_2), \chi(z_3), \ldots, \chi(z_k)),$$
$$\varepsilon^-(z_1, z_2, z_3, \ldots, z_k) = \varepsilon'^-(\chi(z_1), \chi(z_2), \chi(z_3), \ldots, \chi(z_k)),$$

for all $\{z_1, z_2, z_3, \ldots, z_k\} = E_k \subset V$. Since χ is bijective, $\chi^{-1} : V' \to V$ exists and $\chi^{-1}(z') = z$, for all $z' \in V'$. Then $\wedge\sigma_i^{+}\chi^{-1}(z') = \wedge\sigma_i'^{+}(z')$, $\vee\sigma_i^{-}\chi^{-1}(z') = \vee\sigma_i'^{-}(z')$, for all $z' \in V'$ and

$$\varepsilon^{+}(\chi^{-1}(z_1'), \chi^{-1}(z_2'), \chi^{-1}(z_3'), \ldots, \chi^{-1}(z_k')) = \varepsilon'^{+}(z_1', z_2', z_3', \ldots, z_k'),$$
$$\varepsilon^{-}(\chi^{-1}(z_1'), \chi^{-1}(z_2'), \chi^{-1}(z_3'), \ldots, \chi^{-1}(z_k')) = \varepsilon'^{-}(z_1', z_2', z_3', \ldots, z_k'),$$

for all $\{z_1', z_2', z_3', \ldots, z_k'\} = E_k' \subset V'$. Hence we get a bijective mapping $\chi^{-1} : V' \to V$, which is isomorphism from G' to G, i.e., $G \cong G' \Rightarrow G' \cong G$.

iii Transitive: Let $\chi : V \to V'$ and Let $\lambda : V' \to V''$ be an isomorphism of bipolar neutrosophic directed hypergraphs G to G' and G' to G'', respectively, defined by $\chi(z) = z'$ and $\lambda(z') = z''$. Then $\lambda \circ \chi : V \to V''$ is a bijective mapping from G to G'' such that $(\lambda \circ \chi)(z) = \lambda(\chi(z))$, for all $z \in V$. Since $\chi : V \to V'$ is an isomorphism, we have $\wedge\sigma_i^{+}(z) = \wedge\sigma_i'^{+}(\chi(z))$, $\vee\sigma_i^{-}(z) = \vee\sigma_i'^{-}(\chi(z))$, for all $z \in V$ and

$$\varepsilon(z_1, z_2, z_3, \ldots, z_k) = \varepsilon'(\chi(z_1), \chi(z_2), \chi(z_3), \ldots, \chi(z_k)),$$

for all $\{z_1, z_2, z_3, \ldots, z_k\} = E_k \subset V$. Since $\lambda : V' \to V''$ is an isomorphism, we have $\wedge\sigma_i'^{+}(z') = \wedge\sigma_i''^{+}(\lambda(z'))$, $\vee\sigma_i'^{-}(z') = \vee\sigma_i''^{-}(\lambda(z'))$, for all $z' \in V'$ and

$$\varepsilon'(z_1', z_2', z_3', \ldots, z_k') = \varepsilon''(\lambda(z_1'), \lambda(z_2'), \lambda(z_3'), \ldots, \lambda(z_k')),$$

for all $\{z_1', z_2', z_3', \ldots, z_k'\} = E_k' \subset V'$. Thus we have

$$\wedge\sigma_i^{+}(z) = \wedge\sigma_i'^{+}(\chi(z)) = \wedge\sigma_i'^{+}(z') = \wedge\sigma_i''^{+}(\lambda(z')) = \wedge\sigma_i''^{+}(\lambda(\chi(z))),$$

for all $z \in V$, $z' \in V'$, $z'' \in V''$,

$$\begin{aligned}
\varepsilon(z_1, z_2, z_3, \ldots, z_k) &= \varepsilon'(\chi(z_1), \chi(z_2), \chi(z_3), \ldots, \chi(z_k)) \\
&= \varepsilon'(z_1', z_2', z_3', \ldots, z_k') \\
&= \varepsilon''(\lambda(z_1'), \lambda(z_2'), \lambda(z_3'), \ldots, \lambda(z_k')) \\
&= \varepsilon''(\lambda(\chi(z_1)), \lambda(\chi(z_2)), \lambda(\chi(z_3)), \ldots, \lambda(\chi(z_k))),
\end{aligned}$$

for all $\{z_1, z_2, z_3, \ldots, z_k\} = E_k \subset V$, for all $\{z_1', z_2', z_3', \ldots, z_k'\} = E_k' \subset V'$.

Clearly, $\lambda \circ \chi$ is an isomorphism from G to G''. Hence isomorphism of bipolar neutrosophic directed hypergraphs is an equivalence relation.

Theorem 2.3 *A weak isomorphism between bipolar neutrosophic directed hypergraphs is a partial order relation.*

Proof Let $G = (V, \sigma, \varepsilon)$, $G' = (V', \sigma', \varepsilon')$ and $G'' = (V'', \sigma'', \varepsilon'')$ be bipolar neutrosophic directed hypergraphs having underlying sets V, V' and V'', respectively.

i Reflexive: Consider the identity mapping $\chi : V \to V$, such that $\chi(z) = z$, for all $z \in V$. Then χ is bijective homomorphism and satisfies
$\wedge\sigma_i^+(z) = \wedge\sigma_i'^+(\chi(z))$, $\vee\sigma_i^-(z) = \vee\sigma_i'^-(\chi(z))$, for all $z \in V$ and

$$\begin{aligned}
t_\varepsilon^+(z_1, z_2, z_3, \ldots, z_k) &\leq t_{\varepsilon'}^+(\chi(z_1), \chi(z_2), \chi(z_3), \ldots, \chi(z_k)),\\
I_\varepsilon^+(z_1, z_2, z_3, \ldots, z_k) &\leq I_{\varepsilon'}^+(\chi(z_1), \chi(z_2), \chi(z_3), \ldots, \chi(z_k)),\\
f_\varepsilon^+(z_1, z_2, z_3, \ldots, z_k) &\geq f_{\varepsilon'}^+(\chi(z_1), \chi(z_2), \chi(z_3), \ldots, \chi(z_k)),\\
t_\varepsilon^-(z_1, z_2, z_3, \ldots, z_k) &\geq t_{\varepsilon'}^-(\chi(z_1), \chi(z_2), \chi(z_3), \ldots, \chi(z_k)),\\
I_\varepsilon^-(z_1, z_2, z_3, \ldots, z_k) &\geq I_{\varepsilon'}^-(\chi(z_1), \chi(z_2), \chi(z_3), \ldots, \chi(z_k)),\\
f_\varepsilon^-(z_1, z_2, z_3, \ldots, z_k) &\leq f_{\varepsilon'}^-(\chi(z_1), \chi(z_2), \chi(z_3), \ldots, \chi(z_k)).
\end{aligned}$$

Hence χ is a weak isomorphism of bipolar neutrosophic directed hypergraphs to itself. Thus reflexive relation is satisfied.

ii Anti symmetric: Let $\chi : V \to V'$ be a weak isomorphism between G and G' and $\lambda : V' \to V$ be a weak isomorphism between G' and G. Then χ is a bijective mapping $\chi(z) = z'$, satisfying
$\wedge\sigma_i^+(z) = \wedge\sigma_i'^+(\chi(z))$, $\vee\sigma_i^-(z) = \vee\sigma_i'^-(\chi(z))$, for all $z \in V$ and

$$\begin{aligned}
t_\varepsilon^+(z_1, z_2, z_3, \ldots, z_k) &\leq t_{\varepsilon'}^+(\chi(z_1), \chi(z_2), \chi(z_3), \ldots, \chi(z_k)),\\
I_\varepsilon^+(z_1, z_2, z_3, \ldots, z_k) &\leq I_{\varepsilon'}^+(\chi(z_1), \chi(z_2), \chi(z_3), \ldots, \chi(z_k)),\\
f_\varepsilon^+(z_1, z_2, z_3, \ldots, z_k) &\geq f_{\varepsilon'}^+(\chi(z_1), \chi(z_2), \chi(z_3), \ldots, \chi(z_k)),\\
t_\varepsilon^-(z_1, z_2, z_3, \ldots, z_k) &\geq t_{\varepsilon'}^-(\chi(z_1), \chi(z_2), \chi(z_3), \ldots, \chi(z_k)),\\
I_\varepsilon^-(z_1, z_2, z_3, \ldots, z_k) &\geq I_{\varepsilon'}^-(\chi(z_1), \chi(z_2), \chi(z_3), \ldots, \chi(z_k)),\\
f_\varepsilon^-(z_1, z_2, z_3, \ldots, z_k) &\leq f_{\varepsilon'}^-(\chi(z_1), \chi(z_2), \chi(z_3), \ldots, \chi(z_k)),
\end{aligned} \tag{1}$$

for all $\{z_1, z_2, z_3, \ldots, z_k\} = E_i \subset V$ and λ is a bijective mapping $\lambda(z') = z$, satisfying $\wedge\sigma_i^+(z') = \wedge\sigma_i'^+(\lambda(z'))$, $\vee\sigma_i^-(z') = \vee\sigma_i'^-(\lambda(z'))$, for all $z' \in V'$ and

$$\begin{aligned}
t_\varepsilon^+(z_1', z_2', z_3', \ldots, z_k') &\leq t_{\varepsilon'}^+(\lambda(z_1'), \lambda(z_2'), \lambda(z_3'), \ldots, \lambda(z_k')),\\
I_\varepsilon^+(z_1', z_2', z_3', \ldots, z_k') &\leq I_{\varepsilon'}^+(\lambda(z_1'), \lambda(z_2'), \lambda(z_3'), \ldots, \lambda(z_k')),\\
f_\varepsilon^+(z_1', z_2', z_3', \ldots, z_k') &\geq f_{\varepsilon'}^+(\lambda(z_1'), \lambda(z_2'), \lambda(z_3'), \ldots, \lambda(z_k')),\\
t_\varepsilon^-(z_1', z_2', z_3', \ldots, z_k') &\geq t_{\varepsilon'}^-(\lambda(z_1'), \lambda(z_2'), \lambda(z_3'), \ldots, \lambda(z_k')),\\
I_\varepsilon^-(z_1', z_2', z_3', \ldots, z_k') &\geq I_{\varepsilon'}^-(\lambda(z_1'), \lambda(z_2'), \lambda(z_3'), \ldots, \lambda(z_k')),\\
f_\varepsilon^-(z_1', z_2', z_3', \ldots, z_k') &\leq f_{\varepsilon'}^-(\lambda(z_1'), \lambda(z_2'), \lambda(z_3'), \ldots, \lambda(z_k')),
\end{aligned} \tag{2}$$

for all $\{z_1', z_2', z_3', \ldots, z_k'\} = E_i' \subset V'$. The inequalities (1) and (2) hold true only if G and G' contain the same directed hyperedges having same membership degrees. Hence G and G' are equivalent.

iii Transitive: Let $\chi : V \to V'$ and let $\lambda : V' \to V''$ be weak isomorphism of bipolar neutrosophic directed hypergraphs G to G' and G' to G'', respectively, defined by

$\chi(z) = z'$ and $\lambda(z') = z''$. Then $\lambda \circ \chi : V \to V''$ is a bijective mapping from G to G'' such that $(\lambda \circ \chi)(z) = \lambda(\chi(z))$, for all $z \in V$. Since $\chi : V \to V'$ is a weak isomorphism, we have
$\wedge\sigma_i^+(z) = \wedge\sigma_i'^+(\chi(z))$, $\vee\sigma_i^-(z) = \vee\sigma_i'^-(\chi(z))$, for all $z \in V$ and

$$\begin{aligned}
t_\varepsilon^+(z_1, z_2, z_3, \ldots, z_k) &\le t_{\varepsilon'}^+(\chi(z_1), \chi(z_2), \chi(z_3), \ldots, \chi(z_k)),\\
I_\varepsilon^+(z_1, z_2, z_3, \ldots, z_k) &\le I_{\varepsilon'}^+(\chi(z_1), \chi(z_2), \chi(z_3), \ldots, \chi(z_k)),\\
f_\varepsilon^+(z_1, z_2, z_3, \ldots, z_k) &\ge f_{\varepsilon'}^+(\chi(z_1), \chi(z_2), \chi(z_3), \ldots, \chi(z_k)),\\
t_\varepsilon^-(z_1, z_2, z_3, \ldots, z_k) &\ge t_{\varepsilon'}^-(\chi(z_1), \chi(z_2), \chi(z_3), \ldots, \chi(z_k)),\\
I_\varepsilon^-(z_1, z_2, z_3, \ldots, z_k) &\ge I_{\varepsilon'}^-(\chi(z_1), \chi(z_2), \chi(z_3), \ldots, \chi(z_k)),\\
f_\varepsilon^-(z_1, z_2, z_3, \ldots, z_k) &\le f_{\varepsilon'}^-(\chi(z_1), \chi(z_2), \chi(z_3), \ldots, \chi(z_k)),
\end{aligned}$$

for all $\{z_1, z_2, z_3, \ldots, z_k\} = E_k \subset V$. Similarly λ is a weak isomorphism, we have $\wedge\sigma_i'^+(z') = \wedge\sigma_i''^+(\lambda(z'))$, $\vee\sigma_i'^-(z') = \vee\sigma_i''^-(\lambda(z'))$, for all $z' \in V'$ and

$$\begin{aligned}
t_{\varepsilon'}^+(z_1', z_2', z_3', \ldots, z_k') &\le t_{\varepsilon''}^+(\lambda(z_1'), \lambda(z_2'), \lambda(z_3'), \ldots, \lambda(z_k')),\\
I_{\varepsilon'}^+(z_1', z_2', z_3', \ldots, z_k') &\le I_{\varepsilon''}^+(\lambda(z_1'), \lambda(z_2'), \lambda(z_3'), \ldots, \lambda(z_k')),\\
f_{\varepsilon'}^+(z_1', z_2', z_3', \ldots, z_k') &\ge f_{\varepsilon''}^+(\lambda(z_1'), \lambda(z_2'), \lambda(z_3'), \ldots, \lambda(z_k')),\\
t_{\varepsilon'}^-(z_1', z_2', z_3', \ldots, z_k') &\ge t_{\varepsilon''}^-(\lambda(z_1'), \lambda(z_2'), \lambda(z_3'), \ldots, \lambda(z_k')),\\
I_{\varepsilon'}^-(z_1', z_2', z_3', \ldots, z_k') &\ge I_{\varepsilon''}^-(\lambda(z_1'), \lambda(z_2'), \lambda(z_3'), \ldots, \lambda(z_k')),\\
f_{\varepsilon'}^-(z_1', z_2', z_3', \ldots, z_k') &\le f_{\varepsilon''}^-(\lambda(z_1'), \lambda(z_2'), \lambda(z_3'), \ldots, \lambda(z_k')),
\end{aligned}$$

for all $\{z_1', z_2', z_3', \ldots, z_k'\} = E_k' \subset V'$. From the above conditions, we have
$\wedge\sigma_i^+(z) = \wedge\sigma_i'^+(\chi(z)) = \wedge\sigma_i'^+(z') = \wedge\sigma_i''^+(\lambda(z')) = \wedge\sigma_i''^+(\lambda(\chi(z)))$,
$\vee\sigma_i^-(z) = \vee\sigma_i'^-(\chi(z)) = \vee\sigma_i'^-(z') = \vee\sigma_i'^-(z') = \vee\sigma_i''^-(\lambda(\chi(z)))$, for all $z \in V$ and

$$\begin{aligned}
t_\varepsilon^+(z_1, z_2, z_3, \ldots, z_k) &\le t_{\varepsilon'}^+(\chi(z_1), \chi(z_2), \chi(z_3), \ldots, \chi(z_k))\\
&= t_{\varepsilon'}^+(z_1', z_2', z_3', \ldots, z_k')\\
&\le t_{\varepsilon''}^+(\lambda(z_1'), \lambda(z_2'), \lambda(z_3'), \ldots, \lambda(z_k'))\\
&= t_{\varepsilon''}^+(\lambda(\chi(z_1)), \lambda(\chi(z_2)), \lambda(\chi(z_3)), \ldots, \lambda(\chi(z_k))),\\
I_\varepsilon^+(z_1, z_2, z_3, \ldots, z_k) &\le I_{\varepsilon'}^+(\chi(z_1), \chi(z_2), \chi(z_3), \ldots, \chi(z_k))\\
&= I_{\varepsilon'}^+(z_1', z_2', z_3', \ldots, z_k')\\
&\le I_{\varepsilon''}^+(\lambda(z_1'), \lambda(z_2'), \lambda(z_3'), \ldots, \lambda(z_k'))\\
&= I_{\varepsilon''}^+(\lambda(\chi(z_1)), \lambda(\chi(z_2)), \lambda(\chi(z_3)), \ldots, \lambda(\chi(z_k))),\\
f_\varepsilon^+(z_1, z_2, z_3, \ldots, z_k) &\ge f_{\varepsilon'}^+(\chi(z_1), \chi(z_2), \chi(z_3), \ldots, \chi(z_k))\\
&= f_{\varepsilon'}^+(z_1', z_2', z_3', \ldots, z_k')
\end{aligned}$$

$$
\begin{aligned}
&\geq f^{+}_{\varepsilon''}(\lambda(z'_1), \lambda(z'_2), \lambda(z'_3), \ldots, \lambda(z'_k)) \\
&= f^{+}_{\varepsilon''}(\lambda(\chi(z_1)), \lambda(\chi(z_2)), \lambda(\chi(z_3)), \ldots, \lambda(\chi(z_k))).
\end{aligned}
$$

Similarly, we have

$$
\begin{aligned}
t^{-}_{\varepsilon}(z_1, z_2, z_3, \ldots, z_k) &\geq t^{-}_{\varepsilon'}(\chi(z_1), \chi(z_2), \chi(z_3), \ldots, \chi(z_k)) \\
&= t^{-}_{\varepsilon'}(z'_1, z'_2, z'_3, \ldots, z'_k) \\
&\geq t^{-}_{\varepsilon''}(\lambda(z'_1), \lambda(z'_2), \lambda(z'_3), \ldots, \lambda(z'_k)) \\
&= t^{-}_{\varepsilon''}(\lambda(\chi(z_1)), \lambda(\chi(z_2)), \lambda(\chi(z_3)), \ldots, \lambda(\chi(z_k))), \\
I^{-}_{\varepsilon}(z_1, z_2, z_3, \ldots, z_k) &\geq I^{-}_{\varepsilon'}(\chi(z_1), \chi(z_2), \chi(z_3), \ldots, \chi(z_k)) \\
&= I^{-}_{\varepsilon'}(z'_1, z'_2, z'_3, \ldots, z'_k) \\
&\geq I^{-}_{\varepsilon''}(\lambda(z'_1), \lambda(z'_2), \lambda(z'_3), \ldots, \lambda(z'_k)) \\
&= I^{-}_{\varepsilon''}(\lambda(\chi(z_1)), \lambda(\chi(z_2)), \lambda(\chi(z_3)), \ldots, \lambda(\chi(z_k))), \\
f^{-}_{\varepsilon}(z_1, z_2, z_3, \ldots, z_k) &\leq f^{-}_{\varepsilon'}(\chi(z_1), \chi(z_2), \chi(z_3), \ldots, \chi(z_k)) \\
&= f^{-}_{\varepsilon'}(z'_1, z'_2, z'_3, \ldots, z'_k) \\
&\leq f^{-}_{\varepsilon''}(\lambda(z'_1), \lambda(z'_2), \lambda(z'_3), \ldots, \lambda(z'_k)) \\
&= f^{-}_{\varepsilon''}(\lambda(\chi(z_1)), \lambda(\chi(z_2)), \lambda(\chi(z_3)), \ldots, \lambda(\chi(z_k))).
\end{aligned}
$$

for all $\{z_1, z_2, z_3, \ldots, z_k\} = E_k \subset V$, for all $\{z'_1, z'_2, z'_3, \ldots, z'_k\} = E'_k \subset V'$.

Clearly, $\lambda \circ \chi$ is a weak isomorphism from G to G''. Hence weak isomorphism of bipolar neutrosophic directed hypergraphs is a partial order relation.

Remark 2.2 If G and G' are isomorphic bipolar neutrosophic directed hypergraphs, then their vertices preserve degrees but the converse is not true, that is, if degrees are preserved then BNDHGs may or may not be isomorphic.

To check whether the two BNDHGs are isomorphic or not, it is mandatary that they have same number of vertices having same degrees and same number of directed hyperedges.

Remark 2.3
- If two bipolar neutrosophic directed hypergraphs are weak isomorphic then they have same orders but converse may or may not be true.
- If two bipolar neutrosophic directed hypergraphs are co-weak isomorphic then they are of same size but the same size of BNDHGs does not imply to the co-weak isomorphism.
- Any two isomorphic BNDHGs have same order and size but the converse may or may not be true.

Definition 2.18 A *bipolar neutrosophic directed hyperpath* of length k in a bipolar neutrosophic directed hypergraph is defined as a sequence $x_1, \mathrm{E}_1, x_2, \mathrm{E}_2, \ldots, \mathrm{E}_k, x_{k+1}$ of distinct vertices and directed hyperedges such that

1. $\varepsilon(\mathrm{E}_i) > 0, i = 1, 2, 3, \ldots, k,$
2. $x_i, x_{i+1} \in E_i$.

The consecutive pairs (x_i, x_{i+1}) are called the directed arcs of the directed hyperpath.

Definition 2.19 Let s and t be any two arbitrary vertices in a bipolar neutrosophic directed hypergraph and they are connected through a directed hyperpath of length k then the *strength* of that directed hyperpath is $\eta_k(s, t) = (\eta_k^+(s, t), \eta_k^-(s, t))$, where the *positive strength* is defined as

$$\eta_k^+(s,t) = \big\{t_\varepsilon^+(\mathrm{E}_1) \wedge t_\varepsilon^+(\mathrm{E}_2) \wedge t_\varepsilon^+(\mathrm{E}_3) \wedge \cdots \wedge t_\varepsilon^+(\mathrm{E}_k), I_\varepsilon^+(\mathrm{E}_1) \wedge I_\varepsilon^+(\mathrm{E}_2) \wedge I_\varepsilon^+(\mathrm{E}_3) \wedge \cdots \wedge I_\varepsilon^+(\mathrm{E}_k), f_\varepsilon^+(\mathrm{E}_1) \vee f_\varepsilon^+(\mathrm{E}_2) \vee t_\varepsilon^+(\mathrm{E}_3) \vee \cdots \vee t_\varepsilon^+(\mathrm{E}_k)\big\}$$

and the *negative strength* is defined as

$$\eta_k^-(s,t) = \{t_\varepsilon^-(\mathrm{E}_1) \vee t_\varepsilon^-(\mathrm{E}_2) \vee t_\varepsilon^-(\mathrm{E}_3) \vee \cdots \vee t_\varepsilon^-(\mathrm{E}_k), I_\varepsilon^-(\mathrm{E}_1) \vee I_\varepsilon^-(\mathrm{E}_2) \vee I_\varepsilon^-(\mathrm{E}_3) \vee \cdots \vee I_\varepsilon^-(\mathrm{E}_k), f_\varepsilon^-(\mathrm{E}_1) \wedge f_\varepsilon^-(\mathrm{E}_2) \wedge t_\varepsilon^-(\mathrm{E}_3) \wedge \cdots \wedge t_\varepsilon^-(\mathrm{E}_k)\}, x \in \mathrm{E}_1, y \in \mathrm{E}_k,$$

where $\mathrm{E}_1, \mathrm{E}_2, \mathrm{E}_3, \ldots, \mathrm{E}_k$ are directed hyperedges.

The *strength of connectedness* between x and y is defined as

$$\eta^\infty(s,t) = \{\sup t(\eta_k^+(s,t)), \sup I(\eta_k^+(s,t)), \inf f(\eta_k^+(s,t)), \inf t(\eta_k^-(s,t)), \inf I(\eta_k^-(s,t)), \sup f(\eta_k^-(s,t)) | k = 1, 2, 3, \ldots\}.$$

Definition 2.20 A *strong* arc in a bipolar neutrosophic directed hypergraph is defined as $\eta(s, t) \geq \eta^\infty(s, t)$.

Definition 2.21 A bipolar neutrosophic directed hypergraph is said to be connected if $\eta^\infty(s, t) > 0$, for all $s, t \in V$, that is, there exists a bipolar neutrosophic directed hyperpath between each pair of vertices.

Definition 2.22 A *strong* or *effective* bipolar neutrosophic directed hypergraph is defined as

$$\begin{aligned}
t_\varepsilon^+(E_k) &= t_\varepsilon^+(\{x_1, x_2, x_3, \ldots, x_m\}) = \min\{t_{\sigma_i}^+(x_1), t_{\sigma_i}^+(x_2), \ldots, t_{\sigma_i}^+(x_m)\},\\
I_\varepsilon^+(E_k) &= I_\varepsilon^+(\{x_1, x_2, x_3, \ldots, x_m\}) = \min\{I_{\sigma_i}^+(x_1), I_{\sigma_i}^+(x_2), \ldots, I_{\sigma_i}^+(x_m)\},\\
f_\varepsilon^+(E_k) &= f_\varepsilon^+(\{x_1, x_2, x_3, \ldots, x_m\}) = \max\{f_{\sigma_i}^+(x_1), f_{\sigma_i}^+(x_2), \ldots, f_{\sigma_i}^+(x_m)\},\\
t_\varepsilon^-(E_k) &= t_\varepsilon^-(\{x_1, x_2, x_3, \ldots, x_m\}) = \max\{t_{\sigma_i}^-(x_1), t_{\sigma_i}^-(x_2), \ldots, t_{\sigma_i}^-(x_m)\},\\
I_\varepsilon^-(E_k) &= I_\varepsilon^-(\{x_1, x_2, x_3, \ldots, x_m\}) = \max\{I_{\sigma_i}^-(x_1), I_{\sigma_i}^-(x_2), \ldots, I_{\sigma_i}^-(x_m)\},\\
f_\varepsilon^-(E_k) &= f_\varepsilon^-(\{x_1, x_2, x_3, \ldots, x_m\}) = \min\{f_{\sigma_i}^-(x_1), f_{\sigma_i}^-(x_2), \ldots, f_{\sigma_i}^-(x_m)\},
\end{aligned}$$

for all $\{x_1, x_2, x_3, \ldots, x_m\} = E_k \subset V$.

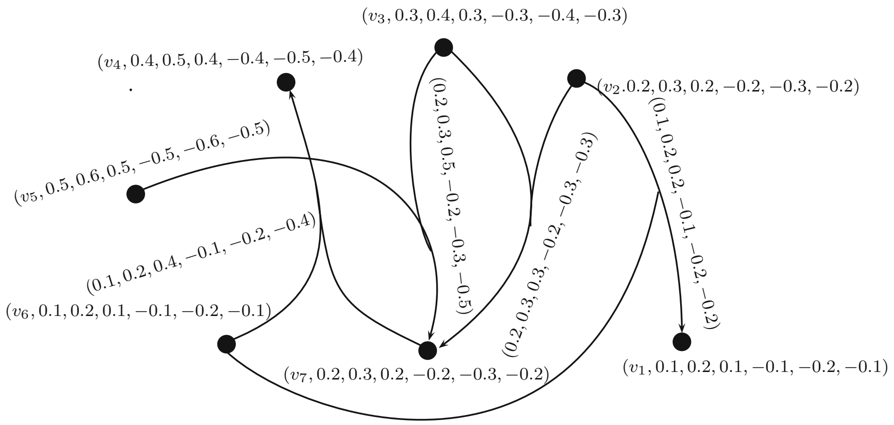

Fig. 5 Strong bipolar neutrosophic directed hypergraph

Example 2.4 Consider a bipolar neutrosophic directed hypergraph $G = (\sigma, \varepsilon)$, as shown in Fig. 5.

Note that,

$$\begin{aligned}
t_\varepsilon^+(E_1) &= t_\varepsilon^+(\{v_1, v_2, v_6\}) = \min\{t_{\sigma_i}^+(v_1), t_{\sigma_i}^+(v_2), t_{\sigma_i}^+(x_6)\},\\
I_\varepsilon^+(E_1) &= I_\varepsilon^+(\{v_1, v_2, v_6\}) = \min\{I_{\sigma_i}^+(v_1), I_{\sigma_i}^+(v_2), I_{\sigma_i}^+(x_6)\},\\
f_\varepsilon^+(E_1) &= f_\varepsilon^+(\{v_1, v_2, v_6\}) = \max\{f_{\sigma_i}^+(v_1), f_{\sigma_i}^+(v_2), f_{\sigma_i}^+(x_6)\},\\
t_\varepsilon^-(E_1) &= t_\varepsilon^-(\{v_1, v_2, v_6\}) = \max\{t_{\sigma_i}^-(v_1), t_{\sigma_i}^-(v_2), t_{\sigma_i}^-(x_6)\},\\
I_\varepsilon^-(E_1) &= I_\varepsilon^-(\{v_1, v_2, v_6\}) = \max\{I_{\sigma_i}^-(v_1), I_{\sigma_i}^-(v_2), I_{\sigma_i}^-(x_6)\},\\
f_\varepsilon^-(E_1) &= f_\varepsilon^-(\{v_1, v_2, v_6\}) = \min\{f_{\sigma_i}^-(v_1), f_{\sigma_i}^-(v_2), f_{\sigma_i}^-(x_6)\}.
\end{aligned}$$

Similarly, for all $\{x_1, x_2, x_3, \dots, x_k\} = E_k \subset V$, we have

$$\begin{aligned}
t_\varepsilon^+(E_k) &= t_\varepsilon^+(\{v_1, v_2, \dots, v_k\}) = \min\{t_{\sigma_i}^+(v_1), t_{\sigma_i}^+(v_2), \dots, t_{\sigma_i}^+(x_k)\},\\
I_\varepsilon^+(E_k) &= I_\varepsilon^+(\{v_1, v_2, \dots, v_k\}) = \min\{I_{\sigma_i}^+(v_1), I_{\sigma_i}^+(v_2), \dots, t_{\sigma_i}^+(x_k)\},\\
f_\varepsilon^+(E_k) &= f_\varepsilon^+(\{v_1, v_2, \dots, v_k\}) = \max\{f_{\sigma_i}^+(v_1), f_{\sigma_i}^+(v_2), \dots, t_{\sigma_i}^+(x_k)\},\\
t_\varepsilon^-(E_k) &= t_\varepsilon^-(\{v_1, v_2, \dots, v_k\}) = \max\{t_{\sigma_i}^-(v_1), t_{\sigma_i}^-(v_2), \dots, t_{\sigma_i}^+(x_k)\},\\
I_\varepsilon^-(E_k) &= I_\varepsilon^-(\{v_1, v_2, \dots, v_k\}) = \max\{I_{\sigma_i}^-(v_1), I_{\sigma_i}^-(v_2), \dots, t_{\sigma_i}^+(x_k)\},\\
f_\varepsilon^-(E_k) &= f_\varepsilon^-(\{v_1, v_2, \dots, v_k\}) = \min\{f_{\sigma_i}^-(v_1), f_{\sigma_i}^-(v_2), \dots, t_{\sigma_i}^+(x_k)\}.
\end{aligned}$$

Hence, G is strong.

Theorem 2.4 *Let G and G' be isomorphic bipolar neutrosophic directed hypergraphs, then G is connected if and only if G' is connected.*

Proof Let $G = (V, \sigma, \varepsilon)$ and $G' = (V', \sigma', \varepsilon')$ be two bipolar neutrosophic directed hypergraphs, where $\varepsilon = \{\varepsilon_1, \varepsilon_2, \varepsilon_3, \dots, \varepsilon_k\}$ and $\varepsilon' = \{\varepsilon'_1, \varepsilon'_2, \varepsilon'_3, \dots, \varepsilon'_k\}$ are directed hyperedges of G and G'. Let $\chi : G \to G'$ be an isomorphism between G and G'. Suppose that G is connected such that

$$\begin{aligned}
0 < \eta^{\infty}(s,t) &= \{\sup t(\eta_k^{+}(s,t)), \sup I(\eta_k^{+}(s,t)), \inf f(\eta_k^{+}(s,t)), \inf t(\eta_k^{-}(s,t)),\\
&\quad \inf \quad I(\eta_k^{-}(s,t)), \sup f(\eta_k^{-}(s,t)) | k = 1,2,3,\ldots\}\\
&= \{\sup \wedge_{i=1}^{k} t_{\varepsilon}^{+}(E_i), \sup \wedge_{i=1}^{k} I_{\varepsilon}^{+}(E_i), \inf \vee_{i=1}^{k} f_{\varepsilon}^{+}(E_i),\\
&\quad \inf \quad \vee_{i=1}^{k} t_{\varepsilon}^{-}(E_i), \inf \vee_{i=1}^{k} I_{\varepsilon}^{-}(E_i), \sup \wedge_{i=1}^{k} f_{\varepsilon}^{-}(E_i) | k=1,2,3,\ldots\}\\
&= \{\sup \wedge_{i=1}^{k} t_{\varepsilon'}^{+}(\chi(E_i)), \sup \wedge_{i=1}^{k} I_{\varepsilon'}^{+}(\chi(E_i)), \inf \vee_{i=1}^{k} f_{\varepsilon'}^{+}(\chi(E_i)),\\
&\quad \inf \quad \vee_{i=1}^{k} t_{\varepsilon'}^{-}(\chi(E_i)), \inf \vee_{i=1}^{k} I_{\varepsilon'}^{-}(\chi(E_i)),\\
&\quad \sup \wedge_{i=1}^{k} f_{\varepsilon'}^{-}(\chi(E_i)) | k = 1,2,3,\ldots\}\\
&= \{\sup t(\eta_k'^{+}(\chi(s),\chi(t))), \sup I(\eta_k'^{+}(\chi(s),\chi(t))),\\
&\quad \inf f(\eta_k'^{+}(\chi(s),\chi(t))), \inf \quad t(\eta_k'^{-}(\chi(s),\chi(t))),\\
&\quad \inf I(\eta_k'^{-}(\chi(s),\chi(t))), \sup f(\eta_k'^{-}(\chi(s),\chi(t))) | k = 1,2,3,\ldots\}\\
&= \eta'^{\infty}(\chi(s),\chi(t)) > 0
\end{aligned}$$

Hence G' is connected. The converse part can be proved by following the same procedure.

Theorem 2.5 *Let $G = (V, \sigma, \varepsilon)$ and $G' = (V', \sigma', \varepsilon')$ be two isomorphic bipolar neutrosophic directed hypergraphs. The arcs in G are strong if and only if their image arcs in G' are strong.*

Proof Let (s,t) be a strong arc in G such that $\eta(s,t) \geq \eta^{\infty}(s,t)$. Since G and G' are isomorphic, then there is a bijective mapping $\chi : G \to G'$ such that

$$\eta^{\infty}(s,t) \leq \eta(s,t) = \eta'(\chi(s),\chi(t)) \Rightarrow \eta'(\chi(s),\chi(t)) \geq \eta^{\infty}(s,t) = \eta'^{\infty}(\chi(s),\chi(t)),$$

which implies that $(\chi(s),\chi(t))$ is a strong arc in G'.

Converse part is trivial.

Theorem 2.6 *Let G be a strong connected bipolar neutrosophic directed hypergraph, then every arc of G is a strong arc.*

Proof Let G be a strong strong bipolar neutrosophic directed hypergraph such that

$$\begin{aligned}
t_{\varepsilon}^{+}(E_k) &= t_{\varepsilon}^{+}(\{x_1,x_2,x_3,\ldots,x_m\}) = \min\{t_{\sigma_i}^{+}(x_1), t_{\sigma_i}^{+}(x_2),\ldots,t_{\sigma_i}^{+}(x_m)\},\\
I_{\varepsilon}^{+}(E_k) &= I_{\varepsilon}^{+}(\{x_1,x_2,x_3,\ldots,x_m\}) = \min\{I_{\sigma_i}^{+}(x_1), I_{\sigma_i}^{+}(x_2),\ldots,I_{\sigma_i}^{+}(x_m)\},\\
f_{\varepsilon}^{+}(E_k) &= f_{\varepsilon}^{+}(\{x_1,x_2,x_3,\ldots,x_m\}) = \max\{f_{\sigma_i}^{+}(x_1), f_{\sigma_i}^{+}(x_2),\ldots,f_{\sigma_i}^{+}(x_m)\},\\
t_{\varepsilon}^{-}(E_k) &= t_{\varepsilon}^{-}(\{x_1,x_2,x_3,\ldots,x_m\}) = \max\{t_{\sigma_i}^{-}(x_1), t_{\sigma_i}^{-}(x_2),\ldots,t_{\sigma_i}^{-}(x_m)\},\\
I_{\varepsilon}^{-}(E_k) &= I_{\varepsilon}^{-}(\{x_1,x_2,x_3,\ldots,x_m\}) = \max\{I_{\sigma_i}^{-}(x_1), I_{\sigma_i}^{-}(x_2),\ldots,I_{\sigma_i}^{-}(x_m)\},\\
f_{\varepsilon}^{-}(E_k) &= f_{\varepsilon}^{-}(\{x_1,x_2,x_3,\ldots,x_m\}) = \min\{f_{\sigma_i}^{-}(x_1), f_{\sigma_i}^{-}(x_2),\ldots,f_{\sigma_i}^{-}(x_m)\},
\end{aligned}$$

for all $\{x_1,x_2,x_3,\ldots,x_m\} = E_k \subset V$ in ε.

There are following two cases:

1. If s and t are connected through only one directed hyperarc (s, t), then $\eta(s, t) = \eta^{\infty}(s, t)$.
2. If s and t are connected by two or more than two hyperpaths, then consider an arbitrary directed hyperpath $s = x_1, E_1, x_2, E_2, \ldots, E_k, x_{k+1} = t$. The strength of the path is

$$\begin{aligned}
\eta(s,t) &= \{\wedge_{i=1}^{k} t_{\varepsilon}^{+}(E_i), \wedge_{i=1}^{k} I_{\varepsilon}^{+}(E_i), \vee_{i=1}^{k} f_{\varepsilon}^{+}(E_i), \vee_{i=1}^{k} t_{\varepsilon}^{-}(E_i), \vee_{i=1}^{k} I_{\varepsilon}^{-}(E_i), \wedge_{i=1}^{k} f_{\varepsilon}^{-}(E_i)\} \\
&= \{\wedge_{i=1}^{k}(\wedge_{j=1}^{m} t_{\sigma_i}^{+}(x_j)), \wedge_{i=1}^{k}(\wedge_{j=1}^{m} I_{\sigma_i}^{+}(x_j)), \vee_{i=1}^{k}(\vee_{j=1}^{m} f_{\sigma_i}^{+}(x_j)), \vee_{i=1}^{k}(\vee_{j=1}^{m} t_{\sigma_i}^{-}(x_j)), \\
&\quad \vee_{i=1}^{k}(\vee_{j=1}^{m} I_{\sigma_i}^{-}(x_j)), \wedge_{i=1}^{k}(\wedge_{j=1}^{m} f_{\sigma_i}^{-}(x_j))\} \\
&= \min t_{\sigma_i}^{+}(x_j), \min I_{\sigma_i}^{+}(x_j), \max f_{\sigma_i}^{+}(x_j), \max t_{\sigma_i}^{-}(x_j), \max I_{\sigma_i}^{-}(x_j), \min f_{\sigma_i}^{-}(x_j) \\
&\qquad \leq [\wedge t_{\sigma_i}^{+}(s)] \wedge [\wedge t_{\sigma_i}^{+}(t)], \leq [\wedge I_{\sigma_i}^{+}(s)] \wedge [\wedge I_{\sigma_i}^{+}(t)], \geq [\vee f_{\sigma_i}^{+}(s)] \vee [\vee f_{\sigma_i}^{+}(t)], \\
&\qquad \geq [\vee t_{\sigma_i}^{-}(s)] \vee [\vee t_{\sigma_i}^{-}(t)], \geq [\vee I_{\sigma_i}^{-}(s))] \vee [\vee I_{\sigma_i}^{-}(t)], \leq [\wedge f_{\sigma_i}^{-}(s)] \wedge [\wedge f_{\sigma_i}^{-}(t)] \\
\eta^{\infty}(s,t) &= \{\sup t(\eta_k^{+}(s,t)), \sup I(\eta_k^{+}(s,t)), \inf f(\eta_k^{+}(s,t)), \inf t(\eta_k^{-}(s,t)), \\
&\quad \inf I(\eta_k^{-}(s,t)), \sup f(\eta_k^{-}(s,t))\} \\
&= \{\sup(\vee t_{\varepsilon}^{+}(E_i)), \sup(\vee I_{\varepsilon}^{+}(E_i)), \inf(\wedge f_{\varepsilon}^{+}(E_i)), \inf(\wedge t_{\varepsilon}^{-}(E_i)), \\
&\quad \inf(\vee I_{\varepsilon}^{-}(E_i)), \sup(\vee f_{\varepsilon}^{-}(E_i))\} \\
&\quad \leq [\wedge t_{\sigma_i}^{+}(s)] \wedge [\wedge t_{\sigma_i}^{+}(t)], \leq [\wedge I_{\sigma_i}^{+}(s)] \wedge [\wedge I_{\sigma_i}^{+}(t)], \geq [\vee f_{\sigma_i}^{+}(s)] \vee [\vee f_{\sigma_i}^{+}(t)], \\
&\quad \geq [\vee t_{\sigma_i}^{-}(s)] \vee [\vee t_{\sigma_i}^{-}(t)], \geq [\vee I_{\sigma_i}^{-}(s))] \vee [\vee I_{\sigma_i}^{-}(t)], \leq [\wedge f_{\sigma_i}^{-}(s)] \wedge [\wedge f_{\sigma_i}^{-}(t)] \\
&= \eta(s,t) \qquad \text{(by using Eq. 3)} \\
\eta^{\infty}(s,t) &\leq \eta(s,t)
\end{aligned} \tag{3}$$

Hence every hyperarc in G is strong.

Theorem 2.7 *Let $G = (\sigma, \varepsilon)$ and $G' = (\sigma', \varepsilon')$ be isomorphic bipolar neutrosophic directed hypergraphs, then G is strong if and only if G' is strong.*

Proof Let $\chi : G \to G'$ be the isomorphism between G and G', such that

$$\begin{aligned}
&\wedge t_{\sigma_i}^{+}(w) = \wedge t_{\sigma_i'}^{+}(\chi(w)), \wedge I_{\sigma_i}^{+}(w) = \wedge I_{\sigma_i'}^{+}(\chi(w)), \vee f_{\sigma_i}^{+}(w) = \vee f_{\sigma_i'}^{+}(\chi(w)), \\
&\vee t_{\sigma_i}^{-}(w) = \vee t_{\sigma_i'}^{-}(\chi(w)), \vee I_{\sigma_i}^{-}(w) = \vee I_{\sigma_i'}^{-}(\chi(w)), \wedge f_{\sigma_i}^{-}(w) = \wedge f_{\sigma_i'}^{-}(\chi(w)),
\end{aligned}$$

for all $w \in V$.

$$\begin{aligned}
t_{\varepsilon}^{+}(E_i) &= t_{\varepsilon}^{+}(w_1, w_2, w_3, \ldots, w_k) = t_{\varepsilon'}^{+}(\chi(w_1), \chi(w_2), \chi(w_3), \ldots, \chi(w_k)), \\
I_{\varepsilon}^{+}(E_i) &= I_{\varepsilon}^{+}(w_1, w_2, w_3, \ldots, w_k) = I_{\varepsilon'}^{+}(\chi(w_1), \chi(w_2), \chi(w_3), \ldots, \chi(w_k)), \\
f_{\varepsilon}^{+}(E_i) &= f_{\varepsilon}^{+}(w_1, w_2, w_3, \ldots, w_k) = f_{\varepsilon'}^{+}(\chi(w_1), \chi(w_2), \chi(w_3), \ldots, \chi(w_k)), \\
t_{\varepsilon}^{-}(E_i) &= t_{\varepsilon}^{-}(w_1, w_2, w_3, \ldots, w_k) = t_{\varepsilon'}^{-}(\chi(w_1), \chi(w_2), \chi(w_3), \ldots, \chi(w_k)), \\
I_{\varepsilon}^{-}(E_i) &= I_{\varepsilon}^{-}(w_1, w_2, w_3, \ldots, w_k) = I_{\varepsilon'}^{-}(\chi(w_1), \chi(w_2), \chi(w_3), \ldots, \chi(w_k)), \\
f_{\varepsilon}^{-}(E_i) &= f_{\varepsilon}^{-}(w_1, w_2, w_3, \ldots, w_k) = f_{\varepsilon'}^{-}(\chi(w_1), \chi(w_2), \chi(w_3), \ldots, \chi(w_k)),
\end{aligned}$$

for all $\{w_1, w_2, w_3, \ldots, w_k\} = E_i \subset V$. Let G be a strong bipolar neutrosophic directed hypergraph and

$$\begin{aligned}
t_{\varepsilon'}^{+}(E_i') &= t_{\varepsilon'}^{+}(\chi(E_i)) = t_{\varepsilon}^{+}(E_i) = \wedge t_{\sigma_i}^{+}(w_i) = \wedge t_{\sigma_i'}^{+}(w_i') \\
I_{\varepsilon'}^{+}(E_i') &= I_{\varepsilon'}^{+}(\chi(E_i)) = I_{\varepsilon}^{+}(E_i) = \wedge I_{\sigma_i}^{+}(w_i) = \wedge I_{\sigma_i'}^{+}(w_i') \\
f_{\varepsilon'}^{+}(E_i') &= f_{\varepsilon'}^{+}(\chi(E_i)) = f_{\varepsilon}^{+}(E_i) = \vee t_{\sigma_i}^{+}(w_i) = \vee t_{\sigma_i'}^{+}(w_i') \\
t_{\varepsilon'}^{-}(E_i') &= t_{\varepsilon'}^{-}(\chi(E_i)) = t_{\varepsilon}^{-}(E_i) = \vee t_{\sigma_i}^{-}(w_i) = \vee t_{\sigma_i'}^{-}(w_i') \\
I_{\varepsilon'}^{-}(E_i') &= I_{\varepsilon'}^{-}(\chi(E_i)) = I_{\varepsilon}^{-}(E_i) = \vee I_{\sigma_i}^{-}(w_i) = \vee I_{\sigma_i'}^{-}(w_i') \\
f_{\varepsilon'}^{-}(E_i') &= f_{\varepsilon'}^{-}(\chi(E_i)) = f_{\varepsilon}^{-}(E_i) = \wedge f_{\sigma_i}^{-}(w_i) = \wedge f_{\sigma_i'}^{-}(w_i')
\end{aligned}$$

Hence G' is a strong bipolar neutrosophic directed hypergraph. The converse part is obvious.

Theorem 2.8 *Let $\chi : G \to G'$ be a co-weak isomorphism between G and G' and G' is strong. Then G is a strong bipolar neutrosophic directed hypergraph.*

Proof Let $\chi : G \to G'$ be a co-weak isomorphism between G and G', which satisfies

$$\wedge t_{\sigma_i}^{+}(x) \leq \wedge t_{\sigma_i'}^{+}(\chi(x)), \wedge I_{\sigma_i}^{+}(x) \leq \wedge I_{\sigma_i'}^{+}(\chi(x)), \vee f_{\sigma_i}^{+}(x) \geq \vee f_{\sigma_i'}^{+}(\chi(x)),$$
$$\vee t_{\sigma_i}^{-}(x) \geq \vee t_{\sigma_i'}^{-}(\chi(x)), \vee I_{\sigma_i}^{-}(x) \geq \vee I_{\sigma_i'}^{-}(\chi(x)), \wedge f_{\sigma_i}^{-}(x) \leq \wedge f_{\sigma_i'}^{-}(\chi(x)),$$

for all $x \in V$.

$$\begin{aligned}
t_{\varepsilon}^{+}(x_1, x_2, x_3, \ldots, x_k) &= t_{\varepsilon'}^{+}(\chi(x_1), \chi(x_2), \chi(x_3), \ldots, \chi(x_k)), \\
I_{\varepsilon}^{+}(x_1, x_2, x_3, \ldots, x_k) &= I_{\varepsilon'}^{+}(\chi(x_1), \chi(x_2), \chi(x_3), \ldots, \chi(x_k)), \\
f_{\varepsilon}^{+}(x_1, x_2, x_3, \ldots, x_k) &= f_{\varepsilon'}^{+}(\chi(x_1), \chi(x_2), \chi(x_3), \ldots, \chi(x_k)), \\
t_{\varepsilon}^{-}(x_1, x_2, x_3, \ldots, x_k) &= t_{\varepsilon'}^{-}(\chi(x_1), \chi(x_2), \chi(x_3), \ldots, \chi(x_k)), \\
I_{\varepsilon}^{-}(x_1, x_2, x_3, \ldots, x_k) &= I_{\varepsilon'}^{-}(\chi(x_1), \chi(x_2), \chi(x_3), \ldots, \chi(x_k)), \\
f_{\varepsilon}^{-}(x_1, x_2, x_3, \ldots, x_k) &= f_{\varepsilon'}^{-}(\chi(x_1), \chi(x_2), \chi(x_3), \ldots, \chi(x_k)),
\end{aligned}$$

for all $\{x_1, x_2, x_3, \ldots, x_k\} = E_i \subset V$. Since G' is strong, then

$$\begin{aligned}
&t_{\varepsilon'}^{+}(\chi(x_1), \chi(x_2), \chi(x_3), \ldots, \chi(x_k)) \\
&\qquad = \min\{t_{\sigma_i'}^{+}(\chi(x_1), t_{\sigma_i'}^{+}(\chi(x_2), t_{\sigma_i'}^{+}(\chi(x_3), \ldots, t_{\sigma_i'}^{+}(\chi(x_k)\} \\
&\qquad = t_{\varepsilon}^{+}(x_1, x_2, x_3, \ldots, x_k) \\
&\qquad \leq \min\{t_{\sigma_i}^{+}(x_1), t_{\sigma_i}^{+}(x_2), t_{\sigma_i}^{+}(x_3), \ldots, t_{\sigma_i}^{+}(x_k)\} \\
&\qquad \leq \min\{t_{\sigma_i'}^{+}(\chi(x_1), t_{\sigma_i'}^{+}(\chi(x_2), t_{\sigma_i'}^{+}(\chi(x_3), \ldots, t_{\sigma_i'}^{+}(\chi(x_k)\}
\end{aligned}$$

$$
\begin{aligned}
&t^{+}_{\varepsilon}(x_1, x_2, x_3, \ldots, x_k)\\
&\qquad = \min\{t^{+}_{\sigma_i}(x_1), t^{+}_{\sigma_i}(x_2), t^{+}_{\sigma_i}(x_3), \ldots, t^{+}_{\sigma_i}(x_k)\}\\
&I^{+}_{\varepsilon'}(\chi(x_1), \chi(x_2), \chi(x_3), \ldots, \chi(x_k))\\
&\qquad = \min\{I^{+}_{\sigma_i'}(\chi(x_1), I^{+}_{\sigma_i'}(\chi(x_2), I^{+}_{\sigma_i'}(\chi(x_3), \ldots, I^{+}_{\sigma_i'}(\chi(x_k)\}\\
&\qquad = I^{+}_{\varepsilon}(x_1, x_2, x_3, \ldots, x_k)\\
&\qquad \leq \min\{I^{+}_{\sigma_i}(x_1), I^{+}_{\sigma_i}(x_2), I^{+}_{\sigma_i}(x_3), \ldots, I^{+}_{\sigma_i}(x_k)\}\\
&\qquad \leq \min\{I^{+}_{\sigma_i'}(\chi(x_1), I^{+}_{\sigma_i'}(\chi(x_2), I^{+}_{\sigma_i'}(\chi(x_3), \ldots, I^{+}_{\sigma_i'}(\chi(x_k)\}\\
&I^{+}_{\varepsilon}(x_1, x_2, x_3, \ldots, x_k)\\
&\qquad = \min\{I^{+}_{\sigma_i}(x_1), I^{+}_{\sigma_i}(x_2), I^{+}_{\sigma_i}(x_3), \ldots, I^{+}_{\sigma_i}(x_k)\}
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
f^{+}_{\varepsilon}(x_1, x_2, x_3, \ldots, x_k) &= \max\{f^{+}_{\sigma_i}(x_1), f^{+}_{\sigma_i}(x_2), f^{+}_{\sigma_i}(x_3), \ldots, f^{+}_{\sigma_i}(x_k)\}\\
t^{-}_{\varepsilon}(x_1, x_2, x_3, \ldots, x_k) &= \max\{t^{-}_{\sigma_i}(x_1), t^{-}_{\sigma_i}(x_2), t^{-}_{\sigma_i}(x_3), \ldots, t^{-}_{\sigma_i}(x_k)\}\\
I^{-}_{\varepsilon}(x_1, x_2, x_3, \ldots, x_k) &= \max\{I^{-}_{\sigma_i}(x_1), I^{-}_{\sigma_i}(x_2), I^{-}_{\sigma_i}(x_3), \ldots, I^{-}_{\sigma_i}(x_k)\}\\
f^{-}_{\varepsilon}(x_1, x_2, x_3, \ldots, x_k) &= \min\{f^{-}_{\sigma_i}(x_1), f^{-}_{\sigma_i}(x_2), f^{-}_{\sigma_i}(x_3), \ldots, f^{-}_{\sigma_i}(x_k)\}
\end{aligned}
$$

Hence G is strong BNDHG.

3 Applications of BNDHGs in decision-making

Decision-making acts as a vital feature of current administration. Decisions are considered very important in this way that they determine both organizational and managerial actions. A decision can be defined as "a series of action which is consciously chosen from among a set of alternatives to achieve a desired result." It is appeared as a balanced commitment to action and a well organized judgment. Problems in almost every conceivable discipline, including decision-making can be solved using graphical models.

1. Affiliation with an apprenticeship group A social group is a unity of two or more humans, sharing similar activities and characteristics, who interact with one another. Social interactions can also occur on the Internet in online communities and these relationships preclude the face-to-face interactions. Different social groups are created on the basis of typical features, including education, apprenticeship, entertainment, tourism, ethics and religion. It is bit difficult for an anonymous user to choose a social group that fulfills his desires and objectives appropriately. We develop a bipolar neutrosophic directed hypergraphical model depicting that how a user can join the most beneficial apprenticeship group by following a step by step procedure. A BNDHG illustrating a group of users as members of different apprenticeship groups is shown in Fig. 6.

If a user wants to select the most appropriate educational group, that is, the most effective one to promote and encourage a specific behavior or outcome, the following

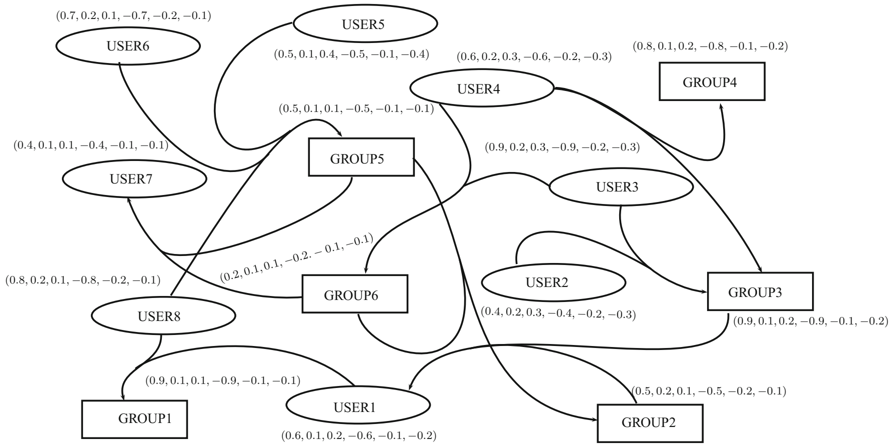

Fig. 6 Representation of members having affiliations with different groups using BNDHG

Table 1 Didactical behavior of users towards apprenticeship groups

Apprenticeship groups	Didactical behavior	Indeterminate behavior	Irrelevant to didactics
GROUP1	0.6	0.1	0.2
GROUP2	0.5	0.1	0.1
GROUP3	0.4	0.2	0.3
GROUP4	0.6	0.2	0.3
GROUP5	0.5	0.1	0.4
GROUP6	0.6	0.2	0.3

Table 2 Prenicious behavior of users towards apprenticeship groups

Apprenticeship groups	Prenicious behavior	Indeterminate behavior	Extraneous behavior
GROUP1	−0.6	−0.1	−0.2
GROUP2	−0.5	−0.1	−0.1
GROUP3	−0.4	−0.2	−0.3
GROUP4	−0.6	−0.2	−0.3
GROUP5	−0.5	−0.1	−0.4
GROUP6	−0.6	−0.2	−0.3

procedure can help him. Firstly, one should think about the collective contribution of members towards the group, which can be find out by means of membership degrees of BN directed hyperedges. The positive and negative contributions of users towards a specific apprenticeship group are given in Tables 1 and 2, respectively.

It can be noted that GROUP1 has 60% didactical behavior, which is maximum among all other groups, 10% indeterminacy and 20% is irrelevant to its objectives. Moreover, it owns −60% of prenicious behavior, which is minimum as compared

Table 3 Educational effects of groups on the users

Apprenticeship groups	Educational effects
GROUP1	(0.9, 0.1, 0.1, −0.9, −0.1, −0.1)
GROUP2	(0.5, 0.2, 0.1, −0.5, −0.2, −0.1)
GROUP3	(0.9, 0.1, 0.2, −0.9, −0.1, −0.2)
GROUP4	(0.8, 0.1, 0.2, −0.8, −0.1, −0.2)
GROUP5	(0.5, 0.5, 0.1, −0.5, −0.5, −0.1)
GROUP6	(0.2, 0.1, 0.1, −0.2, −0.1, −0.1)

Table 4 Educational effects of groups on the users

Apprenticeship groups	In-degrees	Out-degrees
GROUP1	(0.6, 0.1, 0.2, −0.6, −0.1, −0.2)	(0, 0, 0, 0, 0, 0)
GROUP2	(0.2, 0.1, 0.1, −0.2, −0.1, −0.1)	(0.5, 0.1, 0.2, −0.5, −0.1, −0.2)
GROUP3	(1.2, 0.2, 0.6, −1.2, −0.2, −0.6)	(0.5, 0.1, 0.2, −0.5, −0.1, −0.2)
GROUP4	(0.6, 0.1, 0.3, −0.6, −0.1, −0.3)	(0, 0, 0, 0, 0, 0)
GROUP5	(0.5, 0.1, 0.4, −0.5, −0.1, −0.4)	(0.2, 0.1, 0.1, −0.2, −0.1, −0.1)
GROUP6	(0.2, 0.1, 0.3, −0.2, −0.1, −0.3)	(0.4, 0.2, 0.2, −0.4, −0.2, −0.2)

to all other groups and −20% of extraneous behavior. Thus, GROUP1 can be a most appropriate choice for an anonymous user. Secondly, one should do his research on the powerful impacts of all under consideration groups on their members. Degrees of membership of all group vertices depict their effects on their members as given in Table 3.

Note that, GROUP1 has maximum positive effects and minimum negative effects on its members. Thirdly, a person can detect the influence of a group by calculating its in-degree and out-degree as in-degrees interpret the percentage of users joining the group and out-degrees interpret thepercentage of users leaving that group. In-degrees and out-degrees of all Apprenticeship groups are given in Table 4.

Thus, an Apprenticeship group having maximum in-degrees and minimum out-degrees will be the most suitable choice. It can be noted that GROUP1 and GROUP4 both have minimum out-degrees. To handle such type of situations, we will then compare the in-degrees of these two groups. GROUP1 and GROUP4 both have same positive truth membership and positive indeterminacy but the falsity membership value is minimum in case of GROUP1 and the conditions are same in case of negative membership values. Hence GROUP1 will be more suitable than GROUP4. Note that, in all above cases that we have discussed, GROUP1 is the most appropriate choice with the given data. So if any user wants to join an Apprenticeship group, by following the above procedure, he should be affiliate with GROUP1, as this group has maximum positive effects on the didactical behavior of its members and is more closely to the educational objectives. The guide will help to think about selecting a group based on the purpose of someone communication and understanding of the users. It will also help to consider what information is best communicated through different groups.

The method of searching out the most beneficial group is described in the following algorithm.

Algorithm

1. Input the degree of membership of all vertices(users) $x_1, x_2, \ldots, x_n$.
2. Find the positive and negative contributions of users towards groups by calculating the degree of membership of all directed hyperedges as:

$$\begin{aligned}
t_\rho^+(E_k) &\leq \min\{t_{\mu_i}^+(x_1), t_{\mu_i}^+(x_2), \ldots, t_{\mu_i}^+(x_m)\},\\
I_\rho^+(E_k) &\leq \min\{I_{\mu_i}^+(x_1), I_{\mu_i}^+(x_2), \ldots, I_{\mu_i}^+(x_m)\},\\
f_\rho^+(E_k) &\leq \max\{f_{\mu_i}^+(x_1), f_{\mu_i}^+(x_2), \ldots, f_{\mu_i}^+(x_m)\},\\
t_\rho^-(E_k) &\geq \max\{t_{\mu_i}^-(x_1), t_{\mu_i}^-(x_2), \ldots, t_{\mu_i}^-(x_m)\},\\
I_\rho^-(E_k) &\geq \max\{I_{\mu_i}^-(x_1), I_{\mu_i}^-(x_2), \ldots, I_{\mu_i}^-(x_m)\},\\
f_\rho^-(E_k) &\geq \min\{f_{\mu_i}^-(x_1), f_{\mu_i}^-(x_2), \ldots, f_{\mu_i}^-(x_m)\}.
\end{aligned}$$

3. Obtain the most appropriate group as:

$$\max t_\rho^+(E_k), \max I_\rho^+(E_k), \min f_\rho^+(E_k), \min t_\rho^-(E_k), \min I_\rho^-(E_k), \max f_\rho^-(E_k)$$

4. Find the group having strong educational impacts on the users as:

$$\max t_{\mu_i}^+(x_k), \max I_{\mu_i}^+(x_k), \min f_{\mu_i}^+(x_k), \min t_{\mu_i}^-(x_k), \min I_{\mu_i}^-(x_k), \max f_{\mu_i}^-(x_k),$$

   where all x_k are the vertices representing the different groups.
5. Find the positive influence of groups x_k on the users by calculating the in-degrees $D^-(x_k)$ as:

$$\begin{pmatrix}\sum\limits_{x_k\in H(E_k)} t_\varepsilon^+(E_k), \sum\limits_{x_k\in H(E_k)} I_\varepsilon^+(E_k), \sum\limits_{x_k\in H(E_k)} f_\varepsilon^+(E_k),\\ \sum\limits_{x_k\in H(E_k)} t^-\varepsilon(E_k), \sum\limits_{x_k\in H(E_k)} I^-\varepsilon(E_k), \sum\limits_{x_k\in H(E_k)} f^-\varepsilon(E_k)\end{pmatrix}$$

6. Find the negative influence of groups x_k on the users by calculating the out-degrees $D^+(x_k)$ as:

$$\begin{pmatrix}\sum\limits_{x_k\in T(E_k)} t_\varepsilon^+(E_k), \sum\limits_{x_k\in T(E_k)} I_\varepsilon^+(E_k), \sum\limits_{x_k\in T(E_k)} f_\varepsilon^+(E_k),\\ \sum\limits_{x_k\in T(E_k)} t^-\varepsilon(E_k), \sum\limits_{x_k\in T(E_k)} I^-\varepsilon(E_k), \sum\limits_{x_k\in T(E_k)} f^-\varepsilon(E_k)\end{pmatrix}$$

7. Obtain the most effective group as: $(\max D^-(x_k), \min D^+(x_k))$.

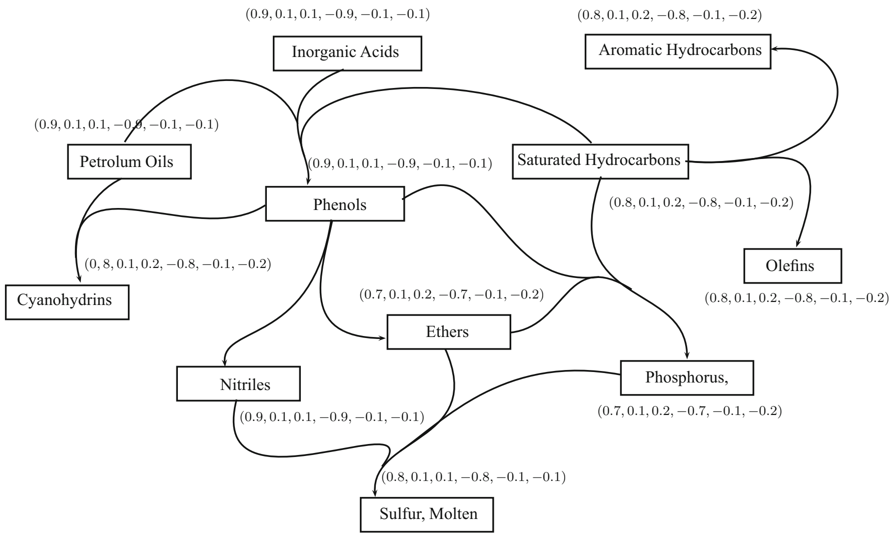

Fig. 7 Safe combinations of compatible chemicals

2. Portrayal of compatible chemicals using isomorphic BNDHGs The formal concept of “isomorphism” captures the informal notion that some objects have “the same structure” if one ignores individual distinctions of “atomic” components of objects. A hypergraph can exists in different forms having the same number of vertices, hyperedges, and also the same connectivity. Such hypergraphs are called isomorphic. Appropriate chemical storage plans are designed to control health and physical dynamite associated with laboratory chemical storage. There are many chemicals which are not compatible to each other and react when they are mixed. The recants can be dangerous in such cases, so care must be taken when attempting to mix or store the chemicals. Here, we describe that if a model representation of compatible groups is given, then by using the isomorphism property we can represent any type of chemicals. Consider the groups of incompatible chemicals, which cannot interact with each other, as the vertices of BNDHG G. Directed hyperedges between the groups represent the safe combinations and absence of hyperedges depicts that the combinations are unsafe. A BNDHG model illustrating the safe combinations is given in Fig. 7.

Membership degrees of each chemical group represent that how they react positively or negatively when are mixed. For example, membership degree of Inorganic Acids $(0.9, 0.1, 0.1, -0.9, -0.1, -0.1)$ depict that these chemicals are 90% compatible, 10% have indeterminacy and 10% have chances to explode. Similarly, negative membership degrees describe the incompatibility of this group.

Now, if we have to represent the compatibilities of chemicals

{Phosphoric acid, Cyclohexane, Dicylcopentadiene, Gasolines, Carbolic oil, Acetone cyanohydrin, Acetonitrile, Diethyl ether, Phosphorus, Sulfur, Benzene} belonging to different groups as mentioned in above BNDHG, we will find out a BNDHG iso-

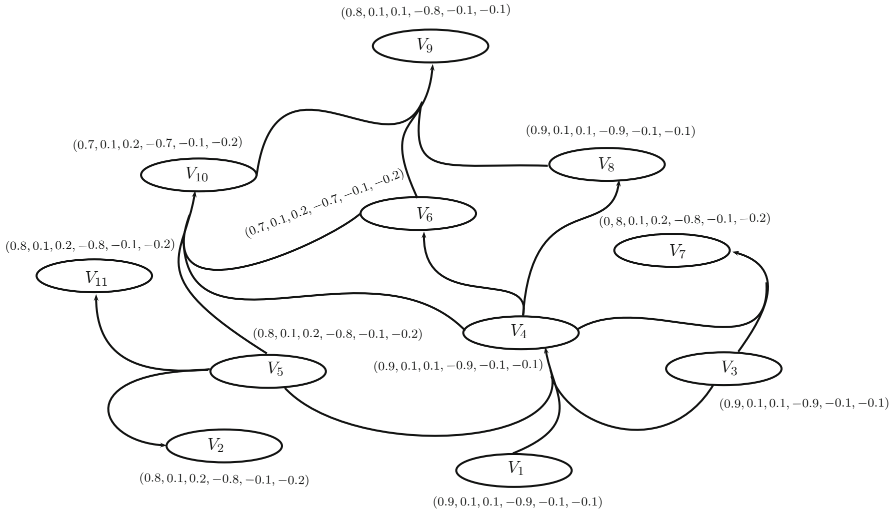

Fig. 8 Isomorphic BNDHG G'

morphic to above. A BNDHG G' isomorphic to the above is given in Fig. 8. Define a bijective mapping $h : G \to G'$, such that

$$\begin{array}{llll}
\text{h(Inorganic Acids)} & = V_1, & \text{h(Aromatic Hydrocarbons)} & = V_2, \\
\text{h(Petroleum Oils)} & = V_3, & \text{h(Phenols)} & = V_4, \\
\text{h(Saturated Hydrocarbons)} & = V_5, & \text{h(Ethers)} & = V_6, \\
\text{h(Cyanohydrins)} & = V_7, & \text{h(Niriles)} & = V_8, \\
\text{h(Sulfur, molten)} & = V_9, & \text{h(Phosphorus)} & = V_{10}, \\
\text{h(Olefins)} & = V_{11}. & &
\end{array}$$

It can be noted that

$$\begin{array}{ll}
t_G^+(InorganicAcids) = t_{G'}^+(V_1), & I_G^+(InorganicAcids) = I_{G'}^+(V_1), \\
f_G^+(InorganicAcids) = f_{G'}^+(V_1), & t_G^-(InorganicAcids) = t_{G'}^-(V_1), \\
I_G^-(InorganicAcids) = I_{G'}^-(V_1), & f_G^-(InorganicAcids) = f_{G'}^-(V_1).
\end{array}$$

Similarly, membership degrees of all groups are equal to their images.

Now determine the relative groups of all given elements and put that elements in corresponding images boxes. For instance, Phosphoric acid belongs to the group of Inorganic acids and the image of Inorganic Acids is V_1. Hence Phosphoric acid will be kept in V_1 box. Similarly, Cyclohexane is an element of Saturated Hydrocarbons, Dicylcopentadiene belongs to Olefins, Gasolines belongs to Petroleum Oils, Carbolic oil is an element of Phenols, Acetone cyanohydrin belongs to Cyanohydrins, Acetonitrile is in Nitriles, Diethyl ether belongs to Ethers, Benzene belongs to Aromatic

Hydrocarbons and Phosphorus, Sulfur are mapped onto themselves. Thus these elements will be positioned at the places of V_5, V_{11}, V_3, V_4, V_7, V_8, V_6, V_2, V_{10} and V_9, respectively. Hence by using the isomorphism property of BNDHGs, we can check the compatibility of chemicals by considering their preimages.

4 Conclusions

Theory of directed hypergraphs has fruitful applications in different fields, including databases, social networking, computer networking and decision making. Bipolar neutrosophic models serve as most powerful tools to discuss and model the problems in many areas, including decision-making, psychology and bipolarity in human behaviors. Bipolar neutrosophic directed hypergraph models are more flexible and applicable as they dissertate neutrosophic behavior positively as well as negatively. In this research paper, we have applied the concept of bipolar neutrosophic sets to the theory of directed hypergraphs. We have discussed the bipolar neutrosophic directed hypergraphs and their certain properties. We aim to widen our research (1) Bipolar fuzzy soft neutrosophic hypergraphs, (2) Interval valued neutrosophic hypergraphs, (3) Fuzzy rough neutrosophic hypergraphs and (4) Bipolar fuzzy rough directed hypergraphs.

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Compliance with ethical standards

Conflict of interest The authors declare that they have no conflict of interest.

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