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A New Type of Single Valued Neutrosophic Covering Rough Set Model

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Abstract: Recently, various types of single valued neutrosophic (SVN) rough set models were presented based on the same inclusion relation. However, there is another SVN inclusion relation in SVN sets. In this paper, we propose a new type of SVN covering rough set model based on the new inclusion relation. Furthermore, the graph and matrix representations of the new SVN covering approximation operators are presented. Firstly, the notion of SVN β^2 -covering approximation space is proposed, which is decided by the new inclusion relation. Then, a type of SVN covering rough set model under the SVN β^2 -covering approximation space is presented. Moreover, there is a corresponding SVN relation rough set model based on a SVN relation induced by the SVN β^2 -covering, and two conditions under which the SVN β^2 -covering can induce a symmetric SVN relation are presented. Thirdly, the graph and matrix representations of the new SVN covering rough set model are investigated. Finally, we propose a novel method for decision making (DM) problems in paper defect diagnosis under the new SVN covering rough set model.

Keywords: single valued neutrosophic set; covering; symmetric relation; graph representation; matrix representation; paper defect diagnosis

1. Introduction

Rough set theory, as a tool to deal with various types of data in data mining, was proposed by Pawlak [1,2] in 1982. Then rough set theory has been extended to generalize rough sets based on other notions such as binary relations [3], neighborhood systems [4], and coverings [5].

Covering-based rough sets [6–9] were proposed to deal with the type of covering data. In application, they have been applied to knowledge reduction [10,11], decision rule synthesis [12,13], and other fields [14,15]. In theory, covering-based rough set theory has been connected with matroid theory [16–18], lattice theory [19,20], and fuzzy set theory [21–23]. Zadeh's fuzzy set theory [24] addresses the problem of how to understand and manipulate imperfect knowledge. It has been used in various applications [25–28]. Recent investigations have attracted more attention on combining covering-based rough set and fuzzy set theories. There are many fuzzy covering rough set models proposed by researchers, such as Ma [29] and Yang et al. [30].

Smarandache [31] and Wang et al. [32] presented single valued neutrosophic (SVN) sets, which can be regarded as an extension of intuitionistic fuzzy sets [33]. Both neutrosophic sets and rough sets can deal with partial and uncertainty information [34]. Therefore, it is necessary to combine them. Recently, Mondal and Pramanik [35] presented the concept of rough neutrosophic set. Yang et al. [36] presented a SVN rough set model based on SVN relations. Wang and Zhang [37] presented two types of SVN covering rough set models. All these SVN rough set models are presented based on an

inclusion relation which is named type-1 inclusion relation and denoted by \subseteq_1 . The definition of \subseteq_1 is shown as follows; for any $A, B \in SVN(U)$,

$$A \subseteq_1 B \text{ iff } T_A(x) \leq T_B(x), I_B(x) \leq I_A(x) \text{ and } F_B(x) \leq F_A(x) \text{ for all } x \in U.$$

Under the type-1 inclusion relation, for two SVN numbers $\alpha = \langle a, b, c \rangle$ and $\beta = \langle d, e, f \rangle$, we have $\alpha \subseteq_1 \beta \Leftrightarrow a \leq d, b \geq e$ and $c \geq f$. The definition of SVN β -covering approximation space is presented as follows (see the work by the authors of [37]).

Let U be a universe and $SVN(U)$ be the SVN power set of U . For a SVN number $\beta = \langle a, b, c \rangle$, we call $\widehat{C} = \{C_1, C_2, \dots, C_m\}$, with $C_i \in SVN(U) (i = 1, 2, \dots, m)$, a SVN β -covering of U , if for all $x \in U$, $C_i \in \widehat{C}$ exists such that $C_i(x) \geq_1 \beta$. We also call (U, \widehat{C}) a SVN β -covering approximation space.

However, there exists another inclusion relation in the work by the authors of [38], which is called type-2 inclusion relation and denoted by \subseteq_2 . The definition of \subseteq_2 is introduced as follows; for any $A, B \in SVN(U)$,

$$A \subseteq_2 B \text{ iff } T_A(x) \leq T_B(x), I_A(x) \leq I_B(x) \text{ and } F_B(x) \leq F_A(x) \text{ for all } x \in U.$$

Under the type-2 inclusion relation, for two SVN numbers $\alpha = \langle a, b, c \rangle$ and $\beta = \langle d, e, f \rangle$, we have $\alpha \subseteq_2 \beta \Leftrightarrow a \leq d, b \leq e$ and $c \geq f$.

In the definition of SVN β -covering approximation space, if $C_i(x) \geq_1 \beta$ is replaced by $C_i(x) \geq_2 \beta$, there will be a new SVN covering approximation space (we call it a SVN β^2 -covering approximation space in this paper). In Example 1 in this paper, we find the following statements.

- (1) Let $\beta = \langle 0.5, 0.1, 0.8 \rangle$. Then \widehat{C} is a SVN β^2 -covering of U , but it is not a SVN β -covering of U .
- (2) Let $\beta = \langle 0.5, 0.3, 0.8 \rangle$. Then \widehat{C} is a SVN β -covering of U , but it is not a SVN β^2 -covering of U .

That is to say, the SVN β^2 -covering approximation space is a new SVN covering approximation space, which is different from the SVN β -covering approximation space. Since different inclusion relations (\subseteq_1 and \subseteq_2) have different union and intersection operations, the SVN β^2 -covering approximation space has different union and intersection operations from the SVN β -covering approximation space. Hence, notions and corresponding SVN covering rough set models of SVN β -covering approximation space do not apply to SVN β^2 -covering approximation space, which is the justification for studying this topic. Therefore, the investigation of the SVN β^2 -covering approximation space and its corresponding SVN covering rough set model is very important. It not only can manage some issues that the SVN β -covering approximation space can not deal with, but also constructs a new type of SVN covering rough set model. This is our motivation of this research.

In this paper, we present some new concepts in SVN β^2 -covering approximation space, as well as their properties. Then the type-2 SVN covering rough set model under the SVN β^2 -covering approximation space is proposed. On the one hand, the graph and matrix representations of the type-2 SVN covering rough set model are investigated respectively. Moreover, some relationships between the type-2 SVN covering rough set model and other SVN rough set models are presented. On the other hand, we present a method to DM problems in paper defect diagnosis, which is an important topic in paper making industries, under the type-2 SVN covering rough set model. Many researchers have studied decision making (DM) problems by rough set models [39–42]. Hence, the proposed DM method is compared with other methods which are presented by Liu [43], Ye [44], Yang et al. [36], and Wang et al. [37] respectively.

The rest of this paper is organized as follows. Section 2 reviews some fundamental definitions about covering-based rough sets and SVN sets. In Section 3, some concepts and properties in SVN β^2 -covering approximation space are studied. The relationship between the SVN β -covering approximation space and the SVN β^2 -covering approximation space is presented. In Section 4, we present the type-2 SVN covering rough set model. Some relationships between the type-2 SVN covering rough set model and other SVN rough set models are presented. Moreover, a SVN relation can be induced by the SVN β^2 -covering, so a corresponding SVN relation rough set model and two conditions

under which the SVN β^2 -covering can induce a symmetric SVN relation are presented. In Section 5, some new graphs and graph operations are presented. Based on this, the graph representation of the type-2 SVN covering approximation operators is shown. In Section 6, some new matrices and matrix operations are also presented, and the matrix representation of the type-2 SVN covering approximation operators is presented. In Section 7, a novel method to paper defect diagnosis is presented under the type-2 SVN covering rough set model. Moreover, the proposed method is compared with other methods. This paper is concluded and further work is indicated in Section 8.

2. Basic Definitions

Suppose U is a nonempty and finite set called universe.

Definition 1. (Covering [45,46]) Let U be a universe and \mathcal{C} be a family of subsets of U . If none of subsets in \mathcal{C} is empty and $\cup \mathcal{C} = U$, then \mathcal{C} is called a covering of U .

The pair (U, \mathcal{C}) is called a covering approximation space.

Definition 2. (SVN set [32]) Let U be a nonempty fixed set. A SVN set A in U is defined as an object of the following form.

$$A = \{ \langle x, T_A(x), I_A(x), F_A(x) \rangle : x \in U \},$$

where $T_A : U \rightarrow [0, 1]$ is called the degree of truth-membership of the element $x \in U$ to A , $I_A : U \rightarrow [0, 1]$ is called the degree of indeterminacy-membership of the element $x \in U$ to A , $F_A(x) : U \rightarrow [0, 1]$ is called the degree of falsity-membership. They satisfy $0 \leq T_A(x) + I_A(x) + F_A(x) \leq 3$ for all $x \in U$. The family of all SVN sets in U is denoted by $SVN(U)$. For convenience, a SVN number is represented by $\alpha = \langle a, b, c \rangle$, where $a, b, c \in [0, 1]$ and $a + b + c \leq 3$.

For the inclusion relation of neutrosophic sets, there are two different definitions in the literature. An original definition is proposed by Smarandache [31,47], we call it type-1 inclusion relation in this paper, denoted by \subseteq_1 . For set theory, union and intersection operations are corresponding to inclusion relation. Hence, there are corresponding union and intersection operations defined as follows; for any $A, B \in SVN(U)$,

- (1) $A \subseteq_1 B$ iff $T_A(x) \leq T_B(x)$, $I_B(x) \leq I_A(x)$ and $F_B(x) \leq F_A(x)$ for all $x \in U$;
- (2) $A \cap_1 B = \{ \langle x, T_A(x) \wedge T_B(x), I_A(x) \vee I_B(x), F_A(x) \vee F_B(x) \rangle : x \in U \}$;
- (3) $A \cup_1 B = \{ \langle x, T_A(x) \vee T_B(x), I_A(x) \wedge I_B(x), F_A(x) \wedge F_B(x) \rangle : x \in U \}$.

Specially, for two SVN numbers, $\alpha = \langle a, b, c \rangle$ and $\beta = \langle d, e, f \rangle$, $\alpha \subseteq_1 \beta \Leftrightarrow a \leq d, b \geq e$ and $c \geq f$.

Under the type-1 inclusion relation, Wang and Zhang [37] presented the definition of SVN β -covering approximation space.

Definition 3. [37] Let U be a universe and $SVN(U)$ be the SVN power set of U . For a SVN number $\beta = \langle a, b, c \rangle$, we call $\widehat{\mathcal{C}} = \{C_1, C_2, \dots, C_m\}$, with $C_i \in SVN(U)$ ($i = 1, 2, \dots, m$), a SVN β -covering of U , if for all $x \in U$, $C_i \in \widehat{\mathcal{C}}$ exists such that $C_i(x) \geq_1 \beta$. We also call $(U, \widehat{\mathcal{C}})$ a SVN β -covering approximation space.

Definition 4. [37] Let $\widehat{\mathcal{C}}$ be a SVN β -covering of U and $\widehat{\mathcal{C}} = \{C_1, C_2, \dots, C_m\}$. For any $x \in U$, the SVN β -neighborhood $\widetilde{\mathbb{N}}_x^\beta$ of x induced by $\widehat{\mathcal{C}}$ can be defined as

$$\widetilde{\mathbb{N}}_x^\beta = \cap_1 \{C_i \in \widehat{\mathcal{C}} : C_i(x) \geq_1 \beta\}. \quad (1)$$

Another one is used in some papers [32,38], we call it type-2 inclusion relation in this paper, denote it by \subseteq_2 . Hence, the type-2 inclusion relation, corresponding union and intersection operations are shown as follows; for any $A, B \in SVN(U)$,

- (1) $A \subseteq_2 B$ iff $T_A(x) \leq T_B(x)$, $I_A(x) \leq I_B(x)$ and $F_B(x) \leq F_A(x)$ for all $x \in U$;
- (2) $A \cap_2 B = \{ \langle x, T_A(x) \wedge T_B(x), I_A(x) \wedge I_B(x), F_A(x) \vee F_B(x) \rangle : x \in U \}$;
- (3) $A \cup_2 B = \{ \langle x, T_A(x) \vee T_B(x), I_A(x) \vee I_B(x), F_A(x) \wedge F_B(x) \rangle : x \in U \}$.

Specially, for two SVN numbers $\alpha = \langle a, b, c \rangle$ and $\beta = \langle d, e, f \rangle$, $\alpha \leq_2 \beta \Leftrightarrow a \leq d, b \leq e$ and $c \geq f$.

For the above inclusion relations of neutrosophic sets, the following operations use the same definition in this paper [32,36].

- (4) $A = B$ iff $A \subseteq_1 B$ and $B \subseteq_1 A$, iff $A \subseteq_2 B$ and $B \subseteq_2 A$;
- (5) $A' = \{ \langle x, F_A(x), 1 - I_A(x), T_A(x) \rangle : x \in U \}$;
- (6) $A \oplus B = \{ \langle x, T_A(x) + T_B(x) - T_A(x) \cdot T_B(x), I_A(x) \cdot I_B(x), F_A(x) \cdot F_B(x) \rangle : x \in U \}$.

3. SVN β^2 -Covering Approximation Space

In this section, the definition of SVN β^2 -covering approximation space is presented. There are two basic concepts—SVN β^2 -covering and SVN β^2 -neighborhood—in this new approximation space.

Definition 5. Let U be a universe and $SVN(U)$ be the SVN power set of U . For a SVN number $\beta = \langle a, b, c \rangle$, we call $\widehat{C} = \{C_1, C_2, \dots, C_m\}$, with $C_i \in SVN(U) (i = 1, 2, \dots, m)$, a SVN β^2 -covering of U , if for all $x \in U$, $C_i \in \widehat{C}$ exists such that $C_i(x) \geq_2 \beta$. We also call (U, \widehat{C}) a SVN β^2 -covering approximation space.

In Definition 5, if $C_i(x) \geq_2 \beta$ is replaced by $C_i(x) \geq_1 \beta$, then $\widehat{C} = \{C_1, C_2, \dots, C_m\}$ is called a SVN β -covering of U in [37]. By the definitions of \geq_1 and \geq_2 , we know if \widehat{C} is a SVN β^2 -covering of U , then \widehat{C} is not necessarily a SVN β -covering. To show the difference between SVN β -covering and SVN β^2 -covering, we use the work presented by the authors of [37] in the following example.

Example 1. Let $U = \{x_1, x_2, x_3, x_4, x_5\}$, $\widehat{C} = \{C_1, C_2, C_3, C_4\}$ and $\beta = \langle 0.5, 0.1, 0.8 \rangle$. We can see that \widehat{C} is a SVN β^2 -covering of U in Table 1, but it is not a SVN β -covering of U .

Table 1. The tabular representation of \widehat{C} in [37].

U	C_1	C_2	C_3	C_4
x_1	$\langle 0.7, 0.2, 0.5 \rangle$	$\langle 0.6, 0.2, 0.4 \rangle$	$\langle 0.4, 0.1, 0.5 \rangle$	$\langle 0.1, 0.5, 0.6 \rangle$
x_2	$\langle 0.5, 0.3, 0.2 \rangle$	$\langle 0.5, 0.2, 0.8 \rangle$	$\langle 0.4, 0.5, 0.4 \rangle$	$\langle 0.6, 0.1, 0.7 \rangle$
x_3	$\langle 0.4, 0.5, 0.2 \rangle$	$\langle 0.2, 0.3, 0.6 \rangle$	$\langle 0.5, 0.2, 0.4 \rangle$	$\langle 0.6, 0.3, 0.4 \rangle$
x_4	$\langle 0.6, 0.1, 0.7 \rangle$	$\langle 0.4, 0.5, 0.7 \rangle$	$\langle 0.3, 0.6, 0.5 \rangle$	$\langle 0.5, 0.3, 0.2 \rangle$
x_5	$\langle 0.3, 0.2, 0.6 \rangle$	$\langle 0.7, 0.3, 0.5 \rangle$	$\langle 0.6, 0.3, 0.5 \rangle$	$\langle 0.8, 0.1, 0.2 \rangle$

Conversely, if \widehat{C} is a SVN β -covering of U , then \widehat{C} is not necessarily a SVN β^2 -covering. In Example 1, suppose $\beta = \langle 0.5, 0.3, 0.8 \rangle$. Then \widehat{C} is a SVN β -covering of U , but it is not a SVN β^2 -covering of U .

By the definition of SVN β -neighborhood, the notion of SVN β^2 -neighborhood is presented in the following definition.

Definition 6. Let \widehat{C} be a SVN β^2 -covering of U and $\widehat{C} = \{C_1, C_2, \dots, C_m\}$. For any $x \in U$, the SVN β^2 -neighborhood $\widetilde{N}_x^{\beta^2}$ of x induced by \widehat{C} can be defined as

$$\widetilde{N}_x^{\beta^2} = \cap_2 \{C_i \in \widehat{C} : C_i(x) \geq_2 \beta\}. \tag{2}$$

Note that $C_i(x)$ is a SVN number $\langle T_{C_i}(x), I_{C_i}(x), F_{C_i}(x) \rangle$. Hence, $C_i(x) \geq_2 \beta$ means $T_{C_i}(x) \geq a$, $I_{C_i}(x) \geq b$ and $F_{C_i}(x) \leq c$, where SVN number $\beta = \langle a, b, c \rangle$.

Remark 1. Let \widehat{C} be a SVN β^2 -covering of U , $\beta = \langle a, b, c \rangle$ and $\widehat{C} = \{C_1, C_2, \dots, C_m\}$. For any $x \in U$,

$$\widetilde{N}_x^{\beta^2} = \cap_2 \{C_i \in \widehat{C} : T_{C_i}(x) \geq a, I_{C_i}(x) \geq b, F_{C_i}(x) \leq c\}. \tag{3}$$

Example 2. Let (U, \widehat{C}) be a SVN β^2 -covering approximation space in Example 1 and $\beta = \langle 0.5, 0.1, 0.8 \rangle$. Then

$$\widetilde{N}_{x_1}^{\beta^2} = C_1 \cap_2 C_2, \widetilde{N}_{x_2}^{\beta^2} = C_1 \cap_2 C_2 \cap_2 C_4, \widetilde{N}_{x_3}^{\beta^2} = C_3 \cap_2 C_4, \widetilde{N}_{x_4}^{\beta^2} = C_1 \cap_2 C_4, \widetilde{N}_{x_5}^{\beta^2} = C_2 \cap_2 C_3 \cap_2 C_4.$$

Hence, all SVN β^2 -neighborhoods are shown in Table 2:

Table 2. The tabular representation of $\widetilde{N}_{x_k}^{\beta^2}$ ($k = 1, 2, 3, 4, 5$).

$\widetilde{N}_{x_k}^{\beta^2}$	x_1	x_2	x_3	x_4	x_5
$\widetilde{N}_{x_1}^{\beta^2}$	$\langle 0.6, 0.2, 0.5 \rangle$	$\langle 0.5, 0.2, 0.8 \rangle$	$\langle 0.2, 0.3, 0.6 \rangle$	$\langle 0.4, 0.1, 0.7 \rangle$	$\langle 0.3, 0.2, 0.6 \rangle$
$\widetilde{N}_{x_2}^{\beta^2}$	$\langle 0.1, 0.2, 0.6 \rangle$	$\langle 0.5, 0.1, 0.8 \rangle$	$\langle 0.2, 0.3, 0.6 \rangle$	$\langle 0.4, 0.1, 0.7 \rangle$	$\langle 0.3, 0.1, 0.6 \rangle$
$\widetilde{N}_{x_3}^{\beta^2}$	$\langle 0.1, 0.1, 0.6 \rangle$	$\langle 0.4, 0.1, 0.7 \rangle$	$\langle 0.5, 0.2, 0.4 \rangle$	$\langle 0.3, 0.3, 0.5 \rangle$	$\langle 0.6, 0.1, 0.5 \rangle$
$\widetilde{N}_{x_4}^{\beta^2}$	$\langle 0.1, 0.2, 0.6 \rangle$	$\langle 0.5, 0.1, 0.7 \rangle$	$\langle 0.4, 0.3, 0.4 \rangle$	$\langle 0.5, 0.1, 0.7 \rangle$	$\langle 0.3, 0.1, 0.6 \rangle$
$\widetilde{N}_{x_5}^{\beta^2}$	$\langle 0.1, 0.1, 0.6 \rangle$	$\langle 0.4, 0.1, 0.8 \rangle$	$\langle 0.2, 0.2, 0.6 \rangle$	$\langle 0.3, 0.3, 0.7 \rangle$	$\langle 0.6, 0.1, 0.5 \rangle$

According to Definitions 3–6, we know that “Let \widehat{C} be a SVN β^2 -covering of U . If \widehat{C} is also a SVN β -covering of U , then $\widetilde{N}_x^{\beta^2}$ and \widetilde{N}_x^β have no inclusion relations (\subseteq_1 and \subseteq_2) for all $x \in U$. To explain this statement, the following example is presented.

Example 3. Let $U = \{x_1, x_2, x_3, x_4, x_5\}$, $\widehat{C} = \{C_1, C_2, C_3, C_4\}$ and $\beta = \langle 0.5, 0.2, 0.8 \rangle$, where \widehat{C} is shown in Table 1 of Example 1. By Definitions 3 and 5, we know that \widehat{C} is a SVN β^2 -covering and also a SVN β -covering of U . Then

$$\begin{aligned} \widetilde{N}_{x_1}^{\beta^2} &= C_1 \cap_2 C_2, \widetilde{N}_{x_2}^{\beta^2} = C_1 \cap_2 C_2, \widetilde{N}_{x_3}^{\beta^2} = C_3 \cap_2 C_4, \widetilde{N}_{x_4}^{\beta^2} = C_4, \widetilde{N}_{x_5}^{\beta^2} = C_2 \cap_2 C_3. \\ \widetilde{N}_{x_1}^\beta &= C_1 \cap_1 C_2, \widetilde{N}_{x_2}^\beta = C_2 \cap_1 C_4, \widetilde{N}_{x_3}^\beta = C_3, \widetilde{N}_{x_4}^\beta = C_1, \widetilde{N}_{x_5}^\beta = C_4. \end{aligned}$$

Hence, all SVN β^2 -neighborhoods and SVN β -neighborhoods are shown in Tables 3, and 4 respectively: By Tables 3 and 4, we see that for all $x_k \in U$ ($k = 1, 2, 3, 4, 5$)

- $\widetilde{N}_{x_k}^{\beta^2} \subseteq_1 \widetilde{N}_{x_k}^\beta$ is not established, since $\widetilde{N}_{x_2}^{\beta^2} \not\subseteq_1 \widetilde{N}_{x_2}^\beta$.
- $\widetilde{N}_{x_k}^\beta \subseteq_1 \widetilde{N}_{x_k}^{\beta^2}$ is not established, since $\widetilde{N}_{x_3}^\beta \not\subseteq_1 \widetilde{N}_{x_3}^{\beta^2}$.
- $\widetilde{N}_{x_k}^{\beta^2} \subseteq_2 \widetilde{N}_{x_k}^\beta$ is not established, since $\widetilde{N}_{x_4}^{\beta^2} \not\subseteq_2 \widetilde{N}_{x_4}^\beta$.
- $\widetilde{N}_{x_k}^\beta \subseteq_2 \widetilde{N}_{x_k}^{\beta^2}$ is not established, since $\widetilde{N}_{x_5}^\beta \not\subseteq_2 \widetilde{N}_{x_5}^{\beta^2}$.

Hence, $\widetilde{N}_{x_k}^{\beta^2}$ and $\widetilde{N}_{x_k}^\beta$ have no inclusion relations (\subseteq_1 and \subseteq_2) for all $x_k \in U$.

In a SVN β^2 -covering approximation space (U, \widehat{C}) , we present the following properties of the SVN β^2 -neighborhood.

Table 3. The tabular representation of $\tilde{N}_{x_k}^{\beta^2}$ ($k = 1, 2, 3, 4, 5$).

$\tilde{N}_{x_k}^{\beta^2}$	x_1	x_2	x_3	x_4	x_5
$\tilde{N}_{x_1}^{\beta^2}$	$\langle 0.6, 0.2, 0.5 \rangle$	$\langle 0.5, 0.2, 0.8 \rangle$	$\langle 0.2, 0.3, 0.6 \rangle$	$\langle 0.4, 0.1, 0.7 \rangle$	$\langle 0.3, 0.2, 0.6 \rangle$
$\tilde{N}_{x_2}^{\beta^2}$	$\langle 0.6, 0.2, 0.5 \rangle$	$\langle 0.5, 0.2, 0.8 \rangle$	$\langle 0.2, 0.3, 0.6 \rangle$	$\langle 0.4, 0.1, 0.7 \rangle$	$\langle 0.3, 0.2, 0.6 \rangle$
$\tilde{N}_{x_3}^{\beta^2}$	$\langle 0.1, 0.1, 0.6 \rangle$	$\langle 0.4, 0.1, 0.7 \rangle$	$\langle 0.5, 0.2, 0.4 \rangle$	$\langle 0.3, 0.3, 0.5 \rangle$	$\langle 0.6, 0.1, 0.5 \rangle$
$\tilde{N}_{x_4}^{\beta^2}$	$\langle 0.1, 0.5, 0.6 \rangle$	$\langle 0.6, 0.1, 0.7 \rangle$	$\langle 0.6, 0.3, 0.4 \rangle$	$\langle 0.5, 0.3, 0.2 \rangle$	$\langle 0.8, 0.1, 0.2 \rangle$
$\tilde{N}_{x_5}^{\beta^2}$	$\langle 0.4, 0.1, 0.5 \rangle$	$\langle 0.4, 0.2, 0.8 \rangle$	$\langle 0.2, 0.2, 0.6 \rangle$	$\langle 0.3, 0.5, 0.7 \rangle$	$\langle 0.6, 0.3, 0.5 \rangle$

Table 4. The tabular representation of $\tilde{N}_{x_k}^{\beta}$ ($k = 1, 2, 3, 4, 5$).

$\tilde{N}_{x_k}^{\beta}$	x_1	x_2	x_3	x_4	x_5
$\tilde{N}_{x_1}^{\beta}$	$\langle 0.6, 0.2, 0.5 \rangle$	$\langle 0.5, 0.3, 0.8 \rangle$	$\langle 0.2, 0.5, 0.6 \rangle$	$\langle 0.4, 0.5, 0.7 \rangle$	$\langle 0.3, 0.3, 0.6 \rangle$
$\tilde{N}_{x_2}^{\beta}$	$\langle 0.1, 0.5, 0.6 \rangle$	$\langle 0.5, 0.2, 0.8 \rangle$	$\langle 0.2, 0.3, 0.6 \rangle$	$\langle 0.4, 0.5, 0.7 \rangle$	$\langle 0.7, 0.3, 0.5 \rangle$
$\tilde{N}_{x_3}^{\beta}$	$\langle 0.4, 0.1, 0.5 \rangle$	$\langle 0.4, 0.5, 0.4 \rangle$	$\langle 0.5, 0.2, 0.4 \rangle$	$\langle 0.3, 0.6, 0.5 \rangle$	$\langle 0.6, 0.3, 0.5 \rangle$
$\tilde{N}_{x_4}^{\beta}$	$\langle 0.7, 0.2, 0.5 \rangle$	$\langle 0.5, 0.3, 0.2 \rangle$	$\langle 0.4, 0.5, 0.2 \rangle$	$\langle 0.6, 0.1, 0.7 \rangle$	$\langle 0.3, 0.2, 0.6 \rangle$
$\tilde{N}_{x_5}^{\beta}$	$\langle 0.1, 0.5, 0.6 \rangle$	$\langle 0.6, 0.1, 0.7 \rangle$	$\langle 0.6, 0.3, 0.4 \rangle$	$\langle 0.5, 0.3, 0.2 \rangle$	$\langle 0.8, 0.1, 0.2 \rangle$

Proposition 1. Let \hat{C} be a SVN β^2 -covering of U and $\hat{C} = \{C_1, C_2, \dots, C_m\}$. Then $\tilde{N}_x^{\beta^2}(x) \geq_2 \beta$ for each $x \in U$.

Proof. For any $x \in U$, $\tilde{N}_x^{\beta^2}(x) = (\bigcap_{C_i(x) \geq_2 \beta} C_i)(x) \geq_2 \beta$. \square

Proposition 2. Let \hat{C} be a SVN β^2 -covering of U and $\hat{C} = \{C_1, C_2, \dots, C_m\}$. For all $x, y, z \in U$, if $\tilde{N}_x^{\beta^2}(y) \geq_2 \beta$, $\tilde{N}_y^{\beta^2}(z) \geq_2 \beta$, then $\tilde{N}_x^{\beta^2}(z) \geq_2 \beta$.

Proof. Let $I = \{1, 2, \dots, m\}$. Since $\tilde{N}_x^{\beta^2}(y) \geq_2 \beta$, for any $i \in I$, if $C_i(x) \geq_2 \beta$, then $C_i(y) \geq_2 \beta$. Since $\tilde{N}_y^{\beta^2}(z) \geq_2 \beta$, for any $i \in I$, $C_i(z) \geq_2 \beta$ when $C_i(y) \geq_2 \beta$. Then, for any $i \in I$, $C_i(x) \geq_2 \beta$ implies $C_i(z) \geq_2 \beta$. Therefore, $\tilde{N}_x^{\beta^2}(z) \geq_2 \beta$. \square

Proposition 3. Let \hat{C} be a SVN β^2 -covering of U and $\hat{C} = \{C_1, C_2, \dots, C_m\}$. For two SVN numbers β_1, β_2 , if $\beta_1 \leq_2 \beta_2 \leq_2 \beta$, then $\tilde{N}_x^{\beta_1} \subseteq_2 \tilde{N}_x^{\beta_2}$ for all $x \in U$.

Proof. For all $x \in U$, since $\beta_1 \leq_2 \beta_2 \leq_2 \beta$, $\{C_i \in \hat{C} : C_i(x) \geq_2 \beta_1\} \supseteq \{C_i \in \hat{C} : C_i(x) \geq_2 \beta_2\}$. Hence, $\tilde{N}_x^{\beta_1} = \bigcap_2 \{C_i \in \hat{C} : C_i(x) \geq_2 \beta_1\} \subseteq_2 \bigcap_2 \{C_i \in \hat{C} : C_i(x) \geq_2 \beta_2\} = \tilde{N}_x^{\beta_2}$ for all $x \in U$. \square

Proposition 4. Let \hat{C} be a SVN β^2 -covering of U . For any $x, y \in U$, $\tilde{N}_x^{\beta^2}(y) \geq_2 \beta$ if and only if $\tilde{N}_y^{\beta^2} \subseteq_2 \tilde{N}_x^{\beta^2}$.

Proof. Suppose the SVN number $\beta = \langle a, b, c \rangle$.

(\Rightarrow): Since $\tilde{N}_x^{\beta^2}(y) \geq_2 \beta$,

$$T_{\tilde{N}_x^{\beta^2}}(y) = T_{\bigcap_2_{C_i(x) \geq a} C_i}(y) = \bigwedge_{T_{C_i}(x) \geq a} T_{C_i}(y) \geq a, I_{\tilde{N}_x^{\beta^2}}(y) = I_{\bigcap_2_{C_i(x) \geq a} C_i}(y) = \bigwedge_{I_{C_i}(x) \geq a} I_{C_i}(y) \geq b,$$

$$I_{\tilde{N}_x^{\beta^2}}(y) = I_{\bigcap_2_{C_i(x) \geq b} C_i}(y) = \bigwedge_{I_{C_i}(x) \geq b} I_{C_i}(y) \geq b,$$

$$F_{\tilde{N}_x^{\beta^2}}(y) = F_{\bigcap_2_{C_i(x) \leq c} C_i}(y) = \bigwedge_{F_{C_i}(x) \leq c} F_{C_i}(y) \leq c$$

and

$$F_{\tilde{N}_x^{\beta^2}}(y) = F_{\bigcap_{\substack{T_{C_i}(x) \geq a \\ I_{C_i}(x) \geq b \\ F_{C_i}(x) \leq c}} C_i}(y) = \bigvee_{\substack{T_{C_i}(x) \geq a \\ I_{C_i}(x) \geq b \\ F_{C_i}(x) \leq c}} F_{C_i}(y) \leq c.$$

Then,

$$\{C_i \in \hat{\mathcal{C}} : T_{C_i}(x) \geq a, I_{C_i}(x) \geq b, F_{C_i}(x) \leq c\} \subseteq \{C_i \in \hat{\mathcal{C}} : T_{C_i}(y) \geq a, I_{C_i}(y) \geq b, F_{C_i}(y) \leq c\}.$$

Therefore, for each $z \in U$,

$$\begin{aligned} T_{\tilde{N}_x^{\beta^2}}(z) &= \bigwedge_{\substack{T_{C_i}(x) \geq a \\ I_{C_i}(x) \geq b \\ F_{C_i}(x) \leq c}} T_{C_i}(z) \geq \bigwedge_{\substack{T_{C_i}(y) \geq a \\ I_{C_i}(y) \geq b \\ F_{C_i}(y) \leq c}} T_{C_i}(z) = T_{\tilde{N}_y^{\beta^2}}(z), \\ I_{\tilde{N}_x^{\beta^2}}(z) &= \bigwedge_{\substack{T_{C_i}(x) \geq a \\ I_{C_i}(x) \geq b \\ F_{C_i}(x) \leq c}} I_{C_i}(z) \geq \bigwedge_{\substack{T_{C_i}(y) \geq a \\ I_{C_i}(y) \geq b \\ F_{C_i}(y) \leq c}} I_{C_i}(z) = I_{\tilde{N}_y^{\beta^2}}(z), \\ F_{\tilde{N}_x^{\beta^2}}(z) &= \bigvee_{\substack{T_{C_i}(x) \geq a \\ I_{C_i}(x) \geq b \\ F_{C_i}(x) \leq c}} F_{C_i}(z) \leq \bigvee_{\substack{T_{C_i}(y) \geq a \\ I_{C_i}(y) \geq b \\ F_{C_i}(y) \leq c}} F_{C_i}(z) = F_{\tilde{N}_y^{\beta^2}}(z). \end{aligned}$$

Hence, $\tilde{N}_y^{\beta^2} \subseteq_2 \tilde{N}_x^{\beta^2}$.

(\Leftarrow): For any $x, y \in U$, since $\tilde{N}_y^{\beta^2} \subseteq_2 \tilde{N}_x^{\beta^2}$,

$$T_{\tilde{N}_x^{\beta^2}}(y) \geq T_{\tilde{N}_y^{\beta^2}}(y) \geq a, I_{\tilde{N}_x^{\beta^2}}(y) \geq I_{\tilde{N}_y^{\beta^2}}(y) \geq b \text{ and } F_{\tilde{N}_x^{\beta^2}}(y) \leq F_{\tilde{N}_y^{\beta^2}}(y) \leq c.$$

Therefore, $\tilde{N}_x^{\beta^2}(y) \geq_2 \beta$. \square

4. A Type of SVN Covering Rough Set Model Based on a New Inclusion Relation

In this section, we propose a type of SVN covering rough set model on the basis of the SVN β^2 -neighborhoods, which is decided by a type-2 inclusion relation. Then, we investigate some properties of the new lower and upper SVN covering approximation operators. Finally, some relationships between this model and some other rough set models are presented.

4.1. Characteristics of the New Type of SVN Covering Rough Set Model Based on the New Inclusion Relation

Definition 7. Let $(U, \hat{\mathcal{C}})$ be a SVN β^2 -covering approximation space. For each $A \in SVN(U)$, where $A = \{ \langle x, T_A(x), I_A(x), F_A(x) \rangle : x \in U \}$, we define the type-2 SVN covering upper approximation $\tilde{\mathcal{C}}^2(A)$ and lower approximation $\tilde{\mathcal{C}}^2(A)$ of A as

$$\begin{aligned} \tilde{\mathcal{C}}^2(A) &= \{ \langle x, \bigvee_{y \in U} [T_{\tilde{N}_x^{\beta^2}}(y) \wedge T_A(y)], \bigvee_{y \in U} [I_{\tilde{N}_x^{\beta^2}}(y) \wedge I_A(y)], \bigwedge_{y \in U} [F_{\tilde{N}_x^{\beta^2}}(y) \vee F_A(y)] \rangle : x \in U \}, \\ \tilde{\mathcal{C}}^2(A) &= \{ \langle x, \bigwedge_{y \in U} [F_{\tilde{N}_x^{\beta^2}}(y) \vee T_A(y)], \bigwedge_{y \in U} [(1 - I_{\tilde{N}_x^{\beta^2}}(y)) \vee I_A(y)], \bigvee_{y \in U} [T_{\tilde{N}_x^{\beta^2}}(y) \wedge F_A(y)] \rangle : x \in U \}. \end{aligned} \tag{4}$$

If $\tilde{\mathcal{C}}^2(A) \neq \tilde{\mathcal{C}}^2(A)$, then A is called the type-2 SVN covering rough set.

Example 4. Let (U, \widehat{C}) be a SVN β^2 -covering approximation space in Example 1, $\beta = \langle 0.5, 0.1, 0.8 \rangle$ and $A = \frac{(0.6, 0.3, 0.5)}{x_1} + \frac{(0.4, 0.5, 0.1)}{x_2} + \frac{(0.3, 0.2, 0.6)}{x_3} + \frac{(0.5, 0.3, 0.4)}{x_4} + \frac{(0.7, 0.2, 0.3)}{x_5}$. Then all SVN β^2 -neighborhoods $\widetilde{N}_{x_k}^{\beta^2}$ ($k = 1, 2, 3, 4, 5$) are shown in Table 2 of Example 2. By Definition 7, we have

$$\begin{aligned} \widetilde{C}^2(A) &= \{ \langle x_1, 0.6, 0.2, 0.5 \rangle, \langle x_2, 0.4, 0.2, 0.6 \rangle, \langle x_3, 0.6, 0.3, 0.5 \rangle, \langle x_4, 0.5, 0.2, 0.6 \rangle, \langle x_5, 0.6, 0.3, 0.5 \rangle \}, \\ \underline{C}^2(A) &= \{ \langle x_1, 0.6, 0.8, 0.5 \rangle, \langle x_2, 0.6, 0.8, 0.4 \rangle, \langle x_3, 0.4, 0.7, 0.5 \rangle, \langle x_4, 0.4, 0.7, 0.4 \rangle, \langle x_5, 0.6, 0.8, 0.3 \rangle \}. \end{aligned}$$

Let the SVN universe set be $U = \{ \langle x, 1, 1, 0 \rangle : x \in U \}$ and the SVN empty set be $\emptyset = \{ \langle x, 0, 0, 1 \rangle : x \in U \}$, which are decided by the type-2 inclusion relation \subseteq_2 . Some basic properties of the type-2 SVN covering upper and lower approximation operators are presented in the following proposition.

Proposition 5. Let \widehat{C} be a SVN β^2 -covering of U . Then the type-2 SVN covering upper and lower approximation operators in Definition 7 satisfy the following properties for all $A, B \in \text{SVN}(U)$.

- (1) $\underline{C}^2(U) = U, \widetilde{C}^2(\emptyset) = \emptyset;$
- (2) $\widetilde{C}^2(A') = (\underline{C}^2(A))', \underline{C}^2(A') = (\widetilde{C}^2(A))';$
- (3) If $A \subseteq_2 B$, then $\underline{C}^2(A) \subseteq_2 \underline{C}^2(B), \widetilde{C}^2(A) \subseteq_2 \widetilde{C}^2(B);$
- (4) $\underline{C}^2(A \cap_2 B) = \underline{C}^2(A) \cap_2 \underline{C}^2(B), \widetilde{C}^2(A \cup_2 B) = \widetilde{C}^2(A) \cup_2 \widetilde{C}^2(B);$
- (5) $\underline{C}^2(A \cup_2 B) \supseteq_2 \underline{C}^2(A) \cup_2 \underline{C}^2(B), \widetilde{C}^2(A \cap_2 B) \subseteq_2 \widetilde{C}^2(A) \cap_2 \widetilde{C}^2(B).$

Proof.(1) Since the SVN universe set is $U = \{ \langle x, 1, 1, 0 \rangle : x \in U \}$ and the SVN empty set is $\emptyset = \{ \langle x, 0, 0, 1 \rangle : x \in U \},$

$$\begin{aligned} \underline{C}^2(U) &= \{ \langle x, \wedge_{y \in U} [F_{\widetilde{N}_x^{\beta^2}}(y) \vee T_U(y)], \wedge_{y \in U} [(1 - I_{\widetilde{N}_x^{\beta^2}}(y)) \vee I_U(y)], \vee_{y \in U} [T_{\widetilde{N}_x^{\beta^2}}(y) \wedge F_U(y)] \rangle : x \in U \} \\ &= \{ \langle x, 1, 1, 0 \rangle : x \in U \} \\ &= U, \end{aligned}$$

and

$$\begin{aligned} \widetilde{C}^2(\emptyset) &= \{ \langle x, \vee_{y \in U} [T_{\widetilde{N}_x^{\beta^2}}(y) \wedge T_{\emptyset}(y)], \vee_{y \in U} [I_{\widetilde{N}_x^{\beta^2}}(y) \wedge I_{\emptyset}(y)], \wedge_{y \in U} [F_{\widetilde{N}_x^{\beta^2}}(y) \vee F_{\emptyset}(y)] \rangle : x \in U \} \\ &= \{ \langle x, 0, 0, 1 \rangle : x \in U \} \\ &= \emptyset; \end{aligned}$$

(2)

$$\begin{aligned} \widetilde{C}^2(A') &= \{ \langle x, \vee_{y \in U} [T_{\widetilde{N}_x^{\beta^2}}(y) \wedge T_{A'}(y)], \vee_{y \in U} [I_{\widetilde{N}_x^{\beta^2}}(y) \wedge I_{A'}(y)], \wedge_{y \in U} [F_{\widetilde{N}_x^{\beta^2}}(y) \vee F_{A'}(y)] \rangle : x \in U \} \\ &= \{ \langle x, \vee_{y \in U} [T_{\widetilde{N}_x^{\beta^2}}(y) \wedge F_A(y)], \vee_{y \in U} [I_{\widetilde{N}_x^{\beta^2}}(y) \wedge (1 - I_A(y))], \wedge_{y \in U} [F_{\widetilde{N}_x^{\beta^2}}(y) \vee T_A(y)] \rangle : x \in U \} \\ &= (\underline{C}^2(A))'. \end{aligned}$$

If we replace A by A' in this proof, we can also prove $\underline{C}^2(A') = (\widetilde{C}^2(A))'$.

(3) Since $A \subseteq_2 B, T_A(x) \leq T_B(x), I_A(x) \leq I_B(x)$ and $F_B(x) \leq F_A(x)$ for all $x \in U$. Therefore

$$\begin{aligned} T_{\underline{C}^2(A)}(x) &= \wedge_{y \in U} [F_{\widetilde{N}_x^{\beta^2}}(y) \vee T_A(y)] \leq \wedge_{y \in U} [F_{\widetilde{N}_x^{\beta^2}}(y) \vee T_B(y)] = T_{\underline{C}^2(B)}(x), \\ I_{\underline{C}^2(A)}(x) &= \wedge_{y \in U} [(1 - I_{\widetilde{N}_x^{\beta^2}}(y)) \vee I_A(y)] \leq \wedge_{y \in U} [(1 - I_{\widetilde{N}_x^{\beta^2}}(y)) \vee I_B(y)] = I_{\underline{C}^2(B)}(x), \\ F_{\underline{C}^2(A)}(x) &= \vee_{y \in U} [T_{\widetilde{N}_x^{\beta^2}}(y) \wedge F_A(y)] \geq \vee_{y \in U} [T_{\widetilde{N}_x^{\beta^2}}(y) \wedge F_B(y)] = F_{\underline{C}^2(B)}(x). \end{aligned}$$

Hence, $\underline{C}^2(A) \subseteq_2 \underline{C}^2(B)$. In the same way, there is $\widetilde{C}^2(A) \subseteq_2 \widetilde{C}^2(B);$

(4) Since

$$\begin{aligned}
 & \underline{\mathbb{C}}^2(A \cap_2 B) \\
 = & \{ \langle x, \wedge_{y \in U} [F_{\tilde{N}_x^{\beta^2}}(y) \vee T_{A \cap_2 B}(y)], \wedge_{y \in U} [(1 - I_{\tilde{N}_x^{\beta^2}}(y)) \vee I_{A \cap_2 B}(y)], \vee_{y \in U} [T_{\tilde{N}_x^{\beta^2}}(y) \wedge F_{A \cap_2 B}(y)] \rangle : x \in U \} \\
 = & \{ \langle x, \wedge_{y \in U} [F_{\tilde{N}_x^{\beta^2}}(y) \vee (T_A(y) \wedge T_B(y))], \wedge_{y \in U} [(1 - I_{\tilde{N}_x^{\beta^2}}(y)) \vee (I_A(y) \wedge I_B(y))], \vee_{y \in U} [T_{\tilde{N}_x^{\beta^2}}(y) \wedge (F_A(y) \vee F_B(y))] \rangle : x \in U \} \\
 = & \{ \langle x, \wedge_{y \in U} [(F_{\tilde{N}_x^{\beta^2}}(y) \vee T_A(y)) \wedge (F_{\tilde{N}_x^{\beta^2}}(y) \vee T_B(y))], \wedge_{y \in U} [((1 - I_{\tilde{N}_x^{\beta^2}}(y)) \vee I_A(y)) \wedge ((1 - I_{\tilde{N}_x^{\beta^2}}(y)) \vee I_B(y))], \vee_{y \in U} [(T_{\tilde{N}_x^{\beta^2}}(y) \wedge F_A(y)) \vee (T_{\tilde{N}_x^{\beta^2}}(y) \wedge F_B(y))] \rangle : x \in U \} \\
 = & \underline{\mathbb{C}}^2(A) \cap_2 \underline{\mathbb{C}}^2(B).
 \end{aligned}$$

Similarly, we can obtain $\tilde{\mathbb{C}}^2(A \cup_2 B) = \tilde{\mathbb{C}}^2(A) \cup_2 \tilde{\mathbb{C}}^2(B)$;

(5) Since $A \subseteq_2 (A \cup_2 B)$, $B \subseteq_2 (A \cup_2 B)$, $(A \cap_2 B) \subseteq_2 A$ and $(A \cap_2 B) \subseteq_2 B$,

$$\underline{\mathbb{C}}^2(A) \subseteq_2 \underline{\mathbb{C}}^2(A \cup_2 B), \underline{\mathbb{C}}^2(B) \subseteq_2 \underline{\mathbb{C}}^2(A \cup_2 B), \tilde{\mathbb{C}}^2(A \cap_2 B) \subseteq_2 \tilde{\mathbb{C}}^2(A) \text{ and } \tilde{\mathbb{C}}^2(A \cap_2 B) \subseteq_2 \tilde{\mathbb{C}}^2(B).$$

Hence, $\underline{\mathbb{C}}^2(A \cup_2 B) \supseteq_2 \underline{\mathbb{C}}^2(A) \cup_2 \underline{\mathbb{C}}^2(B)$, $\tilde{\mathbb{C}}^2(A \cap_2 B) \subseteq_2 \tilde{\mathbb{C}}^2(A) \cap_2 \tilde{\mathbb{C}}^2(B)$.

□

4.2. Relationships between the New Model and Some Other Rough Set Models

In this subsection, we investigate some relationships between the type-2 SVN covering rough set model and other two SVN rough set models respectively. Among these two SVN rough set models, one is a SVN covering rough set model and the other is a SVN relation rough set model.

Wang and Zhang [37] presented the type-1 SVN covering rough set model under a SVN β -covering approximation space, which is related to the type-1 inclusion relation. We consider whether the type-1 SVN covering approximate operators and the type-2 SVN covering approximate operators presented in Section 4.1 have inclusion relations.

Definition 8. [37] Let $(U, \hat{\mathbb{C}})$ be a SVN β -covering approximation space. For each $A \in SVN(U)$, where $A = \{ \langle x, T_A(x), I_A(x), F_A(x) \rangle : x \in U \}$, we define the type-1 SVN covering upper approximation $\tilde{\mathbb{C}}(A)$ and lower approximation $\underline{\mathbb{C}}(A)$ of A as

$$\begin{aligned}
 \tilde{\mathbb{C}}(A) &= \{ \langle x, \vee_{y \in U} [T_{\tilde{N}_x^\beta}(y) \wedge T_A(y)], \vee_{y \in U} [I_{\tilde{N}_x^\beta}(y) \wedge I_A(y)], \wedge_{y \in U} [F_{\tilde{N}_x^\beta}(y) \vee F_A(y)] \rangle : x \in U \}, \\
 \underline{\mathbb{C}}(A) &= \{ \langle x, \wedge_{y \in U} [F_{\tilde{N}_x^\beta}(y) \vee T_A(y)], \wedge_{y \in U} [(1 - I_{\tilde{N}_x^\beta}(y)) \vee I_A(y)], \vee_{y \in U} [T_{\tilde{N}_x^\beta}(y) \wedge F_A(y)] \rangle : x \in U \}.
 \end{aligned}
 \tag{5}$$

If $\tilde{\mathbb{C}}(A) \neq \underline{\mathbb{C}}(A)$, then A is called the type-1 SVN covering rough set.

Let $\hat{\mathbb{C}}$ be a SVN β^2 -covering of U and also be a SVN β -covering of U . By Definitions 7 and 9, we know that the type-1 SVN covering approximate operators ($\tilde{\mathbb{C}}$ and $\underline{\mathbb{C}}$) and the type-2 SVN covering approximate operators ($\tilde{\mathbb{C}}^2$ and $\underline{\mathbb{C}}^2$) are related to all SVN β -neighborhoods (\tilde{N}_x^β , for any $x \in U$) and SVN β^2 -neighborhoods ($\tilde{N}_x^{\beta^2}$, for any $x \in U$), respectively. By Example 3, we know that $\tilde{N}_x^{\beta^2}$ and \tilde{N}_x^β have no inclusion relations (\subseteq_1 and \subseteq_2) for all $x \in U$. Hence, the type-1 SVN covering approximate operators and the type-2 SVN covering approximate operators also have no inclusion relations (\subseteq_1 and \subseteq_2).

In the work by the authors of [36], a SVN relation R on U is defined as $R = \{ \langle (x, y), T_R(x, y), I_R(x, y), F_R(x, y) \rangle : (x, y) \in U \times U \}$, where $T_R : U \times U \rightarrow [0, 1]$, $I_R : U \times U \rightarrow [0, 1]$ and $F_R : U \times U \rightarrow [0, 1]$. If for any $x, y \in U$, $T_R(x, y) = T_R(y, x)$, $I_R(x, y) = I_R(y, x)$ and $F_R(x, y) = F_R(y, x)$, then R is called a symmetric SVN relation.

For a SVN β^2 -covering \widehat{C} of U , one can use the SVN β^2 -covering \widehat{C} induce a SVN relation $R_{\widehat{C}}$ on U as

$$R_{\widehat{C}} = \{ \langle (x, y), T_{R_{\widehat{C}}}(x, y), I_{R_{\widehat{C}}}(x, y), F_{R_{\widehat{C}}}(x, y) \rangle : (x, y) \in U \times U \},$$

where

$$T_{R_{\widehat{C}}}(x, y) = T_{\widetilde{N}_x^{\beta^2}}(y), I_{R_{\widehat{C}}}(x, y) = I_{\widetilde{N}_x^{\beta^2}}(y), F_{R_{\widehat{C}}}(x, y) = F_{\widetilde{N}_x^{\beta^2}}(y) \text{ for any } x, y \in U.$$

The following two propositions present two conditions under which $R_{\widehat{C}}$ is a symmetric SVN relation.

Proposition 6. Let \widehat{C} be a SVN β^2 -covering of U , and $R_{\widehat{C}}$ be the induced SVN relation on U by \widehat{C} . If $\widetilde{N}_x^{\beta^2}(y) = \widetilde{N}_y^{\beta^2}(x)$ for any $x, y \in U$, then $R_{\widehat{C}}$ is a symmetric SVN relation.

Proof. Since $\widetilde{N}_x^{\beta^2}(y) = \widetilde{N}_y^{\beta^2}(x)$ for any $x, y \in U$, $T_{\widetilde{N}_x^{\beta^2}}(y) = T_{\widetilde{N}_y^{\beta^2}}(x)$, $I_{\widetilde{N}_x^{\beta^2}}(y) = I_{\widetilde{N}_y^{\beta^2}}(x)$ and $F_{\widetilde{N}_x^{\beta^2}}(y) = F_{\widetilde{N}_y^{\beta^2}}(x)$. Hence, $T_{R_{\widehat{C}}}(x, y) = T_{R_{\widehat{C}}}(y, x)$, $I_{R_{\widehat{C}}}(x, y) = I_{R_{\widehat{C}}}(y, x)$ and $F_{R_{\widehat{C}}}(x, y) = F_{R_{\widehat{C}}}(y, x)$, i.e., $R_{\widehat{C}}$ is a symmetric SVN relation. \square

Proposition 7. Let \widehat{C} be a SVN β^2 -covering of U , $R_{\widehat{C}}$ be the induced SVN relation on U by \widehat{C} , and $C \in \widehat{C}$. If $|\widehat{C}| = 1$ and $C(x) = C(y)$ for any $x, y \in U$, then $R_{\widehat{C}}$ is a symmetric SVN relation, where $|\widehat{C}|$ denotes the cardinality of \widehat{C} .

Proof. Since $|\widehat{C}| = 1$, C is the only one element of \widehat{C} . Since $C(x) = C(y)$ for any $x, y \in U$, $\widetilde{N}_x^{\beta^2}(y) = \widetilde{N}_y^{\beta^2}(x)$. Hence, $T_{R_{\widehat{C}}}(x, y) = T_{R_{\widehat{C}}}(y, x)$, $I_{R_{\widehat{C}}}(x, y) = I_{R_{\widehat{C}}}(y, x)$ and $F_{R_{\widehat{C}}}(x, y) = F_{R_{\widehat{C}}}(y, x)$, i.e., $R_{\widehat{C}}$ is a symmetric SVN relation. \square

Then, the type-2 SVN covering rough set model defined in Section 4.1 can be viewed as a SVN relation rough set model.

Definition 9. Let \widehat{C} be a SVN β^2 -covering of U , and $R_{\widehat{C}}$ be the induced SVN relation on U by \widehat{C} . For any $A \in SVN(U)$, the upper approximation $\widetilde{R}_{\widehat{C}}(A)$ and lower approximation $\underset{\sim}{R}_{\widehat{C}}(A)$ of A are defined as

$$\begin{aligned} \widetilde{R}_{\widehat{C}}(A) &= \{ \langle x, \bigvee_{y \in U} [T_{R_{\widehat{C}}}(x, y) \wedge T_A(y)], \bigvee_{y \in U} [I_{R_{\widehat{C}}}(x, y) \wedge I_A(y)], \bigwedge_{y \in U} [F_{R_{\widehat{C}}}(x, y) \vee F_A(y)] \rangle : x \in U \}, \\ \underset{\sim}{R}_{\widehat{C}}(A) &= \{ \langle x, \bigwedge_{y \in U} [F_{R_{\widehat{C}}}(x, y) \vee T_A(y)], \bigwedge_{y \in U} [(1 - I_{R_{\widehat{C}}}(x, y)) \vee I_A(y)], \bigvee_{y \in U} [T_{R_{\widehat{C}}}(x, y) \wedge F_A(y)] \rangle : x \in U \}. \end{aligned}$$

Remark 2. Let \widehat{C} be a SVN β^2 -covering of U , and $R_{\widehat{C}}$ be the induced SVN relation on U by \widehat{C} . Then

$$\begin{aligned} \widetilde{R}_{\widehat{C}}(A) &= \widetilde{C}^2(A), \\ \underset{\sim}{R}_{\widehat{C}}(A) &= \underset{\sim}{C}^2(A). \end{aligned}$$

5. Graph Representation of the Type-2 SVN Covering Rough Set Model

In this section, the graph representation of the type-2 SVN covering rough set model is presented. Firstly, some new graphs and graph operations are presented. Then, we show the graph representation of the type-2 SVN covering approximation operators defined in Definition 7. The order of elements in U is given.

A graph is a pair $G = (V, E)$ consisting of a nonempty set V of vertices and a set E of edges such that $E \subseteq U \times U$. We shall often write $V(G)$ for V and $E(G)$ for E , particularly when several graphs are being considered. Two vertices are adjacent if there is an edge with them as ends. A graph $G = (V, E)$

is called bipartite if the vertex set V can be divided into two disjoint sets V_1 and V_2 , such that every edge connects a vertex in V_1 to one in V_2 . One often writes $G = (V_1 \cup V_2, E)$ to denote a bipartite graph whose partition has the partite sets V_1 and V_2 . A complete bipartite graph is a simple bipartite graph such that two vertices are adjacent if and only if they are in different partite sets. A weighted graph is a graph with numerical labels on the edges.

Firstly, the graph representation of the SVN β^2 -covering \widehat{C} is defined in the following definition.

Definition 10. Let \widehat{C} be a SVN β^2 -covering of U with $U = \{x_1, x_2, \dots, x_n\}$ and $\widehat{C} = \{C_1, C_2, \dots, C_m\}$. For any $A \in SVN(U)$, we define a completely weighted bipartite graph $G(A) = (U \cup V, E)$, named completely weighted bipartite graph associated with A , where $V = \{T_A, I_A, F_A\}$, the weight $w(T_A, x_k) = T_A(x_k)$, $w(I_A, x_k) = I_A(x_k)$ and $w(F_A, x_k) = F_A(x_k)$ ($k = 1, 2, \dots, n$). For the SVN β^2 -covering \widehat{C} , there are m completely weighted bipartite graphs $G(C_i)$ ($i = 1, 2, \dots, m$), and all $G(C_i)$ are called the graph representation of the SVN β^2 -covering \widehat{C} .

Example 5. Let (U, \widehat{C}) be a SVN β^2 -covering approximation space in Example 1 and $\beta = \langle 0.5, 0.1, 0.8 \rangle$. Then $G(C_i)$ ($i = 1, 2, 3, 4$) are the graph representation of the SVN β^2 -covering \widehat{C} . All $G(C_i)$ ($i = 1, 2, 3, 4$) are shown in Figures 1 and 2.

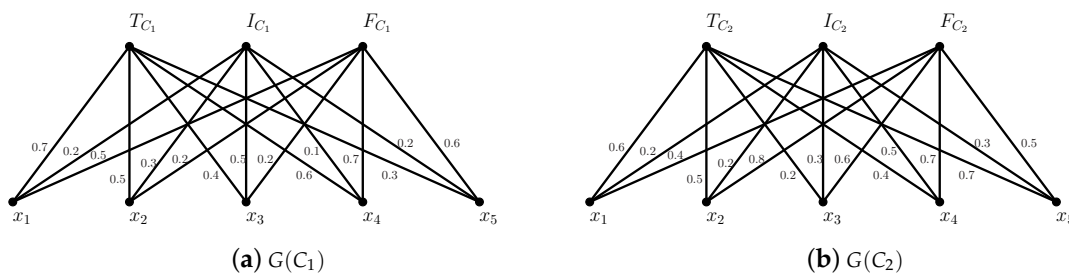


Figure 1. $G(C_1)$ and $G(C_2)$.

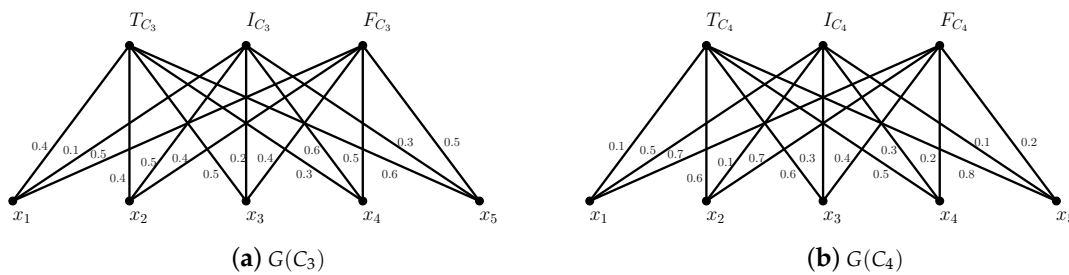


Figure 2. $G(C_3)$ and $G(C_4)$.

An intersection operation about $G(A)$ and $G(B)$ is presented in the following definition, for any $A, B \in SVN(U)$.

Definition 11. Let \widehat{C} be a SVN β^2 -covering of U with $U = \{x_1, x_2, \dots, x_n\}$ and $\widehat{C} = \{C_1, C_2, \dots, C_m\}$. For any $A, B, D \in SVN(U)$, we define a completely weighted bipartite graph $G(D) = G(A) \cap_2 G(B)$ associated with D , where $G(D) = (U \cup \{T_D, I_D, F_D\}, E)$ and

$$w(T_D, x_k) = w(T_A, x_k) \wedge w(T_B, x_k), w(I_D, x_k) = w(I_A, x_k) \wedge w(I_B, x_k) \text{ and } w(F_D, x_k) = w(F_A, x_k) \vee w(F_B, x_k) \text{ (} k = 1, 2, \dots, n \text{)}.$$

Based on Definition 11 and the definition of $A \cap_2 B$, the relationship between $G(A \cap_2 B)$ and $G(A) \cap_2 G(B)$ can be obtained for any $A, B \in SVN(U)$.

Lemma 1. Let \widehat{C} be a SVN β^2 -covering of U . Then $G(A \cap_2 B) = G(A) \cap_2 G(B)$ for any $A, B \in SVN(U)$.

Proof. According to the definition of $A \cap_2 B$ and Definition 11, it is immediate. \square

By Definition 6, $\widetilde{N}_{x_k}^{\beta^2} \in SVN(U)$ for any $x_k \in U$. Hence, any $G(\widetilde{N}_{x_k}^{\beta^2})$ is a completely weighted bipartite graph, which can be represented in the following proposition.

Proposition 8. Let \widehat{C} be a SVN β^2 -covering of U with $U = \{x_1, x_2, \dots, x_n\}$, $\widehat{C} = \{C_1, C_2, \dots, C_m\}$ and $\beta = \langle a, b, c \rangle$. Then

$$G(\widetilde{N}_{x_k}^{\beta^2}) = \cap_2 \{G(C_i) : w(T_{C_i}, x_k) \geq a, w(I_{C_i}, x_k) \geq b, w(F_{C_i}, x_k) \leq c\}.$$

Proof. By Definitions 6 and 11, and Lemma 1, it is immediate. \square

Example 6. Let (U, \widehat{C}) be a SVN β^2 -covering approximation space in Example 1 and $\beta = \langle 0.5, 0.1, 0.8 \rangle$. By Proposition 8, we have

$$G(\widetilde{N}_{x_1}^{\beta^2}) = G(C_1) \cap_2 G(C_2), G(\widetilde{N}_{x_2}^{\beta^2}) = G(C_1) \cap_2 G(C_2) \cap_2 G(C_4), G(\widetilde{N}_{x_3}^{\beta^2}) = G(C_3) \cap_2 G(C_4),$$

$$G(\widetilde{N}_{x_4}^{\beta^2}) = G(C_1) \cap_2 G(C_4), G(\widetilde{N}_{x_5}^{\beta^2}) = G(C_2) \cap_2 G(C_3) \cap_2 G(C_4).$$

Then all $G(\widetilde{N}_{x_k}^{\beta^2})$ are shown in Figures 3, 4, and 5a.

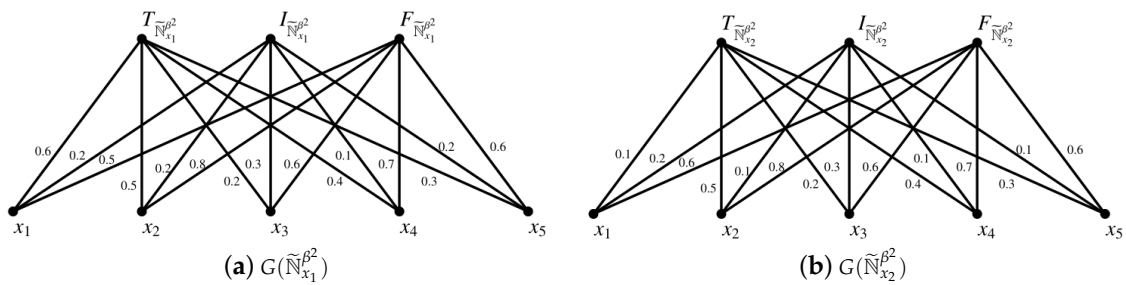


Figure 3. $G(\widetilde{N}_{x_1}^{\beta^2})$ and $G(\widetilde{N}_{x_2}^{\beta^2})$.

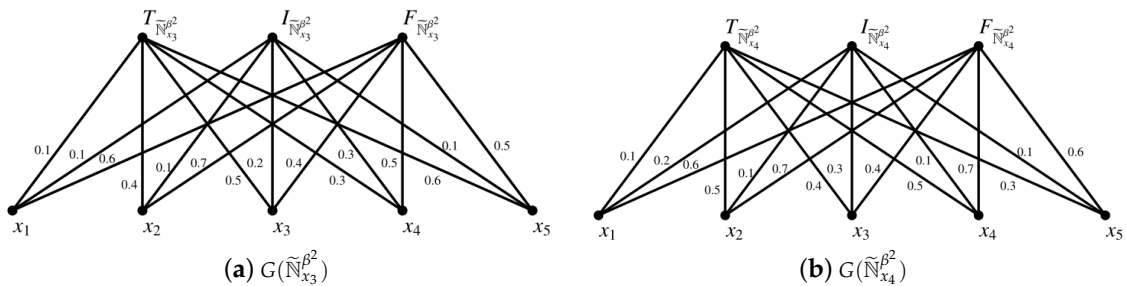


Figure 4. $G(\widetilde{N}_{x_3}^{\beta^2})$ and $G(\widetilde{N}_{x_4}^{\beta^2})$.

Finally, the type-2 SVN covering upper approximation $\widetilde{C}^2(A)$ and lower approximation $\underline{C}^2(A)$ of A are represented by graphs.

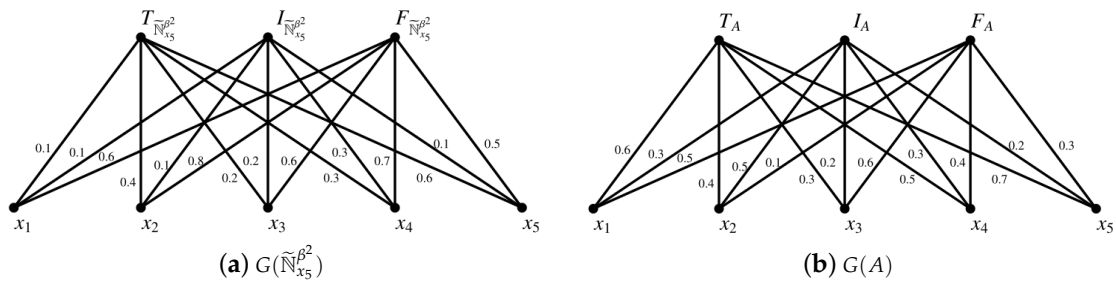


Figure 5. $G(\tilde{N}_{x_5}^{\beta^2})$ and $G(A)$.

Theorem 1. Let \hat{C} be a SVN β^2 -covering of U with $U = \{x_1, x_2, \dots, x_n\}$. For each $A \in SVN(U)$, $G(\tilde{C}^2(A))$ and $G(\underline{C}^2(A))$ are completely weighted bipartite graphs, where $G(\tilde{C}^2(A)) = (U \cup \{T_{\tilde{C}^2(A)}, I_{\tilde{C}^2(A)}, F_{\tilde{C}^2(A)}\}, E_1)$, $G(\underline{C}^2(A)) = (U \cup \{T_{\underline{C}^2(A)}, I_{\underline{C}^2(A)}, F_{\underline{C}^2(A)}\}, E_2)$ and the weight of any edge is listed as follows.

$$\begin{aligned}
 w(T_{\tilde{C}^2(A)}, x_k) &= \bigvee_{j=1}^n [w(T_{\tilde{N}_{x_k}^{\beta^2}}, x_j) \wedge w(T_A, x_j)] \quad (1 \leq k \leq n), \\
 w(I_{\tilde{C}^2(A)}, x_k) &= \bigvee_{j=1}^n [w(I_{\tilde{N}_{x_k}^{\beta^2}}, x_j) \wedge w(I_A, x_j)] \quad (1 \leq k \leq n), \\
 w(F_{\tilde{C}^2(A)}, x_k) &= \bigwedge_{j=1}^n [w(F_{\tilde{N}_{x_k}^{\beta^2}}, x_j) \vee w(F_A, x_j)] \quad (1 \leq k \leq n), \\
 w(T_{\underline{C}^2(A)}, x_k) &= \bigwedge_{j=1}^n [w(F_{\tilde{N}_{x_k}^{\beta^2}}, x_j) \vee w(T_A, x_j)] \quad (1 \leq k \leq n), \\
 w(I_{\underline{C}^2(A)}, x_k) &= \bigwedge_{j=1}^n [(1 - w(I_{\tilde{N}_{x_k}^{\beta^2}}, x_j)) \vee w(I_A, x_j)] \quad (1 \leq k \leq n), \\
 w(F_{\underline{C}^2(A)}, x_k) &= \bigvee_{j=1}^n [w(T_{\tilde{N}_{x_k}^{\beta^2}}, x_j) \wedge w(F_A, x_j)] \quad (1 \leq k \leq n).
 \end{aligned}
 \tag{6}$$

Proof. According to Definition 7, we know $\tilde{C}^2(A) \in SVN(U)$ and $\underline{C}^2(A) \in SVN(U)$ for any $A \in SVN(U)$. Hence, $G(\tilde{C}^2(A))$ and $G(\underline{C}^2(A))$ are completely weighted bipartite graphs by Definition 10. According to Definitions 7 and 10, $G(\tilde{C}^2(A)) = (U \cup \{T_{\tilde{C}^2(A)}, I_{\tilde{C}^2(A)}, F_{\tilde{C}^2(A)}\}, E_1)$, $G(\underline{C}^2(A)) = (U \cup \{T_{\underline{C}^2(A)}, I_{\underline{C}^2(A)}, F_{\underline{C}^2(A)}\}, E_2)$ and the weight of any edge is shown as follows.

$$\begin{aligned}
 w(T_{\tilde{C}^2(A)}, x_k) &= T_{\tilde{C}^2(A)}(x_k) = \bigvee_{j=1}^n [T_{\tilde{N}_{x_k}^{\beta^2}}(x_j) \wedge T_A(x_j)] = \bigvee_{j=1}^n [w(T_{\tilde{N}_{x_k}^{\beta^2}}, x_j) \wedge w(T_A, x_j)] \quad (1 \leq k \leq n), \\
 w(I_{\tilde{C}^2(A)}, x_k) &= I_{\tilde{C}^2(A)}(x_k) = \bigvee_{j=1}^n [I_{\tilde{N}_{x_k}^{\beta^2}}(x_j) \wedge I_A(x_j)] = \bigvee_{j=1}^n [w(I_{\tilde{N}_{x_k}^{\beta^2}}, x_j) \wedge w(I_A, x_j)] \quad (1 \leq k \leq n), \\
 w(F_{\tilde{C}^2(A)}, x_k) &= F_{\tilde{C}^2(A)}(x_k) = \bigwedge_{j=1}^n [F_{\tilde{N}_{x_k}^{\beta^2}}(x_j) \vee F_A(x_j)] = \bigwedge_{j=1}^n [w(F_{\tilde{N}_{x_k}^{\beta^2}}, x_j) \vee w(F_A, x_j)] \quad (1 \leq k \leq n), \\
 w(T_{\underline{C}^2(A)}, x_k) &= T_{\underline{C}^2(A)}(x_k) = \bigwedge_{j=1}^n [F_{\tilde{N}_{x_k}^{\beta^2}}(x_j) \vee T_A(x_j)] = \bigwedge_{j=1}^n [w(F_{\tilde{N}_{x_k}^{\beta^2}}, x_j) \vee w(T_A, x_j)] \quad (1 \leq k \leq n), \\
 w(I_{\underline{C}^2(A)}, x_k) &= I_{\underline{C}^2(A)}(x_k) = \bigwedge_{j=1}^n [(1 - I_{\tilde{N}_{x_k}^{\beta^2}}(x_j)) \vee I_A(x_j)] = \bigwedge_{j=1}^n [(1 - w(I_{\tilde{N}_{x_k}^{\beta^2}}, x_j)) \vee w(I_A, x_j)] \quad (1 \leq k \leq n), \\
 w(F_{\underline{C}^2(A)}, x_k) &= F_{\underline{C}^2(A)}(x_k) = \bigvee_{j=1}^n [T_{\tilde{N}_{x_k}^{\beta^2}}(x_j) \wedge F_A(x_j)] = \bigvee_{j=1}^n [w(T_{\tilde{N}_{x_k}^{\beta^2}}, x_j) \wedge w(F_A, x_j)] \quad (1 \leq k \leq n).
 \end{aligned}$$

□

Example 7. Let (U, \hat{C}) be a SVN β^2 -covering approximation space in Example 1 and $\beta = \langle 0.5, 0.1, 0.8 \rangle$, $A = \frac{(0.6, 0.3, 0.5)}{x_1} + \frac{(0.4, 0.5, 0.1)}{x_2} + \frac{(0.3, 0.2, 0.6)}{x_3} + \frac{(0.5, 0.3, 0.4)}{x_4} + \frac{(0.7, 0.2, 0.3)}{x_5}$. $G(A)$ is shown in Figure 5b. Based on

Theorem 1 and all $G(\tilde{N}_{x_k}^{\beta^2})$ ($k = 1, 2, \dots, 5$) in Example 6 and $G(\tilde{C}^2(A))$ and $G(\mathbb{C}^2(A))$ are obtained in Figure 6.

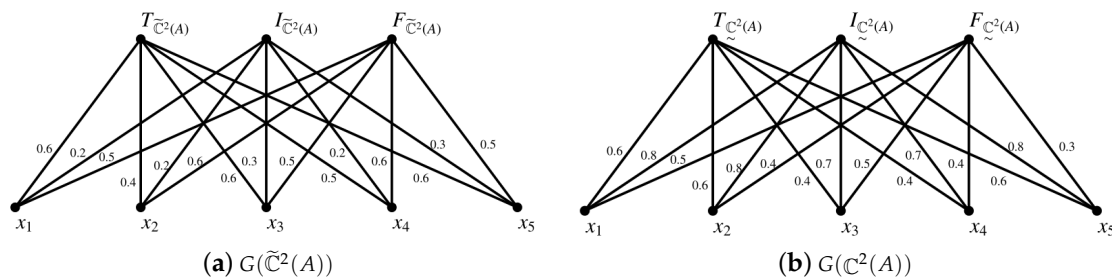


Figure 6. $G(\tilde{C}^2(A))$ and $G(\mathbb{C}^2(A))$.

6. Matrix Representation of the Type-2 SVN Covering Rough Set Model

In this section, the matrix representation of the type-2 SVN covering rough set model is investigated. Firstly, some new matrices and matrix operations are presented. Then, we show the matrix representation of the type-2 SVN approximation operators defined in Definition 7. The order of elements in U is given.

Two new matrices about a SVN β^2 -covering are presented in the following definition.

Definition 12. Let \hat{C} be a SVN β^2 -covering of U with $U = \{x_1, x_2, \dots, x_n\}$ and $\hat{C} = \{C_1, C_2, \dots, C_m\}$. Then $M_{\hat{C}} = (C_j(x_i))_{n \times m}$ is named a matrix representation of \hat{C} , and $M_{\hat{C}}^{\beta^2} = (s_{ij})_{n \times m}$ is called a β^2 -matrix representation of \hat{C} , where

$$s_{ij} = \begin{cases} 1, & C_j(x_i) \geq 2\beta; \\ 0, & \text{otherwise.} \end{cases}$$

Example 8. Let (U, \hat{C}) be a SVN β^2 -covering approximation space in Example 1 and $\beta = \langle 0.5, 0.1, 0.8 \rangle$. Then

$$M_{\hat{C}} = \begin{pmatrix} \langle 0.7, 0.2, 0.5 \rangle & \langle 0.6, 0.2, 0.4 \rangle & \langle 0.4, 0.1, 0.5 \rangle & \langle 0.1, 0.5, 0.6 \rangle \\ \langle 0.5, 0.3, 0.2 \rangle & \langle 0.5, 0.2, 0.8 \rangle & \langle 0.4, 0.5, 0.4 \rangle & \langle 0.6, 0.1, 0.7 \rangle \\ \langle 0.4, 0.5, 0.2 \rangle & \langle 0.2, 0.3, 0.6 \rangle & \langle 0.5, 0.2, 0.4 \rangle & \langle 0.6, 0.3, 0.4 \rangle \\ \langle 0.6, 0.1, 0.7 \rangle & \langle 0.4, 0.5, 0.7 \rangle & \langle 0.3, 0.6, 0.5 \rangle & \langle 0.5, 0.3, 0.2 \rangle \\ \langle 0.3, 0.2, 0.6 \rangle & \langle 0.7, 0.3, 0.5 \rangle & \langle 0.6, 0.3, 0.5 \rangle & \langle 0.8, 0.1, 0.2 \rangle \end{pmatrix}, M_{\hat{C}}^{\beta^2} = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 \end{pmatrix}.$$

In order to calculate all $\tilde{N}_x^{\beta^2}$ (for any $x \in U$) by matrices, the following operation is presented.

Definition 13. Let $A = (a_{ik})_{n \times m}$ and $B = (\langle b_{kj}^+, b_{kj}, b_{kj}^- \rangle)_{1 \leq k \leq m, 1 \leq j \leq l}$ be two matrices. We define $D = A \star B = (\langle d_{ij}^+, d_{ij}, d_{ij}^- \rangle)_{1 \leq i \leq n, 1 \leq j \leq l}$, where

$$\langle d_{ij}^+, d_{ij}, d_{ij}^- \rangle = \langle \wedge_{k=1}^m [(1 - a_{ik}) \vee b_{kj}^+], \wedge_{k=1}^m [(1 - a_{ik}) \vee b_{kj}], 1 - \wedge_{k=1}^m [(1 - a_{ik}) \vee (1 - b_{kj}^-)] \rangle. \quad (7)$$

Based on Definitions 12 and 13, all $\tilde{N}_x^{\beta^2}$ (for any $x \in U$) can be obtained by matrix operations.

Proposition 9. Let \hat{C} be a SVN β^2 -covering of U with $U = \{x_1, x_2, \dots, x_n\}$ and $\hat{C} = \{C_1, C_2, \dots, C_m\}$.

Then

$$M_{\hat{C}}^{\beta^2} \star M_{\hat{C}}^T = (\tilde{N}_{x_i}^{\beta^2}(x_j))_{1 \leq i \leq n, 1 \leq j \leq n}, \quad (8)$$

where $M_{\hat{C}}^T$ is the transpose of $M_{\hat{C}}$

Proof. Suppose $M_C^T = (C_k(x_j))_{m \times n}$, $M_C^{\beta^2} = (s_{ik})_{n \times m}$ and $M_C^{\beta^2} \star M_C^T = (\langle d_{ij}^+, d_{ij}, d_{ij}^- \rangle)_{1 \leq i \leq n, 1 \leq j \leq n}$. Since \widehat{C} is a SVN β^2 -covering of U , for each i ($1 \leq i \leq n$), there exists k ($1 \leq k \leq m$) such that $s_{ik} = 1$.

Then

$$\begin{aligned} & \langle d_{ij}^+, d_{ij}, d_{ij}^- \rangle \\ &= \langle \bigwedge_{k=1}^m [(1 - s_{ik}) \vee T_{C_k}(x_j)], \bigwedge_{k=1}^m [(1 - s_{ik}) \vee I_{C_k}(x_j)], 1 - \bigwedge_{k=1}^m [(1 - s_{ik}) \vee (1 - F_{C_k}(x_j))] \rangle \\ &= \langle \bigwedge_{s_{ik}=1} [(1 - s_{ik}) \vee T_{C_k}(x_j)], \bigwedge_{s_{ik}=1} [(1 - s_{ik}) \vee I_{C_k}(x_j)], 1 - \bigwedge_{s_{ik}=1} [(1 - s_{ik}) \vee (1 - F_{C_k}(x_j))] \rangle \\ &= \langle \bigwedge_{s_{ik}=1} T_{C_k}(x_j), \bigwedge_{s_{ik}=1} I_{C_k}(x_j), 1 - \bigwedge_{s_{ik}=1} (1 - F_{C_k}(x_j)) \rangle \\ &= \langle \bigwedge_{C_k(x_i) \geq 2\beta} T_{C_k}(x_j), \bigwedge_{C_k(x_i) \geq 2\beta} I_{C_k}(x_j), 1 - \bigwedge_{C_k(x_i) \geq 2\beta} (1 - F_{C_k}(x_j)) \rangle \\ &= \langle \bigwedge_{C_k(x_i) \geq 2\beta} T_{C_k}(x_j), \bigwedge_{C_k(x_i) \geq 2\beta} I_{C_k}(x_j), \bigvee_{C_k(x_i) \geq 2\beta} (F_{C_k}(x_j)) \rangle \\ &= (\bigcap_{2C_k(x_i) \geq 2\beta} C_k)(x_j) \\ &= \widetilde{N}_{x_i}^{\beta^2}(x_j), 1 \leq i, j \leq n. \end{aligned}$$

Hence, $M_C^{\beta^2} \star M_C^T = (\widetilde{N}_{x_i}^{\beta^2}(x_j))_{1 \leq i \leq n, 1 \leq j \leq n}$. \square

Example 9. Let (U, \widehat{C}) be a SVN β^2 -covering approximation space in Example 1 and $\beta = \langle 0.5, 0.1, 0.8 \rangle$. According to $M_C^{\beta^2}$ and M_C^T in Example 8, we have

$$\begin{aligned} & M_C^{\beta^2} \star M_C^T \\ &= \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 \end{pmatrix} \star \begin{pmatrix} \langle 0.7, 0.2, 0.5 \rangle & \langle 0.6, 0.2, 0.4 \rangle & \langle 0.4, 0.1, 0.5 \rangle & \langle 0.1, 0.5, 0.6 \rangle \\ \langle 0.5, 0.3, 0.2 \rangle & \langle 0.5, 0.2, 0.8 \rangle & \langle 0.4, 0.5, 0.4 \rangle & \langle 0.6, 0.1, 0.7 \rangle \\ \langle 0.4, 0.5, 0.2 \rangle & \langle 0.2, 0.3, 0.6 \rangle & \langle 0.5, 0.2, 0.4 \rangle & \langle 0.6, 0.3, 0.4 \rangle \\ \langle 0.6, 0.1, 0.7 \rangle & \langle 0.4, 0.5, 0.7 \rangle & \langle 0.3, 0.6, 0.5 \rangle & \langle 0.5, 0.3, 0.2 \rangle \\ \langle 0.3, 0.2, 0.6 \rangle & \langle 0.7, 0.3, 0.5 \rangle & \langle 0.6, 0.3, 0.5 \rangle & \langle 0.8, 0.1, 0.2 \rangle \end{pmatrix}^T \\ &= \begin{pmatrix} \langle 0.6, 0.2, 0.5 \rangle & \langle 0.5, 0.2, 0.8 \rangle & \langle 0.2, 0.3, 0.6 \rangle & \langle 0.4, 0.1, 0.7 \rangle & \langle 0.3, 0.2, 0.6 \rangle \\ \langle 0.1, 0.2, 0.6 \rangle & \langle 0.5, 0.1, 0.8 \rangle & \langle 0.2, 0.3, 0.6 \rangle & \langle 0.4, 0.1, 0.7 \rangle & \langle 0.3, 0.1, 0.6 \rangle \\ \langle 0.1, 0.1, 0.6 \rangle & \langle 0.4, 0.1, 0.7 \rangle & \langle 0.5, 0.2, 0.4 \rangle & \langle 0.3, 0.3, 0.5 \rangle & \langle 0.6, 0.1, 0.5 \rangle \\ \langle 0.1, 0.2, 0.6 \rangle & \langle 0.5, 0.1, 0.7 \rangle & \langle 0.4, 0.3, 0.4 \rangle & \langle 0.5, 0.1, 0.7 \rangle & \langle 0.3, 0.1, 0.6 \rangle \\ \langle 0.1, 0.1, 0.6 \rangle & \langle 0.4, 0.1, 0.8 \rangle & \langle 0.2, 0.2, 0.6 \rangle & \langle 0.3, 0.3, 0.7 \rangle & \langle 0.6, 0.1, 0.5 \rangle \end{pmatrix} \\ &= (N_{x_i}^{\beta^2}(x_j))_{1 \leq i \leq 5, 1 \leq j \leq 5}. \end{aligned}$$

There are two operations in the work by the authors of [37], which can be used to calculate $\widetilde{C}^2(A)$ and $\underline{C}^2(A)$ (for any $A \in SVN(U)$) by matrices.

Definition 14. [37] Let $A = (\langle c_{ij}^+, c_{ij}, c_{ij}^- \rangle)_{m \times n}$ and $B = (\langle d_j^+, d_j, d_j^- \rangle)_{n \times 1}$ be two matrices. We define $C = A \circ B = (\langle e_i^+, e_i, e_i^- \rangle)_{m \times 1}$ and $D = A \diamond B = (\langle f_i^+, f_i, f_i^- \rangle)_{m \times 1}$, where

$$\begin{aligned} \langle e_i^+, e_i, e_i^- \rangle &= \langle \bigvee_{j=1}^n (c_{ij}^+ \wedge d_j^+), \bigvee_{j=1}^n (c_{ij} \wedge d_j), \bigwedge_{j=1}^n (c_{ij}^- \vee d_j^-) \rangle, \\ \langle f_i^+, f_i, f_i^- \rangle &= \langle \bigwedge_{j=1}^n (c_{ij}^- \vee d_j^-), \bigwedge_{j=1}^n [(1 - c_{ij}) \vee d_j], \bigvee_{j=1}^n (c_{ij}^+ \wedge d_j^+) \rangle. \end{aligned} \tag{9}$$

According to Proposition 9 and Definition 14, the set representations of $\widetilde{C}^2(A)$ and $\underline{C}^2(A)$ (for any $A \in SVN(U)$) can be converted to matrix representations. $A = (a_i)_{n \times 1}$ with $a_i = \langle T_A(x_i), I_A(x_i), F_A(x_i) \rangle$ is the vector representation of A . $\widetilde{C}^2(A)$ and $\underline{C}^2(A)$ are also vector representations.

Theorem 2. Let \widehat{C} be a SVN β^2 -covering of U with $U = \{x_1, x_2, \dots, x_n\}$ and $\widehat{C} = \{C_1, C_2, \dots, C_m\}$. Then for any $A \in SVN(U)$,

$$\begin{aligned} \widetilde{C}^2(A) &= (M_C^{\beta^2} \star M_C^T) \circ A, \\ \underset{\sim}{C}^2(A) &= (M_C^{\beta^2} \star M_C^T) \diamond A. \end{aligned} \tag{10}$$

Proof. According to Proposition 9, Definitions 7 and 14, for any x_i ($i = 1, 2, \dots, n$),

$$\begin{aligned} ((M_C^{\beta^2} \star M_C^T) \circ A)(x_i) &= \langle \bigvee_{j=1}^n (T_{\widehat{N}_{x_i}^{\beta^2}}(x_j) \wedge T_A(x_j)), \bigvee_{j=1}^n (I_{\widehat{N}_{x_i}^{\beta^2}}(x_j) \wedge I_A(x_j)), \bigwedge_{j=1}^n (F_{\widehat{N}_{x_i}^{\beta^2}}(x_j) \vee F_A(x_j)) \rangle \\ &= (\widetilde{C}^2(A))(x_i), \end{aligned}$$

and

$$\begin{aligned} ((M_C^{\beta^2} \star M_C^T) \diamond A)(x_i) &= \langle \bigwedge_{j=1}^n (F_{\widehat{N}_{x_i}^{\beta^2}}(x_j) \vee T_A(x_j)), \bigwedge_{j=1}^n [(1 - I_{\widehat{N}_{x_i}^{\beta^2}}(x_j)) \vee I_A(x_j)], \bigvee_{j=1}^n (T_{\widehat{N}_{x_i}^{\beta^2}}(x_j) \wedge F_A(x_j)) \rangle \\ &= (\underset{\sim}{C}^2(A))(x_i). \end{aligned}$$

Hence, $\widetilde{C}^2(A) = (M_C^{\beta^2} \star M_C^T) \circ A, \underset{\sim}{C}^2(A) = (M_C^{\beta^2} \star M_C^T) \diamond A. \quad \square$

Example 10. Let (U, \widehat{C}) be a SVN β^2 -covering approximation space in Example 1, $\beta = \langle 0.5, 0.1, 0.8 \rangle$ and $A = \frac{\langle 0.6, 0.3, 0.5 \rangle}{x_1} + \frac{\langle 0.4, 0.5, 0.1 \rangle}{x_2} + \frac{\langle 0.3, 0.2, 0.6 \rangle}{x_3} + \frac{\langle 0.5, 0.3, 0.4 \rangle}{x_4} + \frac{\langle 0.7, 0.2, 0.3 \rangle}{x_5}$. Then

$$\begin{aligned} &\widetilde{C}^2(A) \\ &= (M_C^{\beta^2} \star M_C^T) \circ A \\ &= \begin{pmatrix} \langle 0.6, 0.2, 0.5 \rangle & \langle 0.5, 0.2, 0.8 \rangle & \langle 0.2, 0.3, 0.6 \rangle & \langle 0.4, 0.1, 0.7 \rangle & \langle 0.3, 0.2, 0.6 \rangle \\ \langle 0.1, 0.2, 0.6 \rangle & \langle 0.5, 0.1, 0.8 \rangle & \langle 0.2, 0.3, 0.6 \rangle & \langle 0.4, 0.1, 0.7 \rangle & \langle 0.3, 0.1, 0.6 \rangle \\ \langle 0.1, 0.1, 0.6 \rangle & \langle 0.4, 0.1, 0.7 \rangle & \langle 0.5, 0.2, 0.4 \rangle & \langle 0.3, 0.3, 0.5 \rangle & \langle 0.6, 0.1, 0.5 \rangle \\ \langle 0.1, 0.2, 0.6 \rangle & \langle 0.5, 0.1, 0.7 \rangle & \langle 0.4, 0.3, 0.4 \rangle & \langle 0.5, 0.1, 0.7 \rangle & \langle 0.3, 0.1, 0.6 \rangle \\ \langle 0.1, 0.1, 0.6 \rangle & \langle 0.4, 0.1, 0.8 \rangle & \langle 0.2, 0.2, 0.6 \rangle & \langle 0.3, 0.3, 0.7 \rangle & \langle 0.6, 0.1, 0.5 \rangle \end{pmatrix} \circ \begin{pmatrix} \langle 0.6, 0.3, 0.5 \rangle \\ \langle 0.4, 0.5, 0.1 \rangle \\ \langle 0.3, 0.2, 0.6 \rangle \\ \langle 0.5, 0.3, 0.4 \rangle \\ \langle 0.7, 0.2, 0.3 \rangle \end{pmatrix} \\ &= \begin{pmatrix} \langle 0.6, 0.2, 0.5 \rangle \\ \langle 0.4, 0.2, 0.6 \rangle \\ \langle 0.6, 0.3, 0.5 \rangle \\ \langle 0.5, 0.2, 0.6 \rangle \\ \langle 0.6, 0.3, 0.5 \rangle \end{pmatrix}, \\ &\underset{\sim}{C}^2(A) \\ &= (M_C^{\beta^2} \star M_C^T) \diamond A \\ &= \begin{pmatrix} \langle 0.6, 0.2, 0.5 \rangle & \langle 0.5, 0.2, 0.8 \rangle & \langle 0.2, 0.3, 0.6 \rangle & \langle 0.4, 0.1, 0.7 \rangle & \langle 0.3, 0.2, 0.6 \rangle \\ \langle 0.1, 0.2, 0.6 \rangle & \langle 0.5, 0.1, 0.8 \rangle & \langle 0.2, 0.3, 0.6 \rangle & \langle 0.4, 0.1, 0.7 \rangle & \langle 0.3, 0.1, 0.6 \rangle \\ \langle 0.1, 0.1, 0.6 \rangle & \langle 0.4, 0.1, 0.7 \rangle & \langle 0.5, 0.2, 0.4 \rangle & \langle 0.3, 0.3, 0.5 \rangle & \langle 0.6, 0.1, 0.5 \rangle \\ \langle 0.1, 0.2, 0.6 \rangle & \langle 0.5, 0.1, 0.7 \rangle & \langle 0.4, 0.3, 0.4 \rangle & \langle 0.5, 0.1, 0.7 \rangle & \langle 0.3, 0.1, 0.6 \rangle \\ \langle 0.1, 0.1, 0.6 \rangle & \langle 0.4, 0.1, 0.8 \rangle & \langle 0.2, 0.2, 0.6 \rangle & \langle 0.3, 0.3, 0.7 \rangle & \langle 0.6, 0.1, 0.5 \rangle \end{pmatrix} \diamond \begin{pmatrix} \langle 0.6, 0.3, 0.5 \rangle \\ \langle 0.4, 0.5, 0.1 \rangle \\ \langle 0.3, 0.2, 0.6 \rangle \\ \langle 0.5, 0.3, 0.4 \rangle \\ \langle 0.7, 0.2, 0.3 \rangle \end{pmatrix} \\ &= \begin{pmatrix} \langle 0.6, 0.8, 0.5 \rangle \\ \langle 0.6, 0.8, 0.4 \rangle \\ \langle 0.4, 0.7, 0.5 \rangle \\ \langle 0.4, 0.7, 0.6 \rangle \\ \langle 0.6, 0.8, 0.3 \rangle \end{pmatrix}. \end{aligned}$$

7. An Application to DM Problems in Paper Defect Diagnosis

Under the type-2 SVN covering rough set model, we present a novel approach to DM problems in paper defect diagnosis in this section.

7.1. The Problem of DM in Paper Defect Diagnosis

Let $U = \{x_k : k = 1, 2, \dots, n\}$ be the set of papers and $V = \{y_i | i = 1, 2, \dots, m\}$ be the m main symptoms (for example, spot, steak, and so on) for a paper defect B . Assume that an inspector R evaluate every paper x_k ($k = 1, 2, \dots, n$).

Assume that the inspector R believes each paper $x_k \in U$ ($k = 1, 2, \dots, n$) has a symptom value C_i ($i = 1, 2, \dots, m$) denoted by $C_i(x_k) = \langle T_{C_i}(x_k), I_{C_i}(x_k), F_{C_i}(x_k) \rangle$, where $T_{C_i}(x_k) \in [0, 1]$ is the degree that inspector R confirms paper x_k has symptom y_i , $I_{C_i}(x_k) \in [0, 1]$ is the degree that inspector R is not sure paper x_k has symptom y_i , $F_{C_i}(x_k) \in [0, 1]$ is the degree that inspector R confirms paper x_k does not have symptom y_i , and $T_{C_i}(x_k) + I_{C_i}(x_k) + F_{C_i}(x_k) \leq 3$.

Let $\beta = \langle a, b, c \rangle$ be the critical value. If any paper $x_k \in U$, there is at least one symptom $y_i \in V$ such that the symptom value C_i for the paper x_k is not less than β (i.e., $C_i(x_k) \geq_2 \beta$), respectively, then $\hat{C} = \{C_1, C_2, \dots, C_m\}$ is a SVN β^2 -covering of U for some SVN number β .

If d is a possible degree, e is an indeterminacy degree and f is an impossible degree of the paper defect B of every paper $x_k \in U$ that is diagnosed by the inspector R , denoted by $A(x_k) = \langle d, e, f \rangle$, then the decision maker (the inspector R) for the DM problem needs to know how to evaluate whether the papers $x_k \in U$ have the paper defect B or not.

7.2. The DM Algorithm

In this subsection, we give an approach for the problem of DM with the above characterizations using the type-2 SVN covering rough set model. According to the characterizations of the DM problem in Section 7.1, we construct the SVN decision information system and present the Algorithm 1 of DM under the framework of the type-2 SVN covering rough set model.

Algorithm 1 The DM algorithm under the type-2 SVN covering rough set model

Input: SVN decision information system (U, \hat{C}, β, A) .

Output: The score ordering for all alternatives.

- **Step 1:** Compute the SVN β^2 -neighborhood $\tilde{N}_x^{\beta^2}$ of x induced by \hat{C} , for all $x \in U$ according to Definition 6;
- **Step 2:** Compute the SVN covering upper approximation $\tilde{C}^2(A)$ and lower approximation $\underline{C}^2(A)$ of A , according to Definition 7;
- **Step 3:** Compute $\tilde{R}_A^2 = \tilde{C}^2(A) \oplus \underline{C}^2(A)$;
- **Step 4:** Compute

$$s(x) = \frac{T_{\tilde{R}_A^2}(x)}{\sqrt{(T_{\tilde{R}_A^2}(x))^2 + (I_{\tilde{R}_A^2}(x))^2 + (F_{\tilde{R}_A^2}(x))^2}};$$

- **Step 5:** Rank all the alternatives $s(x)$ by using the principle of numerical size and select the paper that is more likely to be sick with the paper defect B .
-

According to the above process, we can get the DM according to the ranking. In Step 4, $S(x)$ is the cosine similarity measure between $\tilde{R}_A(x)$ and the ideal solution $(1, 0, 0)$, which is proposed by Ye [44].

7.3. An Applied Example

Example 11. Assume that $U = \{x_1, x_2, x_3, x_4, x_5\}$ is a set of papers. According to the paper defects' symptoms, we write $V = \{y_1, y_2, y_3, y_4\}$ to be four main symptoms (spot, steak, crater, and fracture) for a paper defect B. Assume that the inspector R evaluates every paper x_k ($k = 1, 2, \dots, 5$) as shown in Table 1.

Let $\beta = \langle 0.5, 0.1, 0.8 \rangle$ be the critical value. Then, $\hat{C} = \{C_1, C_2, C_3, C_4\}$ is a SVN β^2 -coverings of U. Assume that the inspector R diagnosed the value $A = \frac{(0.6, 0.3, 0.5)}{x_1} + \frac{(0.4, 0.5, 0.1)}{x_2} + \frac{(0.3, 0.2, 0.6)}{x_3} + \frac{(0.5, 0.3, 0.4)}{x_4} + \frac{(0.7, 0.2, 0.3)}{x_5}$ of the paper defect B of every paper.

Step 1: $\tilde{N}_{x_k}^{\beta^2}$ ($k = 1, 2, 3, 4, 5$) are shown in Table 2.

Step 2:

$$\tilde{C}^2(A) = \{\langle x_1, 0.6, 0.2, 0.5 \rangle, \langle x_2, 0.4, 0.2, 0.6 \rangle, \langle x_3, 0.6, 0.3, 0.5 \rangle, \langle x_4, 0.5, 0.2, 0.6 \rangle, \langle x_5, 0.6, 0.3, 0.5 \rangle\},$$

$$\tilde{C}^2(A) = \{\langle x_1, 0.6, 0.8, 0.5 \rangle, \langle x_2, 0.6, 0.8, 0.4 \rangle, \langle x_3, 0.4, 0.7, 0.5 \rangle, \langle x_4, 0.4, 0.7, 0.4 \rangle, \langle x_5, 0.6, 0.8, 0.3 \rangle\}.$$

Step 3:

$$\begin{aligned} & \tilde{R}_A^2 \\ = & \tilde{C}^2(A) \oplus \tilde{C}^2(A) \\ = & \{\langle x_1, 0.84, 0.16, 0.25 \rangle, \langle x_2, 0.76, 0.16, 0.24 \rangle, \langle x_3, 0.76, 0.21, 0.25 \rangle, \langle x_4, 0.70, 0.14, 0.24 \rangle, \langle x_5, 0.84, 0.24, 0.15 \rangle\}. \end{aligned}$$

Step 4: We can obtain $s(x_k)$ ($k = 1, 2, \dots, 5$) in Table 5.

Step 5: According to the principle of numerical size, we have

$$x_3 < x_4 < x_2 < x_1 < x_5.$$

Therefore, the inspector R diagnoses the paper x_5 as more likely to be sick with the paper defect B.

Table 5. $s(x_k)$ ($k = 1, 2, \dots, 5$).

U	x_1	x_2	x_3	x_4	x_5
$s(x_k)$	0.943	0.935	0.919	0.929	0.948

7.4. A Comparison Analysis

To validate the feasibility of the proposed DM method, a comparative study is conducted with other methods. These methods which were introduced in Liu [43], Ye [44], Yang et al. [36], and Wang et al. [37] are compared with the proposed approach using SVN information system.

Because Table 1 is the same as in the work by the authors of [37] and the counting processes of the methods presented by Liu [43], Ye [44], Yang et al. [36], and Wang et al. [37], are shown in the work by the authors of [37], so we do not show these counting processes in this paper. For Example 11, the results of them are calculated as follows.

- In Liu's method, we suppose the weight vector of the criteria is $\mathbf{w} = (0.35, 0.25, 0.3, 0.1)$ and $\gamma = 1$. Hence, we get

$$s(n_1) = 0.735, s(n_2) = 0.706, s(n_3) = 0.660, s(n_4) = 0.596, s(n_5) = 0.734.$$

According to the cosine similarity degrees $s(n_k)$ ($k = 1, 2, \dots, 5$), we obtain

$$x_4 < x_3 < x_2 < x_5 < x_1.$$

- In Ye's method, we suppose the weight vector of the criteria is $\mathbf{w} = (0.35, 0.25, 0.3, 0.1)$. Then

$$W_1(x_1, A^*) = 0.677, W_2(x_2, A^*) = 0.608, W_3(x_3, A^*) = 0.580, W_4(x_4, A^*) = 0.511, \\ W_5(x_5, A^*) = 0.666.$$

According to all $s(n_{x_k}, n^*)$ ($k = 1, 2, \dots, 5$), we obtain

$$x_4 < x_3 < x_2 < x_5 < x_1.$$

- In Yang’s method, we suppose paper defect $B \in SVN(V)$ and $B = \frac{(0.3,0.6,0.5)}{y_1} + \frac{(0.7,0.2,0.1)}{y_2} + \frac{(0.6,0.4,0.3)}{y_3} + \frac{(0.8,0.4,0.5)}{y_4}$. Let $n^* = \langle 1, 0, 0 \rangle$. We get

$$\widetilde{R}(B) = \{ \langle x_1, 0.6, 0.2, 0.4 \rangle, \langle x_2, 0.6, 0.2, 0.4 \rangle, \langle x_3, 0.6, 0.3, 0.4 \rangle, \langle x_4, 0.5, 0.4, 0.5 \rangle, \langle x_5, 0.8, 0.3, 0.5 \rangle \},$$

$$\widetilde{R}(B) = \{ \langle x_1, 0.5, 0.6, 0.5 \rangle, \langle x_2, 0.3, 0.6, 0.5 \rangle, \langle x_3, 0.3, 0.5, 0.5 \rangle, \langle x_4, 0.6, 0.6, 0.5 \rangle, \langle x_5, 0.6, 0.6, 0.5 \rangle \}.$$

Then,

$$s(n_{x_1}, n^*) = 0.960, s(n_{x_2}, n^*) = 0.951, s(n_{x_3}, n^*) = 0.945, s(n_{x_4}, n^*) = 0.918, s(n_{x_5}, n^*) = 0.948.$$

According to all $s(n_{x_k}, n^*)$ ($k = 1, 2, \dots, 5$), we obtain

$$x_4 < x_3 < x_5 < x_2 < x_1.$$

- In Wang’s method, we do not use $\beta = \langle 0.5, 0.1, 0.8 \rangle$ in Example 11, and the reason is explained later. We suppose $\beta' = \langle 0.5, 0.3, 0.8 \rangle$ in Wang’s method. Then

$$\widetilde{C}(A) = \{ \langle x_1, 0.6, 0.3, 0.5 \rangle, \langle x_2, 0.4, 0.3, 0.6 \rangle, \langle x_3, 0.6, 0.5, 0.5 \rangle, \langle x_4, 0.5, 0.3, 0.6 \rangle, \langle x_5, 0.6, 0.5, 0.5 \rangle \},$$

$$\widetilde{C}(A) = \{ \langle x_1, 0.6, 0.5, 0.5 \rangle, \langle x_2, 0.6, 0.5, 0.4 \rangle, \langle x_3, 0.4, 0.4, 0.5 \rangle, \langle x_4, 0.4, 0.5, 0.4 \rangle, \langle x_5, 0.6, 0.4, 0.3 \rangle \}.$$

Hence,

$$s(x_1) = 0.945, s(x_2) = 0.937, s(x_3) = 0.922, s(x_4) = 0.909, s(x_5) = 0.958.$$

According to all $s(x_k)$ ($k = 1, 2, \dots, 5$), we obtain

$$x_4 < x_3 < x_2 < x_1 < x_5.$$

All results are shown in Table 6 and Figure 7.

Liu [43] and Ye [44] presented the methods by SVN theory. The method developed by Liu [43] is based on the Hammer SVN number aggregation (HSVNNWA) operator, the ranking order will be changed by different w and γ . The parameter γ can be regarded as an attitude of the decision maker’s preferences. For Example 11, we set the weight vector of the criteria is $w = (0.35, 0.25, 0.3, 0.1)$ and $\gamma = 1$, then we obtain $x_4 < x_3 < x_2 < x_5 < x_1$. The method developed by Ye [44] is based on the weighted correlation coefficient $W_k(x_k, A^*)$ or the weighted cosine similarity measure $M_k(x_k, A^*)$, where A^* is the ideal alternative. We can get two ranking orders of x_k ($k = 1, 2, 3, 4, 5$) by the values of $W_k(x_k, A^*)$ and $M_k(x_k, A^*)$, respectively. Then, we find that these two kinds of ranking orders are the same. Hence, we only show $W_k(x_k, A^*)$ in this paper. In Table 6 and Figure 7, there are the same ranking results of their methods.

Table 6. The results utilizing the different methods of Example 11.

Methods	The Final Ranking	The Paper Is Most Sick With the Paper Defect B
Liu [43]	$x_4 < x_3 < x_2 < x_5 < x_1$	x_1
Ye [44]	$x_4 < x_3 < x_2 < x_5 < x_1$	x_1
Yang et al. [36]	$x_4 < x_3 < x_5 < x_2 < x_1$	x_1
Wang et al. [37]	$x_4 < x_3 < x_2 < x_1 < x_5$	x_5
This paper	$x_3 < x_4 < x_2 < x_1 < x_5$	x_5

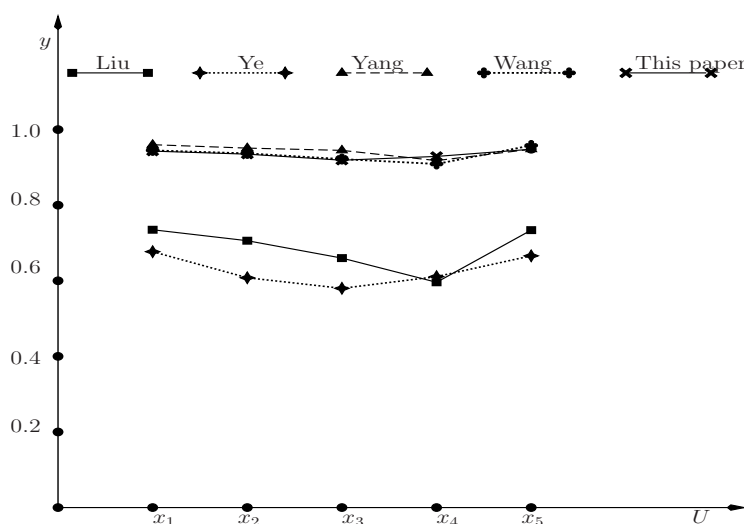


Figure 7. The chat of different values of patient in utilizing different methods in Example 11.

Yang et al. [36] and Wang et al. [37] used different SVN rough set models to make a decision. The method presented by Yang et al. [36] is based on a SVN relation rough set model on two-universes. That is to say, the DM problems with SVN information can be dealt with by Yang’s method when it induces a SVN relation on two-universes. In Example 11, we obtain a SVN relation on two universes from Table 1. The method presented by Wang et al. [37] based on the type-1 SVN covering rough set model. That is to say, the DM problems with SVN information can be dealt with by Wang’s method when it can induce a SVN β -covering. In Example 11, we suppose $\beta = \langle 0.5, 0.1, 0.8 \rangle$. However, \hat{C} is not a SVN β -covering of U when $\beta = \langle 0.5, 0.1, 0.8 \rangle$. Hence, the method presented by Wang et al. can not be used in Example 11 when $\beta = \langle 0.5, 0.1, 0.8 \rangle$. Let’s re-assume $\beta' = \langle 0.5, 0.3, 0.8 \rangle$. Then \hat{C} is a SVN β' -covering of U . Hence, the method presented by Wang et al. can be used in Example 11 when $\beta' = \langle 0.5, 0.3, 0.8 \rangle$.

In this paper, we present the type-2 SVN covering rough set model based on SVN β^2 -coverings. Under the type-2 SVN covering rough set model, a novel method for DM problems with SVN information is presented. The contributions of our proposed method are summarized as follows.

- (1) The DM problems with SVN information can be dealt with by our proposed method when it can induce a SVN β^2 -covering. The method presented by Wang et al. [37] can not be used in Example 11 when $\beta = \langle 0.5, 0.1, 0.8 \rangle$. But our proposed method can deal with Example 11 when $\beta = \langle 0.5, 0.1, 0.8 \rangle$. Hence, our proposed method complements Wang’s.
- (2) It is a new viewpoint to use SVN sets and rough sets in paper defect diagnosis.

Using different methods, the obtained results may be different. To achieve the most accurate results, further diagnosis is necessary for combination with other hybrid methods.

8. Conclusions

This paper investigates a new type of SVN covering rough set model, which can be seen as a new bridge linking SVN sets and covering-based rough sets. Comparing the existing literatures [36,37,48,49], the main contributions of this paper are concluded as follows.

- (1) By introducing some definitions and properties in SVN β^2 -covering approximation spaces, we present the type-2 SVN covering rough set model based on the type-2 inclusion relation. The existing literatures [36,37,48,49] used the type-1 inclusion relation to study the combination of SVN sets and rough sets. Hence, this paper presents a new and interesting viewpoint to study the combination of SVN sets and rough sets.
- (2) It would be tedious and complicated to use set representation to calculate the new SVN covering approximation operators. Therefore, the graph and matrix representations of these new SVN covering approximation operators make it possible to calculate them. We are the first to study the equivalent representation of the SVN rough set model by graph theory. By these graph and matrix representations, calculations will become algorithmic and can be easily implemented by computers.
- (3) Paper defect diagnosis is important in paper making industries. We propose a method to paper defect diagnosis under the type-2 SVN covering rough set model. The proposed DM method is compared with other methods which are presented by Liu [43], Ye [44], Yang et al. [36], and Wang et al. [37], respectively.

Further study will be deserved by the following research topics. On the one hand, the type-2 inclusion relation or graph theory can be considered into other SVN rough set models [34,36,48,49] in future research. On the other hand, neutrosophic sets and related algebraic structures [50–55] will be connected with the research content of this paper in further research.

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