A Novel Extension of Neutrosophic Sets and Its Application in BCK/BCI-Algebras

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ABSTRACT
Generalized neutrosophic set is introduced, and applied it to BCK/BCI-algebras. The notions of generalized neutrosophic subalgebras and generalized neutrosophic ideals in BCK/BCI-algebras are introduced, and related properties are investigated. Characterizations of generalized neutrosophic subalgebra/ideal are considered. Relation between generalized neutrosophic subalgebra and generalized neutrosophic ideal is discussed. In a BCK-algebra, conditions for a generalized neutrosophic subalgebra to be a generalized neutrosophic ideal are provided. Conditions for a generalized neutrosophic set to be a generalized neutrosophic ideal are also provided. Homomorphic image and preimage of generalized neutrosophic ideal are considered.

KEYWORDS: Generalized neutrosophic set, generalized neutrosophic subalgebra, generalized neutrosophic ideal.

1 Introduction

Zadeh (1965) introduced the degree of membership/truth (t) in 1965 and defined the fuzzy set. As a generalization of fuzzy sets, Atanassov (1986) introduced the degree of nonmembership/falsehood (f) in 1986 and defined the intuitionistic fuzzy set. Smarandache introduced the degree of indeterminacy/neutrality (i) as independent component in 1995 (published in 1998) and defined the neutrosophic set on three components

$$\text{(t, i, f)} = \text{(truth, indeterminacy, falsehood)}.$$

For more detail, refer to the site

http://fs.gallup.unm.edu/FlorentinSmarandache.htm.

The concept of neutrosophic set (NS) developed by Smarandache (1999) and Smarandache (2005) is a more general platform which extends the concepts of the classic set and fuzzy set, intuitionistic fuzzy set and interval valued intuitionistic fuzzy set. Neutrosophic set theory is applied to various part (refer to the site http://fs.gallup.unm.edu/neutrosophy.htm). Agboola and Davvaz (2015) introduced the concept of neutrosophic BCI/BCK-algebras, and presented elementary properties of neutrosophic BCI/BCK-algebras. Saeid and Jun (2017) gave relations between an $(\in, \in\vee q)$-neutrosophic subalgebra and a $(q, \in\vee q)$-neutrosophic subalgebra, and discussed characterization of an $(\in, \in\vee q)$-neutrosophic subalgebra by using neutrosophic $\in$-subsets. They provided conditions for an $(\in, \in\vee q)$-neutrosophic subalgebra to be a $(q, \in\vee q)$-neutrosophic subalgebra, and investigated properties on neutrosophic q-subsets and neutrosophic $\in\vee q$-subsets. Jun (2017) considered neutrosophic subalgebras of several types in BCK/BCI-algebras.

In this paper, we consider a generalization of Smarandache's neutrosophic sets. We introduce the notion of generalized neutrosophic sets and apply it to BCK/BCI-algebras. We introduce the notions of generalized neutrosophic subalgebras and generalized neutrosophic ideals in BCK/BCI-algebras, and investigate related properties. We consider characterizations of generalized neutrosophic subalgebra/ideal, and discussed relation between generalized neutrosophic subalgebra and generalized neutrosophic ideal. We provide conditions for a generalized neutrosophic subalgebra to be a generalized neutrosophic ideal in a BCK-algebra. We also provide conditions for a generalized neutrosophic set to be a generalized neutrosophic ideal, and consider homomorphic image and preimage of generalized neutrosophic ideal.

2 PRELIMINARIES

By a *BCI-algebra* we mean an algebra $(X, *, 0)$ of type $(2, 0)$ satisfying the conditions:

(a1) $((x * y) * (x * z)) * (z * y) = 0$,

(a2) $(x * (x * y)) * y = 0$,

(a3) $x * x = 0$,

(a4) $x * y = y * x = 0 \;\Rightarrow\; x = y$,

for all $x, y, z \in X$. If a BCI-algebra X satisfies the condition

(a5) $0 * x = 0$ for all $x \in X$,

then we say that X is a *BCK-algebra*. A partial ordering "$\leq$" on X is defined by

$$(\forall x, y \in X)\,(x \leq y \iff x * y = 0)\,.$$

In a BCK/BCI-algebra X, the following properties are satisfied:

$$(\forall x \in X)\,(x * 0 = x)\,, \tag{2.1}$$

$$(\forall x, y, z \in X)\,((x * y) * z = (x * z) * y)\,. \tag{2.2}$$

A nonempty subset S of a BCK/BCI-algebra X is called a *subalgebra* of X if $x * y \in S$ for all $x, y \in S$. A nonempty subset I of a BCK/BCI-algebra X is called an *ideal* of X if

$$0 \in I, \tag{2.3}$$

$$(\forall x, y \in X)\,(x * y \in I,\ y \in I\ \Rightarrow\ x \in I)\,. \tag{2.4}$$

We refer the reader to the books (Meng & Jun, 1994) and(Huang, 2006) for further information regarding BCK/BCI-algebras.

For any family $\{a_i \mid i \in \Lambda\}$ of real numbers, we define

$$\bigvee\{a_i \mid i \in \Lambda\} := \begin{cases} \max\{a_i \mid i \in \Lambda\} & \text{if } \Lambda \text{ is finite,} \\ \sup\{a_i \mid i \in \Lambda\} & \text{otherwise.} \end{cases}$$

$$\bigwedge\{a_i \mid i \in \Lambda\} := \begin{cases} \min\{a_i \mid i \in \Lambda\} & \text{if } \Lambda \text{ is finite,} \\ \inf\{a_i \mid i \in \Lambda\} & \text{otherwise.} \end{cases}$$

If $\Lambda = \{1, 2\}$, we will also use $a_1 \vee a_2$ and $a_1 \wedge a_2$ instead of $\bigvee\{a_i \mid i \in \Lambda\}$ and $\bigwedge\{a_i \mid i \in \Lambda\}$, respectively.

By a *fuzzy set* in a nonempty set X we mean a function $\mu : X \to [0, 1]$, and the *complement* of μ, denoted by μ^c, is the fuzzy set in X given by $\mu^c(x) = 1 - \mu(x)$ for all $x \in X$. A fuzzy set μ in a BCK/BCI-algebra X is called a *fuzzy subalgebra* of X if $\mu(x * y) \geq \mu(x) \wedge \mu(y)$ for all $x, y \in X$. A fuzzy set μ in a BCK/BCI-algebra X is called a *fuzzy ideal* of X if

$$(\forall x \in X)(\mu(0) \geq \mu(x)), \tag{2.5}$$

$$(\forall x, y \in X)(\mu(x) \geq \mu(x * y) \wedge \mu(y)). \tag{2.6}$$

Let X be a non-empty set. A neutrosophic set (NS) in X (Smarandache, 1999) is a structure of the form:

$$A := \{\langle x; A_T(x), A_I(x), A_F(x)\rangle \mid x \in X\}$$

where $A_T : X \to [0, 1]$ is a truth membership function, $A_I : X \to [0, 1]$ is an indeterminate membership function, and $A_F : X \to [0, 1]$ is a false membership function. For the sake of simplicity, we shall use the symbol $A = (A_T, A_I, A_F)$ for the neutrosophic set

$$A := \{\langle x; A_T(x), A_I(x), A_F(x)\rangle \mid x \in X\}.$$

3 GENERALIZED NEUTROSOPHIC SETS

Definition 3.1. A *generalized neutrosophic set* (GNS) in a non-empty set X is a structure of the form:

$$A := \{\langle x; A_T(x), A_{IT}(x), A_{IF}(x), A_F(x)\rangle \mid x \in X, A_{IT}(x) + A_{IF}(x) \leq 1\}$$

where $A_T : X \to [0,1]$ is a truth membership function, $A_F : X \to [0,1]$ is a false membership function, $A_{IT} : X \to [0,1]$ is an indeterminate membership function which is familiar with truth membership function, and $A_{IF} : X \to [0,1]$ is an indeterminate membership function which is familiar with false membership function.

Example 3.2. Let $X = \{a, b, c\}$ be a set. Then

$$A = \{\langle a; 0.4, 0.6, 0.3, 0.7\rangle, \langle b; 0.6, 0.2, 0.5, 0.7\rangle, \langle c; 0.1, 0.3, 0.5, 0.6\rangle\}$$

is a GNS in X. But

$$B = \{\langle a; 0.4, 0.6, 0.3, 0.7\rangle, \langle b; 0.6, 0.3, 0.9, 0.7\rangle, \langle c; 0.1, 0.3, 0.5, 0.6\rangle\}$$

is not a GNS in X since $B_{IT}(b) + B_{IF}(b) = 0.3 + 0.9 = 1.2 > 1$.

For the sake of simplicity, we shall use the symbol $A = (A_T, A_{IT}, A_{IF}, A_F)$ for the generalized neutrosophic set

$$A := \{\langle x; A_T(x), A_{IT}(x), A_{IF}(x), A_F(x)\rangle \mid x \in X, A_{IT}(x) + A_{IF}(x) \leq 1\}.$$

Note that every GNS $A = (A_T, A_{IT}, A_{IF}, A_F)$ in X satisfies the condition:

$$(\forall x \in X)\, (0 \leq A_T(x) + A_{IT}(x) + A_{IF}(x) + A_F(x) \leq 3)\,.$$

If $A = (A_T, A_{IT}, A_{IF}, A_F)$ is a GNS in X, then $\Box A = (A_T, A_{IT}, A_{IT}^c, A_T^c)$ and $\Diamond A = (A_F^c, A_{IF}^c, A_{IF}, A_F)$ are also GNSs in X.

Example 3.3. Given a set $X = \{0, 1, 2, 3, 4\}$, we know that

$$\begin{aligned} A = \{&\langle 0; 0.4, 0.6, 0.3, 0.7\rangle, \langle 1; 0.6, 0.2, 0.5, 0.7\rangle, \langle 2; 0.1, 0.3, 0.5, 0.6\rangle, \\ &\langle 3; 0.9, 0.1, 0.8, 0.6\rangle, \langle 4; 0.3, 0.6, 0.2, 0.9\rangle\} \end{aligned}$$

is a GNS in X. Then

$$\begin{aligned} \Box A = \{&\langle 0; 0.4, 0.6, 0.4, 0.6\rangle, \langle 1; 0.6, 0.2, 0.8, 0.4\rangle, \langle 2; 0.1, 0.3, 0.7, 0.9\rangle, \\ &\langle 3; 0.9, 0.1, 0.9, 0.1\rangle, \langle 4; 0.3, 0.6, 0.4, 0.7\rangle\} \end{aligned}$$

and

$$\begin{aligned} \Diamond A = \{&\langle 0; 0.3, 0.7, 0.3, 0.7\rangle, \langle 1; 0.3, 0.5, 0.5, 0.7\rangle, \langle 2; 0.4, 0.5, 0.5, 0.6\rangle, \\ &\langle 3; 0.4, 0.2, 0.8, 0.6\rangle, \langle 4; 0.1, 0.8, 0.2, 0.9\rangle\} \end{aligned}$$

are GNSs in X.

4 APPLICATIONS IN BCK/BCI-ALGEBRAS

In what follows, let X denote a BCK/BCI-algebra unless otherwise specified.

Definition 4.1. A GNS $A = (A_T,\ A_{IT},\ A_{IF},\ A_F)$ in X is called a *generalized neutrosophic subalgebra* of X if the following conditions are valid.

$$(\forall x, y \in X) \begin{pmatrix} A_T(x * y) \geq A_T(x) \wedge A_T(y) \\ A_{IT}(x * y) \geq A_{IT}(x) \wedge A_{IT}(y) \\ A_{IF}(x * y) \leq A_{IF}(x) \vee A_{IF}(y) \\ A_F(x * y) \leq A_F(x) \vee A_F(y) \end{pmatrix}. \tag{4.1}$$

Example 4.2. Consider a BCK-algebra $X = \{0, 1, 2, 3\}$ with the Cayley table which is given in Table 1.

Table 1: Cayley table for the binary operation "$*$"

$*$	0	1	2	3
0	0	0	0	0
1	1	0	0	1
2	2	1	0	2
3	3	3	3	0

Then the GNS

$$\begin{aligned} A = \{ &\langle 0; 0.6, 0.7, 0.2, 0.3 \rangle, \langle 1; 0.6, 0.6, 0.3, 0.3 \rangle, \\ &\langle 2; 0.4, 0.5, 0.4, 0.7 \rangle, \langle 3; 0.6, 0.3, 0.6, 0.5 \rangle \} \end{aligned}$$

in X is a generalized neutrosophic subalgebra of X.

Given a GNS $A = (A_T,\ A_{IT},\ A_{IF},\ A_F)$ in X and $\alpha_T,\ \alpha_{IT},\ \beta_F,\ \beta_{IF} \in [0, 1]$, consider the following sets.

$$\begin{aligned} U(T, \alpha_T) &:= \{x \in X \mid A_T(x) \geq \alpha_T\}, \\ U(IT, \alpha_{IT}) &:= \{x \in X \mid A_{IT}(x) \geq \alpha_{IT}\}, \\ L(F, \beta_F) &:= \{x \in X \mid A_F(x) \leq \beta_F\}, \\ L(IF, \beta_{IF}) &:= \{x \in X \mid A_{IF}(x) \leq \beta_{IF}\}. \end{aligned}$$

Theorem 4.3. *If a GNS $A = (A_T,\ A_{IT},\ A_{IF},\ A_F)$ is a generalized neutrosophic subalgebra of X, then the set $U(T, \alpha_T)$, $U(IT, \alpha_{IT})$, $L(F, \beta_F)$ and $L(IF, \beta_{IF})$ are subalgebras of X for all α_T, α_{IT}, β_F, $\beta_{IF} \in [0, 1]$ whenever they are non-empty.*

Proof. Assume that $U(T,\alpha_T)$, $U(IT,\alpha_{IT})$, $L(F,\beta_F)$ and $L(IF,\beta_{IF})$ are nonempty for all α_T, α_{IT}, β_F, $\beta_{IF} \in [0,1]$. Let $x, y \in X$. If $x, y \in U(T,\alpha_T)$, then $A_T(x) \geq \alpha_T$ and $A_T(y) \geq \alpha_T$. It follows that

$$A_T(x * y) \geq A_T(x) \wedge A_T(y) \geq \alpha_T$$

and so that $x * y \in U(T,\alpha_T)$. Hence $U(T,\alpha_T)$ is a subalgebra of X. Similarly, if $x, y \in U(IT,\alpha_{IT})$, then $x * y \in U(IT,\alpha_{IT})$, that is, $U(IT,\alpha_{IT})$ is a subalgebra of X. Suppose that $x, y \in L(F,\beta_F)$. Then $A_F(x) \leq \beta_F$ and $A_F(y) \leq \beta_F$, which imply that

$$A_F(x * y) \leq A_F(x) \vee A_F(y) \leq \beta_F,$$

that is, $x * y \in L(F,\beta_F)$. Hence $L(F,\beta_F)$ is a subalgebra of X. Similarly we can verify that $L(IF,\beta_{IF})$ is a subalgebra of X. $\square$

Corollary 4.4. *If a GNS $A = (A_T,\ A_{IT},\ A_{IF},\ A_F)$ is a generalized neutrosophic subalgebra of X, then the set*

$$A(\alpha_T,\alpha_{IT},\beta_F,\beta_{IF}) := \{x \in X \mid A_T(x) \geq \alpha_T, A_{IT}(x) \geq \alpha_{IT}, A_F(x) \leq \beta_F, A_{IF}(x) \leq \beta_{IF}\}$$

is a subalgebra of X for all α_T, α_{IT}, β_F, $\beta_{IF} \in [0,1]$.

Proof. Straightforward. $\square$

Theorem 4.5. *Let $A = (A_T,\ A_{IT},\ A_{IF},\ A_F)$ be a GNS in X such that $U(T,\alpha_T)$, $U(IT,\alpha_{IT})$, $L(F,\beta_F)$ and $L(IF,\beta_{IF})$ are subalgebras of X for all α_T, α_{IT}, β_F, $\beta_{IF} \in [0,1]$ whenever they are non-empty. Then $A = (A_T,\ A_{IT},\ A_{IF},\ A_F)$ is a generalized neutrosophic subalgebra of X.*

Proof. Assume that $U(T,\alpha_T)$, $U(IT,\alpha_{IT})$, $L(F,\beta_F)$ and $L(IF,\beta_{IF})$ are subalgebras for all α_T, α_{IT}, β_F, $\beta_{IF} \in [0,1]$. If there exist $x, y \in X$ such that

$$A_T(x * y) < A_T(x) \wedge A_T(y),$$

then $x, y \in U(T,t_\alpha)$ and $x * y \notin U(T,t_\alpha)$ for $t_\alpha = A_T(x) \wedge A_T(y)$. This is a contradiction, and so

$$A_T(x * y) \geq A_T(x) \wedge A_T(y)$$

for all $x, y \in X$. Similarly, we can prove

$$A_{IT}(x * y) \geq A_{IT}(x) \wedge A_{IT}(y)$$

for all $x, y \in X$. Suppose that

$$A_{IF}(x * y) > A_{IF}(x) \vee A_{IF}(y)$$

for some $x, y \in X$. Then there exists $f_\beta \in [0, 1)$ such that

$$A_{IF}(x * y) > f_\beta \geq A_{IF}(x) \vee A_{IF}(y),$$

which induces a contradiction since $x, y \in L(IF, f_\beta)$ and $x * y \notin L(IF, f_\beta)$. Thus

$$A_{IF}(x * y) \leq A_{IF}(x) \vee A_{IF}(y)$$

for all $x, y \in X$. Similar way shows that

$$A_F(x * y) \leq A_F(x) \vee A_F(y)$$

for all $x, y \in X$. Therefore $A = (A_T, A_{IT}, A_{IF}, A_F)$ is a generalized neutrosophic subalgebra of X. □

Since $[0, 1]$ is a completely distributive lattice under the usual ordering, we have the following theorem.

Theorem 4.6. *The family of generalized neutrosophic subalgebras of X forms a complete distributive lattice under the inclusion.*

Proposition 4.7. *Every generalized neutrosophic subalgebra $A = (A_T, A_{IT}, A_{IF}, A_F)$ of X satisfies the following assertions:*

(1) $(\forall x \in X)\ (A_T(0) \geq A_T(x),\ A_{IT}(0) \geq A_{IT}(x))$,

(2) $(\forall x \in X)\ (A_{IF}(0) \leq A_{IF}(x),\ A_F(0) \leq A_F(x))$.

Proof. Since $x * x = 0$ for all $x \in X$, it is straightforward. □

Theorem 4.8. *Let $A = (A_T, A_{IT}, A_{IF}, A_F)$ be a GNS in X. If there exists a sequence $\{a_n\}$ in X such that $\lim_{n\to\infty} A_T(a_n) = 1 = \lim_{n\to\infty} A_{IT}(a_n)$ and $\lim_{n\to\infty} A_F(a_n) = 0 = \lim_{n\to\infty} A_{IF}(a_n)$, then $A_T(0) = 1 = A_{IT}(0)$ and $A_F(0) = 0 = A_{IF}(0)$.*

Proof. Using Proposition 4.7, we know that $A_T(0) \geq A_T(a_n)$, $A_{IT}(0) \geq A_{IT}(a_n)$, $A_{IF}(0) \leq A_{IF}(a_n)$ and $A_F(0) \leq A_F(a_n)$ for every positive integer n. It follows that

$$\begin{aligned}
1 &\geq A_T(0) \geq \lim_{n\to\infty} A_T(a_n) = 1,\\
1 &\geq A_{IT}(0) \geq \lim_{n\to\infty} A_{IT}(a_n) = 1,\\
0 &\leq A_{IF}(0) \leq \lim_{n\to\infty} A_{IF}(a_n) = 0,\\
0 &\leq A_F(0) \leq \lim_{n\to\infty} A_F(a_n) = 0.
\end{aligned}$$

Thus $A_T(0) = 1 = A_{IT}(0)$ and $A_F(0) = 0 = A_{IF}(0)$. □

Proposition 4.9. *If every GNS $A = (A_T,\ A_{IT},\ A_{IF},\ A_F)$ in X satisfies:*

$$(\forall x, y \in X)\left(\begin{array}{l} A_T(x*y) \geq A_T(y),\ A_{IT}(x*y) \geq A_{IT}(y) \\ A_{IF}(x*y) \leq A_{IF}(y),\ A_F(x*y) \leq A_F(y) \end{array}\right), \tag{4.2}$$

then $A = (A_T,\ A_{IT},\ A_{IF},\ A_F)$ is constant on X.

Proof. Using (2.1) and (4.2), we have $A_T(x) = A_T(x*0) \geq A_T(0)$, $A_{IT}(x) = A_{IT}(x*0) \geq A_{IT}(0)$, $A_{IF}(x) = A_{IF}(x*0) \leq A_{IF}(0)$, and $A_F(x) = A_F(x*0) \leq A_F(0)$. It follows from Proposition 4.7 that $A_T(x) = A_T(0)$, $A_{IT}(x) = A_{IT}(0)$, $A_{IF}(x) = A_{IF}(0)$ and $A_F(x) = A_F(0)$ for all $x \in X$. Hence $A = (A_T,\ A_{IT},\ A_{IF},\ A_F)$ is constant on X. □

A mapping $f : X \to Y$ of BCK/BCI-algebras is called a *homomorphism* (**?**) if $f(x*y) = f(x)*f(y)$ for all $x, y \in X$. Note that if $f : X \to Y$ is a homomorphism, then $f(0) = 0$. Let $f : X \to Y$ be a homomorphism of BCK/BCI-algebras. For any GNS $A = (A_T,\ A_{IT},\ A_{IF},\ A_F)$ in Y, we define a new GNS $A^f = (A^f_T,\ A^f_{IT},\ A^f_{IF},\ A^f_F)$ in X, which is called the *induced GNS*, by

$$(\forall x \in X)\left(\begin{array}{l} A^f_T(x) = A_T(f(x)),\ A^f_{IT}(x) = A_{IT}(f(x)) \\ A^f_{IF}(x) = A_{IF}(f(x)),\ A^f_F(x) = A_F(f(x)) \end{array}\right). \tag{4.3}$$

Theorem 4.10. *Let $f : X \to Y$ be a homomorphism of BCK/BCI-algebras. If a GNS $A = (A_T,\ A_{IT},\ A_{IF},\ A_F)$ in Y is a generalized neutrosophic subalgebra of Y, then the induced GNS $A^f = (A^f_T,\ A^f_{IT},\ A^f_{IF},\ A^f_F)$ in X is a generalized neutrosophic subalgebra of X.*

Proof. For any $x, y \in X$, we have

$$\begin{aligned} A^f_T(x*y) &= A_T(f(x*y)) = A_T(f(x)*f(y)) \\ &\geq A_T(f(x)) \wedge A_T(f(y)) = A^f_T(x) \wedge A^f_T(y), \end{aligned}$$

$$\begin{aligned} A^f_{IT}(x*y) &= A_{IT}(f(x*y)) = A_{IT}(f(x)*f(y)) \\ &\geq A_{IT}(f(x)) \wedge A_{IT}(f(y)) = A^f_{IT}(x) \wedge A^f_{IT}(y), \end{aligned}$$

$$\begin{aligned} A^f_{IF}(x*y) &= A_{IF}(f(x*y)) = A_{IF}(f(x)*f(y)) \\ &\leq A_{IF}(f(x)) \vee A_{IF}(f(y)) = A^f_{IF}(x) \vee A^f_{IF}(y), \end{aligned}$$

and

$$\begin{aligned} A^f_F(x*y) &= A_F(f(x*y)) = A_F(f(x)*f(y)) \\ &\leq A_F(f(x)) \vee A_F(f(y)) = A^f_F(x) \vee A^f_F(y). \end{aligned}$$

Therefore $A^f = (A^f_T,\ A^f_{IT},\ A^f_{IF},\ A^f_F)$ is a generalized neutrosophic subalgebra of X. □

Theorem 4.11. *Let $f : X \to Y$ be an onto homomorphism of BCK/BCI-algebras and let $A = (A_T,\, A_{IT},\, A_{IF},\, A_F)$ be a GNS in Y. If the induced GNS $A^f = (A_T^f,\, A_{IT}^f,\, A_{IF}^f,\, A_F^f)$ in X is a generalized neutrosophic subalgebra of X, then $A = (A_T,\, A_{IT},\, A_{IF},\, A_F)$ is a generalized neutrosophic subalgebra of Y.*

Proof. Let $x, y \in Y$. Then $f(a) = x$ and $f(b) = y$ for some $a, b \in X$. Then

$$\begin{aligned} A_T(x * y) &= A_T(f(a) * f(b)) = A_T(f(a * b)) = A_T^f(a * b) \\ &\geq A_T^f(a) \wedge A_T^f(b) = A_T(f(a)) \wedge A_T(f(b)) \\ &= A_T(x) \wedge A_T(y), \end{aligned}$$

$$\begin{aligned} A_{IT}(x * y) &= A_{IT}(f(a) * f(b)) = A_{IT}(f(a * b)) = A_{IT}^f(a * b) \\ &\geq A_{IT}^f(a) \wedge A_{IT}^f(b) = A_{IT}(f(a)) \wedge A_{IT}(f(b)) \\ &= A_{IT}(x) \wedge A_{IT}(y), \end{aligned}$$

$$\begin{aligned} A_{IF}(x * y) &= A_{IF}(f(a) * f(b)) = A_{IF}(f(a * b)) = A_{IF}^f(a * b) \\ &\leq A_{IF}^f(a) \vee A_{IF}^f(b) = A_{IF}(f(a)) \vee A_{IF}(f(b)) \\ &= A_{IF}(x) \vee A_{IF}(y), \end{aligned}$$

and

$$\begin{aligned} A_F(x * y) &= A_F(f(a) * f(b)) = A_F(f(a * b)) = A_F^f(a * b) \\ &\leq A_F^f(a) \vee A_F^f(b) = A_F(f(a)) \vee A_F(f(b)) \\ &= A_F(x) \vee A_F(y). \end{aligned}$$

Hence $A = (A_T,\, A_{IT},\, A_{IF},\, A_F)$ is a generalized neutrosophic subalgebra of Y. □

Definition 4.12. A GNS $A = (A_T,\, A_{IT},\, A_{IF},\, A_F)$ in X is called a *generalized neutrosophic ideal* of X if the following conditions are valid.

$$(\forall x \in X) \begin{pmatrix} A_T(0) \geq A_T(x),\ A_{IT}(0) \geq A_{IT}(x) \\ A_{IF}(0) \leq A_{IF}(x), A_F(0) \leq A_F(x) \end{pmatrix}, \tag{4.4}$$

$$(\forall x, y \in X) \begin{pmatrix} A_T(x) \geq A_T(x * y) \wedge A_T(y) \\ A_{IT}(x) \geq A_{IT}(x * y) \wedge A_{IT}(y) \\ A_{IF}(x) \leq A_{IF}(x * y) \vee A_{IF}(y) \\ A_F(x) \leq A_F(x * y) \vee A_F(y) \end{pmatrix}. \tag{4.5}$$

Example 4.13. Consider a BCK-algebra $X = \{0, 1, 2, 3\}$ with the Cayley table which is given in Table 2.

Table 2: Cayley table for the binary operation "$*$"

$*$	0	1	2	3	4
0	0	0	0	0	0
1	1	0	1	0	0
2	2	2	0	0	0
3	3	3	3	0	0
4	4	3	4	1	0

Let

$$\begin{aligned} A = \{&\langle 0; 0.8, 0.7, 0.2, 0.1\rangle, \langle 1; 0.3, 0.6, 0.2, 0.6\rangle, \langle 2; 0.8, 0.4, 0.5, 0.3\rangle, \\ &\langle 3; 0.3, 0.2, 0.7, 0.8\rangle, \langle 4; 0.3, 0.2, 0.7, 0.8\rangle\}. \end{aligned}$$

be a GNS in X. By routine calculations, we know that A is a generalized neutrosophic ideal of X.

Lemma 4.14. *Every generalized neutrosophic ideal $A = (A_T, A_{IT}, A_{IF}, A_F)$ of X satisfies:*

$$(\forall x, y \in X)\left(x \leq y \ \Rightarrow \left\{ \begin{array}{l} A_T(x) \geq A_T(y),\ A_{IT}(x) \geq A_{IT}(y) \\ A_{IF}(x) \leq A_{IF}(y),\ A_F(x) \leq A_F(y) \end{array} \right. \right). \tag{4.6}$$

Proof. Let $x, y \in X$ be such that $x \leq y$. Then $x * y = 0$, and so

$$\begin{aligned} &A_T(x) \geq A_T(x * y) \wedge A_T(y) = A_T(0) \wedge A_T(y) = A_T(y), \\ &A_{IT}(x) \geq A_{IT}(x * y) \wedge A_{IT}(y) A_{IT}(0) \wedge A_{IT}(y) = A_{IT}(y), \\ &A_{IF}(x) \leq A_{IF}(x * y) \vee A_{IF}(y) A_{IF}(0) \vee A_{IF}(y) = A_{IF}(y), \\ &A_F(x) \leq A_F(x * y) \vee A_F(y) A_F(0) \vee A_F(y) = A_F(y). \end{aligned}$$

This completes the proof. □

Lemma 4.15. *Let $A = (A_T, A_{IT}, A_{IF}, A_F)$ be a generalized neutrosophic ideal of X. If the inequality $x * y \leq z$ holds in X, then $A_T(x) \geq A_T(y) \wedge A_T(z)$, $A_{IT}(x) \geq A_{IT}(y) \wedge A_{IT}(z)$, $A_{IF}(x) \leq A_{IF}(y) \vee A_{IF}(z)$ and $A_F(x) \leq A_F(y) \vee A_F(z)$.*

Proof. Let $x, y, z \in X$ be such that $x * y \leq z$, Then $(x * y) * z = 0$, and so

$$\begin{aligned} A_T(x) &\geq \bigwedge\{A_T(x * y), A_T(y)\} \\ &\geq \bigwedge\left\{\bigwedge\{A_T((x * y) * z), A_T(z)\}, A_T(y)\right\} \\ &= \bigwedge\left\{\bigwedge\{A_T(0), A_T(z)\}, A_T(y)\right\} \\ &= \bigwedge\left\{A_T(y), A_T(z)\right\}, \end{aligned}$$

$$\begin{aligned}
A_{IT}(x) &\geq \bigwedge\{A_{IT}(x*y), A_{IT}(y)\} \\
&\geq \bigwedge\left\{\bigwedge\{A_{IT}((x*y)*z), A_{IT}(z)\}, A_{IT}(y)\right\} \\
&= \bigwedge\left\{\bigwedge\{A_{IT}(0), A_{IT}(z)\}, A_{IT}(y)\right\} \\
&= \bigwedge\left\{A_{IT}(y), A_{IT}(z)\right\},
\end{aligned}$$

$$\begin{aligned}
A_{IF}(x) &\leq \bigvee\{A_{IF}(x*y), A_{IF}(y)\} \\
&\leq \bigvee\left\{\bigvee\{A_{IF}((x*y)*z), A_{IF}(z)\}, A_{IF}(y)\right\} \\
&= \bigvee\left\{\bigvee\{A_{IF}(0), A_{IF}(z)\}, A_{IF}(y)\right\} \\
&= \bigvee\left\{A_{IF}(y), A_{IF}(z)\right\},
\end{aligned}$$

and

$$\begin{aligned}
A_{F}(x) &\leq \bigvee\{A_{F}(x*y), A_{F}(y)\} \\
&\leq \bigvee\left\{\bigvee\{A_{F}((x*y)*z), A_{F}(z)\}, A_{F}(y)\right\} \\
&= \bigvee\left\{\bigvee\{A_{F}(0), A_{F}(z)\}, A_{F}(y)\right\} \\
&= \bigvee\left\{A_{F}(y), A_{F}(z)\right\}.
\end{aligned}$$

This completes the proof. $\square$

Proposition 4.16. *Let $A = (A_T,\ A_{IT},\ A_{IF},\ A_F)$ be a generalized neutrosophic ideal of X. If the inequality*

$$(\cdots((x*a_1)*a_2)*\cdots)*a_n = 0$$

holds in X, then

$$\begin{aligned}
A_T(x) &\geq \bigwedge\left\{A_T(a_i) \mid i = 1, 2, \cdots, n\right\}, \\
A_{IT}(x) &\geq \bigwedge\left\{A_{IT}(a_i) \mid i = 1, 2, \cdots, n\right\}, \\
A_{IF}(x) &\leq \bigvee\left\{A_{IF}(a_i) \mid i = 1, 2, \cdots, n\right\}, \\
A_F(x) &\leq \bigvee\left\{A_F(a_i) \mid i = 1, 2, \cdots, n\right\}.
\end{aligned}$$

Proof. It is straightforward by using induction on n and Lemmas 4.14 and 4.15. $\square$

Theorem 4.17. *In a BCK-algebra X, every generalized neutrosophic ideal is a generalized neutrosophic subalgebra.*

Proof. Let $A = (A_T, A_{IT}, A_{IF}, A_F)$ be a generalized neutrosophic ideal of a BCK-algebra X. Since $x * y \leq x$ for all $x, y \in X$, we have $A_T(x * y) \geq A_T(x)$, $A_{IT}(x * y) \geq A_{IT}(x)$, $A_{IF}(x * y) \leq A_{IF}(x)$ and $A_F(x * y) \leq A_F(x)$ by Lemma 4.14. It follows from (4.5) that

$$A_T(x * y) \geq A_T(x) \geq A_T(x * y) \wedge A_T(y) \geq A_T(x) \wedge A_T(y),$$

$$A_{IT}(x * y) \geq A_{IT}(x) \geq A_{IT}(x * y) \wedge A_{IT}(y) \geq A_{IT}(x) \wedge A_{IT}(y),$$

$$A_{IF}(x * y) \leq A_{IF}(x) \leq A_{IF}(x * y) \vee A_{IF}(y) \leq A_{IF}(x) \vee A_{IF}(y),$$

and

$$A_F(x * y) \leq A_F(x) \leq A_F(x * y) \vee A_F(y) \leq A_F(x) \vee A_F(y).$$

Therefore $A = (A_T, A_{IT}, A_{IF}, A_F)$ is a generalized neutrosophic subalgebra of X. □

The converse of Theorem 4.17 is not true. For example, the generalized neutrosophic subalgebra A in Example 4.2 is not a generalized neutrosophic ideal of X since

$$A_T(2) = 0.4 \not\geq 0.6 = A_T(2 * 1) \wedge A_T(1)$$

and/or

$$A_F(2) = 0.7 \not\leq 0.3 = A_F(2 * 1) \vee A_F(1).$$

We give a condition for a generalized neutrosophic subalgebra to be a generalized neutrosophic ideal.

Theorem 4.18. *Let $A = (A_T, A_{IT}, A_{IF}, A_F)$ be a generalized neutrosophic subalgebra of X such that*

$$\begin{aligned} &A_T(x) \geq A_T(y) \wedge A_T(z), \\ &A_{IT}(x) \geq A_{IT}(y) \wedge A_{IT}(z), \\ &A_{IF}(x) \leq A_{IF}(y) \vee A_{IF}(z), \\ &A_F(x) \leq A_F(y) \vee A_F(z) \end{aligned}$$

for all $x, y, z \in X$ satisfying the inequality $x * y \leq z$. Then $A = (A_T, A_{IT}, A_{IF}, A_F)$ is a generalized neutrosophic ideal of X.

Proof. Recall that $A_T(0) \geq A_T(x)$, $A_{IT}(0) \geq A_{IT}(x)$, $A_{IF}(0) \leq A_{IF}(x)$ and $A_F(0) \leq A_F(x)$ for all $x \in X$ by Proposition 4.7. Let $x, y \in X$. Since $x * (x * y) \leq y$, it follows from the hypothesis that

$$\begin{aligned} &A_T(x) \geq A_T(x * y) \wedge A_T(y), \\ &A_{IT}(x) \geq A_{IT}(x * y) \wedge A_{IT}(y), \\ &A_{IF}(x) \leq A_{IF}(x * y) \vee A_{IF}(y), \\ &A_F(x) \leq A_F(x * y) \vee A_F(y). \end{aligned}$$

Hence $A = (A_T, A_{IT}, A_{IF}, A_F)$ is a generalized neutrosophic ideal of X. □

Theorem 4.19. *A GNS $A = (A_T, A_{IT}, A_{IF}, A_F)$ in X is a generalized neutrosophic ideal of X if and only if the fuzzy sets A_T, A_{IT}, A^c_{IF} and A^c_F are fuzzy ideals of X.*

Proof. Assume that $A = (A_T, A_{IT}, A_{IF}, A_F)$ is a generalized neutrosophic ideal of X. Clearly, A_T and A_{IT} are fuzzy ideals of X. For every $x, y \in X$, we have

$$A^c_{IF}(0) = 1 - A_{IF}(0) \geq 1 - A_{IF}(x) = A^c_{IF}(x),$$

$$A^c_F(0) = 1 - A_F(0) \geq 1 - A_F(x) = A^c_F(x),$$

$$\begin{aligned} A^c_{IF}(x) = 1 - A_{IF}(x) &\geq 1 - A_{IF}(x * y) \vee A_{IF}(y) \\ &= \bigwedge\{1 - A_{IF}(x * y), 1 - A_{IF}(y)\} \\ &= \bigwedge\{A^c_{IF}(x * y), A^c_{IF}(y)\} \end{aligned}$$

and

$$\begin{aligned} A^c_F(x) = 1 - A_F(x) &\geq 1 - A_F(x * y) \vee A_F(y) \\ &= \bigwedge\{1 - A_F(x * y), 1 - A_F(y)\} \\ &= \bigwedge\{A^c_F(x * y), A^c_F(y)\}. \end{aligned}$$

Therefore A_T, A_{IT}, A^c_{IF} and A^c_F are fuzzy ideals of X.

Conversely, let $A = (A_T, A_{IT}, A_{IF}, A_F)$ be a GNS in X for which A_T, A_{IT}, A^c_{IF} and A^c_F are fuzzy ideals of X. For every $x \in X$, we have $A_T(0) \geq A_T(x)$, $A_{IT}(0) \geq A_{IT}(x)$,

$$1 - A_{IF}(0) = A^c_{IF}(0) \geq A^c_{IF}(x) = 1 - A_{IF}(x), \text{ that is, } A_{IF}(0) \leq A_{IF}(x)$$

and

$$1 - A_F(0) = A^c_F(0) \geq A^c_F(x) = 1 - A_F(x), \text{ that is, } A_F(0) \leq A_F(x).$$

Let $x, y \in X$. Then

$$A_T(x) \geq A_T(x * y) \wedge A_T(y),$$

$$A_{IT}(x) \geq A_{IT}(x * y) \wedge A_{IT}(y),$$

$$\begin{aligned} 1 - A_{IF}(x) = A^c_{IF}(x) &\geq A^c_{IF}(x * y) \wedge A^c_{IF}(y) \\ &= \bigwedge\{1 - A_{IF}(x * y), 1 - A_{IF}(y)\} \\ &= 1 - \bigvee\{A_{IF}(x * y), A_{IF}(y)\}, \end{aligned}$$

and

$$
\begin{aligned}
1 - A_F(x) = A_F^c(x) &\geq A_F^c(x * y) \wedge A_F^c(y) \\
&= \bigwedge\{1 - A_F(x * y), 1 - A_F(y)\} \\
&= 1 - \bigvee\{A_F(x * y), A_F(y)\},
\end{aligned}
$$

that is, $A_{IF}(x) \leq A_{IF}(x * y) \vee A_{IF}(y)$ and $A_F(x) \leq A_F(x * y) \vee A_F(y)$. Hence $A = (A_T,\ A_{IT},\ A_{IF},\ A_F)$ is a generalized neutrosophic ideal of X. $\square$

Theorem 4.20. *If a GNS $A = (A_T,\ A_{IT},\ A_{IF},\ A_F)$ in X is a generalized neutrosophic ideal of X, then $\Box A = (A_T,\ A_{IT},\ A_{IT}^c,\ A_T^c)$ and $\Diamond A = (A_{IF}^c,\ A_F^c,\ A_F,\ A_{IF})$ are generalized neutrosophic ideals of X.*

Proof. Assume that $A = (A_T,\ A_{IT},\ A_{IF},\ A_F)$ is a generalized neutrosophic ideal of X and let $x, y \in X$. Note that $\Box A = (A_T,\ A_{IT},\ A_{IT}^c,\ A_T^c)$ and $\Diamond A = (A_{IF}^c,\ A_F^c,\ A_F,\ A_{IF})$ are GNSs in X. Let $x, y \in X$. Then

$$
\begin{aligned}
A_{IT}^c(x * y) = 1 - A_{IT}(x * y) &\leq 1 - \bigwedge\{A_{IT}(x), A_{IT}(y)\} \\
&= \bigvee\{1 - A_{IT}(x), 1 - A_{IT}(y)\} \\
&= \bigvee\{A_{IT}^c(x), A_{IT}^c(y)\},
\end{aligned}
$$

$$
\begin{aligned}
A_T^c(x * y) = 1 - A_T(x * y) &\leq 1 - \bigwedge\{A_T(x), A_T(y)\} \\
&= \bigvee\{1 - A_T(x), 1 - A_T(y)\} \\
&= \bigvee\{A_T^c(x), A_T^c(y)\},
\end{aligned}
$$

$$
\begin{aligned}
A_{IF}^c(x * y) = 1 - A_{IF}(x * y) &\geq 1 - \bigvee\{A_{IF}(x), A_{IF}(y)\} \\
&= \bigwedge\{1 - A_{IF}(x), 1 - A_{IF}(y)\} \\
&= \bigwedge\{A_{IF}^c(x), A_{IF}^c(y)\}
\end{aligned}
$$

and

$$
\begin{aligned}
A_F^c(x * y) = 1 - A_F(x * y) &\geq 1 - \bigvee\{A_F(x), A_F(y)\} \\
&= \bigwedge\{1 - A_F(x), 1 - A_F(y)\} \\
&= \bigwedge\{A_F^c(x), A_F^c(y)\}.
\end{aligned}
$$

Therefore $\Box A = (A_T,\ A_{IT},\ A_{IT}^c,\ A_T^c)$ and $\Diamond A = (A_{IF}^c,\ A_F^c,\ A_F,\ A_{IF})$ are generalized neutrosophic ideals of X. $\square$

Theorem 4.21. *If a GNS $A = (A_T,\ A_{IT},\ A_{IF},\ A_F)$ is a generalized neutrosophic ideal of X, then the set $U(T, \alpha_T)$, $U(IT, \alpha_{IT})$, $L(F, \beta_F)$ and $L(IF, \beta_{IF})$ are ideals of X for all α_T, α_{IT}, β_F, $\beta_{IF} \in [0, 1]$ whenever they are non-empty.*

Proof. Assume that $U(T, \alpha_T)$, $U(IT, \alpha_{IT})$, $L(F, \beta_F)$ and $L(IF, \beta_{IF})$ are nonempty for all α_T, α_{IT}, β_F, $\beta_{IF} \in [0, 1]$. It is clear that $0 \in U(T, \alpha_T)$, $0 \in U(IT, \alpha_{IT})$, $0 \in L(F, \beta_F)$ and $0 \in L(IF, \beta_{IF})$. Let $x, y \in X$. If $x * y \in U(T, \alpha_T)$ and $y \in U(T, \alpha_T)$, then $A_T(x * y) \geq \alpha_T$ and $A_T(y) \geq \alpha_T$. Hence

$$A_T(x) \geq A_T(x * y) \wedge A_T(y) \geq \alpha_T,$$

and so $x \in U(T, \alpha_T)$. Similarly, if $x * y \in U(IT, \alpha_T)$ and $y \in U(IT, \alpha_T)$, then $x \in U(IT, \alpha_T)$. If $x * y \in L(F, \beta_F)$ and $y \in L(F, \beta_F)$, then $A_F(x * y) \leq \beta_F$ and $A_F(y) \leq \beta_F$. Hence

$$A_F(x) \leq A_F(x * y) \vee A_F(y) \leq \beta_F,$$

and so $x \in L(F, \beta_F)$. Similarly, if $x * y \in L(IF, \beta_{IF})$ and $y \in L(IF, \beta_{IF})$, then $x \in L(IF, \beta_{IF})$. This completes the proof. □

Theorem 4.22. *Let $A = (A_T,\ A_{IT},\ A_{IF},\ A_F)$ be a GNS in X such that $U(T, \alpha_T)$, $U(IT, \alpha_{IT})$, $L(F, \beta_F)$ and $L(IF, \beta_{IF})$ are ideals of X for all α_T, α_{IT}, β_F, $\beta_{IF} \in [0, 1]$. Then $A = (A_T,\ A_{IT},\ A_{IF},\ A_F)$ is a generalized neutrosophic ideal of X.*

Proof. Let α_T, α_{IT}, β_F, $\beta_{IF} \in [0, 1]$ be such that $U(T, \alpha_T)$, $U(IT, \alpha_{IT})$, $L(F, \beta_F)$ and $L(IF, \beta_{IF})$ are ideals of X. For any $x \in X$, let $A_T(x) = \alpha_T$, $A_{IT}(x) = \alpha_{IT}$, $A_{IF}(x) = \beta_{IF}$ and $A_F(x) = \beta_F$. Since $0 \in U(T, \alpha_T)$, $0 \in U(IT, \alpha_{IT})$, $0 \in L(F, \beta_F)$ and $0 \in L(IF, \beta_{IF})$, we have $A_T(0) \geq \alpha_T = A_T(x)$, $A_{IT}(0) \geq \alpha_{IT} = A_{IT}(x)$, $A_{IF}(0) \leq \beta_{IF} = A_{IF}(x)$ and $A_F(0) \leq \beta_F = A_F(x)$. If there exist $a, b \in X$ such that $A_T(a * b) < A_T(a) \wedge A_T(b)$, then $a, b \in U(T, \alpha_0)$ and $a * b \notin U(T, \alpha_0)$ where $\alpha_0 := A_T(a) \wedge A_T(b)$. This is a contradiction, and hence $A_T(x * y) \geq A_T(x) \wedge A_T(y)$ for all $x, y \in X$. Similarly, we can verify $A_{IT}(x * y) \geq A_{IT}(x) \wedge A_{IT}(y)$ for all $x, y \in X$. Suppose that $A_{IF}(a * b) > A_{IF}(a) \vee A_{IF}(b)$ for some $a, b \in X$. Taking $\beta_0 := A_{IF}(a) \vee A_{IF}(b)$ induces $a, b \in L(IF, \beta_{IF})$ and $a * b \notin L(IF, \beta_{IF})$, a contradiction. Thus $A_{IF}(x * y) \leq A_{IF}(x) \vee A_{IF}(y)$ for all $x, y \in X$. Similarly we have $A_F(x * y) \leq A_F(x) \vee A_F(y)$ for all $x, y \in X$. Consequently, $A = (A_T,\ A_{IT},\ A_{IF},\ A_F)$ is a generalized neutrosophic ideal of X. □

Let Λ be a nonempty subset of $[0, 1]$.

Theorem 4.23. *Let $\{I_t \mid t \in \Lambda\}$ be a collection of ideals of X such that*

(1) $X = \bigcup_{t \in \Lambda} I_t$,

(2) $(\forall s, t \in \Lambda)\,(s > t \iff I_s \subset I_t)$.

Let $A = (A_T,\ A_{IT},\ A_{IF},\ A_F)$ be a GNS in X given as follows:

$$(\forall x \in X)\begin{pmatrix} A_T(x) = \bigvee\{t \in \Lambda \mid x \in I_t\} = A_{IT}(x) \\ A_{IF}(x) = \bigwedge\{t \in \Lambda \mid x \in I_t\} = A_F(x) \end{pmatrix}. \tag{4.7}$$

Then $A = (A_T,\ A_{IT},\ A_{IF},\ A_F)$ is a generalized neutrosophic ideal of X.

Proof. According to Theorem 4.22, it is sufficient to show that $U(T,t)$, $U(IT,t)$, $L(F,s)$ and $L(IF,s)$ are ideals of X for every $t \in [0, A_T(0) = A_{IT}(0)]$ and $s \in [A_{IF}(0) = A_F(0), 1]$. In order to prove $U(T,t)$ and $U(IT,t)$ are ideals of X, we consider two cases:

(i) $t = \bigvee\{q \in \Lambda \mid q < t\}$,

(ii) $t \neq \bigvee\{q \in \Lambda \mid q < t\}$.

For the first case, we have

$$x \in U(T,t) \Longleftrightarrow (\forall q < t)(x \in I_q) \Longleftrightarrow x \in \bigcap_{q<t} I_q,$$
$$x \in U(IT,t) \Longleftrightarrow (\forall q < t)(x \in I_q) \Longleftrightarrow x \in \bigcap_{q<t} I_q.$$

Hence $U(T,t) = \bigcap_{q<t} I_q = U(IT,t)$, and so $U(T,t)$ and $U(IT,t)$ are ideals of X. For the second case, we claim that $U(T,t) = \bigcup_{q \geq t} I_q = U(IT,t)$. If $x \in \bigcup_{q \geq t} I_q$, then $x \in I_q$ for some $q \geq t$. It follows that $A_{IT}(x) = A_T(x) \geq q \geq t$ and so that $x \in U(T,t)$ and $x \in U(IT,t)$. This shows that $\bigcup_{q \geq t} I_q \subseteq U(T,t) = U(IT,t)$. Now, assume that $x \notin \bigcup_{q \geq t} I_q$. Then $x \notin I_q$ for all $q \geq t$. Since $t \neq \bigvee\{q \in \Lambda \mid q < t\}$, there exists $\varepsilon > 0$ such that $(t - \varepsilon, t) \cap \Lambda = \emptyset$. Hence $x \notin I_q$ for all $q > t - \varepsilon$, which means that if $x \in I_q$, then $q \leq t - \varepsilon$. Thus $A_{IT}(x) = A_T(x) \leq t - \varepsilon < t$, and so $x \notin U(T,t) = U(IT,t)$. Therefore $U(T,t) = U(IT,t) \subseteq \bigcup_{q \geq t} I_q$. Consequently, $U(T,t) = U(IT,t) = \bigcup_{q \geq t} I_q$ which is an ideal of X. Next we show that $L(F,s)$ and $L(IF,s)$ are ideals of X. We consider two cases as follows:

(iii) $s = \bigwedge\{r \in \Lambda \mid s < r\}$,

(iv) $s \neq \bigwedge\{r \in \Lambda \mid s < r\}$.

Case (iii) implies that

$$x \in L(IF,s) \Longleftrightarrow (\forall s < r)(x \in I_r) \Longleftrightarrow x \in \bigcap_{s<r} I_r,$$
$$x \in U(F,s) \Longleftrightarrow (\forall s < r)(x \in I_r) \Longleftrightarrow x \in \bigcap_{s<r} I_r.$$

It follows that $L(IF,s) = L(F,s) = \bigcap_{s<r} I_r$, which is an ideal of X. Case (iv) induces $(s, s+\varepsilon) \cap \Lambda = \emptyset$ for some $\varepsilon > 0$. If $x \in \bigcup_{s \geq r} I_r$, then $x \in I_r$ for some $r \leq s$, and so $A_{IF}(x) = A_F(x) \leq r \leq s$, that is, $x \in L(IF,s)$ and $x \in L(F,s)$. Hence $\bigcup_{s \geq r} I_r \subseteq L(IF,s) = L(F,s)$. If $x \notin \bigcup_{s \geq r} I_r$, then $x \notin I_r$ for all $r \leq s$ which implies that $x \notin I_r$ for all $r \leq s + \varepsilon$, that is, if $x \in I_r$ then $r \geq s + \varepsilon$. Hence $A_{IF}(x) = A_F(x) \geq s + \varepsilon > s$, and so $x \notin L(A_{IF}, s) = L(A_F, s)$. Hence $L(A_{IF}, s) = L(A_F, s) = \bigcup_{s \geq r} I_r$ which is an ideal of X. This completes the proof. $\square$

Theorem 4.24. *Let $f : X \to Y$ be a homomorphism of BCK/BCI-algebras. If a GNS $A = (A_T, A_{IT}, A_{IF}, A_F)$ in Y is a generalized neutrosophic ideal of Y, then the new GNS $A^f = (A^f_T, A^f_{IT}, A^f_{IF}, A^f_F)$ in X is a generalized neutrosophic ideal of X.*

Proof. We first have

$$\begin{aligned}
A^f_T(0) &= A_T(f(0)) = A_T(0) \geq A_T(f(x)) = A^f_T(x),\\
A^f_{IT}(0) &= A_{IT}(f(0)) = A_{IT}(0) \geq A_{IT}(f(x)) = A^f_{IT}(x),\\
A^f_{IF}(0) &= A_{IF}(f(0)) = A_{IF}(0) \leq A_{IF}(f(x)) = A^f_{IF}(x),\\
A^f_F(0) &= A_F(f(0)) = A_F(0) \leq A_F(f(x)) = A^f_F(x)
\end{aligned}$$

for all $x \in X$. Let $x, y \in X$. Then

$$\begin{aligned}
A^f_T(x) = A_T(f(x)) &\geq A_T(f(x) * f(y)) \wedge A_T(f(y))\\
&= A_T(f(x * y)) \wedge A_T(f(y))\\
&= A^f_T(x * y) \wedge A^f_T(y),
\end{aligned}$$

$$\begin{aligned}
A^f_{IT}(x) = A_{IT}(f(x)) &\geq A_{IT}(f(x) * f(y)) \wedge A_{IT}(f(y))\\
&= A_{IT}(f(x * y)) \wedge A_{IT}(f(y))\\
&= A^f_{IT}(x * y) \wedge A^f_{IT}(y),
\end{aligned}$$

$$\begin{aligned}
A^f_{IF}(x) = A_{IF}(f(x)) &\leq A_{IF}(f(x) * f(y)) \vee A_{IF}(f(y))\\
&= A_{IF}(f(x * y)) \vee A_{IF}(f(y))\\
&= A^f_{IF}(x * y) \vee A^f_{IF}(y)
\end{aligned}$$

and

$$\begin{aligned}
A^f_F(x) = A_F(f(x)) &\leq A_F(f(x) * f(y)) \vee A_F(f(y))\\
&= A_F(f(x * y)) \vee A_F(f(y))\\
&= A^f_F(x * y) \vee A^f_F(y).
\end{aligned}$$

Therefore $A^f = (A^f_T, A^f_{IT}, A^f_{IF}, A^f_F)$ in X is a generalized neutrosophic ideal of X. □

Theorem 4.25. *Let $f : X \to Y$ be an onto homomorphism of BCK/BCI-algebras and let $A = (A_T, A_{IT}, A_{IF}, A_F)$ be a GNS in Y. If the induced GNS $A^f = (A^f_T, A^f_{IT}, A^f_{IF}, A^f_F)$ in X is a generalized neutrosophic ideal of X, then $A = (A_T, A_{IT}, A_{IF}, A_F)$ is a generalized neutrosophic ideal of Y.*

Proof. For any $x \in Y$, there exists $a \in X$ such that $f(a) = x$. Then

$$\begin{aligned}
A_T(0) &= A_T(f(0)) = A_T^f(0) \geq A_T^f(a) = A_T(f(a)) = A_T(x),\\
A_{IT}(0) &= A_{IT}(f(0)) = A_{IT}^f(0) \geq A_{IT}^f(a) = A_{IT}(f(a)) = A_{IT}(x),\\
A_{IF}(0) &= A_{IF}(f(0)) = A_{IF}^f(0) \leq A_{IF}^f(a) = A_{IF}(f(a)) = A_{IF}(x),\\
A_F(0) &= A_F(f(0)) = A_F^f(0) \leq A_F^f(a) = A_F(f(a)) = A_F(x).
\end{aligned}$$

Let $x, y \in Y$. Then $f(a) = x$ and $f(b) = y$ for some $a, b \in X$. It follows that

$$\begin{aligned}
A_T(x) &= A_T(f(a)) = A_T^f(a)\\
&\geq A_T^f(a * b) \wedge A_T^f(b)\\
&= A_T(f(a * b)) \wedge A_T(f(b))\\
&= A_T(f(a) * f(b)) \wedge A_T(f(b))\\
&= A_T(x * y) \wedge A_T(y),
\end{aligned}$$

$$\begin{aligned}
A_{IT}(x) &= A_{IT}(f(a)) = A_{IT}^f(a)\\
&\geq A_{IT}^f(a * b) \wedge A_{IT}^f(b)\\
&= A_{IT}(f(a * b)) \wedge A_{IT}(f(b))\\
&= A_{IT}(f(a) * f(b)) \wedge A_{IT}(f(b))\\
&= A_{IT}(x * y) \wedge A_{IT}(y),
\end{aligned}$$

$$\begin{aligned}
A_{IF}(x) &= A_{IF}(f(a)) = A_{IF}^f(a)\\
&\leq A_{IF}^f(a * b) \vee A_{IF}^f(b)\\
&= A_{IF}(f(a * b)) \vee A_{IF}(f(b))\\
&= A_{IF}(f(a) * f(b)) \vee A_{IF}(f(b))\\
&= A_{IF}(x * y) \vee A_{IF}(y),
\end{aligned}$$

and

$$\begin{aligned}
A_F(x) &= A_F(f(a)) = A_F^f(a)\\
&\leq A_F^f(a * b) \vee A_F^f(b)\\
&= A_F(f(a * b)) \vee A_F(f(b))\\
&= A_F(f(a) * f(b)) \vee A_F(f(b))\\
&= A_F(x * y) \vee A_F(y).
\end{aligned}$$

Therefore $A = (A_T, A_{IT}, A_{IF}, A_F)$ is a generalized neutrosophic ideal of Y. □

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