A Review On Recent Developments of Neutrosophic Matrix Theory and Open Problems

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Mohammad Abobala

Tishreen University, Faculty of Science, Department of Mathematics, Latakia, Syria

Abstract: This work is dedicated to give the reader a wide review for recent advantages in the algebraic study of neutrosophic matrices, refined neutrosophic matrices, and n-refined neutrosophic matrices.

Key words: neutrosophic matrix, refined neutrosophic matrix, n-refined neutrosophic matgrx, diagonalization, inverse, determinant, AH-linear transformation, neutrosophic vector space

Introduction

Neutrosophy is a new kind of generalized logic proposed by Smarandache [10,21], to deal with indeterminacy in real life and science.

Neutrosophic logic found its way in many branches of human knowledge such as graph theory [33], number theory [3], topology [32], statistics [15], and equations [2].

The neutrosophic algebra was built over the idea of inserting the indeterminacy element I into classical algebraic structures [9]. This idea lead to many concepts such as neutrosophic spaces [11,16,31], neutrosophic modules [12,29,30], neutrosophic rings [18,22], groups [19,39], and functions [8,17,23].

By refining the indeterminacy I into many levels of indeterminacy $I_1,..,I_n$, we get refined and n-refined neutrosophic groups [24], rings [6,36], modules [12,27], and spaces [1,7,13].

In classical algebra, matrices are playing an important role in the theory of vector spaces. They were generalized to neutrosophic matrices [4,5,26], refined neutrosophic matrices [34], and n-refined neutrosophic matrices [39].

Recently, there is an increasing interest in the algebraic properties of these matrices such as diagonalization problem [37], invertibility [37], determinants [37,38,39], and algebraic representations by linear functions [35].

Through this work, we review the recent published developments in the algebraic study of neutrosophic matrices, refined neutrosophic matrices, and n-refined neutrosophic matrices, to provide the interested reader a strong background in this area. Also, we list some of the most important open questions about these matrices, which may represent the future of this branch of studies.

Elementary Properties of Neutrosophic Matrices

Definition 1: [28] Let X be a non-empty fixed set. A neutrosophic set A is an object having the form $\{x, (\mu A(x), \delta A(x), \gamma A(x)): x \in X\}$, where $\mu A(x)$, $\delta A(x)$ and $\gamma A(x)$represent the degree of membership, the degree of indeterminacy, and the degree of non-membership respectively of each element $x \in X$ to the set A .

Definition 2: [9] Let K be a field, the neutrosophic file generated by $\langle K \cup I \rangle$ which is denoted by $K(I) = \langle K \cup I \rangle$.

Definition 3: [9] Classical neutrosophic number has the form $a + bI$ where a, b are real or complex numbers and I is the indeterminacy such that $0 \cdot I = 0$ and $I^2 = I$ which results that

$I^n = I$ for all positive integers n.

Definition 4: [4]

Let $M_{m\times n} = \{(a_{ij}) : a_{ij} \in K(I)\}$,where $K\ (I)$ is a neutrosophic field. We call to be the neutrosophic matrix

Remark 5: [9]

The neutrosophic field $K\ (I)$ is not a field by classical meaning, since I is not invertible.

Definition 6: [37]

Let $M = A + BI$ a neutrosophic n square matrix, where A and B are two n squares matrices, then M is called an invertible neutrosophic n square matrix, if and only if there exists an n square matrix $S = S_1 + S_2 I$, where S_1 and S_2 are two n square matrices such that

$S \cdot M = M \cdot S = U_{n\times n}$, where $U_{n\times n}$ denotes the $n \times n$ identity matrix.

Definition 7: [37]

Let $M = A + BI$ be a neutrosophic n square matrix. The determinant of M is defined as

$.detM = detA + I[det(A + B) - detA]$

Theorem 8: [37]

Let $M = A + BI$ a neutrosophic square $n \times n$ matrix, where A , B are two squares $n \times n$ matrices, then M is invertible if and only if A and $A + B$ are invertible matrices and

$M^{-1} = A^{-1} + I[(A + B)^{-1} - A^{-1}]$.

Proof:

If A and $A+B$ are invertible matrices, then $(A+B)^{-1}$, A^{-1} are existed, and

$M^{-1} = A^{-1} + I[(A+B)^{-1} - A^{-1}]$ exists too. Now to prove $M^{-1}$is the inverse of M,

$MM^{-1} = (A + BI)\cdot ($

$= AA^{-1} + I[A(A + B$

$= U_{n\times n} + I[$

$= U_{n\times n} + I[U_{n\times n} - U_{n\times n}] = U_{n\times n} = M^{-1}M.$

conversely, we suppose that M is invertible, thus there is a matrix $S = S_1 + S_2 I$, with the property $M \cdot S = S \cdot M = U_{n\times n}$.

$MS = (A + BI)(S_1 + S_2 I) = AS_1 + I[(A+B)(S_1 + S_2) - AS_1] = U_{n\times n} + 0_{n\times n}$=$SM$. Hence, we get:

(a)$S_1 A = AS_1 = U_{n\times n}, thus\ A\ is\ invertible\ and\ A^{-1} = S_1.$

(b)$(A+B)(S_1+S_2) - AS_1 = (S_1+S_2)(A+B) - S_1 A = O_{n\times n}$, thus,

$(S_1+S_2)(A+B) = (A+B)(S_1+S_2) = AS_1 = U_{n\times n}$. This implies that $(A+B)$ is invertible.

Theorem 9: [37]

M is invertible matrix if and only if $detM \neq 0$.

Proof:

From **Theorem 8** we find that M is invertible matrix if and only if $A+B, A$ are two invertible matrices, hence $det[A+B] \neq 0, detA \neq 0$ which means

$detM = detA + I[\det(A+B) - detA] \neq 0.$

Example 10: [37]

Consider the following neutrosophic matrix

$M = A + BI = (\begin{matrix} 1 & -1+I \\ I & 2+I \end{matrix})$. Where $A = \begin{pmatrix} 1 & -1 \\ 0 & 2 \end{pmatrix}$, $B = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$.

(a)$detA = 2, A + B = \begin{pmatrix} 1 & 0 \\ 1 & 3 \end{pmatrix}$, $\det(A+B) = 3, detM = 2 + I[3-2] = 2 + I \neq$

$0, hence\ M\ is\ invertible.$

(b) We have $A^{-1} = \begin{pmatrix} 1 & \frac{1}{2} \\ 0 & \frac{1}{2} \end{pmatrix}, (A+B)^{-1} = \begin{pmatrix} 1 & 0 \\ -\frac{1}{3} & \frac{1}{3} \end{pmatrix}, thus\ M^{-1} = (A^{-1}) + I[(A+B)^{-1} - A^{-1}]$

$= \begin{pmatrix} 1 & \frac{1}{2} \\ 0 & \frac{1}{2} \end{pmatrix} + I\begin{pmatrix} 0 & -\frac{1}{2} \\ -\frac{1}{3} & -\frac{1}{6} \end{pmatrix} = \begin{pmatrix} 1 & \frac{1}{2} - \frac{1}{2}I \\ -\frac{1}{3}I & \frac{1}{2} - \frac{1}{6}I \end{pmatrix}.$

(c) We can compute $MM^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = U_{2\times2}$.

Theorem 12: [37]

Let $M = A + BI$ and $N = C + DI$ be two neutrosophic n square matrices, then

(3.7.1) $det(M \cdot N) = detM \cdot detN$.

(3.7.2) $\det(M^{-1}) = (detM)^{-1}$.

(3.7.3) $detM$ =1 if and only if $detA = det(A + B) = 1$.

Proof:

(a) $M \cdot N = A \cdot C + I[B \cdot C + B \cdot D + A \cdot D]$

$= A \cdot C + I[(A + B)(C + D) - A \cdot C]$.

$\det(M \cdot N) = \det(A \cdot C) + I\big[\det\big((A + B)(C + D)\big) - \det(A \cdot C)\big]$,

$= \det A \cdot \det C + I[\det(A + B) \cdot \det(C + D) - \det(A \cdot C)]$,

$= \det A \cdot \det C + I[\det(A + B) \cdot \det(C + D) - \det A \cdot \det C]$,

$= (\det A + I[\det(A + B) - \det A]) \cdot (\det C + I[\det(C + D) - \det C])$,

$= \det M \cdot \det N$.

(b) We have

$\det(MM^{-1}) = \det(U_n$

(c) $detM = 1$ is equivalent to $\det A + I[\det(A + B) - \det A] = 1$, thus it is equivalent to

$detA = det(A + B) = 1$.

Remark: The result in the section (c) can be generalized easily to the following fact:

$detM = detA$ if and only if $detA = det(A + B)$.

Definition 12: [37]

Let $M = A + BI$ be a neutrosophic n square matrix, where A and B are two n square matrices. M is satisfying the orthogonality property if and only if $M \cdot M^T = U_{n \times n}$.

Neutrosophic Eigen Values and Diagonalization Conditions

Definition 13: [37]

Let $M = A + BI$ be a square neutrosophic matrix, we say that M is diagonalizable if and only if there is an invertible neutrosophic matrix $S = C + DI$ such that $S^{-1}MS = D$. Where D is a diagonal neutrosophic matrix$\left(i.e.\ d_{ij} = 0\ \ \forall i \neq j, and\ d_{ij} \neq 0\ \ \forall i = j \right)$.

Theorem 14: [3]

Let $M = A + BI$ be any square neutrosophic matrix. Then M is diagonalizable if and only if $A, A + B$ are diagonalizable.

Remark 15: [37]

If C is the diagonalization matrix of A, and D is the diagonalization matrix of $A + B$, then

$S = C + (D - C)I$is the diagonalization matrix of $M = A + BI$.

Example 16: [37]

Consider the neutrosophic matrix defined in Example 10, we have:

(a) A is a diagonalizable matrix. Its diagonalization matrix is $C = \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}$, the corresponding diagonal matrix is $D_1 = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$, we can see that $C^{-1}AC = D_1$. Also, the diagonalization matrix of $A + B$ is $D = \begin{pmatrix} 1 & 0 \\ -\frac{1}{2} & 1 \end{pmatrix}$, the corresponding diagonal matrix is $D_2 = \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}$. It is easy to check that

$D^{-1}(A + B)D = D_2$.

(b) Since $A, A+B$ are diagonalizable, then M is diagonalizable. The neutrosophic diagonalization matrix of M is $S = C + (D - C)I = \begin{pmatrix} 1 & 1-I \\ -\frac{1}{2}I & -1+2I \end{pmatrix}$. The corresponding diagonal matrix is

$$L = D_1 + I[D_2 - D_1] = \begin{pmatrix} 1 & 0 \\ 0 & 2+I \end{pmatrix}.$$

(c) It is easy to see that $S^{-1} = C^{-1} + I[D^{-1} - C^{-1}] = \begin{pmatrix} 1 & 1-I \\ \frac{1}{2}I & -1+2I \end{pmatrix}$.

(d) We can compute $S^{-1}MS = \begin{pmatrix} 1 & 1-I \\ \frac{1}{2}I & -1+2I \end{pmatrix}\begin{pmatrix} 1 & -1+I \\ I & 2+I \end{pmatrix}\begin{pmatrix} 1 & 1-I \\ -\frac{1}{2}I & -1+2I \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 2+I \end{pmatrix} = L.$

Definition 17: [37]

Let $M = A + BI$ be a n square neutrosophic matrix over the neutrosophic field $F(I)$, we say that $Z = X + YI$ is a neutrosophic Eigen vector if and only if $MZ = (a + bI)Z$. The neutrosophic number $a + bI$ is called the Eigen value of the eigen vector Z.

Theorem 18: [37]

Let $M = A + BI$ be a n square neutrosophic matrix, then $a + bI$ is an eigen value of M if and only if a is an eigen value of A, and $a + b$ is an eigen value of $A + B$. As well as, the eigen vector of $M$is $Z = X + YI$ if and only if X is the corresponding eigen vector of A, and $X + Y$ is the corresponding eigen vector of $A + B$.

Proof:

We suppose that $Z = X + YI$ is an eigen vector of M with the corresponding eigen value $a + bI$, hence $MZ = (a + bI)Z$, this implies

$(A + BI)(X + YI) = (a + bI)(X + YI), thus\ AX + I[(A + B)(X + Y) - AX] = aX + I[(a + b)(X + Y) - aX]$. We get:

$AX = aX, (A + B)(X + Y) = (a + b)(X + Y)$, so that X is an eigen vector of A, $X + Y$ is an eigen vector of $A + B$. The corresponding eigen value of X is a, and the corresponding eigen value of

$X + Y$ is $a + b$.

For the converse, we assume that X is an eigen vector of A with a as the corresponding eigen value, and $X+Y$ is an eigen vector of $A+B$ with $a+b$ as the corresponding eigen value, so that we get $AX=aX, (A+B)(X+Y)=(a+b)(X+Y)$.

Let us compute

$$MZ = (A+BI)(X+$$

$= aX + I[(a+b)(X+Y) - aX] = (a+bI)(X+YI) = (a+bI)Z$. Thus $Z = X + YI$ is an eigen vector of M with $a+bI$ as a neutrosophic eigen value.

Theorem 19: [37]

The eigen values of a neutrosophic matrix $M = A + BI$ can be computed by solving the neutrosophic equation $det(M - (a+bI)\ U_{n\times n}) = 0$.

Example 20: [37]

Consider M the neutrosophic matrix defined in Example 10, we have

(a) The eigen values of the matrix A are {1,2}, and {1,3} for the matrix $A+B$. This implies that the eigen values of the neutrosophic matrix M are

$$\{1 + (3-1)I, 1 + (1$$

(b) If we solved the equation det $(M - (a+bI)U_{n\times n})$=0 has been solved, the same values will be gotten.

(c) The eigen vectors of A are $\{(1,0), (1,-1)\}$,the eigen vectors of $A+B$ are $\{(1,-1/2), (0,1)\}$. Thus, the neutrosophic eigen vectors of M are

$\Big\{(1,0) + I[(0,1) - (1,0)], (1,0) + I\left[\left(1,-\frac{1}{2}\right) - (1,0)\right], (1,-1) + I[(0,1) - (1,-1)], (1,-1) + I\left[\left(1,-\frac{1}{2}\right) - (1,-1)\right]\Big\} = \{(1,0) + I(-1,1), (1,0) + I(0,-1/2), (1,-1) + I(-1,2), (1,-1) + I(0,1/2)\} = \{(1-I, I), (1, -1/2\ I), (1-I, -1+2I), (1, -1+1/2\ I)\}$.

Neutrosophic matrices as linear transformations

Theorem 21: [35]

Let V, W be two vector spaces over the field F with $dim(V) = n, dim(W) = m$, $V(I), W(I)$ be the corresponding neutrosophic vector spaces over the corresponding neutrosophic field $F(I)$. Let $g, h: V \to W$ be two linear transformations, then there exists a neutrosophic linear transformation $f = g + hI: V(I) \to W(I)$, where f is defined as follows: $f(x + yI) = g(x) + [(g + h)(x + y) - g(x)]I$.

Proof:

We define $f = g + hI: V(I) \to W(I)$, where $f(x + yI) = g(x) + [(g + h)(x + y) - g(x)]I$.

f is a linear transformation, that is because:

for every $m = x + yI, n = z + tI \in V(I)$, we have: $f(m + n) = f([x + z] + I[y + t]) = g(x + z) + I[(g + h)(x + y + z + t) - g(x + z)] = (g(x) + [(g + h)(x + y) - g(x)]I) + (g(z) + [(g + h)(z + t) - g(z)]I) = f(m) + f(n)$. On the other hand, consider an arbitrary neutrosophic number $a + bI \in F(I)$, then

$f([a + bI]m) = f([a + bI][x + yI]) = f(ax + I[ay + bx + by]) = f(ax + I[(a + b)(x + y) - ax]) = g(ax) + I[(g + h)[(a + b)(x + y)] - g(ax)] = ag(x) + I[(a + b)(g + h)[x + y] - ag(x)] = (a + bI)(g(x) + I[(g + h)(x + y) - g(x)] = (a + bI)f(m)$. Thus f is a neutrosophic linear transformation.

Definition 22: [35]

The neutrosophic linear transformation f defined in Theorem 21 is called a full AH- linear transformation.

Definition 23: [35]

Let $f = g + hI: V(I) \to W(I)$ be a full AH- linear transformation, $M = A + BI$ be an $n \times m$ neutrosophic matrix over $F(I)$, we call M the neutrosophic matrix of f if and only if $f(x + yI) = M(x + yI)$ for every $x + yI \in V(I)$ **.**

Theorem 24: [35]

Let $f = g + hI: V(I) \to W(I)$ be any full AH- linear transformation, then $M = A + BI$ is the corresponding neutrosophic matrix if and only if A is the matrix of g, B is the matrix of h .

Proof:

We assume that A is the matrix of g, B is the matrix of h, hence $Ax = g(x), By = h(y), (A + B)(x + y) = (g + h)(x + y)$. We have:

$M.(x + yI) = (A + BI)(x + yI) = (Ax + I[Ay + Bx + By]) = (Ax + I[(A + B)(x + y) - Ax]) = g(x) + I[(g + h)(x + y) - g(x)] = f(x + yI)$. Thus M is the neutrosophic matrix of f.

Conversely, suppose that M is the neutrosophic matrix of f, we shall prove that A is the matrix of g and B is the matrix of h.

According to the assumption, we have $M(x+yI)=f(x+yI)$, hence $(Ax+I[(A+B)(x+y)-Ax])=g(x)+I[(g+h)(x+y)-g(x)]$

This implies that $Ax=g(x), (A+B)(x+y)=(g+h)(x+y)$, so that $B(x+y)=h(x+y)$. By considering the arbitrariness of x and y we get that A is the matrix of g and B is the matrix of h.

Example 25: [35]

(a) Let $V(I)=R^2(I)=\{(a,b)+(c,d)I=(a+cI,b+dI); a,b,c,d\in R\}$, consider the following neutrosophic matrix

$M=\begin{pmatrix}1+I & I\\ -I & 2-I\end{pmatrix}$. The corresponding neutrosophic linear transformation is defined as follows:

$f(x+yI)=M.\begin{pmatrix}a+cI\\ b+dI\end{pmatrix}=(a+I[c+a+c+b+d],-aI-cI+2b+2dI-bI-dI)=$
$(a+I[a+2c+b+d],2b+I[-a-c-b+d])=(a,2b)+I(a+2c+b+d,-a-c-b+d)$.

(b) $f=g+hI;\ g(x,y)=(x,2y), h(x,y)=(x+y,-x-y)$. Where $g,h:V\to V$.

Theorem 26: [35]

Let V,W be two vector spaces over the field F, with $dim(V)=n, dim(W)=m$, let $M=A+BI$ be any $n\times m$ neutrosophic matrix over $F(I)$. Then M can be represented by a unique full AH- linear transformation $f=g+hI$, where A is the matrix of g and B is the matrix of h.

Proof:

According to Theorem 24, the neutrosophic matrix $M=A+BI$ can be represented by a neutrosophic full AH-linear transformation $f=g+hI$, where A is the matrix of g and B is the matrix of h. For the uniqueness condition, we suppose that $F=G+HI$ is another linear AH-transformation with the property

$M(x+yI)=F(x+yI)$. We have:

$M.(x+yI)=F(x+yI)=f(x+yI) for\ all\ x+yI\in V(I)$. Thus $F=f$ and f is unique.

The following theorem shows an algorithm to find a basis for the neutrosophic vector space V(I) from any basis of the corresponding classical vector space V.

Theorem 27: [35]

Let V(I) be any neutrosophic vector space over the neutrosophic field F(I), V be its corresponding classical vector space over the field F. Let $S=\{v_1,v_2,\dots,v_n\}$ be a basis of V over F, then

$L = \{l_{ij} = v_i + (v_j - v_i)I;\ 1 \le i, j \le n\}$ is a basis of V(I) over F(I).

Example 28: [35]

It is well known that {x=(1,0), y=(0,1)} is a basis of V=R^2. The corresponding basis of V(I)= $R^2(I)$ is

$\{x, y, x + (y - x)I, y + (x - y)I\} = \{(1,0), (0,1), (1,0) + (-1,1)I, (0,1) + (1,-1)I\}$.

The following theorem shows that every linear transformation between $V(I)$ and $W(I)$ must be a full AH-linear transformation.

Theorem 29: [35]

Let V, W be two vector spaces over the field F, with $dim(V) = n, dim(W) = m$, let $V(I), W(I)$ be the corresponding neutrosophic vector spaces over $F(I)$. Let $f: V(I) \to W(I)$ be any linear transformation, then f is a full AH-linear transformation.

Proof:

Let $f: V(I) \to W(I)$ be any linear transformation, we must prove that there exists two classical linear transformations $g, q: V \to W$, where $f = g + qI$.

Suppose that S =$\{v_1, v_2, \ldots, v_n\}$ is a basis of V, then $L = \{l_{ij} = v_i + (v_j - v_i)I;\ 1 \le i, j \le n\}$ is a basis of V(I). It is known that $f(L) = \{f(v_i + (v_j - v_i)I) = w_i + (w_j - w_i)I;\ w_i, w_j \in W\}$ is a basis of W(I), that is because the direct image of a basis by any linear transformation is a gain a basis.

Define $g: V \to W; g(v_i) = w_i, h: V \to W; h(v_j) = w_j$. It is clear that $f(v_i + (v_j - v_i)I) = g(v_i) + I[h(v_j) - g(v_i)]$. This means that $f = g + qI = g + (h - g)I$. Now, we must prove that $g, q = h - g$ are classical linear transformations.

Let x, y be any two elements of V, we have $x = x + 0I, y = y + 0I \in V(I)$. We have:

$f(x + y) = f([x + 0I] + [y + 0I]) = g(x + y) = g(x) + g(y)$. For any $m \in F, we\ have\ m = m + 0I \in F(I), and\ f([m + 0I][x + 0I]) = f(mx + 0I) = g(mx) = mg(x)$, thus g is a linear transformation.

On the other hand, we have $xI, yI \in V(I), and\ f(xI + yI) = f([x + y]I) = f(0 + [x + y]I) = g(0) + I[(g + q)(x + y) - g(0)] = I[h(x + y)] = h(x)I + h(y)I, thus\ h(x + y) = h(x) + h(y)$.

$f([m + 0I][0 + xI] = f(0 + mxI) = g(0) + I[(g + q)(mx) - g(0)] = I[h(mx)] = mh(x)I, so\ that\ h(mx) = mh(x)$. This implies that g, h are two classical linear transformations, thus g, q are linear transformations, which implies that $f = g + qI$ is a full AH-linear transformation.

Refined neutrosophic Matrices

Definition 31: [34]

Let $A=\begin{pmatrix} a_{11} & \cdots & a_{1m} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nm} \end{pmatrix}$ be an $n \times m$ matrix, if $a_{ij} = x + yI_1 + zI_2 \in R_2(I)$, then it is called an refined neutrosophic matrix. Where $R_2(I)$ is an refined neutrosophic field.

Example 32: [34]

$X = \begin{pmatrix} I_1 & I_1 + I_2 \\ 3 - I_1 & 2I_2 \end{pmatrix}$, is a 2×2 refined neutrosophic matrix.

Remark 33: [34]

(a) If A is an $m \times n$ matrix, then it can be represented as an element of the refined neutrosophic ring of matrices like the following: $A = B + CI_1 + DI_2$. Where D, B, C are classical matrices with elements from the ring R and from size $m \times n$.

For example $A = \begin{pmatrix} 2 + I_1 + 3I_2 & 1 - I_1 - I_2 \\ 3 + 4I_2 & 1 + I_1 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 3 & 1 \end{pmatrix} + \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} I_1 + \begin{pmatrix} 3 & -1 \\ 4 & 0 \end{pmatrix} I_2$.

(b) The addition operation can be defined by using the representation in Remark 3.2 as follows:

$(A + BI_1 + CI_2) + (X + YI_1 + ZI_2) = (A + X) + (B + Y)I_1 + (C + Z)I_2$.

(c) Multiplication can be defined by using the same representation as a special case of multiplication on refined neutrosophic rings as follows:

$(A + BI_1 + CI_2)(X + YI_1 + ZI_2) = (AX) + (AY + BX + BY + BZ + CY)I_1 + (AZ + CZ + CX)I_2$..

This method of multiplication is exactly equivalent to the normal multiplication between matrices; but it is easier to deal with in this way.

Example 34: [34]

Let $X = \begin{pmatrix} I_1 & I_1 + I_2 \\ 3 - I_1 & 2I_2 \end{pmatrix}, Y = \begin{pmatrix} -1 & I_1 \\ 1 + I_2 & 3I_1 \end{pmatrix}$ be two refined neutrosophic matrices over the refined neutrosophic field of reals. We have:

(a) $X = A + BI_1 + CI_2; A = \begin{pmatrix} 0 & 0 \\ 3 & 0 \end{pmatrix}, B = \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}, C = \begin{pmatrix} 0 & 1 \\ 0 & 2 \end{pmatrix}$.

(b) $Y = M + NI_1 + SI_2; M = \begin{pmatrix} -1 & 0 \\ 1 & 0 \end{pmatrix}, N = \begin{pmatrix} 0 & 1 \\ 0 & 3 \end{pmatrix}, S = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$.

(c) $X + Y = \begin{pmatrix} -1 + I_1 & 2I_1 + I_2 \\ 4 - I_1 + I_2 & 3I_1 + 2I_2 \end{pmatrix}$.

(d) $XY = \begin{pmatrix} -I_1 + (I_1 + I_2)(1 + I_2) & I_1 I_1 + (I_1 + I_2)(3I_1) \\ -3 + I_1 + (2I_2)(1 + I_2) & (3 - I_1)(I_1) + (2I_2)(3I_1) \end{pmatrix} = \begin{pmatrix} I_1 + 2I_2 & 7I_1 \\ -3 + I_1 + 4I_2 & 8I_1 \end{pmatrix}$..

(e) If we computed the multiplication using the previous representation, we get:

$AM = \begin{pmatrix} 0 & 0 \\ -3 & 0 \end{pmatrix}, AN = \begin{pmatrix} 0 & 0 \\ 0 & 3 \end{pmatrix}, BM = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, BN = \begin{pmatrix} 0 & 4 \\ 0 & -1 \end{pmatrix}, BS = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, CN = \begin{pmatrix} 0 & 3 \\ 0 & 6 \end{pmatrix}, AS = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, CS = \begin{pmatrix} 1 & 0 \\ 2 & 0 \end{pmatrix}, CM = \begin{pmatrix} 1 & 0 \\ 2 & 0 \end{pmatrix}$..

Hence,

$XY = AM + I_1(AN + BN + BM + BS + CN) + I_2(AS + CS + CM) = \begin{pmatrix} 0 & 0 \\ -3 & 0 \end{pmatrix} + I_1 \begin{pmatrix} 1 & 7 \\ 1 & 8 \end{pmatrix} + I_2 \begin{pmatrix} 2 & 0 \\ 4 & 0 \end{pmatrix} = \begin{pmatrix} I_1 + 2I_2 & 7I_1 \\ -3 + I_1 + 4I_2 & 8I_1 \end{pmatrix}$.

Theorem 35: [34]

The set of all square $n \times n$ refined neutrosophic matrices together make a ring.

Proof:

The proof holds directly from the definition of n-refined neutrosophic rings by taking $n = 2$.

Remark 36: [34]

The identity with respect to multiplication is the normal unitary matrix.

Definition 37: [34]

Let A be a square $n \times n$ refined neutrosophic matrix, then it is called invertible if there exists a refined square $n \times n$ neutrosophic matrix B such that $AB = U_{n\times n}$. Where $U_{n\times n}$ is the unitary classical matrix.

Theorem 38: [34]

Let $X = A + BI_1 + CI_2$ be a square $n \times n$ refined neutrosophic matrix, then it is invertible if and only if $A, A + C, A + B + C$ are invertible. The inverse of X is
$X^{-1} = A^{-1} + ((A + B + C)^{-1} - (A + C)^{-1})I_1 + ((A + C)^{-1} - A^{-1})I_2$.

Definition 39: [34]

We defined the determinant of a square $n \times n$ refined neutrosophic matrix as $detX = detA + [\det(A + B + C) - \det(A + C)]I_1 + [\det(A + C) - detA]I_2$.

This definition is supported by the condition of invertibility.

Theorem 40: [34]

Let $X = A + BI_1 + CI_2$ be a square $n \times n$ refined neutrosophic matrix, we have:

(a) X is invertible if and only if $\det X \neq 0$.

(b) If $Y = M + NI_1 + SI_2$ is a square $n \times n$ refined neutrosophic matrix, then $detXY = detXdetY$.

(c) $detX^{-1} = (detX)^{-1}$.

Proof:

(a) If $detX \neq 0$, this will be equivalent to $detA \neq 0, \det(A + C) \neq 0, \det(A + B + C) \neq 0$, i.e. A, $A + C, A + B + C$ are invertible, thus X is invertible.

(b) $XY = AM + I_1[(A + B + C)(M + N + S) - (A + C)(M + S)] + I_2[(A + C)(M + S) - AM]$.
Hence $detXY = \det(AM) + I_1[\det((A + B + C)(M + N + S))] + I_2[\det((A + C)(M + S))]$=

$detAdetM + I_1[\det(A + B + C)\det(M + N + S)] + I_2[\det(A + C)\det(M + S)]$=

$(detA + I_1[\det(A + B + C) - \det(A + C)] + I_2[\det(A + C) - detA])(detM + I_1[\det(M + N + S) - \det(M + S)] + I_2[\det(M + S) - detM] = detXdetY$.

(c) It holds directly from (b).

Theorem 41: [34]

Let X=$A + BI_1 + CI_2$ be a square $n \times n$ refined neutrosophic matrix, we have:

(a) X is nilpotent if and only if $A, A + C, A + B + C$ are nilpotent.

(b) X is idempotent if and only if $A, A + C, A + B + C$ are idempotent.

Proof:

(a) First of all we will prove that $X^r = A^r + I_1[(A + B + C)^r - (A + C)^r] + I_2[(A + C)^r - A^r]$.

We use the induction, for $r = 1$ it is clear. Suppose that it is true for $r = k$, we prove it for $k + 1$.

$X^{k+1} = X^k X = (A^k + I_1[(A + B + C)^k - (A + C)^k] + I_2[(A + C)^k - A^k])(A + BI_1 + CI_2)$=

$A^{k+1} + I_1[A^k B + (A + B + C)^k A + (A + B + C)^k B + (A + B + C)^k C - (A + C)^k A - (A + C)^k B - (A + C)^k C + (A + C)^k B - A^k B] + I_2[A^k C + (A + C)^k C - A^k C + (A + C)^k A - A^k A]$=

$A^{k+1} + I_1[(A + B + C)^{k+1} - (A + C)^{k+1}] + I_2[(A + C)^{k+1} - A^{k+1}]$.

X is nilpotent if there is a positive integer r such that $X^r = O_{n\times n}$. This is equivalent to

$A^r = (A + C)^r = (A + B + C)^r = O_{n\times n}, which\ implies\ the\ proof.$

(b) The proof is similar to (a).

Example 42: [34]

Consider the following refined neutrosophic matrix $A = \begin{pmatrix} 2+I_1+3I_2 & 1-I_1-I_2 \\ 3+4I_2 & 1+I_1 \end{pmatrix}$, we have:

(a) $A = \begin{pmatrix} 2+I_1+3I_2 & 1-I_1-I_2 \\ 3+4I_2 & 1+I_1 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 3 & 1 \end{pmatrix} + \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} I_1 +$
$\begin{pmatrix} 3 & -1 \\ 4 & 0 \end{pmatrix} I_2$

Where B=$\begin{pmatrix} 2 & 1 \\ 3 & 1 \end{pmatrix}$, C=$\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$, D=$\begin{pmatrix} 3 & -1 \\ 4 & 0 \end{pmatrix}$. $B + D = \begin{pmatrix} 5 & 0 \\ 7 & 1 \end{pmatrix}, B + C + D = \begin{pmatrix} 6 & -1 \\ 7 & 2 \end{pmatrix}$.

(b) $B^{-1} = \begin{pmatrix} -1 & 1 \\ 3 & -2 \end{pmatrix}, (B+D)^{-1} = \begin{pmatrix} \frac{1}{5} & 0 \\ -\frac{7}{5} & 1 \end{pmatrix}, (B+C+D)^{-1} = \begin{pmatrix} \frac{2}{19} & \frac{1}{19} \\ -\frac{7}{19} & \frac{6}{19} \end{pmatrix}$.

(c)
$A^{-1} = B^{-1} + I_1[(B+C+D)^{-1} - (B+D)^{-1}] + I_2[(B+D)^{-1} - B^{-1}] =$
$\begin{pmatrix} -1 & 1 \\ 3 & -2 \end{pmatrix} + I_1 \begin{pmatrix} -\frac{9}{95} & \frac{1}{19} \\ \frac{98}{95} & -\frac{13}{19} \end{pmatrix} + I_2 \begin{pmatrix} \frac{6}{5} & -1 \\ -\frac{22}{5} & 3 \end{pmatrix} = \begin{pmatrix} -1-\frac{9}{95}I_1+\frac{6}{5}I_2 & 1+\frac{1}{19}I_1-I_2 \\ 3+\frac{98}{95}I_1-\frac{22}{5}I_2 & -2-\frac{13}{19}I_1+3I_2 \end{pmatrix}$.

It is easy to find that $A^{-1}A = AA^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

(d) $detB = -1, det(B+D) = 5, det(B+C+D) = 19, detA = -1 + I_1[19-5] + I_2[5-(-1)] =$
$-1 + 14I_1 + 6I_2$.

If we computed the determinant of A by using the classical way, we will get the same result.

Now, we illustrate an example to clarify the application of refined neutrosophic matrices in solving refined neutrosophic algebraic equations defined in [2].

Example 43: [34]

Consider the following system of refined neutrosophic linear equations:

$(2 + I_1 + 3I_2)X + (1 - I_1 - I_2)Y = -I_1$ (*), $(3 + 4I_2)X + (1 + I_1)Y = I_2$(**).

The corresponding refined neutrosophic matrix is $A = \begin{pmatrix} 2+I_1+3I_2 & 1-I_1-I_2 \\ 3+4I_2 & 1+I_1 \end{pmatrix}$.

Since A is invertible, we get the solution of the previous system by computing the product:

$A^{-1}\begin{pmatrix} -I_1 \\ I_2 \end{pmatrix} = \begin{pmatrix} -1-\frac{9}{95}I_1+\frac{6}{5}I_2 & 1+\frac{1}{19}I_1-I_2 \\ 3+\frac{98}{95}I_1-\frac{22}{5}I_2 & -2-\frac{13}{19}I_1+3I_2 \end{pmatrix}\begin{pmatrix} -I_1 \\ I_2 \end{pmatrix} =$
$\begin{pmatrix} I_1\left[1+\frac{9}{95}-\frac{6}{5}+\frac{1}{19}\right] \\ I_1\left[-3-\frac{98}{95}+\frac{22}{5}-\frac{13}{19}\right]+I_2[-2+3] \end{pmatrix} = \begin{pmatrix} -I_1\frac{1}{19} \\ -\frac{6}{19}I_1+I_2 \end{pmatrix}$.Thus $X = -\frac{1}{19}I_1, Y = -\frac{6}{19}I_1 + I_2$

Definition 44: [39]

Let $A=\begin{pmatrix} a_{11} & \dots. & a_{1m} \\ \vdots & \ddots & \vdots \\ a_{n1} & \dots & a_{nm} \end{pmatrix}$ be an $m \times n$ matrix, if $a_{ij} = x + yI_1 + zI_2 + \dots + tI_n \in R_n(I)$, then it is called an n-refined neutrosophic matrix. Where $R_n(I)$ is an n-refined neutrosophic ring.

Remark 45: [39]

If A is an $m \times n$ matrix, then it can be represented as an element of the n-refined neutrosophic ring of matrices like the following: $A = B + CI_1 + DI_2 + \dots + KI_n$. Where $D, B, C, ..K$ are classical matrices with elements from the ring R and from size $m \times n$.

For example $A = \begin{pmatrix} 2 + I_1 + 3I_2 - I_3 & 1 - I_1 - I_2 \\ 3 + 4I_2 + 2I_3 & 1 + I_1 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 3 & 1 \end{pmatrix} + \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} I_1 + \begin{pmatrix} 3 & -1 \\ 4 & 0 \end{pmatrix} I_2 + I_3 \begin{pmatrix} -1 & 0 \\ 2 & 0 \end{pmatrix}$ is a 3-refined neutrosophic matrix.

Remark 47: [39]

The identity with respect to multiplication is the normal unitary matrix.

Definition 48: [39]

Let A be a square $\boldsymbol{m} \times \boldsymbol{m}$ n-refined neutrosophic matrix, then it is called invertible if there exists an n-refined square $\boldsymbol{m} \times \boldsymbol{m}$ neutrosophic matrix B such that $\boldsymbol{AB} = \boldsymbol{U_{m \times m}}$. Where $\boldsymbol{U_{m \times m}}$ is the unitary classical matrix.

Definition 49: [39]

Let X=$A_0 + A_1I_1 + \dots + A_nI_n$ be an n-refined neutrosophic element, we define its canonical sequence as follows:

$M_0 = A_0, M_j = A_0 + A_j + A_{j+1} + .. + A_n; 1 \le j \le n$. For example $M_3 = A_0 + A_3 + A_4 + \dots + A_n$.

Remark 50: [39]

The multiplication operation between two n-refined neutrosophic elements can be represented by the following equation:

$(A_0 + A_1I_1 + \dots + A_nI_n)(B_0 + B_1I_1 + \dots + B_nI_n) = M_0N_0 + (M_nN_n - M_0N_0)I_n + \sum_{i=1}^{n-1}(M_iN_i - M_{i+1}N_{i+1})I_i$, where M_i, N_i are the canonical sequences of $A_0 + A_1I_1 + \dots + A_nI_n$, $B_0 + B_1I_1 + \dots + B_nI_n$ respectively.

Proof:

For $n = 0$, the statement is true easily. Suppose that it is true for $n = k$, we must prove it for $n = k + 1$. We compute the multiplication $L = (A_0 + A_1I_1 + \dots + A_{k+1}I_{k+1})(B_0 + B_1I_1 + \dots + B_{k+1}I_{k+1})$.

$(A_0 + A_1I_1 + \cdots + A_{k+1}I_{k+1})(B_0 + B_1I_1 + \cdots + B_{k+1}I_{k+1}) = (A_0 + A_1I_1 + \cdots + A_kI_k)(B_0 + B_1I_1 + \cdots + B_kI_k) + A_{k+1}I_{k+1}(B_0 + B_1I_1 + \cdots + B_kI_k) + (A_0 + A_1I_1 + \cdots + A_kI_k)B_{k+1}I_{k+1} + A_{k+1}I_{k+1}B_{k+1}I_{k+1}$=$M_0N_0 + (M_kN_k - M_0N_0)I_k + \sum_{i=1}^{k}(M_iN_i - M_{i+1}N_{i+1})I_i + I_1[A_{k+1}B_1 + A_1B_{k+1}] + I_2[A_{k+1}B_2 + A_2B_{k+1}] + \cdots + I_k[A_{k+1}B_k + A_kB_{k+1}] + I_{k+1}[A_0B_{k+1} + A_{k+1}B_0 + A_{k+1}B_{k+1}]$.

Thus, the coefficient of I_{k+1} is
$A_0B_{k+1} + A_{k+1}B_0 + A_{k+1}B_{k+1} = (A_{k+1} + A_0)(B_{k+1} + B_0) - (A_0)(B_0) = M_{k+1}N_{k+1} - M_0N_0$. Also, the coefficient of I_i ; $1 \le i \le k$ is $M_iN_i - M_{i+1}N_{i+1} +$
$A_{k+1}B_i + A_iB_{k+1} =$
$(A_0 + A_i + A_{i+1} + .. + A_k)(B_0 + B_i + B_{i+1} + .. + B_k) - (A_0 + A_{i+1} + A_{i+2} + .. + A_k)(B_0 + B_{i+1} + B_{i+2} + .. + B_k) + A_{k+1}B_i + A_iB_{k+1} = (A_0 + A_i + A_{i+1} + .. + A_k + A_{k+1})(B_0 + B_i + B_{i+1} + .. + B_k + B_{k+1}) - (A_0 + A_{i+1} + A_{i+2} + .. + A_k + A_{k+1})(B_0 + B_{i+1} + B_{i+2} + .. + B_k + B_{k+1}) = M_iN_i - M_{i+1}N_{i+1}$.
Where $1 \le i \le k + 1$. Hence, our proof is complete by induction.

Theorem 51: [39]

Let X=$A_0 + A_1I_1 + \cdots + A_nI_n$ be an n-refined neutrosophic element, then it is invertible if and only if $M_j; 0 \le j \le n$ are invertible. The inverse of X is $X^{-1} = (M_0)^{-1} + ({M_n}^{-1} - {M_0}^{-1})I_n + \sum_{j=1}^{n-1}({M_j}^{-1} - {M_{j+1}}^{-1})I_j = (A_0)^{-1} + (((A_0 + A_1 + \cdots + A_n)^{-1} - ((A_0 + A_2 + \cdots + A_n)^{-1})I_1 + ((A_0 + A_2 + \cdots + A_n)^{-1} - ((A_0 + A_3 + \cdots + A_n)^{-1})I_2 + ((A_0 + A_3 + \cdots + A_n)^{-1} - (A_0 + A_4 + \cdots + A_n)^{-1})I_3 + \cdots + ((A_0 + A_n)^{-1} - (A_0)^{-1})I_n$.

Proof:

X is invertible if and only if there exists Y=$B_0 + B_1I_1 + \cdots + B_nI_n$, where $XY = YX = 1$. By using Remark 4.2, we can write:

$M_0N_0 + (M_nN_n - M_0N_0)I_n + \sum_{i=1}^{n-1}(M_iN_i - M_{i+1}N_{i+1})I_i = 1$. This implies that:

$M_0N_0 = 1$, $M_iN_i - M_{i+1}N_{i+1} = 0$ for all i. Where 0 is the zero element. Hence we get,

$M_iN_i = M_{i+1}N_{i+1} = M_0N_0 = U_{m\times m}$. So that $M_j; 0 \le j \le n$ are invertible.

On the other hand, we put $X^{-1} = (M_0)^{-1} + \left({M_n}^{-1} - {M_0}^{-1}\right)I_n + \sum_{j=1}^{n-1}\left({M_j}^{-1} - {M_{j+1}}^{-1}\right)I_j$, now we compute XX^{-1}

$XX^{-1} = M_0{M_0}^{-1} + \left(M_1{M_1}^{-1} - M_2{M_2}^{-1}\right)I_1 + \left(M_2{M_2}^{-1} - M_3{M_3}^{-1}\right)I_2 + \cdots + (M_n{M_n}^{-1} - M_0{M_0}^{-1})I_n$=1.

Example 52: [39]

Consider $Z(I) = \{a + bI_1 + cI_2; a, b, c \in Z_2\}$ the 2-refined neutrosophic ring of integers, the set of invertible elements in Z_2 is $\{-1,1\}$. Hence the set of all invertible elements in the corresponding 2-

refined neutrosophic ring is $\{1, -1, 1-2I_2, -1+2I_2, 1-2I_1, -1+2I_1, 1+2I_1-2I_2, -1-2I_1+2I_2\}$.

Remark 53: [39]

Let $X = A_0 + A_1I_1 + \cdots + A_nI_n$ be a square $m \times m$ n-refined neutrosophic matrix, then it is invertible if and only if $M_j; 0 \le j \le n$ are invertible. The inverse of X is $X^{-1} = (M_0)^{-1} + (M_n{}^{-1} - M_0{}^{-1})I_n + \sum_{j=1}^{n-1}(M_j{}^{-1} - M_{j+1}{}^{-1})I_j = (A_0)^{-1} + (((A_0 + A_1 + \cdots + A_n)^{-1} - ((A_0 + A_2 + \cdots + A_n)^{-1})I_1 + ((A_0 + A_2 + \cdots + A_n)^{-1} - ((A_0 + A_3 + \cdots + A_n)^{-1})I_2 + ((A_0 + A_3 + \cdots + A_n)^{-1} - (A_0 + A_4 + \cdots + A_n)^{-1})I_3 + \cdots + ((A_0 + A_n)^{-1} - (A_0)^{-1})I_n$.

Definition 54: [39]

We defined the determinant of a square $m \times m$ n-refined neutrosophic matrix as $detX = detA_0 + [\det((A_0 + A_1 + \cdots + A_n) - \det((A_0 + A_2 + \cdots + A_n)]I_1 + [\det((A_0 + A_2 + \cdots + A_n) - \det((A_0 + A_3 + \cdots + A_n)]I_2 + \cdots + [\det(A_0 + A_n) - \det(A_0)]I_n = \det(M_0) + (\det(M_n) - \det(M_0))I_n + \sum_{i=1}^{n-1}(\det(M_i) - \det(M_{i+1}))I_i$.

This definition is supported by the condition of invertibility.

Theorem 55: [39]

Let $X = A_0 + A_1I_1 + \cdots + A_nI_n$ be a square $m \times m$ n-refined neutrosophic matrix, we have:

(a) X is invertible if and only if $\det X \ne 0$.

(b) If $Y = B_0 + B_1I_1 + \cdots + B_nI_n$ is a square $m \times m$ n-refined neutrosophic matrix, then $detXY = detXdetY$.

(c) $detX^{-1} = (detX)^{-1}$.

Proof:

(a) If $detX \ne 0$, this will be equivalent to $detM_j \ne 0\ for\ all\ j$, i.e. M_j are invertible, thus X is invertible according to Theorem 3.3.

(b) $XY = M_0N_0 + (M_nN_n - M_0N_0)I_n + \sum_{i=1}^{n-1}(M_iN_i - M_{i+1}N_{i+1})I_i$. Hence $detXY = \det(M_0N_0) + I_n[det(M_nN_n) - \det(M_0N_0)] + \sum_{i=1}^{n-1}(\det(M_iN_i) - \det(M_{i+1}N_{i+1}))I_i]$=

$detM_0detN_0 + I_n[\det(M_n)\det(N_n) - \det(M_0)\det(N_0)] + \sum_{i=1}^{n-1}(det(M_i)\det(N_i) - det(M_{i+1})\det(N_{i+1}))I_i$=

$[det(M_0) + (det(M_n) - det(M_0))I_n + \sum_{i=1}^{n-1}(det(M_i) - det(M_{i+1}))I_i][det(N_0) + (det(N_n) - det(N_0))I_n + \sum_{i=1}^{n-1}(det(N_i) - det(N_{i+1}))I_i] = detXdetY$.

(c) It holds directly from (b).

Future Research Directions

Here are some of open questions about neutrosophic matrices:

1-) How refined neutrosophic matrices can be represented by AH-linear transformations? Describe these transformations.

2-) How n-refined neutrosophic matrices can be represented by AH-linear transformations? Describe these transformations.

3-) Find an algorithm to compute the eigen values/vectors of n-refined neutrosophic matrices.

4-) Determine the necessary and sufficient conditions for the diagonalization of n-refined neutrosophic matrices.

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