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A Study of Maximal and Minimal Ideals of n-Refined Neutrosophic Rings

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ABSTRACT

If R is a ring, then $R_n(I)$ is called a refined neutrosophic ring. Every AH-subset of $R_n(I)$ has the form $P = \sum_{i=0}^{n} P_i I_i = \{a_0 + a_1 I + \cdots + a_n I_n : a_i \in P_i\}$, where P_i are subsets of the classical ring R. The objective of this paper is to determine the necessary and sufficient conditions on P_i which make P be an ideal of $R_n(I)$. Also, this work introduces a full description of the algebraic structure and form for AH-maximal and minimal ideals in $R_n(I)$.

1. Introduction

Neutrosophy is a new kind of generalized logic proposed by Smarandache [12]. It becomes a useful tool in many areas of science such as number theory [16, 20], solving equations [18, 21], and medical studies [11, 15]. Also, there are many applications of neutrosophic structures in statistics [14], optimization [8], and decision making [7]. On the other hand, neutrosophic algebra began in [4], Smarandache and Kandasamy defined concepts such as neutrosophic groups and neutrosophic rings. These notions were handled widely by Agboola et al. in [6, 10], where homomorphisms and AH-substructures were studied [3, 13, 17].

Recently, there is an arising interest by the generalizations of neutrosophic algebraic structures. Authors proposed n-refined neutrosophic groups [9], rings [1], modules [2, 22], and spaces [5, 19].

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If R is a classical ring, then the corresponding refined neutrosophic ring is defined as follows:

$R_n(I)=\{a_0 + a_1 I + \cdots + a_n I_n \, ; \, a_i \in R\}$.

Addition and multiplication on $R_n(I)$ are defined as:

$\sum_{i=0}^{n} x_i I_i + \sum_{i=0}^{n} y_i I_i = \sum_{i=0}^{n} (x_i + y_i) I_i \, , \sum_{i=0}^{n} x_i I_i \times \sum_{i=0}^{n} y_i I_i = \sum_{i,j=0}^{n} (x_i \times y_j) I_i I_j$.

Where × is the multiplication defined on the ring R and $I_i I_j = I_{\min(i,j)}$.

Every AH-subset of $R_n(I)$ has the form $P = \sum_{i=0}^{n} P_i I_i = \{ a_0 + a_1 I + \cdots + a_n I_n : a_i \in P_i \}$. There is an important question arises here. This question can be asked as follows:

What are the necessary and sufficient conditions on the subsets P_i which make P be an ideal of $R_n(I)$? On the other hand, can we determine the structure of all AH-maximal and minimal ideals in the n-refined neutrosophic ring $R_n(I)$?

Through this paper, we try to answer the previous questions in the case of n-refined neutrosophic rings with unity. All rings through this paper are considered with unity.

2. Preliminaries

Definition 1. [1]. Let $(R,+,\times)$ be a ring and $I_k; 1 \le k \le n$ be n indeterminacies. We define $R_n(I)=\{a_0 + a_1 I + \cdots + a_n I_n \, ; \, a_i \in R\}$ to be n-refined neutrosophic ring. If $n=2$ we get a ring which is isomorphic to 2-refined neutrosophic ring $R(I_1, I_2)$.

Addition and multiplication on $R_n(I)$ are defined as:

$\sum_{i=0}^{n} x_i I_i + \sum_{i=0}^{n} y_i I_i = \sum_{i=0}^{n} (x_i + y_i) I_i \, , \sum_{i=0}^{n} x_i I_i \times \sum_{i=0}^{n} y_i I_i = \sum_{i,j=0}^{n} (x_i \times y_j) I_i I_j$.

Where × is the multiplication defined on the ring R.

It is easy to see that $R_n(I)$ is a ring in the classical concept and contains a proper ring R.

Definition 2. [1]. Let $R_n(I)$ be an n-refined neutrosophic ring, it is said to be commutative if $xy = yx$ for each x , $y \in R_n(I)$, if there is $1 \in R_n(I)$ such $1.x = x.1 = x$, then it is called an n-refined neutrosophic ring with unity.

Theorem 1. [1]. Let $R_n(I)$ be an n-refined neutrosophic ring. Then (a) R is commutative if and only if $R_n(I)$ is commutative, (b) R has unity if and only if $R_n(I)$ has unity, and (c) $R_n(I) = \sum_{i=0}^{n} R I_i = \{\sum_{i=0}^{n} x_i I_i : x_i \in R\}$.

Definition 3. [1]. (a) Let $R_n(I)$ be an n-refined neutrosophic ring and $P = \sum_{i=0}^{n} P_i I_i = \{a_0 + a_1 I + \cdots + a_n I_n : a_i \in P_i\}$ where P_i is a subset of R, we define P to be an AH-subring if P_i is a subring of R for all i, AHSsubring is defined by the condition $P_i = P_j$ for all i, j. (b) P is an AH-ideal if P_i is an two sides ideal of R for all i, the AHS-ideal is defined by the condition $P_i = P_j$ for all i, j. (c) The AH-ideal P is said to be null if $P_i = R$ or $P_i = \{0\}$ for all i.

Definition 4. [1]. Let $R_n(I)$ be an n-refined neutrosophic ring and $P = \sum_{i=0}^{n} P_i I_i$ be an AH-ideal, we define AH-factor $R(I)/P = \sum_{i=0}^{n}(R/P_i)I_i = \sum_{i=0}^{n}(x_i + P_i)I_i \, ; x_i \in R$.

Theorem 2. [1]. Let $R_n(I)$ be an n-refined neutrosophic ring and $P = \sum_{i=0}^{n} P_i I_i$ be an AH-ideal: $R_n(I)/P$ is a ring with the following two binary operations:

$$\sum_{i=0}^{n}(x_i + P_i)I_i + \sum_{i=0}^{n}(y_i + P_i)I_i = \sum_{i=0}^{n}(x_i + y_i + P_i)I_i,$$

$$\sum_{i=0}^{n}(x_i + P_i)I_i \times \sum_{i=0}^{n}(y_i + P_i)I_i = \sum_{i=0}^{n}(x_i \times y_i + P_i)I_i.$$

Definition 5. [1]. (a) Let $R_n(I), T_n(I)$ be two n-refined neutrosophic rings respectively, and $f_R: R \to T$ be a ring homomorphism. We define n-refined neutrosophic AHS-homomorphism as $f: R_n(I) \to T_n(I)$; $f(\sum_{i=0}^{n} x_i I_i) = \sum_{i=0}^{n} f_R(x_i) I_i$, (b) f is an n-refined neutrosophic AHS-isomorphism if it is a bijective n-refined neutrosophic AHS-homomorphism, and (c) AH-Ker $f = \sum_{i=0}^{n} Ker(f_R) I_i = \{\sum_{i=0}^{n} x_i I_i \, ; \, x_i \in Ker\, f_R\}$.

3. Main Discussion

Theorem 3. Let $R_n(\mathrm{I}) = \{a_0 + a_1 I + \cdots + a_n I_n \, ; \, a_i \in R\}$ be any n-refined neutrosophic ring with unity 1. Let $P = \sum_{i=0}^{n} P_i I_i = \{a_0 + a_1 I + \cdots + a_n I_n : a_i \in P_i\}$ be any AH-subset of $R_n(I)$, where P_i are subsets of R. Then P is an ideal of $R_n(I)$ if and only if (a) P_i are classical ideals of R for all I and (b) $P_0 \leq P_k \leq P_{k-1}$. For all $0 < k \leq n$.

Proof. First of all, we assume that (a), (b) are true. We should prove that P is an ideal. Since P_i are classical ideals of R, then they are subgroups of $(R, +)$, hence P is a subgroup of $(R_n(I), +)$. Let $r = r_0 + r_1 I_1 + \cdots + r_n I_n$ be any element of $R_n(I)$, $x = x_0 + x_1 I_1 + \cdots + x_n I_n$ be an arbitrary element of P, where $x_i \in P_i$. We have For $n = 0$, the statement $r.x \in P$ is true clearly. We assume that it is true for $n = k$, we must prove it for $k + 1$.

$$r.x = (r_0 + r_1 I_1 + \cdots + r_k I_k + r_{k+1} I_{k+1})(x_0 + x_1 I_1 + \cdots + x_k I_k + x_{k+1} I_{k+1}) =$$

$$(r_0 + r_1 I_1 + \cdots + r_k I_k)(x_0 + x_1 I_1 + \cdots x_k I_k) + r_{k+1} I_{k+1}(x_0 + \cdots + x_{k+1} I_{k+1}) + (r_0 + \cdots r_k I_k)x_{k+1} I_{k+1}.$$

We remark

$$(r_0 + r_1 I_1 + \cdots + r_k I_k)(x_0 + x_1 I_1 + \cdots x_k I_k) \in P_0 + P_1 I_1 + \cdots + P_k I_k \text{ (by induction hypothesis).}$$

On the other hand, we have

$$r_{k+1} I_{k+1}(x_0 + \cdots + x_{k+1} I_{k+1}) = (r_{k+1} x_0 + r_{k+1} x_{k+1}) I_{k+1} + r_{k+1} x_1 I_1 + \cdots + r_{k+1} x_k I_k.$$

Since all P_i are ideals and $P_0 \leq P_{k+1}$, we have $r_{k+1}x_i \in P_i \; and \; r_{k+1}x_0 + r_{k+1}x_{k+1} \in P_{k+1}$, hence $r_{k+1}I_{k+1}(x_0 + \cdots + x_{k+1}I_{k+1}) \in P$. Also, $(r_0 + \cdots r_kI_k)x_{k+1}I_{k+1} = r_0x_{k+1}I_{k+1} + r_1x_{k+1}I_1 + \cdots + r_kx_{k+1}I_k$. Under the assumption of theorem, we have $r_0x_{k+1} \in P_{k+1}$ and $r_ix_{k+1} \in P_{k+1} \leq P_i$.

For all $1 \leq i \leq k$. Thus P is an ideal.

For the converse, we assume that P is an ideal of $R_n(I)$. We should prove (a) and (b).

It is easy to check that if $P = P_0 + \cdots + P_nI_n$ is a subgroup of $(R_n(I), +)$, then every P_i is a subgroup of (R,+). Now we show that (b) is true.

For every $1 \leq i \leq n$, we have an element I_i, that is because R is a ring with unity, hence. Let x_0 be any element of p_0 , we have $x_0 \in P$, and $x_0I_i \in P$.

Thus $x_0 \in P_i$, which means that $P_0 \leq P_i$ for all $1 \leq i \leq n$.

Also, for every $x_i \in P_i$, we have $x_iI_i \in P$, thus $x_iI_iI_{i-1} = x_iI_{i-1} \in P$, so that $x_i \in P_{i-1}$, which means that $P_i \leq P_{i-1}$ and (b) holds.

Example 1. Let Z be the ring of integers, $Z_3(I) = \{a + bI_1 + cI_2 + dI_3; a, b, c, d \in Z\}$ be the corresponding 3-refined neutrosophic ring, we have:

$$\mathrm{P} = <16> + <2> \mathrm{I}_1 + <4> \mathrm{I}_2 + <8> \mathrm{I}_3 = \{16\mathrm{x} + 2\mathrm{yI}_1 + 4\mathrm{zI}_2 + 8\mathrm{tI}_3; \mathrm{x}, \mathrm{y}, \mathrm{z}, \mathrm{t} \in \mathrm{Z}\}$$

is an ideal of $\mathrm{Z}_3(\mathrm{I})$, that is because, $<16> \leq <8> \leq <4> \leq <2>$.

Now, we are able to describe all AH-maximal and minimal ideals in $R_n(I)$.

Theorem 4. Let R_n(I)={$a_0 + a_1I + \cdots + a_nI_n$; $a_i \in R$} be any n-refined neutrosophic ring with unity 1.

Let P = $\sum_{i=0}^{n} P_iI_i$ = { $a_0 + a_1I + \cdots + a_nI_n$: $a_i \in P_i$} be any ideal of R_n(I). Then (a) non trivial AH-maximal ideals in $R_n(I)$ have the form $P_0 + RI_1 + \cdots + RI_n$, where P_0 is maximal in R and (b) non trivial AH-minimal ideals in $R_n(I)$ have the form P_1I_1, where P_1 is minimal in R.

Proof. (a) assume that P is an AH-maximal ideal on the refined neutrosophic ring $R_n(I)$, hence for every ideal M=($M_0 + M_1I_1 + \cdots + M_nI_n$) with property $P \leq M \leq R_n(I)$, we have M=P or M=$R_n(I)$. This implies that $M_i = R$ or $M_i = P_i$, which means that P_0 is maximal in R. On the other hand, we have $P_0 \leq P_k \leq P_{k-1}$. For all $0 < k \leq n$, thus $P_i \in \{P_0, R\} \; for$ all $1 \leq i \leq n$. Now suppose that there is at least j such that $P_j = P_0$, we get that $P_0 + \cdots + P_jI_j + \cdots RI_n \leq P_0 + RI_1 + \cdots + RI_j + .. + RI_n$, hence P is not maximal. This means that $P_0 + RI_1 + \cdots + RI_n$, where P_0 is maximal in R is the unique form of AH-maximal ideals.

For the converse, we suppose that P_0 is maximal in R and $P_i = R$. For all $1 \leq i \leq n$. Consider M=($M_0 + M_1I_1 + \cdots + M_nI_n$) as an arbitrary ideal of $R_n(I)$ with AH-structure. If $P \leq M \leq R_n$(I), then $P_i \leq M_i \leq R$ and, this means that $P_0 = M_0 \; or \; M_0 = R$, that is because P_0 is maximal.

According to ***Theorem 3***, we have $M_0 \leq M_i \leq M_{i-1}$. Now if $M_0 = R$, we get $M_i = R$, thus M=$R_n(I)$.

If $M_0 = P_0$, we get $M = P$. This implies that *P* is maximal.

(b) It is clear that if P_1 is minimal in R, then P_1I_1 is minimal in $R_n(I)$. For the converse, we assume that *P*= $P_0 + P_1I_1 + \cdots + P_nI_n$ is minimal in $R_n(I)$, consider an arbitrary ideal with AH-structure M= $(M_0 + M_1I_1 + \cdots + M_nI_n)$ of $R_n(I)$ with the property $M \leq P$, we have: $M = \{0\}\ or\ M = P$ which means that $M_1 = P_1\ or\ M_1 = \{0\}$. Hence P_1 is minimal.

According to ***Theorem 3***, we have $M_0 \leq M_k \leq M_{k-1}$ for all *k*. Now, suppose that there is at least *j*≠ 1 such that $P_j \neq \{0\}$, we get $P_jI_j \leq P_0 + P_1I_1 + \cdots + P_nI_n$. Thus *P* is not minimal, which is a contradiction with respect to assumption. Hence any non trivial minimal ideal has the form P_1I_1, where P_1 is minimal in *R*.

Example 2. Let *R*=*Z* be the ring of integers, $Z_n(I) = \{a_0 + a_1I_1 + \cdots + a_nI_n;\ a_i \in Z\}$ be the corresponding n-refined neutrosophic ring, we have

(a) the ideal $P =< 2 > +ZI_1 + \cdots + ZI_n$ is AH-maximal, that is because <2> is maximal in *R* and (b) there is no AH-minimal ideals in $Z_n(I)$, that is because *R* has no minimal ideals.

Example 3. Let *R*= Z_{12} be the ring of integers modulo 12, $Z_{12_n}(I)$ be the corresponding n-refined neutrosophic ring, we have

(a) the ideal $P =< 6 > I_1 = \{0,6I_1\}$ is AH-minimal, that is because <6> is minimal in *R*.

(b) the ideal $Q =< 2 > +Z_{12}I_1 + \cdots + Z_{12}I_n$ is maximal, that is because <2> is maximal in *R*.

Now, we show that ***Theorem 4*** is not available if the ring *R* has no unity, we construct the following example.

Example 4. Consider $2Z_2(I) = \{(2a + 2bI_1 + 2cI_2); a, b, c \in Z\}$ the 2-refined neutrosophic ring of even integers, let $P = (2Z + 4ZI_1 + 4ZI_2) = \{(2a + 4bI_1 + 4cI_2); a, b, c \in Z\}$ be an AH-subset of it. First of all, we show that *P* is an ideal of $2Z_2(I)$. It is easy to see that *(P,+)* is a subgroup. Let $x = (2m + 4nI_1 + 4tI_2)$ be any element of *P*, $r = (2a + 2bI_1 + 2cI_2)$ be any element of $2Z_2(I)$, we have $rx = (4am, +[8an + 4bm + 8bn + 8bt + 8cn] + I_2[8at + 8ct + 4cm]) \in P$. Thus P is an ideal and the inclusion's condition is not available, that is because *2Z* is not contained in *4Z*.

4. Conclusion

In this article, we have found a necessary and sufficient condition for any subset to be an ideal of any n-refined neutrosophic ring with unity. On the other hand, we have characterized the form of maximal and minimal ideals in this class of neutrosophic rings. As a future research direction, we aim to study Köthe's Conjecture on n-refined neutrosophic rings about the structure of nil ideals and the maximality/minimality conditions if R has no unity.

Open Problems

According to our work, we find two interesting open problems.

- Describe the algebraic structure of the group of units of any n-refined neutrosophic ring.
- What are the conditions of AH-maximal and minimal ideals if R has no unity?.

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