α Generalized Closed Sets in Neutrosophic Topological Spaces

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Abstract: *In this paper a new concept of neutrosophic closed sets called neutrosophic α generalized closed sets is introduced and their properties are thoroughly studied and analyzed. Some new interesting theorems based on the newly introduced set are presented.*

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1. Introduction

The concept of neutrosophic sets was first introduced by Florentin Smarandache [3] in 1999 which is a generalization of intuitionistic fuzzy sets by Atanassov [1]. A. A. Salama and S. A. Alblowi [6] introduced the concept of neutrosophic topological spaces after Coker [2] introduced intuitionistic fuzzy topological spaces . Further the basic sets like semi open sets, pre open sets, α open sets and semi-α open sets are introduced in neutrosophic topological spaces and their properties are studied by various authors [4,5]. The purpose of this paper is to introduce and analyze a new concept of neutrosophic closed sets called neutrosophic α generalized closed sets.

2. Preliminaries:

Here in this paper the neutrosophic topological space is denoted by (X, τ). Also the neutrosophic interior, neutrosophic closure of a neutrosophic set A are denoted by NInt(A) and NCl(A). The complement of a neutrosophic set A is denoted by C(A) and the empty and whole sets are denoted by 0_N and 1_N respectively.

Definition 2.1: Let X be a non-empty fixed set. A neutrosophic set (NS) A is an object having the form $A = \{\langle x, \mu_A(x), \sigma_A(x), \nu_A(x)\rangle : x \in X\}$ where $\mu_A(x)$, $\sigma_A(x)$ and $\nu_A(x)$ represent the degree of membership, degree of indeterminacy and the degree of non-membership respectively of each element $x \in X$ to the set A.

A Neutrosophic set $A = \{\langle x, \mu_A(x), \sigma_A(x), \nu_A(x)\rangle : x \in X\}$ can be identified as an ordered triple $\langle \mu_A, \sigma_A, \nu_A\rangle$ in $]^-0, 1^+[$ on X.

Definition 2.2: Let $A = \langle \mu_A, \sigma_A, \nu_A\rangle$ be a NS on X, then the complement C(A) may be defined as

1. $C(A) = \{\langle x, 1-\mu_A(x), 1-\nu_A(x)\rangle : x \in X\}$
2. $C(A) = \{\langle x, \nu_A(x), \sigma_A(x), \mu_A(x)\rangle : x \in X\}$
3. $C(A) = \{\langle x, \nu_A(x), 1-\sigma_A(x), \mu_A(x)\rangle : x \in X\}$

Note that for any two neutrosophic sets A and B,

4. $C(A \cup B) = C(A) \cap C(B)$
5. $C(A \cap B) = C(A) \cup C(B)$

Definition 2.3: For any two neutrosophic sets $A = \{\langle x, \mu_A(x), \sigma_A(x), \nu_A(x)\rangle : x \in X\}$ and $B = \{\langle x, \mu_B(x), \sigma_B(x), \nu_B(x)\rangle : x \in X\}$ we may have

1. $A \subseteq B \Leftrightarrow \mu_A(x) \leq \mu_B(x), \sigma_A(x) \leq \sigma_B(x)$ and $\nu_A(x) \geq \nu_B(x)\ \forall\ x \in X$
2. $A \subseteq B \Leftrightarrow \mu_A(x) \leq \mu_B(x), \sigma_A(x) \geq \sigma_B(x)$ and $\nu_A(x) \geq \nu_B(x)\ \forall\ x \in X$
3. $A \cap B = \langle x, \mu_A(x) \wedge \mu_B(x), \sigma_A(x) \wedge \sigma_B(x), \nu_A(x) \vee \nu_B(x)\rangle$
4. $A \cap B = \langle x, \mu_A(x) \wedge \mu_B(x), \sigma_A(x) \vee \sigma_B(x), \nu_A(x) \vee \nu_B(x)\rangle$
5. $A \cup B = \langle x, \mu_A(x) \vee \mu_B(x), \sigma_A(x) \vee \sigma_B(x), \nu_A(x) \wedge \nu_B(x)\rangle$
6. $A \cup B = \langle x, \mu_A(x) \vee \mu_B(x), \sigma_A(x) \wedge \sigma_B(x), \nu_A(x) \wedge \nu_B(x)\rangle$

Definition 2.4: A neutrosophic topology (NT) on a non-empty set X is a family τ of neutrosophic subsets in X satisfies the following axioms:

(NT_1) $0_N, 1_N \in \tau$

(NT_2) $G_1 \cap G_2 \in \tau$ for any $G_1, G_2 \in \tau$

(NT_3) $\cup G_i \in \tau\ \forall\ \{G_i : i \in J\} \subseteq \tau$

In this case the pair (X, τ) is a neutrosophic topological space (NTS) and any neutrosophic set in τ is known as a neutrosophic open set (NOS) in X. A neutrosophic set A is a neutrosophic closed set (NCS) if and only if its complement C(A) is a neutrosophic open set in X.

Here the empty set (0_N) and the whole set (1_N) may be defined as follows:

(0_1) $0_N = \{\langle x, 0, 0, 1\rangle: x \in X\}$
(0_2) $0_N = \{\langle x, 0, 1, 1\rangle: x \in X\}$
(0_3) $0_N = \{\langle x, 0, 1, 0\rangle: x \in X\}$
(0_4) $0_N = \{\langle x, 0, 0, 0\rangle: x \in X\}$
(1_1) $1_N = \{\langle x, 1, 0, 0\rangle: x \in X\}$
(1_2) $1_N = \{\langle x, 1, 0, 1\rangle: x \in X\}$
(1_3) $1_N = \{\langle x, 1, 1, 0\rangle: x \in X\}$
(1_4) $1_N = \{\langle x, 1, 1, 1\rangle: x \in X\}$

Definition 2.5: Let (X, τ) be a NTS and A = $\{\langle x, \mu_A(x), \sigma_A(x), \nu_A(x)\rangle: x \in X\}$ be a NS in X. Then the neutrosophic interior and the neutrosophic closure of A are defined by

NInt(A) = $\cup$ {G: G is an NOS in X and G $\subseteq$ A}
NCl(A) = $\cap$ {K: K is an NCS in X and A $\subseteq$ K}

Note that for any NS A, NCl(C(A)) = C(NInt(A)) and NInt(C(A)) = C(NCl(A)).

Definition 2.6: A NS A of a NTS X is said to be

(i) a neutrosophic pre-open set (NP-OS) if A $\subseteq$ NInt(NCl(A))
(ii) a neutrosophic semi-open set (NS-OS) if A $\subseteq$ NCl(NInt(A))
(iii) a neutrosophic α-open set (Nα-OS) if A $\subseteq$ NInt(NCl(NInt(A)))
(iv) a neutrosophic semi-α-open set (NS_α-OS) if A $\subseteq$ NCl(αNInt(A))

Definition 2.7: A NS A of a NTS X is said to be

(i) A neutrosophic pre-closed set (NP-CS) if NCl(NInt(A)) $\subseteq$ A
(ii) A neutrosophic semi-closed set (NS-CS) if NInt(NCl(A)) $\subseteq$ A
(iii) A neutrosophic α-closed set (Nα-CS) if NCl(NInt(NCl(A))) $\subseteq$ A
(iv) A neutrosophic semi-α-closed set (NS_α-CS) if NInt(αNCl(A)) $\subseteq$ A

3. α generalized closed sets in neutrosophic topological spaces

In this section we introduce neutrosophic α closure, neutrosophic α interior and α generalized closed set and its respective open set in neutrosophic topological spaces and discuss some of their properties.

Definition 3.1: A NS A in a NTS X is said to be a neutrosophic regular closed set (NRCS) if NCl(NInt(A)) = A and neutrosophic regular open set if NInt(NCl(A)) = A.

Definition 3.2: A NS A in a NTS X is said to be a neutrosophic β closed set (NβCS) if NInt(NCl(NInt(A))) □ A and neutrosophic β open set if A □ NCl(NInt(NCl(A)))

Definition 3.3: Let A be a NS of a NTS (X, τ). Then the neutrosophic α interior and the neutrosophic α closure are defined as

N_αInt(A) = $\cup$ {G: G is a Nα-OS in X and G $\subseteq$ A}
N_αCl(A) = $\cap$ {K: K is a Nα-CS in X and A $\subseteq$ K}

Result 3.4: Let A be a NS in X. Then N_αCl(A) = A $\cup$ NCl(NInt(NCl(A))).

Proof: Since N_αCl(A) is a Nα-CS, NCl(NInt(NCl(N_αCl(A))))$\subseteq$ N_αCl(A) and A $\cup$ NCl(NInt(NCl(A)))$\subseteq$ A $\cup$ NCl(NInt(NCl(N_αCl(A)))) $\subseteq$ A $\cup$ N_αCl(A) = N_αCl(A) ------(i). Now NCl(NInt(NCl(A $\cup$ NCl(NInt(NCl(A)))))) $\subseteq$ NCl(NInt(NCl(A $\cup$ NCl(A)))) = NCl(NInt(NCl(NCl(A)))) = NCl(NInt(NCl(A))) $\subseteq$ A $\cup$ NCl(NInt(NCl(A))). Therefore A $\cup$ NCl(NInt(NCl(A))) is a Nα-CS in X and hence N_αCl(A) $\subseteq$ A $\cup$ NCl(NInt(NCl(A))) ------(ii). From (i) and (ii), N_αCl(A) = A $\cup$ NCl(NInt(NCl(A))).

Definition 3.5: A NS A in a NTS X is said to be a neutrosophic α generalized closed set ($N_{\alpha g}$CS) if N_αCl (A) $\subseteq$ U whenever A $\subseteq$ U and U is a NOS in X. The complement C(A) of a $N_{\alpha g}$CS A is a $N_{\alpha g}$OS in X.

Example 3.6: Let X = {a, b} and $\tau = \{0_N, A, B, 1_N\}$ where A = $\langle$x, (0.5, 0.6), (0.3, 0.2), (0.4, 0.1)$\rangle$ and B = $\langle$x, (0.4, 0.4), (0.4, 0.3), (0.5, 0.4)$\rangle$. Then τ is a NT. Here $\mu_A(a) = 0.5$, $\mu_A(b) = 0.6$, $\sigma_A(a) = 0.3$, $\sigma_A(b) = 0.2$, $\nu_A(a) = 0.4$ and $\nu_A(b) = 0.1$. Also $\mu_B(a) = 0.4$, $\mu_B(b) = 0.4$, $\sigma_B(a) = 0.4$, $\sigma_B(b) = 0.3$, $\nu_B(a) = 0.5$ and $\nu_B(b) = 0.4$. Let M = $\langle$x, (0.5, 0.4), ((0.4, 0.4), (0.4, 0.5)$\rangle$ be any NS in X. Then M $\subseteq$ A where A is a NOS in X. Now N_αCl(M) = M $\cup$ C(B) = C(B) $\subseteq$ A. Therefore M is a $N_{\alpha g}$-CS in X.

Proposition 3.7: Every NCS A is a $N_{\alpha g}$-CS in X but not conversely in general.

Proof: Let A $\subseteq$ U where U is a NOS in X. Now N_αCl(A) = A $\cup$ NCl(NInt(NCl(A))) $\subseteq$ A $\cup$ NCl(A) = A $\cup$ A = A $\subseteq$ U, by hypothesis. Therefore A is a $N_{\alpha g}$-CS in X.

Example 3.8: In Example 3.6, M is a $N_{\alpha g}$-CS in X but not a NCS in X as NCl(M) = C(B) $\neq$ M.

Remark 3.9: Every NS-CS and every $N_{\alpha g}$-CS in a NTS X are independent to each other in general.

Example 3.10: In Example 3.6, M is a $N_{\alpha g}$-CS but not a NS-CS as NInt(NCl(M)) = B $\not\subset$ M.

Example 3.11: Let X = {a, b} and $\tau = \{0_N, A, B, C, 1_N\}$, where A = ⟨x, (0.5, 0.4), (0.3, 0.2), (0.5, 0.6)⟩, B = ⟨x, (0.8, 0.7), (0.4, 0.3), (0.2, 0.3)⟩ and C = ⟨x, (0.2, 0.1), (0.3, 0.2), (0.8, 0.9)⟩. Then τ is a NT. Let M = ⟨x, (0.5, 0.3), (0.3, 0.2), (0.5, 0.7)⟩. Then M is a NS-CS but not a $N_{\alpha g}$-CS as M ⊆ A, B and $N_\alpha Cl(M)$ = M ∪ C(A) = C(A) $\not\subset$ A.

Remark 3.12: Every NP-CS and every $N_{\alpha g}$-CS in a NTS X are independent to each other in general.

Example 3.13: In Example 3.11, M is a NP-CS but not a $N_{\alpha g}$-CS as seen in the respective example.

Example 3.14: Let X = {a, b} and $\tau = \{0_N, A, B, 1_N\}$, where A = ⟨x, (0.5, 0.4), (0.3, 0.2), (0.5, 0.6)⟩ and B = □x, (0.4, 0.3), (0.3, 0.1), (0.6, 0.7)□ □ □Then τ is a NT. Let M = □x, (0.5, 0.5), (0.2, 0.1), (0.4, 0.4)□. Then M is a $N_{\alpha g}$-CS but not a NP-CS as NCl(NInt(M)) = C(A) $\not\subset$ M.

Proposition 3.15: Every Nα-CS A is a $N_{\alpha g}$-CS in X but not conversely in general.

Proof: Let A □ U, where U is a NOS in X. Then $N_\alpha Cl(A)$ = A □ NCl(NInt(NCl(A))) □ A □ A = A □ U, by hypothesis. Hence A is a $N_{\alpha g}$-CS in X.

Example 3.16: In Example 3.6, M is a $N_{\alpha g}$-CS in X but not a Nα-CS as NCl(NInt(NCl(M))) = C(B) $\not\subset$ M.

Proposition 3.17: Every NOS, Nα-OS are $N_{\alpha g}$OS but not conversely in general.

Proof: Obvious.

Example 3.18: In Example 3.6, C(M) is a $N_{\alpha g}$OS but not a NOS, Nα-OS in X.

Remark 3.19: Both NS-OS and NP-OS are independent to $N_{\alpha g}$OS in X in general.

Example 3.20: The above Remark can been proved easily from the Examples 3.10, 3.11 and 3.13, 3.14 respectively.

Proposition 3.21: The union of any two $N_{\alpha g}$CSs is a $N_{\alpha g}$CS in a NTS X.

Proof: Let A and B be any two $N_{\alpha g}$CSs in a NTS X. Let A □ B □ U where U is a NOS in X. Then A □ U and B □ U. Now $N_\alpha Cl$(A □ B) = (A □ B) □ NCl(NInt(NCl(A □ B))) □ (A □ B) □ NCl(NCl(A □ B)) □ (A □ B) □ NCl(A □ B) □ NCl(A □ B) = NCl(A) □ NCl(B) □ U □ U = U, by hypothesis. Hence A □ B is a $N_{\alpha g}$CS in X.

Remark 3.22: The intersection of any two $N_{\alpha g}$CSs need not be a $N_{\alpha g}$CS in a NTS X.

Example 3.23: Let X = {a, b} and $\tau = \{0_N, A, B, 1_N\}$ where A = □x, (0.5, 0.4), (0.3, 0.2), (0.5, 0.6)□□and B = □x, (0.8, 0.7), (0.3, 0.2), (0.2, 0.3)□ □ □Then τ is a NT. Let M = □x, (0.6, 0.9), (0.3, 0.2), (0.4, 0.1)□ and N = □x, (0.9, 0.7), (0.3, 0.2), (0.1, 0.3)□. Then M and N are $N_{\alpha g}$CSs in X but M ∩ N = □x, (0.6, 0.7), (0.3, 0.2), (0.4, 0.3)□ is not a $N_{\alpha g}$CS as M ∩ N ⊆ B and $N_\alpha Cl(M \cap N) = 1_N \not\subset A$.

Proposition 3.24: Let (X, τ) be a NTS. Then for every A ∈ $N_{\alpha g}$C(X) and for every B ∈ NS(X), A ⊆ B ⊆ $N_\alpha Cl(A)$ implies B ∈ $N_{\alpha g}$C(X).

Proof: Let B ⊆ U and U be a NOS in (X, τ). Then since A ⊆ B, A ⊆ U. By hypothesis, B ⊆ $N_\alpha Cl(A)$. Therefore $N_\alpha Cl(B) \subseteq N_\alpha Cl(N_\alpha Cl(A)) = N_\alpha Cl(A)$ ⊆ U, since A is an $N_{\alpha g}$CS in (X, τ). Hence B ∈ $N_{\alpha g}$C(X).

Proposition 3.25: If A is a NOS and a $N_{\alpha g}$CS in (X, τ), then A is a Nα-CS in (X, τ).

Proof: Since A ⊆ A and A is a NOS in (X, τ), by hypothesis, $N_\alpha Cl(A)$ ⊆ A. But A ⊆ $N_\alpha Cl(A)$. Therefore $N_\alpha Cl(A)$ = A. Hence A is a Nα-CS in (X, τ).

Proposition 3.26: Let (X, τ) be a NTS. Then every NS in (X, τ) is a $N_{\alpha g}$CS in (X, τ) if and only if Nα-O(X) = Nα-C(X).

Proof: Necessity: Suppose that every NS in (X, τ) is a $N_{\alpha g}$CS in (X, τ). Let U ∈ NO(X). Then U ∈ Nα-O(X) and by hypothesis, $N_\alpha Cl(U)$ ⊆ U ⊆ $N_\alpha Cl(U)$. This implies $N_\alpha Cl(U)$ = U. Therefore U ∈ Nα-C(X). Hence Nα-O(X) ⊆ Nα-C(X). Let A ∈ Nα-C(X). Then C(A) ∈ Nα-O(X) ⊆ Nα-C(X). That is C(A) ∈ Nα-C(X). Therefore A ∈ Nα-O(X). Hence Nα-C(X) ⊆ Nα-O(X). Thus Nα-O(X) = Nα-C(X).

Sufficiency: Suppose that N□-O(X) = N□-C(X). Let A ⊆ U and U be a NOS in (X, τ). Then

U ∈ N□-O(X) and $N_{□}Cl(A) \subseteq N_{□}Cl(U) = U$, since U ∈ N□-C(X), by hypothesis. Therefore A is an $N_{□g}CS$ in X.

Proposition 3.27: If A is a NOS and a $N_{□g}CS$ in (X, τ), then A is a NROS in (X, τ).

Proof: Let A be a NOS and a $N_{□g}CS$ in (X, τ). Then A is a N□-CS in X. Now NInt(NCl(A)) □ NCl(NInt(NCl(A))) □ A. Since A is a NOS, A = NInt(A) □ NInt(NCl(A)). Hence NInt(NCl(A)) = A and A is a NROS in X.

Definition 3.28: A NS A in (X, τ) is a neutrosophic Q-set (NQ-S) in X if NInt(NCl(A)) = NCl(NInt(A)).

Proposition 3.29: For a NOS A in (X, τ), the following conditions are equivalent:

(i) A is a NCS in (X, τ),
(ii) A is a $N_{□g}CS$ and a NQ-S in (X, τ).

Proof: (i) ⇒ (ii) Since A is a NCS, it is a $N_{□g}CS$ in (X, τ). Now NInt(NCl(A)) = NInt(A) = A = NCl(A) =NCl(NInt(A)), by hypothesis. Hence A is a NQ-S in (X, τ).

(ii) ⇒ (i) Since A is a NOS and a $N_{□g}CS$ in (X, τ), by Theorem 3.27, A is a NROS in (X, τ). Therefore A = NInt(NCl(A)) = NCl(NInt(A)) = NCl(A), by hypothesis. Hence A is a NCS in (X, τ).

Proposition 3.30: Let (X, τ) be a NTS. Then for every A ∈ $N_{□g}O(X)$ and for every B ∈ NS(X), $N_{□}Int(A) \subseteq B \subseteq A$ implies B ∈ $N_{□g}O(X)$.

Proof: Let A be any $N_{□g}OS$ of X and B be any NS of X. By hypothesis $N_{□}IntA) \subseteq B \subseteq A$. Then C(A) is a $N_{□g}CS$ in X and $C(A) \subseteq C(B) \subseteq N_{□}Cl(C(A))$. By Theorem 3.24, C(B) is a $N_{□g}CS$ in (X, τ). Therefore B is a $N_{Ag}OS$ in (X, τ). Hence B ∈ $N_{□g}O(X)$.

Proposition 3.31: Let (X, τ) be a NTS. Then for every A ∈ NS(X) and for every B ∈ NS-O(X), $B \subseteq A \subseteq NInt(NCl(B))$ implies A ∈ $N_{□g}O(X)$.

Proof: Let B be a NS-OS in (X, τ). Then $B \subseteq NCl(NInt(B))$. By hypothesis, $A \subseteq NInt(NCl(B)) \subseteq NInt(NCl(NCl(NInt(B)))) \subseteq NInt(NCl(NInt(B))) \subseteq NInt(NCl(NInt(A)))$. Therefore A is a N□-OS and by Proposition 3.17, A is a $N_{□g}OS$ in (X, τ). Hence A ∈ $N_{□g}O(X)$.

Proposition 3.32: A NS A of a NTS (X, τ) is a $N_{□g}OS$ in (X, τ) if and only if $F \subseteq N_{□}Int(A)$ whenever F is a NCS in (X, τ) and $F \subseteq A$.

Proof: Necessity: Suppose A is a $N_{□g}OS$ in (X, τ). Let F be a NCS in (X, τ) such that $F \subseteq A$. Then C(F) is a NOS and $C(A) \subseteq C(F)$. By hypothesis C(A) is a $N_{□g}CS$ in (X, τ), we have $N_{□}Cl(C(A)) \subseteq C(F)$. Therefore $F \subseteq N_{□}Int(A)$.

Sufficiency: Let U be a NOS in (X, τ) such that $C(A) \subseteq U$. By hypothesis, $C(U) \subseteq N_{□}Int(A)$. Therefore $N_{□}Cl(C(A)) \subseteq U$ and C(A) is an $N_{□g}CS$ in (X, τ). Hence A is a $N_{□g}OS$ in (X, τ).

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