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Algebraic operations on new interval neutrosophic vague sets

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Abstract. The interval neutrosophic set plays an important role to handle indeterminacy and inconsistency of information during decision making process. Recently, the interval neutrosophic vague sets have been proposed as an extension of the neutrosophic sets. Similar with other sets, this newly proposed set have to fulfill some algebraic operations. This paper aims to present algebraic operations for the interval neutrosophic vague sets. Some algebraic operations on interval neutrosophic vague set are introduced. Specifically, algebraic operations of addition, multiplication, scalar multiplication and power for the interval neutrosophic vague sets are presented. In addition, several related examples are also presented together with supporting proofs.

1. Introduction

The uncertain theories such as fuzzy set theory [1], vague set theory [2] and rough set theory [3] are developed to solve imprecise and uncertain information that arise in decision making process. However, the problem with all these theories is they do not handle the indeterminate and inconsistent information. Therefore, Smarandache introduced a new theory namely neutrosophic set (NS in short). The NS is a new mathematical tool for dealing with problems involving incomplete, indeterminate and inconsistent information. A neutrosophic set consists of three membership functions which are truth-membership function (T), indeterminacy-membership (I) function and falsity-membership function (F). All these memberships lied in $\left]0^{-},1^{+}\right[$, the non-standard unit interval [4]. However, this unit interval is difficult to apply in the real applications. Therefore, single valued neutrosophic set (SVNS) was proposed Wang et al., [5]. The operations and relations between two SVNSs are defined namely subset, equality, complement, union and intersection. Meanwhile operations between two SVN-numbers are formulated by Liu and Wang [6]. These operations including addition, multiplication, scalar multiplication and power.

Recently, researchers have shown an interest on research and application of a neutrosophic set. Different sets were rapidly developed and proposed in the literature such as [7-14] etc. These extensions of neutrosophic set have been used in many areas such as aggregation operators, decision making, image processing, information measures, graph and algebraic structures [15]. Karaaslan and Hayat [16] developed the concept of single valued neutrosophic matrices and operations of SVN-matrices were discussed. Meanwhile, Ali and Smrandache [7] proposed a complex neutrosphic set which is an extension of complex fuzzy set. Ali et al., [14] proposed a new notation on interval complex neutrosophic set (ICNS) since complex neutrosophic set cannot deal with unclear and vague information. ICNS is defined along with several set theoretic operations and the operational rules. Ye [17] proposed trapezoidal neutrosophic set based on combination of trapezoidal fuzzy number and a SVNS. Some operational rules, score and accuracy for this set is defined. In line with these developments, the purpose of this paper is to define new algebraic operations on newly defined interval neutrosophic vague set as a novel notation [18]. This study also generalizes the basic properties of these operations such as commutative law and relevant law. This paper is organized as follows. We first present the basic definition of neutrosophic set and single value neutrosophic set that are useful for discussion. We then establish a new concept of interval neutrosophic vague set and define its algebraic operations with illustrative example. We also present some related properties and supporting proofs. Finally, we conclude the paper.

2. Preliminaries

In this section, neutrosophic set and INVS are presented.

2.1 Single Valued Neutrosophic Set (SVNS)

Definition 2.1 [5]
Let X be a universe of discourse. Then a neutrosophic set is defined as follows:

$$A = \{x, \langle T_A(x), I_A(x), F_A(x) \rangle : x \in X\}$$

which is characterized by a truth-membership function, an indeterminacy membership function and falsity-membership function where $T; I; F : X \to]0^-, 1^+[$ and $0^- \leq T_A(x) + I_A(x) + F_A(x) \leq 3^+$.
For application in real scientific and engineering areas, Wang et al., proposed the concept of a single valued neutrosophic set as follows:
Operations between two SVN-numbers are defined by [6]. It is recalled as follows:
Let $x = (T_1, I_1, F_1)$ and $y = (T_2, I_2, F_2)$ be two SVNS, then the operations are defined:

i. $x \oplus y = (T_1 + T_2 - T_1, T_2, I_1\, I_2, F_1\, F_2)$.
ii. $x \otimes y = (T_1 T_2, I_1 + I_2 - I_1\, I_2, F_1 + F_2 - F_1\, F_2)$.
iii. $\lambda x = ((1-(1-T_1)^{\lambda}, I_1^{\lambda}, F_1^{\lambda})$.
iv. $x^{\lambda} = (T_1^{\lambda}, 1-(1-I_1^{\lambda}), 1-(1-F_1^{\lambda}))$.

2.2 Interval Neutrosophic Sets

Definition 2.2 [19]
An interval neutrosophic set (INS) A in X is characterized by truth-membership function $T_A(x)$, indeterminacy membership functions $I_A(x)$ and falsity-membership functions $F_A(x)$ defined as follows:

For two IVNS,

$$A_{INS} = \left\{ x, \left(\left[T_A^L(x), T_A^U(x)\right], \left[I_A^L(x), I_A^U(x)\right], \left[F_A^L(x), F_A^U(x)\right] \right) x \in X \right\} \text{ and}$$
$$B_{INS} = \left\{ x, \left(\left[T_B^L(x), T_B^U(x)\right], \left[I_B^L(x), I_B^U(x)\right], \left[F_B^L(x), F_B^U(x)\right] \right) x \in X \right\}.$$

2.3 Interval Neutrosophic Vague Sets

Definition 2.3 [18]

An interval valued neutrosophic vague set A_{INV} is also known as INVS in the universe of discourse X. An INVS is characterized by truth membership, indeterminacy membership and falsity-membership functions defined as:

$$A_{INV} = \left\{x, \left[\widetilde{T}_A^L(x), \widetilde{T}_A^U(x)\right], \left[\widetilde{I}_A^L(x), \widetilde{I}_A^U(x)\right], \left[\widetilde{F}_A^L(x), \widetilde{F}_A^U(x)\right] > \mid x \in X\right\}$$

$\widetilde{T}_A^L(x) = \left[T^{L^-}, T^{L^+}\right]$, $\widetilde{T}_A^U(x) = \left[T^{U^-}, T^{U^+}\right]$, $\widetilde{I}_A^L(x) = \left[I^{L^-}, I^{L^+}\right]$, $\widetilde{I}_A^U(x) = \left[I^{U^-}, I^{U^+}\right]$ and

$\widetilde{F}_A^L(x) = \left[F^{L^-}, F^{L^+}\right]$, $\widetilde{F}_A^U(x) = \left[F^{U^-}, F^{U^+}\right]$ where

1) $T^{L+} = 1 - F^{L-}, F^{L+} = 1 - T^{L-}$,
2) $T^{U+} = 1 - F^{U-}, F^{U+} = 1 - T^{U-}$,
3) $^-0 \le T^{L-} + T^{U-} + I^{L-} + I^{U-} + F^{L-} + F^{U-} \le 4^+$,
4) $^-0 \le T^{L+} + T^{U+} + I^{L+} + I^{U+} + F^{L+} + F^{U+} \le 4^+$.

Definition 2.4 [18]

Let Φ_{INV} be an INVS of the universe X where $\forall x_i \in X$,

$\widetilde{T}_{\Phi_{INV}}^L(x) = [1,1], \widetilde{T}_{\Phi_{INV}}^U = [1,1]$,

$\widetilde{I}_{\Phi_{INV}}^L(x) = [0,0], \widetilde{I}_{\Phi_{INV}}^U(x) = [0,0]$,

$\widetilde{F}_{\Phi_{INV}}^L(x) = [0,0], \widetilde{F}_{\Phi_{INV}}^U(x) = [0,0]$.

Therefore, Φ_{INV} is called a unit INVS where $1 \le i \le n$.

Definition 2.5 [18]

Let δ_{INV} be an INVS of the universe X where $\forall x_i \in X$,

$\widetilde{T}_{\delta_{INV}}^L(x) = [0,0], \widetilde{T}_{\delta_{INV}}^U = [0,0]$,

$\widetilde{I}_{\delta_{INV}}^L(x) = [1,1], \widetilde{I}_{\delta_{INV}}^U(x) = [1,1]$,

$\widetilde{F}_{\delta_{INV}}^L(x) = [1,1], \widetilde{F}_{\delta_{INV}}^U(x) = [1,1]$.

Therefore, δ_{INV} is called a zero INVS where $1 \le i \le n$.

3. Operations for Interval Neutrosophic Vague Sets

3.1. Basic Algebraic Operations on Interval Neutrosophic Vague Sets

In this section, we introduce some new algebraic operations on INVS based on the operations of SVNS. The algebraic operations on INVS such as addition, multiplication, scalar multiplication and power operations are defined as follows:

Definition 3.1

Let $A = \left\langle \left\{ \left[T_1^{L-}, T_1^{L+}\right], \left[T_1^{U-}, T_1^{U+}\right] \right\}, \left\{ \left[I_1^{L-}, I_1^{L+}\right], \left[I_1^{U-}, I_1^{U+}\right] \right\}, \left\{ \left[F_1^{L-}, F_1^{L+}\right], \left[F_1^{U-}, F_1^{U+}\right] \right\} \right\rangle$ and $B = \left\langle \left\{ \left[T_2^{L-}, T_2^{L+}\right], \left[T_2^{U-}, T_2^{U+}\right] \right\}, \left\{ \left[I_2^{L-}, I_2^{L+}\right], \left[I_2^{U-}, I_2^{U+}\right] \right\}, \left\{ \left[F_2^{L-}, F_2^{L+}\right], \left[F_2^{U-}, F_2^{U+}\right] \right\} \right\rangle$ be two INVS.

The INVS addition denoted as $A \oplus B$ are defined as follows:

$$A \oplus B = \left\langle \left\{ \left[T_1^{L-} + T_2^{L-} - T_1^{L-}T_2^{L-}, T_1^{L+} + T_2^{L+} - T_1^{L+}T_2^{L+}\right], \left[T_1^{U-} + T_2^{U-} - T_1^{U-}T_2^{U-}, T_1^{U+} + T_2^{U+} - T_1^{U+}T_2^{U+}\right] \right\}, \right.$$
$$\left. \left\{ \left[I_1^{L-}I_2^{L-}, I_1^{L+}I_2^{L+}\right], \left[I_1^{U-}I_2^{U-}, I_1^{U+}I_2^{U+}\right] \right\}, \left\{ \left[F_1^{L-}F_2^{L-}, F_1^{L+}F_2^{L+}\right], \left[F_1^{U-}F_2^{U-}, F_1^{U+}F_2^{U+}\right] \right\} \right\rangle.$$

Definition 3.2

Let $A = \left\langle \left\{ \left[T_1^{L-}, T_1^{L+}\right], \left[T_1^{U-}, T_1^{U+}\right] \right\}, \left\{ \left[I_1^{L-}, I_1^{L+}\right], \left[I_1^{U-}, I_1^{U+}\right] \right\}, \left\{ \left[F_1^{L-}, F_1^{L+}\right], \left[F_1^{U-}, F_1^{U+}\right] \right\} \right\rangle$ and $B = \left\langle \left\{ \left[T_2^{L-}, T_2^{L+}\right], \left[T_2^{U-}, T_2^{U+}\right] \right\}, \left\{ \left[I_2^{L-}, I_2^{L+}\right], \left[I_2^{U-}, I_2^{U+}\right] \right\}, \left\{ \left[F_2^{L-}, F_2^{L+}\right], \left[F_2^{U-}, F_2^{U+}\right] \right\} \right\rangle$ be two INVS.

Then, INVS multiplication denoted by $A \otimes B$ is defined as follows:

$$A \otimes B = \left\langle \left\{ \left[T_1^{L-}T_2^{L-}, T_1^{L+}T_2^{L+}\right], \left[T_1^{U-}T_2^{U-}, T_1^{U+}T_2^{U+}\right] \right\}, \left\{ \left[I_1^{L-} + I_2^{L-} - I_1^{L-}I_2^{L-}, I_1^{L+}I_2^{L+} - I_1^{L+}I_2^{L+}\right], \right. \right.$$
$$\left. \left[I_1^{U-} + I_2^{U-} - I_1^{U-}I_2^{U-}, I_1^{U+} + I_2^{U+} - I_1^{U+}I_2^{U+}\right] \right\}, \left\{ \left[F_1^{L-} + F_2^{L-} - F_1^{L-}F_2^{L-}, F_1^{L+} + F_2^{L+} - F_1^{L+}F_2^{L+}\right], \right.$$
$$\left. \left. \left[F_1^{U-} + F_2^{U-} - F_1^{U-}F_2^{U-}, F_1^{U+} + F_2^{U+} - F_1^{U+}F_2^{U+}\right] \right\} \right\rangle.$$

Definition 3.3

Let $A = \left\langle \left\{ \left[T_1^{L-}, T_1^{L+}\right], \left[T_1^{U-}, T_1^{U+}\right] \right\}, \left\{ \left[I_1^{L-}, I_1^{L+}\right], \left[I_1^{U-}, I_1^{U+}\right] \right\}, \left\{ \left[F_1^{L-}, F_1^{L+}\right], \left[F_1^{U-}, F_1^{U+}\right] \right\} \right\rangle$ and $B = \left\langle \left\{ \left[T_2^{L-}, T_2^{L+}\right], \left[T_2^{U-}, T_2^{U+}\right] \right\}, \left\{ \left[I_2^{L-}, I_2^{L+}\right], \left[I_2^{U-}, I_2^{U+}\right] \right\}, \left\{ \left[F_2^{L-}, F_2^{L+}\right], \left[F_2^{U-}, F_2^{U+}\right] \right\} \right\rangle$ be two INVS.

Then, INVS scalar multiplication denoted by λA is defined as follows:

$$\lambda A = \left\langle \left\{ \left[1 - \left(1 - T_1^{L-}\right)^{\lambda}, \left(1 - 1 - T_1^{L+}\right)^{\lambda}\right], \left[1 - \left(1 - T_1^{U-}\right)^{\lambda}, \left(1 - 1 - T_1^{U+}\right)^{\lambda}\right] \right\}, \left\{ \left[\left(I_1^{L-}\right)^{\lambda}, \left(I_1^{L+}\right)^{\lambda}\right], \left[\left(I_1^{U-}\right)^{\lambda}, \left(I_1^{U+}\right)^{\lambda}\right] \right\} \right.$$
$$\left. \left\{ \left[\left(F_1^{L-}\right)^{\lambda}, \left(F_1^{L+}\right)^{\lambda}\right], \left[\left(F_1^{U-}\right)^{\lambda}, \left(F_1^{U+}\right)^{\lambda}\right]; \lambda \in \Re, \lambda > 0 \right\} \right\rangle.$$

Definition 3.4

Let $A = \left\langle \left\{ \left[T_1^{L-}, T_1^{L+}\right], \left[T_1^{U-}, T_1^{U+}\right] \right\}, \left\{ \left[I_1^{L-}, I_1^{L+}\right], \left[I_1^{U-}, I_1^{U+}\right] \right\}, \left\{ \left[F_1^{L-}, F_1^{L+}\right], \left[F_1^{U-}, F_1^{U+}\right] \right\} \right\rangle$ and $B = \left\langle \left\{ \left[T_2^{L-}, T_2^{L+}\right], \left[T_2^{U-}, T_2^{U+}\right] \right\}, \left\{ \left[I_2^{L-}, I_2^{L+}\right], \left[I_2^{U-}, I_2^{U+}\right] \right\}, \left\{ \left[F_2^{L-}, F_2^{L+}\right], \left[F_2^{U-}, F_2^{U+}\right] \right\} \right\rangle$ be two INVS.

Then, INVS power denoted by A^{λ} is defined as follows:

$$A^{\lambda}=\left\langle\left\{\left[\left(T_1^{L-}\right)^{\lambda},\left(T_1^{L+}\right)^{\lambda}\right],\left[\left(T_1^{U-}\right)^{\lambda},\left(T_1^{U+}\right)^{\lambda}\right]\right\},\left\{\left[1-\left(1-I_1^{L-}\right)^{\lambda},1-\left(1-I_1^{L+}\right)^{\lambda}\right],\left[1-\left(1-I_1^{U-}\right)^{\lambda},1-\left(1-I_1^{U+}\right)^{\lambda}\right]\right\}\right.$$
$$\left.\left\{\left[1-\left(1-F_1^{L-}\right)^{\lambda},1-\left(1-F_1^{L+}\right)^{\lambda}\right],\left[1-\left(1-F_1^{U-}\right)^{\lambda},1-\left(1-F_1^{U+}\right)^{\lambda}\right];\lambda\in\Re,\lambda>0\right\}\right..$$

To illustrate these operations, examples are given and we recall Definition 3.1, 3.2, 3.3 and 3.4.

Example 3.1: Let $A=\left\langle\{[0.2,0.5],[0.2,0.3]\},\{[0.1,0.6],[0.3,0.6]\},\{[0.5,0.8],[0.7,0.8]\}:x\in X\right\rangle$ and $B=\left\langle\{[0.2,0.6],[0.4,0.9]\},\{[0.5,0.5],[0.3,0.6]\},\{[0.4,0.8],[0.1,0.6]\}:x\in X\right\rangle$, then the INVS addition operation is given as follows:

By Definition 3.1,

$$A\oplus B=\left\langle\{[0.2+0.2-(0.2)(0.2),0.5+0.6-(0.5)(0.6)],[0.2+0.4-(0.2)(0.4),0.3+0.9-(0.3)(0.9)]\},\right.$$
$$\left.\{[(0.1)(0.5),(0.6)(0.5)],[(0.3)(0.3),(0.6)(0.6)]\},\{[(0.5)(0.4),(0.8)(0.8)],[(0.7)(0.1),(0.8)(0.6)]\}\right\rangle$$
$$=\left\langle\{[0.36,0.8],[0.52,0.93]\},\{[0.05,0.3],[0.09,0.36]\},\{[0.2,0.64],[0.07,0.48]\}\right\rangle,$$

and we check the INVS addition as follows:

From Definition 2.3 we have, $T^{L+}=1-F^{L-}=0.8+0.2=1$, $F^{L^{+}}=1-T^{L^{-}}=0.64+0.36=1$, $F^{U^{-}}=1-T^{U^{+}}=0.07+0.93=1$ and $T^{U^{-}}=1-F^{U^{+}}=0.52+0.48=1$.

Using condition $^{-}0\le T^{L-}+T^{U-}+I^{L-}+I^{U-}+F^{L-}+F^{U-}\le 4^{+}$, therefore, we have $0.36+0.52+0.05+0.09+0.2+0.07=1.29$ and $^{-}0\le T^{L^{+}}+T^{U^{+}}+I^{L^{+}}+I^{U^{+}}+F^{L^{+}}+F^{U^{+}}\le 4^{+}$, therefore, we have $0.8+0.93+0.3+0.36+0.64+0.48=3.51$.

The calculation for INVS multiplication, scalar and power is calculated similarly as follows.

Example 3.2:

Consider Example 3.1. Then by Definition 3.2, the INVS multiplication between A and B as follows:

$$A\otimes B=\left\langle\{[(0.2)(0.2),(0.5)(0.6)],[(0.2)(0.4),(0.3)(0.9)]\},\{[0.1+0.5-(0.1)(0.5)],[0.6+0.5-(0.6)(0.5)]\right.$$
$$[0.3+0.3-(0.3)(0.3)],[0.6+0.6-(0.6)(0.6)]\},\left.\{[0.7+0.1-(0.7)(0.1),0.8+0.6-(0.8)(0.6)]\}\right\rangle$$
$$=\left\langle\{[0.04,0.3],[0.08,0.27]\},\{[0.05,0.8],[0.51,0.84]\},\{[0.7,0.96],[0.73,0.92]\}\right\rangle.$$

Example 3.3:

Consider Example 3.1. Then by Definition 3.3, if $\lambda=2$ we have

$$\lambda A=\left\langle\{[1-(1-0.2)^2,1-(1-0.5)^2],[1-(1-0.2)^2,1-(1-0.3)^2]\},\{[(0.1)^2,(0.6)^2],[(0.3)^2,(0.6)^2]\},\{[(0.5)^2,(0.8)^2],\right.$$
$$\left.[(0.7)^2,(0.8^2)]\}\right\rangle$$
$$=\left\langle\{[0.36,0.75],[0.36,0.51]\},\{[0.01,0.36],[0.09,0.36]\},\{[0.25,0.64],[0.49,0.64]\}\right\rangle.$$

Example 3.4:
Consider Example 3.1. Then by definition 3.4, if $\lambda = 2$ we have

$$A^{\lambda} = \left\langle \left\{ \left[(0.2)^2, (0.5)^2 \right], \left[(0.2)^2, (0.3)^2 \right] \right\}, \left\{ \left[1-(1-0.1)^2, 1-(1-0.6)^2 \right], \left[1-(1-0.3)^2 \right], \left[1-(1-0.6)^2 \right] \right\}, \right.$$
$$\left. \left\{ \left[1-(1-0.5)^2, 1-(1-0.8)^2 \right], \left[1-(1-0.7)^2, 1-(1-0.8)^2 \right] \right\} \right\rangle$$
$$= \left\langle \left\{ [0.04, 0.25], [0.04, 0.09] \right\}, \left\{ [0.36, 0.84], [0.51, 0.84] \right\}, \left\{ [0.75, 0.96], [0.91, 0.96] \right\} \right\rangle.$$

Theorem 1
Let A and B be two INVS and $\lambda . \lambda_1, \lambda_2 > 0$. Then

i. $A \oplus B = B \oplus A$
ii. $A \otimes B = B \otimes A$
iii. $\lambda(A \oplus B) = \lambda A \oplus \lambda B$
iv. $\lambda_1 A \oplus \lambda_2 A = (\lambda_1 \oplus \lambda_2) A$
v. $A^{\lambda_1} \otimes A^{\lambda_2} = A^{\lambda_1 + \lambda_2}$
vi. $A^{\lambda} \otimes B^{\lambda} = (A \otimes B)^{\lambda}$

Proof (i), (ii), (iv) and (vi) are obvious; thus we prove the others.

Proof (iii)
By definition 3.1 and definition 3.3, we have

$$A \oplus B = \left\langle \left\{ \left[T_1^{L-} + T_2^{L-} - (T_1^{L-})(T_2^{L-}), T_1^{L+} + T_2^{L+} - (T_1^{L+})(T_2^{L+}) \right] \right. \right.$$
$$\left. \left[T_1^{U-} + T_2^{U-} - (T_1^{U-})(T_2^{U-}), T_1^{U+} + T_2^{U+} - (T_1^{U+})(T_2^{U+}) \right] \right\}, \left\{ \left[(I_1^{L-}), (I_2^{L-}), (I_1^{L+}), (I_2^{L+}) \right] \right.$$
$$\left. \left[(I_1^{U-}), (I_2^{U-}), (I_1^{U+}), (I_2^{U+}) \right] \right\}, \left\{ \left[(F_1^{L-}), (F_2^{L-}), (F_1^{L+}), (F_2^{L+}) \right], \left[(F_1^{U-}), (F_2^{U-}), (F_1^{U+}), (F_2^{U+}) \right] \right\}.$$

$$\lambda(A \oplus B) = \left\langle \left\{ \left[1 - \left[1 - \left(T_1^{L-} + T_2^{L-} - (T_1^{L-})(T_2^{L-}) \right) \right]^{\lambda}, 1 - \left[1 - \left(T_1^{L+} + T_2^{L+} - (T_1^{L+})(T_2^{L+}) \right) \right]^{\lambda} \right], \right. \right.$$
$$\left. \left[1 - \left[1 - \left(T_1^{U-} + T_2^{U-} - (T_1^{U-})(T_2^{U-}) \right) \right]^{\lambda}, 1 - \left[1 - \left(T_1^{U+} + T_2^{U+} - (T_1^{U+})(T_2^{U+}) \right) \right]^{\lambda} \right] \right\},$$
$$\left\{ \left[\left((I_1^{L-}), (I_2^{L-}) \right)^{\lambda}, \left((I_1^{L+}), (I_2^{L+}) \right)^{\lambda} \right], \left[\left((I_1^{U-}), (I_2^{U-}) \right)^{\lambda}, \left((I_1^{U+}), (I_2^{U+}) \right)^{\lambda} \right] \right\},$$
$$\left. \left\{ \left[\left((F_1^{L-}), (F_2^{L-}) \right)^{\lambda}, \left((F_1^{L+}), (F_2^{L+}) \right)^{\lambda} \right], \left[\left((F_1^{U-}), (F_2^{U-}) \right)^{\lambda}, \left((F_1^{U+}), (F_2^{U+}) \right)^{\lambda} \right] \right\} \right\rangle.$$

We solve truth membership functions:

$$= 1 - \left[1 - T_1^{L-} - T_2^{L-} + (T_1^{L-})(T_2^{L-}) \right]^{\lambda}$$
$$= 1 - \left[(1 - T_1^{L-}) - (T_2^{L-})(1 - T_1^{L-}) \right]^{\lambda}$$
$$= 1 - \left[(1 - T_1^{L-})(1 - T_2^{L-}) \right]^{\lambda}$$

By similar calculation, therefore we get

$$\lambda(A \oplus B) = \left\langle \left\{ \left[1 - \left[(1 - T_1^{L-})(1 - T_2^{L-}) \right]^{\lambda}, 1 - \left[(1 - T_1^{L+})(1 - T_2^{L+}) \right]^{\lambda} \right] \right. \right.$$

$$\left[1-\left[\left(1-T_1^{U-}\right)\left(1-T_2^{U-}\right)\right]^{\lambda},1-\left[\left(1-T_1^{U+}\right)\left(1-T_2^{U+}\right)\right]^{\lambda}\right]\Big\},$$

$$\left\{\left[\left(\left(I_1^{L-}\right)\left(I_2^{L-}\right)\right)^{\lambda},\left(\left(I_1^{L+}\right)\left(I_2^{L+}\right)\right)^{\lambda}\right],\left[\left(\left(I_1^{U-}\right)\left(I_2^{U-}\right)\right)^{\lambda},\left(\left(I_1^{U+}\right)\left(I_2^{U+}\right)\right)^{\lambda}\right]\right\},$$

$$\left\{\left[\left(\left(F_1^{L-}\right)\left(F_2^{L-}\right)\right)^{\lambda},\left(\left(F_1^{L+}\right)\left(F_2^{L+}\right)\right)^{\lambda}\right],\left[\left(\left(F_1^{U-}\right)\left(F_2^{U-}\right)\right)^{\lambda},\left(\left(F_1^{U+}\right)\left(F_2^{U+}\right)\right)^{\lambda}\right]\right\}\Big\rangle.$$

Now $\lambda A\oplus\lambda B$

$$\lambda A\oplus\lambda B=\Big\langle\Big\{\Big[\left(1-\left(1-T_1^{L-}\right)^{\lambda}\right)+\left(1-\left(1-T_2^{L-}\right)^{\lambda}\right)-\left(1-\left(1-T_1^{L-}\right)^{\lambda}\right)\left(1-\left(1-T_2^{L-}\right)^{\lambda}\right),$$

$$\left(1-\left(1-T_1^{L+}\right)^{\lambda}\right)+\left(1-\left(1-T_2^{L+}\right)^{\lambda}\right)-\left(1-\left(1-T_1^{L+}\right)^{\lambda}\right)\left(1-\left(1-T_2^{L+}\right)^{\lambda}\right)\Big],$$

$$=\Big[\left(1-\left(1-T_1^{U-}\right)^{\lambda}\right)+\left(1-\left(1-T_2^{U-}\right)^{\lambda}\right)-\left(1-\left(1-T_1^{U-}\right)^{\lambda}\right)+\left(1-\left(1-T_2^{U-}\right)^{\lambda}\right),$$

$$\left(1-\left(1-T_1^{U+}\right)^{\lambda}\right)+\left(1-\left(1-T_2^{U+}\right)^{\lambda}\right)-\left(1-\left(1-T_1^{U+}\right)^{\lambda}\right)\Big]\Big\}$$

$$\left\{\left[\left(\left(I_1^{L-}\right)^{\lambda}\left(I_2^{L-}\right)^{\lambda},\left(I_1^{L+}\right)^{\lambda}\left(I_2^{L+}\right)^{\lambda}\right)\right],\left[\left(\left(I_1^{U-}\right)^{\lambda}\left(I_2^{U-}\right)^{\lambda},\left(I_1^{U+}\right)^{\lambda}\left(I_2^{U+}\right)^{\lambda}\right)\right]\right\}$$

$$\left\{\left[\left(\left(F_1^{L-}\right)^{\lambda}\left(F_2^{L-}\right)^{\lambda},\left(F_1^{L+}\right)^{\lambda}\left(F_2^{L+}\right)^{\lambda}\right)\right],\left[\left(\left(F_1^{U-}\right)^{\lambda}\left(F_2^{U-}\right)^{\lambda},\left(F_1^{U+}\right)^{\lambda}\left(F_2^{U+}\right)^{\lambda}\right)\right]\right\}\Big\rangle.$$

It is proved for identity and falsity terms, since $\left(\left(I_1^{L-}\right)\left(I_2^{L-}\right)\right)^{\lambda}=\left(I_1^{L-}\right)^{\lambda}\left(I_2^{L-}\right)^{\lambda}$, $\left(\left(I_1^{L+}\right)\left(I_2^{L+}\right)\right)^{\lambda}=\left(I_1^{L+}\right)^{\lambda}\left(I_2^{L+}\right)^{\lambda}$, $\left(\left(F_1^{L-}\right)\left(F_2^{L-}\right)\right)^{\lambda}=\left(F_1^{L-}\right)^{\lambda}\left(F_2^{L-}\right)^{\lambda}$ and $\left(\left(F_1^{L-}\right)\left(F_2^{L-}\right)\right)^{\lambda}=\left(F_1^{L-}\right)^{\lambda}\left(F_2^{L-}\right)^{\lambda}$.

We prove it for truth membership functions

$$=\left(1-\left(1-T_1^{L-}\right)^{\lambda}\right)+\left(1-\left(1-T_2^{L-}\right)^{\lambda}\right)-\left(1-\left(1-T_1^{L-}\right)^{\lambda}\right)\left(1-\left(1-T_2^{L-}\right)^{\lambda}\right)$$

$$=2-\left[\left(1-T_1^{L-}\right)^{\lambda}+\left(1-T_2^{L-}\right)^{\lambda}\right]-\left[1-\left(1-T_2^{L-}\right)^{\lambda}-\left(1-T_1^{L-}\right)^{\lambda}+\left(\left(1-T_1^{L-}\right)^{\lambda}\right)\left(\left(1-T_2^{L-}\right)^{\lambda}\right)\right]$$

$$=2-\left[\left(1-T_1^{L-}\right)^{\lambda}-\left(1-T_2^{L-}\right)^{\lambda}\right]-1+\left(\left(1-T_2^{L-}\right)^{\lambda}+\left(1-T_1^{L-}\right)^{\lambda}\right)-\left(\left(1-T_1^{L-}\right)^{\lambda}\right)\left(\left(1-T_2^{L-}\right)^{\lambda}\right)$$

$$=1-\left(\left(1-T_1^{L-}\right)^{\lambda}\right)\left(\left(1-T_2^{L-}\right)^{\lambda}\right)$$

$$=1-\left[\left(1-T_1^{L-}\right)\left(1-T_2^{L-}\right)\right]^{\lambda}.$$

In the similar manner, hence we get

$$=\left\{\left[1-\left[\left(1-T_1^{L-}\right)\left(1-T_2^{L-}\right)\right]^{\lambda},1-\left[\left(1-T_1^{L+}\right)\left(1-T_2^{L+}\right)\right]^{\lambda}\right],\left[1-\left[\left(1-T_1^{U-}\right)\left(1-T_2^{U-}\right)\right]^{\lambda},1-\left[\left(1-T_1^{U+}\right)\left(1-T_2^{U+}\right)\right]^{\lambda}\right]\right\}$$

Proof (v)

By definition 3.2 and definition 3.4, we have

$$A^{\lambda_1}\otimes A^{\lambda_2}=\Big\langle\Big\{\left[\left(T_1^{L-}\right)^{\lambda_1}\left(T_1^{L-}\right)^{\lambda_2},\left(T_1^{L+}\right)^{\lambda_1}\left(T_1^{L+}\right)^{\lambda_2}\right],\left[\left(T_1^{U-}\right)^{\lambda_1}\left(T_1^{U-}\right)^{\lambda_2},\left(T_1^{U+}\right)^{\lambda_1}\left(T_1^{U+}\right)^{\lambda_2}\right]\Big\},$$

$$\Bigg\{\Bigg[\left(1-\left(1-I_1^{L-}\right)^{\lambda_1}\right)+\left(1-\left(1-I_1^{L-}\right)^{\lambda_2}\right)-\left(1-\left(1-I_1^{L-}\right)^{\lambda_1}\right)\left(1-\left(1-I_1^{L-}\right)^{\lambda_2}\right),$$

$$\left(1-\left(1-I_1^{L+}\right)^{\lambda_1}\right)+\left(1-\left(1-I_1^{L+}\right)^{\lambda_2}\right)-\left(1-\left(1-I_1^{L+}\right)^{\lambda_1}\right)+\left(1-\left(1-I_1^{L+}\right)^{\lambda_2}\right)\Bigg]$$

$$\Bigg[\left(1-\left(1-I_1^{U-}\right)^{\lambda_1}\right)+\left(1-\left(1-I_1^{U-}\right)^{\lambda_2}\right)-\left(1-\left(1-I_1^{U-}\right)^{\lambda_1}\right)\left(1-\left(1-I_1^{U-}\right)^{\lambda_2}\right),$$

$$\left(1-\left(1-I_1^{U+}\right)^{\lambda_1}\right)+\left(1-\left(1-I_1^{U+}\right)^{\lambda_2}\right)-\left(1-\left(1-I_1^{U+}\right)^{\lambda_1}\right)\left(1-\left(1-I_1^{U+}\right)^{\lambda_2}\right)\Bigg]\Bigg\}$$

$$\Bigg\{\Bigg[\left(1-\left(1-F_1^{L-}\right)^{\lambda_1}\right)+\left(1-\left(1-F_1^{L-}\right)^{\lambda_2}\right)-\left(1-\left(1-F_1^{L-}\right)^{\lambda_1}\right)\left(1-\left(1-F_1^{L-}\right)^{\lambda_2}\right),$$

$$\left(1-\left(1-F_1^{L+}\right)^{\lambda_1}\right)+\left(1-\left(1-F_1^{L+}\right)^{\lambda_2}\right)-\left(1-\left(1-F_1^{L+}\right)^{\lambda_1}\right)+\left(1-\left(1-F_1^{L+}\right)^{\lambda_2}\right)\Bigg]$$

$$\Bigg[\left(1-\left(1-F_1^{U-}\right)^{\lambda_1}\right)+\left(1-\left(1-F_1^{U-}\right)^{\lambda_2}\right)-\left(1-\left(1-F_1^{U-}\right)^{\lambda_1}\right)\left(1-\left(1-F_1^{U-}\right)^{\lambda_2}\right),$$

$$\left(1-\left(1-F_1^{U+}\right)^{\lambda_1}\right)+\left(1-\left(1-F_1^{U+}\right)^{\lambda_2}\right)-\left(1-\left(1-F_1^{U+}\right)^{\lambda_1}\right)\left(1-\left(1-F_1^{U+}\right)^{\lambda_2}\right)\Bigg]\Bigg\}.$$

We solve identity membership functions

$$=2\left(\left(1-I_1^{L-}\right)^{\lambda_1}\right)+\left(\left(1-I_1^{L-}\right)^{\lambda_2}\right)-\left(1-\left(1-I_1^{L-}\right)^{\lambda_1}\right)\left(1-\left(1-I_1^{L-}\right)^{\lambda_2}\right)$$

$$=2-\left(\left(1-I_1^{L-}\right)^{\lambda_1}\right)+\left(\left(1-I_1^{L-}\right)^{\lambda_2}\right)-\left(1-\left(1-I_1^{L-}\right)^{\lambda_2}-\left(1-I_1^{L-}\right)^{\lambda_1}+\left(1-I_1^{L-}\right)^{\lambda_1}\left(1-I_1^{L-}\right)^{\lambda_2}\right)$$

$$=2-\left(\left(1-I_1^{L-}\right)^{\lambda_1}\right)-\left(\left(1-I_1^{L-}\right)^{\lambda_2}\right)-1+\left(\left(1-I_1^{L-}\right)^{\lambda_2}+\left(1-I_1^{L-}\right)^{\lambda_1}-\left(1-I_1^{L-}\right)^{\lambda_1}\left(1-I_1^{L-}\right)^{\lambda_2}\right)$$

$$=1-\left(1-I_1^{L-}\right)^{\lambda_1}\left(1-I_1^{L-}\right)^{\lambda_2}.$$

By the similar calculation, hence we get

$$A^{\lambda_1}\otimes A^{\lambda_2}=\Bigg\langle\Bigg\{\Bigg[\left(T_1^{L-}\right)^{\lambda_1},\left(T_1^{L+}\right)^{\lambda_2}\Bigg],\Bigg[\left(T_1^{U-}\right)^{\lambda_1},\left(T_1^{U+}\right)^{\lambda_2}\Bigg]\Bigg\},$$

$$\Bigg\{\Bigg[1-\Big[\left(1-I_1^{L-}\right)^{\lambda_1}\left(1-I_1^{L-}\right)^{\lambda_2}\Big],1-\Big[\left(1-I_1^{L+}\right)^{\lambda_1}\left(1-I_1^{L+}\right)^{\lambda_2}\Big]\Bigg]$$

$$\Bigg[1-\Big[\left(1-I_1^{U-}\right)^{\lambda_1}\left(1-I_1^{U-}\right)^{\lambda_2}\Big],1-\Big[\left(1-I_1^{U+}\right)^{\lambda_1}\left(1-I_1^{U+}\right)^{\lambda_2}\Big]\Bigg]\Bigg\},$$

$$\Bigg\{\Bigg[1-\Big[\left(1-F_1^{L-}\right)^{\lambda_1}\left(1-F_1^{L-}\right)^{\lambda_2}\Big],1-\Big[\left(1-F_1^{L+}\right)^{\lambda_1}\left(1-F_1^{L+}\right)^{\lambda_2}\Big]\Bigg]$$

$$\Bigg[1-\Big[\left(1-F_1^{U-}\right)^{\lambda_1}\left(1-F_1^{U-}\right)^{\lambda_2}\Big],1-\Big[\left(1-I_1^{U+}\right)^{\lambda_1}\left(1-I_1^{U+}\right)^{\lambda_2\,\lambda}\Big]\Bigg]\Bigg\}\Bigg\rangle..$$

Now $A^{\lambda_1+\lambda_2}$ by using definition 3.4, we have

$$A^{\lambda_1+\lambda_2}=\left\langle\left\{\left[T_1^{L-},T_1^{L+}\right],\left[T_1^{U-},T_1^{U+}\right]\right\},\left\{\left[I_1^{L-},I_1^{L+}\right],\left[I_1^{U-},I_1^{U+}\right]\right\},\left\{\left[F_1^{L-},F_1^{L+}\right],\left[F_1^{U-},F_1^{U+}\right]\right\}\right\rangle^{\lambda_1+\lambda_2}$$

$$=\left\langle\left\{\left[\left(T_1^{L-}\right)^{\lambda_1+\lambda_2},\left(T_1^{L+}\right)^{\lambda_1+\lambda_2}\right],\left[\left(T_1^{U-}\right)^{\lambda_1+\lambda_2},\left(T_1^{U+}\right)^{\lambda_1+\lambda_2}\right]\right\},\left\{\left[1-\left(1-I_1^{L-}\right)^{\lambda_1+\lambda_2},1-\left(1-I_1^{L+}\right)^{\lambda_1+\lambda_2}\right],\right.\right.$$
$$\left[1-\left(1-I_1^{U-}\right)^{\lambda_1+\lambda_2},1-\left(1-I_1^{U+}\right)^{\lambda_1+\lambda_2}\right]\right\},\left\{\left[1-\left(1-F_1^{L-}\right)^{\lambda_1+\lambda_2},1-\left(1-F_1^{L+}\right)^{\lambda_1+\lambda_2}\right],\right.$$
$$\left.\left.\left[1-\left(1-F_1^{U-}\right)^{\lambda_1+\lambda_2},1-\left(1-F_1^{U+}\right)^{\lambda_1+\lambda_2}\right]\right\}\right\rangle.$$
$$=\left\langle\left\{\left[\left(T_1^{L-}\right)^{\lambda_1}\left(T_1^{L-}\right)^{\lambda_2},\left(T_1^{L+}\right)^{\lambda_1}\left(T_1^{L+}\right)^{\lambda_2}\right],\left[\left(T_1^{U-}\right)^{\lambda_1}\left(T_1^{U-}\right)^{\lambda_2},\left(T_1^{U+}\right)^{\lambda_1}\left(T_1^{U+}\right)^{\lambda_2}\right]\right\},\right.$$
$$\left\{\left[1-\left[\left(1-I_1^{L-}\right)^{\lambda_1}\left(1-I_1^{L-}\right)^{\lambda_2}\right],1-\left[\left(1-I_1^{L+}\right)^{\lambda_1}\left(1-I_1^{L+}\right)^{\lambda_2}\right]\right],\right.$$
$$\left.\left[1-\left[\left(1-I_1^{U-}\right)^{\lambda_1}\left(1-I_1^{U-}\right)^{\lambda_2}\right],1-\left[\left(1-I_1^{U+}\right)^{\lambda_1}\left(1-I_1^{U+}\right)^{\lambda_2}\right]\right]\right\},$$
$$\left\{\left[1-\left[\left(1-F_1^{L-}\right)^{\lambda_1}\left(1-F_1^{L-}\right)^{\lambda_2}\right],1-\left[\left(1-F_1^{L+}\right)^{\lambda_1}\left(1-F_1^{L+}\right)^{\lambda_2}\right]\right],\right.$$
$$\left.\left.\left[1-\left[\left(1-F_1^{U-}\right)^{\lambda_1}\left(1-F_1^{U-}\right)^{\lambda_2}\right],1-\left[\left(1-F_1^{U+}\right)^{\lambda_1}\left(1-F_1^{U+}\right)^{\lambda_2}\right]\right]\right\}\right\rangle.$$
$$=\left\langle\left\{\left[T_1^{L-},T_1^{L+}\right],\left[T_1^{U-},T_1^{U+}\right]\right\},\left\{\left[I_1^{L-},I_1^{L+}\right],\left[I_1^{U-},I_1^{U+}\right]\right\},\left\{\left[F_1^{L-},F_1^{L+}\right],\left[F_1^{U-},F_1^{U+}\right]\right\}\right\rangle^{\lambda_1+\lambda_2}.$$

$= A^{\lambda_1+\lambda_2}$. This completes the proof.

4. Conclusion
In this paper, we defined some new operations on interval neutrosophic vague set under neutrosophic environment. The basic algebraic operations on interval neutrosophic vague sets namely addition, multiplication, scalar multiplication and power along with illustrative examples were presented. Subsequently, the basic properties of these operations such as commutative law and relevant laws are mathematically proven. This new extension will broaden the fundamental knowledge of existing set theories and subsequently could be applied to real life experiments where truthness, indeterminacy and falsity could be dealth with.

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References
[1] L. A. Zadeh, “Fuzzy sets,” *Inf. Control*, vol. 8, no. 3, pp. 338–353, Jun. 1965.
[2] W.-L. Gau and D. J. Buehrer, “Vague sets,” *IEEE Trans. Syst. Man. Cybern.*, vol. 23, no. 2, pp. 610–614, 1993.
[3] Z. Pawlak, “Rough sets,” *Int. J. Comput. Inf. Sci.*, vol. 11, no. 5, pp. 341–356, Oct. 1982.
[4] U. Rivieccio, “Neutrosophic logics: Prospects and problems,” *Fuzzy Sets Syst.*, vol. 159, no. 14, pp. 1860–1868, 2008.
[5] H. Wang, F. Smarandache, Y. Zhang, and R. Sunderraman, “Single valued neutrosophic set,” in *Multispace and Multistructure*, vol. IV, 2010, pp. 410–413.
[6] P. Liu and Y. Wang, “Multiple attribute decision-making method based on single-valued neutrosophic normalized weighted Bonferroni mean,” *Neural Comput. Appl.*, vol. 25, no. 7–8, pp. 2001–2010, Dec. 2014.

[7] M. Ali and F. Smarandache, “Complex neutrosophic set,” *Neural Comput. Appl.*, vol. 28, no. 7, pp. 1817–1834, Jul. 2017.
[8] I. Deli, M. Ali, and F. Smarandache, “Bipolar neutrosophic sets and their application based on multi-criteria decision making problems,” *Int. Conf. Adv. Mechatron. Syst. ICAMechS*, vol. 2015–Octob, no. April 2016, pp. 249–254, 2015.
[9] S. Alkhazaleh, “Neutrosophic Vague Set Theory,” *Crit. Rev.*, vol. X, no. July, pp. 29–39, 2015.
[10] A. Al-Quran and N. Hassan, “Neutrosophic vague soft expert set theory,” *Artic. J. Intell. Fuzzy Syst.*, vol. 30, no. 6, pp. 3691–3702, 2016.
[11] A. Al-Quran and N. Hassan, “Neutrosophic vague soft multiset for decision under uncertainty,” *Songklanakarin J. Sci. Technolgy*, vol. 40, no. 2, pp. 290–305, 2018.
[12] S. Alias, D. Mohamad, and A. Shuib, “Rough Neutrosophic Multisets Relation with Application in Marketing Strategy,” *Neutrosophic Sets Syst.*, vol. 21, pp. 36–55, 2018.
[13] P. Kumar Maji, “Neutrosophic soft set,” *Ann. Fuzzy Math. Informatics*, vol. 5, no. 1, pp. 157–168, 2013.
[14] M. Ali, L. Q. Dat, L. H. Son, and F. Smarandache, “Interval Complex Neutrosophic Set: Formulation and Applications in Decision-Making,” *Int. J. Fuzzy Syst.*, vol. 20, no. 3, pp. 986–999, 2018.
[15] X. Peng and J. Dai, “A bibliometric analysis of neutrosophic set: two decades review from 1998 to 2017,” *Artificial Intelligence Review*, Springer Netherlands, pp. 1–57, 18-Aug-2018.
[16] F. Karaaslan and K. Hayat, “Some new operations on single-valued neutrosophic matrices and their applications in multi-criteria group decision making,” *Appl. Intell.*, vol. 48, no. 12, pp. 4594–4614, 2018.
[17] J. Ye, “Trapezoidal neutrosophic set and its application to multiple attribute decision-making,” *Neural Comput. Appl.*, vol. 26, no. 5, pp. 1157–1166, Jul. 2015.
[18] H. Hashim, L. Abdullah, and A. Al-Quran, “Interval Neutrosophic Vague Sets,” 2019.
[19] H. Wang, F. Smarandache, Y.-Q. Zhang, and R. Sunderraman, “Interval Neutrosophic Sets and Logic: Theory and Applications in Computing,” in *Hexis, Phoenix*, 2005.