Algebraic Structure of Neutrosophic Duplets in Neutrosophic Rings $\langle Z \cup I \rangle$, $\langle Q \cup I \rangle$ and $\langle R \cup I \rangle$

Vasantha W.B. [1], Ilanthenral Kandasamy [2,*], Florentin Smarandache [3]

[1] School of Computer Science and Engineering, VIT, Vellore 632014, India.

E-mail: vasantha.wb@vit.ac.in

[2] School of Computer Science and Engineering, VIT, Vellore 632014, India.

E-mail: ilanthenral.k@vit.ac.in

[3] Department of Mathematics, University of New Mexico, 705 Gurley Avenue, Gallup, NM 87301, USA.

E-mail: smarand@unm.edu

*Correspondence: Author (ilanthenral.k@vit.ac.in)

Abstract: The concept of neutrosophy and indeterminacy I was introduced by Smarandache, to deal with neutralies. Since then the notions of neutrosophic rings, neutrosophic semigroups and other algebraic structures have been developed. Neutrosophic duplets and their properties were introduced by Florentin and other researchers have pursued this study.In this paper authors determine the neutrosophic duplets in neutrosophic rings of characteristic zero. The neutrosophic duplets of $\langle Z \cup I \rangle$, $\langle Q \cup I \rangle$ and $\langle R \cup I \rangle$; the neutrosophic ring of integers, neutrosophic ring of rationals and neutrosophic ring of reals respectively have been analysed. It is proved the collection of neutrosophic duplets happens to be infinite in number in these neutrosophic rings. Further the collection enjoys a nice algebraic structure like a neutrosophic subring, in case of the duplets collection $\{a - aI | a \in Z\}$ for which $1 - I$ acts as the neutral. For the other type of neutrosophic duplet pairs $\{a - aI, 1 - dI\}$ where $a \in R^+$ and $d \in R$, this collection under component wise multiplication forms a neutrosophic semigroup. Several other interesting algebraic properties enjoyed by them are obtained in this paper.

Keywords: Neutrosophic ring; neutrosophic duplet; neutrosophic duplet pairs; neutrosophic semigroup; neutrosophic subring

1 Introduction

The concept of indeterminacy in the real world data was introduced by Florentin Smarandache [1, 2] as Neutrosophy. Existing neutralities and indeterminacies are dealt by the neutrosophic theory and are applied to real world and engineering problems [3, 4, 5]. Neutrosophic algebraic structures were introduced and studied by [6]. Since then several researchers have been pursuing their research in this direction [7, 8, 9, 10, 11, 12]. Neutrosophic rings [9] and other neutrosophic algebraic structures are elaborately studied in [6, 7, 8, 10].

Related theories of neutrosophic triplet, neutrosophic duplet, and duplet set was studied by Smarandache [13]. Neutrosophic duplets and triplets have interested many and they have studied [14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24]. Neutrosophic duplet semigroup [18], the neutrosophic triplet group [12], classical group of neutrosophic triplet groups[22] and neutrosophic duplets of $\{Z_{pn}, \times\}$ and $\{Z_{pq}, \times\}$ [23] have been recently studied.

Here we mainly introduce the concept of neutrosophic duplets in case of of neutrosophic rings of characteristic zero and study only the algebraic properties enjoyed by neutrosophic duplets, neutrals and neutrosophic duplet pairs.

In this paper we investigate the neutrosophic duplets of the neutrosophic rings $\langle Z \cup I\rangle, \langle Q \cup I\rangle$ and $\langle R \cup I\rangle$. We prove the duplets for a fixed neutral happens to be an infinite collection and enjoys a nice algebraic structure. In fact the collection of neutrals for fixed duplet happens to be infinite in number and they too enjoy a nice algebraic structure.

This paper is organised into five sections , section one is introductory in nature. Important results in this paper are given in section two of this paper. Neutrosophic duplets of the neutrosophic ring $\langle Z \cup I\rangle$, and its properties are analysed in section three of this paper. In the forth section neutrosophic duplets of the rings $\langle Q \cup I\rangle$ and $\langle R \cup I\rangle$; are defined and developed and several theorems are proved. In the final section discussions, conclusions and future research that can be carried out is described.

2 Results

The basic definition of neutrosophic duplet is recalled from [12]. We just give the notations and describe the neutrosophic rings and neutrosophic semigroups [9].

Notation: $\langle Z \cup I\rangle = \{a + bI | a, b \in Z, I^2 = I\}$ is the collection of neutrosophic integers which is a neutrosophic ring of integers. $\langle Q \cup I\rangle = \{a + bI | a, b \in Q, I^2 = I\}$ is the collection of neutrosophic rationals and $\langle R \cup I\rangle = \{a + bI | a, b \in R, I^2 = I\}$ is the collection of neutrosophic reals which are neutrosophic ring of rationals and reals respectively.

Let S be any ring which is commutative and has a unit element 1. Then $\langle S \cup I\rangle = \{a + bI | a, b \in S, I^2 = I, +, \times\}$ be the neutrosophic ring. For more refer [9].

Consider U to be the universe of discourse, and D a set in U, which has a well-defined law $\#$.

Definition 2.1. Consider $\langle a, neut(a)\rangle$, where a, and $neut(a)$ belong to D. It is said to be a neutrosophic duplet if it satisfies the following conditions:

1. $neut(a)$ is not same as the unitary element of D in relation with the law $\#$ (if any);
2. $a \#\ neut(a)$ = neut(a) $\#$ a = a;
3. $anti(a) \notin$ D for which a $\#$ anti(a) = anti(a) $\#$ a = neut(a).

The results proved in this paper are

1. All elements of the form $a - aI$ and $aI - a$ with $1 - I$ as the neutral forms a neutrosophic duplet, $a \in Z^+ \setminus \{0\}$.
2. In fact $B = \{a - aI/a \in Z \setminus \{0\}\} \cup \{0\}$, forms a neutrosophic subring of S.
3. Let $S = \{\langle Q \cup I\rangle, +, \times\}$ be the neutrosophic ring. For every nI with $n \in Q \setminus \{0\}$ we have $a + bI \in \langle Q \cup I\rangle$ with $a + b = 1; a, b \in Q \setminus \{0\}$. such that $\{nI, a + bI\}$ is a neutrosophic duplet.
4. The idempotent $x = 1 - I$ acts as the neutral for infinite collection of elements $a - aI$ where $a \in Q$.

5. For every $a - aI \in S$ where $a \in Q, 1 - dI$ acts as neutrals for $d \in Q$.

6. The ordered pair of neutrosophic duplets $B = \{(nI, m - (m-1)I); n \in R, m \in R \cup \{0\}\}$ forms a neutrosophic semigroup of $S = \langle R \cup I \rangle$ under component wise product.

7. The ordered pair of neutrosophic duplets $D = \{(a - aI, 1 - dI); a \in R^+; d \in R\}$ forms a neutrosophic semigroup under product taken component wise.

3 Neutrosophic duplets of $\langle Z \cup I \rangle$ and its properties

In this section we find the neutrosophic duplets in $\langle Z \cup I \rangle$. Infact we prove there are infinite number of neutrals for any relevant element in $\langle Z \cup I \rangle$. Several interesting results are proved.

First we illustrate some of the neutrosophic duplets in $\langle Z \cup I \rangle$.

Example 3.1. Let S = $\langle Z \cup I \rangle = \{a + bI | a, b \in I, I^2 = I\}$ be the neutrosophic ring. Consider any element $x = 9I \in \langle Z \cup I \rangle$; we see the element $16 - 15I \in \langle Z \cup I \rangle$ is such that $9I \times 16 - 15I = 144I - 135I = 9I = x$. Thus $16 - 15I$ acts as the neutral of $9I$ and $\{9I, 16 - 15I\}$ is a neutrosophic duplet.

Cconsider $15I = y \in \langle Z \cup I \rangle$; $15I \times 16 - 15I = 15I = y$. Thus $\{15I, 16 - 15I\}$ is again a neutrosophic duplet. Let $-9I = s \in \langle Z \cup I \rangle$; $-9I \times 16 - 15I = -144I + 135I = -9I = s$, so $\{-9I, 16 - 15I\}$ is a neutrosophic duplet. Thus $\{\pm 9I, 16 - 15I\}$ happens to be neutrosophic duplets.

Further $nI \in \langle Z \cup I \rangle$ is such that $nI \times 16 - 15I = 16nI - 15nI = nI$. Similarly $-nI \times 16 - 15I = -16nI + 15nI = -nI$. So $\{nI, 16 - 15I\}$ is a neutrosophic duplet for all $n \in Z \setminus \{0\}$. Another natural question which comes to one mind is will $16I - 15$ act as a neutral for nI; $n \in Z \setminus \{0\}$, the answer is yes for $nI \times (16I - 15) = 16nI - 15nI = nI$. Hence the claim.

We call $0I = 0$ as the trivial neutrosophic duplet as $(0, x)$ is a neutrosophic duplet for all $x \in \langle Z \cup I \rangle$.

In view of this example we prove the following theorem.

Theorem 3.2. *Let $S = \langle Z \cup I \rangle = \{a + bI | a, b \in Z, I^2 = I\}$ be a neutrosophic ring. Every $\pm nI \in S; n \in Z \setminus \{0\}$ has infinite number of neutrals of the form*

- $mI - (m-1) = x$
- $m - (m-1)I = y$
- $(m-1) - mI = -x$
- $(m-1)I - mI = -y$

where $m \in Z^+ \setminus \{1, 0\}$.

Proof. Consider $nI \in \langle Z \cup I \rangle$ we see

$$nI \times x = nI[mI - (m-1)] = nnI - nmI + nI = nI.$$

Thus $\{nI, mI - (m-1)\}$ form an infinite collection of neutrosophic duplets for a fixed n and varying $m \in Z^+ \setminus \{0, 1\}$. Proof for other parts (ii), (iii) and (iv) follows by a similar argument. □

Thus in view of the above theorem we can say for any $nI; n \in Z \setminus \{0\}$, n is fixed; we have an infinite collection of neutrals paving way for an infinite collection of neutrosophic duplets contributed by elements $x, y, -x$ and $-y$ given in the theorem. On the other hand for any fixed x or y or $-x$ or $-y$ given in the theorem we have an infinite collection of elements of the form $nI; n \in Z \setminus \{0\}$ such that $\{n, x,$or y or $-x$ or $-y\}$ is a neutrosophic duplet.

Now our problem is to find does these neutrals collection $\{x, y, -x, -y\}$ in theorem satisfy any nice algebraic structure in $\langle Z \cup I \rangle$.

We first illustrate this using some examples before we propose and prove any theorem.

Example 3.3. Let $S = \langle Z \cup I \rangle = \{a + bI | a, b \in Z, I^2 = I\}$ be the ring. $\{S, \times\}$ is a commutative semigroup under product []. Consider the element $x = 5I - 4 \in \langle Z \cup I \rangle$. $5I - 4$ acts as neutral for all elements $nI \in \langle Z \cup I \rangle, n \in Z \setminus \{0\}$. Consider $x \times x = 5I - 4 \times 5I - 4 = 25I - 20I - 20I + 16 = -15I + 16 = x^2$. Now $-15I + 16 \times nI = -15nI + 16nI = nI$. Thus if $\{nI, x\}$ a neutrosophic duplet so is $\{nI, x^2\}$. Consider

$$x^3 = x^2 \times x = (-15I + 16) \times (5I - 4)$$

$$= -75I + 80I + 60I - 64 = 65I - 64 = x^3$$

$$nI \times x^3 = 65nI - 64nI = nI$$

So $\{nI, 65I - 64\} = \{nI, x^3\}$ is a neutrosophic duplet for all $n \in Z \setminus \{0\}$ Consider

$$x^4 = x^3 \times x = 65I - 64 \times 5I - 4$$

$$= 325I - 320I - 260I + 256 = -255I + 256 = x^4$$

Clearly

$$nI \times x^4 = nI \times (-255I + 25) = -255nI + 256nI = nI.$$

So $\{nI, x^4\}$ is a neutrosophic duplet. In fact one can prove for any $nI \in \langle Z \cup I \rangle; n \in Z \setminus \{0\}$ then $x = m - (m-1)I$ is the neutral of nI then $\{nI, x\}, \{nI, x^2\}, \{nI, x^3\}, \ldots, \{nI, x^r\}, \ldots, \{nI, x^t\}; t \in Z^+ \setminus \{0\}$ are all neutrosophic duplets for nI. Thus for any fixed nI there is an infinite collection of neutrals. We see if x is a neutral then the cyclic semigroup generated by x denoted by $\langle x \rangle = \{x, x^2, x^3, \ldots\}$ happens to be a collection of neutrals for $nI \in S$.

Now we proceed onto give examples of other forms of neutrosophic duplets using $\langle Z \cup I \rangle$.

Example 3.4. Let $S = \{\langle Z \cup I \rangle = \{a + bI | a, b \in Z, I^2 = I\}, +, \times\}$ be a neutrosophic ring. We see $x = 1 - I \in S$ such that

$$(1 - I)^2 = 1 - I \times 1 - I = 1 - 2I + I^2 (\because I^2 = I)$$

$$= 1 - I = x.$$

Thus x is an idempotent of S. We see $y = 5 - 5I$ such that

$$y \times x = (5 - 5I) \times (5 - 5I) = 5 - 5I - 5I + 5I = 5 - 5I = y$$

Thus $\{5 - 5I, 1 - I\}$ is a neutrosophic duplets and $1 - I$ is the neutral of $5 - 5I$.

$$y^2 = 5 - 5I \times 5 - 5I = 25 - 25I - 25I + 25I = 25 - 25I$$

We see $\{y^2, 1-I\}$ is again a neutrosophic duplet.

$$y^3 = y \times y^2 = 5 - 5I \times (25 - 25I) = 125 - 125I - 125I + 125I$$
$$= 125 - 125I = y^3$$

Once again $\{y^3, 1-I\}$ is a neutrosophic duplet. In fact we can say for the idempotent $1-I$ the cyclic semigroup $B = \{y, y^2, y^3, \ldots\}$ is such that for every element in B, $1-I$ serves as the neutral.

In view of all these we prove the following theorem.

Theorem 3.5. *Let $S = \{\langle Z \cup I\rangle, +, \times\}$ be the neutrosophic ring.*

1. *$1-I$ is an idempotent of S.*
2. *All elements of the form $a - aI$ and $aI - a$ with $1-I$ as the neutral forms a neutrosophic duplet, $a \in Z^+ \setminus \{0\}$.*
3. *In fact $B = \{a - aI/a \in Z \setminus \{0\}\} \cup \{0\}$, forms a neutrosophic subring of S.*

Proof. 1. Let $x = 1 - I \in S$ to show x is an idempotent of S, we must show $x \times x = x$. We see $1 - I \times 1 - I = 1 - 2I + I^2$ as $I^2 = I$, we get $1 - I \times 1 - I = 1 - I$; hence the claim.

2. Let $a - aI \in S; a \in Z$. $1 - I$ is the neutral of $a - aI$ as $a - aI \times 1 - I = a - aI - aI + aI = a - aI$. Thus $\{a - aI, 1 - I\}$ is a neutrosophic duplet. On similar lines $aI - a$ will also yield a neutrosophic duplet with $1 - I$. Hence the result (ii).

3. Given $B = \{a - aI | a \in Z\}$. To prove B is a group under +. Let $x = a - aI$ and $y = b - bI \in B$; $x + y = a - aI + b - bI = (a + b) - (a + b)I$ as $a + b \in Z$; $a + b - (a + b)I \in B$. So B is closed under the operation +. When $a = 0$ we get $0 - 0I =\in B$ and $a - aI + 0 = a - aI$. 0 acts as the additive identity of B. For every $a - aI \in B$ we have

$$-(a - aI) = (-a) - (-a)I = -a + aI \in B$$

is such that $a - aI + (-a) + aI = 0$ so every $a - aI$ has an additive inverse. Now we show $\{B, \times\}$ is a semigroup under product $\times$.

$$(a - aI) \times (b - bI) = ab - abI - baI + abI = ab - abI \in B.$$

Thus B is a semigroup under product. Clearly $1 - I \in B$. Now we test the distributive law. let $x = a - aI, y = b - bI$ and $z = c - cI \in B$.

$$(a - aI) \times [b - bI + c - cI] = a - aI \times [(b + c) - (b + c)I$$

$$= a(b + c) - aI(b + c) - (b + c)aI + a(b + c)I = a(b + c) - aI(b + c) \in B$$

Thus $\{B, +, \times\}$ is a neutrosophic subring of S. Finally we prove $\langle Z \cup I\rangle$ has neutrosophic duplets of the form $\{a - aI, 1 + dI\}; d \in Z \setminus \{0\}$.

□

Theorem 3.6. *Let $S = \{\langle Z \cup I \rangle = \{a + bI | a, b \in Z, I^2 = I\}, +, \times\}$ be a neutrosophic ring $a + bI \in S$ contributes to a neutrosophic duplet if and only if $a = -b$.*

Proof. Let $a + bI \in S(a \neq 0, b \neq 0)$ be an element which contributes a neutrosophic duplet with $c + dI \in S$. If $\{a + bI, c + dI\}$ is a neutrosophic duplet then $(a + bI) \times (c + dI) = a + bI$, this implies

$$ac + (bd + ad + bc)I = a + bI.$$

This implies $ac = a$ and $bd + ad + bc = b$. $ac = a$ implies $a(c-1) = 0$ since $a \neq 0$ we have $c = 1$. Now in $bd + ad + bc = b$ substitute $c = 1$; it becomes $bd + ad + b = b$ which implies $bd + ad = 0$ that is $(b + a)d = 0$; $d \neq 0$ for if $d = 0$ then $c + dI = 1$ acts as a neutral, for all $a + bI \in S$ which is a trivial neutrosophic duplet. Thus $d \neq 0$, which forces $a + b = 0$ or $a = -b$. hence $a + bI = a - aI$. Now we have to find d. We have $(a - aI)(1 + dI) = a - aI + adI - adI = a - aI$.

This is true for any $d \in Z \setminus \{0\}$. Proof of the converse is direct. □

Next we proceed on to study neutrosophic duplets of $\langle Q \cup I \rangle$ and $\langle R \cup I \rangle$

4 Neutrosophic Duplets of $\langle Q \cup I \rangle$ and $\langle R \cup I \rangle$

In this section we study the neutrosophic duplets of the neutrosophic rings $\langle Q \cup I \rangle = \{a + bI | a, b \in Q, I^2 = I\}$; where Q the field of rationals and $\langle R \cup I \rangle = \{a + bI | a, b, \in R, I^2 = I\}$; where R is the field of reals. We obtain several interesting results in this direction. It is important to note $\langle Z \cup I \rangle \subset \langle Q \cup I \rangle \subset \langle R \cup I \rangle$. Hence all neutrosophic duplets of $\langle Z \cup I \rangle$ will continue to be neutrosophic duplets of $\langle Q \cup I \rangle$ and $\langle R \cup I \rangle$. Our analysis pertains to the existence of other neutrosophic duplets as Z is only a ring where as Q and R are fields. We enumerate many interesting properties related to them.

Example 4.1. Let $S = \{\langle Q \cup I \rangle = \{a + bI | a, b \in Q, I^2 = I\}, +, \times\}$ be the neutrosophic ring of rationals. Consider for any $nI \in$S we have the neutral

$$x = \frac{-7I}{9} + \frac{16}{9} \in S,$$

such that

$$nI \times x = nI \left(\frac{-7I}{9} + \frac{16}{9} \right) = nI.$$

Thus for the element nI the neutral is

$$\frac{-7I}{9} + \frac{16}{9} \in S.$$

We make the following observation

$$\frac{-7}{9} + \frac{16}{9} = 1.$$

In fact all elements of the form $a + bI$ in $\langle Q \cup I \rangle$ with $a + b = 1$; $a, b \in Q \setminus \{0\}$ can act as neutrals for nI. Suppose

$$x = \frac{8I}{9} + \frac{1}{9} \in \langle Q \cup I \rangle$$

then for $nI = y$ we see

$$x \times y = nI \times \left(\frac{8I}{9} + \frac{1}{9}\right) = \frac{8In}{9} + \frac{nI}{9} = nI.$$

Take $x = -9I + 10$ we see

$$x \times y = -9I + 0 \times nI = -9In + 10nI = nI$$

and so on.

However we have proved in section 3 of this paper for any $nI \in \langle Z \cup I \rangle$ the collection of all elements $a + bI \in \langle Z \cup I \rangle$ with $a + b = 1; a, b \in Z \setminus \{0\}$ will act as neutrals of nI.

In view of all these we put forth the following theorem.

Theorem 4.2. *Let $S = \{\langle Q \cup I \rangle, +, \times\}$ be the neutrosophic ring. For every nI with $n \in Q \setminus \{0\}$ we have $a + bI \in \langle Q \cup I \rangle$ with $a + b = 1; a, b \in Q \setminus \{0\}$. such that $\{nI, a + bI\}$ is neutrosophic duplet.*

Proof. Given $nI \in \langle Q \cup I \rangle; n \in Q \setminus \{0\}$, we have to show $a + bI$ is a neutral where $a + b = 1, a, b, \in Q \setminus \{0\}$. consider

$$nI \times (a + bI) = anI + bnI = (a + b)nI = nI$$

as $a + b = 1$. Hence for any fixed $nI \in \langle Q \cup I \rangle$ we have an infinite collection of neutrals. Further the number of such neutrosophic duplets are infinite in number for varying n and varying $a, b \in Q \setminus \{0\}$ with $a + b = 1$. Thus the number of neutrosophic duplets in case of neutrosophic ring $\langle Q \cup I \rangle$ contains all the neutrosophic duplets of $\langle Z \cup I \rangle$ and the number of neutrosophic duplets in $\langle Q \cup I \rangle$ is a bigger infinite than that of the neutrosophic duplets in $\langle Z \cup I \rangle$. Further all $a + bI$ where $a, b \in Q \setminus Z$ with $a + b = 1$ happens to contribute to neutrosophic duplets which are not in $\langle Z \cup I \rangle$. □

Now we proceed on to give other types of neutrosopohic duplets in $\langle Q \cup I \rangle$ using $1 - I$ the idempotent which acts as neutral. Consider

$$x = \frac{5}{3} - \frac{5I}{3} \in \langle Q \cup I \rangle$$

let $y = 1 - I$, we find

$$x \times y = \frac{5}{3} - \frac{5I}{3} \times 1 - I = \frac{5}{3} - \frac{5I}{3} - \frac{5I}{3} + \frac{5I}{3} = \frac{5}{3} - \frac{5I}{3} = x.$$

In view of this we propose the following theorem.

Theorem 4.3. *Let $S = \{\langle Q \cup I \rangle = \{a + bI | a, b \in Q, I^2 = I\}, +, \times\}$ be the neutrosophic ring of rationals.*

1. *The idempotent $x = 1 - I$ acts as the neutral for infinite collection of elements $a - aI$ where $a \in Q$.*

2. *For every $a - aI \in S$ where $a \in Q, 1 - dI$ acts as neutrals for $d \in Q$.*

Proof. Consider any $a - aI = x \in \langle Q \cup I \rangle; a \in Q$ we see for $y = 1 - I$ the idempotent in $\langle Q \cup I \rangle$.

$$x \times y = a - aI \times 1 - I = a - aI - aI + aI = a - aI = x.$$

Thus $1-I$ acts as the neutral for $a-aI$; in fact $\{a-aI, 1_I\}$ is a neutrosophic duplet; for all $a \in Q$. Now consider $s = p - pI$ where $p \in Q$ and $r = 1 - dI \in \langle Q \cup I \rangle; d \in Q$.

$$S \times r = p - pI \times 1 - dI = p - pI - pdI + pdI = p - pI = s$$

Thus $\{p - pI, 1 - dI\}$ are neutrosophic duplets for all $p \in Q$ and $d \in Q$. The collection of neutrosophic duplets which are in $\langle Q \cup I \rangle \setminus \{\langle Z \cup I \rangle\}$ is in fact is of infinite cardinality. □

Next we search of other types of neutrosophic duplets in $\{\langle Q \cup I \rangle\}$. Suppose $a + bI \in \langle Q \cup I \rangle$ and let $c + dI$ be the possible neutral for it, we arrive the conditions on a, b, c and d

$$(a + bI) \times (c + dI) = a + bI$$

$$ac + bc + adI + bdI = a + bI$$

$ac = a$ which is possible if and only if $c = 1$. Hence

$$b + ad + bd = b$$

$$ad + bd = 0$$

$$d(a + b) = 0$$

as $d \neq 0$;

$$a = -b.$$

Thus $a + bI = a - aI$ are only possible elements in $\langle Q \cup I \rangle$which can contribute to neutrosophic duplets and the neutrals associated with them is of the form $1 \pm dI$ and $d \in Q \setminus \{0\}$. Thus we can say even in case of R the field of reals and for the associated neutrosophic ring $\langle R \cup I \rangle$. All results are true in case $\langle Q \cup I \rangle$ and $\langle Z \cup I \rangle$; expect $\langle R \cup I \rangle \setminus \langle Q \cup I \rangle$ has infinite duplets and $\langle R \cup I \rangle$ has infinitely many more neutrosophic duplets than $\langle Q \cup I \rangle$.

The following theorem on real neutrsophic rings is both innovative and intersting.

Theorem 4.4. *Let $S = \langle R \cup I \rangle$ be the real neutrosophic ring. The neutrosophic duplets are contributed only by elements of the form nI and $a - aI$ where $n \in R$ and $a \in R^+$ with neutrals $m - (m-1)I$ and $1 - dI$; $m, d \in R$ respectively.*

Proof. Consider $\{nI, m(m-1)I\}$ the pair

$$nI \times m - (m-1)I = nmI$$

$$-nmI + nI = nI$$

for all $n, m \in R \setminus \{1, 0\}$. Thus $\{nI, m - (m-1)I\}$ is an infinite collection of neutrosophic duplets. We define $(nI, m - (m-1)I)$ as a neutrosophic duplet pair. Consider the pair $\{(a - aI), (1 - dI)\}; a \in R^+, d \in R$. We see

$$a - aI \times 1 - dI = a - aI - daI + adI = a - aI$$

Thus $\{(a - aI), (1 - dI)\}$ forms an infinite collection of neutrosophic duplets. We call $((a - aI), (1 - dI))$ as a neutrosophic duplet pair. Hence the theorem. □

Theorem 4.5. *Let $S = \langle R \cup I \rangle$ be the neutrosophic ring*

1. *The ordered pair of neutrosophic duplets $B = \{(nI, m - (m-1)I); n \in R, m \in R \cup \{0\}\}$ forms a neutrosophic semigroup of $S = \langle R \cup I \rangle$ under component wise product.*
2. *The ordered pair of neutrosophic duplets $D = \{(a - aI, 1 - dI); a \in R^+; d \in R\}$ form a neutrosophic semigroup under product taken component wise.*

Proof. Given $B = \{(nI, m - (m-1)I | n \in R, m \in (R \setminus \{1\})\} \cup (nI, 0) \subseteq (\{\langle R \cup I \rangle\}, \{\langle R \cup I \rangle\})$. To prove B is a neutrosophic semigroup of $(\langle R \cup I \rangle, \langle R \cup I \rangle)$.. For any $x = (nI, (m - (m-1)I)$ and $y = (sI, t - 9t - 1)I) \in B$ we prove $xy = yx \in B$

$$x \times y = xy = (nI, m - (m-1)I \times (sI, t - (t-1)I)$$

$$= (nsI, [m - (m-1)I] \times [t - (t-1)I])$$
$$(nsI, mt - t(m-1)I - m(t-1)I + (m-1)(t-1)I)$$
$$= (nsI, mt - (mt-1)I) \in B$$

It is easily verified $xy = yx$ for all $x, y \in B$. Thus $\{B, \times\}$ is a neutrosophic semigroup of neutrosophic duplet pairs. Consider $x, y \in D$; we show $x \times y \in D$. Let $x = (a - aI, 1 - dI)$ and $y = (b - bI, 1 - cI) \in D$

$$x \times y = (a - aI, 1 - dI) \times (b - bI, 1 - cI)$$

$$= (a - aI \times b - bI, (-aI \times 1 - cI)$$
$$= (ab - abI - abI + abI, 1 - dI - cI + cdI)$$
$$= (ab - abI, 1 - (d + c - cd)I) \in D$$

as $x \times y$ is also in the form of x and y. Hence D the neutrosophic duplet pairs forms a neutrosophic semigroup under component wise product. □

5 Discussions and Conclusions

In this paper the notion of duplets in case neutrosophic rings, $\langle Z \cup I \rangle$, $\langle Q \cup I \rangle$ and $\langle R \cup I \rangle$, have been introduced and analysed. It is proved that the number of neutrosophic duplets in all these three rings happens to be an infinite collection. We further prove there are infinitely many elements for which $1 - I$ happens to be the neutral. Here we establish the duplet pair $\{a - aI, 1 - dI\}; a \in R^+$ and $d \in R$ happen to be a neutrosophic semigroup under component wise product. The collection $\{a - aI\}$ forms a neutrosophic subring $a \in Z$ or Q or R. For future research we want to analyse whether these neutrosophic rings can have neutrosophic triplets and if that collections enjoy some nice algebraic property. Finally we leave it as an open problem to find some applications of these neutrosophic duplets which form an infinite collection.

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