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Application of the Bipolar Neutrosophic Hamacher Averaging Aggregation Operators to Group Decision Making: An Illustrative Example

Muhammad Jamil [1], Saleem Abdullah [2,*], Muhammad Yaqub Khan [1], Florentin Smarandache [3] and Fazal Ghani [2]

1 Department of Mathematics and Statistics, Riphah International University, Islamabad 44000, Pakistan; 10616@students.riu.edu.pk (M.J.); muhammad.yaqub@riphah.edu.pk (M.Y.K.)

2 Department of Mathematics, Abdul Wali Khan University, Mardan 23200, Pakistan; fazalghanimaths@gmail.com

3 Department of Mathematics and Sciences, University of New Mexico, 705 Gurley Ave., Gallup, NM 87301, USA; fsmarandache@gmail.com

* Correspondence: saleemabdullah@awkum.edu.pk

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Abstract: The present study aims to introduce the notion of bipolar neutrosophic Hamacher aggregation operators and to also provide its application in real life. Then neutrosophic set (NS) can elaborate the incomplete, inconsistent, and indeterminate information, Hamacher aggregation operators, and extended Einstein aggregation operators to the arithmetic and geometric aggregation operators. First, we give the fundamental definition and operations of the neutrosophic set and the bipolar neutrosophic set. Our main focus is on the Hamacher aggregation operators of bipolar neutrosophic, namely, bipolar neutrosophic Hamacher weighted averaging (BNHWA), bipolar neutrosophic Hamacher ordered weighted averaging (BNHOWA), and bipolar neutrosophic Hamacher hybrid averaging (BNHHA) along with their desirable properties. The prime gain of utilizing the suggested methods is that these operators progressively provide total perspective on the issue necessary for the decision makers. These tools provide generalized, increasingly exact, and precise outcomes when compared to the current methods. Finally, as an application, we propose new methods for the multi-criteria group decision-making issues by using the various kinds of bipolar neutrosophic operators with a numerical model. This demonstrates the usefulness and practicality of this proposed approach in real life.

Keywords: BNHWA aggregation operator; BNHOWA aggregation operator; BNHHA aggregation operator; score function; accuracy function; certainty function; group decision making

1. Introduction

In the recent era of decision making, there is often incomplete, indeterminate, and inconsistent information. Zadeh introduced the notion of fuzzy set [1], which deals with uncertainty and can be applied in many fields. However, it has a shortcoming, i.e., it only expresses membership value and is unable to express non-membership value. At that point, Atanassov [2] introduced the idea of intuitionistic fuzzy set (IFS) to address issues with the fuzzy set. Every component in IFS is shown by a structured pair, and every pair is portrayed by a membership value (truth-membership) $\zeta_A(p)$ and a non-membership value (falsity-membership) $\text{IO}_A(p)$ that satisfy the conditions $\zeta(p), \text{IO}(p) \in [0,1]$ and $0 \leq \zeta(p), \text{IO}(p) \leq 1$. IFSs can deal with incomplete data but cannot deal with the indeterminate and inconsistent data. Smarandache [3] developed the neutrosophic set (NS) by including an indeterminacy membership value $\acute{\Gamma}(p)$, which is a generality of IFS. NS can deal with information very effectively, i.e., incomplete, indeterminate, and inconsistent. When $\zeta(p) + \Gamma'(p) + \text{IO}(p) < 1$, it shows that this information is indeterminate and when $\zeta(p) + \Gamma'(p) + \text{IO}(p) > 1$, it is inconsistent information.

Single valued neutrosophic set (SVNS), as suggested by Wang et al. [4], was applied to decision making with the conditions $\zeta(p), \Gamma'(p), \text{IO}(p) \in [0,1]$ and $0 \leq \zeta(p) + \Gamma'(p) + \text{IO}(p) \leq 3$. Ye [5] introduced the correlation coefficient and also proposed the comparison method for SVNSs. Wang et al. [6] proposed the interval valued SVNSs to extend the truth, indeterminacy, and false membership to interval values. Ye [7] defined the similarity measures between interval valued neutrosophic sets on the basis of the Hamming and Euclidean distances and also proposed a multiple attribute decision-making method.

Aggregation operators are the important research areas, claiming the attention of today's researchers. Since the proposed theory of IFS, many scientists [8–15] have made essential contributions to the advancement of IFS theory. XU and Yager [12] developed the notion of aggregation operators based on IFS. They also applied these aggregation operators to decision making. Wang and Liu [16], proposed the idea of Einstein aggregation operators. Zhao and Wei [17] built up some of the Einstein hybrid aggregation operators. Fahmi et al. [18–21] developed aggregation operators based on triangular and trapezoidal cubic fuzzy numbers with applications to decision making. Rahman et al. [22–25] proposed aggregation operators on different extension of fuzzy numbers. The bipolar fuzzy set (BFS) [26–28] uses a substitute method to deal with uncertainty in decision making. The bipolar fuzzy set consists of a positive as well as negative membership degree. The membership degree of the bipolar fuzzy set ranges from −1 to 1. BFSs have been useful in various research domains and set theory decision analysis and organizational modeling [29], quantum computing [30], physics and philosophy [31], and graph theory [32]. Bipolar averaging and geometric fuzzy aggregations operators were defined by Gul [33]. Irfan et al. [34] presented the bipolar neutrosophic set with basic operations. They also proposed the comparison method for bipolar neutrosophic sets. Fan et al. [35] developed Heronian mean operators in a bipolar neutrosophic environment. Irfan et al. [36] presented the interval valued bipolar neutrosophic set with applications to pattern recognition. Irfan et al. [37] proposed the interval valued neutrosophic soft set with applications to decision making. Zhan et al. [38] proposed Schweizer-Sklar Muirhead mean aggregation operators based on single-valued neutrosophic set. Ashraf et al. introduced some logarithmic aggregation operators on neutrosophic sets [39].

Hamacher t-norm and t-conorm [40], which are the generalization of algebraic and Einstein t-norm and t-conorm, are more general and flexible. There are many researchers who extended the Hamacher operations to solve multiple attribute decision-making problems combined with other fuzzy environments, such as intuitionistic [41], interval valued intuitionistic [42], hesitant fuzzy [43], hesitant Pythagorean fuzzy [44], bipolar fuzzy numbers [45] and neutrosophic numbers [46,47]. Since the development of this field, there has been no significant research on Hamacher operations and its applicability to bipolar neutrosophic numbers. Here, we extended the Hamacher operations to bipolar neutrosophic numbers to develop bipolar neutrosophic Hamacher aggregation operators for multiple attribute decision-making problems.

Decision making plays a key role in present day management. Necessarily, sound decision making is a basic part of administration. Consciously or unconsciously, a manager makes a decision, or decisions, as it is his or her responsibility as a manager. Decision making has a consequential role as organizational and managerial activities are linked with that decision. A decision is explained as a sequence of actions, intentionally taken from a set of alternates, to accomplish managerial or organizational targets. The decision-making process is an incessant and obligatory part of organization management or activities that are carried out in business. Decisions are made to address the events of all activities related to business and organizations. Decision and economic theories are interconnected with the assumption that field experts make a decision to make the best use out of their personal interest and reasonableness. This, though, doesn't take into consideration the probabilities of intervening factors that make decision making dependent upon the situation. The aforementioned factors play a key role in normalizing decision making for the manager to achieve optimal targets.

Although there is much literature available regarding the present study, the points given below, connected to the bipolar neutrosophic set and its aggregation operator, motivated the researcher to construct a detail and deep inquiry in the present study. Our main rationalizations and are as follows:

(1) Single valued neutrosophic sets (SVNSs) help in dealing with uncertain information in a more reliable way. It is a generalization of classical set, fuzzy set, and intuitionistic fuzzy set etc., and adaptable to the framework in comparison with pre-existing fuzzy sets and its versions.
(2) To deal with uncertain real-life problems, bipolar fuzzy sets are of great value and prove to be helpful in dealing with the positive as well as the negative membership values.
(3) We tried to merge these ideas with the Hamacher aggregation operator and strive to develop a more effective tool to deal with uncertainty in the form of bipolar neutrosophic Haymaker averaging aggregation operators.
(4) The objective of the study was to propose bipolar neutrosophic Hamacher operators and also study its properties. Furthermore, we proposed three aggregation operators, namely bipolar neutrosophic Hamacher weighted averaging operators (BNHWA), and bipolar neutrosophic Hamacher ordered weighted averaging operators (BNHOWA) and bipolar neutrosophic Hamacher hybrid averaging operators (BNHHA). Multi-attribute decision making (MADM) program approach is established based on bipolar neutrosophic numbers
(5) So as to affirm the effectiveness of the proposed method, we applied bipolar neutrosophic numbers to the decision-making problem.
(6) The initial decision matrices were composed of bipolar neutrosophic numbers and transformed into a collective bipolar neutrosophic decision matrix.
(7) The proposed operators probably completely elaborate the vagueness of bipolar neutrosophic Hamacher aggregation operator.

The rest of the study is organized as follows:

- Section 2 comprises the fundamental definitions and their related properties, which are required later in paper.
- In Section 3 we introduce the BNHWA operator, BNHWOA operator and BNHHA operator.
- In Section 4, the new aggregation operators are applied to group decision making and we propose a numerical problem.
- In Section 5, there is a comparison of our method in relation to other methods and concluding remarks are also given.

2. Preliminaries

Fundamental definitions of neutrosophic set are provided in the current section for bipolar fuzzy set, bipolar neutrosophic set, score function, accuracy function, certainty function and Hamacher operations.

Definition 1. *[3] Let P be any fixed set. Then a neutrosophic set (NS) is as follows:*

$$B = \left\{\left(p, \zeta(p), \Gamma'(p), \text{Ю}(p)\right)\middle| p \in P\right\}, \tag{1}$$

where the truth-membership is $\zeta : H \to E$*, the indeterminacy-membership is* $\Gamma' : H \to E$*, and the falsity-membership is* $\text{Ю} : H \to E$*, where* $E =]0^-, 1^+[$*.* $\zeta(p), \Gamma'(p)$ *and* $\text{Ю}(p)$ *are real standard or non-standards subsets of* $]0^-, 1^+[$*, which was proposed by Abraham Robinson in 1966 [48]. There is no restriction on the sum of* $\zeta(p), \Gamma'(p)$ *and* $\text{Ю}(p)$*, so* $0^- \leq \zeta(p) + \Gamma'(p) + \text{Ю}(p) \leq 3^+$*.*

As it is difficult to apply NS in real scientific and engineering areas, Wang et al. [4] proposed the concept of a single valued neutrosophic set (SVNS), which follows.

Definition 2. [4] *Let P be a non-empty set, with an element in P denoted by p, and then the single valued neutrosophic set (SVNS) of A in H is as follows:*

$$A_{NS} = \left\{ \left(p, \zeta(p), \Gamma(p), \text{Ю}(p) \right) \middle| p \in P \right\}, \tag{2}$$

where the truth-membership is $\zeta : H \to N$*, the indeterminacy-membership is* $\Gamma : H \to N$*, and the falsity-membership is* $\text{Ю} : H \to N$*, where* $N = [0, 1]$*. There is one condition, i.e.,*

$$0 \leq \zeta(p) + \Gamma(p) + \text{Ю}(p) \leq 3.$$

The basic operations for two SVNSs are:

$$A_{NS} = \left\{ \left(p, \zeta_A(p), \Gamma_A(p), \text{Ю}_A(p) \right) \middle| p \in P \right\},\ \text{B}_{NS} = \left\{ \left(p, \zeta_B(p), \Gamma_B(p), \text{Ю}_B(p) \right) \middle| p \in P \right\},$$

and are given as follows:

i. The subset $A_{NS} \subseteq B_{NS}$ if, and only if,

$$\zeta_A(p) \leq \zeta_B(p), \Gamma_A(p) \geq \Gamma_B(p), \text{Ю}_A(p) \geq \text{Ю}_B(p).$$

ii. $A_{NS} = B_{NS}$ if, and only if,

$$\zeta_A(p) = \zeta_B(p), \Gamma_A(p) = \Gamma_B(p), \text{Ю}_A(p) = \text{Ю}_B(p).$$

iii. the complement A'_{NS} is

$$A'_{NS} = \left\{ \left(p, \text{Ю}_A(p), 1 - \Gamma_A(p), \zeta_A(p) \right) \middle| p \in P \right\}.$$

iv. the intersection is defined by

$$A_{NS} \cap B_{NS} = \left\{ \left(p, \min\{\zeta_A(p), \zeta_B(p)\}, \max\left\{\Gamma_A(p), \Gamma_B(p)\right\}, \max\{\text{Ю}_A(p), \text{Ю}_B(p)\} \right) \middle| p \in P \right\},\ \text{and}$$

v. the union is defined by

$$A_{NS} \cup B_{NS} = \left\{ \left(p, \max\{\zeta_A(p), \zeta_B(p)\}, \min\left\{\Gamma_A(p), \Gamma_B(p)\right\}, \min\{\text{Ю}_A(p), \text{Ю}_B(p)\} \right) \middle| p \in P \right\},$$

Definition 3. [49] *Let* $u_1 = \left(\zeta_1, \Gamma_1, \text{Ю}_1 \right)$ *and* $u_2 = \left(\zeta_2, \Gamma_2, \text{Ю}_2 \right)$ *be two single-value neutrosophic numbers (SVNNs). Then, the operations for the SVNNs are as follows:*

i. $u_1 + u_2 = \left(\zeta_1 + \zeta_2 - \zeta_1\zeta_2, \Gamma_1\Gamma_2, \text{Ю}_1\text{Ю}_2 \right),$

ii. $u_1 \cdot u_2 = \left(\zeta_1 + \zeta_2, \Gamma_1 + \Gamma_2 - \Gamma_1\Gamma_2, \text{Ю}_1 + \text{Ю}_2 - \text{Ю}_1\text{Ю}_2 \right),$

iii. $\lambda(u_1) = \left(1 - (1 - \zeta_1)^\lambda, \left(\Gamma_1\right)^\lambda, (\text{Ю}_1)^\lambda \right)$ *and*

iv. $(u_1)^\lambda = \left((\zeta_1)^\lambda, 1 - \left(1 - \Gamma_1\right)^\lambda, 1 - (1 - \text{Ю}_1)^\lambda \right),$

where $\lambda > 0$.

Definition 4. [49] *Let* $u_1 = \left(\zeta_1, \Gamma_1, \text{Ю}_1 \right)$ *be a SVNN. Then, the score function* $s(u_1)$ *is as follows:*

$$s(u_1) = \frac{\left(\zeta_1 + 1 - \Gamma_1 + 1 - \text{Ю}_1 \right)}{3}.$$

Definition 5. [49] *Let* $u_1 = \left(\zeta_1, \text{Ґ}_1, \text{Ю}_1\right)$ *be a SVNN. Then, the accuracy function* $a(u_1)$ *is as follows:*

$$\mathrm{a}(u_1) = (\zeta_1 - \text{Ю}_1).$$

Definition 6. [49] *Let* $u_1 = \left(\zeta_1, \text{Ґ}_1, \text{Ю}_1\right)$ *be a SVNN. Then, the certainty function* $c(u_1)$ *is as follows:*

$$c(u_1) = \zeta_1 \,.$$

Definition 7. [49] *Let* $u_1 = \left(\zeta_1, \text{Ґ}_1, \text{Ю}_1\right)$ *and* $u_2 = \left(\zeta_2, \text{Ґ}_2, \text{Ю}_2\right)$ *be two SVNNs. Then, the comparison is as follows:*

i. *if* $s(u_1) > s(u_2)$, *then* u_1 *is greater than* u_2, *denoted by* $u_1 > u_2$,
ii. *if* $s(u_1) = s(u_2)$ *and* $a(u_1) > a(u_2)$, *then* u_1 *is greater than* u_2, *denoted by* $u_1 > u_2$,
iii. *if* $s(u_1) = s(u_2)$, $a(u_1) = a(u_2)$ *and* $c(u_1) > c(u_2)$, *then* u_1 *is greater than* u_2, *denoted by* $u_1 > u_2$ *and*
iv. *if* $s(u_1) = s(u_2)$, $a(u_1) = a(u_2)$ *and* $c(u_1) = c(u_2)$, *then* u_1 *is equal to* u_2, *denoted by* $u_1 = u_2$.

Definition 8. [28] *Let P be a fixed set, and the bipolar fuzzy set is as follows:*

$$F = \left\{\left\langle p, \mu_F^+(p), \eta_F^-(p)\right\rangle \middle| p \in P\right\}, \tag{3}$$

where the positive degree of membership is $\mu_F^+(p) : H \to N$ *and the negative degree of membership is* $\eta_F^-(p) : H \to M$, *where* $N = [1, 0]$ *and* $M = [-1, 0]$.

Definition 9. [34] *A bipolar neutrosophic set (BNS), A in P, is as follows:*

$$A = \left\{\left(p, \zeta^+(p), \text{Ґ}^+(p), \text{Ю}^+(p), \zeta^-(p), \text{Ґ}^-(p), \text{Ю}^-(p)\right) \middle| p \in P\right\}, \tag{4}$$

Let $\zeta^+(p), \text{Ґ}^+(p), \text{Ю}^+(p) = BN^+$ and $\zeta^-(p), \text{Ґ}^-(p), \text{Ю}^-(p) = BN^-$, where, $\zeta^+(p), \text{Ґ}^+(p), \text{Ю}^+(p)$ is the positive degree of truth, the indeterminate and false membership of $p \in P$ and $\zeta^-(p), \text{Ґ}^-(p), \text{Ю}^-(p)$ is the negative degree of truth, the indeterminate and false membership of $p \in P$. Then $BN^+ : H \to N$ and $BN^- : H \to M$, where $N = [1, 0]$ and $M = [-1, 0]$. There are conditions where $0 \leq \zeta^+(p) + \text{Ґ}^+(p) + \text{Ю}^+(p) + \zeta^-(p) + \text{Ґ}^-(p) + \text{Ю}^-(p) \leq 6$.

Example 1. *Let* $P = \{p_1, p_2, p_3\}$, *then*

$$A = \left\{ \begin{array}{c} (p_1, 0.1, 0.5, 0.3, -0.4, -0.5, -0.6), \\ (p_2, 0.2, 0.7, 0.4, -0.3, -0.6, -0.2), \\ (p_3, 0.4, 0.6, 0.7, -0.2, -0.3, -0.1) \end{array} \right\}$$

is a bipolar neutrosophic subset of P.

Basic operations [34], for two bipolar neutrosophic sets (BNSs), are as follows:
Let

$$A_1 = \left\{\left(p, \zeta_1^+(p), \text{Ґ}_1^+(p), \text{Ю}_1^+(p), \zeta_1^-(p), \text{Ґ}_1^-(p), \text{Ю}_1^-(p)\right) \middle| p \in P\right\}$$
$$\text{and } A_2 = \left\{\left(p, \zeta_2^+(p), \text{Ґ}_2^+(p), \text{Ю}_2^+(p), \zeta_2^-(p), \text{Ґ}_2^-(p), \text{Ю}_2^-(p)\right) \middle| p \in P\right\}$$

be two BNSs, then:

i. $A_1 \subseteq A_2$ if, and only if,

$$\zeta_1^+(p) \le \zeta_2^+(p), \text{Ґ}_1^+(p) \le \text{Ґ}_2^+(p), \text{Ю}_1^+(p) \ge \text{Ю}_2^+(p)$$

and

$$\zeta_1^-(p) \ge \zeta_2^-(p), \text{Ґ}_1^-(p) \ge \text{Ґ}_2^-(p), \text{Ю}_1^-(p) \le \text{Ю}_2^-(p),$$

ii. $A_1 = A_2$ if, and only if,

$$\zeta_1^+(p) = \zeta_2^+(p), \text{Ґ}_1^+(p) = \text{Ґ}_2^+(p), \text{Ю}_1^+(p) = \text{Ю}_2^+(p)$$

and

$$\zeta_1^-(p) = \zeta_2^-(p), \text{Ґ}_1^-(p) = \text{Ґ}_2^-(p), \text{Ю}_1^-(p) = \text{Ю}_2^-(p),$$

iii. The union is defined as:

$$(A_1 \cup A_2) = \left\{\left(\max\left(\zeta_1^+(p), \zeta_2^+(p)\right), \frac{\text{Ґ}_1^+(p) + \text{Ґ}_2^+(p)}{2}, \min\left(\text{Ю}_1^+(p), \text{Ю}_2^+(p)\right), \min\left(\zeta_1^-(p), \zeta_2^-(p)\right), \frac{\text{Ґ}_1^-(p) + \text{Ґ}_2^-(p)}{2}, \max\left(\text{Ю}_1^-(p), \text{Ю}_2^-(p)\right)\right)\right\}, \text{ and}$$

iv. The intersection is defined as:

$$(A_1 \cap A_2) = \left\{\left(\min\left(\zeta_1^+(p), \zeta_2^+(p)\right), \frac{\text{Ґ}_1^+(p) + \text{Ґ}_2^+(p)}{2}, \max\left(\text{Ю}_1^+(p), \text{Ю}_2^+(p)\right), \max\left(\zeta_1^-(p), \zeta_2^-(p)\right), \frac{\text{Ґ}_1^-(p) + \text{Ґ}_2^-(p)}{2}, \min\left(\text{Ю}_1^-(p), \text{Ю}_2^-(p)\right)\right)\right\}.$$

Let $A = \left\{\left(p, \zeta^+(p), \text{Ґ}^+(p), \text{Ю}^+(p), \zeta^-(p), \text{Ґ}^-(p), \text{Ю}^-(p)\right) \middle| p \in P\right\}$ and be a BNS. Then the complement A^c is defined as:

$$\zeta_{A^c}^+(p) = \left\{1^+\right\} - \zeta_A^+(p), \text{Ґ}_{A^c}^+(p) = \left\{1^+\right\} - \text{Ґ}_A^+(p), \text{Ю}_{A^c}^+(p) = \left\{1^+\right\} - \text{Ю}_A^+(p)$$

and

$$\zeta_{A^c}^-(p) = \{1^-\} - \zeta_A^-(p), \text{Ґ}_{A^c}^-(p) = \{1^-\} - \text{Ґ}_A^-(p), \text{Ю}_{A^c}^-(p) = \{1^-\} - \text{Ю}_A^-(p).$$

Definition 10. *[34] Let* $u_1 = \left(\zeta_1^+, \text{Ґ}_1^+, \text{Ю}_1^+, \zeta_1^-, \text{Ґ}_1^-, \text{Ю}_1^-\right)$ *and* $u_2 = \left(\zeta_2^+, \text{Ґ}_2^+, \text{Ю}_2^+, \zeta_2^-, \text{Ґ}_2^-, \text{Ю}_2^-\right)$ *be two bipolar neutrosophic numbers (BNNs). Then, the operations for the BNNs are as follows:*

$$u_1 + u_2 = \left(\zeta_1^+ + \zeta_2^+ - \zeta_1^+\zeta_2^+, \text{Ґ}_1^+\text{Ґ}_2^+, \text{Ю}_1^+\text{Ю}_2^+, -\zeta_1^-\zeta_2^-, -\left(-\text{Ґ}_1^- - \text{Ґ}_2^- - \text{Ґ}_1^-\text{Ґ}_2^-\right), -\left(-\text{Ю}_1^- - \text{Ю}_2^- - \text{Ю}_1^-\text{Ю}_2^-\right)\right),$$

$$u_1 \cdot u_2 = \left(\zeta_1^+\zeta_2^+, \text{Ґ}_1^+ + \text{Ґ}_2^+ - \text{Ґ}_1^+\text{Ґ}_2^+, \text{Ю}_1^+ + \text{Ю}_2^+ - \text{Ю}_1^+\text{Ю}_2^+, -\left(-\zeta_1^- - \zeta_2^- - \zeta_1^-\zeta_2^-\right), -\text{Ґ}_1^-\text{Ґ}_2^-, -\text{Ю}_1^-\text{Ю}_2^-\right),$$

$$\lambda(u_1) = \left(1 - \left(1 - \zeta_1^+\right)^\lambda, \left(\text{Ґ}_1^+\right)^\lambda, \left(\text{Ю}_1^+\right)^\lambda, -\left(-\zeta_1^-\right)^\lambda, -\left(-\text{Ґ}_1^-\right)^\lambda, -\left(1 - \left(1 - \left(-\text{Ю}_1^-\right)\right)^\lambda\right)\right),$$

$$(u_1)^\lambda = \left(\left(\zeta_1^+\right)^\lambda, 1 - \left(1 - \text{Ґ}_1^+\right)^\lambda, 1 - \left(1 - \text{Ю}_1^+\right)^\lambda, -\left(1 - \left(1 - \left(-\zeta_1^-\right)\right)^\lambda\right), -\left(-\text{Ґ}_1^-\right)^\lambda, -\left(-\text{Ю}_1^-\right)^\lambda\right).$$

where $\lambda > 0$.

Definition 11. *[34] Let* $u = \left(\zeta^+, \text{Ґ}^+, \text{Ю}^+, \zeta^-, \text{Ґ}^-, \text{Ю}^-\right)$ *be a bipolar neutrosophic number (BNN), then the score function of u is as follows:*

$$S(u) = \frac{1}{6}\left(\zeta^+ + 1 - \text{Ґ}^+ + 1 - \text{Ю}^+ + 1 + \zeta^- - \text{Ґ}^- - \text{Ю}^-\right). \quad (5)$$

Definition 12. *[34] Let $u = \left(\zeta^{+}, \text{Ґ}^{+}, \text{Ю}^{+}, \zeta^{-}, \text{Ґ}^{-}, \text{Ю}^{-}\right)$ be a BNN, then the accuracy function of u is as follows:*

$$a(u) = \zeta^{+} - \text{Ю}^{+} + \zeta^{-} - \text{Ю}^{-}. \tag{6}$$

Definition 13. *[34] Let $u = \left(\zeta^{+}, \text{Ґ}^{+}, \text{Ю}^{+}, \zeta^{-}, \text{Ґ}^{-}, \text{Ю}^{-}\right)$ be a bipolar neutrosophic value, then the certainty function of u is as follows:*

$$c(u) = \zeta^{+} - \text{Ю}^{-}. \tag{7}$$

Definition 14. *[34] Let $u_1 = \left(\zeta_1^{+}, \text{Ґ}_1^{+}, \text{Ю}_1^{+}, \zeta_1^{-}, \text{Ґ}_1^{-}, \text{Ю}_1^{-}\right)$ and $u_2 = \left(\zeta_2^{+}, \text{Ґ}_2^{+}, \text{Ю}_2^{+}, \zeta_2^{-}, \text{Ґ}_2^{-}, \text{Ю}_2^{-}\right)$ be two BNNs, then the comparison method is as follows:*

(i) *If $S(u_1) \succ S(u_2)$, then u_1 is greater than u_2, denoted by $u_1 \succ u_2$,*
(ii) *If $S(u_1) = S(u_2)$, and $a(u_1) \succ a(u_2)$, then u_1 is superior to u_2, denoted by $u_1 \succ u_2$,*
(iii) *If $S(u_1) = S(u_2)$, $a(u_1) = a(u_2)$ and $c(u_1) \succ c(u_2)$, then u_1 is greater than u_2, denoted by $u_1 \succ u_2$ and*
(iv) *If $S(u_1) = S(u_2)$, $a(u_1) = a(u_2)$ and $c(u_1) = c(u_2)$, then u_1 is equal to u_2, denoted by $u_1 = u_2$*

Hamacher [40] proposed a more generalized t-norm and t-conorm. The Hamacher product, $\otimes$, is a t-norm and the Hamacher sum, $\oplus$, is a t-conorm, where:

$$T(a,b) = a \otimes b = \frac{ab}{\ddot{\gamma} + (1-\ddot{\gamma})(a+b-ab)}, \ddot{\gamma} > 0$$
$$T*(a,b) = a \oplus b = \frac{a+b-ab-(1-\ddot{\gamma})ab}{1-(1-\ddot{\gamma})ab}, \ddot{\gamma} > 0$$

when $\ddot{\gamma} = 1$, the Hamacher t-norm and t-conorm will be reduced to algebraic t-norm and *t*-conorm, respectively:

$$T(a,b) = a \otimes b = ab$$
$$T*(a,b) = a \oplus b = a + b - ab$$

when $\ddot{\gamma} = 2$, the Hamacher t-norm and t-conorm will be reduced to the Einstein t-norm and t-conorm, respectively [16]:

$$T(a,b) = a \otimes b = \frac{ab}{1+(1-a)(1-b)}$$
$$T*(a,b) = a \oplus b = \frac{a+b}{1+ab}$$

The following definitions introduce the Hamacher operations of bipolar neutrosophic set, as the notion of the bipolar neutrosophic Hamacher sum, product, scalar multiple and exponential operations are defined.

Definition 15. *Let $u = \left(\zeta^{+}, \text{Ґ}^{+}, \text{Ю}^{+}, \zeta^{-}, \text{Ґ}^{-}, \text{Ю}^{-}\right)$, $u_1 = \left(\zeta_1^{+}, \text{Ґ}_1^{+}, \text{Ю}_1^{+}, \zeta_1^{-}, \text{Ґ}_1^{-}, \text{Ю}_1^{-}\right)$ and $u_2 = \left(\zeta_2^{+}, \text{Ґ}_2^{+}, \text{Ю}_2^{+}, \zeta_2^{-}, \text{Ґ}_2^{-}, \text{Ю}_2^{-}\right)$ be three BNNs values, and $\lambda > 0$ be any real number, then we define basic Hamacher operators with $\ddot{\gamma} > 0$.*

$$u_1 \oplus u_2 = \left(\begin{array}{l} \frac{\zeta_1^{+}+\zeta_2^{+}-\zeta_1^{+}\zeta_2^{+}-(1-\ddot{\gamma})\zeta_1^{+}\zeta_2^{+}}{1-(1-\ddot{\gamma})\zeta_1^{+}\zeta_2^{+}}, \frac{\text{Ґ}_1^{+}\text{Ґ}_2^{+}}{\ddot{\gamma}+(1-\ddot{\gamma})\left(\text{Ґ}_1^{+}+\text{Ґ}_2^{+}-\text{Ґ}_1^{+}\text{Ґ}_2^{+}\right)}, \frac{\text{Ю}_1^{+}\text{Ю}_2^{+}}{\ddot{\gamma}+(1-\ddot{\gamma})\left(\text{Ю}_1^{+}+\text{Ю}_2^{+}-\text{Ю}_1^{+}\text{Ю}_2^{+}\right)}, \\ \frac{-\zeta_1^{-}\zeta_2^{-}}{\ddot{\gamma}+(1-\ddot{\gamma})\left(\zeta_1^{-}+\zeta_2^{-}-\zeta_1^{-}\zeta_2^{-}\right)}, \frac{-\left(-\text{Ґ}_1^{-}-\text{Ґ}_2^{-}-\text{Ґ}_1^{-}\text{Ґ}_2^{-}-(1-\ddot{\gamma})\text{Ґ}_1^{-}\text{Ґ}_2^{-}\right)}{1-(1-\ddot{\gamma})\text{Ґ}_1^{-}\text{Ґ}_2^{-}}, \frac{-\left(-\text{Ю}_1^{-}-\text{Ю}_2^{-}-\text{Ю}_1^{-}\text{Ю}_2^{-}-(1-\ddot{\gamma})\text{Ю}_1^{-}\text{Ю}_2^{-}\right)}{1-(1-\ddot{\gamma})\text{Ю}_1^{-}\text{Ю}_2^{-}} \end{array} \right) \tag{8}$$

$$u_1 \otimes u_2 = \left(\begin{array}{l} \frac{\zeta_1^{+}\zeta_2^{+}}{\ddot{\gamma}+(1-\ddot{\gamma})\left(\zeta_1^{+}+\zeta_2^{+}-\zeta_1^{+}\zeta_2^{+}\right)}, \frac{\text{Ґ}_1^{+}+\text{Ґ}_2^{+}-\text{Ґ}_1^{+}\text{Ґ}_2^{+}-(1-\ddot{\gamma})\text{Ґ}_1^{+}\text{Ґ}_2^{+}}{1-(1-\ddot{\gamma})\text{Ґ}_1^{+}\text{Ґ}_2^{+}}, \frac{\text{Ю}_1^{+}+\text{Ю}_2^{+}-\text{Ю}_1^{+}\text{Ю}_2^{+}-(1-\ddot{\gamma})\text{Ю}_1^{+}\text{Ю}_2^{+}}{1-(1-\ddot{\gamma})\text{Ю}_1^{+}\text{Ю}_2^{+}}, \\ \frac{-\left(-\zeta_1^{-}-\zeta_2^{-}-\zeta_1^{-}\zeta_2^{-}-(1-\ddot{\gamma})\zeta_1^{-}\zeta_2^{-}\right)}{1-(1-\ddot{\gamma})\zeta_1^{-}\zeta_2^{-}}, \frac{-\text{Ґ}_1^{-}\text{Ґ}_2^{-}}{\ddot{\gamma}+(1-\ddot{\gamma})\left(\text{Ґ}_1^{-}+\text{Ґ}_2^{-}-\text{Ґ}_1^{-}\text{Ґ}_2^{-}\right)}, \frac{-\text{Ю}_1^{-}\text{Ю}_2^{-}}{\ddot{\gamma}+(1-\ddot{\gamma})\left(\text{Ю}_1^{-}+\text{Ю}_2^{-}-\text{Ю}_1^{-}\text{Ю}_2^{-}\right)} \end{array} \right) \tag{9}$$

$$\lambda(u)=\left(\begin{array}{c}\frac{\left(1+(\ddot{\gamma}-1)\zeta^{+}\right)^{\lambda}-\left(1-\zeta^{+}\right)^{\lambda}}{\left(1+(\ddot{\gamma}-1)\zeta^{+}\right)^{\lambda}+(\ddot{\gamma}-1)\left(1-\zeta^{+}\right)^{\lambda}},\frac{\ddot{\gamma}\left(\Gamma^{+}\right)^{\lambda}}{\left(1+(\ddot{\gamma}-1)\left(1-\Gamma^{+}\right)\right)^{\lambda}+(\ddot{\gamma}-1)\left(\Gamma^{+}\right)^{\lambda}},\frac{\ddot{\gamma}\left(\text{Ю}^{+}\right)^{\lambda}}{\left(1+(\ddot{\gamma}-1)\left(1-\text{Ю}^{+}\right)\right)^{\lambda}+(\ddot{\gamma}-1)\left(\text{Ю}^{+}\right)^{\lambda}},\\ \frac{-\ddot{\gamma}\left|\zeta^{-}\right|^{\lambda}}{\left(1+(\ddot{\gamma}-1)(1+\zeta^{-})\right)^{\lambda}+(\ddot{\gamma}-1)\left|\zeta^{-}\right|^{\lambda}},\frac{-\ddot{\gamma}\left|\Gamma^{-}\right|^{\lambda}}{\left(1+(\ddot{\gamma}-1)\left(1+\Gamma^{-}\right)\right)^{\lambda}+(\ddot{\gamma}-1)\left|\Gamma^{-}\right|^{\lambda}},\frac{-\left(\left(1+(\ddot{\gamma}-1)\left|\text{Ю}^{-}\right|\right)^{\lambda}-(1+\text{Ю}^{-})^{\lambda}\right)}{\left(1+(\ddot{\gamma}-1)\left|\text{Ю}^{-}\right|\right)^{\lambda}+(\ddot{\gamma}-1)(1+\text{Ю}^{-})^{\lambda}}\end{array}\right) \tag{10}$$

$$(u)^{\lambda}=\left(\begin{array}{c}\frac{\left(\zeta^{+}\right)^{\lambda}}{\left(1+(\ddot{\gamma}-1)\left(1-\zeta^{+}\right)\right)^{\lambda}+(\ddot{\gamma}-1)\left(\zeta^{+}\right)^{\lambda}},\frac{\left(1+(\ddot{\gamma}-1)\Gamma^{+}\right)^{\lambda}-\left(1-\Gamma^{+}\right)^{\lambda}}{\left(1+(\ddot{\gamma}-1)\Gamma^{+}\right)^{\lambda}+(\ddot{\gamma}-1)\left(1-\Gamma^{+}\right)^{\lambda}},\frac{\left(1+(\ddot{\gamma}-1)\text{Ю}^{+}\right)^{\lambda}-\left(1-\text{Ю}^{+}\right)^{\lambda}}{\left(1+(\ddot{\gamma}-1)\text{Ю}^{+}\right)^{\lambda}+(\ddot{\gamma}-1)\left(1-\text{Ю}^{+}\right)^{\lambda}},\\ \frac{-\left(\left(1+(\ddot{\gamma}-1)\left|\zeta^{-}\right|\right)^{\lambda}-(1+\zeta^{-})^{\lambda}\right)}{\left(1+(\ddot{\gamma}-1)\zeta^{-}\right)^{\lambda}+(\gamma-1)(1+\zeta^{-})^{\lambda}},\frac{-\ddot{\gamma}\left|\Gamma^{-}\right|^{\lambda}}{\left(1+(\ddot{\gamma}-1)\left(1+\Gamma^{-}\right)\right)^{\lambda}+(\ddot{\gamma}-1)\left|\Gamma^{-}\right|^{\lambda}},\frac{-\ddot{\gamma}\left|\text{Ю}^{-}\right|^{\lambda}}{\left(1+(\ddot{\gamma}-1)(1+\text{Ю}^{-})\right)^{\lambda}+(\ddot{\gamma}-1)\left|\text{Ю}^{-}\right|^{\lambda}}\end{array}\right) \tag{11}$$

3. Bipolar Neutrosophic Hamacher Aggregation Operators

We propose some properties of the Hamacher aggregation operators in this part of the paper for bipolar neutrosophic Hamacher weighted averaging (BNHWA), bipolar neutrosophic Hamacher ordered weighted averaging (BNHOWA) and bipolar neutrosophic Hamacher hybrid averaging (BNHHA).

Let $u_{\ell}=\left(\zeta_{\ell}^{+},\Gamma_{\ell}^{+},\text{Ю}_{\ell}^{+},\zeta_{\ell}^{-},\Gamma_{\ell}^{-},\text{Ю}_{\ell}^{-}\right)$ be a family of BNNs, where $\ell\in Z$ and $Z=\{1,2,3,\ldots,n\}$.

3.1. Bipolar Neutrosophic HamacherWeighted Averaging Aggregation Operator

Definition 16. *The bipolar neutrosophic Hamacher weighted averaging (BNHWA) operator can be defined as follows:*

$$BNHWA_{\nu}(u_1,u_2,\ldots..,u_n)=\underset{\ell=1}{\overset{n}{\oplus}}(\nu_{\ell}u_{\ell})=\nu_1u_1\oplus\nu_2u_2\oplus\ldots\ldots..\oplus\nu_nu_n \tag{12}$$

where $\nu=(\nu_1,\nu_2,\ldots,\nu_n)^T$ is the weighted vector of u_{ℓ}, such that $\nu_{\ell}>0$ and $\sum\limits_{\ell=1}^{n}\nu_{\ell}=1,\ddot{\gamma}>0$.

Theorem 1. *The (BNHWA) operator gives a bipolar neutrosophic value when:*

$$BNHWA_{\nu}(u_1,u_2,\ldots..,u_n)=$$
$$\left(\begin{array}{c}\frac{\prod_{\ell=1}^{n}(1+(\ddot{\gamma}-1)\zeta_{\ell}^{+})^{\nu_{\ell}}-\prod_{\ell=1}^{n}(1-\zeta_{\ell}^{+})^{\nu_{\ell}}}{\prod_{\ell=1}^{n}(1+(\ddot{\gamma}-1)\zeta_{\ell}^{+})^{\nu_{\ell}}+(\ddot{\gamma}-1)\prod_{\ell=1}^{n}(1-\zeta_{\ell}^{+})^{\nu_{\ell}}},\frac{\ddot{\gamma}\prod_{\ell=1}^{n}\left(\Gamma_{\ell}^{+}\right)^{\nu_{\ell}}}{\prod_{\ell=1}^{n}\left(1+(\ddot{\gamma}-1)\left(1-\Gamma_{\ell}^{+}\right)\right)^{\nu_{\ell}}+(\ddot{\gamma}-1)\prod_{\ell=1}^{n}\left(\Gamma_{\ell}^{+}\right)^{\nu_{\ell}}},\frac{\ddot{\gamma}\prod_{\ell=1}^{n}(\text{Ю}_{\ell}^{+})^{\nu_{\ell}}}{\prod_{\ell=1}^{n}(1+(\ddot{\gamma}-1)(1-\text{Ю}_{\ell}^{+}))^{\nu_{\ell}}+(\ddot{\gamma}-1)\prod_{\ell=1}^{n}(\text{Ю}_{\ell}^{+})^{\nu_{\ell}}},\\ \frac{-\ddot{\gamma}\prod_{\ell=1}^{n}|\zeta_{\ell}^{-}|^{\nu_{\ell}}}{\prod_{\ell=1}^{n}(1+(\ddot{\gamma}-1)(1+\zeta_{\ell}^{-}))^{\nu_{\ell}}+(\ddot{\gamma}-1)\prod_{\ell=1}^{n}|\zeta_{\ell}^{-}|^{\nu_{\ell}}},-\frac{\prod_{\ell=1}^{n}\left(1+(\ddot{\gamma}-1)\left|\Gamma_{\ell}^{-}\right|\right)^{\nu_{\ell}}-\prod_{\ell=1}^{n}\left(1+\Gamma_{\ell}^{-}\right)^{\nu_{\ell}}}{\prod_{\ell=1}^{n}\left(1+(\ddot{\gamma}-1)\left|\Gamma_{\ell}^{-}\right|\right)^{\nu_{\ell}}+(\ddot{\gamma}-1)\prod_{\ell=1}^{n}\left(1+\Gamma_{\ell}^{-}\right)^{\nu_{\ell}}},-\frac{\prod_{\ell=1}^{n}(1+(\ddot{\gamma}-1)|\text{Ю}_{\ell}^{-}|)^{\nu_{\ell}}-\prod_{\ell=1}^{n}(1+\text{Ю}_{\ell}^{-})^{\nu_{\ell}}}{\prod_{\ell=1}^{n}(1+(\ddot{\gamma}-1)|\text{Ю}_{\ell}^{-}|)^{\nu_{\ell}}+(\ddot{\gamma}-1)\prod_{\ell=1}^{n}(1+\text{Ю}_{\ell}^{-})^{\nu_{\ell}}}\end{array}\right) \tag{13}$$

where $\nu=(\nu_1,\nu_2,\ldots,\nu_n)^T$ is the weighted vector of u_{ℓ}, such that $\nu_{\ell}>0$ and $\sum\limits_{\ell=1}^{n}\nu_{\ell}=1,\ddot{\gamma}>0$.

Proof. This theorem can be proved by mathematical induction as follows:

When $n=2$

$$\nu_1u_1=\left(\begin{array}{c}\frac{\left(1+(\ddot{\gamma}-1)\zeta_1^{+}\right)^{\nu_1}-\left(1-\zeta_1^{+}\right)^{\nu_1}}{\left(1+(\ddot{\gamma}-1)\zeta_1^{+}\right)^{\nu_1}+(\ddot{\gamma}-1)\left(1-\zeta_1^{+}\right)^{\nu_1}},\frac{\ddot{\gamma}\left(\Gamma_1^{+}\right)^{\nu_1}}{\left(1+(\ddot{\gamma}-1)\left(1-\Gamma_1^{+}\right)\right)^{\nu_1}+(\ddot{\gamma}-1)\left(\Gamma_1^{+}\right)^{\nu_1}},\frac{\ddot{\gamma}\left(\text{Ю}_1^{+}\right)^{\nu_1}}{\left(1+(\ddot{\gamma}-1)\left(1-\text{Ю}_1^{+}\right)\right)^{\nu_1}+(\ddot{\gamma}-1)\left(\text{Ю}_1^{+}\right)^{\nu_1}},\\ \frac{-\ddot{\gamma}\left|\zeta_1^{-}\right|^{\nu_1}}{\left(1+(\ddot{\gamma}-1)\left(1+\zeta_1^{-}\right)\right)^{\nu_1}+(\ddot{\gamma}-1)\left|\zeta_1^{-}\right|^{\nu_1}},\frac{-\ddot{\gamma}\left|\Gamma_1^{-}\right|^{\nu_1}}{\left(1+(\ddot{\gamma}-1)\left(1+\Gamma_1^{-}\right)\right)^{\nu_1}+(\ddot{\gamma}-1)\left|\Gamma_1^{-}\right|^{\nu_1}},\frac{-((1+((\ddot{\gamma}-1)\left|\text{Ю}_1^{-}\right|))^{\nu_1}-(1+\text{Ю}_1^{-})^{\nu_1})}{\left(1+(\ddot{\gamma}-1)\left|\text{Ю}_1^{-}\right|\right)^{\nu_1}+(\ddot{\gamma}-1)\left(1+\text{Ю}_1^{-}\right)^{\nu_1}}\end{array}\right)$$

$$\nu_2u_2=\left(\begin{array}{c}\frac{\left(1+(\ddot{\gamma}-1)\zeta_2^{+}\right)^{\nu_2}-\left(1-\zeta_2^{+}\right)^{\nu_2}}{\left(1+(\ddot{\gamma}-1)\zeta_2^{+}\right)^{\nu_2}+(\ddot{\gamma}-1)\left(1-\zeta_2^{+}\right)^{\nu_2}},\frac{\ddot{\gamma}\left(\Gamma_2^{+}\right)^{\nu_2}}{\left(1+(\ddot{\gamma}-1)\left(1-\Gamma_2^{+}\right)\right)^{\nu_2}+(\ddot{\gamma}-1)\left(\Gamma_2^{+}\right)^{\nu_2}},\frac{\ddot{\gamma}\left(\text{Ю}_2^{+}\right)^{\nu_2}}{\left(1+(\ddot{\gamma}-1)\left(1-\text{Ю}_2^{+}\right)\right)^{\nu_2}+(\ddot{\gamma}-1)\left(\text{Ю}_2^{+}\right)^{\nu_2}},\\ \frac{-\ddot{\gamma}\left|\zeta_2^{-}\right|^{\nu_2}}{\left(1+(\ddot{\gamma}-1)\left(1+\zeta_2^{-}\right)\right)^{\nu_2}+(\ddot{\gamma}-1)\left|\zeta_2^{-}\right|^{\nu_2}},\frac{-\ddot{\gamma}\left|\Gamma_2^{-}\right|^{\nu_2}}{\left(1+(\ddot{\gamma}-1)\left(1+\Gamma_2^{-}\right)\right)^{\nu_2}+(\ddot{\gamma}-1)\left|\Gamma_2^{-}\right|^{\nu_2}},\frac{-((1+((\ddot{\gamma}-1)\left|\text{Ю}_2^{-}\right|))^{\nu_2}-(1+\text{Ю}_2^{-})^{\nu_2})}{\left(1+(\ddot{\gamma}-1)\left|\text{Ю}_2^{-}\right|\right)^{\nu_2}+(\ddot{\gamma}-1)\left(1+\text{Ю}_2^{-}\right)^{\nu_2}}\end{array}\right)$$

and for

$$\nu_1 u_1 = \left(\begin{array}{c} \frac{(1+(\ddot{\gamma}-1)\zeta_1^+)-(1-\zeta_1^+)}{(1+(\ddot{\gamma}-1)\zeta_1^+)+(\ddot{\gamma}-1)(1-\zeta_1^+)}, \frac{\ddot{\gamma}(\Gamma_1^+)}{(1+(\ddot{\gamma}-1)(1-\Gamma_1^+))+(\ddot{\gamma}-1)(\Gamma_1^+)}, \frac{\ddot{\gamma}(\text{IO}_1^+)}{(1+(\ddot{\gamma}-1)(1-\text{IO}_1^+))+(\ddot{\gamma}-1)(\text{IO}_1^+)}, \\ \frac{-\ddot{\gamma}|\zeta_1^-|}{(1+(\ddot{\gamma}-1)(1+\zeta_1^-))+(\ddot{\gamma}-1)|\zeta_1^-|}, -\frac{(1+((\ddot{\gamma}-1)|\Gamma_1^-|)-(1+\Gamma_1^-)}{(1+(\ddot{\gamma}-1)|\Gamma_1^-|)+(\ddot{\gamma}-1)(1+\Gamma_1^-)}, -\frac{(1+((\ddot{\gamma}-1)|\text{IO}_1^-|)-(1+\text{IO}_1^-)}{(1+(\ddot{\gamma}-1)|\text{IO}_1^-|)+(\ddot{\gamma}-1)(1+\text{IO}_1^-)} \end{array} \right) \quad (14)$$

So it is proved for $n = 1$.

Now when $n = r$ in Equation (14), then

$$BNHWA_\nu(u_1, u_2, \ldots.., u_n) = \left(\begin{array}{c} \frac{\prod_{\ell=1}^{r}(1+(\ddot{\gamma}-1)\zeta_\ell^+)^{\nu_\ell}-\prod_{\ell=1}^{r}(1-\zeta_\ell^+)^{\nu_\ell}}{\prod_{\ell=1}^{r}(1+(\ddot{\gamma}-1)\zeta_\ell^+)^{\nu_\ell}+(\ddot{\gamma}-1)\prod_{\ell=1}^{r}(1-\zeta_\ell^+)^{\nu_\ell}}, \frac{\ddot{\gamma}\prod_{\ell=1}^{r}(\Gamma_\ell^+)^{\nu_\ell}}{\prod_{\ell=1}^{r}(1+(\ddot{\gamma}-1)(1-\Gamma_\ell^+))^{\nu_\ell}+(\ddot{\gamma}-1)\prod_{\ell=1}^{r}(\Gamma_\ell^+)^{\nu_\ell}}, \frac{\ddot{\gamma}\prod_{\ell=1}^{r}(\text{IO}_\ell^+)^{\nu_\ell}}{\prod_{\ell=1}^{r}(1+(\ddot{\gamma}-1)(1-\text{IO}_\ell^+))^{\nu_\ell}+(\ddot{\gamma}-1)\prod_{\ell=1}^{r}(\text{IO}_\ell^+)^{\nu_\ell}}, \\ \frac{-\ddot{\gamma}\prod_{\ell=1}^{r}|\zeta_\ell^-|^{\nu_\ell}}{\prod_{\ell=1}^{r}(1+(\ddot{\gamma}-1)(1+\zeta_\ell^-))^{\nu_\ell}+(\ddot{\gamma}-1)\prod_{\ell=1}^{r}|\zeta_\ell^-|^{\nu_\ell}}, \frac{-\ddot{\gamma}\prod_{\ell=1}^{r}|\Gamma_\ell^-|}{\prod_{\ell=1}^{r}(1+(\ddot{\gamma}-1)(1+\Gamma_\ell^-))^{\nu_\ell}+(\ddot{\gamma}-1)\prod_{\ell=1}^{r}|\Gamma_\ell^-|^{\nu_\ell}}, \frac{-\prod_{j=1}^{k}(1-(\ddot{\gamma}-1)|\text{IO}_\ell^-|)^{\nu_\ell}-\prod_{\ell=1}^{r}(1-\text{IO}_\ell^-)^{\nu_\ell}}{\prod_{\ell=1}^{r}(1-(\ddot{\gamma}-1)|\text{IO}_\ell^-|)^{\nu_\ell}+(\ddot{\gamma}-1)\prod_{\ell=1}^{r}(1+\text{IO}_\ell^-)^{\nu_\ell}} \end{array} \right)$$

This proves that it is true for $n = r$

When $n = r + 1$, then

$$\begin{aligned} &BNHWA_\nu(u_1, u_2, \ldots.., u_n) \\ &= \left(\begin{array}{c} \frac{\prod_{\ell=1}^{r}(1+(\ddot{\gamma}-1)\zeta_\ell^+)^{\nu_\ell}-\prod_{\ell=1}^{r}(1-\zeta_\ell^+)^{\nu_\ell}}{\prod_{\ell=1}^{r}(1+(\ddot{\gamma}-1)\zeta_\ell^+)^{\nu_\ell}+(\ddot{\gamma}-1)\prod_{\ell=1}^{r}(1-\zeta_\ell^+)^{\nu_\ell}}, \frac{\ddot{\gamma}\prod_{\ell=1}^{r}(\Gamma_\ell^+)^{\nu_\ell}}{\prod_{\ell=1}^{r}(1+(\ddot{\gamma}-1)(1-\Gamma_\ell^+))^{\nu_\ell}+(\ddot{\gamma}-1)\prod_{\ell=1}^{r}(\Gamma_\ell^+)^{\nu_\ell}}, \\ \frac{\ddot{\gamma}\prod_{\ell=1}^{r}(\text{IO}_\ell^+)^{\nu_\ell}}{\prod_{\ell=1}^{r}(1+(\ddot{\gamma}-1)(1-\text{IO}_\ell^+))^{\nu_\ell}+(\ddot{\gamma}-1)\prod_{\ell=1}^{r}(\text{IO}_\ell^+)^{\nu_\ell}}, \frac{-\ddot{\gamma}\prod_{\ell=1}^{r}(|\zeta_\ell^-|)^{\nu_\ell}}{\prod_{\ell=1}^{r}(1+(\ddot{\gamma}-1)(1+\zeta_\ell^-))^{\nu_\ell}+(\ddot{\gamma}-1)\prod_{\ell=1}^{r}|\zeta_\ell^-|^{\nu_\ell}}, \\ \frac{-\ddot{\gamma}\prod_{\ell=1}^{r}|\Gamma_\ell^-|}{\prod_{\ell=1}^{r}(1+(\ddot{\gamma}-1)(1+\Gamma_\ell^-))^{\nu_\ell}+(\ddot{\gamma}-1)\prod_{\ell=1}^{r}|\Gamma_\ell^-|^{\nu_\ell}}, -\frac{\prod_{\ell=1}^{r}(1+(\ddot{\gamma}-1)|\text{IO}_\ell^-|)^{\nu_\ell}-\prod_{\ell=1}^{r}(1+\text{IO}_\ell^-)^{\nu_\ell}}{\prod_{\ell=1}^{r}(1+(\ddot{\gamma}-1)|\text{IO}_\ell^-|)^{\nu_\ell}+(\ddot{\gamma}-1)\prod_{\ell=1}^{r}(1+\text{IO}_\ell^-)^{\nu_\ell}} \end{array} \right) \\ &\oplus \left(\begin{array}{c} \frac{(1+(\ddot{\gamma}-1)\zeta_{r+1}^+)^{\nu_{r+1}}-(1-\zeta_{r+1}^+)^{\nu_{r+1}}}{(1+(\ddot{\gamma}-1)\zeta_{r+1}^+)^{\nu_{r+1}}+(\ddot{\gamma}-1)(1-\zeta_{r+1}^+)^{\nu_{r+1}}}, \frac{\ddot{\gamma}(\Gamma_{r+1}^+)^{\nu_{r+1}}}{(1+(\ddot{\gamma}-1)(1-\Gamma_{r+1}^+))^{\nu_{r+1}}+(\ddot{\gamma}-1)(\Gamma_{r+1}^+)^{\nu_{r+1}}}, \\ \frac{\ddot{\gamma}(\text{IO}_{r+1}^+)^{\nu_{r+1}}}{(1+(\ddot{\gamma}-1)(1-\text{IO}_{r+1}^+))^{\nu_{r+1}}+(\ddot{\gamma}-1)(\text{IO}_{r+1}^+)^{\nu_{r+1}}}, \frac{-\ddot{\gamma}|\zeta_{r+1}^-|^{\nu_{r+1}}}{(1+(\ddot{\gamma}-1)(1+\zeta_{r+1}^-))^{\nu_{r+1}}+(\ddot{\gamma}-1)|\zeta_{r+1}^-|^{\nu_{r+1}}}, \\ \frac{-\ddot{\gamma}|\Gamma_{r+1}^-|^{\nu_{r+1}}}{(1+(\ddot{\gamma}-1)(1+\Gamma_{r+1}^-))^{\nu_{r+1}}+(\ddot{\gamma}-1)|\Gamma_{r+1}^-|^{\nu_{r+1}}}, -\frac{(1-(1+(\ddot{\gamma}-1)|\text{IO}_{r+1}^-|))^{\nu_{r+1}}-(1+\text{IO}_{r+1}^-)^{\nu_{r+1}})}{(1+(\ddot{\gamma}-1)|\text{IO}_{r+1}^-|)^{\nu_{r+1}}+(\ddot{\gamma}-1)(1+\text{IO}_{r+1}^-)^{\nu_{r+1}}} \end{array} \right) \\ &= \left(\begin{array}{c} \frac{\prod_{\ell=1}^{r+1}(1+(\ddot{\gamma}-1)\zeta_\ell^+)^{\nu_\ell}-\prod_{\ell=1}^{r+1}(1-\zeta_\ell^+)^{\nu_\ell}}{\prod_{\ell=1}^{r+1}(1+(\ddot{\gamma}-1)\zeta_\ell^+)^{\nu_\ell}+(\ddot{\gamma}-1)\prod_{\ell=1}^{r+1}(1-\zeta_\ell^+)^{\nu_\ell}}, \frac{\ddot{\gamma}\prod_{\ell=1}^{r+1}(\Gamma_\ell^+)^{\nu_\ell}}{\prod_{\ell=1}^{r+1}(1+(\ddot{\gamma}-1)(1-\Gamma_\ell^+))^{\nu_\ell}+(\ddot{\gamma}-1)\prod_{\ell=1}^{r+1}(\Gamma_\ell^+)^{\nu_\ell}}, \\ \frac{\ddot{\gamma}\prod_{\ell=1}^{r+1}(\text{IO}_\ell^+)^{\nu_\ell}}{\prod_{\ell=1}^{r+1}(1+(\ddot{\gamma}-1)(1-\text{IO}_\ell^+))^{\nu_\ell}+(\ddot{\gamma}-1)\prod_{\ell=1}^{r+1}(\text{IO}_\ell^+)^{\nu_\ell}}, \frac{-\ddot{\gamma}\prod_{\ell=1}^{r+1}(|\zeta_\ell^-|)^{\nu_\ell}}{\prod_{\ell=1}^{r+1}(1+(\ddot{\gamma}-1)(1+\zeta_\ell^-))^{\nu_\ell}+(\ddot{\gamma}-1)\prod_{\ell=1}^{r+1}|\zeta_\ell^-|^{\nu_\ell}}, \\ \frac{-\ddot{\gamma}\prod_{\ell=1}^{r+1}|\Gamma_\ell^-|^{\nu_\ell}}{\prod_{\ell=1}^{r+1}(1+(\ddot{\gamma}-1)(1+\Gamma_\ell^-))^{\nu_\ell}+(\ddot{\gamma}-1)\prod_{\ell=1}^{r+1}|\Gamma_\ell^-|^{\nu_\ell}}, -\frac{\prod_{\ell=1}^{r+1}(1+(\ddot{\gamma}-1)|\text{IO}_\ell^-|)^{\nu_\ell}-\prod_{\ell=1}^{r+1}(1+\text{IO}_\ell^-)^{\nu_\ell}}{\prod_{\ell=1}^{r+1}(1+(\ddot{\gamma}-1)|\text{IO}_\ell^-|)^{\nu_\ell}+(\ddot{\gamma}-1)\prod_{\ell=1}^{r+1}(1+\text{IO}_\ell^-)^{\nu_\ell}} \end{array} \right). \end{aligned}$$

Thus, Equation (14) is true for $n = r + 1$, which proves Theorem 1. □

Theorem 2. *(Idempotency) Let* $u_\ell = \left(\zeta_\ell^+, \Gamma_\ell^+, \text{Ю}_\ell^+, \zeta_\ell^-, \Gamma_\ell^-, \text{Ю}_\ell^-\right)$*, where* $\ell \in Z$ *and* $Z = \{1,2,3,\ldots.,n\}$ *be a collection of BNNs are equal, i.e.,* $u_\ell = u$ *for all* ℓ*, then:*

$$BNHWA_\nu(u_1, u_2, \ldots.., u_n) = u.$$

Theorem 3. *(Boundedness) Let* $u^- = \underset{\ell}{min} u_\ell$*,* $u^+ = \underset{\ell}{max} u_\ell$*, then:*

$$u^- \leq BNHWA_\nu(u_1, u_2, \ldots.., u_n) \leq u^+.$$

Theorem 4. *(Monotonicity) Let* $u_\ell = \left(\zeta_\ell^+, \Gamma_\ell^+, \text{Ю}_\ell^+, \zeta_\ell^-, \Gamma_\ell^-, \text{Ю}_\ell^-\right)$*, where* $\ell \in Z$ *and* $Z = \{1,2,3,\ldots,n\}$*, and* $u'_\ell = \left(\zeta'^+_\ell, \Gamma'^+_\ell, \text{Ю}'^+_\ell, \zeta'^-_\ell, \Gamma'^-_\ell, \text{Ю}'^-_\ell\right)$*, where* $\ell \in Z$ *and* $Z = \{1,2,3,\ldots,n\}$ *are two BNNs. If* $u_\ell \leq u'_\ell$*, for all* ℓ*, then:*

$$BNHWA_\nu(u_1, u_2, \ldots.., u_n) \leq BNHWA_\nu\left(u'_1, u'_2, \ldots.., u'_n\right)$$

A discussion of two cases of BNHWA operator follows.

- If $\ddot{\gamma} = 1$, then the BNHWA is converted to the bipolar neutrosophic weighted average (BNWA):

$$BNWA_\nu(u_1, u_2, \ldots.., u_n) = \overset{n}{\underset{\ell=1}{\oplus}} (\nu_\ell u_\ell) = \begin{pmatrix} 1 - \prod_{\ell=1}^{n}\left(1-\zeta_\ell^+\right)^{\nu_\ell}, \prod_{\ell=1}^{n}\left(\Gamma_\ell^+\right)^{\nu_\ell}, \prod_{\ell=1}^{n}\left(\text{Ю}_\ell^+\right)^{\nu_\ell}, \\ -\prod_{\ell=1}^{n}\left|\zeta_\ell^-\right|^{\nu_\ell}, -\left(1-\left(\prod_{\ell=1}^{n}\left(1+\Gamma_\ell^-\right)^{\nu_\ell}\right)\right), -\left(1-\left(\prod_{\ell=1}^{n}\left(1+\text{Ю}_\ell^-\right)^{\nu_\ell}\right)\right) \end{pmatrix}.$$

- If $\ddot{\gamma} = 2$, then the BNHWA is converted to the bipolar neutrosophic Einstein weighted average (BNEWA):

$$BNEWA_\nu(u_1, u_2, \ldots.., u_n) = \overset{n}{\underset{\ell=1}{\oplus}} (\nu_\ell u_\ell) =$$

$$\begin{pmatrix} \frac{\prod_{\ell=1}^{n}\left(1+\zeta_\ell^+\right)^{\nu_\ell} - \prod_{\ell=1}^{n}\left(1-\zeta_\ell^+\right)^{\nu_\ell}}{\prod_{\ell=1}^{n}\left(1+\zeta_\ell^+\right)^{\nu_\ell} + \prod_{\ell=1}^{n}\left(1-\zeta_\ell^+\right)^{\nu_\ell}}, \frac{\prod_{\ell=1}^{n}\left(\Gamma_\ell^+\right)^{\nu_\ell}}{\prod_{\ell=1}^{n}\left(2-\Gamma_\ell^+\right)^{\nu_\ell} + \prod_{\ell=1}^{n}\left(\Gamma_\ell^+\right)^{\nu_\ell}}, \frac{\prod_{\ell=1}^{n}\left(\text{Ю}_\ell^+\right)^{\nu_\ell}}{\prod_{\ell=1}^{n}\left(2-\text{Ю}_\ell^+\right)^{\nu_\ell} + \prod_{\ell=1}^{n}\left(\text{Ю}_\ell^+\right)^{\nu_\ell}}, \\ \frac{-2\prod_{\ell=1}^{n}\left|\zeta_\ell^-\right|^{\nu_\ell}}{\prod_{\ell=1}^{n}\left(2+\zeta_\ell^-\right)^{\nu_\ell} + \prod_{\ell=1}^{n}\left|\zeta_\ell^-\right|^{\nu_\ell}}, -\frac{\prod_{\ell=1}^{n}\left(1+\left|\Gamma_\ell^-\right|\right)^{\nu_\ell} - \prod_{\ell=1}^{n}\left(1+\Gamma_\ell^-\right)^{\nu_\ell}}{\prod_{\ell=1}^{n}\left(1+\left|\Gamma_\ell^-\right|\right)^{\nu_\ell} + \prod_{\ell=1}^{n}\left(1+\Gamma_\ell^-\right)^{\nu_\ell}}, -\frac{\prod_{\ell=1}^{n}\left(1+\left|\text{Ю}_\ell^-\right|\right)^{\nu_\ell} - \prod_{\ell=1}^{n}\left(1+\text{Ю}_\ell^-\right)^{\nu_\ell}}{\prod_{\ell=1}^{n}\left(1+\left|\text{Ю}_\ell^-\right|\right)^{\nu_\ell} + \prod_{\ell=1}^{n}\left(1+\text{Ю}_\ell^-\right)^{\nu_\ell}} \end{pmatrix}.$$

3.2. Bipolar Neutrosophic Hamacher OrderedWeighted Averaging Aggregation Operator

Definition 17. *The bipolar neutrosophic Hamacher ordered weighted averaging (BNHOWA) operator can be defined as follows:*

$$BNHOWA_\nu(u_1, u_2, \ldots\ldots, u_n) = \overset{n}{\underset{\ell=1}{\oplus}}\left(\nu_\ell u_{\rho(\ell)}\right) = \nu_1 u_{\rho(1)} \oplus \nu_2 u_{\rho(2)} \oplus \nu_3 u_{\breve{\sigma}(3)} \oplus \ldots\ldots\ldots.. \oplus \nu_n u_{\rho(n)}, \quad (15)$$

where $(\rho(1), \rho(2), \ldots\ldots\ldots, \rho(n))$ *is a permutation with* $u_{\rho(\ell-1)} \geq u_{\rho(\ell)}$*,* $\forall \ell \in Z$*,* $Z = \{1,2,3,\ldots,n\}$*, and* $\nu = (\nu_1, \nu_2, \ldots, \nu_n)^T$ *is the weighted vector of* u_ℓ *such that* $\nu_\ell > 0$ *and* $\sum_{\ell=1}^{n} \nu_\ell = 1, \ddot{\gamma} > 0.$

Theorem 5. *The (BNHOWA) operator gives a bipolar neutrosophic value:*

$$BNHOWA_{\nu}(u_1, u_2, \ldots.., u_n) = \left(\begin{array}{c} \frac{\prod_{\ell=1}^{n}\left(1+(\ddot{\gamma}-1)\zeta^{+}_{\rho(\ell)}\right)^{\nu_\ell} - \prod_{\ell=1}^{n}\left(1-\zeta^{+}_{\rho(\ell)}\right)^{\nu_\ell}}{\prod_{\ell=1}^{n}\left(1+(\ddot{\gamma}-1)\zeta^{+}_{\rho(\ell)}\right)^{\nu_\ell} + (\ddot{\gamma}-1)\prod_{\ell=1}^{n}\left(1-\zeta^{+}_{\rho(\ell)}\right)^{\nu_\ell}}, \frac{\ddot{\gamma}\prod_{\ell=1}^{n}\left(\Gamma^{+}_{\rho(\ell)}\right)^{\nu_\ell}}{\prod_{\ell=1}^{n}\left(1+(\ddot{\gamma}-1)\left(1-\Gamma^{+}_{\rho(\ell)}\right)\right)^{\nu_\ell} + (\ddot{\gamma}-1)\prod_{\ell=1}^{n}\left(\Gamma^{+}_{\rho(\ell)}\right)^{\nu_\ell}}, \\ \frac{\ddot{\gamma}\prod_{\ell=1}^{n}\left(\text{IO}^{+}_{\rho(\ell)}\right)^{\nu_\ell}}{\prod_{\ell=1}^{n}\left(1+(\ddot{\gamma}-1)\left(1-\text{IO}^{+}_{\rho(\ell)}\right)\right)^{\nu_\ell} + (\ddot{\gamma}-1)\prod_{\ell=1}^{n}\left(\text{IO}^{+}_{\rho(\ell)}\right)^{\nu_\ell}}, \frac{-\ddot{\gamma}\prod_{\ell=1}^{n}\left|\zeta^{-}_{\rho(\ell)}\right|^{\nu_\ell}}{\prod_{\ell=1}^{n}\left(1+(\ddot{\gamma}-1)\left(1+\zeta^{-}_{\rho(\ell)}\right)\right)^{\nu_\ell} + (\ddot{\gamma}-1)\prod_{\ell=1}^{n}\left|\zeta^{-}_{\rho(\ell)}\right|^{\nu_\ell}}, \\ -\frac{\prod_{\ell=1}^{n}\left(1+(\ddot{\gamma}-1)\left|\Gamma^{-}_{\rho(\ell)}\right|\right)^{\nu_\ell} - \prod_{\ell=1}^{n}\left(1+\Gamma^{-}_{\rho(\ell)}\right)^{\nu_\ell}}{\prod_{\ell=1}^{n}\left(1+(\ddot{\gamma}-1)\left|\Gamma^{-}_{\rho(\ell)}\right|\right)^{\nu_\ell} + (\ddot{\gamma}-1)\prod_{\ell=1}^{n}\left(1+\Gamma^{-}_{\rho(\ell)}\right)^{\nu_\ell}}, -\frac{\prod_{\ell=1}^{n}\left(1+(\ddot{\gamma}-1)\left|\text{IO}^{-}_{\rho(\ell)}\right|\right)^{\nu_\ell} - \prod_{\ell=1}^{n}\left(1+\text{IO}^{-}_{\rho(\ell)}\right)^{\nu_\ell}}{\prod_{\ell=1}^{n}\left(1+(\ddot{\gamma}-1)\left|\text{IO}^{-}_{\rho(\ell)}\right|\right)^{\nu_\ell} + (\ddot{\gamma}-1)\prod_{\ell=1}^{n}\left(1+\text{IO}^{-}_{\rho(\ell)}\right)^{\nu_\ell}} \end{array} \right) \quad (16)$$

where $(\rho(1), \rho(2), \ldots\ldots\ldots, \rho(n))$ *is a permutation with* $u_{\rho(\ell-1)} \geq u_{\rho(\ell)}$, $\forall \ell \in Z$, $Z = \{1, 2, 3, \ldots, n\}$ *and* $\nu = (\nu_1, \nu_2, \ldots, \nu_n)^T$ *is the weighted vector of* u_ℓ *such that* $\nu_\ell > 0$ *and* $\sum_{\ell=1}^{n} \nu_\ell = 1, \ddot{\gamma} > 0$.

Proof. The theorem is straightforward. □

Theorem 6. *(Idempotency) Let* $u_\ell = \left(\zeta^{+}_{\ell}, \Gamma^{+}_{\ell}, \text{IO}^{+}_{\ell}, \zeta^{-}_{\ell}, \Gamma^{-}_{\ell}, \text{IO}^{-}_{\ell}\right)$, *where* $\ell \in Z$ *and* $Z = \{1, 2, 3, \ldots, n\}$ *are a collection of equal BNNs, i.e.,* $u_\ell = u$ *for all* ℓ, *then:*

$$BNHOWA_{\nu}(u_1, u_2, \ldots.., u_n) = u.$$

Theorem 7. *(Boundedness) Let* $u^{-} = \min_{\ell} u_\ell$, $u^{+} = \max_{\ell} u_\ell$, *then:*

$$u^{-} \leq BNHOWA_{\nu}(u_1, u_2, \ldots.., u_n) \leq u^{+}.$$

Theorem 8. *(Monotonicity) Let* $u_\ell = \left(\zeta^{+}_{\ell}, \Gamma^{+}_{\ell}, \text{IO}^{+}_{\ell}, \zeta^{-}_{\ell}, \Gamma^{-}_{\ell}, \text{IO}^{-}_{\ell}\right)$, *where* $\ell \in Z$ *and* $Z = \{1, 2, 3, \ldots, n\}$ *and* $u'_\ell = \left(\zeta'^{+}_{\ell}, \Gamma'^{+}_{\ell}, \text{IO}'^{+}_{\ell}, \zeta'^{-}_{\ell}, \Gamma'^{-}_{\ell}, \text{IO}'^{-}_{\ell}\right)$, *where* $\ell \in Z$ *and* $Z = \{1, 2, 3, \ldots, n\}$ *are two BNNs. If* $u_\ell \leq u'_\ell$, *for all* ℓ, *then:*

$$BNHOWA_{\varpi}(u_1, u_2, \ldots.., u_n) \leq BNHOWA_{\varpi}\left(u'_1, u'_2, \ldots.., u'_n\right).$$

Now, we discuss two cases of the BNHOWA operator:

- If $\ddot{\gamma} = 1$, the BNHOWA is converted to the bipolar neutrosophic ordered weighted average (BNOWA):

$$BNOWA_{\nu}(u_1, u_2, \ldots.., u_n) = \oplus_{\ell=1}^{n}\left(\nu_\ell u_{\rho(\ell)}\right) = \left(\begin{array}{c} 1 - \prod_{\ell=1}^{n}\left(1 - \zeta^{+}_{\rho(\ell)}\right)^{\nu_\ell}, \prod_{\ell=1}^{n}\left(\Gamma^{+}_{\rho(\ell)}\right)^{\nu_\ell}, \prod_{\ell=1}^{n}\left(\text{IO}^{+}_{\rho(\ell)}\right)^{\nu_\ell}, \\ -\prod_{\ell=1}^{n}\left|\zeta^{-}_{\rho(\ell)}\right|^{\nu_\ell}, -\left(1 - \left(\prod_{\ell=1}^{n}\left(1 + \Gamma^{-}_{\rho(\ell)}\right)^{\nu_\ell}\right)\right), -\left(1 - \left(\prod_{\ell=1}^{n}\left(1 + \text{IO}^{-}_{\rho(\ell)}\right)^{\nu_\ell}\right)\right) \end{array} \right).$$

- If $\ddot{\gamma} = 2$, the BNHOWA is converted to the bipolar neutrosophic Einstein ordered weighted average (BNEOWA):

$$BNEOWA_{\nu}(u_1, u_2, \ldots.., u_n) = \overset{n}{\underset{\ell=1}{\oplus}}\left(\nu_\ell u_{\rho(\ell)}\right)$$
$$= \left(\begin{array}{c} \frac{\prod_{\ell=1}^{n}\left(1+\zeta^{+}_{\rho(\ell)}\right)^{\nu_\ell} - \prod_{\ell=1}^{n}\left(1-\zeta^{+}_{\rho(\ell)}\right)^{\nu_\ell}}{\prod_{\ell=1}^{n}\left(1+\zeta^{+}_{\rho(\ell)}\right)^{\nu_\ell} + \prod_{\ell=1}^{n}\left(1-\zeta^{+}_{\rho(\ell)}\right)^{\nu_\ell}}, \frac{2\prod_{\ell=1}^{n}\left(\Gamma^{+}_{\rho(\ell)}\right)^{\nu_\ell}}{\prod_{\ell=1}^{n}\left(2-\Gamma^{+}_{\rho(\ell)}\right)^{\nu_\ell} + \prod_{\ell=1}^{n}\left(\Gamma^{+}_{\rho(\ell)}\right)^{\nu_\ell}}, \frac{2\prod_{\ell=1}^{n}\left(\text{Ю}^{+}_{\rho(\ell)}\right)^{\nu_\ell}}{\prod_{\ell=1}^{n}\left(2-\text{Ю}^{+}_{\rho(\ell)}\right)^{\nu_\ell} + \prod_{\ell=1}^{n}\left(\text{Ю}^{+}_{\rho(\ell)}\right)^{\nu_\ell}}, \\ \frac{-2\prod_{\ell=1}^{n}\left|\zeta^{-}_{\rho(\ell)}\right|^{\nu_\ell}}{\prod_{\ell=1}^{n}\left(2+\zeta^{-}_{\rho(\ell)}\right)^{\nu_\ell} + \prod_{\ell=1}^{n}\left|\zeta^{-}_{\rho(\ell)}\right|^{\nu_\ell}}, -\frac{\prod_{\ell=1}^{n}\left(1+\left|\Gamma^{-}_{\rho(\ell)}\right|\right)^{\nu_\ell} - \prod_{\ell=1}^{n}\left(1+\Gamma^{-}_{\rho(\ell)}\right)^{\nu_\ell}}{\prod_{\ell=1}^{n}\left(1+\left|\Gamma^{-}_{\rho(\ell)}\right|\right)^{\nu_\ell} + \prod_{\ell=1}^{n}\left(1+\Gamma^{-}_{\rho(\ell)}\right)^{\nu_\ell}}, -\frac{\prod_{\ell=1}^{n}\left(1+\left|\text{Ю}^{-}_{\rho(\ell)}\right|\right)^{\nu_\ell} - \prod_{\ell=1}^{n}\left(1+\text{Ю}^{-}_{\rho(\ell)}\right)^{\nu_\ell}}{\prod_{\ell=1}^{n}\left(1+\left|\text{Ю}^{-}_{\rho(\ell)}\right|\right)^{\nu_\ell} + \prod_{\ell=1}^{n}\left(1+\text{Ю}^{-}_{\rho(\ell)}\right)^{\nu_\ell}} \end{array}\right).$$

3.3. Bipolar Neutrosophic Hamacher HybridAveraging Aggregation Operator

Definition 18. *The bipolar neutrosophic Hamacher hybrid averaging (BNHHA) operator can be defined as follows:*

$$BNHHA_{w,\nu}(u_1, u_2, \ldots\ldots, u_n) = \overset{n}{\underset{\ell=1}{\oplus}}\left(\nu_\ell \dot{u}_{\rho(\ell)}\right) = \nu_1\dot{u}_{\rho(1)} \oplus \nu_2\dot{u}_{\rho(2)} \oplus \nu_3\dot{u}_{\rho(3)} \oplus \ldots\ldots\ldots. \oplus \nu_n\dot{u}_{\rho(n)} \tag{17}$$

where $w = (w_1, w_2, \ldots, w_n)$ is a weighting vector of $u_\ell(\ell \in Z)$, $Z = \{1, 2, 3, \ldots, n\}$, such that $w_\ell \in [0, 1]$, $\sum_{\ell=1}^{n} w_\ell = 1$ and $\dot{u}_{\rho(\ell)}$ is the ℓ-th largest element of the bipolar neutrosophic argument,$\dot{u}_\ell\left(\dot{u}_\ell = (n\nu_\ell)u_\ell, \ell = 1, 2, \ldots, n\right)$ and also $\nu = (\nu_1, \nu_2, \ldots, \nu_n)$ are weighting vectors of bipolar neutrosophic arguments $u_\ell(\ell \in Z)$, $Z = \{1, 2, 3, \ldots, n\}$, such that $\nu_\ell \in [0, 1]$, $\sum_{\ell=1}^{n} \nu_\ell = 1$, where n is the balancing coefficient. Note that BNHHA reduces to BNHWA if $w = \left(\frac{1}{n}, \frac{1}{n}, \ldots.., \frac{1}{n}\right)^T$ and BNHOWA operator if:

$$\nu = \left(\frac{1}{n}, \frac{1}{n}, \ldots.., \frac{1}{n}\right).$$

Theorem 9. *The (BNHHA) operator returns a bipolar neutrosophic value when:*

$$BNHHA_{w,\nu}(u_1, u_2, \ldots.., u_n) = \begin{array}{c} \frac{\prod_{\ell=1}^{n}\left(1+(\ddot{\gamma}-1)\dot{\zeta}^{+}_{\rho(\ell)}\right)^{\nu_\ell} - \prod_{\ell=1}^{n}\left(1-\dot{\zeta}^{+}_{\rho(\ell)}\right)^{\nu_\ell}}{\prod_{\ell=1}^{n}\left(1+(\ddot{\gamma}-1)\dot{\zeta}^{+}_{\rho(\ell)}\right)^{\nu_\ell} + (\ddot{\gamma}-1)\prod_{\ell=1}^{n}\left(1-\dot{\zeta}^{+}_{\rho(\ell)}\right)^{\nu_\ell}}, \frac{\ddot{\gamma}\prod_{\ell=1}^{n}\left(\dot{\Gamma}^{+}_{\rho(\ell)}\right)^{\nu_\ell}}{\prod_{\ell=1}^{n}\left(1+(\ddot{\gamma}-1)\left(1-\dot{\Gamma}^{+}_{\rho(\ell)}\right)\right)^{\nu_\ell} + (\ddot{\gamma}-1)\prod_{\ell=1}^{n}\left(\dot{\Gamma}^{+}_{\rho(\ell)}\right)^{\nu_\ell}}, \\ \frac{\ddot{\gamma}\prod_{\ell=1}^{n}\left(\dot{\text{Ю}}^{+}_{\rho(\ell)}\right)^{\nu_\ell}}{\prod_{\ell=1}^{n}\left(1+(\ddot{\gamma}-1)\left(1-\dot{\text{Ю}}^{+}_{\rho(\ell)}\right)\right)^{\nu_\ell} + (\ddot{\gamma}-1)\prod_{\ell=1}^{n}\left(\dot{\text{Ю}}^{+}_{\rho(\ell)}\right)^{\nu_\ell}}, \frac{-\ddot{\gamma}\prod_{\ell=1}^{n}\left|\dot{\zeta}^{-}_{\rho(\ell)}\right|^{\nu_\ell}}{\prod_{\ell=1}^{n}\left(1+(\ddot{\gamma}-1)\left(1+\dot{\zeta}^{-}_{\rho(\ell)}\right)\right)^{\nu_\ell} + (\ddot{\gamma}-1)\prod_{\ell=1}^{n}\left|\dot{\zeta}^{-}_{\rho(\ell)}\right|^{\nu_\ell}}, \\ -\frac{\prod_{\ell=1}^{n}\left(1+(\ddot{\gamma}-1)\left|\dot{\Gamma}^{-}_{\rho(\ell)}\right|\right)^{\nu_\ell} - \prod_{\ell=1}^{n}\left(1+\dot{\Gamma}^{-}_{\rho(\ell)}\right)^{\nu_\ell}}{\prod_{\ell=1}^{n}\left(1+(\ddot{\gamma}-1)\left|\dot{\Gamma}^{-}_{\rho(\ell)}\right|\right)^{\nu_\ell} + (\ddot{\gamma}-1)\prod_{\ell=1}^{n}\left(1+\dot{\Gamma}^{-}_{\rho(\ell)}\right)^{\nu_\ell}}, -\frac{\prod_{\ell=1}^{n}\left(1+(\ddot{\gamma}-1)\left|\dot{\text{Ю}}^{-}_{\rho(\ell)}\right|\right)^{\nu_\ell} - \prod_{\ell=1}^{n}\left(1+\dot{\text{Ю}}^{-}_{\rho(\ell)}\right)^{\nu_\ell}}{\prod_{\ell=1}^{n}\left(1+(\ddot{\gamma}-1)\left|\dot{\text{Ю}}^{-}_{\rho(\ell)}\right|\right)^{\nu_\ell} + (\ddot{\gamma}-1)\prod_{\ell=1}^{n}\left(1+\dot{\text{Ю}}^{-}_{\rho(\ell)}\right)^{\nu_\ell}} \end{array} \tag{18}$$

where $w = (w_1, w_2, \ldots, w_n)$ is the weighting vector of $u_\ell(\ell \in Z)$, $Z = \{1, 2, 3, \ldots, n\}$, such that $w_\ell \in [0, 1]$, $\sum_{\ell=1}^{n} w_\ell = 1$ and $\dot{u}_{\rho(\ell)}$ is the ℓ -th largest element of the bipolar neutrosophic arguments, $\dot{u}_\ell\left(\dot{u}_\ell = (n\nu_\ell)u_\ell, \ell = 1, 2, \ldots, n\right)$ and also $\nu = (\nu_1, \nu_2, \ldots, \nu_n)$ are the weighting vector of bipolar neutrosophic arguments $u_\ell(\ell \in Z)$, $Z = \{1, 2, 3, \ldots, n\}$, such that $\nu_\ell \in [0, 1]$, $\sum_{\ell=1}^{n} \nu_\ell = 1$, where n is the balancing coefficient,$\ddot{\gamma} > 0$.

Proof. The theorem is straightforward. □

Now, we discuss two cases of BNHOWA:

- If $\ddot{\gamma} = 1$, the BNHHA is converted to the bipolar neutrosophic hybrid averaging (BNHA):

$$BNHA_{w,v}(u_1, u_2, \ldots.., u_n) = \overset{n}{\underset{\ell=1}{\oplus}}\left(v_\ell \dot{u}_{\rho(\ell)}\right) = \left(\begin{array}{c} 1 - \prod_{\ell=1}^{n}\left(1 - \dot{\zeta}^{+}_{\rho(\ell)}\right)^{v_\ell}, \prod_{\ell=1}^{n}\left(\dot{\Gamma}^{+}_{\rho(\ell)}\right)^{v_\ell}, \prod_{\ell=1}^{n}\left(\dot{\text{Ю}}^{+}_{\rho(\ell)}\right)^{v_\ell}, \\ -\prod_{\ell=1}^{n}\left|\dot{\zeta}^{-}_{\rho(\ell)}\right|^{v_\ell}, -\left(1 - \left(\prod_{\ell=1}^{n}\left(1 + \dot{\Gamma}^{-}_{\rho(\ell)}\right)^{v_\ell}\right)\right), -\left(1 - \left(\prod_{\ell=1}^{n}\left(1 + \dot{\text{Ю}}^{-}_{\rho(\ell)}\right)^{v_\ell}\right)\right) \end{array}\right).$$

- If $\ddot{\gamma} = 2$, the BNHHA is converted to the bipolar neutrosophic Einstein hybrid averaging (BNEHA):

$$BNEHA_{w,v}(u_1, u_2, \ldots.., u_n) = \overset{n}{\underset{\ell=1}{\oplus}}\left(v_\ell \dot{u}_{\rho(\ell)}\right)$$

$$= \left(\begin{array}{c} \dfrac{\prod_{\ell=1}^{n}\left(1+\dot{\zeta}^{+}_{\rho(\ell)}\right)^{v_\ell} - \prod_{\ell=1}^{n}\left(1-\dot{\zeta}^{+}_{\rho(\ell)}\right)^{v_\ell}}{\prod_{\ell=1}^{n}\left(1+\dot{\zeta}^{+}_{\rho(\ell)}\right)^{v_\ell} + \prod_{\ell=1}^{n}\left(1-\dot{\zeta}^{+}_{\rho(\ell)}\right)^{v_\ell}}, \dfrac{2\prod_{\ell=1}^{n}\left(\dot{\Gamma}^{+}_{\rho(\ell)}\right)^{v_\ell}}{\prod_{\ell=1}^{n}\left(2-\dot{\Gamma}^{+}_{\rho(\ell)}\right)^{v_\ell} + \prod_{\ell=1}^{n}\left(\dot{\Gamma}^{+}_{\rho(\ell)}\right)^{v_\ell}}, \dfrac{2\prod_{\ell=1}^{n}\left(\dot{\text{Ю}}^{+}_{\rho(\ell)}\right)^{v_\ell}}{\prod_{\ell=1}^{n}\left(2-\dot{\text{Ю}}^{+}_{\rho(\ell)}\right)^{v_\ell} + \prod_{\ell=1}^{n}\left(\dot{\text{Ю}}^{+}_{\rho(\ell)}\right)^{v_\ell}}, \\ \dfrac{-2\prod_{\ell=1}^{n}\left|\dot{\zeta}^{-}_{\rho(\ell)}\right|^{v_\ell}}{\prod_{\ell=1}^{n}\left(2+\dot{\zeta}^{-}_{\rho(\ell)}\right)^{v_\ell} + \prod_{\ell=1}^{n}\left|\dot{\zeta}^{-}_{\rho(\ell)}\right|^{v_\ell}}, \dfrac{\prod_{\ell=1}^{n}\left(1+\left|\dot{\Gamma}^{-}_{\rho(\ell)}\right|\right)^{v_\ell} - \prod_{\ell=1}^{n}\left(1+\dot{\Gamma}^{-}_{\rho(\ell)}\right)^{v_\ell}}{\prod_{\ell=1}^{n}\left(1+\left|\dot{\Gamma}^{-}_{\rho(\ell)}\right|\right)^{v_\ell} + \prod_{\ell=1}^{n}\left(1+\dot{\Gamma}^{-}_{\rho(\ell)}\right)^{v_\ell}}, \dfrac{\prod_{\ell=1}^{n}\left(1+\left|\dot{\text{Ю}}^{-}_{\rho(\ell)}\right|\right)^{v_\ell} - \prod_{\ell=1}^{n}\left(1+\dot{\text{Ю}}^{-}_{\rho(\ell)}\right)^{v_\ell}}{\prod_{\ell=1}^{n}\left(1+\left|\dot{\text{Ю}}^{-}_{\rho(\ell)}\right|\right)^{v_\ell} + \prod_{\ell=1}^{n}\left(1+\dot{\text{Ю}}^{-}_{\rho(\ell)}\right)^{v_\ell}} \end{array}\right).$$

4. An Application of the Bipolar Neutrosophic Hamacher Averaging Aggregation Operators to Group Decision Making

In this section, we apply the bipolar neutrosophic Hamacher averaging aggregation operators to the multiple attribute group decision-making problems in which the attribute weights take the form of crisp numbers and the attribute values take the form of BNNs.

Algorithm 1: Bipolar Neutrosophic Group Decision Making Problems

Let $G = \{G_1, G_2, \ldots, G_m\}$ be the set of m alternatives, $L = \{L_1, L_2, \ldots.., L_n\}$ be the set of n attributes or criterions, and $D = \{D_1, D_2, \ldots, D_k\}$ be the finite k decision makers. Let $v = (v_1, v_2, \ldots, v_n)^T$ be the weighted vector of the decision makers $\overline{D^s}(s = 1, 2, \ldots, k)$, such that $v_\ell \in [0, 1]$ and $\sum_{\ell=1}^{n} v_\ell = 1$. Let $w = (w_1, w_2, \ldots, w_n)^T$ be the weighted vector of the attribute set $L = \{L_1, L_2, \ldots.., L_n\}$ such that $w_\ell \in [0, 1]$ and $\sum_{\ell=1}^{n} w_\ell = 1$. An alternative of the criterion is assessed by the decision maker and the values are represented by bipolar neutrosophic values, where $u^{(s)}_{\widehat{i}\widehat{j}} = \left[\left(\zeta^{+}_{\widehat{i}\widehat{j}}, \Gamma^{+}_{\widehat{i}\widehat{j}}, \text{Ю}^{+}_{\widehat{i}\widehat{j}}, \zeta^{-}_{\widehat{i}\widehat{j}}, \Gamma^{-}_{\widehat{i}\widehat{j}}, \text{Ю}^{-}_{\widehat{i}\widehat{j}}\right)\right]_{m\times n}$ is the decision matrix provided by the decision maker (Tables 1–3) and $u^{(s)}_{\widehat{i}\widehat{j}}$ is a bipolar neutrosophic number (Table 4) for alternative $G_{\widehat{i}}$, associated with criterion $L_{\widehat{i}}$. The condition $\zeta^{+}_{\widehat{i}\widehat{j}}, \Gamma^{+}_{\widehat{i}\widehat{j}}, \text{Ю}^{+}_{\widehat{i}\widehat{j}}, \zeta^{-}_{\widehat{i}\widehat{j}}, \Gamma^{-}_{\widehat{i}\widehat{j}}$ and $\text{Ю}^{-}_{\widehat{i}\widehat{j}} \in [0, 1]$ is such that
$0 \le \zeta^{+}_{\widehat{i}\widehat{j}} + \Gamma^{+}_{\widehat{i}\widehat{j}} + \text{Ю}^{+}_{\widehat{i}\widehat{j}} + \zeta^{-}_{\widehat{i}\widehat{j}} + \Gamma^{-}_{\widehat{i}\widehat{j}} + \text{Ю}^{-}_{\widehat{i}\widehat{j}} \le 6$ for $\widehat{i} = 1, 2, \ldots., m$ and $\widehat{j} = 1, 2, \ldots., n$.

Step 1: Construct the decision matrix $\overline{D^s} = \left[u^{(s)}_{\widehat{i}\widehat{j}}\right]_{m\times n}$ $(s = 1, 2, \ldots, k)$ for the decision.

Step 2: Compute $BNHWA_v\left(r_{\widehat{i}1}, r_{\widehat{i}2}, \ldots.., r_{\widehat{i}n}\right)$ for each $\widehat{i} = 1, 2, 3, \ldots., m$:

$$r_{\widehat{i}} = \left(\zeta^{+}_{\widehat{i}}, \Gamma^{+}_{\widehat{i}}, \text{Ю}^{+}_{\widehat{i}}, \zeta^{-}_{\widehat{i}}, \Gamma^{-}_{\widehat{i}}, \text{Ю}^{-}_{\widehat{i}}\right) =$$

$$BNHWA_v\left(r_{\widehat{i}1}, r_{\widehat{i}2}, \ldots.., r_{\widehat{i}n}\right) = \overset{n}{\underset{\ell=1}{\oplus}}(v_\ell r_{i\ell})$$

$$= \left(\begin{array}{c} \dfrac{\prod_{\ell=1}^{n}(1+(\ddot{\gamma}-1)\zeta^{+}_{\ell})^{v_\ell} - \prod_{\ell=1}^{n}(1-\zeta^{+}_{\ell})^{v_\ell}}{\prod_{\ell=1}^{n}(1+(\ddot{\gamma}-1)\zeta^{+}_{\ell})^{v_\ell} + (\ddot{\gamma}-1)\prod_{\ell=1}^{n}(1-\zeta^{+}_{\ell})^{v_\ell}}, \dfrac{\ddot{\gamma}\prod_{\ell=1}^{n}(\Gamma^{+}_{\ell})^{v_\ell}}{\prod_{\ell=1}^{n}(1+(\ddot{\gamma}-1)(1-\Gamma^{+}_{\ell}))^{v_\ell} + (\ddot{\gamma}-1)\prod_{\ell=1}^{n}(\Gamma^{+}_{\ell})^{v_\ell}}, \dfrac{\ddot{\gamma}\prod_{\ell=1}^{n}(\text{Ю}^{+}_{\ell})^{v_\ell}}{\prod_{\ell=1}^{n}(1+(\ddot{\gamma}-1)(1-\text{Ю}^{+}_{\ell}))^{v_\ell} + (\ddot{\gamma}-1)\prod_{\ell=1}^{n}(\text{Ю}^{+}_{\ell})^{v_\ell}}, \\ \dfrac{-\ddot{\gamma}\prod_{\ell=1}^{n}|\zeta^{-}_{\ell}|^{v_\ell}}{\prod_{\ell=1}^{n}(1+(\ddot{\gamma}-1)(1+\zeta^{-}_{\ell}))^{v_\ell} + (\ddot{\gamma}-1)\prod_{\ell=1}^{n}|\zeta^{-}_{\ell}|^{v_\ell}}, -\dfrac{\prod_{\ell=1}^{n}(1+(\ddot{\gamma}-1)|\Gamma^{-}_{\ell}|)^{v_\ell} - \prod_{\ell=1}^{n}(1+\Gamma^{-}_{\ell})^{v_\ell}}{\prod_{\ell=1}^{n}(1+(\ddot{\gamma}-1)|\Gamma^{-}_{\ell}|)^{v_\ell} + (\ddot{\gamma}-1)\prod_{\ell=1}^{n}(1+\Gamma^{-}_{\ell})^{v_\ell}}, -\dfrac{\prod_{\ell=1}^{n}(1+(\ddot{\gamma}-1)|\text{Ю}^{-}_{\ell}|)^{v_\ell} - \prod_{\ell=1}^{n}(1+\text{Ю}^{-}_{\ell})^{v_\ell}}{\prod_{\ell=1}^{n}(1+(\ddot{\gamma}-1)|\text{Ю}^{-}_{\ell}|)^{v_\ell} + (\ddot{\gamma}-1)\prod_{\ell=1}^{n}(1+\text{Ю}^{-}_{\ell})^{v_\ell}} \end{array}\right)$$

Step 3: Calculate the scores of $S\left(r_{\widehat{i}}\right)$ for the $\left(\widehat{i} = 1, 2, 3, \ldots., m\right)$.

Step 4: Rank all the software systems of $BNHWA_v\left(u_{\widehat{i}1}, u_{\widehat{i}2}, \ldots.., u_{\widehat{i}n}\right)$ according to the scores values.

Step 5: Select the best alternative(s).

Illustrative Example

We considered an issue, taken from Deli [29], as an application for the proposed method in the present paper. The issue given is that an investment company wants to make some investments in the best possible options. There are four types of companies $G_{\overset{\frown}{i}}\left(\overset{\frown}{i} = 1, 2, 3 \ldots\ldots\ldots m\right)$ that are available as alternatives, namely G_1: computer company, G_2: food company, G_3: car company, and G_4: arms company, to invest money. The investment company takes into account four attributes to evaluate the alternatives: L_1: risk, L_2: growth, L_3: environmental impact, and L_4: performance. We utilized the bipolar neutrosophic numbers to assess the four possible alternatives of $G_{\overset{\frown}{i}}\left(\overset{\frown}{i} = 1, 2, 3, 4\right)$ under the four criteria. The weight vector of the attributes is $\nu = (\frac{1}{4}, \frac{1}{5}, \frac{3}{10}, \frac{1}{4})^T$. There are three experts, i.e., $\overline{D^s}(s = 1, 2, 3)$, from a group of decision makers, whose weight vector is $\nu = (\frac{3}{10}, \frac{3}{10}, \frac{2}{5})^T$. The expert opion about the companies based on atrribute are given in Tables 1–3.

Step 1: Decision matrices

Table 1. Bipolar Neutrosophic Decision Matrix, D_1.

	L_1	L_2	L_3	L_4
G_1	(0.5,0.4,0.3, −0.7, −0.5, −0.6)	(0.1, 0.5, 0.4, −0.2, −0.4, −0.7)	(0.3, 0.7, 0.4−0.6, −0.4, −0.6)	(0.2, 0.4, 0.7, −0.3, −0.5, −0.1)
G_2	(0.2, 0.6, 0.4, −0.3, −0.6, −0.8)	(0.5, 0.7, 0.6, −0.3, −0.4, −0.5)	(0.5, 0.5, 0.1, −0.7, −0.4, −0.8)	(0.6, 0.5, 0.4, −0.5, −0.4, −0.6)
G_3	(0.4, 0.5, 0.3, −0.5, −0.6, −0.7)	(0.8, 0.9, 0.2, −0.7, −0.4, −0.6)	(0.2, 0.6, 0.5, −0.5, −0.4, −0.7)	(0.5, 0.7, 0.3, −0.5, −0.4, −0.2)
G_4	(0.7, 0.6, 0.5, −0.6, −0.5, −0.4)	(0.5, 0.7, 0.6, −0.6, −0.3, −0.5)	(0.3, 0.1, 0.8, −0.9, −0.5, −0.6)	(0.2, 0.5, 0.7, −0.4, −0.5, −0.8)

Table 2. Bipolar Neutrosophic Decision Matrix, D_2.

	L_1	L_2	L_3	L_4
G_1	(0.2, 0.5,0.3, −0.4, −0.6, −0.5)	(0.4, 0.3, 0.7, −0.5, −0.4, −0.6)	(0.5, 0.7, 0.3, −0.4, −0.7, −0.6)	(0.1, 0.4, 0.6, −0.3, −0.4, −0.2)
G_2	(0.5, 0.6,0.4, −0.2, −0.4, −0.5)	(0.5, 0.1, 0.6, −0.6, −0.4, −0.2)	(0.3, 0.5, 0.4, −0.1, −0.4, −0.6)	(0.5, 0.3, 0.4−0.7, −0.4, −0.5)
G_3	(0.7, 0.4, 0.5, −0.4, −0.5, −0.6)	(0.7, 0.2, 0.4, −0.3, −0.5, −0.1)	(0.1, 0.7, 0.5, −0.4, −0.3, −0.8)	(0.4, 0.3, 0.5, −0.7, −0.4, −0.3)
G_4	(0.3, 0.4, 0.5, −0.7, −0.1, −0.3)	(0.8, 0.2, 0.1, −0.5, −0.3, −0.4)	(0.5, 0.2, 0.4, −0.1, −0.4, −0.7)	(0.4, 0.3, 0.7, −0.5, −0.2, −0.6)

Table 3. Bipolar Neutrosophic Decision Matrix, D_3.

	L_1	L_2	L_3	L_4
G_1	(0.4, 0.5, 0.1, −0.6, −0.4, −0.5)	(0.4, 0.3, 0.6, −0.1, −0.6, −0.3)	(0.2, 0.6, 0.3, −0.7, −0.2, −0.5)	(0.2, 0.1, 0.8, −0.9, −0.2, −0.3)
G_2	(0.5, 0.3, 0.2, −0.4, −0.1, −0.6)	(0.4, 0.5, 0.3, −0.7, −0.2, −0.3)	(0.5, 0.3, 0.4, −0.5, −0.4, −0.6)	(0.8, 0.6, 0.2, −0.1, −0.5, −0.4)
G_3	(0.3, 0.4, 0.6, −0.7, −0.2, −0.4)	(0.2, 0.9, 0.1, −0.4, −0.5, −0.6)	(0.3, 0.7, 0.2, −0.5, −0.3, −0.4)	(0.4, 0.1, 0.6, −0.3, −0.4, −0.1)
G_4	(0.5, 0.3, 0.6, −0.6, −0.3, −0.5)	(0.5, 0.6, 0.2, −0.5, −0.3, −0.6)	(0.4, 0.3, 0.7, −0.8, −0.4, −0.7)	(0.7, 0.4, 0.5, −0.5, −0.3, −0.4)

Step 2: We computed $BNHWA_\nu\left(r_{\overset{\frown}{i}1}, r_{\overset{\frown}{i}2}, \ldots.., r_{\overset{\frown}{i}n}\right)$ for $\ddot{\gamma} = 2$: The collective bipolar neutrosophic decision matrix is given in Table 4.

$$r_{\overset{\frown}{i}} = \left(\zeta^+_{\overset{\frown}{i}}, \Gamma^+_{\overset{\frown}{i}}, \mathrm{IO}^+_{\overset{\frown}{i}}, \zeta^-_{\overset{\frown}{i}}, \Gamma^-_{\overset{\frown}{i}}, \mathrm{IO}^-_{\overset{\frown}{i}}\right) =$$

$$BNHWA_\nu\left(r_{\overset{\frown}{i}1}, r_{\overset{\frown}{i}2}, \ldots.., r_{\overset{\frown}{i}n}\right) = \overset{n}{\underset{\ell=1}{\oplus}} (\nu_\ell r_{i\ell})$$

$$= \left(\begin{array}{c} \frac{\prod_{\ell=1}^{n}(1+(\ddot{\gamma}-1)\zeta^+_\ell)^{\nu_\ell} - \prod_{\ell=1}^{n}(1-\zeta^+_\ell)^{\nu_\ell}}{\prod_{\ell=1}^{n}(1+(\ddot{\gamma}-1)\zeta^+_\ell)^{\nu_\ell} + (\ddot{\gamma}-1)\prod_{\ell=1}^{n}(1-\zeta^+_\ell)^{\nu_\ell}}, \frac{\ddot{\gamma}\prod_{\ell=1}^{n}\left(\Gamma^+_\ell\right)^{\nu_\ell}}{\prod_{\ell=1}^{n}\left(1+(\ddot{\gamma}-1)\left(1-\Gamma^+_\ell\right)\right)^{\nu_\ell} + (\ddot{\gamma}-1)\prod_{\ell=1}^{n}\left(\Gamma^+_\ell\right)^{\nu_\ell}}, \frac{\ddot{\gamma}\prod_{\ell=1}^{n}(\mathrm{IO}^+_\ell)^{\nu_\ell}}{\prod_{\ell=1}^{n}(1+(\ddot{\gamma}-1)(1-\mathrm{IO}^+_\ell))^{\nu_\ell} + (\ddot{\gamma}-1)\prod_{\ell=1}^{n}(\mathrm{IO}^+_\ell)^{\nu_\ell}}, \\ \frac{-\ddot{\gamma}\prod_{\ell=1}^{n}|\zeta^-_\ell|^{\nu_\ell}}{\prod_{\ell=1}^{n}(1+(\ddot{\gamma}-1)(1+\zeta^-_\ell))^{\nu_\ell} + (\ddot{\gamma}-1)\prod_{\ell=1}^{n}|\zeta^-_\ell|^{\nu_\ell}}, -\frac{\prod_{\ell=1}^{n}\left(1+(\ddot{\gamma}-1)\left|\Gamma^-_\ell\right|\right)^{\nu_\ell} - \prod_{\ell=1}^{n}\left(1+\Gamma^-_\ell\right)^{\nu_\ell}}{\prod_{\ell=1}^{n}\left(1+(\ddot{\gamma}-1)\left|\Gamma^-_\ell\right|\right)^{\nu_\ell} + (\ddot{\gamma}-1)\prod_{\ell=1}^{n}\left(1+\Gamma^-_\ell\right)^{\nu_\ell}}, -\frac{\prod_{\ell=1}^{n}(1+(\ddot{\gamma}-1)|\mathrm{IO}^-_\ell|)^{\nu_\ell} - \prod_{\ell=1}^{n}(1+\mathrm{IO}^-_\ell)^{\nu_\ell}}{\prod_{\ell=1}^{n}(1+(\ddot{\gamma}-1)|\mathrm{IO}^-_\ell|)^{\nu_\ell} + (\ddot{\gamma}-1)\prod_{\ell=1}^{n}(1+\mathrm{IO}^-_\ell)^{\nu_\ell}} \end{array} \right)$$

Step 3: We calculated the score function, for $\ddot{\gamma} = 2$:

$$r_1 = (0.2993, 0.3560, 0.4013, -0.4544, -0.4369, -0.4585),$$
$$r_2 = (0.5060, 0.4265, 0.3217, -0.3607, -0.3854, -0.5668),$$
$$r_3 = (0.4150, 0.4765, 0.3629, -0.4780, -0.4010, -0.4920),$$
$$r_4 = (0.4989, 0.3360, 0.5028, -0.5272, -0.3517, -0.5683),$$

$$S\left(r_{\widehat{i}}\right) = \frac{1}{6}\left(\zeta^{+} + 1 - \Gamma^{\prime +} + 1 - \text{IO}^{+} + 1 + \zeta^{-} - \Gamma^{\prime -} - \text{IO}^{-}\right)\text{and}$$

$$S(r_1) = 0.4972, S(r_2) = 0.5582, S(r_3) = 0.4984, S(r_4) = 0.5088.$$

Step 4: We calculated the scores for $\ddot{\gamma} = 2$, which gave:

$$G_2 > G_4 > G_3 > G_1$$

Step 5: Thus, the best option was G_2

Table 4. Collective Bipolar Neutrosophic Decision Matrix R.

	L_1	L_2
G₁	(0.3757,0.4683,0.1962, −0.5615, −0.4946, −0.5317)	(0.3155,0.3520,0.5616, −0.2056, −0.4865, −0.5313)
G₂	(0.4181,0.4622,0.3062, −0.3001, −0.3585, −0.6479)	(0.4614,0.3623,0.4622, −0.5292, −0.3233, −0.3359)
G₃	(0.4708,0.4283,0.4669, −0.5406, −0.4250, −0.5632)	(0.5855,0.6202,0.1902, −0.4417, −0.4712, −0.4741)
G₄	(0.5174,0.4072,0.5386, −0.6291, −0.3083, −0.4134)	(0.6132,0.4687,0.2350, −0.5288, −0.3000, −0.5147)
	L_3	L_4
G₁	(0.3264,0.3931,0.3276, −0.5709, −0.4174, −0.5255)	(0.1703,0.2364,0.7077, −0.4916, −0.3566, −0.2115)
G₂	(0.4441,0.4108,0.2709, −0.3623, −0.4000, −0.6722)	(0.6708,0.4669,0.3061, −0.3107, −0.4413, −0.4946)
G₃	(0.2115,0.6692,0.3536, −0.4683, −0.3308, −0.6406)	(0.4312,0.2639,0.4669, −0.4597, −0.4000, −0.1914)
G₄	(0.4029,0.1931,0.6264, −0.4973, −0.4312, −0.6724)	(0.4891,0.3942,0.6154, −0.4683, −0.3359, −0.6088)

5. Comparison with the Different Methods

There are various tools utilized by researchers so far in decision making. Chen et al. [50] utilized FSs, and, later on, Atanassov [2] utilized intuitionistic FSs, Dubois et al. [51] utilized BFSs, Zavadskas et al. [52] utilized NSs, Ali et al. [53] utilized bipolar neutrosophic soft sets, and Irfan et al. [34] utilized BNSs and so many others have studied decision making. In this paper, we applied the bipolarity to the neutrosophic sets via Hamacher operators. If $\ddot{\gamma} = 1$, then our proposed model corresponded to the same BNSs of the decision making as Irfan et al. [34].

The advantage of our proposed methods was that the decision maker could choose different values of $\ddot{\gamma}$ in accordance with their preferences (Table 5). Generally, when the values of $\ddot{\gamma} = 1, 2$, are used, they form algebraic aggregation operators and Einstein aggregation operators. The aggregation operators suggested in this paper were more general and flexible in accordance with the different values of $\ddot{\gamma}$, keeping in view the above computation and analysis, it is derived that although the overall rating values of the alternatives are varying by using different values of $\ddot{\gamma}$, the ranking orders of the alternatives are slightly contrastive (Table 5). However, the most desirable investment company is G_2.

Table 5. Ranking of the four alternatives for the different values of $\ddot{\gamma}$.

$\ddot{\gamma}$	Aggregation Operators	Ranking
1	*BNHWA*	$G_2 > G_4 > G_1 > G_3$
1.5	*BNHWA*	$G_2 > G_4 > G_1 > G_3$
2	*BNHWA*	$G_2 > G_4 > G_3 > G_1$
2.5	*BNHWA*	$G_2 > G_4 > G_3 > G_1$
3	*BNHWA*	$G_2 > G_4 > G_1 > G_3$

6. Conclusions

The purpose of this paper was to study the different bipolar neutrosophic aggregation operators, based on Hamacher t-norms and *t*-conorms, and their application to multiple criteria group decision making where the criteria are bipolar neutrosophic values. Motivated by the Hamacher operations, we have proposed bipolar neutrosophic Hamacher aggregation operators. Firstly, we have introduced bipolar neutrosophic Hamacher aggregation operators, as well as their desirable properties. These aggregation operators were bipolar neutrosophic Hamacher weighted averaging (BNHWA), bipolar neutrosophic Hamacher ordered weighted averaging (BNHOWA), and bipolar neutrosophic Hamacher hybrid averaging (BNHHA). When $\ddot{\gamma} = 1$, the bipolar neutrosophic Hamacher averaging operators reduced to the bipolar neutrosophic averaging aggregation operator and for $\ddot{\gamma} = 2$, the bipolar neutrosophic Hamacher averaging operators transformed to the bipolar neutrosophic Einstein averaging aggregation operators. Finally, we have introduced a method for multi-attribute group decision making. A descriptive example of opting for the best company or alternative to investing money was provided. The results in this paper showed that our proposed methods were more effective and practical in real life. In our future study, we are determined to extend the proposed models to other domains and applications, such as risk analysis, pattern recognition, and so on.

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