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Bipolar Neutrosophic Dombi Aggregation Operators with Application in Multi-attribute Decision Making Problems

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ABSTRACT In this paper, Dombi t-norm (TN) and Dombi t-conorm (TCN) are used to generate more complex, flexible and feasible operation rules by managing a parameter in bipolar neutrosophic fuzzy (BNF) environment. We introduce the notion of bipolar neutrosophic Dombi weighted geometric aggregation (BNDWGA) and bipolar neutrosophic Dombi ordered weighted geometric aggregation (BNDOWGA) operators. We discuss their different properties along with proofs and also investigate multi-attribute decision making (MADM) methods on basis of propose aggregation operators under bipolar neutrosophic fuzzy (BNF) environment. We propose an algorithm of selection of cultivating crop to explain the proposed methods under bipolar neutrosophic fuzzy (BNF) environment. The effect of parameter on our proposed bipolar neutrosophic Dombi aggregation operators are discussed graphically. Moreover, the comparison of our proposed methods with existing methods to test their suitable preferences are discussed in detail.

INDEX TERMS Bipolar neutrosophic set, Bipolar Neutrosophic Dombi Aggregation operators, Decision-making environment.

I. INTRODUCTION

In fuzzy theory, a newly defined model generally overcome drawbacks of previously defined models. Due to vagueness and uncertainties issues in many daily life problems, the routine mathematics is not always available. To deal with such issues, various procedures such as hypothesis of probability, rough set hypothesis, and fuzzy set hypothesis have been considered as alternative models and to avoid vulnerabilities as well. Unfortunately, most of the alternative such mathematics have their own down sides and drawbacks. For instance, most of the words like, genuine, lovely, best, renowned are not measurable and are, in fact, ambiguous. The criteria for words like wonderful, best, renowned etc., fluctuate from individual to individual. To handel such type of ambiguous and uncertain information, Zadeh [1] initiated

the study of possibility based on participation function that assigns an enrollment grade in $[0, 1]$. FSs have only one membership degree and cannot handle complex problem. The concept of intuitionistic fuzzy set(IFS) was introduced by Atonasso [2]. IFS is used for a highly flexible description of uncertain information. IFS includes both membership degree and non-membership degree. And thereafter, the concept of interval-valued intuitionistic fuzzy set (IVIFS) was introduced by Atanassov and Gargov [3]. IVIFS is a generalization of IFS. In practical decision-making problems, IFS and IVIFS can not represent inconsistent information. The concept of neutrosophic set (NS) was introduced by Smarandache [4]. NS includes truth membership degree, indeterminacy membership degree, and falsity membership degree to characterize incomplete, inconsistent and uncertain

information. Wang et al. [5, 6] introduced the concept of interval neutrosophic set(INS) and the concept of single-valued neutrosophic set(SVNS) to apply NS in practical decision-making problems.

In practical decision-making problems, when we want to take decision then we think about positive and negative effect. Positive information explains why is acceptable, permitted, possible, described, or satisfactory. On contrary, negative information explains why is rejected, forbidden, or impossible discussed by Gruber in [7]. Satisfactory or acceptable perceptions are positive preferences, while unsatisfactory or unacceptable perceptions are negative preferences. Positive preferences relate to desires, while negative preferences relate to the constraints investigated by Bistarelli et al. [8]. For example, when the decision maker (DM) examines an object, he/she may explain why they considers (less or more) satisfactory or acceptable; on contrary, he/her may also explain why they considers unsatisfactory or unacceptable because of different constraints discussed by Wang et al. [9]. Zhang [10] introduced the concept of bipolar fuzzy set (BFS) consisting of positive membership degree and negative membership degree to describe the above information. Due to the advantages of BFS and NS, Deli et al. [11] introduced the concept of bipolar neutrosophic set (BNS) which describes fuzzy, bipolar, inconsistent and uncertain information. Dey et al.[12] proposed the bipolar neutrosophic TOPSIS(BN-TOPSIS) method to solve MADM problems under bipolar neutrosophic fuzzy environment. Zhang et al. [9] proposed the methods based on the Frank choquet Bonferroni Mean Operators to solve MADM problems under bipolar neutrosophic fuzzy environment. In [35-37] discussed some aggregation operators on different models.

The MADM model refers to make decisions when there are multiple but a finite list of alternatives and attributes. Dombi [13] introduced the new triangular norms which are Dombi TN and Dombi TCN. Dombi TN and Dombi TCN show the good flexibility with operational parameter. Until now, Dombi operations have not extended to aggregate bipolar neutrosophic fuzzy environment. In this paper, we investigate some new aggregation operators based on the combination of bipolar neutrosophic numbers(BNNs) and Dombi operations. We have proposed some bipolar neutrosophic Dombi aggregation operators to aggregate bipolar neutrosophic fuzzy information, and developed MADM methods based on BNDWGA and BNDOWGA operators to solve MADM problems with bipolar neutrosophic fuzzy information.

The remainder of the paper is as follows. In Section 2, the review of some basic definitions and concepts are given to be used in this paper. In Section 3, we investigate the notion of Dombi operations of bipolar neutrosophic numbers (BNNs). In Section 4, we propose BNDWGA and BNDOWGA operators and their properties in detail. In Section 5, the comprehensive MADM methods based on proposed bipolar neutro-

sophic Dombi aggregation operators are proposed. In Section 6, provides a numerical example of selection of cultivating crop. In Section 7, we discuss the parametric analysis and comparative analysis with existing methods.

II. PRELIMINARIES

In this section, we present a brief survey of few fundamentals of different sorts of sets which will be utilized in sequel.

A. BIPOLAR NEUTROSOPHIC SET AND BIPOLAR NEUTROSOPHIC NUMBER

Definition 1. [11] Let U be a fixed set. Then, BNS N can be defined as follows $N(u) = \langle \tau_N^+(u), \omega_N^+(u), \mathcal{U}_N^+(u), \tau_N^-(u), \omega_N^-(u), \mathcal{U}_N^-(u) \rangle$, where $\tau_N^+(u), \omega_N^+(u), \mathcal{U}_N^+(u) : U \rightarrow [0, 1]$ and $\tau_N^-(u), \omega_N^-(u), \mathcal{U}_N^-(u) : U \rightarrow [-1, 0]$. The positive membership degrees $\tau_N^+(u), \omega_N^+(u), \mathcal{U}_N^+(u)$ are the truth membership, indeterminacy membership degree and falsity membership degree of an element $u \in U$ corresponding to BNS N and the negative membership degrees $\tau_N^-(u), \omega_N^-(u), \mathcal{U}_N^-(u)$ denote the truth membership degree, indeterminacy membership degree and falsity membership degree of an element $u \in U$ to some implicit counter property corresponding to a BNS N .

In particular, if U has only one element, then $N(u) = \langle \tau_N^+(u), \omega_N^+(u), \mathcal{U}_N^+(u), \tau_N^-(u), \omega_N^-(u), \mathcal{U}_N^-(u) \rangle$ is called bipolar neutrosophic number(BNN). For convenience, BNN $N = \langle \tau_N^+(u), \omega_N^+(u), \mathcal{U}_N^+(u), \tau_N^-(u), \omega_N^-(u), \mathcal{U}_N^-(u) \rangle$ is also denoted as $N = \langle \tau_N^+, \omega_N^+, \mathcal{U}_N^+, \tau_N^-, \omega_N^-, \mathcal{U}_N^- \rangle$.

Deli et al. [11] defined the algebraic operations of BNNs which are as follows:

Definition 2. Let $\tilde{\aleph}_1 = \langle \tau_1^+, \omega_1^+, \mathcal{U}_1^+, \tau_1^-, \omega_1^-, \mathcal{U}_1^- \rangle$ and $\tilde{\aleph}_2 = \langle \tau_2^+, \omega_2^+, \mathcal{U}_2^+, \tau_2^-, \omega_2^-, \mathcal{U}_2^- \rangle$ be two BNNs. Then, algebraic operations of BNNs are defined as follows:

- (1) $\tilde{\aleph}_1 \oplus \tilde{\aleph}_2 = \langle \tau_1^+ + \tau_2^+ - \tau_1^+ \tau_2^+, \omega_1^+ \omega_2^+, \mathcal{U}_1^+ \mathcal{U}_2^+, -\tau_1^+ \tau_2^+, -(\omega_1^- + \omega_2^- - \omega_1^- \omega_2^-), -(\mathcal{U}_1^- + \mathcal{U}_2^- - \mathcal{U}_1^- \mathcal{U}_2^-) \rangle$,
- (2) $\tilde{\aleph}_1 \otimes \tilde{\aleph}_2 = \langle \tau_1^+ \tau_2^+, \omega_1^+ + \omega_2^+ - \omega_1^+ \omega_2^+, \mathcal{U}_1^+ + \mathcal{U}_2^+ - \mathcal{U}_1^+ \mathcal{U}_2^+, -(\tau_1^- + \tau_2^- - \tau_1^- \tau_2^-), -\omega_1^- \omega_2^-, -\mathcal{U}_1^- \mathcal{U}_2^- \rangle$,
- (3) $\ell \cdot \tilde{\aleph}_1 = \langle 1 - (1 - \tau_1^+)^{\ell}, (\omega_1^+)^{\ell}, (\mathcal{U}_1^+)^{\ell}, (-\tau_1^-)^{\ell}, -(1 - (1 - (\omega_1^-)^{\ell})), -(1 - (1 - (\mathcal{U}_1^-)^{\ell})) \rangle$ ($\ell > 0$),
- (4) $\tilde{\aleph}_1^{\ell} = \langle (\tau_1^+)^{\ell}, 1 - (1 - \omega_1^+)^{\ell}, 1 - (1 - \mathcal{U}_1^+)^{\ell}, -(1 - (1 - (\tau_1^-)^{\ell})), -(\omega_1^-)^{\ell}, -(\mathcal{U}_1^-)^{\ell} \rangle$ ($\ell > 0$).

For comparing two BNNs, Deli et al. [11] developed a comparison method which consists of the score function, accuracy function and certainty function.

Definition 3. [11] Let $\tilde{\aleph}_1 = \langle \tau_1^+, \omega_1^+, \mathcal{U}_1^+, \tau_1^-, \omega_1^-, \mathcal{U}_1^- \rangle$ be BNN, then the score function $s(\aleph_1)$, accuracy function $a(\aleph_1)$ and certainty function $c(\aleph_1)$ are defined as:

$$s(\tilde{\aleph}_1) = \frac{\tau_1^+ + 1 - \omega_1^+ + 1 - \mathcal{U}_1^+ + 1 + \tau_1^- - \omega_1^- - \mathcal{U}_1^-}{6}$$

$$a(\tilde{\aleph}_1) = (\tau_1^+ - \mathcal{U}_1^+) + (\tau_1^- - \mathcal{U}_1^-) \tag{2}$$

$$c(\tilde{\aleph}_1) = \tau_1^+ - \mathcal{U}_1^- \tag{3}$$

The comparison method of BNNs can be obtained based on Equations (1) – (3) as follows.

Definition 4. [11] Let $\tilde{N}_1 = \langle \tau_1^+, \omega_1^+, \cup_1^+, \tau_1^-, \omega_1^-, \cup_1^- \rangle$ and $\tilde{N}_2 = \langle \tau_2^+, \omega_2^+, \cup_2^+, \tau_2^-, \omega_2^-, \cup_2^- \rangle$ be two BNNs, therefore

- (1) if $s(\tilde{N}_1) > s(\tilde{N}_2)$, then $\tilde{N}_1 > \tilde{N}_2$,
- (2) if $s(\tilde{N}_1) = s(\tilde{N}_2)$ and $a(\tilde{N}_1) > a(\tilde{N}_2)$, then $\tilde{N}_1 > \tilde{N}_2$,
- (3) if $s(\tilde{N}_1) = s(\tilde{N}_2)$, $a(\tilde{N}_1) = a(\tilde{N}_2)$ and $c(\tilde{N}_1) > c(\tilde{N}_2)$, then $\tilde{N}_1 > \tilde{N}_2$,
- (4) if $s(\tilde{N}_1) = s(\tilde{N}_2)$, $a(\tilde{N}_1) = a(\tilde{N}_2)$ and $c(\tilde{N}_1) = c(\tilde{N}_2)$, then $\tilde{N}_1 \sim \tilde{N}_2$.

B. DOMBI OPERATIONS

Dombi product and Dombi sum are special cases of TN and TCN respectively, and are given in the following definition.

Definition 5. [13] Let N_1 and N_2 be any two real numbers, then Dombi TN and Dombi TCN are defined in the following expressions:

$$N_1 \otimes_D N_2 = Dom(N_1, N_2) = \frac{1}{1 + \{(\frac{1-N_1}{N_1})^\lambda + (\frac{1-N_2}{N_2})^\lambda\}^{\frac{1}{\lambda}}} \quad (4)$$

and

$$N_1 \oplus_D N_2 = Dom^*(N_1, N_2) = 1 - \frac{1}{1 + \{(\frac{N_1}{1-N_1})^\lambda + (\frac{N_2}{1-N_2})^\lambda\}^{\frac{1}{\lambda}}} \quad (5)$$

where $\lambda \geq 1$ and $(N_1, N_2) \in [0, 1] \times [0, 1]$. Some special cases can be easily proved.

- (1) if $\lambda \rightarrow 1$, then $N_1 \oplus_D N_2 \rightarrow \frac{N_1 + N_2 - 2N_1 N_2}{1 - N_1 N_2}$ and $N_1 \otimes_D N_2 \rightarrow \frac{N_1 N_2}{N_1 + N_2 - N_1 N_2}$,
- (2) If $\lambda \rightarrow \infty$, then $N_1 \oplus_D N_2 \rightarrow \max(N_1, N_2)$ and $N_1 \otimes_D N_2 \rightarrow \min(N_1, N_2)$. Dombi sum and Dombi product are reduced to simple max-operator and simple min-operator, respectively.

III. DOMBI OPERATIONS OF BNNs

In this section, we define the Dombi operations of BNNs and discuss their properties in detail.

Definition 6. Let $\tilde{N}_1 = \langle \tau_1^+, \omega_1^+, \cup_1^+, \tau_1^-, \omega_1^-, \cup_1^- \rangle$ and $\tilde{N}_2 = \langle \tau_2^+, \omega_2^+, \cup_2^+, \tau_2^-, \omega_2^-, \cup_2^- \rangle$ be two BNNs and $\lambda \geq 1$. Then, Dombi sum and Dombi product of two BNNs \tilde{N}_1 and \tilde{N}_2 are denoted by $\tilde{N}_1 \otimes_D \tilde{N}_2$ and $\tilde{N}_1 \oplus_D \tilde{N}_2$ respectively and defined as follows:

$$\begin{aligned}
 (1) \quad \tilde{N}_1 \oplus_D \tilde{N}_2 &= \left\langle \frac{1}{1 + \{(\frac{\tau_1^+}{1-\tau_1^+})^\lambda + (\frac{\tau_2^+}{1-\tau_2^+})^\lambda\}^{\frac{1}{\lambda}}}, \frac{1}{1 + \{(\frac{1-\omega_1^+}{\omega_1^+})^\lambda + (\frac{1-\omega_2^+}{\omega_2^+})^\lambda\}^{\frac{1}{\lambda}}}, \frac{1}{1 + \{(\frac{1-\cup_1^+}{\cup_1^+})^\lambda + (\frac{1-\cup_2^+}{\cup_2^+})^\lambda\}^{\frac{1}{\lambda}}}, \right. \\
 &\quad \left. \frac{-1}{1 + \{(\frac{1+\tau_1^-}{-\tau_1^-})^\lambda + (\frac{1+\tau_2^-}{-\tau_2^-})^\lambda\}^{\frac{1}{\lambda}}}, \frac{1}{1 + \{(\frac{-\omega_1^-}{1+\omega_1^-})^\lambda + (\frac{-\omega_2^-}{1+\omega_2^-})^\lambda\}^{\frac{1}{\lambda}}}, \frac{1}{1 + \{(\frac{-\cup_1^-}{1+\cup_1^-})^\lambda + (\frac{-\cup_2^-}{1+\cup_2^-})^\lambda\}^{\frac{1}{\lambda}}} \right\rangle \\
 (2) \quad \tilde{N}_1 \otimes_D \tilde{N}_2 &= \left\langle \frac{1}{1 + \{(\frac{1-\tau_1^+}{\tau_1^+})^\lambda + (\frac{1-\tau_2^+}{\tau_2^+})^\lambda\}^{\frac{1}{\lambda}}}, 1 - \frac{1}{1 + \{(\frac{\omega_1^+}{1-\omega_1^+})^\lambda + (\frac{\omega_2^+}{1-\omega_2^+})^\lambda\}^{\frac{1}{\lambda}}}, \right. \\
 &\quad \left. 1 - \frac{1}{1 + \{(\frac{\cup_1^+}{1-\cup_1^+})^\lambda + (\frac{\cup_2^+}{1-\cup_2^+})^\lambda\}^{\frac{1}{\lambda}}}, \frac{-1}{1 + \{(\frac{-\tau_1^-}{1+\tau_1^-})^\lambda + (\frac{-\tau_2^-}{1+\tau_2^-})^\lambda\}^{\frac{1}{\lambda}}}, \right. \\
 &\quad \left. \frac{-1}{1 + \{(\frac{1+\omega_1^-}{-\omega_1^-})^\lambda + (\frac{1+\omega_2^-}{-\omega_2^-})^\lambda\}^{\frac{1}{\lambda}}}, \frac{-1}{1 + \{(\frac{1+\cup_1^-}{-\cup_1^-})^\lambda + (\frac{1+\cup_2^-}{-\cup_2^-})^\lambda\}^{\frac{1}{\lambda}}} \right\rangle \\
 (3) \quad \ell_D \tilde{N}_1 &= \left\langle \frac{1}{1 + \{\ell(\frac{\tau_1^+}{1-\tau_1^+})^\lambda\}^{\frac{1}{\lambda}}}, \frac{1}{1 + \{\ell(\frac{1-\omega_1^+}{\omega_1^+})^\lambda\}^{\frac{1}{\lambda}}}, \right. \\
 &\quad \left. \frac{1}{1 + \{\ell(\frac{1-\cup_1^+}{\cup_1^+})^\lambda\}^{\frac{1}{\lambda}}}, \frac{-1}{1 + \{\ell(\frac{1+\tau_1^-}{-\tau_1^-})^\lambda\}^{\frac{1}{\lambda}}}, \right. \\
 &\quad \left. \frac{-1}{1 + \{\ell(\frac{-\omega_1^-}{1+\omega_1^-})^\lambda\}^{\frac{1}{\lambda}}}, \frac{-1}{1 + \{\ell(\frac{-\cup_1^-}{1+\cup_1^-})^\lambda\}^{\frac{1}{\lambda}}} \right\rangle (\ell > 0), \\
 (4) \quad \tilde{N}_1^{\wedge \ell_D} &= \left\langle \frac{1}{1 + \{\ell(\frac{1-\tau_1^+}{\tau_1^+})^\lambda\}^{\frac{1}{\lambda}}}, 1 - \frac{1}{1 + \{\ell(\frac{\omega_1^+}{1-\omega_1^+})^\lambda\}^{\frac{1}{\lambda}}}, \right. \\
 &\quad \left. 1 - \frac{1}{1 + \{\ell(\frac{\cup_1^+}{1-\cup_1^+})^\lambda\}^{\frac{1}{\lambda}}}, \frac{-1}{1 + \{\ell(\frac{-\tau_1^-}{1+\tau_1^-})^\lambda\}^{\frac{1}{\lambda}}}, \right. \\
 &\quad \left. \frac{-1}{1 + \{\ell(\frac{1+\omega_1^-}{-\omega_1^-})^\lambda\}^{\frac{1}{\lambda}}}, \frac{-1}{1 + \{\ell(\frac{1+\cup_1^-}{-\cup_1^-})^\lambda\}^{\frac{1}{\lambda}}} \right\rangle (\ell > 0).
 \end{aligned}$$

Theorem 1. Let $\tilde{N}_1 = \langle \tau_1^+, \omega_1^+, \cup_1^+, \tau_1^-, \omega_1^-, \cup_1^- \rangle$ and $\tilde{N}_2 = \langle \tau_2^+, \omega_2^+, \cup_2^+, \tau_2^-, \omega_2^-, \cup_2^- \rangle$ be two BNNs and let $\tilde{e} = \tilde{N}_1 \oplus_D \tilde{N}_2$, $\tilde{f} = \tilde{N}_1 \otimes_D \tilde{N}_2$, $\tilde{g} = \ell_D \tilde{N}_1$ ($\ell > 0$) and

$\tilde{h} = \tilde{N}_1^{\wedge \ell}$ ($\ell > 0$). Therefore, \tilde{e} , \tilde{f} , \tilde{g} and \tilde{h} are also BNNs.

As Theorem 1 can be easily verified. Therefore, the proof is omitted here. The properties of Dombi operations of BNNs are defined as follows.

Theorem 2. Let $\tilde{N}_1 = \langle \tau_1^+, \omega_1^+, \upsilon_1^+, \tau_1^-, \omega_1^-, \upsilon_1^- \rangle$ and $\tilde{N}_2 = \langle \tau_2^+, \omega_2^+, \upsilon_2^+, \tau_2^-, \omega_2^-, \upsilon_2^- \rangle$ be two BNNs and $\ell, \ell_1, \ell_2 > 0$. Then, the following properties can be proven easily.

- (a) $\tilde{N}_1 \oplus_D \tilde{N}_2 = \tilde{N}_2 \oplus_D \tilde{N}_1$,
- (b) $\tilde{N}_1 \otimes_D \tilde{N}_2 = \tilde{N}_2 \otimes_D \tilde{N}_1$,
- (c) $\ell \cdot_D (\tilde{N}_1 \oplus_D \tilde{N}_2) = \ell \cdot_D \tilde{N}_2 \oplus_D \ell \cdot_D \tilde{N}_1$,
- (d) $(\tilde{N}_1 \otimes_D \tilde{N}_2)^{\wedge \ell} = \tilde{N}_1^{\wedge \ell} \otimes_D \tilde{N}_2^{\wedge \ell}$,
- (e) $(\ell_1 + \ell_2) \cdot_D \tilde{N}_1 = \ell_1 \cdot_D \tilde{N}_1 \oplus_D \ell_2 \cdot_D \tilde{N}_1$,
- (f) $\tilde{N}_1^{\wedge \ell_1 + \ell_2} = \tilde{N}_1^{\wedge \ell_1} \otimes_D \tilde{N}_1^{\wedge \ell_2}$.

IV. BIPOLAR NEUTROSOPHIC DOMBI AGGREGATION OPERATORS

In this section, we will propose the bipolar neutrosophic Dombi weighted geometric aggregation (BNDWGA) and the bipolar neutrosophic Dombi ordered weighted geometric aggregation (BNDOWGA) operators and discuss different properties of these aggregation operators (AOs).

A. BIPOLAR NEUTROSOPHIC DOMBI WEIGHTED GEOMETRIC AGGREGATION OPERATOR

Definition 7. Let $\tilde{N}_\alpha = \langle \tau_\alpha^+, \omega_\alpha^+, \upsilon_\alpha^+, \tau_\alpha^-, \omega_\alpha^-, \upsilon_\alpha^- \rangle$ ($\alpha = 1, 2, 3, \dots, r$) be a family of BNNs. A mapping BNDWGA: $U^r \rightarrow U$ is called BNDWGA operator, if it satisfies

$$\begin{aligned} \text{BNDWGA}(\tilde{N}_1, \tilde{N}_2, \dots, \tilde{N}_r) &= \bigotimes_{\alpha=1}^r \tilde{N}_\alpha^{\psi_\alpha} \\ &= \tilde{N}_1^{\wedge \psi_1} \otimes_D \tilde{N}_2^{\wedge \psi_2} \otimes_D \dots \otimes_D \tilde{N}_r^{\wedge \psi_r} \end{aligned}$$

where $\psi = (\psi_1, \psi_2, \dots, \psi_r)^T$ is the weight vector of \tilde{N}_α , $\psi_\alpha \in [0, 1]$ and $\sum_{\alpha=1}^r \psi_\alpha = 1$.

Theorem 3. Let $\tilde{N}_\alpha = \langle \tau_\alpha^+, \omega_\alpha^+, \upsilon_\alpha^+, \tau_\alpha^-, \omega_\alpha^-, \upsilon_\alpha^- \rangle$ ($\alpha = 1, 2, 3, \dots, r$) be a family of BNNs and $\psi = (\psi_1, \psi_2, \dots, \psi_r)^T$ be the weight vector of \tilde{N}_α , $\psi_\alpha \in [0, 1]$ and $\sum_{\alpha=1}^r \psi_\alpha = 1$. Then, the value aggregated by using BNDWGA operator is still a BNN, which is calculated by using the following formula

$$\text{BNDWGA}(\tilde{N}_1, \tilde{N}_2, \dots, \tilde{N}_r) = \tilde{N}_1^{\wedge \psi_1} \otimes_D \tilde{N}_2^{\wedge \psi_2} \otimes_D \dots \otimes_D \tilde{N}_r^{\wedge \psi_r}$$

$$= \left\langle \frac{1}{1 + \{\sum_{\alpha=1}^r \psi_\alpha (\frac{1-\tau_\alpha^+}{\tau_\alpha^+})^\lambda\}^{\frac{1}{\lambda}}}, 1 - \frac{1}{1 + \{\sum_{\alpha=1}^r \psi_\alpha (\frac{\omega_\alpha^+}{1-\omega_\alpha^+})^\lambda\}^{\frac{1}{\lambda}}}, \right. \\ \left. 1 - \frac{1}{1 + \{\sum_{\alpha=1}^r \psi_\alpha (\frac{\upsilon_\alpha^+}{1-\upsilon_\alpha^+})^\lambda\}^{\frac{1}{\lambda}}}, \frac{1}{1 + \{\sum_{\alpha=1}^r \psi_\alpha (\frac{-\tau_\alpha^-}{1+\tau_\alpha^-})^\lambda\}^{\frac{1}{\lambda}}}, -1, \right. \\ \left. \frac{-1}{1 + \{\sum_{\alpha=1}^r \psi_\alpha (\frac{1+\omega_\alpha^-}{-\omega_\alpha^-})^\lambda\}^{\frac{1}{\lambda}}}, \frac{-1}{1 + \{\sum_{\alpha=1}^r \psi_\alpha (\frac{1+\upsilon_\alpha^-}{-\upsilon_\alpha^-})^\lambda\}^{\frac{1}{\lambda}}} \right\rangle \quad (6)$$

Proof. If $r = 2$ based on the Definition 7 of BNNs for Dombi operations, the following result can be obtained:

$$\begin{aligned} \text{BNDWGA}(\tilde{N}_1, \tilde{N}_2) &= \tilde{N}_1^{\wedge \psi_1} \otimes_D \tilde{N}_2^{\wedge \psi_2} \\ &= \left\langle \frac{1}{1 + \{\psi_1 (\frac{1-\tau_1^+}{\tau_1^+})^\lambda + \psi_2 (\frac{1-\tau_2^+}{\tau_2^+})^\lambda\}^{\frac{1}{\lambda}}}, \right. \\ &1 - \frac{1}{1 + \{\psi_1 (\frac{\omega_1^+}{1-\omega_1^+})^\lambda + \psi_2 (\frac{\omega_2^+}{1-\omega_2^+})^\lambda\}^{\frac{1}{\lambda}}}, \\ &1 - \frac{1}{1 + \{\psi_1 (\frac{\upsilon_1^+}{1-\upsilon_1^+})^\lambda + \psi_2 (\frac{\upsilon_2^+}{1-\upsilon_2^+})^\lambda\}^{\frac{1}{\lambda}}}, \\ &\frac{1}{1 + \{\psi_1 (\frac{-\tau_1^-}{1+\tau_1^-})^\lambda + \psi_2 (\frac{-\tau_2^-}{1+\tau_2^-})^\lambda\}^{\frac{1}{\lambda}}} - 1, \\ &\frac{-1}{1 + \{\psi_1 (\frac{1+\omega_1^-}{-\omega_1^-})^\lambda + \psi_2 (\frac{1+\omega_2^-}{-\omega_2^-})^\lambda\}^{\frac{1}{\lambda}}}, \\ &\left. \frac{-1}{1 + \{\psi_1 (\frac{1+\upsilon_1^-}{-\upsilon_1^-})^\lambda + \psi_2 (\frac{1+\upsilon_2^-}{-\upsilon_2^-})^\lambda\}^{\frac{1}{\lambda}}} \right\rangle, \\ &= \left\langle \frac{1}{1 + \{\sum_{\alpha=1}^2 \psi_\alpha (\frac{1-\tau_\alpha^+}{\tau_\alpha^+})^\lambda\}^{\frac{1}{\lambda}}}, 1 - \frac{1}{1 + \{\sum_{\alpha=1}^2 \psi_\alpha (\frac{\omega_\alpha^+}{1-\omega_\alpha^+})^\lambda\}^{\frac{1}{\lambda}}}, \right. \\ &1 - \frac{1}{1 + \{\sum_{\alpha=1}^2 \psi_\alpha (\frac{\upsilon_\alpha^+}{1-\upsilon_\alpha^+})^\lambda\}^{\frac{1}{\lambda}}}, \frac{1}{1 + \{\sum_{\alpha=1}^2 \psi_\alpha (\frac{-\tau_\alpha^-}{1+\tau_\alpha^-})^\lambda\}^{\frac{1}{\lambda}}} - 1, \\ &\left. \frac{-1}{1 + \{\sum_{\alpha=1}^2 \psi_\alpha (\frac{1+\omega_\alpha^-}{-\omega_\alpha^-})^\lambda\}^{\frac{1}{\lambda}}}, \frac{-1}{1 + \{\sum_{\alpha=1}^2 \psi_\alpha (\frac{1+\upsilon_\alpha^-}{-\upsilon_\alpha^-})^\lambda\}^{\frac{1}{\lambda}}} \right\rangle. \end{aligned}$$

If $r = s$ is based on the Equation (6), then we have got the following equation:

$$\begin{aligned} \text{BNDWGA}(\tilde{N}_1, \tilde{N}_2, \dots, \tilde{N}_s) &= \bigotimes_{\alpha=1}^s \tilde{N}_\alpha^{\psi_\alpha} \\ &= \left\langle \frac{1}{1 + \{\sum_{\alpha=1}^s \psi_\alpha (\frac{1-\tau_\alpha^+}{\tau_\alpha^+})^\lambda\}^{\frac{1}{\lambda}}}, 1 - \frac{1}{1 + \{\sum_{\alpha=1}^s \psi_\alpha (\frac{\omega_\alpha^+}{1-\omega_\alpha^+})^\lambda\}^{\frac{1}{\lambda}}}, \right. \\ &1 - \frac{1}{1 + \{\sum_{\alpha=1}^s \psi_\alpha (\frac{\upsilon_\alpha^+}{1-\upsilon_\alpha^+})^\lambda\}^{\frac{1}{\lambda}}}, \frac{1}{1 + \{\sum_{\alpha=1}^s \psi_\alpha (\frac{-\tau_\alpha^-}{1+\tau_\alpha^-})^\lambda\}^{\frac{1}{\lambda}}} - 1, \\ &\left. \frac{-1}{1 + \{\sum_{\alpha=1}^s \psi_\alpha (\frac{1+\omega_\alpha^-}{-\omega_\alpha^-})^\lambda\}^{\frac{1}{\lambda}}}, \frac{-1}{1 + \{\sum_{\alpha=1}^s \psi_\alpha (\frac{1+\upsilon_\alpha^-}{-\upsilon_\alpha^-})^\lambda\}^{\frac{1}{\lambda}}} \right\rangle. \end{aligned}$$

If $r = s + 1$, then there is following result:

$$\begin{aligned} \text{BNDWGA}(\tilde{N}_1, \tilde{N}_2, \dots, \tilde{N}_s, \tilde{N}_{s+1}) &= \bigotimes_{\alpha=1}^s \tilde{N}_\alpha^{\psi_\alpha} \otimes_D \tilde{N}_{s+1}^{\psi_{s+1}} \\ &= \left\langle \frac{1}{1 + \{\sum_{\alpha=1}^s \psi_\alpha (\frac{1-\tau_\alpha^+}{\tau_\alpha^+})^\lambda\}^{\frac{1}{\lambda}}}, 1 - \frac{1}{1 + \{\sum_{\alpha=1}^s \psi_\alpha (\frac{\omega_\alpha^+}{1-\omega_\alpha^+})^\lambda\}^{\frac{1}{\lambda}}}, \right. \\ &1 - \frac{1}{1 + \{\sum_{\alpha=1}^s \psi_\alpha (\frac{\upsilon_\alpha^+}{1-\upsilon_\alpha^+})^\lambda\}^{\frac{1}{\lambda}}}, \frac{1}{1 + \{\sum_{\alpha=1}^s \psi_\alpha (\frac{-\tau_\alpha^-}{1+\tau_\alpha^-})^\lambda\}^{\frac{1}{\lambda}}} - 1, \\ &\frac{1}{1 + \{\sum_{\alpha=1}^s \psi_\alpha (\frac{1+\omega_\alpha^-}{-\omega_\alpha^-})^\lambda\}^{\frac{1}{\lambda}}}, \frac{1}{1 + \{\sum_{\alpha=1}^s \psi_\alpha (\frac{1+\upsilon_\alpha^-}{-\upsilon_\alpha^-})^\lambda\}^{\frac{1}{\lambda}}} \left. \right\rangle \\ &\otimes_D \left\langle \frac{1}{1 + \{\psi_{s+1} (\frac{1-\tau_{s+1}^+}{\tau_{s+1}^+})^\lambda\}^{\frac{1}{\lambda}}}, 1 - \frac{1}{1 + \{\psi_{s+1} (\frac{\omega_{s+1}^+}{1-\omega_{s+1}^+})^\lambda\}^{\frac{1}{\lambda}}}, \right. \\ &1 - \frac{1}{1 + \{\psi_{s+1} (\frac{\upsilon_{s+1}^+}{1-\upsilon_{s+1}^+})^\lambda\}^{\frac{1}{\lambda}}}, \frac{1}{1 + \{\psi_{s+1} (\frac{-\tau_{s+1}^-}{1+\tau_{s+1}^-})^\lambda\}^{\frac{1}{\lambda}}} - 1, \\ &\frac{1}{1 + \{\psi_{s+1} (\frac{1+\omega_{s+1}^-}{-\omega_{s+1}^-})^\lambda\}^{\frac{1}{\lambda}}}, \frac{1}{1 + \{\psi_{s+1} (\frac{1+\upsilon_{s+1}^-}{-\upsilon_{s+1}^-})^\lambda\}^{\frac{1}{\lambda}}} \left. \right\rangle \\ &= \left\langle \frac{1}{1 + \{\sum_{\alpha=1}^{s+1} \psi_\alpha (\frac{1-\tau_\alpha^+}{\tau_\alpha^+})^\lambda\}^{\frac{1}{\lambda}}}, 1 - \frac{1}{1 + \{\sum_{\alpha=1}^{s+1} \psi_\alpha (\frac{\omega_\alpha^+}{1-\omega_\alpha^+})^\lambda\}^{\frac{1}{\lambda}}}, \right. \\ &1 - \frac{1}{1 + \{\sum_{\alpha=1}^{s+1} \psi_\alpha (\frac{\upsilon_\alpha^+}{1-\upsilon_\alpha^+})^\lambda\}^{\frac{1}{\lambda}}}, \frac{1}{1 + \{\sum_{\alpha=1}^{s+1} \psi_\alpha (\frac{-\tau_\alpha^-}{1+\tau_\alpha^-})^\lambda\}^{\frac{1}{\lambda}}} - 1, \\ &\left. \frac{-1}{1 + \{\sum_{\alpha=1}^{s+1} \psi_\alpha (\frac{1+\omega_\alpha^-}{-\omega_\alpha^-})^\lambda\}^{\frac{1}{\lambda}}}, \frac{-1}{1 + \{\sum_{\alpha=1}^{s+1} \psi_\alpha (\frac{1+\upsilon_\alpha^-}{-\upsilon_\alpha^-})^\lambda\}^{\frac{1}{\lambda}}} \right\rangle. \end{aligned}$$

Hence, Theorem 3 is true for $r = s + 1$. Thus, the Equation (6) holds for all r . \square

The BNDWGA operator has the following properties:

(1) **Idempotency:** Let all the BNNs be $\tilde{N}_\alpha = \langle \tau_\alpha^+, \omega_\alpha^+, \cup_\alpha^+, \tau_\alpha^-, \omega_\alpha^-, \cup_\alpha^- \rangle = \tilde{N}(\alpha = 1, 2, 3, \dots, r)$, where $\psi = (\psi_1, \psi_2, \dots, \psi_r)^T$ be the weight vector of $\tilde{N}_\alpha, \psi_\alpha \in [0, 1]$ and $\sum_{\alpha=1}^r \psi_\alpha = 1$. Then, $BNDWGA(\tilde{N}_1, \tilde{N}_2, \dots, \tilde{N}_r) = \tilde{N}$.

(2) **Monotonicity:** Let $\tilde{N}_\alpha (\alpha = 1, 2, \dots, r)$ and $\tilde{N}'_\alpha (\alpha = 1, 2, \dots, r)$ be two families of BNNs, where $\psi = (\psi_1, \psi_2, \dots, \psi_r)^T$ be the weight vector of \tilde{N}_α and $\tilde{N}'_\alpha, \psi_\alpha \in [0, 1]$ and $\sum_{\alpha=1}^r \psi_\alpha = 1$. For all α , if $\tilde{N}_\alpha \geq \tilde{N}'_\alpha$, then

$$BNDWGA(\tilde{N}_1, \tilde{N}_2, \dots, \tilde{N}_r) \geq BNDWGA(\tilde{N}'_1, \tilde{N}'_2, \dots, \tilde{N}'_r).$$

(3) **Boundedness:** Let $\tilde{N}_\alpha = \langle t_\alpha^+, \omega_\alpha^+, \cup_\alpha^+, \tau_\alpha^-, \omega_\alpha^-, \cup_\alpha^- \rangle (\alpha = 1, 2, 3, \dots, r)$ be a family of BNNs, where $\psi = (\psi_1, \psi_2, \dots, \psi_r)^T$ be the weight vector of $\tilde{N}_\alpha, \psi_\alpha \in [0, 1]$ and $\sum_{\alpha=1}^r \psi_\alpha = 1$. Therefore, we have $BNDWGA(\tilde{N}^-, \tilde{N}^-, \dots, \tilde{N}^-) \leq BNDWGA(\tilde{N}_1, \tilde{N}_2, \dots, \tilde{N}_r) \leq BNDWGA(\tilde{N}^+, \tilde{N}^+, \dots, \tilde{N}^+)$,

$$\text{where } \tilde{N}^- = \langle \tau_{\tilde{N}^-}^+, \omega_{\tilde{N}^-}^+, \cup_{\tilde{N}^-}^+, \tau_{\tilde{N}^-}^-, \omega_{\tilde{N}^-}^-, \cup_{\tilde{N}^-}^- \rangle = \begin{cases} \langle \min(\tau_1^+, \tau_2^+, \dots, \tau_r^+), \max(\omega_1^+, \omega_2^+, \dots, \omega_r^+), \\ \max(\cup_1^+, \cup_2^+, \dots, \cup_r^+), \max(\tau_1^-, \tau_2^-, \dots, \tau_r^-), \\ \min(\omega_1^-, \omega_2^-, \dots, \omega_r^-), \min(\cup_1^-, \cup_2^-, \dots, \cup_r^-) \rangle \end{cases}$$

and

$$\tilde{N}^+ = \langle \tau_{\tilde{N}^+}^+, \omega_{\tilde{N}^+}^+, \cup_{\tilde{N}^+}^+, \tau_{\tilde{N}^+}^-, \omega_{\tilde{N}^+}^-, \cup_{\tilde{N}^+}^- \rangle = \begin{cases} \langle \max(\tau_1^+, \tau_2^+, \dots, \tau_r^+), \min(\omega_1^+, \omega_2^+, \dots, \omega_r^+), \\ \min(\cup_1^+, \cup_2^+, \dots, \cup_r^+), \min(\tau_1^-, \tau_2^-, \dots, \tau_r^-), \\ \max(\omega_1^-, \omega_2^-, \dots, \omega_r^-), \max(\cup_1^-, \cup_2^-, \dots, \cup_r^-) \rangle. \end{cases}$$

Proof. (1) Since $\tilde{N}_\alpha = \langle \tau_\alpha^+, \omega_\alpha^+, \cup_\alpha^+, \tau_\alpha^-, \omega_\alpha^-, \cup_\alpha^- \rangle = \tilde{N}(\alpha = 1, 2, 3, \dots, r)$.

Then, the following result can be obtained by using Equation (6).

$$BNDWGA(\tilde{N}_1, \tilde{N}_2, \dots, \tilde{N}_r) = \left\langle \frac{1}{1 + \{\sum_{\alpha=1}^r \psi_\alpha (\frac{1 - \tau_\alpha^+}{1 - \omega_\alpha^+})^\lambda\}^{\frac{1}{\lambda}}}, 1 - \frac{1}{1 + \{\sum_{\alpha=1}^r \psi_\alpha (\frac{\omega_\alpha^+}{1 - \omega_\alpha^+})^\lambda\}^{\frac{1}{\lambda}}}, 1 - \frac{1}{1 + \{\sum_{\alpha=1}^r \psi_\alpha (\frac{\cup_\alpha^+}{1 - \cup_\alpha^+})^\lambda\}^{\frac{1}{\lambda}}}, \frac{1}{1 + \{\sum_{\alpha=1}^r \psi_\alpha (\frac{-\tau_\alpha^-}{1 + \tau_\alpha^-})^\lambda\}^{\frac{1}{\lambda}}}, \frac{-1}{1 + \{\sum_{\alpha=1}^r \psi_\alpha (\frac{1 + \omega_\alpha^-}{-\omega_\alpha^-})^\lambda\}^{\frac{1}{\lambda}}}, \frac{-1}{1 + \{\sum_{\alpha=1}^r \psi_\alpha (\frac{1 + \cup_\alpha^-}{-\cup_\alpha^-})^\lambda\}^{\frac{1}{\lambda}}} \right\rangle$$

$$= \left\langle \frac{1}{1 + \{(\frac{1 - \tau_1^+}{1 - \omega_1^+})^\lambda\}^{\frac{1}{\lambda}}}, 1 - \frac{1}{1 + \{(\frac{\omega_1^+}{1 - \omega_1^+})^\lambda\}^{\frac{1}{\lambda}}}, 1 - \frac{1}{1 + \{(\frac{\cup_1^+}{1 - \cup_1^+})^\lambda\}^{\frac{1}{\lambda}}}, \frac{1}{1 + \{(\frac{-\tau_1^-}{1 + \tau_1^-})^\lambda\}^{\frac{1}{\lambda}}}, \frac{-1}{1 + \{(\frac{1 + \omega_1^-}{-\omega_1^-})^\lambda\}^{\frac{1}{\lambda}}}, \frac{-1}{1 + \{(\frac{1 + \cup_1^-}{-\cup_1^-})^\lambda\}^{\frac{1}{\lambda}}} \right\rangle$$

$$= \langle \tau^+, \omega^+, \cup^+, \tau^-, \omega^-, \cup^- \rangle = \tilde{N}.$$

Hence, $BNDWGA(\tilde{N}_1, \tilde{N}_2, \dots, \tilde{N}_r) = \tilde{N}$ holds.

(2) The property is obvious based on the Equation (6).

(3) Let $\tilde{N}^- = \langle \tau_{\tilde{N}^-}^+, \omega_{\tilde{N}^-}^+, \cup_{\tilde{N}^-}^+, \tau_{\tilde{N}^-}^-, \omega_{\tilde{N}^-}^-, \cup_{\tilde{N}^-}^- \rangle$ and $\tilde{N}^+ = \langle \tau_{\tilde{N}^+}^+, \omega_{\tilde{N}^+}^+, \cup_{\tilde{N}^+}^+, \tau_{\tilde{N}^+}^-, \omega_{\tilde{N}^+}^-, \cup_{\tilde{N}^+}^- \rangle$. There are the following inequalities:

$$\begin{aligned} & \frac{1}{1 + \{\sum_{\alpha=1}^r \psi_\alpha (\frac{1 - \tau_\alpha^+}{1 - \omega_\alpha^+})^\lambda\}^{\frac{1}{\lambda}}} \leq \frac{1}{1 + \{\sum_{\alpha=1}^r \psi_\alpha (\frac{1 - \tau_\alpha^+}{\tau_\alpha^+})^\lambda\}^{\frac{1}{\lambda}}} \\ & \leq \frac{1}{1 + \{\sum_{\alpha=1}^r \psi_\alpha (\frac{1 - \tau_\alpha^+}{\tau_\alpha^+})^\lambda\}^{\frac{1}{\lambda}}}, \\ & 1 - \frac{1}{1 + \{\sum_{\alpha=1}^r \psi_\alpha (\frac{\omega_\alpha^+}{1 - \omega_\alpha^+})^\lambda\}^{\frac{1}{\lambda}}} \leq 1 - \frac{1}{1 + \{\sum_{\alpha=1}^r \psi_\alpha (\frac{\omega_\alpha^+}{1 - \omega_\alpha^+})^\lambda\}^{\frac{1}{\lambda}}} \\ & \leq 1 - \frac{1}{1 + \{\sum_{\alpha=1}^r \psi_\alpha (\frac{\omega_\alpha^+}{1 - \omega_\alpha^+})^\lambda\}^{\frac{1}{\lambda}}}, \\ & 1 - \frac{1}{1 + \{\sum_{\alpha=1}^r \psi_\alpha (\frac{\cup_\alpha^+}{1 - \cup_\alpha^+})^\lambda\}^{\frac{1}{\lambda}}} \leq 1 - \frac{1}{1 + \{\sum_{\alpha=1}^r \psi_\alpha (\frac{\cup_\alpha^+}{1 - \cup_\alpha^+})^\lambda\}^{\frac{1}{\lambda}}} \\ & \leq 1 - \frac{1}{1 + \{\sum_{\alpha=1}^r \psi_\alpha (\frac{\cup_\alpha^+}{1 - \cup_\alpha^+})^\lambda\}^{\frac{1}{\lambda}}}, \\ & \frac{1}{1 + \{\sum_{\alpha=1}^r \psi_\alpha (\frac{-\tau_\alpha^-}{1 + \tau_\alpha^-})^\lambda\}^{\frac{1}{\lambda}}} - 1 \leq \frac{1}{1 + \{\sum_{\alpha=1}^r \psi_\alpha (\frac{-\tau_\alpha^-}{1 + \tau_\alpha^-})^\lambda\}^{\frac{1}{\lambda}}} - 1 \\ & \leq \frac{1}{1 + \{\sum_{\alpha=1}^r \psi_\alpha (\frac{-\tau_\alpha^-}{1 + \tau_\alpha^-})^\lambda\}^{\frac{1}{\lambda}}} - 1, \\ & \frac{-1}{1 + \{\sum_{\alpha=1}^r \psi_\alpha (\frac{1 + \omega_\alpha^-}{-\omega_\alpha^-})^\lambda\}^{\frac{1}{\lambda}}} \leq \frac{-1}{1 + \{\sum_{\alpha=1}^r \psi_\alpha (\frac{1 + \omega_\alpha^-}{-\omega_\alpha^-})^\lambda\}^{\frac{1}{\lambda}}} \\ & \leq \frac{-1}{1 + \{\sum_{\alpha=1}^r \psi_\alpha (\frac{1 + \omega_\alpha^-}{-\omega_\alpha^-})^\lambda\}^{\frac{1}{\lambda}}}, \\ & \frac{-1}{1 + \{\sum_{\alpha=1}^r \psi_\alpha (\frac{1 + \cup_\alpha^-}{-\cup_\alpha^-})^\lambda\}^{\frac{1}{\lambda}}} \leq \frac{-1}{1 + \{\sum_{\alpha=1}^r \psi_\alpha (\frac{1 + \cup_\alpha^-}{-\cup_\alpha^-})^\lambda\}^{\frac{1}{\lambda}}} \\ & \leq \frac{-1}{1 + \{\sum_{\alpha=1}^r \psi_\alpha (\frac{1 + \cup_\alpha^-}{-\cup_\alpha^-})^\lambda\}^{\frac{1}{\lambda}}}. \end{aligned}$$

Hence, $BNDWGA(\tilde{N}^-, \tilde{N}^-, \dots, \tilde{N}^-)$

$\leq BNDWGA(\tilde{N}_1, \tilde{N}_2, \dots, \tilde{N}_r) \leq BNDWGA(\tilde{N}^+, \tilde{N}^+, \dots, \tilde{N}^+)$ holds. \square

Example 1. Let $\tilde{N}_1 = \langle 0.6, 0.7, 0.3, -0.6, -0.3, -0.5 \rangle$, $\tilde{N}_2 = \langle 0.5, 0.4, 0.6, -0.6, -0.7, -0.3 \rangle$, $\tilde{N}_3 = \langle 0.6, 0.7, 0.4, -0.9, -0.7, -0.7 \rangle$ and $\tilde{N}_4 = \langle 0.2, 0.6, 0.8, -0.6, -0.3, -0.9 \rangle$ be four BNNs and let the weight vector of BNNs $\tilde{N}_\alpha (\alpha = 1, 2, 3, 4)$ be $\psi = (\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{8})^T$. $\psi_1 = \frac{1}{2}$, $\psi_2 = \frac{1}{4}$, $\psi_3 = \frac{1}{8}$ and $\psi_4 = \frac{1}{8}$ are the weights of $\tilde{N}_\alpha (\alpha = 1, 2, 3, 4)$ such that $\sum_{\alpha=1}^4 \psi_\alpha = 1$. Then, by Theorem 3, for $\lambda = 3$

$$BNDWGA(\tilde{N}_1, \tilde{N}_2, \tilde{N}_3, \tilde{N}_4) =$$

$$\left\langle \frac{1}{1 + \{(\frac{1}{2}(\frac{1-0.6}{1-0.6})^3 + \frac{1}{4}(\frac{1-0.5}{1-0.5})^3 + \frac{1}{8}(\frac{1-0.6}{1-0.6})^3 + \frac{1}{8}(\frac{1-0.2}{1-0.2})^3)\}^{\frac{1}{3}}}, 1 - \frac{1}{1 + \{(\frac{1}{2}(\frac{0.7}{1-0.7})^3 + \frac{1}{4}(\frac{0.4}{1-0.4})^3 + \frac{1}{8}(\frac{0.7}{1-0.7})^3 + \frac{1}{8}(\frac{0.6}{1-0.6})^3)\}^{\frac{1}{3}}}, 1 - \frac{1}{1 + \{(\frac{1}{2}(\frac{0.3}{1-0.3})^3 + \frac{1}{4}(\frac{0.6}{1-0.6})^3 + \frac{1}{8}(\frac{0.4}{1-0.4})^3 + \frac{1}{8}(\frac{0.8}{1-0.8})^3)\}^{\frac{1}{3}}}, \frac{1}{1 + \{(\frac{1}{2}(\frac{0.6}{1+0.6})^3 + \frac{1}{4}(\frac{0.6}{1+0.6})^3 + \frac{1}{8}(\frac{0.9}{1+0.9})^3 + \frac{1}{8}(\frac{0.6}{1-0.6})^3)\}^{\frac{1}{3}}}, \frac{-1}{1 + \{(\frac{1}{2}(\frac{1-0.3}{1-0.3})^3 + \frac{1}{4}(\frac{1-0.7}{1-0.7})^3 + \frac{1}{8}(\frac{1-0.7}{1-0.7})^3 + \frac{1}{8}(\frac{1-0.3}{1-0.3})^3)\}^{\frac{1}{3}}}, \frac{-1}{1 + \{(\frac{1}{2}(\frac{1-0.5}{1-0.5})^3 + \frac{1}{4}(\frac{1-0.3}{1-0.3})^3 + \frac{1}{8}(\frac{1-0.7}{1-0.7})^3 + \frac{1}{8}(\frac{1-0.9}{1-0.9})^3)\}^{\frac{1}{3}}} \right\rangle$$

$$BNDWGA(\tilde{N}_1, \tilde{N}_2, \tilde{N}_3, \tilde{N}_4) = \langle 0.3294, 0.6706, 0.6747, -0.8198, -0.3336, -0.393 \rangle.$$

B. BIPOLAR NEUTROSOPHIC DOMBI ORDERED WEIGHTED GEOMETRIC AGGREGATION OPERATOR

Definition 8. Let $\tilde{N}_\alpha = \langle \tau_\alpha^+, \omega_\alpha^+, \upsilon_\alpha^+, \tau_\alpha^-, \omega_\alpha^-, \upsilon_\alpha^- \rangle$ ($\alpha = 1, 2, 3, \dots, r$) be a family of BNNs. A mapping BNDOWGA: $U^r \rightarrow U$ is called BNDOWGA operator, if it satisfies

$$BNDOWGA(\tilde{N}_1, \tilde{N}_2, \dots, \tilde{N}_r) = \bigotimes_{\beta=1}^r \tilde{N}_{\sigma(\beta)}^{\psi_\beta} = \tilde{N}_{\sigma(1)}^{\psi_1} \otimes_D \tilde{N}_{\sigma(2)}^{\psi_2} \otimes_D \dots \otimes_D \tilde{N}_{\sigma(r)}^{\psi_r}$$

where σ is permutation that orders the elements: $\tilde{N}_{\sigma(1)} \geq \tilde{N}_{\sigma(2)} \geq \dots \geq \tilde{N}_{\sigma(r)}$. where $\psi = (\psi_1, \psi_2, \dots, \psi_r)^T$ is the weight vector of $\tilde{N}_{\sigma(\beta)}$ ($\beta = 1, 2, \dots, r$), $\psi_\beta \in [0, 1]$ and $\sum_{\beta=1}^r \psi_\beta = 1$.

Theorem 4. Let $\tilde{N}_\alpha = \langle \tau_\alpha^+, \omega_\alpha^+, \upsilon_\alpha^+, \tau_\alpha^-, \omega_\alpha^-, \upsilon_\alpha^- \rangle$ ($\alpha = 1, 2, 3, \dots, r$) be a family of BNNs and $\psi = (\psi_1, \psi_2, \dots, \psi_r)^T$ be the weight vector of \tilde{N}_α , $\psi_\alpha \in [0, 1]$ and $\sum_{\alpha=1}^r \psi_\alpha = 1$. Then, the value aggregated by using BNDOWGA operator is still a BNN, which is calculated by using the following formula

$$BNDOWGA(\tilde{N}_1, \tilde{N}_2, \dots, \tilde{N}_r) = \tilde{N}_{\sigma(1)}^{\psi_1} \otimes_D \tilde{N}_{\sigma(2)}^{\psi_2} \otimes_D \dots \otimes_D \tilde{N}_{\sigma(r)}^{\psi_r} = \left\langle \frac{1}{1+\{\sum_{\beta=1}^r \psi_\beta (\frac{1-\tau_\beta^+}{\tau_\beta^+})^\lambda\}^{\frac{1}{\lambda}}}, 1 - \frac{1}{1+\{\sum_{\beta=1}^r \psi_\beta (\frac{\omega_\beta^+}{1-\omega_\beta^+})^\lambda\}^{\frac{1}{\lambda}}}, 1 - \frac{1}{1+\{\sum_{\beta=1}^r \psi_\beta (\frac{\upsilon_\beta^+}{1-\upsilon_\beta^+})^\lambda\}^{\frac{1}{\lambda}}}, \frac{1}{1+\{\sum_{\beta=1}^r \psi_\beta (\frac{-\tau_\beta^-}{1+\tau_\beta^-})^\lambda\}^{\frac{1}{\lambda}}}, \frac{1}{1+\{\sum_{\beta=1}^r \psi_\beta (\frac{-\omega_\beta^-}{-\omega_\beta^-})^\lambda\}^{\frac{1}{\lambda}}}, \frac{1}{1+\{\sum_{\beta=1}^r \psi_\beta (\frac{1+\upsilon_\beta^-}{-\upsilon_\beta^-})^\lambda\}^{\frac{1}{\lambda}}} \right\rangle. \tag{7}$$

Proof. If $r = 2$ based on the Definition 8 of BNNs for Dombi operations, the following result can be obtained:

$$BNDOWGA(\tilde{N}_1, \tilde{N}_2) = \tilde{N}_{\sigma(1)}^{\psi_1} \otimes_D \tilde{N}_{\sigma(2)}^{\psi_2} = \left\langle \frac{1}{1+\{\psi_1 (\frac{1-\tau_1^+}{\tau_1^+})^\lambda + \psi_2 (\frac{1-\tau_2^+}{\tau_2^+})^\lambda\}^{\frac{1}{\lambda}}}, 1 - \frac{1}{1+\{\psi_1 (\frac{\omega_1^+}{1-\omega_1^+})^\lambda + \psi_2 (\frac{\omega_2^+}{1-\omega_2^+})^\lambda\}^{\frac{1}{\lambda}}}, 1 - \frac{1}{1+\{\psi_1 (\frac{\upsilon_1^+}{1-\upsilon_1^+})^\lambda + \psi_2 (\frac{\upsilon_2^+}{1-\upsilon_2^+})^\lambda\}^{\frac{1}{\lambda}}}, \frac{1}{1+\{\psi_1 (\frac{-\tau_1^-}{1+\tau_1^-})^\lambda + \psi_2 (\frac{-\tau_2^-}{1+\tau_2^-})^\lambda\}^{\frac{1}{\lambda}}}, \frac{1}{1+\{\psi_1 (\frac{-\omega_1^-}{-\omega_1^-})^\lambda + \psi_2 (\frac{-\omega_2^-}{-\omega_2^-})^\lambda\}^{\frac{1}{\lambda}}}, \frac{1}{1+\{\psi_1 (\frac{1+\upsilon_1^-}{-\upsilon_1^-})^\lambda + \psi_2 (\frac{1+\upsilon_2^-}{-\upsilon_2^-})^\lambda\}^{\frac{1}{\lambda}}} \right\rangle,$$

$$= \left\langle \frac{1}{1+\{\sum_{\beta=1}^r \psi_\beta (\frac{1-\tau_\beta^+}{\tau_\beta^+})^\lambda\}^{\frac{1}{\lambda}}}, 1 - \frac{1}{1+\{\sum_{\beta=1}^r \psi_\beta (\frac{\omega_\beta^+}{1-\omega_\beta^+})^\lambda\}^{\frac{1}{\lambda}}}, 1 - \frac{1}{1+\{\sum_{\beta=1}^r \psi_\beta (\frac{\upsilon_\beta^+}{1-\upsilon_\beta^+})^\lambda\}^{\frac{1}{\lambda}}}, \frac{1}{1+\{\sum_{\beta=1}^r \psi_\beta (\frac{-\tau_\beta^-}{1+\tau_\beta^-})^\lambda\}^{\frac{1}{\lambda}}}, \frac{1}{1+\{\sum_{\beta=1}^r \psi_\beta (\frac{-\omega_\beta^-}{-\omega_\beta^-})^\lambda\}^{\frac{1}{\lambda}}}, \frac{1}{1+\{\sum_{\beta=1}^r \psi_\beta (\frac{1+\upsilon_\beta^-}{-\upsilon_\beta^-})^\lambda\}^{\frac{1}{\lambda}}} \right\rangle.$$

If $r = s$, based on the Equation (7), then we have got the following equation:

$$BNDOWGA(\tilde{N}_1, \tilde{N}_2, \dots, \tilde{N}_s) = \bigotimes_{\beta=1}^s \tilde{N}_{\sigma(\beta)}^{\psi_\beta} = \left\langle \frac{1}{1+\{\sum_{\beta=1}^s \psi_\beta (\frac{1-\tau_\beta^+}{\tau_\beta^+})^\lambda\}^{\frac{1}{\lambda}}}, 1 - \frac{1}{1+\{\sum_{\beta=1}^s \psi_\beta (\frac{\omega_\beta^+}{1-\omega_\beta^+})^\lambda\}^{\frac{1}{\lambda}}}, 1 - \frac{1}{1+\{\sum_{\beta=1}^s \psi_\beta (\frac{\upsilon_\beta^+}{1-\upsilon_\beta^+})^\lambda\}^{\frac{1}{\lambda}}}, \frac{1}{1+\{\sum_{\beta=1}^s \psi_\beta (\frac{-\tau_\beta^-}{1+\tau_\beta^-})^\lambda\}^{\frac{1}{\lambda}}}, \frac{1}{1+\{\sum_{\beta=1}^s \psi_\beta (\frac{-\omega_\beta^-}{-\omega_\beta^-})^\lambda\}^{\frac{1}{\lambda}}}, \frac{1}{1+\{\sum_{\beta=1}^s \psi_\beta (\frac{1+\upsilon_\beta^-}{-\upsilon_\beta^-})^\lambda\}^{\frac{1}{\lambda}}} \right\rangle.$$

If $r = s + 1$, then there is following result:

$$BNDOWGA(\tilde{N}_1, \tilde{N}_2, \dots, \tilde{N}_s, \tilde{N}_{s+1}) = \bigotimes_{\beta=1}^s \tilde{N}_{\sigma(\beta)}^{\psi_\beta} \otimes \tilde{N}_{\sigma(s+1)}^{\psi_{s+1}} = \left\langle \frac{1}{1+\{\sum_{\beta=1}^s \psi_\beta (\frac{1-\tau_\beta^+}{\tau_\beta^+})^\lambda\}^{\frac{1}{\lambda}}}, 1 - \frac{1}{1+\{\sum_{\beta=1}^s \psi_\beta (\frac{\omega_\beta^+}{1-\omega_\beta^+})^\lambda\}^{\frac{1}{\lambda}}}, 1 - \frac{1}{1+\{\sum_{\beta=1}^s \psi_\beta (\frac{\upsilon_\beta^+}{1-\upsilon_\beta^+})^\lambda\}^{\frac{1}{\lambda}}}, \frac{1}{1+\{\sum_{\beta=1}^s \psi_\beta (\frac{-\tau_\beta^-}{1+\tau_\beta^-})^\lambda\}^{\frac{1}{\lambda}}}, \frac{1}{1+\{\sum_{\beta=1}^s \psi_\beta (\frac{-\omega_\beta^-}{-\omega_\beta^-})^\lambda\}^{\frac{1}{\lambda}}}, \frac{1}{1+\{\sum_{\beta=1}^s \psi_\beta (\frac{1+\upsilon_\beta^-}{-\upsilon_\beta^-})^\lambda\}^{\frac{1}{\lambda}}} \right\rangle \otimes_D \left\langle \frac{1}{1+\{\psi_{s+1} (\frac{1-\tau_{s+1}^+}{\tau_{s+1}^+})^\lambda\}^{\frac{1}{\lambda}}}, 1 - \frac{1}{1+\{\psi_{s+1} (\frac{\omega_{s+1}^+}{1-\omega_{s+1}^+})^\lambda\}^{\frac{1}{\lambda}}}, 1 - \frac{1}{1+\{\psi_{s+1} (\frac{\upsilon_{s+1}^+}{1-\upsilon_{s+1}^+})^\lambda\}^{\frac{1}{\lambda}}}, \frac{1}{1+\{\psi_{s+1} (\frac{-\tau_{s+1}^-}{1+\tau_{s+1}^-})^\lambda\}^{\frac{1}{\lambda}}}, \frac{1}{1+\{\psi_{s+1} (\frac{-\omega_{s+1}^-}{-\omega_{s+1}^-})^\lambda\}^{\frac{1}{\lambda}}}, \frac{1}{1+\{\psi_{s+1} (\frac{1+\upsilon_{s+1}^-}{-\upsilon_{s+1}^-})^\lambda\}^{\frac{1}{\lambda}}} \right\rangle = \left\langle \frac{1}{1+\{\sum_{\beta=1}^{s+1} \psi_\beta (\frac{1-\tau_\beta^+}{\tau_\beta^+})^\lambda\}^{\frac{1}{\lambda}}}, 1 - \frac{1}{1+\{\sum_{\beta=1}^{s+1} \psi_\beta (\frac{\omega_\beta^+}{1-\omega_\beta^+})^\lambda\}^{\frac{1}{\lambda}}}, 1 - \frac{1}{1+\{\sum_{\beta=1}^{s+1} \psi_\beta (\frac{\upsilon_\beta^+}{1-\upsilon_\beta^+})^\lambda\}^{\frac{1}{\lambda}}}, \frac{1}{1+\{\sum_{\beta=1}^{s+1} \psi_\beta (\frac{-\tau_\beta^-}{1+\tau_\beta^-})^\lambda\}^{\frac{1}{\lambda}}}, \frac{1}{1+\{\sum_{\beta=1}^{s+1} \psi_\beta (\frac{-\omega_\beta^-}{-\omega_\beta^-})^\lambda\}^{\frac{1}{\lambda}}}, \frac{1}{1+\{\sum_{\beta=1}^{s+1} \psi_\beta (\frac{1+\upsilon_\beta^-}{-\upsilon_\beta^-})^\lambda\}^{\frac{1}{\lambda}}} \right\rangle.$$

Hence, Theorem 4 is true for $r = s + 1$. Thus, Equation (7) holds for all r . \square

The BNDOWGA operator also has the following properties:

- (1) **Idempotency:** Let all the BNNs be $\tilde{N}_\alpha = \langle \tau_\alpha^+, \omega_\alpha^+, \upsilon_\alpha^+, \tau_\alpha^-, \omega_\alpha^-, \upsilon_\alpha^- \rangle = \tilde{N}$ ($\alpha = 1, 2, 3, \dots, r$), where $\psi = (\psi_1, \psi_2, \dots, \psi_r)^T$ be the weight vector of \tilde{N}_α , $\psi_\alpha \in [0, 1]$ and $\sum_{\alpha=1}^r \psi_\alpha = 1$. Then, $BNDOWGA(\tilde{N}_1, \tilde{N}_2, \dots, \tilde{N}_r) = \tilde{N}$.
- (2) **Monotonicity:** Let \tilde{N}_α ($\alpha = 1, 2, \dots, r$) and \tilde{N}'_α ($\alpha = 1, 2, \dots, r$) be two families of BNNs, where $\psi = (\psi_1, \psi_2, \dots, \psi_r)^T$ be the weight vector of \tilde{N}_α and \tilde{N}'_α , $\psi_\alpha \in [0, 1]$ and $\sum_{\alpha=1}^r \psi_\alpha = 1$. For all α , if $\tilde{N}_\alpha \geq \tilde{N}'_\alpha$, then $BNDOWGA(\tilde{N}_1, \tilde{N}_2, \dots, \tilde{N}_r) \geq BNDOWGA(\tilde{N}'_1, \tilde{N}'_2, \dots, \tilde{N}'_r)$.

(3) **Boundedness:** Let $\tilde{N}_\alpha = \langle \tau_\alpha^+, \omega_\alpha^+, \mathcal{U}_\alpha^+, \tau_\alpha^-, \omega_\alpha^-, \mathcal{U}_\alpha^- \rangle$ ($\alpha = 1, 2, 3, \dots, r$) be a family of BNNs, where $\psi = (\psi_1, \psi_2, \dots, \psi_r)^T$ be the weight vector of \tilde{N}_α , $\psi_\alpha \in [1, 0]$ and $\sum_{\alpha=1}^r \psi_\alpha = 1$. Therefore, we have $\text{BNDOWGA}(\tilde{N}_1^-, \tilde{N}_2^-, \dots, \tilde{N}_r^-) \leq \text{BNDOWGA}(\tilde{N}_1^+, \tilde{N}_2^+, \dots, \tilde{N}_r^+) \leq \text{BNDOWGA}(\tilde{N}_1^+, \tilde{N}_2^+, \dots, \tilde{N}_r^+)$, where $\tilde{N}^- = \langle \tau_{\tilde{N}^-}^+, \omega_{\tilde{N}^-}^+, \mathcal{U}_{\tilde{N}^-}^+, \tau_{\tilde{N}^-}^-, \omega_{\tilde{N}^-}^-, \mathcal{U}_{\tilde{N}^-}^- \rangle$

$$= \begin{cases} \langle \min(\tau_1^+, \tau_2^+, \dots, \tau_r^+), \max(\omega_1^+, \omega_2^+, \dots, \omega_r^+), \\ \max(\mathcal{U}_1^+, \mathcal{U}_2^+, \dots, \mathcal{U}_r^+), \max(\tau_1^-, \tau_2^-, \dots, \tau_r^-), \\ \min(\omega_\alpha^-, \omega_\alpha^-, \dots, \omega_\alpha^-), \min(\mathcal{U}_1^-, \mathcal{U}_2^-, \dots, \mathcal{U}_r^-) \rangle & \text{and} \\ \tilde{N}^+ = \langle \tau_{\tilde{N}^+}^+, \omega_{\tilde{N}^+}^+, \mathcal{U}_{\tilde{N}^+}^+, \tau_{\tilde{N}^+}^-, \omega_{\tilde{N}^+}^-, \mathcal{U}_{\tilde{N}^+}^- \rangle \\ = \begin{cases} \langle \max(\tau_1^+, \tau_2^+, \dots, \tau_r^+), \min(\omega_1^+, \omega_2^+, \dots, \omega_r^+), \\ \min(\mathcal{U}_1^+, \mathcal{U}_2^+, \dots, \mathcal{U}_r^+), \min(\tau_1^-, \tau_2^-, \dots, \tau_r^-), \\ \max(\omega_\alpha^-, \omega_\alpha^-, \dots, \omega_\alpha^-), \max(\mathcal{U}_1^-, \mathcal{U}_2^-, \dots, \mathcal{U}_r^-) \rangle. \end{cases} \end{cases}$$

Proof. (1) Since $\tilde{N}_\alpha = \langle \tau_\alpha^+, \omega_\alpha^+, \mathcal{U}_\alpha^+, \tau_\alpha^-, \omega_\alpha^-, \mathcal{U}_\alpha^- \rangle = \tilde{N}(\alpha = 1, 2, 3, \dots, r)$. Then, the following result can be obtained by using Equation (7):

$$\text{BNDOWGA}(\tilde{N}_1, \tilde{N}_2, \dots, \tilde{N}_r) = \left\langle \begin{aligned} & \frac{1}{1 + \{\sum_{\beta=1}^r \psi_\beta (\frac{1-\tau_\beta^+}{\tau_\beta^+})^\lambda\}^{\frac{1}{\lambda}}}, \\ & 1 - \frac{1}{1 + \{\sum_{\beta=1}^r \psi_\beta (\frac{\omega_\beta^+}{1-\omega_\beta^+})^\lambda\}^{\frac{1}{\lambda}}}, \\ & 1 - \frac{1}{1 + \{\sum_{\beta=1}^r \psi_\beta (\frac{\mathcal{U}_\beta^+}{1-\mathcal{U}_\beta^+})^\lambda\}^{\frac{1}{\lambda}}}, \\ & \frac{1}{1 + \{\sum_{\beta=1}^r \psi_\beta (\frac{-\tau_\beta^-}{1+\tau_\beta^-})^\lambda\}^{\frac{1}{\lambda}}} - 1, \\ & \frac{-1}{1 + \{\sum_{\beta=1}^r \psi_\beta (\frac{1+\omega_\beta^-}{-\omega_\beta^-})^\lambda\}^{\frac{1}{\lambda}}}, \\ & \frac{-1}{1 + \{\sum_{\beta=1}^r \psi_\beta (\frac{1+\mathcal{U}_\beta^-}{-\mathcal{U}_\beta^-})^\lambda\}^{\frac{1}{\lambda}}} \end{aligned} \right\rangle$$

$$= \left\langle \begin{aligned} & \frac{1}{1 + \{(\frac{1-\tau^+}{\tau^+})^\lambda\}^{\frac{1}{\lambda}}}, \\ & 1 - \frac{1}{1 + \{(\frac{\omega^+}{1-\omega^+})^\lambda\}^{\frac{1}{\lambda}}}, \\ & 1 - \frac{1}{1 + \{(\frac{\mathcal{U}^+}{1-\mathcal{U}^+})^\lambda\}^{\frac{1}{\lambda}}}, \\ & \frac{1}{1 + \{(\frac{-\tau^-}{1+\tau^-})^\lambda\}^{\frac{1}{\lambda}}} - 1, \\ & \frac{-1}{1 + \{(\frac{1+\omega^-}{-\omega^-})^\lambda\}^{\frac{1}{\lambda}}}, \\ & \frac{-1}{1 + \{(\frac{1+\mathcal{U}^-}{-\mathcal{U}^-})^\lambda\}^{\frac{1}{\lambda}}} \end{aligned} \right\rangle$$

$$= \langle \tau^+, \omega^+, \mathcal{U}^+, \tau^-, \omega^-, \mathcal{U}^- \rangle = \tilde{N}.$$

Hence, $\text{BNDOWGA}(\tilde{N}_1, \tilde{N}_2, \dots, \tilde{N}_r) = \tilde{N}$ holds.

(2) The property is obvious based on the Equation (7).

(3) Let $\tilde{N}^- = \langle \tau_{\tilde{N}^-}^+, \omega_{\tilde{N}^-}^+, \mathcal{U}_{\tilde{N}^-}^+, \tau_{\tilde{N}^-}^-, \omega_{\tilde{N}^-}^-, \mathcal{U}_{\tilde{N}^-}^- \rangle$ and $\tilde{N}^+ = \langle \tau_{\tilde{N}^+}^+, \omega_{\tilde{N}^+}^+, \mathcal{U}_{\tilde{N}^+}^+, \tau_{\tilde{N}^+}^-, \omega_{\tilde{N}^+}^-, \mathcal{U}_{\tilde{N}^+}^- \rangle$. There are the following inequalities:

$$\begin{aligned} & \frac{1}{1 + \{\sum_{\beta=1}^r \psi_\beta (\frac{1-\tau_\beta^+}{\tau_\beta^+})^\lambda\}^{\frac{1}{\lambda}}} \leq \frac{1}{1 + \{\sum_{\beta=1}^r \psi_\beta (\frac{1-\tau_\beta^+}{\tau_\beta^+})^\lambda\}^{\frac{1}{\lambda}}} \\ & \leq \frac{1}{1 + \{\sum_{\beta=1}^r \psi_\beta (\frac{1-\tau_\beta^+}{\tau_\beta^+})^\lambda\}^{\frac{1}{\lambda}}}, \\ & 1 - \frac{1}{1 + \{\sum_{\beta=1}^r \psi_\beta (\frac{\omega_\beta^+}{1-\omega_\beta^+})^\lambda\}^{\frac{1}{\lambda}}} \leq 1 - \frac{1}{1 + \{\sum_{\beta=1}^r \psi_\beta (\frac{\omega_\beta^+}{1-\omega_\beta^+})^\lambda\}^{\frac{1}{\lambda}}} \\ & \leq 1 - \frac{1}{1 + \{\sum_{\beta=1}^r \psi_\beta (\frac{\omega_\beta^+}{1-\omega_\beta^+})^\lambda\}^{\frac{1}{\lambda}}}, \\ & 1 - \frac{1}{1 + \{\sum_{\beta=1}^r \psi_\beta (\frac{\mathcal{U}_\beta^+}{1-\mathcal{U}_\beta^+})^\lambda\}^{\frac{1}{\lambda}}} \leq 1 - \frac{1}{1 + \{\sum_{\beta=1}^r \psi_\beta (\frac{\mathcal{U}_\beta^+}{1-\mathcal{U}_\beta^+})^\lambda\}^{\frac{1}{\lambda}}} \\ & \leq 1 - \frac{1}{1 + \{\sum_{\beta=1}^r \psi_\beta (\frac{\mathcal{U}_\beta^+}{1-\mathcal{U}_\beta^+})^\lambda\}^{\frac{1}{\lambda}}}, \\ & \frac{1}{1 + \{\sum_{\beta=1}^r \psi_\beta (\frac{-\tau_\beta^-}{1+\tau_\beta^-})^\lambda\}^{\frac{1}{\lambda}}} - 1 \\ & \leq \frac{1}{1 + \{\sum_{\beta=1}^r \psi_\beta (\frac{-\tau_\beta^-}{1+\tau_\beta^-})^\lambda\}^{\frac{1}{\lambda}}} - 1 \\ & \leq \frac{1}{1 + \{\sum_{\beta=1}^r \psi_\beta (\frac{-\tau_\beta^-}{1+\tau_\beta^-})^\lambda\}^{\frac{1}{\lambda}}} - 1, \\ & \frac{-1}{1 + \{\sum_{\beta=1}^r \psi_\beta (\frac{1+\omega_\beta^-}{-\omega_\beta^-})^\lambda\}^{\frac{1}{\lambda}}} \leq \frac{-1}{1 + \{\sum_{\beta=1}^r \psi_\beta (\frac{1+\omega_\beta^-}{-\omega_\beta^-})^\lambda\}^{\frac{1}{\lambda}}} \leq \\ & \frac{-1}{1 + \{\sum_{\beta=1}^r \psi_\beta (\frac{1+\omega_\beta^-}{-\omega_\beta^-})^\lambda\}^{\frac{1}{\lambda}}}, \\ & \frac{-1}{1 + \{\sum_{\beta=1}^r \psi_\beta (\frac{1+\mathcal{U}_\beta^-}{-\mathcal{U}_\beta^-})^\lambda\}^{\frac{1}{\lambda}}} \leq \frac{-1}{1 + \{\sum_{\beta=1}^r \psi_\beta (\frac{1+\mathcal{U}_\beta^-}{-\mathcal{U}_\beta^-})^\lambda\}^{\frac{1}{\lambda}}} \leq \\ & \frac{-1}{1 + \{\sum_{\beta=1}^r \psi_\beta (\frac{1+\mathcal{U}_\beta^-}{-\mathcal{U}_\beta^-})^\lambda\}^{\frac{1}{\lambda}}}. \end{aligned}$$

Hence, $\text{BNDOWGA}(\tilde{N}^-, \tilde{N}^-, \dots, \tilde{N}^-) \leq \text{BNDOWGA}(\tilde{N}_1, \tilde{N}_2, \dots, \tilde{N}_r) \leq \text{BNDOWGA}(\tilde{N}^+, \tilde{N}^+, \dots, \tilde{N}^+)$ holds. \square

Example 2. Let $\tilde{N}_1 = \langle 0.6, 0.7, 0.3, -0.6, -0.3, -0.5 \rangle$, $\tilde{N}_2 = \langle 0.5, 0.4, 0.6, -0.6, -0.7, -0.3 \rangle$, $\tilde{N}_3 = \langle 0.6, 0.7, 0.4, -0.9, -0.7, -0.7 \rangle$ and $\tilde{N}_4 = \langle 0.2, 0.6, 0.8, -0.6, -0.3, -0.9 \rangle$ be four BNNs and let the weight vector of BNNs \tilde{N}_α ($\alpha = 1, 2, 3, 4$) be $\psi = (\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{8})^T$. $\psi_1 = \frac{1}{2}$, $\psi_2 = \frac{1}{4}$, $\psi_3 = \frac{1}{8}$ and $\psi_4 = \frac{1}{8}$ are the weights of \tilde{N}_α ($\alpha = 1, 2, 3, 4$) such that $\sum_{\alpha=1}^4 \psi_\alpha = 1$. Then, by Theorem(4), for $\lambda = 3$

$$\text{BNDOWGA}(\tilde{N}_1, \tilde{N}_2, \tilde{N}_3, \tilde{N}_4) = \left\langle \begin{aligned} & \frac{1}{1 + \{(\frac{1}{2}(\frac{1-0.6}{0.6})^3 + \frac{1}{4}(\frac{1-0.5}{0.5})^3 + \frac{1}{8}(\frac{1-0.6}{0.6})^3 + \frac{1}{8}(\frac{1-0.2}{0.2})^3)\}^{\frac{1}{3}}}, \\ & 1 - \frac{1}{1 + \{(\frac{1}{2}(\frac{0.7}{1-0.7})^3 + \frac{1}{4}(\frac{0.4}{1-0.4})^3 + \frac{1}{8}(\frac{0.7}{1-0.7})^3 + \frac{1}{8}(\frac{0.6}{1-0.6})^3)\}^{\frac{1}{3}}}, \\ & 1 - \frac{1}{1 + \{(\frac{1}{2}(\frac{0.4}{1-0.4})^3 + \frac{1}{4}(\frac{0.6}{1-0.6})^3 + \frac{1}{8}(\frac{0.3}{1-0.3})^3 + \frac{1}{8}(\frac{0.8}{1-0.8})^3)\}^{\frac{1}{3}}}, \\ & \frac{1}{1 + \{(\frac{1}{2}(\frac{0.9}{1+0.9})^3 + \frac{1}{4}(\frac{0.6}{1+0.6})^3 + \frac{1}{8}(\frac{0.6}{1+0.6})^3 + \frac{1}{8}(\frac{0.6}{1+0.6})^3)\}^{\frac{1}{3}}} - 1, \\ & \frac{-1}{1 + \{(\frac{1}{2}(\frac{1-0.7}{0.7})^3 + \frac{1}{4}(\frac{1-0.7}{0.7})^3 + \frac{1}{8}(\frac{1-0.3}{0.3})^3 + \frac{1}{8}(\frac{1-0.3}{0.3})^3)\}^{\frac{1}{3}}}, \\ & \frac{-1}{1 + \{(\frac{1}{2}(\frac{1-0.7}{0.7})^3 + \frac{1}{4}(\frac{1-0.3}{0.3})^3 + \frac{1}{8}(\frac{1-0.5}{0.5})^3 + \frac{1}{8}(\frac{1-0.9}{0.9})^3)\}^{\frac{1}{3}}} \end{aligned} \right\rangle,$$

$$\text{BNDOWGA}(\tilde{N}_1, \tilde{N}_2, \tilde{N}_3, \tilde{N}_4) = \langle 0.3294, 0.6706, 0.6753, -0.8774, -0.4034, -0.4008 \rangle.$$

V. MODEL FOR MADM USING BIPOLAR NEUTROSOPHIC FUZZY INFORMATION

In this section, we discuss two comprehensive MADM methods are extended based on the proposed BNDWGA and BNDOWGA operators.

For MADM model with bipolar neutrosophic fuzzy information, let $A = \{A_1, A_2, \dots, A_r\}$ be a set of alternatives and $C = \{C_1, C_2, \dots, C_r\}$ be a set of attributes. Let $\psi = (\psi_1, \psi_2, \dots, \psi_r)^T$ be the weight vector of attributes such that $\psi_\beta > 0$, $\sum_{\beta=1}^r \psi_\beta = 1$ ($\beta = 1, 2, \dots, r$) and ψ_β refers to the weight of attribute C_β . Suppose that $N = (\tilde{N}_{\alpha\beta})_{s \times r} = (\tau_{\alpha\beta}^+, \omega_{\alpha\beta}^+, \upsilon_{\alpha\beta}^+, \tau_{\alpha\beta}^-, \omega_{\alpha\beta}^-, \upsilon_{\alpha\beta}^-)_{s \times r}$ ($\alpha = 1, 2, \dots, s$) ($\beta = 1, 2, \dots, r$) is BNN decision matrix, where $\tau_{\alpha\beta}^+, \omega_{\alpha\beta}^+, \upsilon_{\alpha\beta}^+$ indicates the truth membership degree, indeterminacy membership degree and falsity membership degree of alternative A_α under attribute C_β with respect to positive preferences and $\tau_{\alpha\beta}^-, \omega_{\alpha\beta}^-, \upsilon_{\alpha\beta}^-$ indicates the truth membership degree, indeterminacy membership degree and falsity membership degree of alternative A_α under attribute C_β with respect to negative preferences. We have conditions $\tau_{\alpha\beta}^+, \omega_{\alpha\beta}^+, \upsilon_{\alpha\beta}^+, \tau_{\alpha\beta}^-, \omega_{\alpha\beta}^-, \upsilon_{\alpha\beta}^- \in [0, 1]$ such that $0 \leq \tau_{\alpha\beta}^+ + \omega_{\alpha\beta}^+ + \upsilon_{\alpha\beta}^+ - \tau_{\alpha\beta}^- - \omega_{\alpha\beta}^- - \upsilon_{\alpha\beta}^- \leq 6$ for ($\alpha = 1, 2, \dots, s$) ($\beta = 1, 2, \dots, r$).

An algorithm is constructed based on proposed bipolar neutrosophic Dombi aggregation operators which solves MADM problems.

Algorithm

Step 1 Collect information on the bipolar neutrosophic evaluation.

Step 2 Calculate score and the accuracy values of collected information.

The score values $s(\tilde{N}_{\alpha\beta})$ and accuracy values $a(\tilde{N}_{\alpha\beta})$ of alternatives A_α can be calculated by using Equations (1) and (2).

Step 3 The compressions method in Definition 4 to reorder information on evaluation under each attribute.

The comparison method is used to reorder $\tilde{N}_{\alpha\beta}$.

Step 4 Derive the collective BNN \tilde{N}_α ($\alpha = 1, 2, \dots, s$) for the alternative A_α ($\alpha = 1, 2, \dots, s$).

Method (1) Utilize BNDWGA operator to calculate the collective BNN for each alternative, then

$$\tilde{N}_\alpha = \text{BNDWGA}(\tilde{N}_{\alpha 1}, \tilde{N}_{\alpha 2}, \dots, \tilde{N}_{\alpha r}) = \bigotimes_{\beta=1}^r \tilde{N}_{\alpha\beta}^{\psi_\beta}$$

$$= \left\langle \left(\frac{1}{1 + \{\sum_{\beta=1}^r \psi_\beta (\frac{1 - \tau_{\alpha\beta}^+}{\tau_{\alpha\beta}^+})^\lambda\}^{\frac{1}{\lambda}}}, 1 - \frac{1}{1 + \{\sum_{\beta=1}^r \psi_\beta (\frac{\omega_{\alpha\beta}^+}{1 - \omega_{\alpha\beta}^+})^\lambda\}^{\frac{1}{\lambda}}}, \right. \right.$$

$$\left. \left. 1 - \frac{1}{1 + \{\sum_{\beta=1}^r \psi_\beta (\frac{\upsilon_{\alpha\beta}^+}{1 - \upsilon_{\alpha\beta}^+})^\lambda\}^{\frac{1}{\lambda}}}, \frac{1}{1 + \{\sum_{\beta=1}^r \psi_\beta (\frac{-\tau_{\alpha\beta}^-}{1 + \tau_{\alpha\beta}^-})^\lambda\}^{\frac{1}{\lambda}}}, -1, \right. \right.$$

$$\left. \left. \frac{-1}{1 + \{\sum_{\beta=1}^r \psi_\beta (\frac{1 + \omega_{\alpha\beta}^-}{-\omega_{\alpha\beta}^-})^\lambda\}^{\frac{1}{\lambda}}}, \frac{-1}{1 + \{\sum_{\beta=1}^r \psi_\beta (\frac{1 + \upsilon_{\alpha\beta}^-}{-\upsilon_{\alpha\beta}^-})^\lambda\}^{\frac{1}{\lambda}}} \right\rangle, \quad (8)$$

where $\psi = (\psi_1, \psi_2, \dots, \psi_r)$ is the weight vector such that $\psi_\beta \in [0, 1]$ and $\sum_{\beta=1}^r \psi_\beta = 1$.

Method (2) Utilize BNDOWGA operator to calculate the collective BNN for each alternative, then

$$\tilde{N}_\alpha = \text{BNDOWGA}(\tilde{N}_{\alpha 1}, \tilde{N}_{\alpha 2}, \dots, \tilde{N}_{\alpha r}) = \bigotimes_{\gamma=1}^r \tilde{N}_{\sigma(\alpha\gamma)}^{\psi_\gamma}$$

$$= \left\langle \left(\frac{1}{1 + \{\sum_{\gamma=1}^r \psi_\gamma (\frac{1 - \tau_{\alpha\sigma(\gamma)}^+}{\tau_{\alpha\sigma(\gamma)}^+})^\lambda\}^{\frac{1}{\lambda}}}, \right. \right.$$

$$\left. \left. 1 - \frac{1}{1 + \{\sum_{\gamma=1}^r \psi_\gamma (\frac{\omega_{\alpha\sigma(\gamma)}^+}{1 - \omega_{\alpha\sigma(\gamma)}^+})^\lambda\}^{\frac{1}{\lambda}}}, \right. \right.$$

$$\left. \left. 1 - \frac{1}{1 + \{\sum_{\gamma=1}^r \psi_\gamma (\frac{\upsilon_{\alpha\sigma(\gamma)}^+}{1 - \upsilon_{\alpha\sigma(\gamma)}^+})^\lambda\}^{\frac{1}{\lambda}}}, \frac{1}{1 + \{\sum_{\gamma=1}^r \psi_\gamma (\frac{-\tau_{\alpha\sigma(\gamma)}^-}{1 + \tau_{\alpha\sigma(\gamma)}^-})^\lambda\}^{\frac{1}{\lambda}}}, -1, \right. \right.$$

$$\left. \left. \frac{-1}{1 + \{\sum_{\gamma=1}^r \psi_\gamma (\frac{1 + \omega_{\alpha\sigma(\gamma)}^-}{-\omega_{\alpha\sigma(\gamma)}^-})^\lambda\}^{\frac{1}{\lambda}}}, \frac{-1}{1 + \{\sum_{\gamma=1}^r \psi_\gamma (\frac{1 + \upsilon_{\alpha\sigma(\gamma)}^-}{-\upsilon_{\alpha\sigma(\gamma)}^-})^\lambda\}^{\frac{1}{\lambda}}} \right\rangle, \quad (9)$$

where σ is permutation that orders the elements: $\tilde{N}_{\sigma(\alpha 1)} \geq \tilde{N}_{\sigma(\alpha 2)} \geq \dots \geq \tilde{N}_{\sigma(\alpha r)}$. where $\psi = (\psi_1, \psi_2, \dots, \psi_r)^T$ is the weight vector such that $\psi_\gamma \in [0, 1]$ and $\sum_{\gamma=1}^r \psi_\gamma = 1$.

Step 5 Calculate the score values $s(\tilde{N}_\alpha)$ ($\alpha = 1, 2, \dots, s$) of BNNs \tilde{N}_α ($\alpha = 1, 2, \dots, s$) to rank all the alternatives A_α ($\alpha = 1, 2, \dots, s$) and then select favorable one(s). If score values \tilde{N}_α and \tilde{N}_β of BNNs are equal, then we calculate accuracy values $a(\tilde{N}_\alpha)$ and $a(\tilde{N}_\beta)$ of BNNs \tilde{N}_α and \tilde{N}_β , respectively and then rank the alternatives A_α and A_β as accuracy values $a(\tilde{N}_\alpha)$ and $a(\tilde{N}_\beta)$.

Step 6 Rank all the alternatives A_α ($\alpha = 1, 2, \dots, s$) and select favorable one(s).

Step 7 End.

VI. NUMERICAL EXAMPLE

A numerical example of the selection of cultivating crop taken from Deli et al. [10] is provided here. Furthermore, parametric analysis and comparative analysis confirm the flexibility and effectiveness of the proposed methods. In order to increase the production of agriculture, an agriculture department considers the selecting a crop for cultivating in the farm. Four cultivating crops A_1, A_2, A_3 , and A_4 are selected for further evaluation through preliminary screening. The agriculture department decided to invite four group experts to evaluate information. The expert group consists of investment experts, land experts, weather experts and labour experts. The four cultivating crops are evaluated by experts on the basis of four attributes or criteria: cost (C_1), nutrition of land (C_2), effect of weather (C_3), labor (C_4). These attributes are interactive and interlinked.

Algorithm

Step 1 Collect information on bipolar neutrosophic evaluation.

The information collected from expert discussion on evaluation is given in Table 1.

TABLE 1. Bipolar neutrosophic evaluation information.

	C_1	C_2	C_3	C_4
A_1	$\langle 0.5, 0.7, 0.2, -0.7, -0.3, -0.6 \rangle$	$\langle 0.4, 0.4, 0.5, -0.7, -0.8, -0.4 \rangle$	$\langle 0.7, 0.7, 0.5, -0.8, -0.7, -0.6 \rangle$	$\langle 0.1, 0.5, 0.7, -0.5, -0.2, -0.8 \rangle$
A_2	$\langle 0.9, 0.7, 0.5, -0.7, -0.7, -0.1 \rangle$	$\langle 0.7, 0.6, 0.8, -0.7, -0.5, -0.1 \rangle$	$\langle 0.9, 0.4, 0.6, -0.1, -0.7, -0.5 \rangle$	$\langle 0.5, 0.2, 0.7, -0.5, -0.1, -0.9 \rangle$
A_3	$\langle 0.3, 0.4, 0.2, -0.6, -0.3, -0.7 \rangle$	$\langle 0.2, 0.2, 0.2, -0.4, -0.7, -0.4 \rangle$	$\langle 0.9, 0.5, 0.5, -0.6, -0.5, -0.2 \rangle$	$\langle 0.7, 0.5, 0.3, -0.4, -0.2, -0.2 \rangle$
A_4	$\langle 0.9, 0.7, 0.2, -0.8, -0.6, -0.1 \rangle$	$\langle 0.3, 0.5, 0.2, -0.5, -0.5, -0.2 \rangle$	$\langle 0.5, 0.4, 0.5, -0.1, -0.7, -0.2 \rangle$	$\langle 0.4, 0.2, 0.8, -0.5, -0.5, -0.6 \rangle$

Step 2 Calculate score and accuracy values of collected information.

For each alternative A_α under attribute C_β , the score values $s(\tilde{N}_{\alpha\beta})$ and accuracy values $a(\tilde{N}_{\alpha\beta})$ can be calculated based on Equations (1) and (2). The score values $s(\tilde{N}_{\alpha\beta})$ and accuracy values $a(\tilde{N}_{\alpha\beta})$ are shown in Tables 2 and 3, respectively.

TABLE 2. Score values $s(\tilde{N}_{\alpha\beta})$.

	C_1	C_2	C_3	C_4
A_1	0.4667	0.5000	0.5000	0.4000
A_2	0.4667	0.3667	0.6667	0.5167
A_3	0.5167	0.5833	0.5000	0.4833
A_4	0.4833	0.4667	0.5667	0.5000

TABLE 3. Accuracy values $a(\tilde{N}_{\alpha\beta})$.

	C_1	C_2	C_3	C_4
A_1	0.2000	-0.4000	0	-0.3000
A_2	-0.2000	-0.7000	0.7000	0.2000
A_3	0.2000	0	0	0.2000
A_4	0	-0.2000	0.1000	-0.3000

Step 3 Reorder information on evaluation under each attribute.

The comparison method based on Definition 4 is used to reorder $\tilde{N}_{\alpha\beta}$.

TABLE 4. Reordering bipolar neutrosophic evaluation information by using comparison method based on Definition 4.

	C_1	C_2	C_3	C_4
A_1	$\langle 0.7, 0.7, 0.5, -0.8, -0.7, -0.6 \rangle$	$\langle 0.4, 0.4, 0.5, -0.7, -0.8, -0.4 \rangle$	$\langle 0.5, 0.7, 0.2, -0.7, -0.3, -0.6 \rangle$	$\langle 0.1, 0.5, 0.7, -0.5, -0.2, -0.8 \rangle$
A_2	$\langle 0.9, 0.4, 0.6, -0.1, -0.7, -0.5 \rangle$	$\langle 0.5, 0.2, 0.7, -0.5, -0.1, -0.9 \rangle$	$\langle 0.9, 0.7, 0.5, -0.7, -0.7, -0.1 \rangle$	$\langle 0.7, 0.6, 0.8, -0.7, -0.5, -0.1 \rangle$
A_3	$\langle 0.2, 0.2, 0.2, -0.4, -0.7, -0.4 \rangle$	$\langle 0.3, 0.4, 0.2, -0.6, -0.3, -0.7 \rangle$	$\langle 0.9, 0.5, 0.5, -0.6, -0.5, -0.2 \rangle$	$\langle 0.7, 0.5, 0.3, -0.4, -0.2, -0.2 \rangle$
A_4	$\langle 0.5, 0.4, 0.5, -0.1, -0.7, -0.2 \rangle$	$\langle 0.4, 0.2, 0.8, -0.5, -0.5, -0.6 \rangle$	$\langle 0.9, 0.7, 0.2, -0.8, -0.6, -0.1 \rangle$	$\langle 0.3, 0.5, 0.2, -0.5, -0.5, -0.2 \rangle$

Step 4 Derive the collective BNN \tilde{N}_α ($\alpha = 1, 2, \dots, s$) for the alternative A_α ($\alpha = 1, 2, \dots, s$). Method (1) utilize BNDWGA operator using Eq. (8) and supporting $\lambda = 7$ to calculate the collective BNN for each alternative, then

$$\begin{aligned} \tilde{N}_1 &= \langle 0.1301, 0.6857, 0.6345, -0.7517, -0.2494, -0.4480 \rangle; \\ \tilde{N}_2 &= \langle 0.5735, 0.6795, 0.7667, -0.6913, -0.1301, -0.1038 \rangle; \\ \tilde{N}_3 &= \langle 0.2324, 0.4546, 0.4264, -0.5839, -0.2494, -0.2335 \rangle; \\ \tilde{N}_4 &= \langle 0.3424, 0.6788, 0.7482, -0.7837, -0.5323, -0.1092 \rangle. \end{aligned}$$

Method (2) utilize BNDOWGA operator using Eq. (9) and supporting $\lambda = 7$ to calculate the collective BNN for each alternative, then

$$\begin{aligned} \tilde{N}_1 &= \langle 0.1301, 0.6857, 0.6345, -0.7841, -0.2512, -0.4480 \rangle; \\ \tilde{N}_2 &= \langle 0.5493, 0.6357, 0.7496, -0.6569, -0.1193, -0.1193 \rangle; \\ \tilde{N}_3 &= \langle 0.2160, 0.4527, 0.4264, -0.5661, -0.2506, -0.2335 \rangle; \\ \tilde{N}_4 &= \langle 0.3626, 0.6343, 0.7664, -0.7482, -0.5342, -0.1298 \rangle. \end{aligned}$$

Step 5 Calculate the score values $s(\tilde{N}_\alpha)$ ($\alpha = 1, 2, \dots, s$) of BNNs \tilde{N}_α ($\alpha = 1, 2, \dots, s$) for each alternatives A_α ($\alpha = 1, 2, \dots, s$). The score values $s(\tilde{N}_\alpha)$ is calculated by using Equation (1).

Method 1 The following score values are obtained by using the BNDWGA operator.

$$s(\tilde{N}_1) = 0.2926; s(\tilde{N}_2) = 0.2783; s(\tilde{N}_3) = 0.3751; s(\tilde{N}_4) = 0.2955.$$

Method 2 The following score values are obtained by using the BNDOWGA operator.

$$s(\tilde{N}_1) = 0.2875; s(\tilde{N}_2) = 0.2910; s(\tilde{N}_3) = 0.3758; s(\tilde{N}_4) = 0.3129.$$

Step 6 Rank all the alternatives A_α ($\alpha = 1, 2, \dots, s$) and select favorable one(s).

The alternative can be ranked in descending order based on the comparison method, and favorable alternative can be selected.

Method 1 The following ranking order based on score values is obtained by using BNDWGA operator: $A_3 \succ A_4 \succ A_1 \succ A_2$. Thus, A_3 is favorable.

Method 2 The following ranking order based on the score values is obtained by using BNDOWGA operator: $A_3 \succ A_4 \succ A_2 \succ A_1$. Thus, A_3 is favorable.

Step 7 End.

VII. PARAMETRIC ANALYSIS AND COMPARATIVE ANALYSIS

This section describes effect of parametric λ on decision making results and comparison between proposed methods and existing methods.

A. ANALYSIS ON THE EFFECT OF PARAMETER λ ON DECISION MAKING RESULTS

This subsection discusses the effect of parameter λ in detail. First, effect of parameter λ on the proposed operators is as follows.

Table 5 shows that the corresponding ranking orders with

TABLE 5. Ranking orders with parameter of BNDWGA operator.

λ	$s(\tilde{N}_1)$	$s(\tilde{N}_2)$	$s(\tilde{N}_3)$	$s(\tilde{N}_4)$	BNDWGA
1	0.4089	0.3886	0.4729	0.4105	$A_3 \succ A_4 \succ A_1 \succ A_2$
2	0.3641	0.3389	0.4387	0.3570	$A_3 \succ A_1 \succ A_4 \succ A_2$
3	0.3372	0.3134	0.4162	0.3305	$A_3 \succ A_1 \succ A_4 \succ A_2$
4	0.3200	0.2988	0.4007	0.3158	$A_3 \succ A_1 \succ A_4 \succ A_2$
5	0.3081	0.2895	0.3896	0.3066	$A_3 \succ A_1 \succ A_4 \succ A_2$
6	0.2993	0.2830	0.3814	0.3002	$A_3 \succ A_4 \succ A_1 \succ A_2$
7	0.2926	0.2783	0.3751	0.2955	$A_3 \succ A_4 \succ A_1 \succ A_2$
8	0.2874	0.2748	0.3702	0.2920	$A_3 \succ A_4 \succ A_1 \succ A_2$
9	0.2832	0.2720	0.3662	0.2892	$A_3 \succ A_4 \succ A_1 \succ A_2$
10	0.2799	0.2698	0.3630	0.2869	$A_3 \succ A_4 \succ A_1 \succ A_2$

respect to the BNDWGA operator are changed as the value of λ changing from 1 to 10.

Table 5 shows that the ranking order is slightly different

when the value of λ is changed for the BNDWGA operator, but the corresponding favorable alternative remains identical. For $1 \leq \lambda \leq 10$, the ranking orders are shown as $A_3 \succ A_4 \succ A_1 \succ A_2$ and $A_3 \succ A_1 \succ A_4 \succ A_2$. As a result, the favorable alternative is A_3 as shown in shown in Figure 1.

Table 6 shows that the corresponding ranking orders with

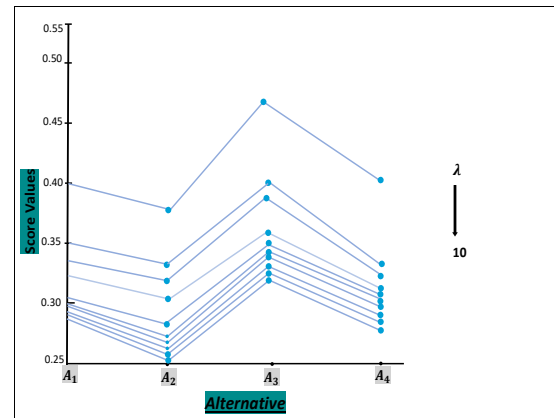


FIGURE 1. Behaviour of BNDOWGA Operator.

TABLE 6. Ranking orders with parameter of BNDOWGA operator.

λ	$s(\tilde{N}_1)$	$s(\tilde{N}_2)$	$s(\tilde{N}_3)$	$s(\tilde{N}_4)$	BNDOWGA
1	0.4134	0.4447	0.4871	0.4542	$A_3 \succ A_4 \succ A_2 \succ A_1$
2	0.3608	0.3750	0.4486	0.4002	$A_3 \succ A_4 \succ A_2 \succ A_1$
3	0.3319	0.3405	0.4223	0.3670	$A_3 \succ A_4 \succ A_2 \succ A_1$
4	0.3142	0.3203	0.4042	0.3455	$A_3 \succ A_4 \succ A_2 \succ A_1$
5	0.3022	0.3070	0.3917	0.3309	$A_3 \succ A_4 \succ A_2 \succ A_1$
6	0.2937	0.2977	0.3826	0.3206	$A_3 \succ A_4 \succ A_2 \succ A_1$
7	0.2875	0.2910	0.3758	0.3129	$A_3 \succ A_4 \succ A_2 \succ A_1$
8	0.2827	0.2858	0.3706	0.3071	$A_3 \succ A_4 \succ A_2 \succ A_1$
9	0.2790	0.2818	0.3665	0.3026	$A_3 \succ A_4 \succ A_2 \succ A_1$
10	0.2760	0.2785	0.3632	0.2989	$A_3 \succ A_4 \succ A_2 \succ A_1$

respect to the BNDOWGA operators are changed as the value of λ changing from 1 to 10.

Table 6 shows that ranking order is stable and the corresponding favorable alternative remains identical, when the value of λ is changed for BNDOWGA operator. For, $1 \leq \lambda \leq 10$, the corresponding ranking order is $A_3 \succ A_4 \succ A_2 \succ A_1$, then favorable alternative is A_3 . As a result, favorable stable alternative is A_3 as shown in Figure 2.

B. COMPARATIVE ANALYSIS

In this subsection, a comparative analysis of the proposed methods based on proposed bipolar neutrosophic Dombi aggregation operators with existing methods are discuss in detail. The ranking orders obtained by the proposed methods show that A_3 is favorable alternative. In similar to A_ψ ,

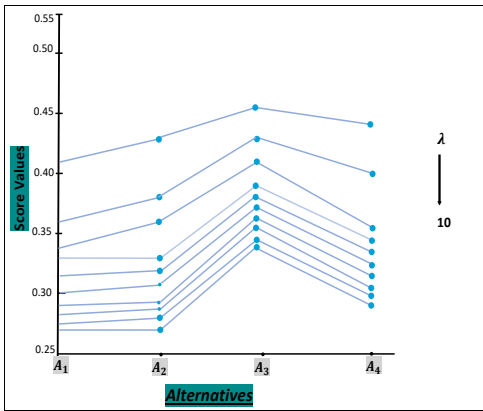


FIGURE 2. Behaviour of BNDOWGA Operator.

TABLE 7. Ranking Orders Obtained by Different Methods.

Methods	Rankings
A_{ψ} operator [11]	$A_3 \succ A_4 \succ A_2 \succ A_1$
G_{ψ} operator [11]	$A_3 \succ A_4 \succ A_2 \succ A_1$
BN-TOPSIS method(ψ_1) [12]	$A_4 \succ A_2 \succ A_3 \succ A_1$
BN-TOPSIS method(ψ_2) [12]	$A_3 \succ A_2 \succ A_4 \succ A_1$
FBNCWBM operator ($s, t = 1$) [9]	$A_3 \succ A_4 \succ A_2 \succ A_1$
FBNCGBM operator ($s, t = 1$) [9]	$A_3 \succ A_4 \succ A_2 \succ A_1$
BNDWGA operator ($\lambda = 7$)	$A_3 \succ A_4 \succ A_1 \succ A_2$
BNDOWGA operator ($\lambda = 7$)	$A_3 \succ A_4 \succ A_2 \succ A_1$

G_{ψ} , BN-TOPSIS(ψ_1) and BN-TOPSIS(ψ_2) methods, the proposed method based on proposed BNDWGA operator does not consider the interaction and interrelation among attributes. In contrast to A_{ψ} , G_{ψ} , BN-TOPSIS(ψ_1) and BN-TOPSIS(ψ_2) method, the proposed BNDWGA operator can choose the appropriate parameters according to the preferences of the DMs. In similar to FBNCWBM ($s, t = 1$) and FBNCGBM ($s, t = 1$) operators, the proposed BNDWGA operator can select the appropriate parameters according to preferences of the DMs. In contrast to FBNCWBM ($s, t = 1$) and FBNCGBM ($s, t = 1$) operators, the proposed BNDWGA operator does not consider the interaction and interrelation among attributes. Some MADM problems are independent of attributes. Then, DMs can use the proposed method based on the proposed BNDWGA operator to solve the MADM problems which are independent of attributes. In contrast to A_{ψ} , G_{ψ} , BN-TOPSIS(ψ_1) and BN-TOPSIS(ψ_2) methods, the proposed method based on proposed BNDOWGA operator considers the interaction and interrelationship among the attributes by using ordered weighted geometric aggregation operator and can choose proper parameters according to preferences of DMs. In similar to FBNCWBM ($s, t = 1$) and FBNCGBM ($s, t = 1$) operators, the proposed BNDOWGA operator can choose proper parameters according to preferences of DMs and can consider the interaction

TABLE 8. Characteristic comparison of different methods

Methods	Flexible measure easier
A_{ψ} operator [11]	No
G_{ψ} operator [11]	No
BN-TOPSIS method(ψ_1) [12]	No
BN-TOPSIS method(ψ_2) [12]	No
FBNCWBM operator ($s, t = 1$) [9]	Yes
FBNCGBM operator ($s, t = 1$) [9]	Yes
BNDWGA operator ($\lambda = 7$)	Yes
BNDOWGA operator ($\lambda = 7$)	Yes

and interrelationship among the attributes. Some MADM problems are dependent of attributes. Then, DMs can use the proposed method based on the proposed BNDOWGA operator to solve these MADM problem which are dependent of attributes. Thus, the proposed methods based on the proposed bipolar neutrosophic Dombi aggregation operators are more reliable and flexible. The DMs can use these proposed methods based on the proposed bipolar neutrosophic Dombi aggregation operators according to their requirements in practical MADM problems.

CONCLUSION

BNSs describe fuzzy, bipolar, inconsistent and uncertain information. BNDWGA and BNDOWGA operators were proposed based on Dombi operations to make sure that the bipolar neutrosophic Dombi aggregation operators are reliable and flexible. Furthermore, a numerical example was given to verify proposed methods. The reliability and flexibility of the proposed methods were further illustrated through a parameter analysis and a comparative analysis with existing methods.

The participation of this paper is as follows. First, BNSs were used to present the decision-making information based on evaluation. Second, Dombi operations were put forward to bipolar neutrosophic fuzzy environments. Third, BNDWGA and BNDOWGA operators are proposed under the bipolar neutrosophic fuzzy environment. Fourth, MADM methods based on proposed bipolar neutrosophic Dombi aggregation operators were developed. Finally, the flexibility and reliability of proposed methods were verified by the numerical example.

In future, BNDWGA and BNDOWGA operators can be put forward to other fuzzy environments such as bipolar neutrosophic soft expert sets and bipolar interval neutrosophic sets. We can develop Pythagorean fuzzy interaction power Bonferroni mean aggregation operators in multiple attribute decision making and some new aggregation operators under single-valued neutrosophic 2-tuple linguistic environment and their applications to multiattribute group decision making in further work.

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COMPETING INTERESTS

All authors are here with confirm that there are no competing interests between them.

REFERENCES

- [1] L.A. Zadeh, "Fuzzy sets", *Inf. and Cont.*, vol. 1, 8(3), pp. 338-353, Jun 1965.
- [2] K. T. Atanassov, "Intuitionistic fuzzy sets," *Fuzzy Sets Syst.*, vol. 20, no. 1, pp. 87-96, Jan. 1986.
- [3] K.T. Atanassov, G. Gargov, "Interval valued intuitionistic fuzzy sets". *Fuzzy Sets Syst.*, vol. 33, no. 1, pp. 343-349, Oct. 1989.
- [4] F. Smarandache, "A unifying field in logics: neutrosophic logic", *Multiple-valued Log.*, vol. 8, no. 3, pp. 489-503, 1999.
- [5] H. Wang, P. Madiraju, Y. Zhang, R. Sunderraman, "Interval neutrosophic sets", *Mathematics* vol.1, pp. 274-277, Sep. 2004.
- [6] H. Wang, F. Smarandache, Y. Zhang, R. Sunderraman, "Single valued neutrosophic sets". *Rev. Air Force Acad.*, vol. 17, pp. 10-14, May 2010.
- [7] J. Gruber, E. H. Seigel, A. L. Purcell, H. A. Earls, G. Cooper, L. F. Barrett, "Unseen positive and negative affective information influences social perception in bipolar I disorder and healthy adults", *J. Affect. Disord.*, vol. 192, pp. 191-198, 2015.
- [8] S. Bistarelli, M.S. Pini, K.B. Venable, "Modelling and solving bipolar preference problems", *Ercim Workshop on constraints Lisbona Giugno*, vol. 23, no. 4, pp. 545-575, May 2015.
- [9] L. Wang, H. Zhang, J. Wang, "Frank Choquet Bonferroni Mean Operators of Bipolar Neutrosophic Sets and Their Application to Multi-criteria Decision-making Problems", *In. Fuzzy Syst.*, vol. 20, no.1, pp. 13-28, Jan. 2018.
- [10] W. R. Zhang, "Bipolar fuzzy sets and relations a computational framework for cognitive modeling and multiagent decision analysis", *In the workshop on fuzzy information processing society bi conference*, pp. 305-309, 1994.
- [11] I. Deli, M. Ali, F. Smarandache, "Bipolar neutrosophic sets and their application based on multi-criteria decision making problems", *Int. Conf. Adv. Mech. Syst.*, pp. 249-254, 2015.
- [12] P. Dey, S. Pramanik, B.C. Giri, "Toposis for solving multi-attribute decision making problems under bipolar neutrosophic environment", *Pons asbl, Brus., Eur. Uni.*, pp. 65-77, Nov. 2016.
- [13] J. A. Dombi, "General class of fuzzy operators, the demorgan class of fuzzy operators and fuzziness measures induced by fuzzy operators", *Fuzzy Sets Syst.*, vol. 8, pp. 149-163, Aug. 1982.
- [14] R.R. Yager, "Prioritized aggregation operators", *Int. J App. Reasoning*, vol. 48, pp. 263-274, Apr. 2008.
- [15] R.R. Yager, J. Kacprzyk, "The Ordered Weighted Averaging Operators", *Theory and Appl.*, pp. 15-28, 1997.
- [16] Z. S. Xu, Q. L. Da, "The ordered weighted geometric averaging operators", *Int. J Intell. Syst.*, vol. 17, no. 7, pp. 709-716, July 2002.
- [17] Chiclana, Francisco, H. Francisco H. V. Enrique, "The ordered weighted geometric operator: properties and application in MCDM Problems", *In Proc. 8th Conf. Inform. Processing and Management of Uncertainty in Know. Syst. IPMU*, pp. 985-991, 2000.
- [18] Herrera, Francisco, H. V. Enrique, C. Francisco, A study of the origin and uses of the ordered weighted geometric operator in multicriteria decision making, *Int. J Intell. Syst.*, vol. 18, no. 6, pp. 689-707, 2003.
- [19] R. R. Yager, "Weighted triangular norms using generating functions", *Int. J Intell. Syst.*, vol. 19, no. 3, pp. 217-231, Mar. 2004.
- [20] R.R. Yager, "Structures generated from weighted fuzzy intersection and union", *J of the Chinese Fuzzy Syst. Ass.*, vol. 2, pp. 37-58, Apr 1996.
- [21] J. Chen, Y. Jun, "Some single-valued neutrosophic Dombi weighted aggregation operators for multiple attribute decision-making", *Symmetry*, vol. 9, no. 6, pp. 82, June 2017.
- [22] M. Akram, F. Smarandache, "Decision-making with bipolar neutrosophic TOPSIS and bipolar neutrosophic ELECTRE-I", *Axioms*, vol. 7, no. 2, pp. 33, Jun 2018.
- [23] P. Liu, Y. Wang, "Interval neutrosophic prioritized OWA operator and its application to multiple attribute decision making", *J Syst. Sci. and Comp.*, vol. 29, no. 3, pp. 681-697, June 2016.
- [24] C. Jana, M. Pal, J. Wang, "Bipolar fuzzy Dombi prioritized aggregation operators in multiple attribute decision making", *Soft Comp.*, vol. 24, no.5, pp. 3631-3646, Mar. 2020.
- [25] J. Ye, "Prioritized aggregation operators of trapezoidal intuitionistic fuzzy sets and their application to multicriteria decision-making", *Neural Comp. and App.*, vol. 25, no.6, pp. 1447-1454, Nov. 2014.
- [26] G. Wei, W. Yu, "Some single-valued neutrosophic dombi prioritized weighted aggregation operators in multiple attribute decision making", *J Intell. and Fuzzy Syst.*, vol. 35, no. 2, pp. 2001-2013, Jan. 2018.
- [27] C. Jana, M. Pal, J. Wang, "Bipolar fuzzy Dombi aggregation operators and its application in multiple-attribute decision-making process", *J of Amb. Intelli. and Human. Comp.*, vol. 10, no. 9, pp. 3533-3549, Sep. 2019.
- [28] C. Jana, "Picture fuzzy Dombi aggregation operators: Application to MADM process", *Appl. Soft Comp.*, vol. 74, pp. 99-109, Jan. 2019.
- [29] P. Liu, Y. Wang, "Interval neutrosophic prioritized OWA operator and its application to multiple attribute decision making", *J Syst. Sci. and Comp.*, vol. 29, no.3, pp. 681-697, June 2016.
- [30] J. Ye, "Prioritized aggregation operators of trapezoidal intuitionistic fuzzy sets and their application to multicriteria decision-making", *Neural Comp. and App.*, vol. 25, no. 6, pp. 1447-1454, Nov. 2014.
- [31] Z. Xu, R. R. Yager, "Some geometric aggregation operators based on intuitionistic fuzzy sets", *Int. J gen. syst.*, vol. 35, no. 4, pp. 417-433, Aug. 2006.
- [32] R. R. Yager, "On ordered weighted averaging aggregation operators in multicriteria decisionmaking", *IEEE Tran. on syst., Man and Cyb.*, vol. 18, no. 1, pp. 183-190, Jan. 1988.
- [33] X. H. Wu, J.Q. Wang, J. J. Peng, X. H. Chen, "Cross-entropy and prioritized aggregation operator with simplified neutrosophic sets and their application in multi-criteria decision-making problems", *Int. J of Fuzzy Syst.*, vol. 18, no. 6, pp. 1104-1116, Dec. 2016.
- [34] S. Ashraf S, S. Abdullah T. Mahmood, "Spherical fuzzy Dombi aggregation operators and their application in group decision making problems", *J of Ambient Intell. and Human. Comput.*, vol. 8, pp. 1-9, June 2019.
- [35] S. Ashraf, S. Abdullah, M. Aslam, "Symmetric sum based aggregation operators for spherical fuzzy information: Application in multi-attribute group decision making problem", *J Intell. and Fuzzy Syst.*, vol 38, no. 4, pp.5241-5255, 2020.
- [36] H. Garg, "Some picture fuzzy aggregation operators and their applications to multicriteria decision-making", *Arab J Sci. Eng.*, vol. 42, no. 12, pp. 5275-5290, 2017.



Fuzzy Theory.

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