Bipolar Neutrosophic Projection Based Models for Multi-attribute Decision Making Problems

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Abstract: ***Bipolar neutrosophic sets are the extension of neutrosophic sets and are based on the idea of positive and negative preferences of information. Projection measure is a useful apparatus for modeling real life decision making problems. In the paper, we have defined projection, bidirectional projection and hybrid projection measures between bipolar neutrosophic sets and the proposed measures are then applied to multi-attribute decision making problems. The ratings of performance values of the alternatives with respect to the attributes are expressed by bipolar neutrosophic values. We calculate projection, bidirectional projection, and hybrid projection measures between each alternative and ideal alternative with bipolar neutrosophic information and then all the alternatives are ranked to identify best option. Finally, a numerical example is provided to demonstrate the applicability and effectiveness of the developed method. Comparison analysis with other existing methods is also provided.***

Keywords: Bipolar neutrosophic sets, projection measure, bidirectional projection measure, hybrid projection measure, multi-attribute decision making.

1. Introduction

For describing and managing indeterminate and inconsistent information, Smarandache [1] introduced neutrosophic sets which has three independent components namely truth membership degree (T), indeterminacy membership degree (I) and falsity membership degree (F) where T, I, F lie in$]^{-}0, 1^{+}[$. Later, Wang et al. [2] proposed single valued neutrosophic sets (SVNSs) to deal real decision making problems where T, I, F lie in [0, 1].

In 1994, Zhang [3], [4] grounded the notion of bipolar fuzzy sets by extending the concept of fuzzy sets. The value of membership degree of an element of bipolar fuzzy set belongs to [-1, 1]. With reference to a bipolar fuzzy set, the membership degree zero of an element reflects that the element is irrelevant to the corresponding property, the membership degree belongs to (0,1] of an element reflects that the element somewhat satisfies the property, and the membership degree belongs to [−1,0) of an element reflects that the element somewhat satisfies the implicit counter-property.

Deli et al. [5] introduced the concept of bipolar neutrosophic sets (BNSs) by combining the concept of bipolar fuzzy sets and neutrosophic sets. With reference to a bipolar neutrosophic set Q, the positive membership degrees $T_Q^+(x)$, $I_Q^+(x)$, $F_Q^+(x)$ represent respectively the truth membership, indeterminate membership and falsity membership of an element $x \in X$ corresponding to the bipolar neutrosophic set Q and the negative membership degree $T_Q^-(x)$, $I_Q^-(x)$, $F_Q^-(x)$ denotes the truth membership, indeterminate membership and false membership of an element $x \in X$ to some implicit counter-property corresponding to the bipolar neutrosophic set Q.

Projection measure is a useful decision making device as it takes into account the distance as well as the included angle for measuring the closeness degree between two objects [6, 7]. Yue [6] and Zhang et al. [7] studied projection based multi attribute decision making in crisp environment i.e. projections are defined on ordinary numbers or crisp numbers. Yue [8] further investigated a new multi attribute group decision making (MAGDM) method based on determining the weights of the decision makers by employing projection technique with interval data. Yue and Jia [9] established a methodology for MAGDM based on a new normalized projection measure, in which the attribute values are provided by decision makers in hybrid form with crisp values and interval data.

Xu and Da [10] and Xu [11] studied projection method for decision making in uncertain environment with preference information. Wei [12] discussed a multi attribute decision making (MADM) method based on the projection technique, in which the attribute values are presented in terms of intuitionistic fuzzy numbers. Zhang et al. [13] proposed a grey relational projection method for MADM based on intuitionistic trapezoidal fuzzy number. Zeng et al. [14] investigated projections on interval valued intuitionistic fuzzy numbers and we developed model and algorithm to the MAGDM problems with interval-valued intuitionistic fuzzy information. Xu and Hu [15] developed two projection based models for MADM in intuitionistic fuzzy environment and interval valued intuitionistic fuzzy environment. Sun [16] presented a group decision making method based on projection method and score function under interval valued intuitionistic fuzzy environment. Tsao and Chen [17] developed a novel projection based compromising method for multi criteria decision making method in interval valued intuitionistic fuzzy environment.

In neutrosophic environment Chen and Ye [18] developed projection based model of neutrosophic numbers and presented MADM method to select clay-bricks in construction field. Bidirectional projection measure [19] considers the distance and included angle between two vectors *x, y* and also considers the bidirectional projection measure between two vectors *x, y*. Ye [19] defined bidirectional projection measure as an improvement of the general projection measure of SVNSs to overcome the drawback of the general projection measure. In the same study, Ye [19] developed multi attribute decision making method for selecting problems of mechanical design schemes under a single-valued neutrosophic environment. Ye [20] also presented bidirectional projection method for multiple attribute group decision making with neutrosophic numbers.

Ye [21] defined credibility – induced interval neutrosophic weighted arithmetic averaging operator and credibility – induced interval neutrosophic weighted geometric averaging operator and developed the projection measure based ranking method for multi-attribute decision making (MADM) problems with interval neutrosophic information and credibility information. Dey et al. [22] proposed a new approach to neutrosophic soft MADM using grey relational projection method. Dey et al. [23] defined weighted projection measure with interval neutrosophic assessments and applied the proposed concept to solve MADM problems with interval valued neutrosophic information.

In the field of bipolar neutrosophic environment, Deli et al. [5] defined score, accuracy, and certainty functions in order to compare BNSs and developed bipolar neutrosophic weighted average (BNWA) and bipolar neutrosophic weighted geometric (BNWG) operators to obtain collective bipolar neutrosophic information. In the same study, Deli et al. [5] also proposed a multi-criteria decision making (MCDM) approach on the basis of score, accuracy, and certainty functions and BNWA, BNWG operators. Deli and Subas [24] presented a single valued bipolar neutrosophic MCDM through correlation coefficient similarity measure. Şahin et al. [25] provided a MCDM method based on Jaccard similarity measure of BNS. Uluçay et al. [26] defined Dice similarity, weighted Dice similarity, hybrid vector similarity, weighted hybrid vector similarity measures under BNSs and developed MCDM methods based on the proposed similarity measures. Dey et al. [27] defined Hamming and Euclidean distance measures to compute the distance between BNSs and investigated a TOPSIS approach to derive the most desirable alternative.

In this study, we define projection, bidirectional projection and hybrid projection measures under bipolar neutrosophic information. Then, we develop three algorithms for solving MADM problems with bipolar neutrosophic assessments.

We organize the rest of the paper in the following way. In Section 2, we recall several useful definitions concerning SVNSs and BNSs. Section 3 defines projection, bidirectional projection and hybrid projection measures between BNSs. Section 4 is devoted to present three models for solving MADM under bipolar neutrosophic environment. In Section 5, we solve a decision making problem with bipolar neutrosophic information on the basis of the proposed measures. Comparison analysis is provided to demonstrate the feasibility and flexibility of the proposed methods in Section 6. Finally, the last Section provides conclusions and future scope of work.

2. Basic concepts regarding SVNSs and BNSs

In this Section, we provide some basic definitions regarding SVNSs, BNSs which are useful for the construction of the paper.

2.1 Single valued neutrosophic Sets [2]

Let X be a universal space of points with a generic element of X denoted by x, then a SVNS P is characterized by a truth membership function $T_P(x)$, an indeterminate membership function $I_P(x)$ and a falsity membership function $F_P(x)$. A SVNS P is expressed in the following way.

$$P = \{x, \langle T_P(x), I_P(x), F_P(x) \rangle \mid x \in X\}$$

where, $T_P(x)$, $I_P(x)$, $F_P(x): X \to [0, 1]$ and $0 \le T_P(x) + I_P(x) + F_P(x) \le 3$ for each point $x \in X$.

2.2 Bipolar Neutrosophic Set [5]

Consider X be a universal space of objects, then a BNS Q in X is presented as follows:

$$Q = \{x, \langle T_Q^+(x), I_Q^+(x), F_Q^+(x), T_Q^-(x), I_Q^-(x), F_Q^-(x) \rangle \mid x \in X\},$$

where $T_Q^+(x)$, $I_Q^+(x)$, $F_Q^+(x): X \to [0, 1]$ and $T_Q^-(x)$, $I_Q^-(x)$, $F_Q^-(x): X \to [-1, 0]$. The positive membership degrees $T_Q^+(x)$, $I_Q^+(x)$, $F_Q^+(x)$ denote the truth membership, indeterminate membership, and falsity membership functions of an element $x \in X$ corresponding to a BNS Q and the negative membership degrees $T_Q^-(x)$, $I_Q^-(x)$, $F_Q^-(x)$ denote the truth membership, indeterminate membership, and falsity membership of an element $x \in X$ to several implicit counter property associated with a BNS Q. For convenience, a bipolar neutrosophic value (BNV) is presented as $\tilde{q} = < T_Q^+, I_Q^+, F_Q^+, T_Q^-, I_Q^-, F_Q^- >$.

Definition 1 [5]. Let, $Q_1 = \{x, \langle T_{Q_1}^+(x), I_{Q_1}^+(x), F_{Q_1}^+(x), T_{Q_1}^-(x), I_{Q_1}^-(x), F_{Q_1}^-(x) \rangle \mid x \in X\}$ and $Q_2 = \{x, \langle T_{Q_2}^+(x), I_{Q_2}^+(x), F_{Q_2}^+(x), T_{Q_2}^-(x), I_{Q_2}^-(x), F_{Q_2}^-(x) \rangle \mid x \in X\}$ be two BNSs. Then $Q_1 \subseteq Q_2$ if and only if

$T_{Q_1}^+(x) \le T_{Q_2}^+(x)$, $I_{Q_1}^+(x) \le I_{Q_2}^+(x)$, $F_{Q_1}^+(x) \ge F_{Q_2}^+(x)$; $T_{Q_1}^-(x) \ge T_{Q_2}^-(x)$, $I_{Q_1}^-(x) \ge I_{Q_2}^-(x)$, $F_{Q_1}^-(x) \le F_{Q_2}^-(x)$ for all $x \in X$.

Definition 2 [5]. Let, $Q_1 = \{x, \langle T_{Q_1}^+(x), I_{Q_1}^+(x), F_{Q_1}^+(x), T_{Q_1}^-(x), I_{Q_1}^-(x), F_{Q_1}^-(x) \rangle \mid x \in X\}$ and $Q_2 = \{x, \langle T_{Q_2}^+(x), I_{Q_2}^+(x), F_{Q_2}^+(x), T_{Q_2}^-(x), I_{Q_2}^-(x), F_{Q_2}^-(x) \rangle \mid x \in X\}$ be two BNSs. Then $Q_1 = Q_2$ if and only if

$T_{Q_1}^+(x) = T_{Q_2}^+(x)$, $I_{Q_1}^+(x) = I_{Q_2}^+(x)$, $F_{Q_1}^+(x) = F_{Q_2}^+(x)$; $T_{Q_1}^-(x) = T_{Q_2}^-(x)$, $I_{Q_1}^-(x) = I_{Q_2}^-(x)$, $F_{Q_1}^-(x) = F_{Q_2}^-(x)$ for all $x \in X$.

Definition 3 [5]. Let, $Q = \{x, \langle T_Q^+(x), I_Q^+(x), F_Q^+(x), T_Q^-(x), I_Q^-(x), F_Q^-(x) \rangle \mid x \in X\}$ be a BNS. The complement of Q is represented by Q^c and is defined as follows:

$T_{Q^c}^+(x) = \{1^+\} - T_Q^+(x)$, $I_{Q^c}^+(x) = \{1^+\} - I_Q^+(x)$, $F_{Q^c}^+(x) = \{1^+\} - F_Q^+(x)$;

$T_{Q^c}^-(x) = \{1^-\} - T_Q^-(x)$, $I_{Q^c}^-(x) = \{1^-\} - I_Q^-(x)$, $F_{Q^c}^-(x) = \{1^-\} - F_Q^-(x)$.

Definition 4. Let, Q_1 = {x, $\left\langle T^{+}_{Q_1}(x), I^{+}_{Q_1}(x), F^{+}_{Q_1}(x), T^{-}_{Q_1}(x), I^{-}_{Q_1}(x), F^{-}_{Q_1}(x) \right\rangle$ | $x \in X$} and Q_2 = {x, $\left\langle T^{+}_{Q_2}(x), I^{+}_{Q_2}(x), F^{+}_{Q_2}(x), T^{-}_{Q_2}(x), I^{-}_{Q_2}(x), F^{-}_{Q_2}(x) \right\rangle$ | $x \in X$} be two BNSs. Their union $Q_1 \cup Q_2$ is defined as follows:

$Q_1 \cup Q_2$ = {Max $(T^{+}_{Q_1}(x), T^{+}_{Q_2}(x))$, Min $(I^{+}_{Q_1}(x), I^{+}_{Q_2}(x))$, Min $(F^{+}_{Q_1}(x), F^{+}_{Q_2}(x))$, Min $(T^{-}_{Q_1}(x), T^{-}_{Q_2}(x))$, Max $(I^{-}_{Q_1}(x), I^{-}_{Q_2}(x))$, Max $(F^{-}_{Q_1}(x), F^{-}_{Q_2}(x))$}, $\forall\ x \in X$.

Their intersection $Q_1 \cap Q_2$ is defined as follows:

$Q_1 \cap Q_2$ = {Min $(T^{+}_{Q_1}(x), T^{+}_{Q_2}(x))$, Max $(I^{+}_{Q_1}(x), I^{+}_{Q_2}(x))$, Max $(F^{+}_{Q_1}(x), F^{+}_{Q_2}(x))$, Max $(T^{-}_{Q_1}(x), T^{-}_{Q_2}(x))$,Min $(I^{-}_{Q_1}(x), I^{-}_{Q_2}(x))$, Min $(F^{-}_{Q_1}(x), F^{-}_{Q_2}(x))$}, $\forall\ x \in X$.

Definition 5 [5]. Let $\tilde{q}_1$ = < $T^{+}_{Q_1}, I^{+}_{Q_1}, F^{+}_{Q_1}, T^{-}_{Q_1}, I^{-}_{Q_1}, F^{-}_{Q_1}$ > and $\tilde{q}_2$ = < $T^{+}_{Q_2}, I^{+}_{Q_2}, F^{+}_{Q_2}, T^{-}_{Q_2}, I^{-}_{Q_2}, F^{-}_{Q_2}$ > be two BNNs, then

i. $\beta . \tilde{q}_1$ = < $1 - (1 - T^{+}_{Q_1})^{\beta}$, $(I^{+}_{Q_1})^{\alpha}$, $(F^{+}_{Q_1})^{\beta}$, $-(-T^{-}_{Q_1})^{\beta}$, $-(-I^{-}_{Q_1})^{\beta}$, $-(1 - (1 - (-F^{-}_{Q_1}))^{\beta})$ >;

ii. $(\tilde{q}_1)^{\beta}$ = < $(T^{+}_{Q_1})^{\beta}$, $1 - (1 - I^{+}_{Q_1})^{\beta}$, $1 - (1 - F^{+}_{Q_1})^{\beta}$, $-(1 - (1 - (-T^{-}_{Q_1}))^{\beta})$, $-(-I^{-}_{Q_1})^{\beta}$, $(-F^{-}_{Q_1})^{\beta}$ >;

iii. $\tilde{q}_1 + \tilde{q}_2$ = < $T^{+}_{Q_1} + T^{+}_{Q_2} - T^{+}_{Q_1} . T^{+}_{Q_2}$, $I^{+}_{Q_1} . I^{+}_{Q_2}$, $F^{+}_{Q_1} . F^{+}_{Q_2}$, $-T^{-}_{Q_1} . T^{-}_{Q_2}$, $-(-I^{-}_{Q_1} - I^{-}_{Q_2} - I^{-}_{Q_1} . I^{-}_{Q_2})$, $-(-F^{-}_{Q_1} - F^{-}_{Q_2} - F^{-}_{Q_1} . F^{-}_{Q_2})$ >;

iv. $\tilde{q}_1 . \tilde{q}_2$ = < $T^{+}_{Q_1} . T^{+}_{Q_2}$, $I^{+}_{Q_1} + I^{+}_{Q_2} - I^{+}_{Q_1} . I^{+}_{Q_2}$, $F^{+}_{Q_1} + F^{+}_{Q_2} - F^{+}_{Q_1} . F^{+}_{Q_2}$, $-(-T^{-}_{Q_1} - T^{-}_{Q_2} - T^{-}_{Q_1} . T^{-}_{Q_2})$, $-I^{-}_{Q_1} . I^{-}_{Q_2}$, $-F^{-}_{Q_1} . F^{-}_{Q_2}$ >

where $\beta > 0$.

3. Projection, bidirectional projection and hybrid projection measures of BNSs

This Section proposes a general projection, a bidirectional projection and a hybrid projection measures for BNSs.

Definition 6. Consider $X = (x_1, x_2, \ldots, x_m)$ be a finite universe of discourse and Q be a BNS in X, then modulus of Q is defined as follows:

$$\| Q \| = \sqrt{\sum_{j=1}^{m} \alpha_j^2} = \sqrt{\sum_{j=1}^{m} [(T^{+}_{Q_j})^2 + (I^{+}_{Q_j})^2 + (F^{+}_{Q_j})^2 + (T^{-}_{Q_j})^2 + (I^{-}_{Q_j})^2 + (F^{-}_{Q_j})^2]}$$

where $\alpha_j = \left\langle T^{+}_{Q_j}(x), I^{+}_{Q_j}(x), F^{+}_{Q_j}(x), T^{-}_{Q_j}(x), I^{-}_{Q_j}(x), F^{-}_{Q_j}(x) \right\rangle$, j = 1, 2, …, m.

Definition 7 [10, 28]. Consider $u = (u_1, u_2, \ldots, u_m)$ and $v = (v_1, v_2, \ldots, v_m)$ be two vectors, then the projection of vector u onto vector v can be defined as follows:

$$Proj\ (u)_v = \| u \| \text{Cos}\ (u, v) = \sqrt{\sum_{j=1}^{m} u_j^2} \times \frac{\sum_{j=1}^{m}(u_j v_j)}{\sqrt{\sum_{j=1}^{m} u_j^2} \times \sqrt{\sum_{j=1}^{m} v_j^2}} = \frac{\sum_{j=1}^{m}(u_j v_j)}{\sqrt{\sum_{j=1}^{m} v_j^2}}$$

where, *Proj* $(u)_v$ represents that the closeness of u and v in magnitude.

Definition 8. Consider $X = (x_1, x_2, \ldots, x_m)$ be a finite universe of discourse and R, S be two BNSs in X, then

$$Proj\ (R)_S = \| R \| \text{Cos}\ (R, S) = \frac{1}{\| S \|}\ (R.S)$$

is called the projection of R on S, where

$$\|R\| = \sqrt{\sum_{i=1}^{m}[(T_R^+)^2(x_i) + (I_R^+)^2(x_i) + (F_R^+)^2(x_i) + (T_R^-)^2(x_i) + (I_R^-)^2(x_i) + (F_R^-)^2(x_i)]},$$

$$\|S\| = \sqrt{\sum_{i=1}^{m}[(T_S^+)^2(x_i) + (I_S^+)^2(x_i) + (F_S^+)^2(x_i) + (T_S^-)^2(x_i) + (I_S^-)^2(x_i) + (F_S^-)^2(x_i)]},$$

and $R.S = \sum_{i=1}^{m} [T_R^+(x_i)T_S^+(x) + I_R^+(x_i)I_S^+(x_i) + F_R^+(x_i)F_S^+(x_i) + T_R^-(x_i)T_S^-(x_i) + I_R^-(x_i)I_S^-(x_i) + F_R^-(x_i)F_S^-(x_i)]$.

Example 1. R = < 0.5, 0.3, 0.2, -0.2, -0.1, -0.05 >, S = < 0.7, 0.3, 0.1, -0.4, -0.2, -0.3 > be two BNSs in X, then the projection of R on S is obtained as follows:

$$Proj\ (R)_S = \frac{1}{\| S \|}\ (R.S)$$

$$= \frac{(0.5)(0.7) + (0.3)(0.3) + (0.2)(0.1) + (-0.2)(-0.4) + (-0.1)(-0.2) + (-0.05)(-0.3)}{\sqrt{(0.7)^2 + (0.3)^2 + (0.1)^2 + (-0.4)^2 + (-0.2)^2 + (-0.3)^2}}$$

= 0.612952

The bigger value of *Proj* $(R)_S$ reflects that R and S are closer to each other.

However, in single valued neutrosophic environment, Ye [20] observed that for two vectors α and β, the general projection measure cannot describe accurately the degree of α close to β. We also notice that the general projection incorporated by Xu [11] is not reasonable in several cases under bipolar neutrosophic setting, for example let, $\alpha = \beta$ = < $a, a, a, -a, -a, -a$ > and γ = < $2a, 2a, 2a, -2a, -2a, -2a$ >, then *Proj* $(\alpha)_\beta$ = 2.44949 $\|a\|$ and *Proj* $(\gamma)_\beta$ = 4.898979 $\|a\|$. This shows that β is much closer to γ than α which is not true because $\alpha = \beta$. Ye [20] opined that α is equal to β whenever *Proj* $(\alpha)_\beta$ and *Proj* $(\beta)_\alpha$ should be equal to 1. Therefore, Ye [20] proposed an alternative method called bidirectional projection measure to overcome the limitation of general projection measure as given below.

Definition 9 [20]. Consider x and y be two vectors, then the bidirectional projection between x and y is defined as follows:

$$B\text{-}proj\ (x, y) = \frac{1}{1 + \left| \frac{x.y}{\| x \|} - \frac{x.y}{\| y \|} \right|} = \frac{\| x \| \| y \|}{\| x \| \| y \| + | \| x \| - \| y \| | x.y}$$

where ||x||, ||y|| denote the modulus of x and y respectively, and x. y is the inner product between x and y.
Here, *B-Proj* (x, y) = 1 if and only if x = y and $0 \le$ *B-Proj* (x, y) ≤ 1, i.e. bidirectional projection is a normalized measure.

Definition 10. Consider R = $\left\langle T_R^+(x_i), I_R^+(x_i), F_R^+(x_i), T_R^-(x_i),\ I_R^-(x_i), F_R^-(x_i) \right\rangle$ and S = $\left\langle T_S^+(x_i), I_S^+(x_i), F_S^+(x_i), T_S^-(x_i), I_S^-(x_i), F_S^-(x_i) \right\rangle$ be two BNSs in $X = (x_1, x_2, \ldots, x_m)$, then the bidirectional projection measure between R and S is defined as follows:

$$B\text{-}Proj\,(R, S) = \frac{1}{1+\left|\dfrac{R.S}{\|R\|}-\dfrac{R.S}{\|S\|}\right|} = \frac{\|R\|\,\|S\|}{\|R\|\,\|S\|+|\,\|R\|-\|S\|\,|\,R.S}$$

where

$$\|R\| = \sqrt{\sum_{i=1}^{m}[(T_R^+)^2(x_i)+(I_R^+)^2(x_i)+(F_R^+)^2(x_i)+(T_R^-)^2(x_i)+(I_R^-)^2(x_i)+(F_R^-)^2(x_i)]}$$

, ||S|| =

$$\sqrt{\sum_{i=1}^{m}[(T_S^+)^2(x_i)+(I_S^+)^2(x_i)+(F_S^+)^2(x_i)+(T_S^-)^2(x_i)+(I_S^-)^2(x_i)+(F_S^-)^2(x_i)]}$$ and

$$R.S = \sum_{i=1}^{m}[T_R^+(x_i)T_S^+(x)+I_R^+(x_i)I_S^+(x_i)+F_R^+(x_i)F_S^+(x_i)+T_R^-(x_i)T_S^-(x_i)+I_R^-(x_i)I_S^-(x_i) + F_R^-(x_i)F_S^-(x_i)]$$

Proposition 1. Let *B-Proj* $(R)_S$ be a bidirectional projection measure between BNSs R and S, then we have

1. $0 \le$ *B-Proj* $(R)_S \le 1$;
2. *B-Proj* $(R)_S$ = *B-Proj* $(S)_R$;
3. *B-Proj* $(R)_S$ = 1 for $R = S$.

Proof.

1. *B-Proj* (R, S) = 0 if and only if either $\|R\| = 0$ or $\|S\| = 0$ i.e. when either R = (0, 0, 0, 0, 0, 0) or S = (0, 0, 0, 0, 0, 0) which is trivial case. For two non-zero vectors R and S, $\|R\|\|S\| + |\,\|R\| - \|S\|\,|\,R.S \ge \|R\|\|S\|$, obviously, *B-Proj* ($R$, S) ≤ 1.
2. From definition, $R.S$ = $S.R$, therefore, *B-Proj*$(R)_S = \dfrac{\|R\|\|S\|}{\|R\|\|S\|+|\|R\|-\|S\||R.S} = \dfrac{\|S\|\|R\|}{\|S\|\,\|R\|+|\,\|S\|-\|R\|\,|S.R}$ = *B-Proj* (S, R).
3. Obviously, *B-Proj*$(R)_S$ = 1, only when $\|R\| = \|S\|$ i. e. when $T_R^+(x_i) = T_S^+(x_i)$, $I_R^+(x_i) = I_S^+(x_i)$, $F_R^+(x_i) = F_S^+(x_i)$, $T_R^-(x_i) = T_S^-(x_i)$, $I_R^-(x_i) = I_S^-(x_i)$, $F_R^-(x_i) = F_S^-.(x_i)$

   This completes the proof.

Example 2. Assume that R = < 0.5, 0.3, 0.2, -0.2, -0.1, -0.05 >, S = < 0.7, 0.3, 0.1, -0.4, -0.2, -0.3 > be two BNSs in the universe of discourse X, then the bidirectional projection measure between R on S is computed as given below.

$$B\text{-}Proj\ (R, S) = \frac{(0.6576473).(0.9380832)}{(0.6576473).(0.9380832)+|0.9380832-06576473|(0.575)}$$

= 0.7927845

Definition 11. Let $R = \left\langle T_R^+(x_i), I_R^+(x_i), F_R^+(x_i), T_R^-(x_i), I_R^-(x_i), F_R^-(x_i) \right\rangle$ and $S = \left\langle T_S^+(x_i), I_S^+(x_i), F_S^+(x_i), T_S^-(x_i), I_S^-(x_i), F_S^-(x_i) \right\rangle$ be two BNSs in $X = (x_1, x_2, \ldots, x_m)$, then the hybrid projection measure is defined as the combination of projection measure and bidirectional projection measure. The hybrid projection measure between R and S is represented as follows:

$$Hyb\text{-}Proj\ (R, S) = \rho\ Proj\ (R)_S + (1 - \rho)\ B\text{-}Proj\ (R, S)$$

$$= \rho \frac{R.S}{\|S\|} + (1 - \rho) \frac{\|R\|\,\|S\|}{\|R\|\,\|S\| + |\,\|R\| - \|S\|\,|\,R.S}$$

where

$$\|R\| = \sqrt{\sum_{i=1}^{m}[(T_R^+)^2(x_i)+(I_R^+)^2(x_i)+(F_R^+)^2(x_i)+(T_R^-)^2(x_i)+(I_R^-)^2(x_i)+(F_R^-)^2(x_i)]},$$

$$\|S\| = \sqrt{\sum_{i=1}^{m}[(T_S^+)^2(x_i)+(I_S^+)^2(x_i)+(F_S^+)^2(x_i)+(T_S^-)^2(x_i)+(I_S^-)^2(x_i)+(F_S^-)^2(x_i)]},$$ and

$$R.S = \sum_{i=1}^{m}[T_R^+(x_i)T_S^+(x)+I_R^+(x_i)I_S^+(x_i)+F_R^+(x_i)F_S^+(x_i)+T_R^-(x_i)T_S^-(x_i)+I_R^-(x_i)I_S^-(x_i)+F_R^-(x_i)F_S^-(x_i)]$$

where $0 \le \rho \le 1$.

Example 3. Assume that R = < 0.5, 0.3, 0.2, -0.2, -0.1, -0.05 >, S = < 0.7, 0.3, 0.1, -0.4, -0.2, -0.3 > be two BNSs in the universe of discourse X, then the hybrid projection measure between R on S with ρ = 0.7 is calculated as given below.

Hyb-Proj (R, S) = (0.7). (0.612952) + (1 - 0.7). (0.7927845) = 0.6669018.

4. Projection, bi-directional projection and hybrid projection based decision making methods for MADM problems with bipolar neutrosophic information

In this Section, we develop projection based decision making models to MADM problems with bipolar neutrosophic assessments. Consider $E = \{E_1, E_2, \ldots, E_m\}$, ($m \ge 2$) be a discrete set of m feasible alternatives, $F = \{F_1, F_2, \ldots, F_n\}$, ($n \ge 2$) be a set of attributes under consideration and $w = (w_1, w_2, \ldots, w_n)^T$ be the weight vector of the attributes such that $0 \le w_j \le 1$ and $\sum_{j=1}^{n} w_j = 1$. Now, we provide three algorithms for MADM problems involving bipolar neutrosophic information.

4.1. Algorithm 1.

Step 1. The rating of evaluation value of alternative E_i (i = 1, 2, …, m) for the predefined attribute F_j (j = 1, 2, …, n) is presented by the decision maker in terms of BNVs and the bipolar neutrosophic decision matrix is constructed as given below.

$$\langle q_{ij} \rangle_{m\times n} = \begin{bmatrix} q_{11} & q_{12} & \cdots & q_{1n} \\ q_{21} & q_{22} & \cdots & q_{2n} \\ . & . & \cdots & . \\ . & . & \cdots & . \\ q_{m1} & q_{m2} & \cdots & q_{mn} \end{bmatrix}$$

where q_{ij} = < $(T_{ij}^{+}, I_{ij}^{+}, F_{ij}^{+}, T_{ij}^{-}, I_{ij}^{-}, F_{ij}^{-})$ > with $T_{ij}^{+}, I_{ij}^{+}, F_{ij}^{+}$, $-T_{ij}^{-}$, $-I_{ij}^{-}$, $-F_{ij}^{-} \in$ [0, 1] and $0 \le T_{ij}^{+} + I_{ij}^{+} + F_{ij}^{+} - T_{ij}^{-} - I_{ij}^{-} - F_{ij}^{-} \le 6$ for i = 1, 2, …, m; j = 1, 2, …, n.

Step 2. We formulate bipolar weighted decision matrix by multiplying weights w_j of the attributes as follows:

$$w_j \otimes \langle q_{ij} \rangle_{m\times n} = \langle z_{ij} \rangle_{m\times n} = \begin{bmatrix} z_{11} & z_{12} & \cdots & z_{1n} \\ z_{21} & z_{22} & \cdots & z_{2n} \\ . & . & \cdots & . \\ . & . & \cdots & . \\ z_{m1} & z_{m2} & \cdots & z_{mn} \end{bmatrix}$$

where $z_{ij} = w_j . q_{ij}$ = < $1 - (1 - T_{ij}^{+})^{w_j}$, $(I_{ij}^{+})^{w_j}$, $(F_{ij}^{+})^{w_j}$, $-(-T_{ij}^{-})^{w_j}$, $-(-I_{ij}^{-})^{w_j}$, $-(1 - (1 - (-F_{ij}^{-}))^{w_j})$ > = < $\mu_{ij}^{+}, \nu_{ij}^{+}, \omega_{ij}^{+}, \mu_{ij}^{-}, \nu_{ij}^{-}, \omega_{ij}^{-}$ > with $\mu_{ij}^{+}, \nu_{ij}^{+}, \omega_{ij}^{+}$, $-\mu_{ij}^{-}$, $-\nu_{ij}^{-}$, $-\omega_{ij}^{-} \in$ [0, 1] and $0 \le \mu_{ij}^{+} + \nu_{ij}^{+} + \omega_{ij}^{+} - \mu_{ij}^{-} - \nu_{ij}^{-} - \omega_{ij}^{-} \le 6$ for i = 1, 2, …, m; j = 1, 2, …, n.

Step 3. We identify the bipolar neutrosophic positive ideal solution (BNPIS) [15, 16] as follows:

$z^{PIS} = \langle e_j^{+}, f_j^{+}, g_j^{+}, e_j^{-}, f_j^{-}, g_j^{-} \rangle$ = < [{ $\underset{i}{\mathrm{Max}}(\mu_{ij}^{+}) \,|\, j \in \sigma$ }; { $\underset{i}{\mathrm{Min}}(\mu_{ij}^{+}) \,|\, j \in \varsigma$ }],

[{ $\underset{i}{\mathrm{Min}}(\nu_{ij}^{+}) \,|\, j \in \sigma$ }; { $\underset{i}{\mathrm{Max}}(\nu_{ij}^{+}) \,|\, j \in \varsigma$ }], [{ $\underset{i}{\mathrm{Min}}(\omega_{ij}^{+}) \,|\, j \in \sigma$ };

{ $\underset{i}{\mathrm{Max}}(\omega_{ij}^{-}) \,|\, j \in \varsigma$ }], [{ $\underset{i}{\mathrm{Min}}(\mu_{ij}^{-}) \,|\, j \in \sigma$ }; { $\underset{i}{\mathrm{Max}}(\mu_{ij}^{-}) \,|\, j \in \varsigma$ }], [{ $\underset{i}{\mathrm{Max}}(\nu_{ij}^{-}) \,|\, j \in \sigma$ };

{ $\underset{i}{\mathrm{Min}}(\nu_{ij}^{-}) \,|\, j \in \varsigma$ }],[{ $\underset{i}{\mathrm{Max}}(\omega_{ij}^{-}) \,|\, j \in \sigma$ };

{ $\underset{i}{\mathrm{Min}}(\omega_{ij}^{-}) \,|\, j \in \varsigma$ }] >, j = 1, 2, …, n, where σ and ς are benefit and cost type attributes respectively.

Step 4. Determine the projection measure between z^{PIS} and $Z^{i} = \langle z_{ij} \rangle_{m\times n}$ for all i = 1, 2, …, m; j = 1, 2, …, n by using the following Eq.

$$Proj\ (Z^i)_{z^{PIS}} = \frac{\sum_{j=1}^{n}[\mu_{ij}^+ e_j^+ + \nu_{ij}^+ f_j^+ + \omega_{ij}^+ g_j^+ + \mu_{ij}^- e_j^- + \nu_{ij}^- f_j^- + \omega_{ij}^- g_j^-]}{\sqrt{\sum_{j=1}^{n}[(e_j^+)^2 + (f_j^+)^2 + (g_j^+)^2 + (e_j^-)^2 + (f_j^-)^2 + (g_j^-)^2]}}$$

Step 5. Rank the alternatives in a descending order based on the projection measure $Proj\ (Z^i)_{z^{PIS}}$ for i = 1, 2, …, *m* and bigger value of $Proj\ (Z^i)_{z^{PIS}}$ determines the best alternative.

4.2. Algorithm 2.

Step 1. Give the bipolar neutrosophic decision matrix $\langle q_{\text{ij}} \rangle_{\text{m}\times\text{n}}$, i = 1, 2, …, *m*; j = 1, 2, …, *n*.

Step 2. Construct weighted bipolar neutrosophic decision matrix $\langle z_{\text{ij}} \rangle_{m\times n}$, i = 1, 2, …, *m*; j = 1, 2, …, *n*.

Step 3. Determine $z^{\text{PIS}} = \langle e_{\text{j}}^+, f_{\text{j}}^+, g_{\text{j}}^+, e_{\text{j}}^-, f_{\text{j}}^-, g_{\text{j}}^- \rangle$; j = 1, 2, …, *n*.

Step 4. Compute the bidirectional projection measure between z^{PIS} and Z^{i} $= \langle z_{\text{ij}} \rangle_{\text{m}\times\text{n}}$ for all i = 1, 2, …, *m*; j = 1, 2, …, *n* by using the Eq. as given below.

$$\text{B-Proj }(Z^{\text{i}}, z^{\text{PIS}}) = \frac{\| Z^i \| \| z^{PIS} \|}{\| Z^i \| \| z^{PIS} \| + | \| Z^i \| - \| z^{PIS} \| | Z^i . z^{PIS}}$$

where $\| Z^i \| = \sqrt{\sum_{j=1}^{n}[(\mu_{ij}^+)^2 + (\nu_{ij}^+)^2 + (\omega_{ij}^+)^2 + (\mu_{ij}^-)^2 + (\nu_{ij}^-)^2 + (\omega_{ij}^-)^2]}$, i = 1, 2, …, *m*.

$\| z^{PIS} \| = \sqrt{\sum_{j=1}^{n}[(e_j^+)^2 + (f_j^+)^2 + (g_j^+)^2 + (e_j^-)^2 + (f_j^-)^2 + (g_j^-)^2]}$ and

$Z^i . z^{PIS} = \sum_{j=1}^{n}[\mu_{ij}^+ e_j^+ + \nu_{ij}^+ f_j^+ + \omega_{ij}^+ g_j^+ + \mu_{ij}^- e_j^- + \nu_{ij}^- f_j^- + \omega_{ij}^- g_j^-]$, i = 1, 2, …, *m*.

Step 5. According to the bidirectional projection measure *B-Proj* (Z^{i}, z^{PIS}) for i = 1, 2, …, *m* alternatives are ranked and bigger value of *B-Proj* (Z^{i}, z^{PIS}) reflects the best option.

4.3. Algorithm 3.

Step 1. Construct the bipolar neutrosophic decision matrix $\langle q_{\text{ij}} \rangle_{m\times n}$, i = 1, 2, …, *m*; j = 1, 2, …, *n*.

Step 2. Formulate the weighted bipolar neutrosophic decision matrix $\langle z_{\text{ij}} \rangle_{m\times n}$, i = 1, 2, …, *m*; j = 1, 2, …, *n*.

Step 3. Identify $z^{\text{PIS}} = \langle e_{\text{j}}^+, f_{\text{j}}^+, g_{\text{j}}^+, e_{\text{j}}^-, f_{\text{j}}^-, g_{\text{j}}^- \rangle$, j = 1, 2, …, *n*.

Step 4. By combining projection measure *Proj* $(Z^i)_{z^{PIS}}$ and bidirectional projection measure *B-Proj* (Z^{i}, z^{PIS}), we calculate the hybrid projection measure between z^{PIS} and Z^{i} = $\langle \mathrm{z}_{\mathrm{ij}} \rangle_{\mathrm{m\times n}}$ for all i = 1, 2, …, *m*; j = 1, 2, …, *n* as follows.

$$Hyb\text{-}Proj\,(Z^i, z^{PIS}) = \rho\; Proj\;(Z^i)_{z^{PIS}} + (1 - \rho)\; B\text{-}Proj\,(Z^i, z^{PIS})$$

$$= \rho\,\frac{Z^i.z^{PIS}}{\|z^{PIS}\|} + (1 - \rho)\frac{\|Z^i\|\,\|z^{PIS}\|}{\|Z^i\|\,\|z^{PIS}\| + |\,\|Z^i\| - \|z^{PIS}\|\,|\,Z^i.z^{PIS}}$$

where $\|Z^i\| = \sqrt{\sum_{j=1}^{n}[(\mu_{ij}^{+})^2 + (\nu_{ij}^{+})^2 + (\omega_{ij}^{+})^2 + (\mu_{ij}^{-})^2 + (\nu_{ij}^{-})^2 + (\omega_{ij}^{-})^2]}$, i = 1, 2, …, *m*,

$$\|z^{PIS}\| = \sqrt{\sum_{j=1}^{n}[(e_j^{+})^2 + (f_j^{+})^2 + (g_j^{+})^2 + (e_j^{-})^2 + (f_j^{-})^2 + (g_j^{-})^2]},$$

$Z^i.z^{PIS} = \sum_{j=1}^{n}[\mu_{ij}^{+}e_j^{+} + \nu_{ij}^{+}f_j^{+} + \omega_{ij}^{+}g_j^{+} + \mu_{ij}^{-}e_j^{-} + \nu_{ij}^{-}f_j^{-} + \omega_{ij}^{-}g_j^{-}]$, i = 1, 2, …, *m*, with $0 \le \rho \le 1$.

Step 5. We rank all the alternatives in accordance with the hybrid projection measure *Hyb-Proj* (Z^{i}, z^{PIS}) and greater value of *Hyb-Proj* (Z^{i}, z^{PIS}) implies the better alternative.

5. A numerical example

Consider the problem studied in [5, 27] where a customer desires to purchase a car. Suppose four types of car (alternatives) *E*i, (i = 1, 2, 3, 4) are taken into consideration in the decision making situation. Four attributes namely Fuel economy (F_1), Aerod (F_2), Comfort (F_3), Safety (F_4) is considered to evaluate the alternatives. Assume the weight vector [5] of the attribute is given by $w = (w_1, w_2, w_3, w_4)$ = (0.5, 0.25, 0.125, 0.125).

Method 1. The proposed projection measure based decision making with bipolar neutrosophic information for car selection is presented in the following steps:

Step 1: Construct the bipolar neutrosophic decision matrix

The bipolar neutrosophic decision matrix $\langle q_{\mathrm{ij}} \rangle_{\mathrm{m\times n}}$ presented by the decision maker as given below (see Table 1).

Table 1. The bipolar neutrosophic decision matrix

	F_1	F_2	F_3	F_4
E_1	<0.5, 0.7, 0.2, -0.7, -0.3, -0.6>	<0.4, 0.5, 0.4, -0.7, -0.8, -0.4>	<0.7, 0.7, 0.5, -0.8, -0.7, -0.6>	<0.1, 0.5, 0.7, -0.5, -0.2, -0.8>

E_2	<0.9, 0.7, 0.5, -0.7, -0.7, -0.1>	<0.7, 0.6, 0.8, -0.7, -0.5, -0.1>	<0.9, 0.4, 0.6, -0.1, -0.7, -0.5>	<0.5, 0.2, 0.7, -0.5, -0.1, -0.9>
E_3	<0.3, 0.4, 0.2, -0.6, -0.3, -0.7>	<0.2, 0.2, 0.2, -0.4, -0.7, -0.4>	<0.9, 0.5, 0.5, -0.6, -0.5, -0.2>	<0.7, 0.5, 0.3, -0.4, -0.2, -0.2>
E_4	<0.9, 0.7, 0.2, -0.8, -0.6, -0.1>	<0.3, 0.5, 0.2, -0.5, -0.5, -0.2>	<0.5, 0.4, 0.5, -0.1, -0.7, -0.2>	<0.2, 0.4, 0.8, -0.5, -0.5, -0.6>

Step 2. Construction of weighted bipolar neutrosophic decision matrix

The weighted decision matrix $\langle z_{ij} \rangle_{m\times n}$ is obtained by multiplying weights of the attributes to the bipolar neutrosophic decision matrix as follows (see Table 2).

Table 2. The weighted bipolar neutrosophic decision matrix

	F_1	F_2	F_3	F_4
E_1	<0.293, 0.837, 0.447, -0.837, -0.548, -0.368>	<0.120, 0.795, 0.841, -0.915, -0.946, -0.120>	<0.140, 0.956, 0.917, -0.972, -0.956, -0.108>	<0.013, 0.917, 0.956, -0.917, -0.818, -0.182 >
E_2	<0.684, 0.837, 0.707, -0.837, -0.837, -0.051>	<0.260, 0.880, 0.946, -0.915, -0.841, -0.026>	<0.250, 0.892, 0.938, -0.750, -0.956, -0.083>	<0.083, 0.818, 0.956, -0.917, -0.750, -0.250>
E_3	<0.163, 0.632, 0.447, -0.774, -0.548, -0.452>	<0.054, 0.669, 0.669, -0.795, -0.915, -0.120>	<0.250, 0.917, 0.917, -0.938, -0.917, -0.028>	<0.140, 0.917, 0.860, -0.892, -0.818, -0.028>
E_4	<0.648, 0.837, 0.447, -0.894, -0.774, -0.051>	<0.085, 0.841, 0.669, -0.841, -0.841, -0.054>	<0.083, 0.892, 0.917, -0.750, -0.956, -0.028>	<0.062, 0.818, 0.972, -0.917, -0.917, -0.108>

Step 3. Selection of BNPIS

The BNRPIS (z^{PIS}) $=\langle e_j^+, f_j^+, g_j^+, e_j^-, f_j^-, g_j^- \rangle$, (j = 1, 2, 3, 4) is computed from the weighted decision matrix as follows:

$\langle e_1^+, f_1^+, g_1^+, e_1^-, f_1^-, g_1^- \rangle$ = < 0.684, 0.632, 0.447, -0.894, -0.548, -0.051 >;

$\langle e_2^+, f_2^+, g_2^+, e_2^-, f_2^-, g_2^- \rangle$ = < 0.26, 0.669, 0.669, -0.915, -0.841, -0.026 >;

$\langle e_3^+, f_3^+, g_3^+, e_3^-, f_3^-, g_3^- \rangle$ = < 0.25, 0.892, 0.917, -0.972, -0.917, -0.028 >;

$\langle e_4^+, f_4^+, g_4^+, e_4^-, f_4^-, g_4^- \rangle$ = < 0.14, 0.818, 0.86, -0.917, -0.75, -0.028 >.

Step 4. Determination of weighted projection measure

The projection measure between positive ideal bipolar neutrosophic solution z^{PIS} and each weighted decision matrix $\langle z_{ij} \rangle_{m\times n}$ can be obtained as follows:

Proj $(Z^1)_{z^{PIS}}$ = 3.4214, *Proj* $(Z^2)_{z^{PIS}}$ = 3.4972, *Proj* $(Z^3)_{z^{PIS}}$ = 3.1821, *Proj* $(Z^4)_{z^{PIS}}$ = 3.3904.

Step 5. Rank the alternatives

We observe that *Proj* $(Z^2)_{z^{PIS}} >$ *Proj* $(Z^1)_{z^{PIS}} >$ *Proj* $(Z^4)_{z^{PIS}} >$ *Proj* $(Z^3)_{z^{PIS}}$. Therefore, the ranking order of the cars is $E_2 \succ E_1 \succ E_4 \succ E_3$ and hence, E_2 is the best alternative for the customer.

Method 2. The proposed bidirectional projection measure based decision making for car selection is presented as follows:

Step 1. Same as Method 1

Step 2. Same as Method 1

Step 3. Same as Method 1

Step 4. Calculation of bidirectional projection measure

The bidirectional projection measure between positive ideal bipolar neutrosophic solution z^{PIS} and each weighted decision matrix $\langle z_{ij} \rangle_{m\times n}$ can be determined as given below.

B-Proj (Z^1, z^{PIS}) = 0.8556, *B-Proj* (Z^2, z^{PIS}) = 0.8101, *B-Proj* (Z^3, z^{PIS}) = 0.9503, *B-Proj* (Z^4, z^{PIS}) = 0.8969.

Step 5. Ranking the alternatives

Here, we notice that *B-Proj* $(Z^3, z^{PIS}) >$ *B-Proj* $(Z^4, z^{PIS}) >$ *B-Proj* $(Z^1, z^{PIS}) >$ *B-Proj* (Z^2, z^{PIS}) and therefore, the ranking order of the alternatives is obtained as $E_3 \succ E_4 \succ E_1 \succ E_2$. Hence, E_3 is the best choice among the alternatives.

Method 3. The proposed hybrid projection measure based MADM with bipolar neutrosophic information is provided as follows:

Step 1. Same as Method 1

Step 2. Same as Method 1

Step 3. Same as Method 1

Step 4. Computation of hybrid projection measure

The hybrid projection measures for different values of $\rho \in [0, 1]$ and the ranking order are shown in the following Table 3

Table 3. Results of hybrid projection measure for different valus of ρ

Similarity measure	ρ	Measure values	Ranking order
Hyb-Proj (Z^i, z^{PIS})	0.25	*Hyb-Proj* $(Z^1, z^{PIS}) = 1.4970$ *Hyb-Proj* $(Z^2, z^{PIS}) = 1.4819$ *Hyb-Proj* $(Z^3, z^{PIS}) = 1.5082$ *Hyb-Proj* $(Z^4, z^{PIS}) = 1.5203$	$E_4 > E_3 > E_1 > E_2$
Hyb-Proj (Z^i, z^{PIS})	0. 50	*Hyb-Proj* $(Z^1, z^{PIS}) = 2.1385$ *Hyb-Proj* $(Z^2, z^{PIS}) = 2.1536$ *Hyb-Proj* $(Z^3, z^{PIS}) = 2.0662$ *Hyb-Proj* $(Z^4, z^{PIS}) = 2.1436$	$E_2 > E_4 > E_1 > E_3$
Hyb-Proj (Z^i, z^{PIS})	0. 75	*Hyb-Proj* $(Z^1, z^{PIS}) = 2.7800$ *Hyb-Proj* $(Z^2, z^{PIS}) = 2.8254$ *Hyb-Proj* $(Z^3, z^{PIS}) = 2.6241$ Hyb-Proj $(Z^4, z^{PIS}) = 2.7670$	$E_2 > E_1 > E_4 > E_2$
Hyb-Proj (Z^i, z^{PIS})	0. 90	*Hyb-Proj* $(Z^1, z^{PIS}) = 3.1648$ *Hyb-Proj* $(Z^2, z^{PIS}) = 3.2285$ *Hyb-Proj* $(Z^3, z^{PIS}) = 2.9589$ *Hyb-Proj* $(Z^4, z^{PIS}) = 3.1410$	$E_2 > E_1 > E_4 > E_3$

6. Comparative analysis

In the Section, we compare the results obtained from the proposed methods with the results derived from other existing methods under bipolar neutrosophic environment to show the effectiveness of the developed methods.

Dey et al. [27] assume that the weights of the attributes are not identical and weights are fully unknown to the decision maker. Dey et al. [27] formulated maximizing deviation model under bipolar neutrosophic assessment to compute unknown weights of the attributes as w = (0.2585, 0.2552, 0.2278, 0.2585). By considering w = (0.2585, 0.2552, 0.2278, 0.2585), the proposed projection measure are shown as follows:

Proj $(Z^1)_{z^{PIS}} = 3.3954$, *Proj* $(Z^2)_{z^{PIS}} = 3.3872$, *Proj* $(Z^3)_{z^{PIS}} = 3.1625$, *Proj* $(Z^4)_{z^{PIS}} = 3.2567$.

Since, *Proj* $(Z^1)_{z^{PIS}} >$ *Proj* $(Z^2)_{z^{PIS}} >$ *Proj* $(Z^4)_{z^{PIS}} >$ *Proj* $(Z^3)_{z^{PIS}}$, therefore the ranking order of the four alternatives is given by $E_1 \succ E_2 \succ E_4 \succ E_3$. Thus, E_1 is the best choice for the customer.

Now, by taking w = (0.2585, 0.2552, 0.2278, 0.2585), the bidirectional projection measure are calculated as given below.
$B\text{-}Proj\ (Z^1, z^{PIS}) = 0.8113$, $B\text{-}Proj\ (Z^2, z^{PIS}) = 0.8111$, $B\text{-}Proj\ (Z^3, z^{PIS}) = 0.9854$, $B\text{-}Proj\ (Z^4, z^{PIS}) = 0.9974$.

Since, $B\text{-}Proj\ (Z^4, z^{PIS}) > B\text{-}Proj\ (Z^3, z^{PIS}) > B\text{-}Proj\ (Z^1, z^{PIS}) > B\text{-}Proj\ (Z^2, z^{PIS})$, consequently the ranking order of the four alternatives is given by $E_4 \succ E_3 \succ E_1 \succ E_2$ and hence, E_4 is obviously the best option for the customer.

Also, by taking w = (0.2585, 0.2552, 0.2278, 0.2585), the proposed hybrid projection measures for different values of $\rho \in [0, 1]$ and the ranking order are revealed in the following Table 4.

Table 4. Results of hybrid projection measure for different valus of ρ

Similarity measure	ρ	Measure values	Ranking order
$Hyb\text{-}Proj\ (Z^i, z^{PIS})$	0.25	$Hyb\text{-}Proj\ (Z^1, z^{PIS}) = 1.4573$ $Hyb\text{-}Proj\ (Z^2, z^{PIS}) = 1.4551$ $Hyb\text{-}Proj\ (Z^3, z^{PIS}) = 1.5297$ $Hyb\text{-}Proj\ (Z^4, z^{PIS}) = 1.5622$	$E_4 > E_3 > E_1 > E_2$
$Hyb\text{-}Proj\ (Z^i, z^{PIS})$	0. 50	$Hyb\text{-}Proj\ (Z^1, z^{PIS}) = 2.1034$ $Hyb\text{-}Proj\ (Z^2, z^{PIS}) = 2.0991$ $Hyb\text{-}Proj\ (Z^3, z^{PIS}) = 2.0740$ $Hyb\text{-}Proj\ (Z^4, z^{PIS}) = 2.1270$	$E_4 > E_1 > E_2 > E_3$
$Hyb\text{-}Proj\ (Z^i, z^{PIS})$	0. 75	$Hyb\text{-}Proj\ (Z^1, z^{PIS}) = 2.4940$ $Hyb\text{-}Proj\ (Z^2, z^{PIS}) = 2.7432$ $Hyb\text{-}Proj\ (Z^3, z^{PIS}) = 2.6182$ $\text{Hyb-Proj}\ (Z^4, z^{PIS}) = 2.6919$	$E_2 > E_4 > E_3 > E_1$
$Hyb\text{-}Proj\ (Z^i, z^{PIS})$	0. 90	$Hyb\text{-}Proj\ (Z^1, z^{PIS}) = 3.1370$ $Hyb\text{-}Proj\ (Z^2, z^{PIS}) = 3.1296$ $Hyb\text{-}Proj\ (Z^3, z^{PIS}) = 2.9448$ $Hyb\text{-}Proj\ (Z^4, z^{PIS}) = 3.0308$	$E_1 > E_2 > E_4 > E_3$

Deli et al. [5] assume the weight vector of the attributes as w = (0.5, 0.25, 0.125, 0.125) and the ranking order based on score values is presented as follows:

$$E_3 \succ E_4 \succ E_2 \succ E_1$$

Thus, E_3 was the most desirable alternative.

Dey et al. [27] employed maximizing deviation method to find unknown attribute weights as w = (0.2585, 0.2552, 0.2278, 0.2585). The ranking order of the alternatives is presented based on the relative closeness coefficient as given below.

$$E_3 \succ E_2 \succ E_4 \succ E_1$$

Obviously, E_3 was the most suitable option for the customer.

Dey et al. [27] also consider the weight vector of the attributes as w = (0.5, 0.25, 0.125, 0.125), then by using TOPSIS method, the ranking order of the cars is represented as follows:

$$E_4 \succ E_2 \succ E_3 \succ E_1.$$

So, E_4 would be the most preferable alternative for the buyer. We observe that different projection measures provide different ranking results and the projection measure is weight sensitive. Therefore, decision maker should choose the projection measure and weights of the attributes in the decision making context according to his/her needs, desires and practical condition

7. Conclusion

In this paper, we have defined projection, bidirectional projection measures between bipolar neutrosophic sets. Further, we have defined a hybrid projection measure by combining projection and bidirectional projection measures. Through these projection measures we have developed three algorithms for multi-attribute decision making models under bipolar neutrosophic environment for choosing the best alternative. Finally, a car selection problem has been provided to show the flexibility and applicability of the proposed methods. Furthermore, comparison analysis of the proposed methods with the other existing methods has also been demonstrated. The proposed algorithms can be extended to interval bipolar neutrosophic environment. In future, we shall apply projection, bidirectional projection, and hybrid projection measures of interval bipolar neutrosophic sets for group decision making, medical diagnosis, weaver selection, pattern recognition problems.

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