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Fundamental Homomorphism Theorems for Neutrosophic Extended Triplet Groups

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Received: 17 July 2018; Accepted: 31 July 2018; Published: 3 August 2018

Abstract: In classical group theory, homomorphism and isomorphism are significant to study the relation between two algebraic systems. Through this article, we propose neutro-homomorphism and neutro-isomorphism for the neutrosophic extended triplet group (NETG) which plays a significant role in the theory of neutrosophic triplet algebraic structures. Then, we define neutro-monomorphism, neutro-epimorphism, and neutro-automorphism. We give and prove some theorems related to these structures. Furthermore, the Fundamental homomorphism theorem for the NETG is given and some special cases are discussed. First and second neutro-isomorphism theorems are stated. Finally, by applying homomorphism theorems to neutrosophic extended triplet algebraic structures, we have examined how closely different systems are related.

Keywords: neutro-monomorphism; neutro-epimorphism; neutro-automorphism; fundamental neutro-homomorphism theorem; first neutro-isomorphism theorem; and second neutro -isomorphism theorem

1. Introduction

Groups are finite or infinite set of elements which are vital to modern algebra equipped with an operation (such as multiplication, addition, or composition) that satisfies the four basic axioms of closure, associativity, the identity property, and the inverse property. Groups can be found in geometry studied by "Felix klein in 1872" [1], characterizing phenomenality like symmetry and certain types of transformations. Group theory, firstly introduced by "Galois" [2], with the study of polynomials has applications in physics, chemistry, and computer science, and also puzzles like the Rubik's cube as it may be expressed utilizing group theory. Homomorphism is both a monomorphism and an epimorphism maintaining a map between two algebraic structures of the same type (such as two groups, two rings, two fields, two vector spaces) and isomorphism is a bijective homomorphism defined as a morphism, which has an inverse that is also morphism. Accordingly, homomorphisms are effective in analyzing and calculating algebraic systems as they enable one to recognize how intently distinct systems are associated. Similar to the classical one, neuro-homomorphism is the transform between two neutrosophic triplet algebraic objects N and H. That is, if elements in N satisfy some algebraic equation involving binary operation "*", their images in H satisfy the same algebraic equation. A neutro-isomorphism identifies two algebraic objects with one another. The most common use of neutro-homomorphisms and neutro-isomorphisms in this study is to deal with homomorphism theorems which allow for the identification of some neutrosophic triplet quotient objects with certain other neutrosophic triplet subgroups, and so on.

The neutrosophic logic and a neutrosophic set, firstly made known by Florentin Smarandache [3] in 1995, has been widely applied to several scientific fields. This study leads to a new direction, exploration, path of thinking to mathematicians, engineers, computer scientists, and many other researchers, so the area of study grew extremely and applications were found in many areas of neutrosophic logic and sets such as computational modelling [4], artificial intelligence [5], data mining [6], decision making problems [7], practical achievements [8], and so forth. Florentin Smarandache and Mumtazi Ali investigated the neutrosophic triplet group and neutrosophic triplet as expansion of matter plasma, nonmatter plasma, and antimatter plasma [9,10]. By using the concept of neutrosophic theory Vasantha and Smarandache introduced neutrosophic algebraic systems and *N*-algebraic structures [11] and this was the first neutrosofication of algerbraic structures. The characterization of cancellable weak neutrosophic duplet semi-groups and cancellable NTG are investigated [12] in 2017. Florentin Smarandache and Mumtaz Ali examined the applications of the neutrosophic triplet field and neutrosophic triplet ring [13,14] in 2017. Şahin Mehmet and Abdullah Kargın developed the neutrosophic triplet normed space and neutrosophic triplet inner product [15,16]. The neutrosophic triplet G-module and fixed point theorem for NT partial metric space are given in literature [17,18]. Similarity measures of bipolar neutrosophic sets and single valued triangular neutrosophic numbers and their appliance to multi-attribute group decision making investigated in [19,20]. By utilizing distance-based similarity measures, refined neutrosophic hierchical clustering methods are achieved in [21]. Single valued neutrosophic sets to deal with pattern recognition problems are given with their application in [22]. Neutrosophic soft lattices and neutrosophic soft expert sets are analyzed in [23,24]. Centroid single valued neutrosophic numbers and their applications in MCDM is considered in [25]. Bal Mikail, Moges Mekonnen Shalla, and Necati Olgun reviewed neutrosophic triplet cosets and quotient groups [26] by using the concept of NET in 2018. The concepts concerning neutrosophic sets and neutrosophic modules are described in [27,28], respectively. A method to handle MCDM problems under the SVNSs are introduced in [29]. Bipolar neutrosophic soft expert set theory and its basic operations are defined in [30].

The other parts of a paper is coordinated thusly. Subsequently, through the literature analysis in the first section and preliminaries in the second section, we investigated neutro-monomorphism, neutro-epimorphism, neutro-isomorphism, and neutro-automorphism in Section 3 and a fundamental homomorphism theorem for NETG in Section 4. We give and prove the first neutro-isomorphism theorem for NETG in Section 5, and then the second neutro-isomorphism theorem for NETG is given in Section 6. Finally, results are given in Section 7.

2. Preliminaries

In this section, we provide basic definitions, notations and facts which are significant to develop the paper.

2.1. Neutrosophic Extended Triplet

Let U be a universe of discourse, and $(N, *)$ a set included in it, endowed with a well-defined binary law $*$.

Definition 1 ([3]). *The set N is called a neutrosophic extended triplet set if for any $x \in N$ there exist $e^{neut(x)} \in N$ and $e^{anti(x)} \in N$. Thus, a neutrosophic extended triplet is an object of the form $(x, e^{neut(x)}, e^{anti(x)})$ where $e^{neut(x)}$ is extended neutral of x, which can be equal or different from the classical algebraic unitary element if any, such that*

$$x * e^{neut(x)} = e^{neut(x)} * x = x$$

and $e^{anti(x)} \in N$ is the extended opposite of x such that

$$x * e^{anti(x)} = e^{anti(x)} * x = e^{neut(x)}$$

In general, for each $x \in N$ there are many existing $e^{neut(x)'s}$ and $e^{anti(x)'s}$.

Theorem 1 ([11]). *Let $(N, *)$ be a commutative NET with respect to $*$ and $a, b \in N$;*

(i) $neut(a) * neut(b) = neut(a * b)$;
(ii) $anti(a) * anti(b) = anti(a * b)$;

Theorem 2 ([11]). *Let (N, $*$) be a commutative NET with respect to $*$ and $a \in N$;*

(i) $neut(a) * neut(a) = neut(a)$;
(ii) $anti(a) * neut(a) = neut(a) * anti(a) = anti(a)$

2.2. NETG

Definition 2 ([3]). *Let (N, $*$) be a neutrosophic extended triplet set. Then (N, $*$) is called a NETG, if the following classical axioms are satisfied.*

(a) *(N, $*$) is well defined, i.e., for any $x, y \in N$ one has $x * y \in N$.*
(b) *(N, $*$) is associative, i.e., for any $x, y, z \in N$ one has $x * (y * z) = (x * y) * z$.*

We consider, that the extended neutral elements replace the classical unitary element as well the extended opposite elements replace the inverse element of classical group. Therefore, NETGs are not a group in classical way. In the case when NETG enriches the structure of a classical group, since there may be elements with more extended opposites.

2.3. Neutrosophic Extended Triplet Subgroup

Definition 3 ([26]). *Given a NETG (N, $*$), a neutrosophic triplet subset H is called a neutrosophic extended triplet subgroup of N if it itself forms a neutrosophic extended triplet group under $*$. Explicity this means*

(1) *The extended neutral element $e^{neut(x)}$ lies in H.*
(2) *For any $x, y \in H, x * y \in H$.*
(3) *If $x \in H$ then $e^{anti(x)} \in H$.*

In general, we can show $H \leq N$ as $x \in H$ and then $e^{anti(x)} \in H$, i.e $x * e^{anti(x)} = e^{neut(x)} \in H$.

Definition 4. *Suppose that N is NETG and $H_1, H_2 \leq N.H_1$ and H_2 are called neutrosophic triplet conjugates of N if $n \in N$ thereby $H_1 = nH_2(anti(n))$.*

2.4. Neutro-Homomorphism

Definition 5 ([26]). *Let (N_1, $*$) and (N_2, $\circ$) be two NETGs. A mapping $f : N_1 \to N_2$ is called a neutro-homomorphism if*

(a) *For any $x, y \in N$, we have*

$$f(x * y) = f(x) \circ f(y)$$

(b) *If $(x, neut(x), anti(x)$ is a neutrosophic extended triplet from N_1, then*

$$f(neut(x)) = neut(f(x))$$

and

$$f(anti(x)) = anti(f(x)).$$

Definition 6 ([26]). *Let f: $N_1 \rightarrow N_2$ be a neutro-homomorphism from a NETG (N_1, $*$) to a NETG (N_2, $\circ$). The neutrosophic triplet image of f is*

$$Im(f) = \{f(g) : g \in N_1, *\}.$$

Definition 7 ([26]). *Let f: $N_1 \rightarrow N_2$ be a neutro-homomorphism from a NETG (N_1, $*$) to a NETG (N_2, $\circ$) and B $\subseteq N_2$. Then*

$$f^{-1}(B) = \{x \in N_1 : f(x) \in B\}$$

is the neutrosophic triplet inverse image of B under f.

Definition 8 ([26]). *Let $f : N_1 \rightarrow N_2$ be a neutro-homomorphism from a NETG (N_1, $*$) to a NETG (N_2, $\circ$). The neutrosophic triplet kernel of f is a subset*

$$Ker(f) = \{x \in N_1 : f(x) = neut(x)\}\ of\ N_1,$$

where neut(x) denotes the neutral element of N_2.

Definition 9. *The neutrosophic triplet kernel of ϕ is called the neutrosophic triplet center of NETG N and it is denoted by Z(N). Explicitly,*

$$\begin{aligned} Z(N) &= \{a \in N : \varphi_a = neut_N\} \\ &= \{a \in N : ab(anti(a)) = b, \forall b \in N\} \\ &= \{a \in N : ab = ba, \forall b \in N\}. \end{aligned}$$

Hence Z(N) is the neutrosophic triplet set of elements in N that commute with all elements in N. Note that obviously Z(N) is a neutrosophic triplet. We have $Z(N) = N$ in the case that N is abelian.

Definition 10 ([26]). *Let N be a NETG and $H \subseteq N. \forall x \in N$, the set $xh/h \in H$ is called neutrosophic triplet coset denoted by xH. Analogously,*

$$Hx = hx/h \in H$$

and

$$(xH)anti(x) = (xh)anti(x)/h \in H.$$

When $h \leq N, xH$ is called the left neutrosophic triplet coset of H in N containing x, and Hx is called the right neutrosophic triplet coset of H in N containing x. $\mid xH \mid$ and $\mid Hx \mid$ are used to denote the number of elements in xH and Hx, respectively.

2.5. Neutrosophic Triplet Normal Subgroup and Quotient Group

Definition 11 ([26]). *A neutrosophic extended triplet subgroup H of a NETG of N is called a neutrosophic triplet normal subgroup of N if $aH(anti(a)) \subseteq H, \forall x \in N$ and we denote it as $H \trianglelefteq N$ and $H \triangleleft N$ if $H \neq N$.*

Example 1. *Let N be NETG. {neut} $\triangleleft$ N and N $\trianglelefteq$ N.*

Definition 12 ([26]). *If N is a NETG and $H \trianglelefteq N$ is a neutrosophic triplet normal subgroup, then the neutrosophic triplet quotient group N/H has elements $xH : x \in N$, the neutrosophic triplet cosets of H in N, and operation $(xH)(yH) = (xy)H$.*

3. Neutro-Monomorphism, Neutro-Epimorphism, Neutro-Isomorphism, Neutro-Automorphism

In this section, we define neutro-monomorphism, neutro-epimorphism, neutro-isomorphism, and neutro-automorphism. Then, we give and some important theorems related to them.

3.1. Neutro-Monomorphism

Definition 13. *Assume that (N_1, $*$) and (N_2, $\circ$) be two NETG's. If a mapping $f : N_1 \to N_2$ of NETG is only one to one (injective) f is called neutro-monomorphism.*

Theorem 3. *Let (N_1, $*$) and (N_2, $\circ$) be two NETG's. $\varphi : N_1 \to N_2$ is a neutro-monomorphism of NETG if and only if $ker\varphi = \{neut_{N1}\}$.*

Proof. Assume φ is injective. If $a \in \ker\varphi$, then

$$\varphi(a) = neut_{N2} = \varphi(neut_{N1}), \forall a \in N_1$$

and hence by injectivity $a = neut_{N1}$. Conversely, assume $\ker\varphi = \varphi(neut_{N1})$. Let $a,b \in N_1$ such that $\varphi(a) = \varphi(b)$. We need to show that $a = b$.

$$\begin{aligned} neut_H &= \varphi(b)anti(\varphi(a)) \\ &= \varphi(b)\varphi(anti(a)) \\ &= \varphi(b(anti(a))). \end{aligned}$$

Thus $b(anti(a))) \in ker\varphi$, and hence, by assumption $ker\varphi = \varphi(neut_{N1})$. We conclude that $b(anti(a))) = neut_{N1}, i.e., \ a = b$. □

Definition 14. *Let (N_1, $*$) and (N_2, $\circ$) be two NETG's. If a mapping $f : N_1 \to N$ is only onto (surjective) f is called neutro-epimorphism.*

Theorem 4. *Let N and H be two NETG's. If $\varphi : N \to H$ is a neutro-homomorphism of NETG, then so is φ^{-1}: $H \to N$.*

Proof. Let $x = \varphi(a), y = \varphi(b), \forall a, b \in N$ and $\forall x, y \in H$. So $a = anti(\varphi(x)), b = anti(\varphi(y))$. Now

$$\begin{aligned} anti(xy) &= \varphi(\varphi(a)\varphi(b)) \\ &= anti(\varphi(ab) = ab \\ &= anti(\varphi(x))anti(\varphi(y)). \end{aligned}$$

□

Theorem 5. *Let N be NETG and $a, b \in N$. The map $\phi : N \to AutN$. Then, $a \to \varphi_a$, is a neutro-homomorphism.*

Proof. For any fixed $n \in N$, we have

$$\begin{gathered} \varphi_{ab}(N) = abn(anti(ab)) = abn(anti(a))anti(b) \\ = \varphi_a(bn(anti(b)) = \varphi_a\varphi_b(n), \\ \text{So } \varphi_{ab} = \varphi_a\varphi_b, i.e., \phi(ab) = \phi(a)\phi(b). \end{gathered}$$

It is in fact has anti-neutral element i.e., $\varphi(anti(n)) = anti(\varphi_n)$. Since $\varphi_n anti(\varphi_n(a)) = n(anti(n)an)anti(n) = a$, and so φ_n is injective. □

Theorem 6. *Let $f : N \to H$ be a neutro-homomorphism of NETG N and H. For $h \in H$ and $x \in f^{-1}(h), f^{-1}(h) = x \in kerf$.*

Proof. (1) Let's show that $f^{-1}(h) \subseteq x\ kerf$. If $x \in f^{-1}(h)$, then $f(x) = h$ and $b \in f^{-1}(h)$, then $f(b) = h$. If $f(x) = f(y)$, then:

$$anti(f(x))f(x) = anti(f(x))f(b)(\text{by theorem 1})$$

$$neut_H = f(anti(x))f(b)\ (\text{by definition 1})$$

$$\Rightarrow anti(x)b \in kerf.$$

For at least $k \in kerf, anti(x)b = k$. If $b = xk$, then,

$$b \in xkerf \Rightarrow f^{-1}(h) \subseteq xkerf \tag{1}$$

(2) Let's show that $xkerf \subseteq f^{-1}(h)$. Let $b \in xkerf$. For at least $k \in kerf, b = xk$

$$\Rightarrow f(b) = f(xk) = f(x)f(k) = h\ neut_H = h$$

If $f^{-1}(h) = b$ and $b \in f^{-1}(h)$, then

$$xkerf \subseteq f^{-1}(h) \tag{2}$$

by (1) and (2), we obtain $xkerf = f^{-1}(h)$.
□

Theorem 7. *Let $\varphi : N_1 \to N_2$ be a neutro-homomorphism of NETG N_1 and N_2.*

(1) $If\ H_2 \trianglelefteq N_2$, then $\varphi^{-1}(H_2) \trianglelefteq N_1$.
(2) $If H_1 \trianglelefteq N_1$ *and* φ *is a* $neutro-epimorhism$ *then* $\varphi(H_1) \trianglelefteq N_2$.

Proof. (1) If $x \in \varphi^{-1}(H_2)$ and $a \in N_1$, then $\varphi(x) \in H_2$ and so $\varphi((ax)(anti(a)) = \varphi(a)\varphi(x)anti(\varphi(a)) \in H_2$. Since H_2 is neutrosophic triplet normal subgroup. We conclude $ax(anti(a)) \in \varphi^{-1}(H_2)$.

(2) Since H_1 is neutrosophic triplet normal subgroup, we have $\varphi(a)\varphi(H_1)anti(\varphi(a)) \subseteq \varphi(H_1)$. Since we assume φ is surjective, every $b \in N_2$ can be written as $b = \varphi(a), a \in N_1$. Therefore, $b\varphi(H_1)anti(b) \in \varphi(H_1)$.
□

Theorem 8 ([26]). *Let $f : N \to H$ be a neutro-homomorphism from a NETG N to a NETG H. $Kerf \triangleleft N$.*

Theorem 9. *Let N be NETG and $H \trianglelefteq N$. The map $\varphi : N \to N/H,\ n \to nH,$ is a neutro-homomorphism with neutrosophic triplet kernel $ker\,\varphi = H$.*

Proof. We have $\varphi(ab) = (ab)H = (aH)(bH) = \varphi(a)\varphi(a)$, so ϕ is a neutro-homomorphism. As to the neutrosophic triplet kernel, $a \in ker\varphi \Leftrightarrow \varphi(a) = H$ (since H is neutral in N/H) $\Leftrightarrow aH = H$ (by definition of ϕ) $\Leftrightarrow a \in H$. □

Theorem 10. *Let N be NETG and $H \subseteq N$ be a non-empty neutrosophic extended triplet subset. Then $H \trianglelefteq N$, if and only if there exists a neutro-homomorphism $\varphi : N_1 \to N_2$ with $H = ker\varphi$.*

Proof. Its straight forward. □

3.2. Neutro-Isomorphism

Definition 15. *Let (N_1, $*$) and (N_2, $\circ$) be two NETGs. If a mapping $f : N_1 \to N_2$ neutro-homomorphism is one to one and onto f is called neutro-isomorphism. Here, N_1 and N_2 are called neutro-isomorphic and denoted as $N_1 \cong N_2$.*

Theorem 11. *Let (N_1, $*$) and (N_2, $\circ$) be two NETG's. If $f : N_1 \to N_2$ is a neutro-isomorphism of NETG's, then so is $f^{-1} : N_2 \to N_1$.*

Proof. It is obvious to show that f is one to one and onto. Now let's show that f is neutro-homomorphism. Let $x = \varphi(a), y = \varphi(b), \forall a, b \in N_1, \forall x, y \in N_2$ and so, $a = anti(\varphi(x)), b = anti(\varphi(y))$. Now $anti(xy) = anti(\varphi(\varphi(a)\varphi(b))) = anti(\varphi(\varphi(ab))) = ab = anti(\varphi(x))anti(\varphi(y))$. □

3.3. Neutro-Automorphism.

Definition 16. *Let (N_1, $*$) and (N_2, $\circ$) be two NETG'S. If a mapping $f : N_1 \to N_2$ is one to one and onto f is called neutro-automorphism.*

Definition 17. *Let N be NETG. $\varphi \in AutN$ is called a neutro-inner automorphism if there is a $n \in N$ such that $\varphi = \varphi_n$.*

Proposition 1. *Let N be a NETG. For $a \in N, f_a : N \to N$ such that $x \to ax(anti(a)$ is a neutro-automorphism $(AutN)$.*

Proof. (1) $\forall x, y \in N$, we have to show that

$$f(x) = f(y) \Rightarrow x = y.ax(anti(a)) = ay(anti(a)) \Rightarrow ax(anti(a))a = ay(anti(a))a \Rightarrow ax(neut(a)) = ay(neut(a)) \Rightarrow$$

Therefore, f is one to one.

(2) $\forall x, y \in N$, we have to show that

$$f(x) = ax(anti(a)) = y.ax(anti(a))a = ya \Rightarrow ax(neut(a)) = ya \Rightarrow ax = ya \Rightarrow anti(a)ax = anti(a)ya \Rightarrow neut(a)x =$$

So, f is onto. Therefore, f_a is a neutro-automorphism.

□

Lemma 1. *Let a be an element of NETG N such that* $a^2 = a$. *Then* $a = neut(a)$.

Proof. We have

$$\begin{aligned} &= (anti(a) * a) * a \text{ for } anti(a) \in N \text{ (anti axiom)} \\ &= anti(a) * a^2 \text{ (associativity axiom)} \\ &= anti(a) * a \text{ (by assumption)} \\ &= neut(a) \text{ (by definition of anti)} \end{aligned}$$

□

Theorem 12. *Let N be NETG and* $H_1, H_2 \leq N$. *Then the neutrosophic extended triplet set* $H_1H_2 = \{ab : a \in H_1, b \in H_2\}$ *is a neutrosophic extended triplet subgroup in the case that* $H_1H_2 = H_2H_1$.

Proof. Suppose H_1H_2 is a neutrosophic extended triplet subgroup. Then, for all $a \in H_1, b \in H_2$, we have $anti(a)anti(b) \in H_1H_2$, *i.e.*, $H_2H_1 \subseteq H_1H_2$. But also for $h \in H_1H_2$ we find $a \in H_1, b \in H_2$ thereby $anti(h) = ab$, and then $h = anti(b)anti(a) \in H_2H_1$. So $H_1H_2 \subseteq H_2H_1$, that's, $H_1H_2 = H_2H_1$. On the other hand, assume that $H_1H_2 = H_2H_1$. Then $\forall a, a' \in H_1, b, b' \in H_2$ we have $aba'b' \in aH_2H_1b' = aH_1H_2b' = H_1H_2$. Furthermore, $\forall a \in H_1, b \in H_2$ we have $anti(ab) = anti(b)anti(a) \in H_2H_1 = H_1H_2$. □

4. Fundamental Theorem of Neutro-Homomorphism

The fundamental theorem of neutro-homomorphism relates the structure of two objects between which a neutrosophic kernel and image of the neutro-homomorphism is given. It is also significant to prove neutro-isomorphism theorems. In this section, we give and prove the fundamental theorem of neutro-homomorphism. Then, we discuss a few special cases. Finally, we give examples by using NETG.

Theorem 13. *Let* N_1, N_2 *be NETG's and* $\phi : N_1 \to N_2$ *be a neutro-homomorphism. Then,* $N_1/ker(\phi) \cong im(\phi)$. *Furthermore if* φ *is neutro-epimorphism, then*

$$N_1/ker\phi \cong N_2.$$

$$\begin{tikzcd} N_1 \arrow[rr, "\phi"] \arrow[dr, "\varphi"'] & & im(\phi) \\ & N_1/ker(\phi) \arrow[ur, "i"'] & \end{tikzcd}$$

Proof. We will construct an explicit map $i : N_1/ker(\phi) \to im(\phi)$ and prove that it is a neutro-isomorphism and well defined. Since $ker(\varphi)$ is neutrosophic triplet normal subgroup of N_1. Let $K = ker(\phi)$, and recall that $N_1/K = \{aK : a \in N_1\}$. Define $i : N_1/K \to im(\phi), i : nK \to \phi(n), n \in N_1$. Thus, we need to check the following conditions.

(1) i is well defined
(2) i is injective
(3) i is surjective
(4) i is a neutro-homomorphism

(1) We must show that if $aK = bK$, then $i(aK) = (bK)$. Suppose $aK = bK$. We have $aK = bK \Rightarrow anti(b)aK = K \Rightarrow anti(b)a \in K$. Here, $neut_{(n2)} = \phi(anti(b)a) = \phi(anti(b)\phi(a) = anti(\phi(b))\phi(a) \Rightarrow \phi(a) = \phi(b)$. Hence, $i(aK) = \phi(a) = \phi(b) = i(bK)$. Therefore, it is well defined.

(2) We must show that $i(aK) = i(bK) \Rightarrow aK = bK$. Suppose that $i(aK) = i(bK)$. Then

$$i(aK) = i(bK) \Rightarrow aK = bK.$$

$$\Rightarrow \phi(anti(b))\ \phi(a) = neut_{(n2)} \Rightarrow \phi(anti(b)a) = neut_{(n2)} \Rightarrow anti(b)a \in K$$

$$\Rightarrow anti(b)aK = K\ (aN_2 = N_2 \Leftrightarrow a \in N_2).$$

Thus, i is injective.

(3) We must show that for any element in the domain (N_1/K) gets mapped to it by i. let's pick any element $\phi(a) \in im(\phi)$. By definition, $i(aK) = \phi(a)$, hence i is surjective.

(4) We must show that $i(aK\,bK) = i(aK)i(bK).i(aK\,bK) = i(abK)(aK\,bK = abK) = \phi(ab) = \phi(a)\phi(b) = i(aK\,bK) = i(aK)i(bK)$. Thus, i is a neutro-homomorphism.

In summary, since $i : N_1/K \to im(\phi)$ is a well-defined neutro-homomorphism that is injective and surjective. Therefore, it is a neutro-isomorphism. *i.e.*, $N_1/K \cong im(\phi)$, and the fundamental theorem of neutro-homomorphism is proven. □

Corollary 1 (A Few Special Cases of Fundamental Theorem of Neutro-homomorphism).

- *Let $N = (1, 1, 1)$ be a trivial neutrosophic extended triplet. If φ: $N_1 \to N_2$ is an embedding, then neutrosophic $ker(\varphi) = \{neut(1) = 1N_1\}$. The Theorem 12 says that $im(\varphi) \cong \{N_1/1N_1\} \cong N_1$.*
- *If φ: $N_1 \to N_2$ is a map $\varphi(n) = neut(1) = 1N_2$ for all $n_2 \in N_1$, then neutrosophic $ker(\varphi) = N_1$, so Theorem 13 says that $1N_2 = im(\varphi) \cong N_1/N_1$.*

Example 2. *The neutrosophic extended triplet alternating group A_n (the neutrosophic extended triplet subgroup of even permutation in NETG S_n) has index 2 in S_n.*

Solution. *To prove that $[S_n:A_n] = 2$. We will construct a surjective neutro-homomorphism ϕ: $S_n \to Z_2$ with neutrosophic triplet $ker\phi = A_n$. Here the neutrosophic extended triplets of Z_2 are (0, 0, 0) and (1, 1, 1). If this is achieved, it would follow that $S_n/A_n \cong Z_2$, so $|S_n/A_n| = |Z_2| = 2$, and therefore $[S_n:A_n] = |S_n/A_n| = 2$, as desired. Define ϕ: $S_n \to Z_2$ by* $\phi(f) = \begin{cases} [0]\ if\ f\ is\ even \\ [1]\ if\ f\ is\ odd \end{cases}$

By construction ϕ is surjective. To prove that ϕ is a neutro-homomorphism we need to show that $\phi(x) + \phi(y) = \phi(xy)$, $\forall x, y \in S_n$. Here if x and y are both even or both odd, then xy is even. If x is even and y is odd, or if x is odd and y is even, then xy is odd. Let us see these four different cases as follows:

(1) x and y are both even. Then xy is also even. So, $\phi(x) = \phi(y) = \phi(xy) = [0]$. Since $[0] + [0] = [0]$ holds.

(2) x is even, and y is odd. Then xy is odd. So, $\phi(x) + \phi(y) = [0] + [1] = [1] = \phi(xy)$.

(3) x is odd, and y is even. This case is analogous to case 2.

(4) x and y are both odd. Then xy is even, so $\phi(x) + \phi(y) = [1] + [1] = [0] = \phi(xy)$. Thus, we verified that ϕ is a neutro-homomorphism. Finally, neutrosophic trplet $ker\phi = \{x \in S_n: \phi(x) = [0]_2\}$ is the neutrosophic extended triplet set of all even permutations, so neutrosophic triet $ker\varphi = A_n$.

5. First Neutro-Isomorphism Theorem

The first neutro-isomorphism theorem relates two neutrosophic triplet quotient groups involving products and intersections of neutrosophic extended triplet subgroups. In this section, we give and prove the first neutro-isomorphism theorem. Finally, we give an example by using NETG.

Theorem 14. *Let N be NETG and H, K be two neutrosophic extended triplet subgroup of N and H is a neutrosophic triplet normal in K. Then*

(a) *HK is neutrosophic triplet subgroup of N.*
(b) *$H \bigcap K$ is neutrosophic triplet normal subgroup in K.*
(c) $\frac{HK}{H} \cong \frac{\mathrm{K}}{H \bigcap K}$

Proof. (a) Let $xy \in HK$. If $x = h_1k_1$ and $y = h_2k_2, h_1h_2 \in H$ and $k_1, k_2 \in K$. Consider

$$\begin{aligned} x(anti(y)) = (h_1k_1) \quad & anti(h_2k_2) \\ &= (h_1k_1)anti(k_2)anti(h_2) \\ &= h_1(k_1(anti(k_2)))anti(h_2), (k_3 = k_1(anti(k2)) : k_3 \in K \\ &= h_1k_3(anti(h_2)) \\ &= h_1k_3(anti(h_2))anti(k_3)k_3 \\ &= h_1k_3(anti(h_2))anti(k_3)k_3 \\ &= h_1h_2k_3 \; because \; H \lhd kso \; h_3 = k_3(anti(h_2))anti(k_3) \in H \\ &\Rightarrow x(anti(y) = h_4k_3 \in HK, (h_4 = h_1h_2) \\ &\Rightarrow HK \text{ is NETG of } N. \end{aligned}$$

(b) We have to prove $H \cap K$ is neutrosophic triplet normal subgroup in k or $H \cap K \lhd k$. Let $x \in H \cap K$ and $x \in K$. If $x \in H$ and $x \in K$, then $kx(anti(k)) \in H$ because $H \lhd k$ and $kx(anti(k)) \in K$ because $xk \in K$. Thus, $kx(anti(k)) \in H \cap K$. Since $H \cap K \lhd k$.
(c) $\frac{HK}{H} \cong \frac{K}{H \bigcap K}$. Let $H \bigcap K = D$, so $\frac{K}{D} = \frac{K}{H \bigcap K}$. Now let's define a mapping φ: $HK \rightarrow \frac{K}{D}$ by $\phi(hk) = KD$.

1. φ is well defined

$$h_1k_1 = h_2k_2, h_1h_2 \in H \text{ and } k_1k_2 \in K$$

$$\begin{aligned} & k_1h'_1 = k_2h'_2 \\ & \Rightarrow anti(k_2)k_1h'_1 = h'_2 \\ & \Rightarrow anti(k_2)k_1 = h'_2(anti(h_1)), h'_2(anti(h_1)) \in H \\ & \Rightarrow anti(k_2)k_1 \in H, \text{ but } anti(k_2)k_1 \in K \\ & \Rightarrow anti(k_2)k_1 \in H \cap K = D \\ & \Rightarrow anti(k_2)k_1 \in D \\ & \Rightarrow anti(k_2)k_1D = D \\ & \Rightarrow k_1D = k_2D \\ & \Rightarrow \phi(h_1k_1) = \phi(h_2k_2). \end{aligned}$$

2. φ is neutro-homomorphism.

$$\begin{aligned} \Phi(h_1k_1.h_2k_2) &= \phi(h_1(k_1h_2)k_2 \\ &= \phi(h_1h2'k_1k_2) \\ &= K_1k_2D \\ &= k_1Dk_2D \\ &= \phi(h_1k_1).\phi(h_2k_2) \end{aligned}$$

3. φ is onto.

Since for every $KD \in K/D, \ni neut.k \in HK$ under φ such that $\phi(neut.k) = KD$. Hence, φ is onto. Now by Theorem 13,

$$HK/Ker\phi \cong K/D$$

Now it is enough to prove that $ker\phi = H$. Let $h \in H, h(neut) \in HK$. Thus

$$\begin{aligned}\phi(h) &= \phi(h.neut) = neut.D = D\\ &\Rightarrow \phi(h) = D\\ &\Rightarrow h \in ker\phi.i.eH \subseteq ker\phi\end{aligned}$$

Conversly, $hk \in ker\phi$, where $h \in H$ and $k \in K$. If $\phi(hk) = D$, then

$$\begin{aligned}KD = D \quad &\Rightarrow k \in D = H \cap K\\ &\Rightarrow h \in H \text{ and } k \in K\\ &\Rightarrow hk \subseteq H\\ &\Rightarrow ker\phi \subseteq H. \text{ Thus } H = ker\phi\end{aligned}$$

by (1) $\frac{HK}{H} \cong \frac{K}{H \bigcap K}$. □

Example 3. *Let N be NETG. Neutro-isomorphism theorems are for instance useful in the calculation of NETG orders, since neutro-isomorphic groups have the same order. If $H \leq N$ and $K \trianglelefteq N$ so that HK is finite, then Lagrange's theorem [26] in neutrosophic triplet with theorem 13 yield*

$$\begin{aligned}|HK|/|K| &= |HK:K|\\ &= |HK/K|\\ &= |H/H \textstyle\bigcap K|\\ &= |H:H \textstyle\bigcap K|\\ &= |H|/|H \textstyle\bigcap K|, \text{ that is}\\ |HK| &= |H|\,|K|/|H \textstyle\bigcap K|\end{aligned}$$

6. Second Neutro-Isomorphism Theorem

The second neutro- isomorphism theorem is extremely useful in analyzing the neutrosophic extended normal subgroups of a neutrosophic triplet quotient group. In this section, we give and prove the second neutro-homomorphism theorem for NETG.

Theorem 15. *Let N be a NETG. Let H and K be neutrosophic triplet normal subgroup of N with $K \subseteq H$. Then $H/K \lhd N/K$ and $N/KH/K \cong N/H$*

Proof. Consider the natural map *Ψ:N→N/H*. The neutrosophic triplet kernel, *H* contains *K.* Thus, by the universal property of N/K, it follows that there is a neutro-homomorphism $N/K \to N/H$. This map is clearly surjective. In fact, it sends the neutrosophic triplet left coset nK to the neutrosophic triplet left coset nH. Now suppose that nK is in the neutrosophic triplet kernel. Then the neutrosophic triplet left coset nH is the neutral neutrosophic triplet coset, that is, $nH = H$, so that $n \in H$. Thus the neutrosophic triplet kernel consists of those neutrosophic triplet left cosets of the form nK, for $n \in H$, that is, H/K.

1. Ψ is well defined. Let $ak = bk$.

$$\begin{gathered} anti(b)ak = k \\ anti(b)a \in k \\ \Rightarrow K \lhd H \\ anti(b)a \in H \\ aH = bH(anti(b)aH = H) \\ \Psi(ak) = \Psi(bk) \end{gathered}$$

2. Ψ is neutro-homomorphism

$$\begin{aligned} &a_k, b_k \in N/K \\ &\Psi(a_k b_k) = \Psi(abk) = abH = aHbH = \Psi(ak)\Psi(bk). \end{aligned}$$

3. Ψ is onto
   For all $y = aH \in N/H, x = ak \in N/K \Rightarrow \Psi(x) = y$.
4. kerΨ = H/K
   The neutral element of N/H is H. Therefore

$$\begin{aligned} &ker\ \Psi\text{: } \{xk \in N/K : \Psi(xk) = H\} \\ &= \{xk \in N/K : \Psi(xk) = xH = H\} \\ &= \{xk \in N/K : x \in H\} \\ &= \{xk \in H/K\} \\ &= H/K. \end{aligned}$$

By Theorem 13 $N/KH/K \cong N/H$.

□

7. Conclusions

This paper is mainly focused on fundamental homomorphism theorems for neutrosophic extended triplet groups. We gave and proved the fundamental theorem of neutro-homomorphism, as well as first and second neutro-isomorphism theorems explained for NETG. Furthermore, we define neutro-monomorphism, neutro-epimorphism, neutro-automorphism, inner neutro-automorphism, and center for neutrosophic extended triplets. Finally, by applying them to neutrosophic algebraic structures, we have examined how closely different systems are related. By using the concept of a fundamental theorem of neutro-homomorphism and neutro-isomorphism theorems, the relation between neutrosophic algebraic structures (neutrosophic triplet ring, neutrosophic triplet field, neutrosophic triplet vector space, neutrosophic triplet normed space, neutrosophic modules, etc.) can be studied and the field of study in neutrosophic algebraic structures will be extended.

Author Contributions: M.Ç. defined and studied preliminaries and the fundamental theorem of neutro-homomorphism theorem with examples and wrote part of the paper. M.M.S. defined and studied first and second neutro-isomorphism theorems with example. The revision and submission of the paper was completed by N.O.

Funding: This research received no external funding.

Conflicts of Interest: The authors declare no conflict of interest.

References

1. Klein, F. His "Erlanger Programm". In *History and Philosophy of Modern Mathematics*; University of Minnesota Press: Minneapolis, MN, USA, 1988; Volume 11, p. 145.
2. Schneps, L.; Lochak, P. (Eds.) *Around Grothendieck's Esquisse D'un Programme*; Cambridge University Press: Cambridge, UK, 1997; Volume 1.
3. Smarandache, F. *Neutrosophy: Neutrosophic Probability, Set, and Logic: Analytic Synthesis & Synthetic Analysis*; American Research Press: Santa Fe, NM, USA, 1998.
4. Solis, A.R.; Panoutsos, G. Granular computing neural-fuzzy modelling: A neutrosophic approach. *Appl. Soft Comput.* **2013**, *13*, 4010–4021. [CrossRef]
5. Khatibi, V.; Montazer, G.A. Intuitionistic fuzzy set vs. fuzzy set application in medical pattern recognition. *Artif. Intell. Med.* **2009**, *47*, 43–52. [CrossRef] [PubMed]
6. Ye, J. Single-valued neutrosophic minimum spanning tree and its clustering method. *Int. J. Intell. Syst.* **2014**, *23*, 311–324. [CrossRef]
7. Ye, J. Multicriteria decision-making method using the correlation coefficient under single-valued neutrosophic environment. *Int. J. Gen. Syst.* **2013**, *42*, 386–394. [CrossRef]
8. Smarandache, F.; Ye, J. Summary of the Special Issue "Neutrosophic Information Theory and Applications" at "Information" Journal. *Information* **2018**, *9*, 49. [CrossRef]
9. Smarandache, F.; Ali, M. Neutrosophic triplet group. *Neural Comput. Appl.* **2018**, *29*, 595–601. [CrossRef]
10. Smarandache, F.; Ali, M. Neutrosophic triplet as extension of matter plasma, unmatter plasma, and antimatter plasma. Available online: http://adsabs.harvard.edu/abs/2016APS..GECHT6110S (accessed on 1 August 2018).
11. Kandasamy, W.B.; Smarandache, F. Some Neutrosophic Algebraic Structures and Neutrosophic N-Algebraic structures. Available online: https://books.google.com.tr/books/about/Some_Neutrosophic_Algebraic_Structures_a.html?id=ICTeBAAAQBAJ&redir_esc=y (accessed on 1 August 2018).
12. Zhang, X.; Smarandache, F.; Liang, X. Neutrosophic Duplet Semi-Group and Cancellable Neutrosophic Triplet Groups. *Symmetry* **2017**, *9*, 275. [CrossRef]
13. Smarandache, F.; Ali, M. Neutrosophic Triplet Field used in Physical Applications. *Bull. Am. Phys. Soc.* **2017**, *7*, 62.
14. Smarandache, F.; Ali, M. Neutrosophic Triplet Ring and its Applications (Log Number: NWS17-2017-000062). In Proceedings of the 18th Annual Meeting of the APS Northwest Section, Pacific University, Forest Grove, OR, USA, 1–3 June 2017.
15. Şahin, M.; Kargın, A. Neutrosophic triplet normed space. *Open Phys.* **2017**, *15*, 697–704. [CrossRef]
16. Şahin, M.; Kargın, A. Neutrosophic triplet inner product space. *Neutrosophic Oper. Res.* **2017**, *2*, 193–215.
17. Smarandache, F.; Şahin, M.; Kargın, A. Neutrosophic Triplet G-Module. *Mathematics* **2018**, *6*, 53. [CrossRef]
18. Şahin, M.; Kargın, A.; Çoban, M.A. Fixed Point Theorem for Neutrosophic Triplet Partial Metric Space. *Symmetry* **2018**, *7*, 240. [CrossRef]
19. Uluçay, V.; Deli, I.; Şahin, M. Similarity measures of bipolar neutrosophic sets and their application to multiple criteria decision making. *Neural Comput. Appl.* **2018**, *29*, 739–748. [CrossRef]
20. Şahin, M.; Kargın, A.; Smarandache, F. Generalized Single Valued Triangular Neutrosophic Numbers and Aggregation Operators for Application to Multi-attribute Group Decision Making. In *New Trends in Neutrosophic Theory and Applications*; International Knowledge Press: Manchester, UK, 2017.
21. Memet, S.; Orhan, E.; Vakkas, U.; Harun, D. Refined Neutrosophic Hierchical Clustering Methods. *Asian J. Math. Comput. Res.* **2017**, *15*, 283–295.
22. Sahin, M.; Olgun, N.; Uluçay, V.; Kargın, A.; Smarandache, F. A New Similarity Measure Based on Falsity Value between Single Valued Neutrosophic Sets Based on the Centroid Points of Transformed Single Valued Neutrosophic Numbers with Applications to Pattern Recognition. *Neutrosophics Sets Syst.* **2017**, *15*, 35–52.
23. Uluçay, V.; Şahin, M.; Olgun, N.; Kilicman, A. On neutrosophic soft lattices. *Afr. Matematika* **2017**, *28*, 379–388. [CrossRef]
24. Sahin, M.; Alkhazaleh, S.; Ulucay, V. Neutrosophic soft expert sets. *Appl. Math.* **2015**, *6*, 116–127. [CrossRef]
25. Sahin, M.; Ecemis, O.; Uluçay, V.; Kargın, A. Some New Generalized Aggregation Operators Based on Centroid Single Valued Triangular Neutrosophic Numbers and Their Applications in Multi-Attribute Decision Making. *Asian J. Math. Comput. Res.* **2017**, *16*, 63–84.

26. Bal, M.; Shalla, M.M.; Olgun, N. Neutrosophic Triplet Cosets and Quotient Groups. *Symmetry* **2018**, *10*, 126. [CrossRef]
27. Olgun, N.; Bal, M. Neutrosophic Modules. In *Neutrosophic Operational Research*; Pons Publishing House: Brussels, Belgium, 2017; Volume 2, pp. 181–192.
28. Broumi, S.; Bakali, A.; Talea, M.; Smarandache, F.; Uluçay, V.; Sahin, M.; Dey, A.; Dhar, M.; Tan, R.P.; Bahnasse, A.; et al. Neutrosophic Sets: An Overview. Available online: http://www.rxiv.org/pdf/1805.0007v1.pdf (accessed on 1 August 2018).
29. Şahin, M.; Uluçay, V.; Acıoglu, H. Some weighted arithmetic operators and geometric operators with SVNSs and their application to multi-criteria decision making problems. Available online: https://zenodo.org/record/1237954#.W2M2u9IzbIU (accessed on 1 August 2018).
30. Şahin, M.; Uluçay, V.; Broumi, S. Bipolar Neutrosophic Soft Expert Set Theory. Available online: https://books.google.com.tr/books/about/Bipolar_Neutrosophic_Soft_Expert_Set_The.html?id=aAhZDwAAQBAJ&redir_esc=y (accessed on 1 August 2018).