"INTERVAL TYPE-2 FUZZY SETS AND INTERVAL NEUTROSOPHIC SETS IN INTELLIGENT SYSTEMS"

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BONAFIDE CERTIFICATE

This is to certify that the thesis entitled "**INTERVAL TYPE-2 FUZZY SETS AND INTERVAL NEUTROSOPHIC SETS IN INTELLIGENT SYSTEMS**" is the bonafide work of **Ms. M. LATHA MAHESWARI** who carried out the research under my supervision during the period of 2015 - 2019. Certified further, that to the best of my knowledge the work reported herein does not form part of any other thesis or dissertation on the basis of which a degree or award was conferred on an earlier occasion on this or any other scholar.

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DECLARATION

I hereby declare that the thesis entitled "**INTERVAL TYPE-2 FUZZY SETS AND INTERVAL NEUTROSOPHIC SETS IN INTELLIGENT SYSTEMS**" submitted by me to Hindustan Institute of Technology and Science, for the award of the degree of Doctor of Philosophy in Mathematics is a record of bonafide research work carried out by me under the guidance of **Dr. D. Nagarajan** during the period of 2015 – 2019 and has not formed the basis for the award of any degree, diploma, associateship, fellowship, titles in this or any other University or other similar institution of higher learning.

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ABSTRACT

In this thesis, interval type-2 fuzzy sets (IT2FSs) and interval neutrosophic sets (INSs) have been considered for all the proposed concepts. Fusion of information is an essential task to get the optimized solution for any real world problem. In this task, aggregation operators are playing an important role in all the fields. Since most of the realistic problems have uncertainty in nature, one can use the logic of fuzzy and neutrosophic theory. For the entire proposed concepts interval based logic has been used as it handles more uncertainty. Finding a shortest path in a network system is an important work for all the transactions or transportations. The study has proposed (i) derived aggregation operators based on Schweizer Sklar triangular norms. The same concept has been extended to interval valued neutrosophic environment using Schweizer Sklar triangular norms. A new score function has been introduced for interval neutrosophic numbers. Also the role of the proposed aggregation operators in traffic control management has been analyzed as the mathematical properties are representing the qualities of the system. Furthermore, a comparative analysis has been done for different types of sets with their advantages and limitations, (ii) Dombi interval valued neutrosophic graph (IVNG) and its operations. Here Cartesian and composition products have been derived and the validity has been verified with the numerical validation, (iii) Blockchain single and interval valued neutrosophic graphs are proposed and applied in transaction of Bitcoins in Blockchain

technology, (iv) new aggregation operators for triangular interval type-2 fuzzy numbers using Yager triangular norms with their desirable properties. Also edge detection is done for the image which is based on digital imaging and communications in medicine (DICOM) using MATLAB 2015a under triangular interval type-2 fuzzy environment. Further the role of the mathematical properties of the proposed aggregation operators has been related to qualities of the medical image processing, (v) automatic detergent intake has been analyzed in a washing machine using interval type-2 fuzzy logic controller. Also the transportation delay and stability of the system have been analyzed using four different defuzzification methods.

Keywords: Aggregation Operators, DICOM Image, Triangular Norms, Edge Detection, Neutrosophic Graph, Dombi Interval Valued Neutrosophic Graph, Block chain Technology, Interval Type-2 Fuzzy Logic Controller, and Washing Machine.

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LIST OF SYMBOLS

$\mu_M(x)$	-	Membership Function of the element x
$\overline{M}$	-	Type-2 Fuzzy Set
$LMF_{\overline{M}}(x)$	-	Lower Membership Function
$UMF_{\overline{M}}(x)$	-	Upper Membership Function
$\underline{l_M}$	-	Left Lower Membership Value
$\overline{l_M}$	-	Left Upper Membership Value
$\underline{r_M}$	-	Right lower Membership Value
$\overline{r_M}$	-	Right Upper Membership Value
c_M	-	Centre Membership Value
$TN(a,b)$	-	t-norm of the elements a and b
$TCN(a,b)$	-	t-conorm of the elements a and b
$\oplus$	-	Sum
$\otimes$	-	Product
S	-	Sum of the Terms
$\wp$	-	Product of the Terms
$G_1(V_1,E_1)$	-	Crisp Graph with vertex V_1 and edge set E_1
$G_2(V_2,E_2)$	-	Crisp Graph with vertex V_2 and edge set E_2
V	-	Neutrosophic Vertex Set
E	-	Neutrosophic Edge Set

G_{N}	-	Neutrosophic Graph
$\mathrm{G}_{\mathrm{N}_1} \times \mathrm{G}_{\mathrm{N}_2}$	-	Cartesian Product of Neutrosophic Graphs
$\mathbf{G}_{\mathbf{N}1} \circ \mathbf{G}_{\mathbf{N}2}$	-	Composite Product of Neutrosophic Graphs
$\mathbf{G}_{\mathbf{N}}(\lambda,\delta)$	-	Dombi Neutrosophic Graph
$\underset{Y}{\otimes}$	-	Yager t-norm
$\underset{Y}{\oplus}$	-	Yager t-conorm
$\underset{H}{\otimes}$	-	Hamacher t-norm
$\underset{H}{\oplus}$	-	Hamacher t-conorm
$T^{L}(x)$	-	Lower Truth Membership Function of the element
$T^{U}(x)$	-	Upper Truth Membership Function of the element
$\xi(\mathrm{G})$	-	Minimum Degree of Single Valued Neutrosophic Graph
$\eta(\mathrm{G})$	-	Maximum Degree of Single Valued Neutrosophic Graph
$td(v_i)$	-	Total Degree of Single Valued Neutrosophic Graph
$\sum_{i=1}^{n} = \mathrm{S}$	-	Sum of the terms
$\prod_{i=1}^{2} = \wp$	-	Product of terms

LIST OF ABBREVIATIONS

AFNN	-	Adaptive Fuzzy Neural Network
AHP SWOT	-	Analytic Hierarchy Process Strengths, Weaknesses, Opportunities and Threats
ANP TOPSIS	-	Analytic Hierarchy Process the Technique for Order of Preference by Similarity to Ideal Solution
AP	-	Antecedent Part
AI	-	Artificial Intelligence
AO	-	Aggregation Operator
BC	-	Blockchain
BCFG	-	Blockchain Fuzzy Graph
BCSVNG	-	Blockchain Single Valued Neutrosophic Graph
BCIVNG	-	Blockchain Interval Valued Neutrosophic Graph
BCT	-	Blockchain Technology
CP	-	Consequent Part
CS	-	Control System
CT	-	Computed Tomography
DDE	-	Delay Differential Equation
DFGs	-	Dombi Fuzzy Graphs
DMP	-	Decision Making Problem
DICOM	-	Digital Imaging and Communications in Medicine
DIVNG	-	Dombi Interval Valued Neutrosophic Graph
DIVNEG	-	Dombi Interval Valued Neutrosophic Edge Graph
DIVNEGs	-	Dombi Interval Valued Neutrosophic Edge Graphs
DSVNG	-	Dombi Single Valued Neutrosophic Graph
ED	-	Edge Detection

EU	-	Existence and Uniqueness
EUT	-	Existence and Uniqueness Theorem
EUTs	-	Existence and Uniqueness Theorems
FDCS	-	Fuzzy Delayed Control System
FDDE	-	Fuzzy Delay Differential Equation
FG	-	Fuzzy Graph
FGs	-	Fuzzy Graphs
FIP	-	Fuzzy Image Processing
FIE	-	Fuzzy Inference Engine
FIS	-	Fuzzy Inference System
FL	-	Fuzzy Logic
FVs	-	Fuzzy Variables
FRs	-	Fuzzy Rules
FLC	-	Fuzzy Logic Controller
FLCS	-	Fuzzy Logic Control System
FLCSs	-	Fuzzy Logic Control Systems
FLS	-	Fuzzy Logic System
FM	-	Fuzzy Morphology
FMF	-	Fuzzy Membership Function
FMFs	-	Fuzzy Membership Functions
FN	-	Fuzzy Number
FNs	-	Fuzzy Numbers
FOU	-	Footprint of Uncertainty
FS	-	Fuzzy Set
FSs	-	Fuzzy Sets
FST	-	Fuzzy Sumuder Transform
FVIE	-	Fuzzy Volterra Integrodifferential Equation

FVIEs	-	Fuzzy Volterra Integrodifferential Equations
HI	-	History Function
IDS	-	Image Display System
IE	-	Image Enrichment
IFG	-	Intuitionistic Fuzzy Graph
IFGs	-	Intuitionistic Fuzzy Graphs
INNs	-	Interval Neutrosophic Numbers
INSs	-	Interval Neutrosophic Sets
INSSWA	-	Interval Neutrosophic Schweizer Sklar Weighted Arithmetic
INSSWG	-	Interval Neutrosophic Schweizer Sklar Weighted Geometric
INWAAO	-	Interval Neutrosophic Weighted Arithmetic Averaging Operator
INWGAO	-	Interval Neutrosophic Weighted Geometric Averaging Operator
IP	-	Image Processing
IT2FL	-	Interval Type-2 Fuzzy logic
IT2FLC	-	Interval Type-2 Fuzzy Logic Control
IT2FLS	-	Interval Type-2 Fuzzy Logic System
IT2FLSs	-	Interval Type-2 Fuzzy Logic Systems
IT2 FPID	-	Interval Type-2 Fuzzy PID
IT2TSFM	-	Interval Type-2 Takagi Sugeno Fuzzy Model
IV	-	Input Variable
IVs	-	Input Variables
IVFGs	-	Interval Valued Fuzzy Graphs
IVNG	-	Interval Valued Neutrosophic Graph

IVNGs	-	Interval Valued Neutrosophic Graphs
IVNN	-	Interval Valued Neutrosophic Number
IVNNs	-	Interval Valued Neutrosophic Numbers
IVNS	-	Interval Valued Neutrosophic Set
IVNSs	-	Interval Valued Neutrosophic Sets
LMI	-	Linear Matrix Inequality
MADMP	-	Multi Attribute Decision Making Problem
MAGDM	-	Multi Attribute Group Decision Making
MAGDMP	-	Multi Attribute Group Decision Making Problem
MF	-	Membership Function
MFs	-	Membership Functions
MFM	-	Mathematical Fuzzy Morphology
MG	-	Morphological Gradient
MIP	-	Medical Image Processing
MM	-	Mathematical Morphology
MOs	-	Morphological Operations
MOGA	-	Multi Objective Genetic Algorithm
MRI	-	Magnetic Resonance Imaging
NLEs	-	Neutrosophic Linear Equations
NM	-	Neutrosophic Matrix
NN	-	Neutrosophic Number
NNs	-	Neutrosophic Numbers
NS	-	Neutrosophic Set
NSs	-	Neutrosophic Sets
NG	-	Neutrosophic Graph
NGs	-	Neutrosophic Graphs
OLs	-	Operational Laws

OV	-	Output Variable
OVs	-	Output Variables
PACSs	-	Picture Archiving and Communication Systems
PID	-	Proportional Integrated Derivative
SF	-	Score Function
SGD	-	Strongly Generalized Differentiability
SVNG	-	Single Valued Neutrosophic Graph
SVNGs	-	Single Valued Neutrosophic Graphs
SVNN	-	Single Valued Neutrosophic Number
SVNNs	-	Single Valued Neutrosophic Numbers
SVNS	-	Single Valued Neutrosophic Set
SVNSs	-	Single Valued Neutrosophic Sets
SNSs	-	Simplified Neutrosophic Set
SNWAAO	-	Simplified Neutrosophic Weighted Arithmetic Averaging Operator
SNWGAO	-	Simplified Neutrosophic Weighted Geometric Averaging Operator
SV	-	Score Value
SVs	-	Score Values
TD	-	Type of Dirt
t-conorm	-	Triangular Conorm
TIT2FS	-	Triangular Interval Type-2 Fuzzy Set
TIT2FSs	-	Triangular Interval Type-2 Fuzzy Sets
TIT2FN	-	Triangular Interval Type-2 Fuzzy Number
TIT2FNs	-	Triangular Interval Type-2 Fuzzy Numbers
TIT2FYWA	-	Triangular Interval Type-2 Fuzzy Yager Weighted Arithmetic

TIT2FYWG	-	Triangular Interval Type-2 Fuzzy Yager Weighted Geometric
TIT2SSWA	-	Triangular Interval Type-2 Schweizer Sklar Weighted Arithmetic
TIT2SSWG	-	Triangular Interval Type-2 Schweizer Sklar Weighted Geometric
TIT2WA	-	Triangular Interval Type-2 Weighted Arithmetic
TIT2WG	-	Triangular Interval Type-2 Weighted Geometric
TMS	-	Traffic Management System
TN	-	Triangular Norm
TNs	-	Triangular Norms
TNNs	-	Trapezoidal Neutrosophic Numbers
TSFM	-	Takagi-Sugeno Fuzzy Model
t-norm	-	Triangular Norm
T1FLC	-	Type-1 Fuzzy Logic Controller
T1FS	-	Type-1 Fuzzy Set
T1FSs	-	Type-1 Fuzzy Sets
T2FL	-	Type-2 Fuzzy Logic
T2FS	-	Type-2 Fuzzy Set
T2FSs	-	Type-2 Fuzzy Sets
T1FLS	-	Type-1 Fuzzy Logic System
T1FLSs	-	Type-1 Fuzzy Logic Systems
T2FLS	-	Type-2 Fuzzy Logic System
T2FLSs	-	Type-2 Fuzzy Logic Systems
WC	-	Weight of Cloths
WM	-	Washing Machine

CHAPTER-1

INTRODUCTION

1.0 INTRODUCTION

1.1 RESEARCH CONTEXT

Fuzzy set and its generalization called neutrosophic set have been implemented in different areas of engineering, science, and technology. However, type-1 fuzzy sets and neutrosophic sets play a vital role in dealing with uncertainty of the real world problems, handling interval data is beyond the scope of these environments.

Many people accept that allocating an exact number to the opinion of the experts is too contrary, and assigning an interval of values is more practical. Hence the reason of choosing interval based fuzzy sets and neutrosophic sets in this study.

The search for meticulously connecting real world with crisp mathematics requires development of better methodologies in areas like information fusion, image processing, decision making and control system to deal with impreciseness.

Further, this study mainly focuses on the theoretical part of interval type-2 fuzzy sets and interval neutrosophic sets and its application as well in real world problems such as operational laws, aggregation operators and new score function of the mentioned environments and their application in traffic control management, image processing and control system, single and interval valued neutrosophic graphs using triangular norms, single and interval valued neutrosophic graphs in Blockchain technology.

Hence, this study strengthens the theoretical and application part of the fuzzy and neutrosophic environments which were left outmoded over the past few years.

1.2 AGGREGATION OPERATORS OF INTERVAL TYPE-2 FUZZY SETS AND INTERVAL NEUTROSOPHIC SETS IN TRAFFIC CONTROL MANAGEMENT

This chapter introduces the aggregation operators under interval type-2 fuzzy environment and interval neutrosophic environment and applied them in a traffic control management. Also new score function of the interval neutrosophic number has been introduced.

The theory of triangular norms provides the mathematical properties, and these properties represent the crucial qualities of the control system (CS), such as stability (Gupta and Qi, 1991). Issues related to traffic congestion is regularly experienced in daily life. Controlling traffic signals is one of the areas in which fuzzy logic (FL) is the most popularly employed in transportation engineering. Traffic congestion affects the safety of the people, disrupts routine everyday activities and the quality of lifestyle and leads to a commercial, natural and health burden for the government and related organizations (Niittymaki and Pursula, 2000).

Traffic control aims to reduce the negative effects of traffic by establishing intelligent models to correct state calculation, control and forecasting. Control problems have attracted considerable attention in the control community (Li *et al.,* 2006). As real-world problems in nature often involve uncertainty, FL has been applied successfully to deal with impreciseness. This theory is based on the concepts of degree to deal with uncertainties in a field of knowledge. This logic agrees with linguistic and imprecise traffic data as well as in modeling signal timings.

Modeling the control is the basic principle of fuzzy signal control with respect to a human expert's knowledge. The model of the fuzzy controller needs an expert's knowledge and experience in the traffic control field in developing the linguistic protocol that produces the input of the control in the system. As FL exploits linguistic information, reproduces human thinking and captures the uncertainty of the real-world problems, it is successful in producing good performance for various practical problems (Castro, 1995).

A fuzzy logic system (FLS) works with the use of IF-THEN rules, where the knowledge will be often uncertain. It is very useful for decoy approximation. If the antecedent and consequent parts are type-1 fuzzy sets (T1FSs), then the system is called type-1 fuzzy logic system (T1FLS), whereas in type-2 fuzzy logic system (T2FLS), the antecedent or consequent set will be of type-2 fuzzy sets (T2FSs). The membership function of T2FS is a three-dimensional one, which includes upper and lower membership functions, and the area between them is called the footprint of uncertainty (FOU) (Karnik *et al.*, 1999).

For a set of regional linear models, the Takagi Sugeno model will be used for an optimized output (Wen *et al.,* 2015). As noise is nonlinear, systems that use traditional logic and electronics fail to deal with the complex nature of the signal in terms of algorithms and circuits. FL is the most appropriate method to describe imprecise characteristics accurately (Jarrah and Shaout, 2007). Traffic congestion problems may arise owing to different conditions such as insufficient number of lanes, broken road surface, high volume of vehicles, irrational allocation of signaling system and poor visibility of the road. Furthermore, traffic congestion increases the level of pollution, as in most of the cases; the engines of the vehicles are left running.

To mitigate these problems, a new methodology has been implemented by accompanying the automated sensor approach in the system of traffic signaling. In early years, and at present, in some places, traffic is controlled by the usage of hand signs by the traffic police, signals and markings. This impreciseness cannot be dealt with by type-1 fuzzy as it is precise in nature, whereas type-2 fuzzy, an extension of type-1 fuzzy, can handle a high level of uncertainty (Almaraaashi *et al.*, 2002).

The approach of type-2 fuzzy will overcome the consequence of time delay in control systems (Li *et al.*, 2014). Hence, fuzzy logic controllers have been applied for controlling several physical processes successfully (Patel, 2014). T1FS induces a unique membership value to every set element between 0 and 1 and is useful to model knowledge, but it fails to deal with special uncertainties such as different opinions of experts on the same concept. It also cannot deal with only degree of truth and is not able to minimize the noise. In spite of these shortcomings, different opinions may produce different membership functions, therefore a model can be designed using T2FS (Comas *et al.*, 2014).

The triangular norms namely t-norms and t-conorms are preferable operators for controlling the system as it satisfies commutativity, idempotency, monotonicity and boundary conditions, which represent the qualities of a good system and generalize conjunction and disjunction respectively. Fuzzy conjunction is used for the system to decide the particular decision in a given period of time. Additionally, it is an operation between two memberships degrees, which describes two fuzzy sets (FSs) treated as a premise in an inference system.

To control the inference, t-norms use some parameters in the system, and may have different behavior based on the parameters (Qin and Liu, 2014). Interval valued neutrosophic sets (IVNSs) are determined by an interval membership grade, interval indeterminacy grade and interval nonmembership grade (Broumi and Smarandache, 2015).

The generalization of intuitionistic fuzzy is the neutrosophic set with the indeterminate reasoning, and interval valued neutrosophic set is the general case of single-valued neutrosophic environment. Using these concepts, the uncertainty of the problem can be dealt effectively, as the neutrosophic concept can also deal with indeterminacy. Interval valued sets, especially neutrosophic sets, handle indeterminacy with the lower and upper membership functions, and hence uncertainty in a real world problem can be solved in an optimized way (Singh *et al.*, 2016).

By considering the determinate part and indeterminacy, a neutrosophic number can be formed and interchanged in the form of an interval number. Using this concept and the operational laws of neutrosophic matrices, traffic flow can be identified in each intersection by considering neutrosophic linear equations, and is an effective way of finding traffic flow (Ye, 2017). In addition, some models have been designed to avoid accidents, and unwanted situations while inspecting and collecting information about individuals (Mayouf *et al*., 2018).

1.3 INTERVAL VALUED NEUTROSOPHIC GRAPH USING TRIANGULAR NORMS

This chapter introduces the definition of single and interval valued neutrosophic graphs using Dombi triangular norms. Also the Cartesian and composite products of two Dombi interval valued neutrosophic graphs have been derived.

The generalized Dombi operator family was introduced by Dombi and it has been applied to speech Recognition Task (Gosztolya *et al.*, 2009). Graph theory plays a vital role in different fields especially in physics, biology and computer science engineering, deal with complex networks (Pavlopoulos *et al.*, 2011). It is also used to solve various optimization problems in transportation where network is nothing but the logical sequence of the method and visualization possibility, permits surveys to be afforded. It is able to respond for two equations simultaneously for the purpose and procedure (Stoilova and Nedelchev, 2012).

Permanent growth of the population is one of the main faces of modern cities and it is the reason for building new roads and highways to avoid traffic problems and for stress free life of the people (Dave and Jhala, 2014). A fuzzy set can be described mathematically by assigning a value, a grade of membership to each possible individual in the universe of discourse. This grade of membership associates a degree to which that individual either is similar or appropriate with the concept performed by the fuzzy set (FS).

A fuzzy subset of a set X is a mapping from membership to non-membership and is defined by $\eta : X \to [0,1]$ continuous rather than unexpected (Dey *et al.*, 2012). Fuzzy relations are popular and important in the fields of computer networks, decision making, neural network and expert systems. (Broumi *et al.*, 2016). Direct and indirect relationship will also be considered in graph theory (Gomes *et* *al.,* 2012). Model of relation is nothing but a graph and it is a comfortable way of describing the information about the objects in the connection (Dey *et al.,* 2012).

While there is impreciseness in the statement or its communication or in both, fuzzy graph model can be designed for getting an optimized output. Maximizing the utility of the application is always done by the researchers during the constructing of a model with a key characteristics reliability, complexity and impreciseness (Sunitha and Mathew, 2013).

Among these, impreciseness plays an important role in maximizing the utility of the model. This situation can be described by fuzzy sets and fuzzy graphs, since the real world problems can have the partial information (Dey *et al.,* 2013). Performance evaluation can be described as modeling and application and should be done before using them by using graph theory (Lopes *et al.*, 2013). When the system is huge and complex, it is challenging to extract the information about the system using classical graph theory and at this junction fuzzy graph can be used to examine the system (Prabha *et al.*, 2014).

A situation in which goods is shifted from one location to another can be dealt by graphs. In water supply, water users and pipe joins are vertices and pipelines are edges (Bisen, 2017). Graph theory is a branch of combinatorics (Dhavaseelan *et al.,* 2015). Fuzzy graph is the fuzzy correlation of various graph theoretic notions such as cycles, paths, trees and connectedness and set some of their properties (Manjusha and Sunitha, 2015).

The term degree of membership (DM) and describe the notion of FS to handle with impreciseness. Atanassov (1986) introduced intuitionistic fuzzy set (IFS) by including the degree of non-membership in the concept of FS as an independent component. Samarandache (2006) introduced neutrosophic set (NS) by finding the term degree of indeterminacy from the logical point of view as an independent component to handle with imprecise, indeterminate and unpredictable information which are exist in the real world problems.

The NSs are defined by truth, indeterminacy and false membership functions which are taking the values in the real standard interval (Akram and Sitara, 2017). Wang *et al.* (2005) proposed the concept of single valued neutrosophic sets (SVNSs) and interval valued neutrosophic sets (IVNSs) as well, where the three membership functions (MFs) are independent and takes value in the unit interval [0,1] (Broumi *et al.*, 2016). If uncertainty exists in the set of vertices or edges or both then the model becomes a fuzzy graph (FG).

FGs can be established by considering the vertex and edge sets as fuzzy, in the same way one can model interval valued fuzzy graphs (IVFGs), intuitionistic fuzzy graphs (IFGs), neutrosophic graphs (NGs), single valued Neutrosophic graphs (SVNGs) and interval valued neutrosophic graphs (IVNGs) (Broumi *et al.*, 2016). Network of the brain is a neutrosophic graph especially strong neutrosophic graph (Kandasamy *et al.*, 2016).

Intelligent transport systems is a universal aspect gets the attention of worldwide interest from professionals in transportation, political decision makers and computerized industry (Bhagat and Patel, 2016). It is developed by understanding the progress of the road traffic in the interval of time and communication between the participants and structural elements available in the situation (Oberoi *et al.*, 2017).

Graph theory defines the relationship between various individuals and has got many number of applications in different fields namely database theory, modern sciences and technology, neural networks, data mining cluster analysis, expert systems image capturing and control theory (Sahin, 2017). The strength of the relationship in social networks can be analyzed by fuzzy graph theory and it has important potential (Jain *et al.*, 2017). While the network is large, analysis and evaluation of traffic will be very challenging one for the network managers and it can be done using dynamic bandwidth (Sethi and Behera, 2017). Indeterminacy of the object or edge or both cannot be handled by fuzzy, intuitionistic fuzzy, bipolar fuzzy or interval valued fuzzy graphs and hence neutrosophic graphs have been introduced (Quek *et al.*, 2018).

Triangular norms in the structure of probabilistic metric spaces, triangular norms play an important role in the application of fuzzy logic such as fuzzy graph and decision making process (Ashraf *et al.*, 2018). Some of the real world applications can be modeled in a better way with triangular norms especially t-norm than using minimum operations. Using this concept awareness of tracking in person for networks is possible (Mordeson and Mathew, 2018).

1.4 BLOCKCHAIN NEUTROSOPHIC GRAPH

This chapter introduces Blockchain single and interval valued neutrosophic graph and applied in Blockchain technology for Bitcoin transaction.

A completely peer-to-peer form of electronic cash will permit payments through online and direct transaction can be done from one participant to another without facing any financial organization. If a central party wants to avoid double spending, then the main gain will be lost even though digital signatures contribute part of the solution. This issue was the reason to bargain a solution to this problem based on the peer-to-peer network (Satoshi, 2008).

For the direct transaction of two willing parties without having a trusted third party, an electronic system using cryptographic proof (signaling code) can be used. Fuzzy graphs are playing an important role in a network where impreciseness exists on the vertices and edges. Yeh and Banh also proposed the fuzzy graph independently and examined various connectedness theories (Yeh and Banh, 1975).

The universal problems namely sustainable development or transformation of assets can be dealt with effectively by Blockchain technology (BCT) than the existing financial systems. The financial sector acquires in various operative costs for the smooth and effective functioning of the entire system. These costs consist of time and money needed for investment in the framework, electricity cost spent for operation and from automated teller machines, consumption of water and gas by the employees and wastage production (Ober *et al.*, 2013).

There is also no possibility of creating fiat currency without costs. In order to give assurance on a regular basis in the quality standards for the bank notes in circulation, the used ones are shredded. To find an overview of the overall cost of an existing financial system, the cost for the production of coins and noted will be included. Whereas in BCT, one needs only to connect to the network and do not obtain the electricity cost for any source. Also the production of the crypto currency (a digitalized currency, where the encoding method is applied to control the production of currency and funds transference verification) (Decker and Wattenhofer, 2013).

Platforms of Central banking, improvement of business processing, automotive ownership, sharing of health information, deals and voting can be potentially replaced by BCT and it plays an important role, in political components namely governmental interference, control leadership, and taxation. BCT is also very useful in Exchange rates of currency market growth and monetary as an economic component.

Further it is very helpful in social components namely environmental situation, culture, the behavior of the customer and demand. In the same way, BCT has a potential action in modern technologies and tendency (Fleder *et al.*, 2014). BCT permits an emerging set of participants to continue with a secure and alter proof ledger for all the activities without having a third trusted party. Here, transactions are not actually documented but instead, every participant keeps a provincial copy of the ledger which is a related listing of blocks and they comprise agreed transactions (Bonneau *et al.*, 2015).

Crypto currency is nothing a Bitcoin which is a universal payment system and also the initial decentralized digital currency since the system works without a single administrator or central bank. Bitcoins made as a payment for a process called mining and can be exchanged for different currencies. To solve the double spending problem, Blockchain for the Bitcoin has been an appropriate choice without the help of the trusted third party as a central server. Blockchain transactions will be done on the interchangeable ledger data saved at every node (Arockiaraj and Charumathi, 2018).

A Blockchain network can be seen as a reliable computer whose private states are auditable by anyone. A ledger of transactions may call as a Blockchain. Generally, a physical ledger will be maintained by a centralized party whereas in Blockchain is a distributed ledger which locates on the device of every participant. Bitcoins are believable and best used (Ramkumar, 2018).

A Fuzzy Set (FS) can be described mathematically by assigning a value, a grade of membership to every desirable person in the universe of discourse. This grade of membership associates a degree for the participant is either identical or appropriate to the approach performed by FS. Fuzzy relationships are popular and essential in the fields of computer chains, decision making, neural network, expert systems. Direct relationship and also indirect relationship will be considered in graph theory. Model of relation is nothing but a graph which is a comfortable way of describing information about the connection between two objects. In the graph, points and relations are defined by vertices and edges respectively.

While impreciseness exists in the statement of the phenomenon or in the communication or both, the fuzzy graph model can be designed for getting an optimized output. Maximizing the Utility of the application is always done by the researchers during the constructing of a model with a key characteristics reliability, complexity, and impreciseness. Among these, impreciseness is a considerable one in maximizing the utility of the technique. This situation can be described by fuzzy sets (Chan and Olmsted, 2017).

NSs are defined by the membership functions of truth, indeterminacy, and falsity whose values take from the real standard interval. The theoretical concept of single valued neutrosophic sets (SVNSs) and interval valued neutrosophic sets (IVNSs) as well have been proposed as a generalization of neutrosophic sets. If uncertainty exists in vertices or edges set or both then the structure turns into a fuzzy graph. It can be established by taking the vertex and edge sets as FSs, in the same way, one can model any other types of fuzzy graphs (Broumi *et al.*, 2016).

Graph theory defines the relationship between various individuals and has got any numbers of applications in different fields namely database theory, modern discipline and technology, neural networks, data scooping cluster analysis, knowledge systems image capturing and control theory. Handling Indeterminacy on the object or edge or both cannot be handled by FGs and hence NGs have been introduced (Jain *et al.*, 2017).

A new perspective for neutrosophic theory and its applications also proposed. There are many methods have been proposed under single valued neutrosophic, interval valued neutrosophic and neutrosophic environments by collaborating with other methods such as TOPSIS, DEMATEL, VIKOR. Further, NS cross entropy, hyperbolic sine similarity measure, hybrid binary algorithm similarity measure method, and single valued co neutrosophic graphs play an important role in decision making.

In the fuzzy graph, all the edges are represented by fuzzy numbers (FNs) and that may be interval valued fuzzy number (IVFN) also. Whereas in the neutrosophic graph the edges are represented by single valued neutrosophic numbers (SVNNs) (Basset *et al.*, 2018).

1.5 AGGREGATION OPERATORS IN EDGE DETECTION ON DICOM IMAGE

This chapter introduces new aggregation operators and derived their properties for triangular interval type-2 fuzzy numbers under interval type-2 fuzzy environment using Yager triangular norms. Also edge detection has been made for DICOM image using MATLAB 2015a program for interval type-2 fuzzy logic.

An image is defined as a function of two variables x and y in real world problems i.e., f(x, y), where f is an amplitude of the image at the real coordinates x and y. It is generally a collection of objects as the images have sub images or region of interest where important information of the medical image stored without any distortion. The amplitude of the image is called gray level or intensity of the image at that point.

If the amplitude values are finite and discrete quantities then the image is called a digital image. This digital image is a finite set of digital values and represented by a two dimensional image. It is a construction of a finite number of elements called picture elements, image elements and pixels and each of which has a specific location and value. An image is usually exploited by noise during transmission and recovery. Using image denoising, this problem can be sorted out by removing an additional noise of the image while go for retention.

The scan image of magnetic resonance imaging (MRI) has been used for the edge detection. For number of reasons, medical images are transferred such as teleconference purpose among the clinicians, distant learning of medical personal and interdisciplinary exchange between radiologists for consultative aspiration. Data storage and transmission are also should be taken care while maintaining the information of the patient.

Most of the hospitals and health care systems concern about data storage and transmission like administrative documents, information of the patient, medical images and graphs (Scholl *et al*., 2011). In two major ways medical image data occurs such as a large amount of image data from thousands of images like in picture archiving and communication systems and a large amount of data from a single data set. Both are applied in practice. Since the amount of medical image data is steadily increasing, fast extracting and indicator techniques are needed for narrowing the gap between the numerical nature and the linguistic meaning of images (Jain and Aggarwal, 2012).

As MRI uses energetic magnetic fields without ionizing emission, there won't be any irreversible biological effects on the human body, for example ultrasound scans use high frequency sound waves and MRI uses radio waves together with strong magnetic fields whereas conventional X-ray, computed tomography (CT) and molecular imaging use ionizing radiation. In order to simplify the usage and recapturing of the data and avoid mishandling and loss of data, medical images and patient information among the above data, need to be organized in a convenient manner.

Data hiding approach can be used to overcome the capacity problem and to shorten storage and exchange cost to hide patient information with medical images and also for authentication. This technique claim large amount of patient information in a single image rather over several individuals. To speed up and enrich the operation of the analysis of the medical images, medical image processing tool can be used (Tibrewala and Malviya, 2015).

Image processing is a mode of information processing where input and outputs are images. This technique generally process images as two dimensional signals and employ typical signal processing approach to the images. This process partitioned into two branches of image processing such as digital image processing and medical image processing. Medical imaging is nothing but the visualization of the body parts, organs used for clinical diagnosis, disease monitoring and treatment. This technique captures the areas of radiology, optimal imaging and nuclear medicine. Ultrasound imaging and MRI are also comes under medical imaging (Lee and Liew, 2016).

Different image processing (IP) captures various details and hence it is challengeable one for the researchers to develop smooth algorithms to excerpt the details from the image and create a fused image. Medical image fusion is used to consolidate the essential features of the medical image into a single image for improving the clinical accuracy to take better decisions (Myna and Prakash, 2015).

In this way, MRI data captures the details of soft tissues. Image fusion will reduce the requirement of the memory and produce an improved image than the individual images. Before doing image processing, image pre-process should be done for images at the lowest level of consideration. Here both input and output are bright images. For every real image there is another pixel with the same brightness value, if this condition is satisfied distorted pixels can be selected from the image and restored as an average value of neighboring pixels (Shengagavalli and Ramar, 2013).

In the fields of research and engineering, image processing performs a key function to increase the effectiveness of the fields and enrich the quality of an image as well. FL technique provides better results for image consolidation from visual point of view. The image has an edge while the intensity deviation between the adjacent pixels is large. Edge detection is a crucial aspect of digital image processing which is used most often in image segmentation based on sudden changes in intensity.

Edge detection (ED) itself is a key research work in image processing, pattern recognition, image analysis, biomedical image segmentation and techniques of computer vision. In a simpler manner, it is used to detect sharp edge and local changes in intensity. The pixels at which sudden change in the intensity of the image function is called edge pixels and edges are sets of combined edge pixels (Misra and Sinha, 2014).

Fuzzy logic is an explicit way to describe about human's everyday perceptions. Fuzzy image processing (FIP) is the process of applying FS and it depends on membership values, rule base and inference engine. Usually the boundaries of any image will be uncertain and image segmentation and extraction are very difficult, also crucial to the diagnosis inflammation. It is to be noted that the quality of an image depends on the type and direction of the digital camera. By using fuzzy logic these problems can be dealt effectively as it handles uncertainty in an efficient manner (Fazel *et al.*, 2009).

A function from a universal set to the unit interval [0,1] and the membership of every element lies between 0 and 1 is called a fuzzy set. The representation and the process of fuzzy image processing depend on the selected fuzzy approach and the problem to be solved. Human communication is the basis of fuzzy logic and is built on the design of qualitative statement which is used everyday language (Wang, 2000). FIP has three main stages such as image fuzzification, adapting membership values and defuzzification. It is a collection of fuzzy set theory, fuzzy logic and fuzzy measure theory.

Due to unavailability of fuzzy hardware, fuzzification and defuzzification steps had been carried out in this type of image processing. Once the data of the image is converted from gray level to membership plane, the membership values will be modified by proper fuzzy approach using clustering, rule based technique or integration technique during fuzzification. Decoding of the conclusion is called defuzzification and then produces an output image. Hence an essential part of FIP is modifying the fuzzy membership values.

Fuzzy image enrichment is based on gray level mapping into a fuzzy plane. The aim of the present study to produce an image of higher contrast than the original image by applying larger weight to the gray levels which are closer to the mean gray level farther from the mean. Fuzzy image enrichment process analyzes about the FSs and the variables used with the membership functions. The gray scale images can be enhanced using fuzzy intensification factors by modifying the membership valued which applied on the images and the enhanced images will have a nice quality (Kousik *et al.,* 2012).

The triangular norms like t-norm and t-conorm commenced from the studies of probabilistic metric spaces and are used for furnishing formal definitions of union and intersection of FSs. Maximum and minimum operators have been used in almost all the cases for those definitions. It can be done with other triangular norms as well. These norms play as an AO by choosing maximum and minimum operator as a special choice and have to accomplish the same set of mathematical requirements.

Aggregation operators have to satisfy some mathematical properties like Commutativity, monotonicity, associativity and neutrality and these properties represent the essentials of image processing (Gupta and Qi, 1991). The fuzzy mathematical morphology is an inference of binary morphology using fuzzy sets technique. Its basic tools are called morphological operations (MOs) which are applied to an image P and adjusted through a structural element Q, hose shape and size are to be chosen in order to examine the structure of P.

The primary MOs are erosion, opening and closing. Building a fuzzy morphology (FM) can be done by fuzzy operators like conjunctors (t- norms) and implications and the most applied are the t-norms and their residual implications. The mathematical property called idempotency is represented for opening and closing operators in fuzzy image processing.

According to the multi scale morphological gradients, the operators have been designed to address the segmentation problem of images. Detecting an edge and the boundary of the image objects is the aim of the gradient. Selection of structural element and the gradient depends on the geometry of the objects. There are two morphological filters have been used in image processing namely opening and closing, where opening is an anti extensive and closing is an extensive morphological filters. Both are used for removing unwanted objects of an image. Opening filter is represented by maximum operator whereas closing is represented by minimum operator. Gray level image is opened by structural element which removes the light area of less size than the element whereas the closing operator removes the dark area and lighten those areas. Choosing an appropriate size and shape of the structural elements is very important to filter and to remove few elements from the original image. Fuzzy system has taken the responsibility of fundamental methodology to describe and process impreciseness and uncertainty in the linguistic information (Manuel *et al*., 2009).

Edge detection using fuzzy set is proposed here, where an image is considered as a fuzzy set and pixels are taken as elements of FS. The following are the characteristics of edge detector. Identifying less number of improper edges and detection of real edge supposed to be maximum, the spotted pixels should be adjacent to the actual image, and error of getting a single image should be less and design an edge detector which performs well in various contexts.

The purpose of edge detection is to detect the pixels in the image which coincide to the edges of the objects seen. It is usually done by comparing with a threshold that marks the pixel whether belonging to an edge or not. Filtering is also an important task in image processing. There are two types of filtering used such as horizontal and vertical. The horizontal filer gives the high frequency components through horizontal edges in the image similarly for vertical filter.

Using these two filters linear features like roads, railways, path of the river through horizontal and vertical directions can be extracted (Shengagavalli and Ramar, 2013). The mathematical properties of the AO for using Yager triangular norms (TNs) have been proved as it covers all the continuous t-norms by changing the parameter whereas other major triangular norms fail to do so. Frank triangular norms also cover all the continuous t-norms and hence they have been used in this work (Hirota *et al*., 2004).

1.6 WASHING MACHINE WITH INTERVAL TYPE-2 FUZZY LOGIC

This chapter analyzes the automatic detergent intake of interval type-2 fuzzy washing machine. Also transportation delay and stability have been analyzed by considering four different types of defuzzification methods.

In control system, there are two major types such as proportional integrated derivative (PID) and fuzzy logic control systems (FLCSs) which are widely applied in the fields of engineering activities. Though PID controllers are universally used in the mechanism of control operations, it may exhibits a poor performance for nonlinear systems due to its insufficient knowledge of the parameters.

FLCS is an efficient model with several attributing qualities such as the reduced oscillations, better stability, small overshoot and faster settling time. Fuzzy logic is a mathematical tool to deal with parameter uncertainties. Linguistic constraints such as “many”, “low”, “medium”, “often” and “few” can be represented by a mechanism that is provided by fuzzy theory. The approximate reasoning can be dealt with by applying the FLSs and it is observed that the response of the system is better than PID with less control complexity.

In FLCS, a linguistic control strategy is converted into an automatic mode based on expert knowledge. FL provides control strategies for the low order dynamics with weak non linearities (Lupulescu and Abbas, 2012). A type of differential equation in which the derivative of the uncertain function at a certain time is given in terms of the values of the function at preceding times is called a delay differential equation (DDE) and is also called time delay systems. The delay will occur due to the usage of certain physical properties such as transmission of a signal and measurements of system variable.

The fuzzy rules are generally used in fuzzy control system and are often applied to capture the uncertainty lies in human reasoning (Boulkroune *et al.,* 2010). The conventional control system needs an explicit model of the physical phenomenon whereas FL deals with problem based on the knowledge of the human operators (Arqub *et al.*, 2016). The theory of fuzzy provides a sufficient way for handling the uncertain criteria.

In the washing machine (WM), stability analysis is an important task with respect to uncertain parameters. It is established on the spinning operation under peculiar condition for stress and it is the main source of the WM (Roseline, 2016). There is an interval type-2 fuzzy logic (IT2FL) toolbox that is available to get a desired output. Problem formation is based on if-then rule in terms of linguistic variable. Many defuzzification methods are being applied to maintain the stability of the system and also for getting a crisp output (Stammingera *et al.,* 2018).

CHAPTER-2

REVIEW OF LITERATURE

2.0 REVIEW OF LITERATURE

2. (a) AGGREGATION OPERATORS OF INTERVAL TYPE-2 FUZZY SETS AND INTERVAL NEUTROSOPHIC SETS IN TRAFFIC CONTROL MANAGEMENT

Zadeh (1965) developed fuzzy set theory and fuzzy logic to handle linguistic and emotional uncertainty. In conventional logic the number 400 is an integer, whereas 400.8 is not. Depending on situation and emotional opinion, the same number could be considered large, very large, small, very small and so on. Therefore, the number 400 could be considered large to a certain degree, very large to another and so on. Hence there may have many linguistic values of one linguistic variable, which are true to some degree and lies between 0 and 1. In classical set theory, an element either belongs to the set or not.

Classical set is often called crisp set. In fuzzy set theory, an element with the grade of membership in the interval [0, 1] and all the membership grades together form the membership function. Further, arithmetic of fuzzy numbers called fuzzy arithmetic is a generalization of interval arithmetic. Since fuzzy set allows degree of membership for an element of the universal element, intervals are considered at various levels in [0, 1]. It plays a vital role in many of the applications namely decision making, approximate reasoning, fuzzy control, optimization and statistics with imprecise probabilities.

Gupta and Qi (1991) had proposed the theoretical concepts of t-norms and methods of fuzzy reasoning. Castro (1995) proved that fuzzy logic controllers (FLCs) are global approximations. Karnik *et al.* (1999) presented type-2 fuzzy logic systems, which can deal with more uncertainties. Niittymaki and Pursula (2000) proved that signal control using fuzzy logic can be an efficient controlling method for signalized intersections.

Wei *et al.* (2001) have presented a traffic signal control management by using fuzzy logic and MOGA. Wu and Mendel (2002) applied imprecise bounds in the model of interval type-2 fuzzy logic systems (IT2FLSs). Wang *et al.* (2005) introduced the concept of interval neutrosophic sets based on truth value. Aguero and Vargas (2005) concluded that the dynamic structure of distribution networks using T2FLSs.

Wang *et al.* (2005) presented, in detail, the theoretical concepts about the interval based neutrosophic set and its application in computing. Smarandache (2006) proved that a neutrosophic set is the logical reasoning and generalization of intuitionistic fuzzy set. Li *et al.* (2006) proposed a different method for predicting traffic using type-2 fuzzy logic. Jarrah and Shaout (2007) proposed a volume control of motor vehicles using fuzzy logic. Ozek and Akpolat (2008) proposed an operating system for the type-2 fuzzy logic tool box.

Petrescu *et al.* (2008) proposed a fuzzy control design for an independent vehicle governing system, where the design is replaced by some local linear systems which are defined over the given points and the union of these systems inclined a Takagi Sugeno model. Algreer and Ali (2008) accomplished position control using fuzzy logic. Wang *et al.* (2010) introduced the theory of single valued neutrosophic sets (SVNSs).

Almaraashi *et al.* (2012) constructed a generalized T2FLSs using interval type-2 setting and artificial strengthening. Tellez *et al.* (2012) proposed the idea of T2FLSs using parametric representation. Li *et al.* (2014) described mathematical properties such as monotonicity of IT2FLSs. Blaho *et al.* (2014) used type-2 fuzzy logic in diminishing the collision of impreciseness in a chain of control systems. Singhala *et al.* (2014) developed a temperature control system by using fuzzy logic.

Patel (2014) explained that the situations and methods in which fuzzy logic can be applied. Comas *et al.* (2014) defined the measures to determine the degree of truth and the theoretical background of the decision support system. Qin and Liu (2014) proposed the Frank triangular norms for triangular interval type-2 fuzzy set (TIT2FS) and applied it in a decision making process. Ye (2014) improved the correlation coefficient of SVNSs, examined their properties and extended the concept to interval neutrosophic sets (INSs). They also applied the proposed concepts in decision making problems.

Ye (2014) generalized Jaccard, dice and cosine similarity measures in vector space and presented three vector similarity measures between simplified neutrosophic sets (SNSs). They also used it in a decision making problem. Ye (2014) proposed the concept of SNS, its operational laws and two aggregation operators, namely, simplified neutrosophic weighted arithmetic average operator (SNWAAO) and SNW geometric averaging operator (SNWGAO) and applied them in a decision making problem.

Singh *et al.* (2015) used comparative analysis of neural network and fuzzy algorithm in implementing ACC. Shafeek (2015) designed an autopilot to control the header of an aircraft using PD-like type-1and type-2 fuzzy logic controllers. Wen *et al.* (2015) proposed an intelligent signal controller using T2FL and NSGAII. Lafta and Hassan (2015) introduced the mobile automation control using fuzzy logic. Broumi and Smarandache (2015) proposed interval valued neutrosophic soft rough sets. Smarandache (2015) explained undoubtedly about the neutrosophic theory symbolically. Ye (2015) proposed the ranking method on possibility degree for INNs from the probability aspect.

Ruspini (2015) conveyed that, at the conceptual level, the clustering problem or its formulation in terms of distances or similarities could not be solved within the structure of classical set theory. In conventional formulation of the problem, a finite set will be partitioned into non-intersecting subsets in a such a way that the points which are closer to each other are in the same subset of the partition and which are far from each other are in different subsets. Fuzzy partition of the sample set overcome this problem by classifying the points as belonging to a subset to a degree expressed by a number between 0 and 1.

Poyen *et al.* (2016) designed a dynamic traffic signal system based on density where the signal timing changes automatically on sensing traffic. Sharma and Sahu (2016) reviewed fuzzy logic based traffic signal control. Singh *et al.* (2016) analyzed an uncertainty for the provided many valued context. Ye (2016) proposed new exponential laws of INSs; interval neutrosophic weighted exponential aggregation operator and its dual operator and applied them in a decision-making problem for global supplier selection.

Ye (2016) introduced a credibility induced INWA averaging operator (INWAAO) and credibility induced INWG averaging operator (INWGAO) and examined their properties. They also presented a measure of projection between INNs and its ranking method and applied it in a decision making problem. Bouyahia *et al.* (2017) used fuzzy switching linear models to present real time traffic smoothing from GPS spare measures. Chen and Ye (2017) derived the mathematical properties of Dombi triangular norms based on a SVNS and applied them in a decision making method.

Singh (2017) has discovered some of the important hidden patterns in the interval valued neutrosophic context. Ye (2017) presented the concepts of neutrosophic linear equations (NLEs) and, neutrosophic matrix (NM), and proposed NM operations for the first time. They also introduced some solving methods on NMs. Lakshmi *et al.* (2018) proposed an intelligent system for traffic control to enable emergency vehicles to pass without any disruptions.

Noormohammadpour and Raghavendra (2018) have brought out the important characteristic of traffic control in data centers. Shi and Ye (2018) have derived the Dombi aggregation operators of neutrosophic cubic sets and applied them in a decision making process. Liu and Wang (2018) proposed interval valued intuitionist fuzzy Schweizer-Sklar power aggregation operators and applied them in a decision making problem for supplier selection.

Broumi *et al.* (2018) discussed the lack of knowledge partially for [0, 1] using IVNSs. Mayouf *et al.* (2018) developed an accident management system applicable for cellular technology in public transportation. Sumia and Ranga (2018) proposed a new intelligent traffic management system (TMS). Ankam *et* *al.* (2018) designed a new TMS for the benefit of vehicle owners to carry the documents such as license and insurance during investigation by the authorities. Tarek *et al*. (2019) applied fog calculation for optimized strategy traffic control. Mannhardt and Landmark (2019) proposed mining railway traffic control logs.

It is observed from the literature that there is no contribution of work related to traffic control management using aggregation operators under interval type-2 fuzzy and interval neutrosophic environments. Also it is yet to introduce a score function for interval neutrosophic numbers. Hence the scope of the present study.

2. (b) INTERVAL VALUED NEUTROSOPHIC GRAPH USING TRIANGULAR NORMS

Dombi (2009) introduced the generalized Dombi operator family and the multiplicative utility function. Gosztolya *et al.* (2009) applied the Generalized Dombi Operator Family to the Speech Recognition Task. Pavlopoulos *et al.* (2011) examined biological networks using graph theory. Stoilova and Nedelchev (2012) established a computerized technique to solve network problems using graph theory.

Dey *et al.* (2012) proposed an operation on a complement of a fuzzy graph. Gomes *et al.* (2012) offered functional consistency of an input gene network. Dey *et al.* (2012) proposed a program for coloring the vertex of a fuzzy graph. Dey *et al.* (2013) applied vertex coloring function of a fuzzy graph to the traffic light problem.

Sunitha and Mathew (2013) reviewed the theory of FG. Dave and Jhala (2014) introduced a graph for the problem and circular arcs and applied in TMS. Dhavaseelan *et al.* (2015) described certain types of neutrosophic graphs. Manjusha and Sunitha (2015) proposed strong domination number using membership values of strong arcs in FGs. Broumi *et al.* (2016) gave introduction to bipolar SVNG theory. Broumi *et al.* (2016) proposed an isolated IVNG.

Broumi *et al.* (2016) applied interval valued neutrosophic concept in decision making problem to invest the money in the best company. Broumi *et al.* (2016) proved the necessary and sufficient condition for a neutrosophic graph to be an isolated SVNG. Furthermore, Broumi *et al.* (2016) examined the properties of different types of degrees, size and order of SVNGs and proposed the definition of regular SVNG.

Kandasamy *et al.* (2016) proposed a strong NGs and sub graph topological subspaces. Bhagat and Patel (2016) presented a complete study of all existing Intelligent Transport systems namely research models and open systems. Bisen (2017) applied the concept of graph in traffic control management in city and airport. Sahin (2017) applied graph theory concepts to single valued neutrosophic graphs and examine a new type of graph model and concluded the result to crisp graphs, FGs and IFGs and characterized their properties.

Jain *et al.* (2017) examined asymmetrical partnership using FG and detect hidden connections in Facebook. Sethi and Behera (2017) proposed an optimized algorithm using the approach of rating of web pages and it assigns a minimum approved bandwidth to every connected user. Akram and Sitara (2017) represented a graph model based on interval valued neutrosophic sets. Oberoi *et* *al.* (2017) proposed a dimensional modeling of traffic in urban road using graph theory.

Broumi *et al.* (2017) proposed uniform SVNG. Quek *et al.* (2018) proposed some of the results on the graph theory for complex NSs. Ashraf *et al.* (2018) proposed Dombi fuzzy graphs (DFGs) and proved the standard operations on Dombi fuzzy graphs. Marapureddy (2018) proposed fuzzy graph of semi group. Mordeson and Mathew (2018) proposed t-norm fuzzy graphs and discussed the importance of t-norm in network system. Khan *et al*. (2019) introduced complex neutrosophic graph with Laplacian energy. Akram and Shahzadi (2019) introduced bipolar neutrosophic graphs.

It is found that neutrosophic graphs using triangular norms have not yet been introduced and hence introducing interval valued neutrosophic graph using Dombi triangular norms and Cartesian and composite products of interval valued neutrosophic graphs are the scope of the study.

2.(c) BLOCKCHAIN NEUTROSOPHIC GRAPH

Yeh and Bang (1975) proposed fuzzy relations, FGs and applied them in cluster analysis. Satoshi (2008) presented a solution to the problem of double spending using a peer-to-peer network. Leroy (2009) portrayed the evolution and proof of linguistic care of an accumulator back end. Dey *et al.* (2012) have done a vertex colouring of a FG. Dey *et al.* (2013) applied the concept of FG in light control in TCM. Ober *et al.* (2013) proposed a model and obscurity of the Bitcoin transaction graph.

Decker and Wattenhofer (2013) examined knowledge reproduction in the network of Bitcoin. Fleder *et al.* (2014) linked Bitcoin public keys to real people and commented about the public transaction graph and hence done a graph analysis scheme to find and compiled activity of known as well as unknown users. Stanfill and Wholey (2014) proposed a transactional graph on the basis of computation with error management. Ye (2014) proposed aggregation operators under a simplified neutrosophic environment and employed them in a DMP. Biswas *et al.* (2014) introduced a new methodology for dealing with unknown weight information and applied in a decision making problem (DMP).

Biswas *et al.* (2014) proposed a grey relational analysis based on entropy under a single valued neutrosophic setting and applied in a decision making process with a multi attribute. Mondal and Pramanik (2015) introduced a design for brick selection established on grey comparative analysis for decision making under neutrosophic setting. Mondal and Pramanik (2015) proposed a tangent similarity measure and applied in multiple attribute DMP (MADMP) based on neutrosophic concepts. Biswas *et al.* (2015) imported the cosine similarity measure with trapezoidal NNs (TNNs) and applied in a DMP.

Broumi *et al.* (2015) introduced an extended TOPSIS methodology using uncertain linguistic variables under interval neutrosophic environment. Greaves and Au (2015) investigated the prognostic power of the Blockchain network using lineaments on the future price of Bitcoin. Pilkington (2015) clarified the main ethics behind the BCT and a few of its application of cutting edge. Bonneau *et al.* (2015) analyzed invisibility problems in Bitcoin and contribute an evaluation plan for private enlarging proposals and contributed a new intuition on language disintermediation protocols. Biswas *et al.* (2016) proposed the TOPSIS methodology under the single valued neutrosophic setting for multi attribute group decision making (MAGDM).

Biswas *et al.* (2016) proposed AOs for triangular fuzzy neutrosophic set information and used in DMP. Biswas *et al.* (2016) introduced a ranking method based on value and ambiguity index using single valued TNNs and applied in DMP. Eyal *et al.* (2016) designed a Blockchain protocol called Bitcoin next generation. Broumi *et al.* (2016) introduced operational laws on IVNGs. Broumi *et al.* (2016) proposed the formulas to find degree, size, and order of SVNG. Pramanik *et al.* (2017) proposed hybrid similarity measures under a neutrosophic environment and applied them in a DMP.

Dalapati *et al.* (2017) introduced IN cross entropy for an interval neutrosophic set environment and applied in MAGDMP. Broumi *et al.* (2017) proposed uniform single valued neutrosophic graphs. Cocco *et al.* (2017) paid attention to the threats and opportunities of carrying out a Blockchain mechanism across banking. Jeoseph *et al.* (2017) reviewed the approval and future use of BCT. Chan and Olmsted (2017) proposed a design for prevailing transactions from Ethereum into a graph database namely leveraging graph computer.

Illgner (2017) proposed a Blockchain to fix all Blockchains. Swan and Filippi (2017) explained about the philosophy of BCT. Banuelos *et al.* (2017) proposed an advanced method to implement business developments on top of commodity BCT. Dinh *et al.* (2017) surveyed the case of the art targeting private Blockchain where the parties are authenticated. Desai (2017) analyzed industry application and have legal perspectives for BCT.

Jain *et al.* (2017) analyzed asymmetrical associations using FG to find hidden connections on Facebook. Raikwar *et al.* (2018) proposed a framework of BC for insurance processes. Ramkumar (2018) proposed the BC integrity framework. Hill (2018) presented a review on BC. Arockiaraj and Charumathi (2018) introduced the BCFG and its concepts and properties. Halaburda (2018) answered for the question, BC transformation without BC. Gupta and Sadoghi (2018) explained about BC process in a detailed manner.

Ramkumar (2018) accomplished a large scale measure in BC. Asraf *et al.* (2018) proposed Dombi fuzzy graphs (DFGs). Marapureddy (2018) introduced a FG for the semi group. Quek *et al.* (2018) introduced a few of the results for complex NSs on graph theory. Smarandache and Pramanik (2018) introduced a new perspective to neutrosophic theory and its applications. Basset *et al.* (2018) proposed an extended neutrosophic AHP SWOT analysis for critical planning and decision making. Basset *et al.* (2018) proposed the association rule mining algorithm to analyze big data. Basset *et al.* (2018) introduced a group ANP TOPSIS framework under a hybrid neutrosophic setting for a supplier selection problem.

Basset *et al.* (2018) presented a hybrid approach of NSs and the DEMATEL method to enhance the criteria for supplier selection. Pramanik *et al.* (2018) proposed NS cross entropy under a single valued neutrosophic environment and applied in a MAGDM problem. Biswas *et al.* (2018) proposed a neutrosophic TOPSIS method and solved group DMP. Pramanik and Mallick (2018) proposed a VIKOR method using trapezoidal neutrosophic numbers and solved the MAGDM problem using the proposed method.

Biswas *et al.* (2018) solved the MADMP by introducing distance measure using interval trapezoidal NNs. Biswas *et al.* (2018) introduced a TOPSIS strategy for solving the MADMP with trapezoidal numbers. Biswas *et al.* (2018) solved the MAGDM problem using the expected value of neutrosophic trapezoidal numbers. Mondal *et al.* (2018) introduced hyperbolic sine similarity measure based MADM strategy under a single valued neutrosophic environment. Mondal *et al.* (2018) proposed a hybrid binary algorithm similarity measure under SVNS assessments for the MAGDMP. Dhavaseelan *et al.* (2018) proposed SVNGs. Lopes and Pereira (2019) analyzed about the opportunities of the BCT in health care. Cao (2019) studied about BCT with energy internet.

Though Blockchain technology has been used by applying a graph and fuzzy graph, it is yet to be applied the neutrosophic concept. This present study proposed Blockchain single and interval valued neutrosophic graphs and applied in Blockchain technology for the transaction of Bitcoins.

2.(d) AGGREGATION OPERATORS IN EDGE DETECTION ON DICOM IMAGE

Image processing (IP) is an interesting and growing research field in pattern recognition, medical IP, remote sensing and industrial applications. The following studies have done by researchers till date. Franke *et al*. (2000) presented a new operations on IP established on Dubois and Prade fuzzy triangular norms. Guo and Watson (2003) started probationary results of applying Modified Conjugate Directional Filtering in medical IP (MIP). Hirota *et al.* (2004) adopted fuzzy relational equations in compression and reconstruction of an image.

Hidalgo *et al.* (2009) placed left continuous conjunctive uninorms in the assiduity of the review on a mathematical morphology (MM). Zarandi *et al*. (2009) proposed a new path using a digital camera over a microscope to identify the sore in digital images. Khaire and Thakur (2012) identified an edge using FS. Exact calculations of extended logical operations on fuzzy truth values spotlighted on the procedure of dimensional domain for image enrichment (IE) using FL. Kamra and Rani (2012) have done ED without inducing threshold value by using fuzzy inference system (FIS).

Mondal *et al*. (2012) started an innovative technique for segmenting and eradication of an image using functional deficient of any exterior interference. Dammavalam *et al.* (2012) proffered a new method to merge the images from various sources using FL in enhancement of the quality of the images. Jain and Aggarwal (2012) have designed an algorithm called matrix scanning to find the noise on the pixels of the image to salvage the border size. Gupta and Chela (2013) found a novel hybrid approach to IE using artificial neural network and FL. Hernandez *et al.* (2013) evolutes an algorithm for edge evulsing using FL theory.

Shenbagavalli and Ramar (2013) evolved ED using FL for satellite images. Mishra and Sinha (2014) have found a contemporary path for the contrast enrichment of the color image using escalation operator based on FL. Koul *et al*. (2014) Demonstrated to prove the application of FSs to the image parameters. Jeon (2014) portrayed a novel path for the enrichment of the color image using fuzzy membership functions (FMFs). Gangwar *et al.* (2014) used encoding and decoding system to compress and to reconstruct the pixel of an image via Huffman and rough FSs respectively.

Khleaf *et al*. (2015) started with fuzzy hyperbolic threshold for the endowment of an image contrast. Prasanna and Rai (2015) examined the applications of FL in IM. Myna and Prakash (2015) gave the fusion methodology for an image with FL. Walad and Shetty (2015) have done the edge detection by implementing fuzzy based MATLAB algorithm. Amza and Cicic (2015) utilized the technique of artificial intelligence (AI) to detect the defect on the X-ray images automatically.

Tibrewala and Malviya (2015) depicted the working design of the fuzzy knowledge based system to identify the latent and mislaid objects. Mankar *et al*. (2015) developed an algorithm for image segmentation by AI and digital IP. Khoon and Chuin (2016) chronicle a software tool for MIP. Ferdyansah *et al*. (2016) used IP and FL design to model automated texture inspection. Salve *et al*. (2017) perceived the human age through an ED without resolving the threshold value. Sornam *et al.* (2017) analyzed the freshness of the fish using IP via clustering approach and cited their components in the wavelet transformation domain.

Vardar *et al*. (2017) recognized the pattern using fuzzy rules. Rao *et al.* (2017) made hybrid image classification by Ant Colony Optimization (ACO) for textured and non-textured images. Sarath and Sreejith (2017) established and correlated two kinds of image enrichment using FL. Rios *et al.* (2017) proposed a decisive method for analyzing an image using FIS. Castillo *et al.* (2017) surveyed the application of IP using T2FS. Sheikh and Khan (2017) gave a detailed depiction of reproduction of images using FL. Sheikh and Khan (2017) disposed a structure of simulation of ED using FL. Rao *et al.* (2018) found a hybrid image classification for analyzing the composite and non-composite images.

Jain *et al.* (2018) verified the signature using FL adaptive resonance theory I. Kenjharayoobchandio and Yasarayaz (2018) established an algorithm for ED and agitation of a digital image is determined. Myna and Prakash (2018) found a new technique on interval T2FL connected with discrete wavelet transformation. Dutta (2018) used intuitionistic FL in image fusion technique. Khunkhet and Remsungnen (2018) classified unmilled jasmine rice 105 using IP and FL. Jeon e*t al*. (2018) used fuzzy and rough sets and initiated a real time IP. Privezentsev *et al.* (2018) constructed a strange method of ED situated on representation of fuzzy image and pixels.

ED on DICOM image has been done using FIS in MATLAB with interval type-2 fuzzy environment to obtain simple fuzzy rules based edge detection approach. Here the smallest attainable 2 x 2 screen that moves smoothly over an entire image pixel by pixel. This system focused on edge pixels using fuzzy rules. Thanki and Kothari (2019) studied about the applications in the field of digital image processing. Thanki and Kothari (2019) analyzed color image processing.

It is noted that edge detection on DICOM image using the MATLAB program under triangular interval type-2 fuzzy logic with the help of triangular norms has not been done and hence the motivation of the present study.

2.(e) WASHING MACHINE WITH INTERVAL TYPE-2 FUZZY LOGIC

Castro *et al.* (2007) introduced IT2FL toolbox. The authors in, Farahi and Barati (2011) investigated about the first order linear fuzzy time delays dynamical system with the usage of alpha level sets. Ge and Zhu (2012) proposed a method to solve an uncertain DDE and proved the existence and uniqueness of theorems (EUTs) by using Lipschitz condition and condition of linear gain by Banach Fixed Point Theorem.

Lupulescu and Abbas (2012) used Liu process and proved a local EUTs for fuzzy delay differential equation (FDDE) and developed a continuous dependence of the result corresponding to an initial data. Gaurav and Kaur (2012) compared traditional PID controller and fuzzy logic controller for controlling liquid flow. Liu and Fei (2012) attained the solution of the EU to unbiased uncertain DDEs.
Lata and Kumar (2013) introduced a new analytical method for solving fuzzy linear DEs of order n. Maan *et al.* (2013) used adapting fuzzy parameter and proposed a fuzzy delay predator prey system.

Boulkroune *et al.* (2010) investigated the fuzzy flexible control algorithm for multivariable system with uncertainty. Zulkefi and Maan (2014) proposed the EUTs for fuzzy time delay dynamical system. Mahdi (2014) decisive optimal parameters to develop an ephemeral feedback of the PID control system. Wang *et al.* (2014) enforced linear matrix inequality (LMI) in stabilizing the fractional order chaotic systems with uncertain parameters. Demetgul *et al.* (2014) designed a WM using FL. Qin and Liu (2014) proposed Frank AOs for TIT2FNs and applied them in DMP.

Narayanamoorthy and Yookesh (2015) proposed an algorithm of the approximate mechanism for solving FDDE via Adomian decomposition method. Cano *et al.* (2015) inclined an established statement to solve initial value problems for linear fuzzy interval Des using a generic representation under the consolidation of differences. Min *et al.* (2015) imported and examined two set of system of fuzzy differential formations.

Sai and Reddy (2015) applied FL in power station. Arqub *et al.* (2015) examined existence and uniqueness theorems and the properties of assured nonlinear fuzzy Volterra integrodifferential equations (FVIEs) under SGD. Arqub (2015) has proposed emulating kernel Hilbert space method to get an explicit and analytical solutions of FVIE. Arqub *et al.* (2016) commenced a new approach to solve FDE established on the reproducing kernel theory, inquiring, closed solutions for the equations of order two, boundary value problems for two points based on strongly generalized differentiability (SGD).

Arqub *et al.* (2016) inquired some ideal results on fuzzy fractional differential equations. Habib and Akram (2015) used adaptive fuzzy neural network (AFNN) to design system for an execution of the decision for WM. Momani *et al.* (2016) proposed existence and uniqueness of fuzzy solutions for the non-linear second order FVIEs. Agarwal *et al.* (2016) prompted a control system established on improved FL for WMs and proved that it's taking less time for washing. Oo and Soe (2016) proposed a system for automatic filling of water and detergent for WM. Roseline (2016) suggested a WM along with decay of power and water. Masood (2017) designed and established a smart WM based on FL.

Agarwal *et al.* (2017) applied fuzzy sumuder transform (FST) method to approximate the solutions of fuzzy differential equations. Stammingera *et al.* (2018) developed a method to evaluate the enduring performance of WM. Masood (2018) applied FL in the model of smart WM. Agarwal (2018) presented fuzzy logic control of WM.

It is noticed that, fuzzy logic controllers have been used to design the washing machine so far. Also IT2FL WM has been designed and washing time has been analyzed for various parameters. But detergent intake, transportation delay and stability of the interval type-2 fuzzy logic washing machine have not been analyzed till date.

2.1 LACUNA IDENTIFIED

The ambience analysis, review of literature and reflection of objective demand brought the case that researchers and scientists have not studied “Interval Type-2 Fuzzy Sets and Interval Neutrosophic Sets in Intelligent Systems” till date. Hence, the purpose and scope of the present study.

Though there are some areas have been developed, theoretical concepts and their application were not examined. Hence the present study investigates one of the application areas of fuzzy logic and neutrosophic logic to fill this gap.

It is observed from the literature that there is no contribution of work related to traffic control management using aggregation operators under interval type-2 fuzzy and interval neutrosophic environments. Also it is yet to introduce a score function for interval neutrosophic numbers. Therefore, the scope of the present study is motivated as these concepts have more applications in control systems and network based problems.

As the triangular norms based neutrosophic graphs have not yet been studied, the review of literature is the motivation of introducing interval valued neutrosophic graph using Dombi triangular norms and Cartesian and composite products of interval valued neutrosophic graphs.

Though Blockchain technology has been used by applying a graph and fuzzy graph, it is yet to be applied the neutrosophic concept. This present study proposed Blockchain single and interval valued neutrosophic graphs and applied in Blockchain technology for the transaction of Bitcoins.

Edge detection on DICOM image using the MATLAB program under triangular interval type-2 fuzzy logic with the help of triangular norms has not been done and hence the motivation of the present study.

Fuzzy logic controllers have been used to design the washing machine so far. Also IT2FL WM has been designed and washing time has been analyzed for various parameters. But detergent intake, transportation delay and stability of the interval type-2 fuzzy logic washing machine have not been analyzed till date.

2.2 OBJECTIVES

The present study is attempted to develop the aggregation operators under interval type-2 fuzzy and interval neutrosophic environments and applied in intelligent systems such as control system, image processing and decision making process.

To establish the operational laws and aggregation operators under interval type-2 fuzzy sets and interval neutrosophic sets using Schweizer and Sklar triangular norms. Further applied the proposed aggregation operators for controlling traffic flow on roads by considering traffic congestion in terms of triangular interval type-2 fuzzy numbers and interval neutrosophic numbers.

To propose single and interval valued neutrosophic graphs based on Dombi and Hamacher triangular norms. Also to introduce the Cartesian and composite products of proposed graphs with numerical validation.

To propose Blockchain single and interval valued neutrosophic graph and finding the degree, total degree, minimum and maximum degree of Blockchain single valued neutrosophic graph.

To develop aggregation operators under triangular interval type-2 fuzzy environment using Yager triangular norms with their desirable properties. Edge detection to be done using MATLAB 2015a program based on interval type-2 fuzzy logic.

To analyze detergent intake of the interval type-2 fuzzy logic washing machine. Also to analyze transportation delay and stability analysis to be done by applying different defuzzification methods.

CHAPTER-3

MATERIALS AND METHODS

3.0 MATERIALS AND METHODS

In the following section the preliminaries of fuzzy sets, neutrosophic sets, image processing and fuzzy control system are described in this section.

Definition 3.1 Fuzzy Set (FS)

A function $M : X \to [0,1]$ is said to be a fuzzy set which is an extended version of a crisp set and is defined by $M = \{(x, \mu_M(x)) : x \in X\}$, where X is the universal set and $\mu_M(x)$ is the membership function (MF) of the element x.

Definition 3.2 Membership Function

A grade of truth of the elements in the FS defined by a function from the elemental set to their grade is called a membership function. The membership of an element $x \in X$ in fuzzy set M is denoted by $\mu_M(x)$.

Definition 3.3 Type-2 Fuzzy Set (T2FS)

A fuzzy set is called T2FS, if $\overline{M}$ is defined by $\overline{M} = \{((x,v), \mu_{\overline{M}}(x,v))\}, \forall x \in X, \forall v \in J_x \subseteq [0,1]$, where $\mu_{\overline{M}}(x,v)$ is called the Type-2 MF and $0 \le \mu_{\overline{M}}(x,v) \le 1$. Also J_x is called primary membership function (PMF) of x. For each primary membership value (PMV) there exist a secondary membership value (SMV) which characterize the possibility of PMV and takes values in [0, 1] for generalized T2FS and takes only the value 1 for Interval T2FS (IT2FS).

Definition 3.4 Interval Type-2 Fuzzy Set (IT2FS)

A FS is said to be interval type-2 fuzzy set (IT2FS), if all $\mu_{\overline{M}}(x,v)$ are equal to 1.

Definition 3.5 Fuzzy Number (FN)

A Fuzzy Number is a convex fuzzy set on the real line with bounded support R such that $\exists x_0 \in R, \mu_M\left(x_0\right)=1$ and μ_M is piece wise continuous.

Definition 3.6 Convex Set

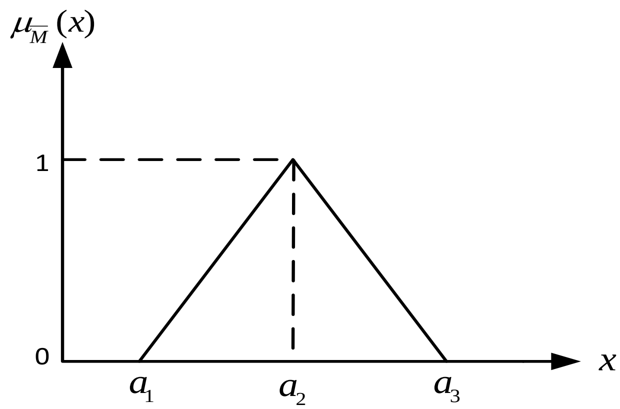
If for all $x_1, x_2, x_3 \in X$, $\mu_M\left(x_2\right) \geq \min\left(\mu_M\left(x_1\right), \mu_M\left(x_3\right)\right)$ then the set M is called a convex set i.e., either monotonically increases or decreases in membership functions either side of its maximum

Definition 3.7 Triangular Membership Function

Triangular shape membership functions whose precise appearance is determined by the values a_1, a_2 and a_3. The triangular membership function of an element x is defined by for a triangular fuzzy set (Figure 3.1).

$$\mu_{\overline{M}}(x) = \begin{cases} 0, & x < a_1 \\ \dfrac{x - a_1}{a_2 - a_1}, & a_1 \leq x \leq a_2 \\ \dfrac{a_3 - x}{a_3 - a_a}, & a_2 \leq x \leq a_3 \\ 0, & x > a_3 \end{cases} \tag{3.1}$$

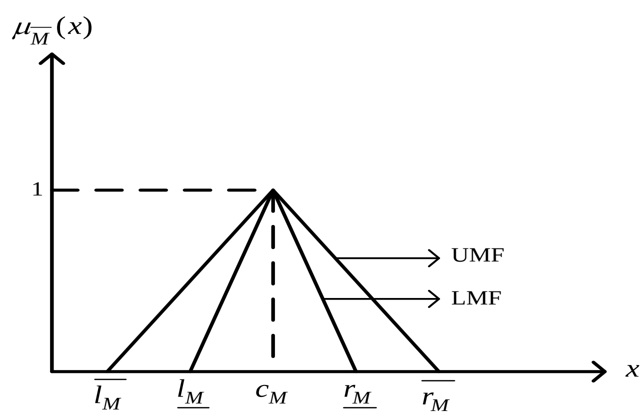
Figure 3.1: Triangular Fuzzy Set



Definition 3.8 Triangular Interval Type-2 Fuzzy Set (TIT2FS)

TIT2FS is defined $\overline{M} = \left(\left[\underline{l_M}, \overline{l_M}\right], c_M, \left[\underline{r_M}, \overline{r_M}\right]\right)$, be a TIT2FS defined on X (Figure 3.2).

Figure 3.2: Triangular Interval Type-2 Fuzzy Set



The upper and lower membership functions of TIT2FS are defined as follows

$$LMF_{\overline{M}}(x) = \begin{cases} \dfrac{x - \overline{l_M}}{c_M - \overline{l_M}} & , \quad \overline{l_M} \leq x < c_M \\ 1 & , \quad x = c_M \\ \dfrac{x - \underline{r_M}}{c_M - \underline{r_M}} & , \quad c_M \leq x < \underline{r_M} \\ 0 & , \quad otherwise \end{cases} \tag{3.2}$$

$$UMF_{\overline{M}}(x)=\begin{cases}\dfrac{x-\underline{l_M}}{c_M-\underline{l_M}} & , \quad \underline{l_M}\le x<c_M\\ 1 & , \quad x=c_M\\ \dfrac{x-\overline{r_M}}{c_M-\overline{r_M}} & , \quad c_M\le x<\overline{r_M}\\ 0 & , \quad otherwise\end{cases} \tag{3.3}$$

where $\underline{l_M},\overline{l_M},c_M,\underline{r_M},\overline{r_M}$ are the measuring points on TIT2FS satisfying $0\le\underline{l_M}\le\overline{l_M}\le c_M\le\underline{r_M}\le\overline{r_M}\le 1$. If we consider X as a set of real numbers, a TIT2FS in X is called TIT2FN. The footprint of uncertainty (FOU) is the area between lower and upper membership functions. If $\underline{l_M}=\overline{l_M},\underline{r_M}=\overline{r_M}$, then $UMF_{\overline{M}}(x)=LMF_{\overline{M}}(x)$ for all the values of x in X, then the TIT2FS will become Type-1 case.

Definition 3.9 Aggregation Operator (AO)

Let $(A_\alpha)_{\alpha\in[0,1]}$ be a family of AOs which is non-decreasing and B be any aggregation operator. Then $A_B:\underset{n\in N}{\mathrm{U}}[0,1]^n\to[0,1]$ defined by $A_B(\tau_1,\tau_2,...,\tau_n)=A_{B(\tau_1,\tau_2,...,\tau_n)}(\tau_1,\tau_2,...,\tau_n)$ is also an AO.

Definition 3.10 T-Norm

Let $T:[0,1]\times[0,1]\to[0,1]$. T is a t-norm iff for all $a,b,c\in[0,1]$ the following properties are satisfied.

(i). $TN(a,b)=TN(b,a)$ (Commutativity)

(ii). $TN(a,b)\le TN(a,c)$ if $b\le c$ (monotonicity) (iii). $TN\left(a,TN\left(b,c\right)\right)\leq TN\left(TN\left(a,b\right),c\right)$(Associativity) and

(iv). $TN\left(x,1\right)=x$

The dual of the t-norm is called t-conorm.

Definition 3.11 Triangular Interval Type-2 Weighted Geometric Operator

Let $\overline{T}_i=\left(\left[\underline{l_{T_i}},\overline{l_{T_i}}\right],m_{T_i},\left[\underline{r_{T_i}},\overline{r_{T_i}}\right]\right),i=1,2,...,n$ be a set of TIT2FNs and let $TIT2WG:\Delta^n\rightarrow\Delta$, if $TIT2WG_\beta\left(\overline{T_1},\overline{T_2},...,\overline{T_n}\right)=\overline{T_1}^{\beta_1}\otimes\overline{T_2}^{\beta_2}\otimes.....\otimes\overline{T_n}^{\beta_n}$ (3.4)

then the function TIT2WG is called triangular interval type-2 weighted geometric operator and β is the vector of $\overline{T_i}=,i=1,2,...,n$ and $\sum_{i=1}^{n}\beta_i=1$. As a particular case, if $\beta=\left(1/n,1/n,...,1/n\right)^T$ then TIT2WG operator is abridged to triangular interval type-2 geometric averaging operator and is defined by $TIT2WG_\beta\left(\overline{T_1},\overline{T_2},...,\overline{T_n}\right)=\left(\overline{T_1}\otimes\overline{T_2}\otimes...\otimes\overline{T_n}\right)^{\frac{1}{n}}$.

Definition 3.12 Triangular Interval Type-2 Weighted Arithmetic Operator

Let $\overline{T}_i=\left(\left[\underline{l_{T_i}},\overline{l_{T_i}}\right],m_{T_i},\left[\underline{r_{T_i}},\overline{r_{T_i}}\right]\right),i=1,2,...,n$ be a set of TIT2FNs and let $TIT2WA:\Delta^n\rightarrow\Delta$, if

$TIT2WA_\beta\left(\overline{T_1},\overline{T_2},...,\overline{T_n}\right)=\beta_1\overline{T_1}\oplus\beta_2\overline{T_2}\oplus.....\oplus\beta_n\overline{T_n}$ (3.5)

then the function TIT2WA is called triangular interval type-2 weighted arithmetic operator and β is the vector of $\overline{T_i}=,i=1,2,...,n$ and $\sum_{i=1}^{n}\beta_i=1$. As a particular case, if $\beta=\left(1/n,1/n,...,1/n\right)^T$ then TIT2WA operator is abridged to triangular interval type-2 arithmetic averaging operator and is defined by $TIT2WA_{\beta}\left(\overline{T_1},\overline{T_2},...,\overline{T_n}\right)=\frac{1}{n}\left(\overline{T_1}\oplus\overline{T_2}\oplus...\oplus\overline{T_n}\right)$.

Definition 3.13 Method of Mathematical Induction

An excellent proof strategy for proving the theorems easily is called mathematical induction method. Here the first proposition called base of induction proved first. Next we prove the k th proposition and if it is true then (k+1) th proposition is also true called the induction step.

Definition 3.14 Schweizer and Sklar Triangular Norms

T-norm: $TN(p,q)=p\otimes q=1-\left[(1-p)^{\varphi}+(1-q)^{\varphi}-(1-p)^{\varphi}(1-q)^{\varphi}\right]^{\frac{1}{\varphi}}$ (3.6)

T-conorm: $TCN(p,q)=p\oplus q=\left(p^{\varphi}+q^{\varphi}-p^{\varphi}q^{\varphi}\right)^{\frac{1}{\varphi}}$, $p,q\in[0,1]^2$ (3.7)

where $\varphi>0$ is the parameter.

Definition 3.15 Traffic Data interms of Fuzzy Numbers

Collection of traffic data and projections thereon traffic volumes are essential requirements for road development planning and management schemes. For the present study, traffic data have been considered as a triangular interval type-2 fuzzy numbers and interval neutrosophic numbers to deal uncertainty and indeterminacy as well as an alternative tool for the same purpose.

Definition 3.16 Dombi Triangular Norms

Dombi product or t-norm $\otimes$ and t-conorm $\oplus$ are defined by

$$TN(x,y) = x \underset{D}{\otimes} y = \frac{1}{1+\left[\left(\frac{1-x}{y}\right)^{\xi}+\left(\frac{1-y}{y}\right)^{\xi}\right]^{1/\xi}}, \xi > 0 \tag{3.8}$$

$$TCN(x,y) = x \underset{D}{\oplus} y = \frac{1}{1+\left[\left(\frac{1-x}{y}\right)^{-\xi}+\left(\frac{1-y}{y}\right)^{-\xi}\right]^{1/-\xi}}, \xi > 0 \tag{3.9}$$

This triangular norm contains the product, Hamacher operators, and Einstein operators and as the limiting case, minimum and maximum operators can be obtained. Multivariable case can be dealt easily by the new form of Hamacher family. Dombi operators have flexible parameters and hence the success rate will be greater one.

Definition 3.17 Hamacher Triangular Norms

Hamacher product or t-norm and sum or t-conorm are represented by $\underset{H}{\otimes}$ and $\underset{H}{\oplus}$ respectively and is described as,

$$T(x,y) = x \underset{H}{\otimes} y = \frac{xy}{\xi+(1-\xi)(x+y-xy)}, \xi > 0 \tag{3.10}$$

$$TC(x,y) = x \underset{H}{\oplus} y = \frac{x+y+(\xi-2)xy}{1+(\xi-1)xy}, \xi > 0 \tag{3.11}$$

Definition 3.18 Special Cases of Dombi and Hamacher Triangular Norms

If we replace $\xi = 0$ in Hamacher family of triangular norms and $\xi = 1$ in Dombi family of triangular norms then

$$T(x,y) = x \otimes y = \frac{xy}{(x+y-xy)}\ , \tag{3.12}$$

$$TC(x,y) = x \oplus y = \frac{x+y-2xy}{(1-xy)}\,. \tag{3.13}$$

Definition 3.19 Graph

A mathematical system $G=(V,E)$ is called a graph, where $V=V(G)$, a vertex set and $E=E(G)$ is an edge set. Here, undirected graph has been considered and hence every edge is considered as an unordered pair of different vertices.

Definition 3.20 Fuzzy Graph

Let V be a non-empty finite set, λ be a fuzzy subsets on V and δ be a fuzzy subsets on $\mathrm{V}\times\mathrm{V}$. The pair $\mathrm{G}=(\lambda,\delta)$ is a fuzzy graph over the set V if $\delta(x,y)\le\min\{\lambda(x),\lambda(y)\}$ for all $(x,y)\in\mathrm{V}\times\mathrm{V}$ where λ is a fuzzy vertex and δ is a fuzzy edge. Where: A mapping $\lambda:\mathrm{V}\to[0,1]$ is called a fuzzy subset of V, where V is the non-empty set. A mapping $\delta:\mathrm{V}\times\mathrm{V}\to[0,1]$ is a fuzzy relation on λ of V If $\delta(x,y)\le\min\{\lambda(x),\lambda(y)\}$. If $\delta(x,y)=\min\{\lambda(x),\lambda(y)\}$ then G is a strong fuzzy graph.

Definition 3.21 Dombi Fuzzy Graph

A pair $DG=(\lambda,\delta)$ is a Dombi fuzzy graph if

$$\delta(xy)\le\frac{\lambda(x)\lambda(y)}{\lambda(x)+\lambda(x)-\lambda(x)\lambda(y)} \tag{3.14}$$

for all $x, y \in \mathrm{V}$ where the Dombi fuzzy vertex set, $\lambda : \mathrm{V} \to [0,1]$ is a fuzzy subset in V and the Dombi fuzzy edge set, $\delta : \mathrm{V} \times \mathrm{V} \to [0,1]$ is a symmetric fuzzy relation on λ.

Definition 3.22 Interval Neutrosophic Set (INS)

Let U be the non-empty set. An interval neutrosophic set B is defined as follows. $B = \{x, \langle T(x), I(x), F(x) \rangle | x \in B\}$, where the intervals

$T(x) = [T^L(x), T^U(x)] \subseteq [0,1]$, $I(x) = [I^L(x), I^U(x)] \subseteq [0,1]$, $F(x) = [F^L(x), F^U(x)] \subseteq [0,1]$ for $x \in U$ are the grades of the truth membership, indeterminacy membership and false membership respectively.

Definition 3.23 Interval Neutrosophic Numbers

Let $X = \{x_1, x_2, ..., x_n\}$ be an INS, where $x_j = \langle [T_j^L, T_j^U], [I_j^L, I_j^U], [F_j^L, F_j^U] \rangle$ for $j = 1, 2, 3, ..., n$ is a collection of INNs and $T_j^L, T_j^U, I_j^L, I_j^U, F_j^L, F_j^U \in [0,1]$, $\upsilon > 0$ and $\delta > 0$.

Definition 3.24 Single Valued Neutrosophic Graph (SVNG)

A pair $\mathrm{G_N} = (P, Q)$ is SVNG with elemental set V. Where:

Degree of truth membership, indeterminacy membership and falsity membership of the element $x_i \in \mathbf{V}$ are defined by $T_P : \mathrm{V} \to [0,1]$, $I_P : \mathrm{V} \to [0,1]$ and $F_P : \mathrm{V} \to [0,1]$ respectively and $0 \le T_P(x_i) + I_P(x_i) + F_P(x_i) \le 3, \forall x_i \in \mathbf{V}, i = 1, 2, 3, ..., n$

Degree of truth membership, indeterminacy membership and falsity membership of the edge $(x_i, y_j) \in \mathbf{E}$ are denoted by $T_Q : \mathrm{E} \subseteq \mathrm{V} \times \mathrm{V} \to [0,1]$, $I_Q : \mathrm{E} \subseteq \mathrm{V} \times \mathrm{V} \to [0,1]$ and $F_Q : \mathrm{E} \subseteq \mathrm{V} \times \mathrm{V} \to [0,1]$ respectively and are defined by

$$T_Q\left(\{x_i, y_j\}\right) \le \min\left[T_P(x_i), T_P(y_j)\right] \quad (3.15)$$

$$I_Q\left(\{x_i, y_j\}\right) \ge \max\left[I_P(x_i), I_P(y_j)\right] \quad (3.16)$$

$$F_Q\left(\{x_i, y_j\}\right) \ge \max\left[F_P(x_i) F_P(y_j)\right] \quad (3.17)$$

where $0 \le T_Q\left(\{x_i, y_j\}\right) + I_Q\left(\{x_i, y_j\}\right) + F_Q\left(\{x_i, y_j\}\right) \le 3, \forall \{x_i, y_j\} \in \mathbf{E}(i, j = 1, 2, ..., n)$.

Also P is a single valued neutrosophic vertex of V and Q is a single valued Neutrosophic edge set of $\mathbf{E}$. Q is a symmetric single valued Neutrosophic relation on P.

Definition 3.25 Interval Valued Neutrosophic Graph (IVNG)

A pair $\mathrm{G_N} = (P, Q)$ is IVNG, where $P = \left\langle \left[T_P^L, T_P^U\right], \left[I_P^L, I_P^U\right], \left[F_P^L, F_P^U\right] \right\rangle$, an IVN is set on V and $Q = \left\langle \left[T_Q^L, T_Q^U\right], \left[I_Q^L, I_Q^U\right], \left[F_Q^L, F_Q^U\right] \right\rangle$ is an IVN edge set on $\mathbf{E}$ satisfying the following conditions:

1. Degree of truth membership, indeterminacy membership and falsity membership of the element $x_i \in \mathbf{V}$ are defined by $T_P^L : \mathrm{V} \to [0,1]$, $T_P^U : \mathrm{V} \to [0,1]$, $I_P^L : \mathrm{V} \to [0,1]$, $I_P^L : \mathrm{V} \to [0,1]$, $I_P^U : \mathrm{V} \to [0,1]$ and $F_P^L : \mathrm{V} \to [0,1], F_P^U : \mathrm{V} \to [0,1]$ respectively and $0 \le T_P(x_i) + I_P(x_i) + F_P(x_i) \le 3, \forall x_i \in \mathbf{V}, i = 1, 2, 3, ..., n$

2. Degree of truth membership, indeterminacy membership and falsity membership of the edge $(x_i, y_j) \in \mathbf{E}$ are denoted by $T_Q^L : \mathrm{V} \times \mathrm{V} \to [0,1]$, $T_Q^U : \mathrm{V} \times \mathrm{V} \to [0,1]$ $I_Q^L : \mathrm{V} \times \mathrm{V} \to [0,1]$, $I_Q^U : \mathrm{V} \times \mathrm{V} \to [0,1]$ and $F_Q^L : \mathrm{V} \times \mathrm{V} \to [0,1]$, $F_Q^U : \mathrm{V} \times \mathrm{V} \to [0,1]$ respectively and are defined by

$$T_Q^L\left(\{x_i, y_j\}\right) \le \min\left[T_P^L(x_i), T_P^L(y_j)\right] \quad (3.18)$$

$$T_Q^U\left(\{x_i, y_j\}\right) \le \min\left[T_P^U(x_i), T_P^U(y_j)\right] \quad (3.19)$$

$$I_Q^L\left(\{x_i, y_j\}\right) \geq \max\left[T_P^L(x_i), T_P^L(y_j)\right] \tag{3.20}$$

$$I_Q^U\left(\{x_i, y_j\}\right) \geq \max\left[I_P^U(x_i), I_P^U(y_j)\right] \tag{3.21}$$

$$F_Q^L\left(\{x_i, y_j\}\right) \geq \max\left[F_P^L(x_i), F_P^L(y_j)\right] \tag{3.22}$$

$$F_Q^U\left(\{x_i, y_j\}\right) \geq \max\left[F_P^U(x_i), F_P^U(y_j)\right] \tag{3.23}$$

where $0 \leq T_Q\left(\{x_i, y_j\}\right) + I_Q\left(\{x_i, y_j\}\right) + F_Q\left(\{x_i, y_j\}\right) \leq 3, \forall \{x_i, y_j\} \in \mathbf{E}(i, j = 1, 2, \ldots, n)$

Definition 3.26 Standard Products of graphs

Consider two $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ then the following products are defined by

Direct Product : $E(G_1 \text{g} G_2) = \{(x_1, x_2)(y_1, y_2) | x_1 y_1 \in E_1 \,\&\, x_2 y_2 \in E_1\}$

Cartesian Product :

$E(G_1 \times G_2) = \{(x_1, x_2)(y_1, y_2) | x_1 = y_1 \,\&\, x_2 y_2 \in E_2, or\ x_1 y_1 \in E_1 \,\&\, x_2 = y_2\}$

Strong Product:

$E(G_1 \times G_2) = \{(x_1, x_2)(y_1, y_2) | x_1 = y_1 \,\&\, x_2 y_2 \in E_2, or\ x_1 y_1 \in E_1 \,\&\, x_2 y_2 \in E_2\}$

Dombi fuzzy Cartesian Product:

$$(\lambda_1 \times \lambda_2)(x_1, x_2) = \frac{\lambda_1(x_1)\lambda_2(x_2)}{\lambda_1(x_1) + \lambda_2(x_2) - \lambda_1(x_1)\lambda_2(x_2)}, \forall (x_1, x_2) \in \mathbf{V}_1 \times \mathbf{V}_2$$

$$(\delta_1 \times \delta_2)((x, x_2)(x, y_2)) = \frac{\lambda_1(x)\delta_2(x_2 y_2)}{\lambda_1(x) + \delta_2(x_2 y_2) - \lambda_1(x_1)\delta_2(x_2 y_2)}, \forall x \in \mathbf{V}_1, x_2 y_2 \in \mathbf{E}_2$$

$$(\delta_1 \times \delta_2)((x_1, z)(y_1, z)) = \frac{\lambda_2(z)\delta_1(x_1 y_1)}{\lambda_2(z) + \delta_1(x_1 y_1) - \lambda_2(z)\delta_1(x_1 y_1)}, \forall z \in \mathbf{V}_2, x_1 y_1 \in \mathbf{E}_1$$

Defnition 3.27 Experimental Data Set Considered for interval valued neutrosophic graph

The elements for the vertex set and edge set are the interval valued neutrosophic numbers adopted from Broumi *et al*. (2016).

Definition 3.28 Bitcoins

Bitcoins are the digital currency and worldwide payment system and are believable and best used when, there are a series of transaction, need to be recorded and need to be verified with respect to purity of the information and the order of the events.

Definition 3.29 Block chain

A Block chain is a network and can be seen as a reliable computer whose private states are auditable by anyone. It can also be defined such as cryptographic approach for modeling an unalterable append only public ledger, it includes a methodology for obtaining an open general agreement on each entry and ledger entries are mappings of the states of processes by the Blockchain network.

Uses of Block chain

A uniform approach to execute a variety of application processes. Also reliable and efficient Low upward approaches for users namely states with query application and audit correctness of changes of states.

Definition 3.30 Block chain Fuzzy Graph (BCFG)

The pair $\mathrm{G}=(\lambda,\delta)$ is a BCFG, where λ is a fuzzy vertex set and δ is symmetric on λ such that $\delta(a,b)\leq \min\{\lambda(a),\lambda(b)\}$, $\forall\ a,b\in \mathrm{V}$ with the following criterion.

1. If $i \neq j$ then $\sum \left[\delta\left(a_i, b_j\right) \leq \min\left(\lambda\left(a_i\right), \lambda\left(b_j\right)\right)\right] = 1$ (3.24)

2. If $i \neq j$ then $\sum \left[\delta\left(a_i, b_j\right) \leq \max\left(\lambda\left(a_i\right), \lambda\left(b_j\right)\right)\right] = 1$ (3.25)

3. If $i = j$ then $\sum \left[\delta\left(a_i, b_j\right) \leq \min\left(\lambda\left(a_i\right), \lambda\left(b_j\right)\right)\right] = 0$ (3.26)

Definition 3.31 Yager Triangular Norms

$\underset{Y}{\otimes}$ is Yager product (T Norm) and $\underset{Y}{\oplus}$ is a Yager sum (T conorm) and are defined as follows.

$$r \underset{Y}{\otimes} s = \max\left(1 - [(1-r)^{\eta} + (1-s)^{\eta}]^{\frac{1}{\eta}}, 0\right), \eta > 0, \; for\, all \;\; r, s \in [0,1]^2 \tag{3.27}$$

$$r \underset{Y}{\oplus} s = \min\left(\left(r^{\eta} + s^{\eta}\right)^{\frac{1}{\eta}}, 1\right), \eta > 0, \; for\, all \;\; r, s \in [0,1]^2 \tag{3.28}$$

Definition 3.32 Reason of using Yager Triangular Norms

Using triangular interval type-2 weighted arithmetic and geometric aggregation operators; triangular interval type-2 fuzzy weighted arithmetic and triangular interval type-2 fuzzy weighted geometric operators based on Yager triangular norms are derived with their desirable properties due to the flexibility of interval type-2 triangular fuzzy numbers. Also Yager triangular norms contain maximum and minimum operators and hence it will be useful for the operations in image processing for image lightening and thickening.

Definition 3.33 DICOM Image

Digital Imaging and Communication in Medicine (DICOM) is the accepted and typical format for the transmission and management of medical imaging report and relevant data. It is used for collecting and transferring medical images and allowing the combination of medical imaging tools such as servers, scanners, printers, workstations, network hardware and Picture Archiving and Communication Systems (PACSs) from different manufacturers.

Definition 3.34 Edge Detection (ED)

ED is one of the important processes in image processing. Since most of the images have poor brightness, the proper decision cannot be taken in checking the existence of an edge in an image. Edges may be enriched before carrying out the edge detection. While taking off an edge, due to ambiguity, fuzzy method (type-1) may be useful and may not find better edges. At this junction T2FS is useful as it handles more uncertainties.

Definition 3.35 Algorithm of Edge Detection

Edge detection of the image contains the following steps.

Step 1: Read an m x n image.

Step 2: The first set of 25 pixels of a 3 x 3 window is selected central pixel having values (2, 2).

Step 3: Once the initialization is over, the pixel values exposed to the fuzzy conditions for the existence of an image according to the fuzzy rule.

Step 4: After exposed the pixel values to the fuzzy conditions in type-2 fuzzy set setting, produce an intermediate image.

Step 5: Check whether all the pixels are checked it not then, first check the horizontal coordinate pixels once it is done for all the pixels then vertical pixels have to be checked or else the horizontal pixel is enhanced to retrieve the next set of pixels of a window.

Step 6: After highlighted an edge, the unwanted parts will be removed and obtain an output image corresponding to the input image.

Definition 3.36 Morphology

Morphology is a non-linear image processing technique and is used to shape the image features. Here also T2FS plays an important role to get better results.

Definition 3.37 Mathematical Fuzzy Morphology (MFM)

It is a fuzzy logic approach to generalize a binary morphology into MFM.

Definition 3.38 Morphological Operations

Morphological Operations which are being used in image processing are Erosion (Maximum)**,** Dilation (Minimum) and Opening and Closing (Idempotency). Morphological operations are the basic tools to modify an image over the structural aspect of another image. To study the structure of the first image, size and shape of another image should be considered according to the first image.

Definition 3.39 Morphological Operators

Morphological operators convert the original image into another one by interacting with the other images of the structural elements like absolute size and shape. Also it provides a systematic approach for characterization in many applications like object segmentation, edge detection and suppression of noise. The goal of MFM to develop the binary morphological operators to gray level images.

Definition 3.40 Morphological Gradient (MG)

MG is useful to detect an edge and act as a first approximation to a morphological segmentation. MG is the discrepancy between (a) dilation and erosion, (b) dilation and the original image and (c) original image and its erosion

Definition 3.41 Role of Mathematical Properties of triangular norms in image processing

For constructing Fuzzy Morphology (FM), we use Conjunctions and Implications. Among these two, we used conjunction (t-norm) here and from the below, the representation of mathematical properties in image processing has been explained. This shows the role of mathematical properties of t-norm in image processing.

Commutativity: The result of image display system (IDS) application on two successive points P and Q is the same as applying on them in inverse order, since the value of flapped points is the sum of values of all data diluted on that point and therefore the operator is commutative.

Monotonicity: If the brightness of P is less than or equal to Q then all the data points in brightness of P is less than or equal to brightness with respect to the corresponding data points of brightness of Q. Therefore for any point n, the brightness appeared from P is $n+aP$, where a is proportional to inverse of distance. Similarly, the brightness appears from Q is $n+aQ$. Since $a>0$, the brightness of appeared from P is less than or equal to that of from Q.

Associativity: Assume that P, Q and R are the sources of light going to affect to the point n by IDS. For every source, IDS increases the brightness with respect to the distance regardless of other sources. On the point n, the order of applying IDS does not affect the distance. Sum of effects of P, Q and R is the value of n. Therefore, the operator is associative.

Idempotency: This property and its generalization is used for the morphological operation opening and closing.

Neutrality of 0: Consider a pyramid of height 0, sum of this with others does not influence them. Therefore, 0 is the neutral element.

Definition 3.42 Material used for Edge Detection on DICOM Image

The data set (Figure. 3.3) is the montage of the images in a single file and is from a patient MRI. This MRI which is in the 3D form is converted to 2D form (DICOM) using MATLAB2015a. The 3D format consists of 25 DICOM file formats; the montage of the images is obtained as a single frame. Out of these 25 DICOM images a clear full image is chosen (Figure. 3.4). Using Dilation corrosion method, the gradient is identified. The edge detection is performed through triangular norms using MATLAB 2015a.

Figure 3.3: Montage of the Image

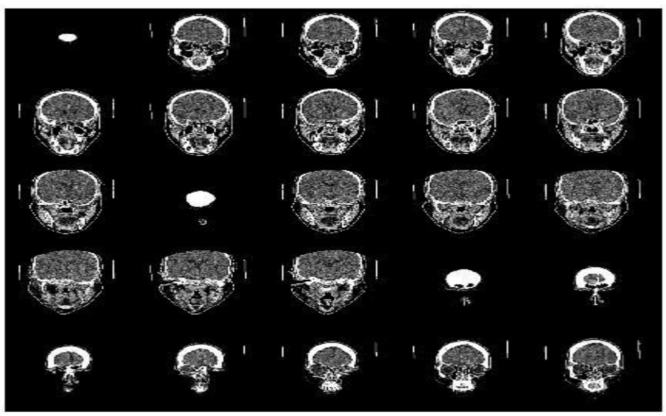
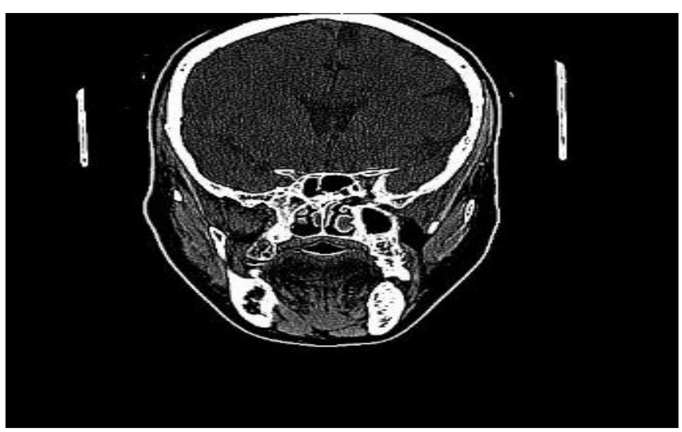


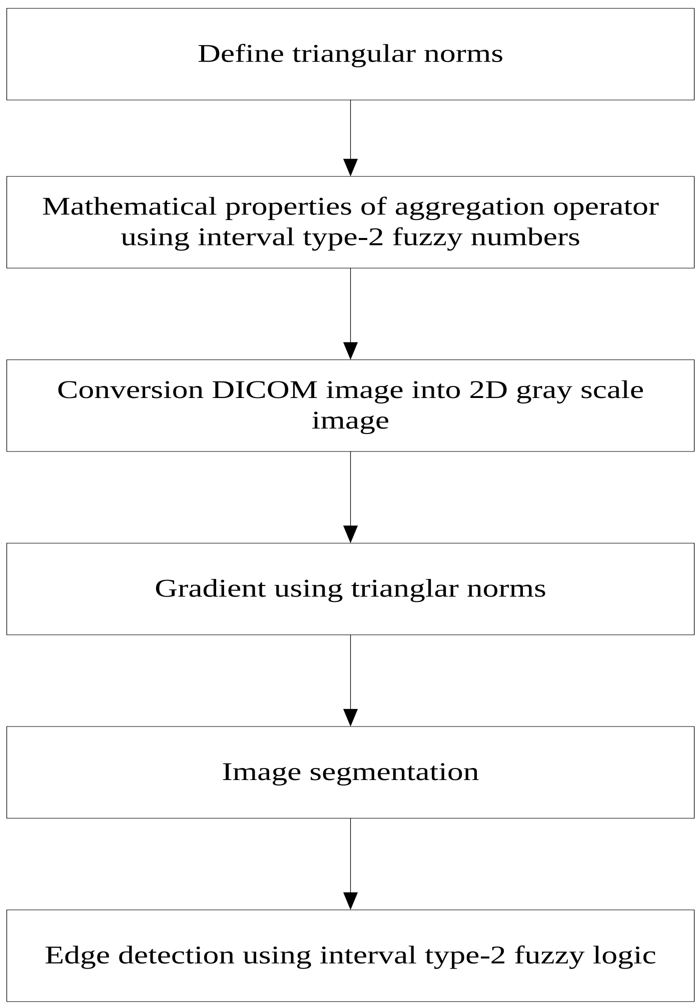
Figure 3.4: Clear Image from Montage



Definition 3.43 Architecture of edge detection on DICOM Image using Triangular Norms

The design for the process of edge detection on DICOM image using aggregation operators under triangular interval type-2 fuzzy environment is described (Figure 3.5).

Figure 3.5: Architecture of Edge Detection on DICOM Image

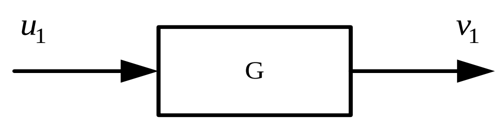


Definition 3.44 Linear System

A system is called linear, when the output of the system is proportional to its input (Figure 3.6). Here, changes in input affects the output according to small or large changes.

Also, the final response induced by two or more stimuli is the sum of the responses caused by individual stimulus called superposition property. This kind of systems has static linearity as well as sinusoidal fidelity. Hence the linear control system satisfies homogeneous and additive property.

Figure 3.6: Linear System

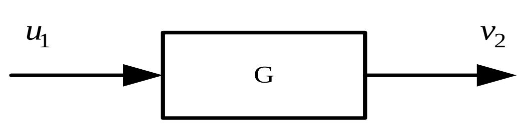


Where the input u_1 is proportional to the output v_1.

Definition 3.45 Non-linear System

A system is called nonlinear when the output does not directly vary with respect to the input (Figure 3.7). Here the superposition property is not applied. Non-linear systems do not have static linearity as well as Sinusoidal Fidelity.

Figure 3.7: Non-Linear System



Where the input u_1 is not proportional to the output v_2.

Definiiton 3.46 Reason of Using Fuzzy Logic

In controlling the system, even though neural network and genetic algorithms are performing well in real world applications, the fuzzy logic is the most applicable technique as it has the convenience of obtaining solution where human operators can perceive and apply their skill in constructing the controller with automatic operations.

Definition 3.47 Fuzzy Logic Control System (FLCS)

A system does not have authentic models for the reason of uncertainty and absence of pure knowledge. Measurements of uncertainty do not naturally have stochastic explosion or noise designs. Considering the above points, researchers are motivated to use fuzzy logic in control systems to handle uncertainty. Control system works with fuzzy logic is called fuzzy logic control system or fuzzy logic controller which contributes the conversion of linguistic control approach using IF-THEN rules.

Definition 3.48 Traffic Signal Control

It is a pre-timed or induced or flexible control and is described as follows.

Pre-timed Controllers: These types of controllers decide the signal timings in advance which are collected from earlier pattern of traffic and repeat the same.

Actuated Controllers: These will identify the moving and interrupted traffic on each lane towards cross roads and estimate the duration of the signal phase.

Adaptive Controllers: These identify the entire cross roads and modify the signal phase and response timings to real time traffic.

Definition 3.49 Levels of Signal Control

The fuzziness of signal control can be classified into three levels namely input, control and output level and are described as follows.

Input Level: Here a partial picture of the succeeding traffic environment will be done using measurements.

Control Level: At this level there will be various possibilities and unable to deciding the right or the best possibility since the relationship of source and reaction of the signal control cannot be explained

Output Level: Here the exact criteria of the control are not known such as extension gap.

Definition 3.50 Fuzzy Logic in Traffic Control System

As fuzzy logic is theoretically easy to understand, adaptable, lenient with uncertain data which can design nonlinear functions of inconsistent complexity and also can be built with the knowledge of the experts, and flexible with traditional control approach, thus a fuzzy logic based control system has been a successful pursuit to implement intelligence in a traffic control system.

A nonlinear mapping of an input data set to scalar output is called a fuzzy logic system. It consists of four parts namely fuzzifier, fuzzy rules, inference engine and defuzzifier. The fuzzy system converts the input to the output. Hereby, the linguistic values are divided into fuzzy sets and traffic flow can be defined as low, high and medium. The degree of addiction to every fuzzy set is shown by membership functions.

The input value of the fuzzy system may exist in more than one fuzzy set. The corresponding numeric values to fuzzy set are called fuzzification and the reverse is called defuzzification. Fuzzy IF-THEN rules are the main logic of the inference system and involve vague reasoning. Fuzzy rules are well defined using an expert's knowledge, hence a mathematical model is not necessary for the objects and the system is very flexible.

The parameters of the membership functions and its values, operators, fuzzy rules, defuzzification and other parameters can be modified according to the desired result. IT2FLSs are used to recognize control laws to minimize control errors.

The output of this system, called the control signal, is supposed to be monotonic with respect to the error and/or variation of the error called inputs of the system. The fuzzy control is found to be preferable in complicated problems with multi-objective decisions. Various traffic flows contest for the same time and space and various preferences are frequently set to different flows or vehicle groups.

Definition 3.51 Role of Membership Functions

Application of membership function is an essential role in the stage of fuzzification and defuzzification of the fuzzy logic system to calculate the nonfuzzy input values to fuzzy linguistic terms and for the converse. It is used to measure the linguistic term.

An amazing characteristic of the fuzzy logic lying in the fuzzification of the numerical value is that it need not be fuzzified using only one membership function, and hence the value can be described by different sets at a particular time.

Definition 3.52 Premises of Fuzzy Logic Controller

To design a fuzzy logic controller, the following premises need to be taken care. (i). Input, output and state variables should be accessible for the measurement and controlling aspiration. (ii). Knowledge body which has linguistic rules and a set of input-output data from which rules can be excerpted, supposed to be existed. (iii). There must be a solution. (iv). A good enough solution supposed to be expected rather than an optimized one. (v). Fuzzy Logic Controller must be modeled with in sufficient range of rigor. (vi). Issues regarding stability and optimality must be taken care while designing the controller.

Definition 3.53 Fuzzy Interval

An interval of continuous real numbers defined by two portraits respectively attained by an increasing and decreasing component of the membership functions is called fuzzy interval. A continuous number from this interval takes an element in every alpha cut.

Definition 3.54 Type-2 Fuzzy Logic Systems (T2FLSs)

T2FLSs have advantages of type-1 FLSs as well as captures high level of uncertainties, producing complex input-output functions and better results as well. Here the computational complexity is more and theoretical analysis is difficult. IT2FS reduces all the difficulties discussed above and hence it has been used in designing and control fields. Using primary and secondary membership functions uncertainties can be dealt effectively in easy manner and stability of the system also can be retained.

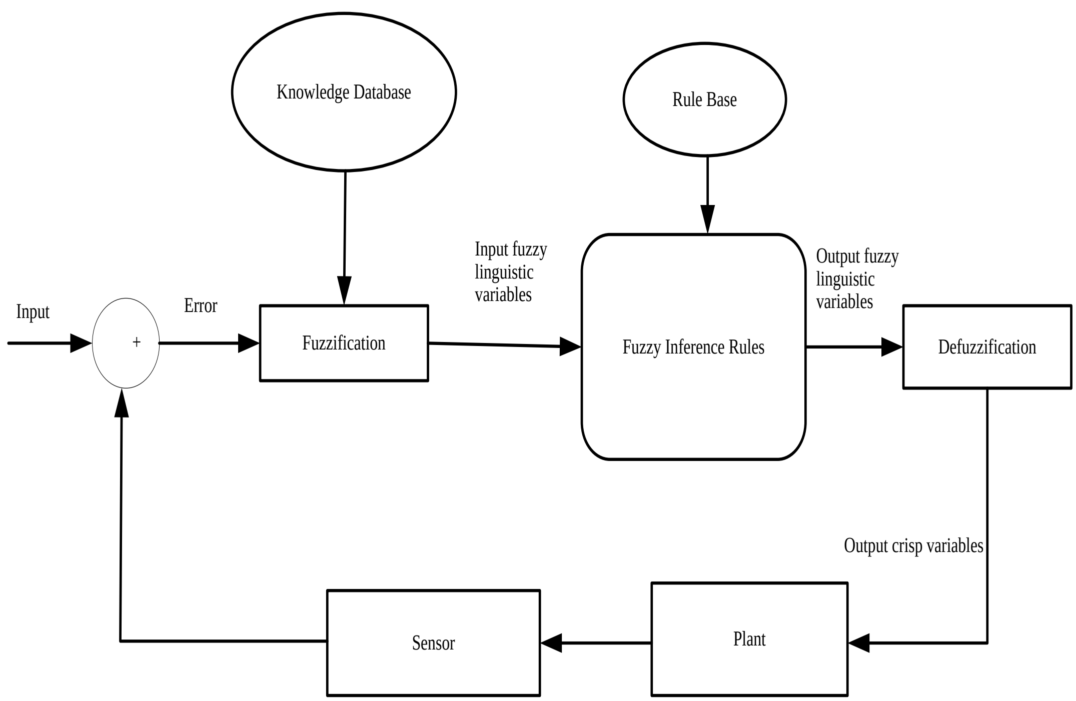
Definition 3.55 Components of Fuzzy Logic System (FLS)

Rule base, Fuzzy Inference Engine (FIE), Fuzzifier and Defuzzifier are the main components of the FLS. These four components are used to choose fuzzy rule which shows the human thinking, judgment and perception, to combine rules to develop a scaling from crisp inputs to T2FS as outputs, Gaussian Fuzzifier to simplify the computation in the FIE when the MFs in the IF-THEN rules are Gaussian and a mapping from fuzzy set to crisp point and calculates the crisp Output respectively.

Definition 3.56 Architecture of Fuzzy Logic Controller (FLC)

The design of the fuzzy logic controller (Figure 3.8) is given designed to under stand the stages and process of the system under fuzzy environment.

Figure 3.8: Design of Fuzzy Logic Controller



Definition 3.57 Stability of a Control System

Stability is an essential factor for all the control systems. If the output of the system is under control then the system is known as a stable one. Otherwise it is unstable. Usually stable system gives a constrained output for a given constrained input.

Definition 3.58 Types of Stability

There are three types of systems based on stability namely absolutely stable (stability exists for every dimension of the system component values), conditionally stable (stable only for a certain dimension of the system component values) and marginally stable (only by producing an output signal along with consistent amplitude and frequency of vibrations for bounded input).

Definition 3.59 Interval Type-2 Takagi-Sugeno Fuzzy Model (IT2TSFM) based Control System (CS)

IT2TSFM based CS is described as follows. Consider p rules for the system where antecedent parts are type-2 fuzzy sets and the consequent part is a crisp number and the rule is defined by Rule i : If $f_1(x(t))$ is $\overline{M}_1^i$ and,…,and $f_\psi(x(t))$ is $\overline{M}_\psi^i$ then $\dot{x}(t) = A_i x(t) + B_i u(t)$, where $\overline{M}_\alpha^i$ is an Interval type-2 fuzzy set of rule i corresponding to the function $f_\alpha(x(t))$, $\alpha = 1, 2, 3, ..., \psi$, $i = 1, 2, 3, ..., p$, ψ is a positive integer, $A_i \in \Re^{n\times n}$ and $B_i \in \Re^{n\times m}$ are constant system and input matrices respectively. $x(t) \in \Re^{n\times 1}$ is the system state vector and $u(t) \in \Re^{m\times 1}$ is the input vector. The firing strength of the i th rule if the following interval sets: $\omega_i(x(t)) = \left[\omega_i^L(x(t)), \omega_i^U(x(t))\right]$, $i = 1, 2, 3, ..., p.$ Here

$\omega_i^L(x(t)) = \underline{\mu}_{\overline{M}_1^i}\left(f_1(x(t))\right) \times \underline{\mu}_{\overline{M}_2^i}\left(f_2(x(t))\right) \times ... \times \underline{\mu}_{\overline{M}_\psi^i}\left(f_\psi(x(t))\right) \geq 0,$

$$\omega_i^U(x(t)) = \bar{\mu}_{\overline{M}_1^i}(f_1(x(t))) \times \bar{\mu}_{\overline{M}_2^i}(f_2(x(t))) \times \ldots \times \bar{\mu}_{\overline{M}_\psi^i}(f_\psi(x(t))) \geq 0,$$

$\underline{\mu}_{\overline{M}_\alpha^i}(f_\alpha(x(t))) \in [0,1]$, $\bar{\mu}_{\overline{M}_\alpha^i}(f_\alpha(x(t))) \in [0,1]$ denotes the upper and lower grades of membership governed by the lower and upper membership functions respectively.

Definition 3.60 Delay Differential Equation (DDE)

A DDE problem has the form

$$\dot{r}(t) = f(t, \mathrm{r}(t-\tau_1), \ldots, \mathrm{r}(t-\tau_n)),\ t \geq t_0 \tag{3.29}$$

$$r(t) = \gamma(t),\ t \leq t_0, \tag{3.30}$$

where $r(t)$ is the physical quantity which changes over the time and depend also on the past and $\phi(t)$ is the history function (HF). There may be one or more solutions depending on the regularity of HF.

DDE is distinguished for its prompt outcome from the moment of its existence. It is applied widely to model the genetic aspects like after- effect and to intend the case that the velocity of the system leans not only on the state at inclined instant but also on the prehistory of the path until this instant.

Differential equations with plays a vital role in system designing in the field of biology, physics, engineering and other sciences whereas ordinary differential equations fails to deal the system which has imprecise knowledge. Delay differential systems are playing an essential portrayal in various control methods. Mathematical models can be made closer to the real fact using delay equations.

Definition 3.61 Types of delays

Generally there are four types of delays defined.

(i). Constant delay (τ =constant)

(ii). Time reliant delay $(\tau = \tau(t))$

(iii). State dependent delay $\left(\tau=\tau\left(t,r\left(t\right)\right)\right)$

(iv). Neutral delay $\left(\tau=\tau\left(t,r\left(t\right),\dot{r}\left(t\right)\right)\right)$

Definition 3.62 Fuzzy Delay Differential Equation (FDDE)

Let $\wp$ be the Liu process and for some given functions $\mathrm{A},\mathrm{B}:[0,\infty)\times\wp_q\to R$ the FDDE is

$$dX\left(t\right)=A\left(t,X_t\right)dt+B\left(t,X_t\right)d\wp\left(t\right),\ t\geq\tau \tag{3.31}$$

where $X\left(t\right)=\gamma\left(t-\tau\right),\tau-q\leq t\leq\tau.$

If the given functions are continuous then $X:\left[\tau-q,\mathrm{b}\right)\times\Omega\to R$ is a solution of FDDE if and only if

$$X\left(t\right)=\begin{cases}\phi\left(t-\tau\right) & ,\quad \tau-q\leq t\leq\tau\\ \gamma\left(0\right)+\int_{\tau}^{t}A\left(s,X_s\right)ds+\int_{\tau}^{t}B\left(s,X_s\right)d\wp\left(s\right) & ,\quad \tau\leq t\leq b\end{cases} \tag{3.32}$$

The solution is an unsure process X_t that contents the above equation identically in time t. Since many of the real world problems contain uncertainty in nature, the dynamical systems are designed by FDDE. If the basic design of the miniature depends upon the subjective preferences then fuzzy differential equation is the one way of incorporating the fuzziness exists in the system.

Definition 3.63 Stability analysis for DDE

Consider the Hutchison equation (logistic delay equation)

$\dot{H}\left(t\right)=H\left(t\right)\left[1-H\left(t\tau\right)\right]$ which has two steady states $H_0^*=0,H^*=1$. If linearized at $H^*:H\left(t\right)=1+H(t)$ then $\frac{dH\left(t\right)}{dt}\approx-H\left(t-\tau\right)$. The solutions of $H\left(t\right)$ in the form

$H(t)=e^{\lambda t}$ is the characteristic equation $\lambda=-e^{-\lambda\tau}$. Here the zeros of λ are called the characteristic roots and the conditions for stability are,

(i). the fixed point solution P^* is stable if and only if $\mathrm{Re}(\lambda)<0, \forall\lambda$.

(ii). If the roots are of complex form $\lambda=\mu+i\omega$ then

$$\mu=-e^{-\mu\tau}\cos(\omega\tau), \omega=e^{-\mu\tau}\sin(\omega\tau).$$

(iii). Here stability is affected by the values of τ. When $\mu(\tau)$ first crosses the imaginary axis will lead to stability loss.

(iv).The condition for stability is $0<\tau<\frac{\pi}{2}$.

(v). when $\tau>\frac{\pi}{2}$, oscillatory behavior will take place.

Definition 3.64 Fuzzy Delayed Control System (FDCS)

For each $s=1,2,...,r$, a FDCS is defined by the plant rule

$$s: IF\ \delta_1(t) is\, M_{1s}\ AND ... AND\, \delta_p(t) is\, M_{hs}$$

$$THEN\ \dot{x}(t)=E_s\, x(t)+F_s\, x(t-\tau_s(t))+G_s u(t) \tag{3.33}$$

where $\delta_1(t),...,\delta_h(t)$ are the proposition variables and every $M_{is}\ (i=1,2,...,\mathrm{h}; s=1,2,...,r)$ is the FS with respect to $\delta_i(t)$ and s. Let $M_{is}[\delta_i(t)]$ be the MF of M_{is} at the position $\delta_i(t)$ and is defined by

$$\omega_s[\delta(t)]=\prod_{i=1}^{h} M_{is}[\delta_i(t)], s=1,2,...,r. \tag{3.34}$$

The resulting delayed FCS is concluded as the weighted average of the local model for the given pair of $\left[x(t),u(t)\right]$ is

$$\dot{x}(t)=\frac{\sum_{s=1}^{r}\omega_s\left[\delta(t)\right]\left[E_s x(t)+F_s x\left(t-\tau_s(t)\right)+G_s u(t)\right]}{\sum_{s=1}^{r}\omega_s\left[\delta(t)\right]},$$

$$=\sum_{s=1}^{r}k_s\left[\delta(t)\right]\left[E_s x(t)+F_s x\left(t-\tau_s(t)\right)+G_s u(t)\right],\ t\geq 0 \tag{3.35}$$

Where $$k_s\left[\delta(t)\right]=\frac{\omega_s\left[\delta(t)\right]}{\sum_{i=1}^{r}\omega_i\left[\delta(t)\right]}, \tag{3.36}$$

$$k_s\left[\delta(t)\right]\geq 0,\sum_{s=1}^{r}k_s\left[\delta(t)\right]=1 \tag{3.37}$$

In the local system $\dot{x}(t)$,

$$x(t)=\left[x_1(t),x_2(t),...,x_n(t)\right]^T \tag{3.38}$$

$$E_s=\left(e_{ij}^s\right)_{n\times n},F_s=\left(f_{ij}^s\right)_{n\times n},G_s=\left(g_{ij}^s\right)_{n\times m} \tag{3.39}$$

Using MFs, the delayed local systems are evenly linked to design a universal nonlinear delayed FDCS. For every system of delayed FCS, there must exist a constant $\tau>0$ such that $0\leq\tau_s(t)\leq\tau$, $s=1,2,...,r$. Assume the initial value for every solution to be $x(t)=\gamma(t),t\in\left[-\tau,0\right]$. The vector continuous function is $\delta(t)=\left[\delta_1(t),\delta_2(t),...,\delta_n(t)\right]^T$ and

$$\|\gamma\|=\sup_{-\tau\leq\theta\leq 0}\sqrt{\gamma_1^2(\theta)+\gamma_2^2(\theta)+...+\gamma_n^2(\theta)} \tag{3.40}$$

Definition 3.65 Conditions for the stable system

For a real matrix M, $\lambda_{\max}(M)$ and $\lambda_{\min}(M)$ are larger and smaller Eigen values of M respectively. The spectral norm is defined by

$$\|M\| = \sqrt{\lambda_{\max}\left(M^T M\right)} \tag{3.41}$$

For a matrix Y

(i). if $Y > 0$ for $Y \in R^{n\times n}$, then Y is real symmetric positive definite (Stability occurs but unstable)

(ii). if $Y \geq 0$ for $Y \in R^{n\times n}$, then Y is real symmetric positive semi definite (Unstable)

(iii). If Y is not clearly declared then all the matrices considered to have appropriate range for algebraic operations (Stable)

Definition 3.66 Construction of Fuzzy Logic

There are three distinct types of structures in constructing fuzzy logic namely fuzzy sets (FSs), fuzzy variables (FVs) and fuzzy rules (FRs). A FS is decisive by a MF of a FV which generates values between 0 and 1. The relationship between the antecedent and consequent FVs will actuate FR and are generally described by ordinary language etymological terms. A rule accomplice an action about linguistic variables to a consequence is called fuzzy IF-THEN rule.

Definition 3.67 Fuzzy Logic Controller

Fuzzy engine has the tendency of imitating human behavior. To establish practical control systems, it is found that Fuzzy logic is the most fruitful automation which empowers control engineers to progress control planning in a potential manner.

A system which is convoluted, poorly defined, nonlinear or time shifting can be handled by fuzzy logic controller (FLC). There is requirement to have a mathematical exemplary of the control system and therefore implementation of FLC is comparably accessible and it has three different parts namely fuzzifier, rule base and defuzzifier.

The performance of each case is described as follows.

Fuzzifier converts the numeric value into FSs and hence the process is called fuzzification. An essential component of the FLC is the inference engine, which executes all logic administrations. The rule base dwells with MFs and control rules.

Finally the defuzzifier revamps the FS into a numeric value as one should get a numeric output and hence the procedure is called defuzzification. In this study, Takagi-Sugeno method for fuzzification and fuzzy clustering defuzzification method for defuzzification have been used.

Definition 3.68 Type-1 Fuzzy Logic Controller

In T1FLC, input variables are mapped into FSs (sets of MFs). The MF is the graphical portrayal of the degree of every input and ultimately completes an output response.

The nature of the MFs is supposedly based on numerical character and the extension of MFs. Fuzzy rule will be followed in the form of IF-THEN statement.

Here, the membership values of an input will be considered to carry the aspect for regulating their effect on the fuzzy output as sets in the final result. Once the function that runs the system are fixed, proportioned and connected then the crisp output will be occurred after defuzzification.

Definition 3.69 Type-2 Fuzzy Logic Controller

In type-2 fuzzy logic (T2FL) there are two types of membership grades (MGs) namely primary and secondary which takes values from 0 to 1. For every primary grade there is a secondary MG which defines the possibilities of primary MGs.

Generally in the interval type-2 fuzzy logic system (IT2FLS) both primary and secondary MGs will be the interval sets. Since T2FL system has an intensive computational part it is widely applied in control system for the smooth performance.

T2FLC system has two kinds of MFs namely lower and upper which are represented by T1FS. T2FS also meant for interval based settings. The interval between superior and inferior MFs called footprint of uncertainty (FOU) and which represents the uncertainty level of the problem.

Definition 3.70 Fuzzy Inference System (FIS)

To handle a system with uncertain inputs or uncertain system, ordinary mathematical mechanism like differential equation is not suitable as the human decision is always uncertain in nature. By disparity, the subjective conditions of human knowledge and reasoning technique can be designed by FIS without engaging explicitly with quantitative investigation called FIS or fuzzy modeling or fuzzy identification.

The core part of FIS is IF-THEN rules. Which is also called fuzzy conditional statements expressed by the form IF P THEN Q , where P and Q are the fuzzy set labels represented by membership functions (MFs) appropriately. FRs also engaged the sequence of IF-THEN statements.

By considering the following statement, if pressure is high then volume is small. Where the pressure and volume in antecedent part (AP) are lingual variables, high and small in the consequent part (CP) are the linguistic values or labels defined by MFs.

By using Takagi-Sugeno Fuzzy Model (TSFM), another fuzzy IF-THEN rule for the opposing force on a moving substance can be expressed by, if velocity is high, then force $k * (velocity)^2$. In this statement the AP is again high which is fuzzy and the CP is a non-fuzzy equation of an input variable velocity. Both of the above cases have been applied in modeling as well as in controlling of the system extensively.

The complex nonlinear system can be represented by fuzzy sets and the fuzzy reasoning uses an impressive method called TSFM. Moreover, it is an interesting and a smooth model since it allows an easy association of local linear system for designing the global nonlinear systems by using MFs. In the field of fuzzy control, some designing methods were introduced along with an expansion of fuzzy systems.

Definition 3.71 Effects of delays in control system

The existence of delays usually become the source of instability and deteriorating performance of the systems and therefore the stability of the system depends on the delay. Equipment of the system, transmission of the signal, size of the system variable, actuators, sensors and the field networks are reasons for getting delays in the control system.

Time delay: A system with time delay performs as a class of infinite dimensional system and used to depict the reproduction and transit development or population gesture. Non-linear systems with time delay can be analyzed by the typical method called linearization and fuzzy logic theory for stabilizing the system.

The designed controllers assure asymptotic stability, where the stability occurs while the component of the control system is operating but also when component failure exists as well. Usually time delay occurs due to the failing in transit and delay in collaborating the assessment sensor and actuator of the control.

It is a serious problem and shall be dealt with appropriate measure otherwise it will affect the stability of the system. Since the outcome and effects are bound to happen determined by some time interval, so the delay in economic activity is natural. This kind of delay often occurs in biological system, network systems, rolling mill systems and metallurgical processing systems.

Neutral delay: The system with neutral delay also affects the stability and therefore it becomes a more complex issue as it affects the derivative of the delayed state. It can be handled by using Linear Matrix Inequality (LMI), which is able to solve stability analysis and stabilization. Using LMI, the stability of the system can be analyzed for FCSs with bounded fuzzy delays. Here the results are independent of the length of the time delay. Neutral DDE can be applied when the dynamical system established on present and past states and involve derivatives with delay.

State delay: The delay in state and input of the variable also induces complex behavior of the system and obviously leads to stability problems.

Sensor delay: Sensor delay is called as a system along with outpouring controller and unanimity feedback although employing an output sensor with τ seconds late in releasing an output.

Actuator delay: If the delay arises while transfer the output of the controller then the delay is called actuator delay

Definition 3.72 Fuzzy Logic in Washing Machine

In our daily life, WM is one of the important appliances for household chores. The amount of washing time consuming is always based on the factors such as the type, the grade of dirt and also the number of cloths. This will be regulated by the sensors. But there is no standard mathematical concurrence between these parameters. In the early time, we tend to set an inexact time put to allocate for the running of WM, so we had the experience of error based on it. The controller regulates the outside input system as shown by sensors. This design can be recorded by fuzzy logic to help the system in making decisions similar to human behaviors by using interval type-2 fuzzy logic controller (IT2FLC).

Definition 3.73 Saturation Time

The saturation time of stain in water and for vice versa will be calculated by an imprecise manner and an absurdity of water level with the help of visible sensor. The inferred visual sensor is measured in the basis of fuzzy rule. The time of saturation can be effectively measured using IT2FLC.

Definition 3.74 Introduction of the inputs in interval type-2 fuzzy logic washing machine

The weight of the cloths decided by the knowledge of the washing machine user. The more amounts of the cloths necessitates the more consumption of the detergent. The type of dirt is determined by the saturation time, the time to reach saturation. The point of saturation is a point at which there are no further detectable changes in the color of water. The type of dirt determines the quality/nature of dirt such as the excess level of grease, the medium level of grease or no content of grease at all.

Greasy cloths take large amount of detergent according to clearness/ the purity level of the water since greasy type needs more amount of detergent for an effective washing process than the dirt of other types. Hence, the straight forward sensor system can supply us the necessary input for the interval type-2 fuzzy logic controller.

CHAPTER-4

RESULT

4.0 RESULT

4.1 INTERVAL TYPE-2 FUZZY SETS AND INTERVAL NEUTROSOPHIC SETS IN TRAFFIC CONTROL MANAGEMENT

In this section, traffic control management has been analyzed under IT2FS and INS environments. Operational laws (OLs) and aggregation operators have been derived with their properties in detail using SS triangular norms (TNs). Further traffic control management has been analyzed by considering traffic flow by TIT2FNs and INNs.

4.1a OPERATIONAL LAWS UNDER INTERVAL TYPE-2 FUZZY PERSPECTIVE

The following operational laws have been described using TIT2FNs based on SS TNs.

Consider three TIT2FNs $\overline{F}, \overline{F_1}, \overline{F_2}$ and $\varphi > 0$.

4.1a.1 ADDITION

$$\overline{F_1} \oplus \overline{F_2} = \left\langle \left[\left(sum\left(\underline{l_{F_i}}\right)^{\varphi} - prod\left(\underline{l_{F_i}}\right)^{\varphi} \right)^{\frac{1}{\varphi}}, \left(sum\left(\overline{l_{F_i}}\right)^{\varphi} - prod\left(\overline{l_{F_i}}\right)^{\varphi} \right)^{\frac{1}{\varphi}} \right], \right.$$

$$\left(sum\left(c_{F_i}\right)^{\varphi} - prod\left(c_{F_i}\right)^{\varphi} \right)^{\frac{1}{\varphi}},$$

$$\left. \left[\left(sum\left(\underline{r_{F_i}}\right)^{\varphi} - prod\left(\underline{r_{F_i}}\right)^{\varphi} \right)^{\frac{1}{\varphi}}, \left(sum\left(\overline{r_{F_i}}\right)^{\varphi} - prod\left(\overline{r_{F_i}}\right)^{\varphi} \right)^{\frac{1}{\varphi}} \right] \right\rangle \qquad (4.1)$$

where $\sum_{i=1}^{2} = \mathrm{S}$, $\prod_{i=1}^{2} = \wp$

4.1a.1(i) NUMERICAL VALIDATION

If $\overline{F_1} = \langle [0.4, 0.5], 0.6, [0.7, 0.8] \rangle$ and $\overline{F_2} = \langle [0.5, 0.6], 0.7, [0.8, 0.9] \rangle$ be any two TIT2FNs and if $\varphi = 2$ then

$$\overline{F_1} \oplus \overline{F_2} = \left\langle \left[\left((0.4)^2 + (0.5)^2 - (0.4)^2 . (0.5)^2 \right)^{1/2}, \left((0.5)^2 + (0.6)^2 - (0.5)^2 . (0.6)^2 \right)^{1/2} \right], \right.$$

$$\left((0.6)^2 + (0.7)^2 - (0.6)^2 . (0.7)^2 \right)^{1/2},$$

$$\left. \left[\left((0.7)^2 + (0.8)^2 - (0.7)^2 . (0.8)^2 \right)^{1/2}, \left((0.8)^2 + (0.9)^2 - (0.8)^2 . (0.9)^2 \right)^{1/2} \right] \right\rangle$$

$$= \langle [0.6096, 0.7211], 0.8207, [0.9035, 0.9652] \rangle = \text{TIT2FN}$$

4.1a.2 MULTIPLICATION

$$\overline{F_1} \otimes \overline{F_2} = \left\langle \left[1 - \left(\mathrm{S}\left(1 - \underline{l_{F_i}}\right)^{\varphi} - \wp\left(1 - \underline{l_{F_i}}\right)^{\varphi} \right)^{\frac{1}{\varphi}}, 1 - \left(\mathrm{S}\left(1 - \overline{l_{F_i}}\right)^{\varphi} - \wp\left(1 - \overline{l_{F_i}}\right)^{\varphi} \right)^{\frac{1}{\varphi}} \right], \right.$$

$$1 - \left(\mathrm{S}\left(1 - c_{F_i}\right)^{\varphi} - \wp\left(1 - c_{F_i}\right)^{\varphi} \right)^{\frac{1}{\varphi}},$$

$$\left. \left[1 - \left(\mathrm{S}\left(1 - \underline{r_{F_i}}\right)^{\varphi} - \wp\left(1 - \underline{r_{F_i}}\right)^{\varphi} \right)^{\frac{1}{\varphi}}, 1 - \left(\mathrm{S}\left(1 - \overline{r_{F_i}}\right)^{\varphi} - \wp\left(1 - \overline{r_{F_i}}\right)^{\varphi} \right)^{\frac{1}{\varphi}} \right] \right\rangle \quad (4.2)$$

where $\sum_{i=1}^{2} = \mathrm{S}$, $\prod_{i=1}^{2} = \wp$

4.1a.2(i) NUMERICAL VALIDATION

$$\overline{F_1}\otimes\overline{F_2}=\left\langle\left[1-\left((1-0.4)^2+(1-0.5)^2-(1-0.4)^2.(1-0.5)^2\right)^{1/2},\right.\right.$$

$$\left.1-\left((1-0.5)^2+(1-0.6)^2-(1-0.5)^2.(1-0.6)^2\right)^{1/2}\right]$$

$$1-\left((1-0.6)^2+(1-0.7)^2-(1-0.6)^2.(1-0.7)^2\right)^{1/2},$$

$$\left[1-\left((1-0.7)^2+(1-0.8)^2-(1-0.7)^2.(1-0.8)^2\right)^{1/2},\right.$$

$$\left.\left.1-\left((1-0.8)^2+(1-0.9)^2-(1-0.8)^2.(1-0.9)^2\right)^{1/2}\right]\right\rangle$$

$$=\langle[0.2789,0.3917],0.5146,[0.6445,0.777]\rangle=\text{TIT2FN}$$

4.1a.3 MULTIPLICATION BY AN ORDINARY NUMBER

$$v.\overline{F_1}=\left\langle\left[\left(\left(\underline{l_{F_1}}\right)^{\varphi}\right)^{\frac{v}{\varphi}},\left(\left(\overline{l_{F_1}}\right)^{\varphi}\right)^{\frac{v}{\varphi}}\right],\left(\left(c_{F_1}\right)^{\varphi}\right)^{\frac{v}{\varphi}},\left[\left(\left(\underline{r_{F_1}}\right)^{\varphi}\right)^{\frac{v}{\varphi}},\left(\left(\overline{r_{F_1}}\right)^{\varphi}\right)^{\frac{v}{\varphi}}\right]\right\rangle \tag{4.3}$$

Here, v is an ordinary number.

4.1a.3(i) NUMERICAL VALIDATION

Let $\overline{F_1}=\langle[0.4,0.5],0.6,[0.7,0.8]\rangle$ and $v=0.3$

$$0.3.\overline{F_1}=\left\langle\left[\left((0.4)^2\right)^{\frac{0.3}{2}},\left((0.5)^2\right)^{\frac{0.3}{2}}\right],\left((0.6)^2\right)^{\frac{0.3}{2}},\left[\left((0.7)^2\right)^{\frac{0.3}{2}},\left((0.8)^2\right)^{\frac{0.3}{2}}\right]\right\rangle$$

$$=\langle[0.2789,0.3917],0.5146,[0.6445,0.777]\rangle=\text{TIT2FN}$$

4.1a.4 POWER OPERATION

$$\overline{F_1}^{\,v} = \left\langle \left[1-\left(\left(1-\underline{l}_{F_1}\right)^{\varphi}\right)^{\frac{v}{\varphi}}, 1-\left(\left(1-\overline{l_{F_1}}\right)^{\varphi}\right)^{\frac{v}{\varphi}} \right], \right.$$

$$\left. 1-\left(\left(1-c_{F_1}\right)^{\varphi}\right)^{\frac{v}{\varphi}}, \left[1-\left(\left(1-\underline{r}_{F_1}\right)^{\varphi}\right)^{\frac{v}{\varphi}}, 1-\left(\left(1-\overline{r_{F_1}}\right)^{\varphi}\right)^{\frac{v}{\varphi}} \right] \right\rangle \tag{4.4}$$

4.1a. 4(i) NUMERICAL VALIDATION

Let $v = 0.3$ and $\overline{F_1}$

$$\overline{F_1}^{\,0.3} = \left\langle \left[1-\left(\left(1-0.4\right)^{2}\right)^{\frac{0.3}{2}}, 1-\left(\left(1-0.5\right)^{2}\right)^{\frac{0.3}{2}} \right], 1-\left(\left(1-0.6\right)^{2}\right)^{\frac{0.3}{2}}, \right.$$

$$\left. \left[1-\left(\left(1-0.7\right)^{2}\right)^{\frac{0.3}{2}}, 1-\left(\left(1-0.8\right)^{2}\right)^{\frac{0.3}{2}} \right] \right\rangle$$

$$= \left\langle [0.1421, 0.1877], 0.2403, [0.3032, 0.383] \right\rangle = \text{TIT2FN}$$

4.1a. 5 PROPOSED THEOREMS USING TIT2SSWG OPERATOR

Here, the SS operator under triangular interval type-2 setting has been developed and proposed as a triangular interval type-2 Schweizer and Sklar weighted geometric (TIT2SSWG) operator based on SS triangular norms.

4.1a.5(i) Theorem

Let $\overline{F_i} = \left(\left[\underline{l}_{F_i}, \overline{l_{F_i}}\right], c_{F_i}, \left[\underline{r}_{F_i}, \overline{r_{F_i}}\right]\right)$, $i = 1, 2, \ldots, n$ be a set of TIT2FNs; then their aggregated value using TIT2SSWG operator is still a TIT2FN, $0 \le \underline{l}_{F_i} \le \overline{l_{F_i}} \le c_{F_i} \le \underline{r}_{F_i} \le \overline{r_{F_i}} \le 1, i = 1, 2, \ldots, n$ and

$$TIT2SSWG_{\varpi}\left(\overline{F_1},\overline{F_2},...,\overline{F_n}\right)$$

$$=\left\langle\left[1-\left(\mathrm{S}\left(1-\underline{l_{F_i}}\right)^{\varphi}-\wp\left(1-\underline{l_{F_i}}\right)^{\varphi}\right)^{\frac{\varpi_i}{\varphi}},1-\left(\mathrm{S}\left(1-\overline{l_{F_i}}\right)^{\varphi}-\wp\left(1-\overline{l_{F_i}}\right)^{\varphi}\right)^{\frac{\varpi_i}{\varphi}}\right],\right.$$

$$1-\left(\mathrm{S}\left(1-c_{F_i}\right)^{\varphi}-\wp\left(1-c_{F_i}\right)^{\varphi}\right)^{\frac{\varpi_i}{\varphi}},$$

$$\left.\left[1-\left(\mathrm{S}\left(1-\underline{r_{F_i}}\right)^{\varphi}-\wp\left(1-\underline{r_{F_i}}\right)^{\varphi}\right)^{\frac{\varpi_i}{\varphi}},1-\left(\mathrm{S}\left(1-\overline{r_{F_i}}\right)^{\varphi}-\wp\left(1-\overline{r_{F_i}}\right)^{\varphi}\right)^{\frac{\varpi_i}{\varphi}}\right]\right\rangle$$

where $\sum_{i=1}^{n}=\mathrm{S}$, $\prod_{i=1}^{n}=\wp$

Proof:

By mathematical induction method, we prove this theorem.

For $n=2$

Consider the power operation

$$\overline{F}^{v_1}=\left\langle\left[1-\left(\left(1-\underline{l_F}\right)^{\varphi}\right)^{\frac{v_1}{\varphi}},1-\left(\left(1-\overline{l_F}\right)^{\varphi}\right)^{\frac{v_1}{\varphi}}\right],1-\left(\left(1-c_F\right)^{\varphi}\right)^{\frac{v_1}{\varphi}},\left[1-\left(\left(1-\underline{r_F}\right)^{\varphi}\right)^{\frac{v_1}{\varphi}},1-\left(\left(1-\overline{r_F}\right)^{\varphi}\right)^{\frac{v_1}{\varphi}}\right]\right\rangle$$

$$\overline{F}^{v_2}=\left\langle\left[1-\left(\left(1-\underline{l_F}\right)^{\varphi}\right)^{\frac{v_2}{\varphi}},1-\left(\left(1-\overline{l_F}\right)^{\varphi}\right)^{\frac{v_2}{\varphi}}\right],1-\left(\left(1-c_F\right)^{\varphi}\right)^{\frac{v_2}{\varphi}},\left[1-\left(\left(1-\underline{r_F}\right)^{\varphi}\right)^{\frac{v_2}{\varphi}},1-\left(\left(1-\overline{r_F}\right)^{\varphi}\right)^{\frac{v_2}{\varphi}}\right]\right\rangle$$

$$TIT2WG_{\varpi}\left(\overline{F_1},\overline{F_2}\right)=\overline{F_1}^{\varpi_1}\otimes\overline{F_1}^{\varpi_2}$$

$$=\left\langle\left[1-\left(\mathrm{S}\left(1-\underline{l_{F_i}}\right)^{\varphi}-\wp\left(1-\underline{l_{F_i}}\right)^{\varphi}\right)^{\frac{\varpi_i}{\varphi}},1-\left(\mathrm{S}\left(1-\overline{l_{F_i}}\right)^{\varphi}-\wp\left(1-\overline{l_{F_i}}\right)^{\varphi}\right)^{\frac{\varpi_i}{\varphi}}\right],\right.$$

$$1-\left(S\left(1-c_{F_i}\right)^{\varphi}-\wp\left(1-c_{F_i}\right)^{\varphi}\right)^{\frac{\varpi_i}{\varphi}},$$

$$\left.\left[1-\left(S\left(1-\underline{r_{F_i}}\right)^{\varphi}-\wp\left(1-\underline{r_{F_i}}\right)^{\varphi}\right)^{\frac{\varpi_i}{\varphi}},1-\left(S\left(1-\overline{r_{F_i}}\right)^{\varphi}-\wp\left(1-\overline{r_{F_i}}\right)^{\varphi}\right)^{\frac{\varpi_i}{\varphi}}\right]\right\rangle$$

where $\sum_{i=1}^{2}= S$, $\prod_{i=1}^{2}=\wp$

For $n=k$

$$TIT2SSWG_{\varpi}\left(\overline{F_1},\overline{F_2},...,\overline{F_k}\right)=\overline{F_1}^{\varpi_1}\otimes\overline{F_2}^{\varpi_2}\otimes...\otimes\overline{F_k}^{\varpi_k}$$

$$=\left\langle\left[1-\left(S\left(\underline{l_{F_i}}\right)^{\varphi}-\wp\left(\underline{l_{F_i}}\right)^{\varphi}\right)^{\frac{\varpi_i}{\varphi}},1-\left(S\left(\overline{l_{F_i}}\right)^{\varphi}-\wp\left(\overline{l_{F_i}}\right)^{\varphi}\right)^{\frac{\varpi_i}{\varphi}}\right],1-\left(S\left(1-c_{F_i}\right)^{\varphi}-\wp\left(1-c_{F_i}\right)^{\varphi}\right)^{\frac{\varpi_i}{\varphi}},\right.$$

$$\left.\left[1-\left(S\left(1-\underline{r_{F_i}}\right)^{\varphi}-\wp\left(1-\underline{r_{F_i}}\right)^{\varphi}\right)^{\frac{\varpi_i}{\varphi}},1-\left(S\left(1-\overline{r_{F_i}}\right)^{\varphi}-\wp\left(1-\overline{r_{F_i}}\right)^{\varphi}\right)^{\frac{\varpi_i}{\varphi}}\right]\right\rangle$$

where $\sum_{i=1}^{k}=S$ and $\prod_{i=1}^{k}=\wp$

For $n=k+1$

$$TIT2SSWG_{\varpi}\left(\overline{F_1},\overline{F_2},...,\overline{F_{k+1}}\right)=\left(\overline{F_1}^{\varpi_1}\otimes\overline{F_2}^{\varpi_2}\otimes...\otimes\overline{F_k}^{\varpi_k}\right)\otimes\overline{F_{k+1}}^{\varpi_{k+1}}$$

$$=\left\langle\left[1-\left(S\left(\underline{l_{F_i}}\right)^{\varphi}-\wp\left(\underline{l_{F_i}}\right)^{\varphi}\right)^{\frac{\varpi_i}{\varphi}},1-\left(S\left(\overline{l_{F_i}}\right)^{\varphi}-\wp\left(\overline{l_{F_i}}\right)^{\varphi}\right)^{\frac{\varpi_i}{\varphi}}\right],1-\left(S\left(1-c_{F_i}\right)^{\varphi}-\wp\left(1-c_{F_i}\right)^{\varphi}\right)^{\frac{\varpi_i}{\varphi}},\right.$$

$$\left[1-\left(\mathrm{S}\left(1-\underline{r_{F_i}}\right)^{\varphi}-\wp\left(1-\underline{r_{F_i}}\right)^{\varphi}\right)^{\frac{\varpi_i}{\varphi}},1-\left(\mathrm{S}\left(1-\overline{r_{F_i}}\right)^{\varphi}-\wp\left(1-\overline{r_{F_i}}\right)^{\varphi}\right)^{\frac{\varpi_i}{\varphi}}\right]\Bigg\rangle$$

$$\otimes\left\langle\left[1-\left(\left(1-\underline{l_{F_{k+1}}}\right)^{\varphi}\right)^{\frac{\varpi_{k+1}}{\varphi}},1-\left(\left(1-\overline{l_{S_{k+1}}}\right)^{\varphi}\right)^{\frac{\varpi_{k+1}}{\varphi}}\right],1-\left(\left(1-c_{F_{k+1}}\right)^{\varphi}\right)^{\frac{\varpi_{k+1}}{\varphi}},\right.$$

$$\left.\left[1-\left(\left(1-\underline{r_{F_{k+1}}}\right)^{\varphi}\right)^{\frac{\varpi_{k+1}}{\varphi}},1-\left(\left(1-\overline{r_{F_{k+1}}}\right)^{\varphi}\right)^{\frac{\varpi_{k+1}}{\varphi}}\right]\right\rangle$$

$$=\left\langle\left[1-\left(\mathrm{S}\left(1-\underline{l_{F_i}}\right)^{\varphi}-\wp\left(1-\underline{l_{F_i}}\right)^{\varphi}\right)^{\frac{\varpi_i}{\varphi}},1-\left(\mathrm{S}\left(1-\overline{l_{F_i}}\right)^{\varphi}-\wp\left(1-\overline{l_{F_i}}\right)^{\varphi}\right)^{\frac{\varpi_i}{\varphi}}\right],\right.$$

$$1-\left(\mathrm{S}\left(1-c_{F_i}\right)^{\varphi}-\wp\left(1-c_{F_i}\right)^{\varphi}\right)^{\frac{\varpi_i}{\varphi}},$$

$$\left.\left[1-\left(\mathrm{S}\left(1-\underline{r_{F_i}}\right)^{\varphi}-\wp\left(1-\underline{r_{F_i}}\right)^{\varphi}\right)^{\frac{\varpi_i}{\varphi}},1-\left(\mathrm{S}\left(1-\overline{r_{F_i}}\right)^{\varphi}-\wp\left(1-\overline{r_{F_i}}\right)^{\varphi}\right)^{\frac{\varpi_i}{\varphi}}\right]\right\rangle$$

where $\sum_{i=1}^{k+1}=\mathrm{S}$ and $\prod_{i=1}^{k+1}=\wp$

Hence, the result is true for all values of n.

4.1a.5(ii) NUMERICAL VALIDATION

For $n=2$ the calculation has been given and the computation is similar for all the values of n. In addition, consider the weight vector $\varpi_1=0.55$ and $\varpi_2=0.45$. Without loss of generality, take $\varphi=2$ throughout the work.

$$TIT2SSWG_{\varpi}\left(\overline{F_1},\overline{F_2}\right)=\overline{F_1}^{0.55}\otimes\overline{F_2}^{0.45}$$

$$=\left\langle\left[1-\left((1-0.4)^2+(1-0.5)^2-(1-0.4)^2.(1-0.5)^2\right)^{\frac{0.55+0.45}{2}},\right.\right.$$

$$\left.1-\left((1-0.5)^2+(1-0.6)^2-(1-0.5)^2.(1-0.6)^2\right)^{\frac{0.55+0.45}{2}}\right]$$

$$1-\left((1-0.6)^2+(1-0.7)^2-(1-0.6)^2.(1-0.7)^2\right)^{\frac{0.55+0.45}{2}},$$

$$\left[1-\left((1-0.7)^2+(1-0.8)^2-(1-0.7)^2.(1-0.8)^2\right)^{\frac{0.55+0.45}{2}},\right.$$

$$\left.\left.1-\left((1-0.8)^2+(1-0.9)^2-(1-0.8)^2.(1-0.9)^2\right)^{\frac{0.55+0.45}{2}}\right]\right\rangle$$

$$=\langle[0.48,0.63],0.7644,[0.8736,0.9504]\rangle=\text{TIT2FN}$$

4.1a.5(iii) Theorem (Idempotency)

If all $\overline{F_i},\ i=1,2,...,n$ are equal, i.e., $\overline{F_i}=\overline{F}$ then $TIT2SSWG_{\varpi}\left(\overline{F_1},\overline{F_2},...,\overline{F_n}\right)=\overline{F}$.

Proof:

$TIT2SSWG_{\varpi}\left(\overline{F_1},\overline{F_2},...,\overline{F_n}\right)$

$$=\left\langle\left[1-\left(\mathrm{S}\left(1-\underline{l_{F_i}}\right)^{\varphi}-\wp\left(1-\underline{l_{S_t}}\right)^{\varphi}\right)^{\frac{\varpi_i}{\varphi}},1-\left(\mathrm{S}\left(1-\overline{l_{S_t}}\right)^{\varphi}-\wp\left(1-\overline{l_{S_t}}\right)^{\varphi}\right)^{\frac{\varpi_i}{\varphi}}\right],\right.$$

$$1-\left(\mathrm{S}\left(1-m_{S_t}\right)^{\varphi}-\wp\left(1-m_{S_t}\right)^{\varphi}\right)^{\frac{\varpi_i}{\varphi}},$$

$$\left[1-\left(\mathrm{S}\left(1-\underline{r}_{F_i}\right)^{\varphi}-\wp\left(1-\underline{r}_{F_i}\right)^{\varphi}\right)^{\frac{\varpi_i}{\varphi}},1-\left(\mathrm{S}\left(1-\overline{r}_{F_i}\right)^{\varphi}-\wp\left(1-\overline{r}_{F_i}\right)^{\varphi}\right)^{\frac{\varpi_i}{\varphi}}\right]\right\rangle$$

$$=\left\langle\left[1-\left(\left(1-\underline{l}_{F_i}\right)^{\varphi}\right)^{\frac{\sum_{i=1}^{n}\varpi_i}{\varphi}},1-\left(\left(1-\overline{l}_{F_i}\right)^{\varphi}\right)^{\frac{\sum_{i=1}^{n}\varpi_i}{\varphi}}\right],\left(1-\left(1-c_{F_i}\right)^{\varphi}\right)^{\frac{\sum_{i=1}^{n}\varpi_i}{\varphi}},\right.$$

$$\left.\left[1-\left(\left(1-\underline{r}_{F_i}\right)^{\varphi}\right)^{\frac{\sum_{i=1}^{n}\varpi_i}{\varphi}},1-\left(\left(1-\overline{r}_{F_i}\right)^{\varphi}\right)^{\frac{\sum_{i=1}^{n}\varpi_i}{\varphi}}\right]\right\rangle$$

$$=\left\langle\left[1-\left(\left(1-\underline{l}_{F_i}\right)^{\varphi}\right)^{\frac{1}{\varphi}},1-\left(\left(1-\overline{l}_{F_i}\right)^{\varphi}\right)^{\frac{1}{\varphi}}\right],1-\left(\left(1-m_{S_t}\right)^{\varphi}\right)^{\frac{1}{\varphi}},\right.$$

$$\left.\left[1-\left(\left(1-\underline{r}_{F_i}\right)^{\varphi}\right)^{\frac{1}{\varphi}},1-\left(\left(1-\overline{r}_{F_i}\right)^{\varphi}\right)^{\frac{1}{\varphi}}\right]\right\rangle$$

$$=\left\langle\left[\left(1-\underline{l}_{F_i}\right)^{\varphi},\left(1-\overline{l}_{F_i}\right)^{\varphi}\right],\left(1-c_{F_i}\right)^{\varphi},\left[\left(1-\underline{r}_{F_i}\right)^{\varphi},\left(1-\overline{r}_{F_i}\right)^{\varphi}\right]\right\rangle$$

$$=\left\langle\left[\left(1-\underline{l}_{F_i}\right),\left(1-\overline{l}_{F_i}\right)\right],\left(1-c_{F_i}\right),\left[\left(1-\underline{r}_{F_i}\right),\left(1-\overline{r}_{F_i}\right)\right]\right\rangle$$

$$=\left\langle\left[\underline{l}_{F_i},\overline{l}_{F_i}\right],c_{F_i},\left[\underline{r}_{F_i},\overline{r}_{F_i}\right]\right\rangle$$

$$=\left\langle\left[\underline{l}_{F},\overline{l}_{F}\right],c_{F},\left[\underline{r}_{F},\overline{r}_{F}\right]\right\rangle$$

$=\overline{F}$, where $\sum_{i=1}^{n}=\mathrm{S}$ and $\prod_{i=1}^{n}=\wp$

4.1a.5(iv) Theorem

If $\overline{S}_{n+1} = \left(\left[\underline{l_{S_{n+1}}}, \overline{l_{S_{n+1}}}\right], m_{S_{n+1}}, \left[\underline{r_{S_{n+1}}}, \overline{r_{S_{n+1}}}\right]\right)$ is a TIT2FN on X then,

$TIT2SSWG_{\varpi}\left(\overline{F_1}\otimes\overline{S_{n+1}}, \overline{F_2}\otimes\overline{F_{n+1}}, ..., \overline{F_n}\otimes\overline{F_{n+1}}\right) = TIT2SSWG_{\varpi}\left(\overline{F_1}, \overline{F_2}, ..., \overline{F_n}\right)\otimes\overline{F_{n+1}}$.

Proof:

$$\text{Since, } \overline{F_i}\otimes\overline{F_{n+1}} = \left\langle \left[1-\left(\mathrm{S}^*\left(1-\underline{l_{F_t}}\right)^{\varphi} - \wp^*\left(1-\underline{l_{F_t}}\right)^{\varphi}\right)^{\frac{1}{\varphi}}, 1-\left(\mathrm{S}^*\left(1-\overline{l_{F_t}}\right)^{\varphi} - \wp^*\left(1-\overline{l_{F_t}}\right)^{\varphi}\right)^{\frac{1}{\varphi}}\right],\right.$$

$$1-\left(\mathrm{S}^*\left(1-c_{F_t}\right)^{\varphi} - \wp^*\left(1-c_{F_t}\right)^{\varphi}\right)^{\frac{1}{\varphi}},$$

$$\left.\left[1-\left(\mathrm{S}^*\left(1-\underline{r_{F_t}}\right)^{\varphi} - \wp^*\left(1-\underline{r_{F_t}}\right)^{\varphi}\right)^{\frac{1}{\varphi}}, 1-\left(\mathrm{S}^*\left(1-\overline{r_{F_t}}\right)^{\varphi} - \wp^*\left(1-\overline{r_{F_t}}\right)^{\varphi}\right)^{\frac{1}{\varphi}}\right]\right\rangle$$

where $\mathrm{S}^* = \sum_{t=i,n+1}$, $\wp^* = \prod_{t=i,n+1}$

Consider the LHS,

$$TIT2SSWG_{\varpi}\left(\overline{F_1}\otimes\overline{S_{n+1}}, \overline{F_2}\otimes\overline{F_{n+1}}, ..., \overline{F_n}\otimes\overline{F_{n+1}}\right)$$

$$= \left\langle \left[1-\left(\mathrm{S}\left(\mathrm{S}^*\left(1-\underline{l_{F_i}}\right)^{\varphi}\right) - \wp\left(\wp^*\left(1-\underline{l_{F_i}}\right)^{\varphi}\right)\right)^{\frac{\varpi_i}{\varphi}}, 1-\left(\mathrm{S}\left(\mathrm{S}^*\left(1-\overline{l_{F_i}}\right)\right)^{\xi} - \wp\left(\wp^*\left(1-\overline{l_{F_i}}\right)\right)^{\varphi}\right)^{\frac{\varpi_i}{\varphi}}\right],\right.$$

$$1-\left(\mathrm{S}\left(\mathrm{S}^*\left(1-c_{F_i}\right)^{\varphi}\right) - \wp\left(\wp^*\left(1-c_{F_i}\right)^{\varphi}\right)\right)^{\frac{\varpi_i}{\varphi}},$$

$$\left.\left[1-\left(\mathrm{S}\left(\mathrm{S}^*\left(1-\underline{r_{F_i}}\right)^{\varphi}\right) - \wp\left(\wp^*\left(1-\underline{r_{F_i}}\right)^{\varphi}\right)\right)^{\frac{\varpi_i}{\varphi}}, 1-\left(\mathrm{S}\left(\mathrm{S}^*\left(1-\overline{r_{F_i}}\right)\right)^{\xi} - \wp\left(\wp^*\left(1-\overline{r_{F_i}}\right)\right)^{\varphi}\right)^{\frac{\varpi_i}{\varphi}}\right]\right\rangle$$

$$=\left\langle\left[1-\left(\mathrm{S}\left(1-\underline{l_{F_i}}\right)^{\varphi}+\left(1-\underline{l_{F_{n+1}}}\right)^{\varphi}-\wp\left(1-\underline{l_{F_i}}\right)^{\varphi}\left(1-\underline{l_{F_{n+1}}}\right)^{\varphi}\right)^{\frac{\varpi_i}{\varphi}},\right.\right.$$

$$\left.1-\left(\mathrm{S}\left(1-\overline{l_{F_i}}\right)^{\varphi}+\left(1-\overline{l_{F_{n+1}}}\right)^{\varphi}-\wp\left(1-\overline{l_{F_i}}\right)^{\varphi}\left(1-\overline{l_{F_{n+1}}}\right)^{\varphi}\right)^{\frac{\varpi_i}{\varphi}}\right],$$

$$1-\left(\mathrm{S}\left(1-c_{F_i}\right)^{\varphi}+\left(1-c_{F_{n+1}}\right)^{\varphi}-\wp\left(1-c_{F_i}\right)^{\varphi}\left(1-c_{F_{n+1}}\right)^{\varphi}\right)^{\frac{\varpi_i}{\varphi}},$$

$$\left[1-\left(\mathrm{S}\left(1-\underline{r_{F_i}}\right)^{\varphi}+\left(1-\underline{r_{F_{n+1}}}\right)^{\varphi}-\wp\left(1-\underline{r_{F_t}}\right)^{\varphi}\left(1-\underline{r_{F_{n+1}}}\right)^{\varphi}\right)^{\frac{\varpi_i}{\varphi}},\right.$$

$$\left.\left.1-\left(\mathrm{S}\left(1-\overline{r_{F_i}}\right)^{\varphi}+\left(1-\overline{r_{F_{n+1}}}\right)^{\varphi}-\wp\left(1-\overline{r_{F_i}}\right)^{\varphi}\left(1-\overline{r_{F_{n+1}}}\right)^{\varphi}\right)^{\frac{\varpi_i}{\varphi}}\right]\right\rangle$$

$$=\left\langle\left[1-\left(\mathrm{S}\left(\left(1-\underline{l_{F_i}}\right)^{\varphi}+\left(1-\underline{l_{F_{n+1}}}\right)^{\varphi}\right)-\wp\left(\left(1-\underline{l_{F_i}}\right)^{\varphi}+\left(1-\underline{l_{F_{n+1}}}\right)^{\varphi}\right)\right)^{\frac{\varpi_i}{\varphi}},\right.\right.$$

$$\left.1-\left(\mathrm{S}\left(\left(1-\overline{l_{F_i}}\right)^{\varphi}+\left(1-\overline{l_{F_{n+1}}}\right)^{\varphi}\right)-\wp\left(\left(1-\overline{l_{F_i}}\right)^{\varphi}+\left(1-\overline{l_{F_{n+1}}}\right)^{\varphi}\right)\right)^{\frac{\varpi_i}{\varphi}}\right],$$

$$1-\left(\mathrm{S}\left(\left(1-c_{F_i}\right)^{\varphi}+\left(1-c_{F_{n+1}}\right)^{\varphi}\right)-\wp\left(\left(1-c_{F_i}\right)^{\varphi}+\left(1-c_{F_{n+1}}\right)^{\varphi}\right)\right)^{\frac{\varpi_i}{\varphi}},$$

$$\left[1-\left(\mathrm{S}\left(\left(1-\underline{r_{F_i}}\right)^{\varphi}+\left(1-\underline{r_{F_{n+1}}}\right)^{\varphi}\right)-\wp\left(\left(1-\underline{r_{F_i}}\right)^{\varphi}+\left(1-\underline{r_{F_{n+1}}}\right)^{\varphi}\right)\right)^{\frac{\varpi_i}{\varphi}},\right.$$

$$1-\left(\mathrm{S}\left(\left(1-\overline{r_{F_i}}\right)^{\varphi}+\left(1-\overline{r_{F_{n+1}}}\right)^{\varphi}\right)-\wp\left(\left(1-\overline{r_{F_i}}\right)^{\varphi}+\left(1-\overline{r_{F_{n+1}}}\right)^{\varphi}\right)\right)^{\frac{\varpi_i}{\varphi}}\Bigg]\Bigg\rangle \tag{4.5}$$

RHS= $TIT2SSWG_{\varpi}\left(\overline{F_1},\overline{F_2},...,\overline{F_n}\right)\otimes\overline{F_{n+1}}$

$$=\Bigg\langle\Bigg[1-\left(\mathrm{S}\left(1-\underline{l_{F_i}}\right)^{\varphi}-\wp\left(1-\underline{l_{F_i}}\right)^{\varphi}\right)^{\frac{\varpi_i}{\varphi}},\left(\mathrm{S}\left(1-\overline{l_{F_i}}\right)^{\varphi}-\wp\left(1-\overline{l_{F_i}}\right)^{\varphi}\right)^{\frac{\varpi_i}{\varphi}}\Bigg],$$

$$1-\left(\mathrm{S}\left(1-c_{F_i}\right)^{\varphi}-\wp\left(1-c_{F_i}\right)^{\varphi}\right)^{\frac{\varpi_i}{\varphi}},$$

$$\Bigg[1-\left(\mathrm{S}\left(1-\underline{r_{F_i}}\right)^{\varphi}-\wp\left(1-\underline{r_{F_i}}\right)^{\varphi}\right)^{\frac{\varpi_i}{\varphi}},\left(\mathrm{S}\left(1-\overline{r_{F_i}}\right)^{\varphi}-\wp\left(1-\overline{r_{F_i}}\right)^{\varphi}\right)^{\frac{\varpi_i}{\varphi}}\Bigg]\Bigg\rangle\oplus$$

$$\Bigg\langle\Bigg[1-\left(\left(1-\underline{l_{F_{n+1}}}\right)^{\varphi}\right)^{\frac{\varpi_i}{\varphi}},1-\left(\left(1-\overline{l_{F_{n+1}}}\right)^{\varphi}\right)^{\frac{\varpi_i}{\varphi}}\Bigg],\left(1-\left(1-c_{F_{n+1}}\right)^{\varphi}\right)^{\frac{\varpi_i}{\varphi}},$$

$$\Bigg[1-\left(\left(1-\underline{r_{F_{n+1}}}\right)^{\varphi}\right)^{\frac{\varpi_i}{\varphi}},1-\left(\left(1-\overline{r_{F_{n+1}}}\right)^{\varphi}\right)^{\frac{\varpi_i}{\varphi}}\Bigg]\Bigg\rangle$$

$$=\Bigg\langle\Bigg[1-\left(\mathrm{S}\left(\left(1-\underline{l_{F_i}}\right)^{\varphi}+\left(1-\underline{l_{F_{n+1}}}\right)^{\varphi}\right)-\wp\left(\left(1-\underline{l_{F_i}}\right)^{\varphi}+\left(1-\underline{l_{F_{n+1}}}\right)^{\varphi}\right)\right)^{\frac{\varpi_i}{\varphi}},$$

$$1-\left(\mathrm{S}\left(\left(1-\overline{l_{F_i}}\right)^{\varphi}+\left(1-\overline{l_{F_{n+1}}}\right)^{\varphi}\right)-\wp\left(\left(1-\overline{l_{F_i}}\right)^{\varphi}+\left(1-\overline{l_{F_{n+1}}}\right)^{\varphi}\right)\right)^{\frac{\varpi_i}{\xi}}\Bigg],$$

$$1-\left(\mathrm{S}\left(\left(1-c_{F_i}\right)^{\varphi}+\left(1-c_{F_{n+1}}\right)^{\varphi}\right)-\wp\left(\left(1-c_{F_i}\right)^{\varphi}+\left(1-c_{F_{n+1}}\right)^{\varphi}\right)\right)^{\frac{\varpi_i}{\varphi}}$$

$$\left[1-\left(\mathrm{S}\left(\left(1-\underline{r_{F_i}}\right)^{\varphi}+\left(1-\underline{r_{F_{n+1}}}\right)^{\varphi}\right)-\wp\left(\left(1-\underline{r_{F_i}}\right)^{\varphi}+\left(1-\underline{r_{F_{n+1}}}\right)^{\varphi}\right)\right)^{\frac{\varpi_i}{\varphi}},\right.$$

$$\left.1-\left(\mathrm{S}\left(\left(1-\overline{r_{F_i}}\right)^{\varphi}+\left(1-\overline{r_{F_{n+1}}}\right)^{\varphi}\right)-\wp\left(\left(1-\overline{r_{F_i}}\right)^{\varphi}+\left(1-\overline{r_{F_{n+1}}}\right)^{\varphi}\right)\right)^{\frac{\varpi_i}{\varphi}}\right]\Bigg\rangle \qquad (4.6)$$

From Equation (4.5) and Equation (4.6), the theorem holds.

4.1a.5(v) Theorem

If $k>0$, $\overline{F}_{n+1}=\left(\left[\underline{l_{F_{n+1}}},\overline{l_{F_{n+1}}}\right],c_{F_{n+1}},\left[\underline{r_{F_{n+1}}},\overline{r_{F_{n+1}}}\right]\right)$ is a TIT2FN on X then,

$$TIT2SSWG_{\varpi}\left(\overline{F_1}^{k},\overline{F_2}^{k},...,\overline{F_n}^{k}\right)=\left(TIT2SSWG_{\varpi}\left(\overline{F_1},\overline{F_2},...,\overline{F_n}\right)\right)^{k}$$

Proof:

$$k.\overline{S_i}=\left\langle\left[1-\left(\mathrm{S}\left(1-\underline{l_{F_i}}\right)^{\varphi}\right)^{\frac{k}{\varphi}},1-\left(\wp\left(1-\overline{l_{F_i}}\right)^{\varphi}\right)^{\frac{k}{\varphi}}\right],1-\left(\left(1-c_{F_i}\right)^{\varphi}\right)^{\frac{k}{\varphi}},\right.$$

$$\left.\left[1-\left(\mathrm{S}\left(1-\underline{r_{F_i}}\right)^{\varphi}\right)^{\frac{k}{\varphi}},1-\left(\wp\left(1-\overline{r_{F_i}}\right)^{\varphi}\right)^{\frac{k}{\varphi}}\right]\right.$$

LHS= $TIT2SSWG_{\varpi}\left(\overline{F_1}^{k},\overline{F_2}^{k},...,\overline{F_n}^{k}\right)$

$$=\left\langle\left[1-\left(\left[\mathrm{S}\left(\mathrm{S}\left(1-\underline{l_{F_i}}\right)^{\varphi}\right)-\wp\left(\wp\left(1-\underline{l_{F_i}}\right)^{\varphi}\right)\right]^{k}\right)^{\frac{\varpi_i}{\varphi}},1-\left(\left[\mathrm{S}\left(\mathrm{S}\left(1-\overline{l_{F_i}}\right)^{\xi}\right)-\wp\left(\wp\left(1-\overline{l_{F_i}}\right)^{\varphi}\right)\right]^{k}\right)^{\frac{\varpi_i}{\varphi}}\right],\right.$$

$$1-\left[\left(\left[\mathrm{S}\left(\mathrm{S}\left(1-c_{F_i}\right)^{\varphi}\right)\right]\right)^{k}-\left(\left[\wp\left(\wp\left(1-c_{F_i}\right)^{\varphi}\right)\right]\right)^{k}\right]^{\frac{\varpi_i}{\varphi}},$$

$$\left[1-\left(\left[\mathrm{S}\left(\mathrm{S}\left(1-\underline{r_{F_i}}\right)^{\varphi}\right)-\wp\left(\wp\left(1-\underline{r_{F_i}}\right)^{\varphi}\right)\right]^{k}\right)^{\frac{\varpi_i}{\varphi}},\right.$$

$$\left.1-\left(\left[\mathrm{S}\left(\mathrm{S}\left(1-\overline{r_{F_i}}\right)^{\xi}\right)-\wp\left(\wp\left(1-\overline{r_{F_i}}\right)^{\varphi}\right)\right]^{k}\right)^{\frac{\varpi_i}{\varphi}}\right]\right\rangle$$

$$=\left\langle\left[1-\left(\mathrm{S}\left(1-\underline{l_{F_i}}\right)^{\varphi}-\wp\left(1-\underline{l_{F_i}}\right)^{\varphi}\right)^{\frac{k\varpi_i}{\varphi}},1-\left(\mathrm{S}\left(1-\overline{l_{F_i}}\right)^{\varphi}-\wp\left(1-\overline{l_{F_i}}\right)^{\varphi}\right)^{\frac{k\varpi_i}{\varphi}}\right],\right.$$

$$1-\left(\mathrm{S}\left(1-c_{F_i}\right)^{\varphi}-\wp\left(1-c_{F_i}\right)^{\varphi}\right)^{\frac{k\varpi_i}{\varphi}},$$

$$\left.\left[1-\left(\mathrm{S}\left(1-\underline{r_{F_i}}\right)^{\varphi}-\wp\left(1-\underline{r_{F_i}}\right)^{\varphi}\right)^{\frac{k\varpi_i}{\varphi}},1-\left(\mathrm{S}\left(1-\overline{r_{F_i}}\right)^{\varphi}-\wp\left(1-\overline{r_{F_i}}\right)^{\varphi}\right)^{\frac{k\varpi_i}{\varphi}}\right]\right\rangle \qquad (4.7)$$

$$\text{RHS} = \left(TIT2SSWG_{\varpi}\left(\overline{F_1},\overline{F_2},...,\overline{F_n}\right)\right)^{k}$$

$$=\left\langle\left[1-\left(\mathrm{S}\left(1-\underline{l_{F_i}}\right)^{\varphi}-\wp\left(1-\underline{l_{F_i}}\right)^{\varphi}\right)^{\frac{k\varpi_i}{\varphi}},1-\left(\mathrm{S}\left(1-\overline{l_{F_i}}\right)^{\varphi}-\wp\left(1-\overline{l_{F_i}}\right)^{\varphi}\right)^{\frac{k\varpi_i}{\varphi}}\right],\right.$$

$$1-\left(\mathrm{S}\left(1-c_{F_i}\right)^{\varphi}-\wp\left(1-c_{F_i}\right)^{\varphi}\right)^{\frac{k\varpi_i}{\varphi}},$$

$$\left.\left[1-\left(\mathrm{S}\left(1-\underline{r_{F_i}}\right)^{\varphi}-\wp\left(1-\underline{r_{F_i}}\right)^{\varphi}\right)^{\frac{k\varpi_i}{\varphi}},1-\left(\mathrm{S}\left(1-\overline{r_{F_i}}\right)^{\varphi}-\wp\left(1-\overline{r_{F_i}}\right)^{\varphi}\right)^{\frac{k\varpi_i}{\varphi}}\right]\right\rangle$$

$$
=\left\langle\left[1-\left(S\left(1-\underline{l}_{F_i}\right)^{\varphi}-\wp\left(1-\underline{l}_{F_i}\right)^{\varphi}\right)^{\frac{k\varpi_i}{\varphi}},1-\left(S\left(1-\overline{l_{F_i}}\right)^{\varphi}-\wp\left(1-\overline{l_{F_i}}\right)^{\varphi}\right)^{\frac{k\varpi_i}{\varphi}}\right],\right.
$$

$$
1-\left(S\left(1-c_{F_i}\right)^{\varphi}-\wp\left(1-c_{F_i}\right)^{\varphi}\right)^{\frac{k\varpi_i}{\varphi}},
$$

$$
\left.\left[1-\left(S\left(1-\underline{r}_{F_i}\right)^{\varphi}-\wp\left(1-\underline{r}_{F_i}\right)^{\varphi}\right)^{\frac{k\varpi_i}{\varphi}},1-\left(S\left(1-\overline{r_{F_i}}\right)^{\varphi}-\wp\left(1-\overline{r_{F_i}}\right)^{\varphi}\right)^{\frac{k\varpi_i}{\varphi}}\right]\right\rangle \qquad (4.8)
$$

From Equation (4.7) and Equation (4.8),

$$
TIT2SSWG_{\varpi}\left(\overline{F_1}^k,\overline{F_2}^k,...,\overline{F_n}^k\right)=\left(TIT2SSWG_{\varpi}\left(\overline{F_1},\overline{F_2},...,\overline{F_n}\right)\right)^k
$$

4.1a.5(vi) Theorem (Stability)

If $\overline{F}_{n+1}=\left(\left[\underline{l}_{F_{n+1}},\overline{l_{F_{n+1}}}\right],c_{F_{n+1}},\left[\underline{r}_{F_{n+1}},\overline{r_{F_{n+1}}}\right]\right)$ is a TIT2FN on X. If $k>0$ then,

$$
TIT2SSWG_{\varpi}\left(\overline{F_1}^k\otimes\overline{F_{n+1}},\overline{F_2}^k\otimes\overline{F_{n+1}},...,\overline{F_n}^k\otimes\overline{F_{n+1}}\right)
$$
$$
=\left(TIT2SSWG_{\varpi}\left(\overline{F_1},\overline{F_2},...,\overline{F_n}\right)\right)^k\otimes\overline{F_{n+1}}.
$$

Proof:

From Theorem 4.1a.5(v) and Theorem 4.1a.5(vi),

$$
TIT2SSWG_{\varpi}\left(\overline{F_1}^k\otimes\overline{F_{n+1}},\overline{F_2}^k\otimes\overline{F_{n+1}},...,\overline{F_n}^k\otimes\overline{F_{n+1}}\right)=\left(TIT2SSWG_{\varpi}\left(\overline{F_1},\overline{F_2},...,\overline{F_n}\right)\right)^k\otimes\overline{F_{n+1}}
$$

is true.

4.1a.5(vii) Theorem

Let $\overline{F_i} = \left(\left[\underline{l_{F_i}}, \overline{l_{F_i}}\right], c_{F_i}, \left[\underline{r_{F_i}}, \overline{r_{F_i}}\right]\right),\ i = 1, 2, \ldots, n$ be a set of TIT2FNs $0 \le \underline{l_{F_i}} \le \overline{l_{F_i}} \le c_{F_i} \le \underline{r_{F_i}} \le \overline{r_{F_i}} \le 1, i = 1, 2, \ldots, n$; then their aggregated value using TIT2SSWA operator is still a TIT2FN, and

$$TIT2SSWA_{\varpi}\left(\overline{F_1}, \overline{F_2}, \ldots, \overline{F_n}\right)$$

$$= \left\langle \left[\left(\mathrm{S}\left(\underline{l_{F_i}}\right)^{\varphi} - \wp\left(\underline{l_{F_i}}\right)^{\varphi}\right)^{\frac{\varpi_i}{\varphi}}, \left(\mathrm{S}\left(\overline{l_{F_i}}\right)^{\varphi} - \wp\left(\overline{l_{F_i}}\right)^{\varphi}\right)^{\frac{\varpi_i}{\varphi}}\right], \left(\mathrm{S}\left(c_{F_i}\right)^{\varphi} - \wp\left(c_{F_i}\right)^{\varphi}\right)^{\frac{\varpi_i}{\varphi}},\right.$$

$$\left.\left[\left(\mathrm{S}\left(1 - \underline{r_{F_i}}\right)^{\varphi} - \wp\left(\underline{r_{F_i}}\right)^{\varphi}\right)^{\frac{\varpi_i}{\varphi}}, \left(\mathrm{S}\left(\overline{r_{F_i}}\right)^{\varphi} - \wp\left(\overline{r_{F_i}}\right)^{\varphi}\right)^{\frac{\varpi_i}{\varphi}}\right]\right\rangle, \text{ where } \sum_{i=1}^{n} = \mathrm{S},\ \prod_{i=1}^{n} = \wp$$

Proof of this theorem and the following theorems can be performed in a similar way as for theorems using IT2SSWG operator.

4.1a.5(viii) If all $\overline{F_i},\ i = 1, 2, \ldots, n$ are equal, i.e., $\overline{F_i} = \overline{F}$ then

$$TIT2SSWA_{\varpi}\left(\overline{F_1}, \overline{F_2}, \ldots, \overline{F_n}\right) = \overline{F}.$$

4.1a.5(ix) $TIT2SSWA_{\varpi}\left(\overline{F_1} \oplus \overline{F_{n+1}}, \overline{F_2} \oplus \overline{F_{n+1}}, \ldots, \overline{F_n} \oplus \overline{F_{n+1}}\right)$

$$= TIT2SSWA_{\varpi}\left(\overline{F_1}, \overline{F_2}, \ldots, \overline{F_n}\right) \oplus \overline{F_{n+1}}.$$

4.1a.5(x) $TIT2SSWA_{\varpi}\left(k.\overline{F_1}, k.\overline{F_2}, \ldots, k.\overline{F_n}\right) = k.TIT2SSWA_{\varpi}\left(\overline{F_1}, \overline{F_2}, \ldots, \overline{F_n}\right)$.

4.1a.5(xi) $TIT2SSWA_{\varpi}\left(k.\overline{F_1} \oplus \overline{F_{n+1}}, k.\overline{F_2} \oplus \overline{F_{n+1}}, \ldots, k.\overline{F_n} \oplus \overline{F_{n+1}}\right)$

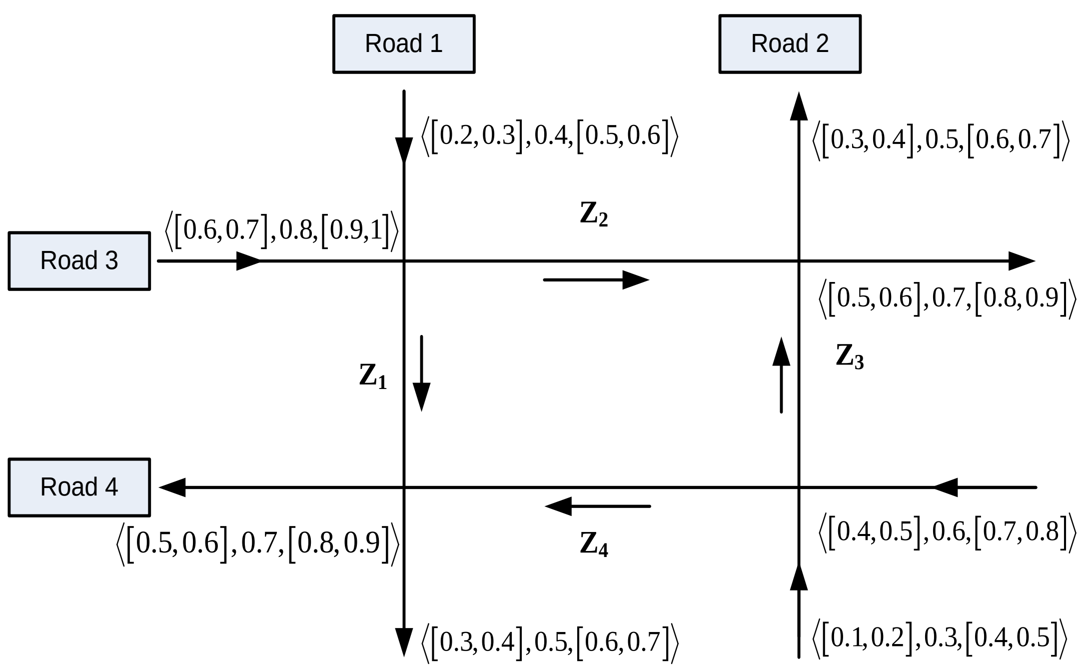
$$= k.TIT2SSWA_{\varpi}\left(\overline{F_1}, \overline{F_2}, \ldots, \overline{F_n}\right) \oplus \overline{F_{n+1}}.$$

4.1a.6 TRAFFIC FLOW CONTROL USING TIT2SSWA OPERATOR

The traffic flow of the junction is considered during rush hour on a working day (Figure 4.1). The arrow marks represent the direction of the flow in each direction. The average number of vehicles per hour coming in and departing at each intersecting point is taken as TIT2FNs instead of crisp numbers.

The aim of this work is to identify the junction that has a higher number of vehicles (traffic) that need to be cleared first using the score value of the IT2FNs, and the result can be concluded based on the greater score value.

Figure 4.1: Traffic flow on the road with four junctions using Triangular Interval Type-2 Fuzzy Numbers



Using Theorem 4.1a.5(vii),

$$TIT2SSWA_{\varpi}(Z_1,Z_2)=(0.45)Z_1\oplus(0.55)Z_2$$

$$=\left\langle\left[\left((0.6)^2+(0.2)^2-(0.6)^2.(0.2)^2\right)^{\frac{0.45+0.55}{2}},\left((0.7)^2+(0.3)^2-(0.7)^2.(0.3)^2\right)^{\frac{0.45+0.55}{2}}\right],\right.$$

$$\left((0.8)^2+(0.4)^2-(0.8)^2.(0.4)^2\right)^{\frac{0.45+0.55}{2}},$$

$$\left.\left[\left((0.9)^2+(0.5)^2-(0.9)^2.(0.5)^2\right)^{\frac{0.45+0.55}{2}},\left((1)^2+(0.6)^2-(1)^2.(0.6)^2\right)^{\frac{0.45+0.55}{2}}\right]\right\rangle$$

$$=\langle[0.62,0.73],0.84,[0.92,1]\rangle$$

Similarly,

$$TIT2SSWA_{\varpi}(Z_2,Z_3)=(0.45)Z_2\oplus(0.55)Z_3=\langle[0.56,0.68],0.78,[0.87,0.9]\rangle$$

$$TIT2SSWA_{\varpi}(Z_3,Z_4)=(0.45)Z_3\oplus(0.55)Z_4=\langle[0.41,0.53],0.65,[0.75,0.85]\rangle$$

$$TIT2SSWA_{\varpi}(Z_4,Z_1)=(0.45)Z_4\oplus(0.55)Z_1=\langle[0.56,0.68],0.79,[0.88,0.95]\rangle$$

Finding the formula for score function of TIT2FNs, the following score values have been obtained.

$$SV(Z_1,Z_2)=\left(\frac{0.62+1}{2}+1\right)\times\frac{0.62+0.73+0.92+1+4(0.84)}{8}=1.5$$

Similarly, $SV(Z_2,Z_3)=1.33$, $SV(Z_3,Z_4)=1.05$, $SV(Z_4,Z_1)=1.37$

From the score values, the junction between Z_1 and Z_2 has a higher value, and therefore it is recommended that this junction has more traffic and may be cleared first.

4.1b OPERATIONAL LAWS FOR INNs USING SS TRINAGULAR NORMS NEUTROSOPHIC PERSPECTIVE

The concept of IT2FSs can be extended to INSs. Fuzzy sets handle only truth and false membership grades whereas neutrosophic sets handle not only truth and false membership grades but also indeterminacy grade, extension of the above results would provide an efficient way of handling uncertainties existing in the real-world problems. The above mentioned theorems have been extended to an interval neutrosophic setting.

The following OLs have been described using INNs based on SS TNs.

Consider two INNs and non zero parameter δ.

4.1b.1 ADDITION

$$
\begin{aligned}
x_1 \oplus x_2 = \Bigg\{ & \Bigg[\left(\left(T_1^L\right)^{\delta} + \left(T_2^L\right)^{\delta} - \left(T_1^L\right)^{\delta} \left(T_2^L\right)^{\delta} \right)^{1/\delta}, \left(\left(T_1^U\right)^{\delta} + \left(T_2^U\right)^{\delta} - \left(T_1^U\right)^{\delta} \left(T_2^U\right)^{\delta} \right)^{1/\delta} \Bigg], \\
& \Bigg[1 - \left(\left(1 - I_1^L\right)^{\delta} + \left(1 - I_2^L\right)^{\delta} - \left(1 - I_1^L\right)^{\delta} \left(1 - I_2^L\right)^{\delta} \right)^{1/\delta}, \\
& \quad 1 - \left(\left(1 - I_1^U\right)^{\delta} + \left(1 - I_2^U\right)^{\delta} - \left(1 - I_1^U\right)^{\delta} \left(1 - I_2^U\right)^{\delta} \right)^{1/\delta} \Bigg], \\
& \Bigg[1 - \left(\left(1 - F_1^L\right)^{\delta} + \left(1 - F_2^L\right)^{\delta} - \left(1 - F_1^L\right)^{\delta} \left(1 - I_2^L\right)^{\delta} \right)^{1/\delta}, \\
& \quad 1 - \left(\left(1 - F_1^U\right)^{\delta} + \left(1 - F_2^U\right)^{\delta} - \left(1 - F_1^U\right)^{\delta} \left(1 - F_2^U\right)^{\delta} \right)^{1/\delta} \Bigg] \Bigg\}.
\end{aligned} \tag{4.9}
$$

4.1b.1(i) NUMERICAL VALIDATION

If $x_1 = \langle [0.7, 0.8], [0.0, 0.1], [0.1, 0.2] \rangle$ and $x_2 = \langle [0.4, 0.5], [0.2, 0.3], [0.3, 0.4] \rangle$ are the two INNs and $\delta = 2$ then

$$x_1 \oplus x_2 = \left\{ \left[\left((0.7)^2 + (0.4)^2 - (0.7)^2 .(0.4)^2 \right)^{1/2}, \left((0.8)^2 + (0.5)^2 - (0.8)^2 .(0.5)^2 \right)^{1/2} \right], \right.$$

$$\left[1 - \left((1-0)^2 + (1-0.2)^2 - (1-0)^2 .(1-0.2)^2 \right)^{1/2}, \right.$$

$$\left. 1 - \left((1-0.1)^2 + (1-0.3)^2 - (1-0.1)^2 .(1-0.3)^2 \right)^{1/2} \right],$$

$$\left[1 - \left((1-0.1)^2 + (1-0.3)^2 - (1-0.1)^2 .(1-0.3)^2 \right)^{1/2}, \right.$$

$$\left. \left. 1 - \left((1-0.2)^2 + (1-0.4)^2 - (1-0.2)^2 .(1-0.4)^2 \right)^{1/2} \right] \right\}$$

$$x_1 \oplus x_2 = \{ [0.76, 0.85], [0.11, 0.13], [0.05, 0.12] \} = \text{INN}$$

4.1b.2 MULTIPLICATION

$$x_1 \otimes x_2 = \left\{ \left[1 - \left(\left(1 - T_1^L\right)^\delta + \left(1 - T_2^L\right)^\delta - \left(1 - T_1^L\right)^\delta \left(1 - T_2^L\right)^\delta \right)^{1/\delta} \right. \right.$$

$$\left. 1 - \left(\left(1 - T_1^U\right)^\delta + \left(1 - T_2^U\right)^\delta - \left(1 - T_1^U\right)^\delta \left(1 - T_2^U\right)^\delta \right)^{1/\delta} \right],$$

$$\left[\left(\left(I_1^L\right)^\delta + \left(I_2^L\right)^\delta - \left(I_1^L\right)^\delta \left(I_2^L\right)^\delta \right)^{1/\delta}, \left(\left(I_1^U\right)^\delta + \left(I_2^U\right)^\delta - \left(I_1^U\right)^\delta \left(I_2^U\right)^\delta \right)^{1/\delta} \right],$$

$$\left. \left[\left(\left(F_1^L\right)^\delta + \left(F_2^L\right)^\delta - \left(F_1^L\right)^\delta \left(I_2^L\right)^\delta \right)^{1/\delta}, \left(\left(F_1^U\right)^\delta + \left(F_2^U\right)^\delta - \left(F_1^U\right)^\delta \left(F_2^U\right)^\delta \right)^{1/\delta} \right] \right\}. \quad (4.10)$$

4.1b.2 (i) NUMERICAL VALIDATION

$$x_1 \otimes x_2 = \left\{ \left[1-\left((1-0.7)^2 + (1-0.4)^2 - (1-0.7)^2 . (1-0.4)^2 \right)^{1/2}, \right. \right.$$

$$\left. 1-\left((1-0.8)^2 + (1-0.5)^2 - (1-0.8)^2 . (1-0.5)^2 \right)^{1/2} \right],$$

$$\left[\left((0)^2 + (0.2)^2 - (0)^2 . (0.2)^2 \right)^{1/2}, \left((0.1)^2 + (0.3)^2 - (0.1)^2 . (0.3)^2 \right)^{1/2} \right],$$

$$\left. \left[\left((0.1)^2 + (0.3)^2 - (0.1)^2 . (0.3)^2 \right)^{1/2}, \left((0.2)^2 + (0.4)^2 - (0.2)^2 . (0.4)^2 \right)^{1/2} \right] \right\}$$

$$x_1 \otimes x_2 = \{[0.35, 0.47], [0.2, 0.31], [0.3148, 0.44]\} = \text{INN}$$

4.1b.3 MULTIPLICATION BY AN ORDINARY NUMBERS

$$g.x_1 = \left\{ \left[\left(g\left(T_1^L\right)^{\delta} \right)^{1/\delta}, \left(g\left(T_1^U\right)^{\delta} \right)^{1/\delta} \right], \left[1-\left(g\left(1-I_1^L\right)^{\delta} \right)^{1/\delta},\ 1-\left(g\left(1-I_1^U\right)^{\delta} \right)^{1/\delta} \right], \right.$$

$$\left. \left[1-\left(g\left(1-F_1^L\right)^{\delta} \right)^{1/\delta}, 1-\left(g\left(1-F_1^U\right)^{\delta} \right)^{1/\delta} \right] \right\} \tag{4.11}$$

4.1b.3(i) NUMERICAL VALIDATION

Consider $g = 0.2$

$$(0.2).x_1 = \left\{ \left[\left(0.2(0.7)^2 \right)^{1/2}, \left(0.2(0.8)^2 \right)^{1/2} \right], \left[1-\left(0.2(1-0.0)^2 \right)^{1/2}, 1-\left(0.2(1-0.1)^2 \right)^{1/2} \right], \right.$$

$$\left. \left[1-\left(0.2(1-0.1)^2 \right)^{1/2}, 1-\left(0.2(1-0.2)^2 \right)^{1/2} \right] \right\}.$$

$$= \{[0.3130, 357], [0.55, 0.5976], [0.5976, 0.6422]\} = \text{INN}$$

4.1b.4 POWER OPERATION

$$x_1^{\,g} = \left\{ \left[1-\left(g\left(1-T_1^L\right)^{\delta} \right)^{1/\delta}, 1-\left(g\left(1-T_1^U\right)^{\delta} \right)^{1/\delta} \right], \left[\left(g\left(I_1^L\right)^{\delta} \right)^{1/\delta},\ \left(g\left(I_1^U\right)^{\delta} \right)^{1/\delta} \right], \right.$$

$$\left. \left[\left(g\left(F_1^L\right)^{\delta} \right)^{1/\delta}, \left(g\left(F_1^U\right)^{\delta} \right)^{1/\delta} \right] \right\} \tag{4.12}$$

4.1b.4 (i) NUMERICAL VALIDATION

Consider $g = 0.2$

$$x_1^{0.2} = \left\{ \left[1-\left(0.2\left(1-0.7\right)^{2} \right)^{1/2}, 1-\left(0.2\left(1-0.8\right)^{2} \right)^{1/2} \right], \left[\left(0.2\left(0.0\right)^{2} \right)^{1/2},\ \left(0.2\left(0.1\right)^{2} \right)^{1/2} \right], \right.$$

$$\left. \left[\left(0.2\left(0.1\right)^{2} \right)^{1/2}, \left(0.2\left(0.2\right)^{2} \right)^{1/2} \right] \right\}.$$

$$= \left\{ \left[0.8658, 0.9106\right], \left[0.0, 0.0447\right], \left[0.0447, 0.0894\right] \right\} = \text{INN}$$

4.1b.5 PROPOSED SCORE FUNCTION

For ranking INNs, a new score function of an interval neutrosophic number (INN) is proposed and defined by

$$SF\left(\overline{F}\right) = \frac{1}{2}\left[\left(T_F^L + T_F^U\right) - \left(I_F^L I_F^U\right) + \left(I_F^U - 1\right)^2 + F_F^U \right] \tag{4.13}$$

where, $\overline{F}$ is the interval neutrosophic number.

4.1b.6 PROPOSED THEOREMS USING INSSWG OPERATOR

Mathematical properties of the INSSWA operator have been derived as theorem in this section.

4.1b.6(i) Theorem

Let $x_j = \left\langle \left[T_j^L, T_j^U\right], \left[I_j^L, I_j^U\right], \left[F_j^L, F_j^U\right] \right\rangle$, $j = 1,2,3,...,n$ be a collection of INNs and their weight vector is $\upsilon = \left(\upsilon_1, \upsilon_2, ..., \upsilon_n\right)$, $\upsilon_i \in [0,1]$ and $\sum_{j=1}^{n} \upsilon_j = 1$. Then, the aggregated value of the interval neutrosophic Schweizer and Sklar weighted geometric (INSSWG) operator is still an INN and

$$INSSWG_{\upsilon}\left(x_1, x_2, ..., x_n\right)$$

$$= \Bigg\{ \left[1 - \left(\left(\mathrm{S}\left(\upsilon_j \left(1 - T_j^L\right)^{\delta}\right) - \wp\left(\upsilon_j \left(1 - T_j^L\right)^{\delta}\right) \right)^{1/\delta}, 1 - \left(\mathrm{S}\left(\upsilon_j \left(1 - T_j^U\right)^{\delta}\right) - \wp\left(\upsilon_j \left(1 - T_j^U\right)^{\delta}\right) \right)^{1/\delta} \right) \right],$$

$$\left[\left(\mathrm{S}\left(\upsilon_{j\mathrm{i}}\left(I_j^L\right)^{\delta}\right) - \wp\left(\upsilon_j \left(I_j^L\right)^{\delta}\right) \right)^{1/\delta}, \left(\mathrm{S}\left(\upsilon_{j\mathrm{i}}\left(I_j^U\right)^{\delta}\right) - \wp\left(\upsilon_j \left(I_j^U\right)^{\delta}\right) \right)^{1/\delta} \right],$$

$$\left[\left(\mathrm{S}\left(\upsilon_j \left(F_j^L\right)^{\delta}\right) - \wp\left(\upsilon_j \left(F_j^L\right)^{\delta}\right) \right)^{1/\delta}, \left(\mathrm{S}\left(\upsilon_j \left(F_j^U\right)^{\delta}\right) - \wp\left(\upsilon_j \left(F_j^U\right)^{\delta}\right) \right)^{1/\delta} \right] \Bigg\},$$

where $\sum_{i=1}^{n} = \mathrm{S}$, $\prod_{i=1}^{n} = \wp$.

Proof:

Mathematical induction is used to prove this theorem.

When $n = 2$, using SS triangular norms we get,

$$INSSWG_{\upsilon}\left(x_1, x_2\right) = x_1^{\upsilon_1} \otimes x_2^{\upsilon_2}$$

$$
\begin{aligned}
&= \Bigg\{ \Bigg[1-\left(\upsilon_1\left(1-T_1^L\right)^{\delta}+\upsilon_2\left(1-T_2^L\right)^{\delta}-\upsilon_1\left(1-T_1^L\right)^{\delta}\upsilon_2\left(1-T_2^L\right)^{\delta}\right)^{1/\delta}, \\
&\qquad 1-\left(\upsilon_1\left(1-T_1^U\right)^{\delta}+\upsilon_2\left(1-T_2^U\right)^{\delta}-\upsilon_1\left(1-T_1^U\right)^{\delta}\upsilon_2\left(1-T_2^U\right)^{\delta}\right)^{1/\delta} \Bigg] \\
&\qquad \Bigg[\left(\upsilon_1\left(I_1^L\right)^{\delta}+\upsilon_2\left(I_2^L\right)^{\delta}-\upsilon_1\left(I_1^L\right)^{\delta}\upsilon_2\left(I_2^L\right)^{\delta}\right)^{1/\delta}, \\
&\qquad\quad \left(\upsilon_1\left(I_1^U\right)^{\delta}+\upsilon_2\left(I_2^U\right)^{\delta}-\upsilon_1\left(I_1^U\right)^{\delta}\upsilon_2\left(I_2^U\right)^{\delta}\right)^{1/\delta} \Bigg], \\
&\qquad \Bigg[\left(\upsilon_1\left(F_1^L\right)^{\delta}+\upsilon_2\left(F_2^L\right)^{\delta}-\upsilon_1\left(F_1^L\right)^{\delta}\upsilon_2\left(I_2^L\right)^{\delta}\right)^{1/\delta}, \\
&\qquad\quad \left(\upsilon_1\left(F_1^U\right)^{\delta}+\upsilon_2\left(F_2^U\right)^{\delta}-\upsilon_1\left(F_1^U\right)^{\delta}\upsilon_2\left(F_2^U\right)^{\delta}\right)^{1/\delta} \Bigg] \Bigg\}. \\
&= \Bigg\{ \Bigg[1-\left(\mathrm{S}\left(\upsilon_j\left(1-T_j^L\right)^{\delta}\right)-\wp\left(\upsilon_j\left(1-T_j^L\right)^{\delta}\right)\right)^{1/\delta}, 1-\left(\mathrm{S}\left(\upsilon_j\left(1-T_j^U\right)^{\delta}\right)-\wp\left(\upsilon_j\left(1-T_j^U\right)^{\delta}\right)\right)^{1/\delta} \Bigg], \\
&\qquad \Bigg[\left(\mathrm{S}\left(\upsilon_{ji}\left(I_j^L\right)^{\delta}\right)-\wp\left(\upsilon_j\left(I_j^L\right)^{\delta}\right)\right)^{1/\delta}, \left(\mathrm{S}\left(\upsilon_{ji}\left(I_j^U\right)^{\delta}\right)-\wp\left(\upsilon_j\left(I_j^U\right)^{\delta}\right)\right)^{1/\delta} \Bigg], \\
&\qquad \Bigg[\left(\mathrm{S}\left(\upsilon_j\left(F_j^L\right)^{\delta}\right)-\wp\left(\upsilon_j\left(F_j^L\right)^{\delta}\right)\right)^{1/\delta}, \left(\mathrm{S}\left(\upsilon_j\left(F_j^U\right)^{\delta}\right)-\wp\left(\upsilon_j\left(F_j^U\right)^{\delta}\right)\right)^{1/\delta} \Bigg] \Bigg\}
\end{aligned}
$$

where $\sum_{j=1}^{2} = \mathrm{S}$, $\prod_{j=1}^{2} = \wp$.

When $n = k$, use $\sum_{j=1}^{k} = \mathrm{S}$, $\prod_{j=1}^{k} = \wp$.

$$INSSWG_{\upsilon}\left(x_1,x_2,...,x_k\right)=x_1^{\upsilon_1}\otimes x_2^{\upsilon_2}\otimes...\otimes x_k^{\upsilon_k}$$

$$=\left\{\left[1-\left(\left(\mathrm{S}\left(\upsilon_j\left(1-T_j^L\right)^{\delta}\right)-\wp\left(\upsilon_j\left(1-T_j^L\right)^{\delta}\right)\right)^{1/\delta},1-\left(\mathrm{S}\left(\upsilon_j\left(1-T_j^U\right)^{\delta}\right)-\wp\left(\upsilon_j\left(1-T_j^U\right)^{\delta}\right)\right)^{1/\delta}\right)\right],\right.$$

$$\left[\left(\mathrm{S}\left(\upsilon_{j\mathrm{i}}\left(I_j^L\right)^{\delta}\right)-\wp\left(\upsilon_j\left(I_j^L\right)^{\delta}\right)\right)^{1/\delta},\left(\mathrm{S}\left(\upsilon_{j\mathrm{i}}\left(I_j^U\right)^{\delta}\right)-\wp\left(\upsilon_j\left(I_j^U\right)^{\delta}\right)\right)^{1/\delta}\right],$$

$$\left.\left[\left(\mathrm{S}\left(\upsilon_j\left(F_j^L\right)^{\delta}\right)-\wp\left(\upsilon_j\left(F_j^L\right)^{\delta}\right)\right)^{1/\delta},\left(\mathrm{S}\left(\upsilon_j\left(F_j^U\right)^{\delta}\right)-\wp\left(\upsilon_j\left(F_j^U\right)^{\delta}\right)\right)^{1/\delta}\right]\right\}$$

When $n=k+1$, $INSSWG_{\upsilon}\left(x_1,x_2,...,x_k,x_{k+1}\right)=\left(x_1^{\upsilon_1}\otimes x_2^{\upsilon_2}\otimes...\otimes x_k^{\upsilon_k}\right)\otimes x_{k+1}^{\upsilon_{k+1}}$

$$=\left\langle\left\{\left[1-\left(\left(\mathrm{S}\left(\upsilon_i\left(1-T_i^L\right)^{\delta}\right)-\wp\left(\upsilon_i\left(1-T_i^L\right)^{\delta}\right)\right)^{1/\delta},1-\left(\mathrm{S}\left(\upsilon_i\left(1-T_i^U\right)^{\delta}\right)-\wp\left(\upsilon_i\left(1-T_i^U\right)^{\delta}\right)\right)^{1/\delta}\right)\right],\right.\right.$$

$$\left[\left(\mathrm{S}\left(\upsilon_{i\mathrm{i}}\left(I_i^L\right)^{\delta}\right)-\wp\left(\upsilon_i\left(I_i^L\right)^{\delta}\right)\right)^{1/\delta},\left(\mathrm{S}\left(\upsilon_{i\mathrm{i}}\left(I_i^U\right)^{\delta}\right)-\wp\left(\upsilon_i\left(I_i^U\right)^{\delta}\right)\right)^{1/\delta}\right],$$

$$\left.\left.\left[\left(\mathrm{S}\left(\upsilon_j\left(F_j^L\right)^{\delta}\right)-\wp\left(\upsilon_j\left(F_j^L\right)^{\delta}\right)\right)^{1/\delta},\left(\mathrm{S}\left(\upsilon_j\left(F_j^U\right)^{\delta}\right)-\wp\left(\upsilon_j\left(F_j^U\right)^{\delta}\right)\right)^{1/\delta}\right]\right\}\right\rangle\otimes\upsilon_{k+1}x_{k+1}$$

$$=\left\{\left[1-\left(\sum_{j=1}^{k+1}\left(\upsilon_j\left(1-T_j^L\right)^{\delta}\right)-\prod_{j=1}^{k+1}\left(\upsilon_j\left(1-T_j^L\right)^{\delta}\right)\right)^{1/\delta},1-\left(\sum_{j=1}^{k+1}\left(\upsilon_j\left(1-T_j^U\right)^{\delta}\right)-\prod_{j=1}^{k+1}\left(\upsilon_j\left(1-T_j^U\right)^{\delta}\right)\right)^{1/\delta}\right],\right.$$

$$\left[\left(\sum_{j=1}^{k+1}\left(\upsilon_{j\mathrm{i}}\left(I_j^L\right)^{\delta}\right)-\prod_{j=1}^{k+1}\left(\upsilon_j\left(I_j^L\right)^{\delta}\right)\right)^{1/\delta},\left(\sum_{j=1}^{k+1}\left(\upsilon_{j\mathrm{i}}\left(I_j^U\right)^{\delta}\right)-\prod_{j=1}^{k+1}\left(\upsilon_j\left(I_j^U\right)^{\delta}\right)\right)^{1/\delta}\right],$$

$$\left.\left[\left(\sum_{j=1}^{k+1}\left(\upsilon_j\left(F_j^L\right)^{\delta}\right)-\prod_{j=1}^{k+1}\left(\upsilon_j\left(F_j^L\right)^{\delta}\right)\right)^{1/\delta},\left(\sum_{j=1}^{k+1}\left(\upsilon_j\left(F_j^U\right)^{\delta}\right)-\prod_{j=1}^{k+1}\left(\upsilon_j\left(F_j^U\right)^{\delta}\right)\right)^{1/\delta}\right]\right\}$$

Hence, the theorem is true for all values of n.

4.1b.6(ii) NUMERICAL VALIDATION

For the case $n=2$, consider the same x_1 and x_2, and consider the weight vectors $\upsilon_1 = 0.55$ and $\upsilon_2 = 0.45$

$$INSSWG_{\upsilon}\left(x_1, x_2\right) = x_1^{0.55} \otimes x_2^{0.45}$$

$$= \left\{\left[1-\left(0.55(1-0.7)^2+0.45(1-0.4)^2-\left(0.55(1-0.7)^2\right)\left(0.45(1-0.4)^2\right)\right)^{1/2},\right.\right.$$

$$\left.1-\left(0.55(1-0.8)^2+0.45(1-0.5)^2-\left(0.55(1-0.8)^2\right)\left(0.45(1-0.5)^2\right)\right)^{1/2}\right],$$

$$\left[\left(0.55(0)^2+0.45(0.1)^2-\left(0.55(0)^2\right)\left(0.45(0.1)^2\right)\right)^{1/2},\right.$$

$$\left.\left(0.55(0.1)^2+0.45(0.3)^2-\left(0.55(0.1)^2\right)\left(0.45(0.3)^2\right)\right)^{1/2}\right],$$

$$\left[\left(0.55(0.1)^2+0.45(0.3)^2-\left(0.55(0.1)^2\right)\left(0.45(0.3)^2\right)\right)^{1/2},\right.$$

$$\left.\left.\left(0.55(0.2)^2+0.45(0.4)^2-\left(0.55(0.2)^2\right)\left(0.45(0.4)^2\right)\right)^{1/2}\right]\right\}$$

$$= \left\{[0.5489, 0.6367], [0.0671, 0.2140], [0.2140, 0.3040]\right\}.$$

4.1b.6(iii) Theorem

If $\upsilon = (1/n, 1/n, ..., 1/n)$ then, $INSSWG_{\upsilon}\left(x_1, x_2, ..., x_n\right)$

$$= \left\{\left[1-\left(\left(\mathrm{S}\left(\frac{1}{n}\left(1-T_j^L\right)^{\delta}\right)-\wp\left(\frac{1}{n}\left(1-T_j^L\right)^{\delta}\right)\right)^{1/\delta}, 1-\left(\mathrm{S}\left(\frac{1}{n}\left(1-T_j^U\right)^{\delta}\right)-\wp\left(\frac{1}{n}\left(1-T_j^U\right)^{\delta}\right)\right)^{1/\delta}\right)\right],\right.$$

$$\left[\left(S\left(\frac{1}{n}{}_{\mathrm{i}}\left(I_j^L\right)^\delta\right)-\wp\left(\frac{1}{n}\left(I_j^L\right)^\delta\right)\right)^{1/\delta},\left(S\left(\frac{1}{n}{}_{\mathrm{i}}\left(I_j^U\right)^\delta\right)-\wp\left(\frac{1}{n}\left(I_j^U\right)^\delta\right)\right)^{1/\delta}\right],$$

$$\left[\left(S\left(\frac{1}{n}\left(F_j^L\right)^\delta\right)-\wp\left(\frac{1}{n}\left(F_j^L\right)^\delta\right)\right)^{1/\delta},\left(S\left(\frac{1}{n}\left(F_j^U\right)^\delta\right)-\wp\left(\frac{1}{n}\left(F_j^U\right)^\delta\right)\right)^{1/\delta}\right]\Bigg\}$$

where $\sum_{i=1}^{n} = S, \prod_{i=1}^{n} = \wp$.

Therefore, INSSWG operator reduces into an interval neutrosophic Schweizer and Sklar weighted arithmetic (INSSWA) operator when the weight vector $\upsilon = (1/n, 1/n, ..., 1/n)$.

4.1b.6(iv) Theorem (Idempotency)

Let $x_j = \left\langle \left[T_j^L, T_j^U\right], \left[I_j^L, I_j^U\right], \left[F_j^L, F_j^U\right] \right\rangle$, $j = 1, 2, 3, ..., n$ be a collection of INNs and if $x_j = x$, then $INSSWG_\upsilon\left(x_1, x_2, ..., x_n\right) = x.$

Proof:

$INSSWG_\upsilon\left(x_1, x_2, ..., x_n\right)$

$$=\left\{\left[1-\left(\left(S\left(\upsilon_i\left(1-T_i^L\right)^\delta\right)-\wp\left(\upsilon_i\left(1-T_i^L\right)^\delta\right)\right)^{1/\delta},1-\left(S\left(\upsilon_i\left(1-T_i^U\right)^\delta\right)-\wp\left(\upsilon_i\left(1-T_i^U\right)^\delta\right)\right)^{1/\delta}\right)\right],\right.$$

$$\left[\left(S\left(\upsilon_{i\mathrm{i}}\left(I_i^L\right)^\delta\right)-\wp\left(\upsilon_i\left(I_i^L\right)^\delta\right)\right)^{1/\delta},\left(S\left(\upsilon_{i\mathrm{i}}\left(I_i^U\right)^\delta\right)-\wp\left(\upsilon_i\left(I_i^U\right)^\delta\right)\right)^{1/\delta}\right],$$

$$\left[\left(S\left(\upsilon_j\left(F_j^L\right)^\delta\right)-\wp\left(\upsilon_j\left(F_j^L\right)^\delta\right)\right)^{1/\delta},\left(S\left(\upsilon_j\left(F_j^U\right)^\delta\right)-\wp\left(\upsilon_j\left(F_j^U\right)^\delta\right)\right)^{1/\delta}\right]\Bigg\}$$

$$=\left\{\left[1-\left(\left(1-T^L\right)^\delta\right)^{1/\delta},1-\left(\left(1-T^U\right)^\delta\right)^{1/\delta}\right],\left[\left({}_{\mathrm{i}}\left(I^L\right)^\delta\right)^{1/\delta},\left({}_{\mathrm{i}}\left(I^U\right)^\delta\right)^{1/\delta}\right],\left[\left({}_{\mathrm{i}}\left(F^L\right)^\delta\right)^{1/\delta},\left({}_{\mathrm{i}}\left(F^U\right)^\delta\right)^{1/\delta}\right]\right\}$$

$$=\left\{\left[1-\left(\left(1-T^{L}\right)\right),1-\left(\left(1-T^{U}\right)\right)\right],\left[\left(\mathrm{i}\left(I^{L}\right)\right),\left(\mathrm{i}\left(I^{U}\right)\right)\right],\left[\left(\mathrm{i}\left(F^{L}\right)\right),\left(\mathrm{i}\left(F^{U}\right)\right)\right]\right\}$$

$$=\left\{\left[1-T^{L},1-T^{U}\right],\left[I^{L},I^{U}\right],\left[F^{L},F^{U}\right]\right\}$$

$$=\left\{\left[T^{L},T^{U}\right],\left[I^{L},I^{U}\right],\left[F^{L},F^{U}\right]\right\}$$

$$=x$$

Hence, the theorem is proved.

Numerical computation can be performed in a similar way (Theorem 4.1b.6 (i)).

4.1b.6(v) Theorem (Boundedness)

Let $x_j, j=1,2,...,n$ be a collection of INNs and let

$$x^{-}=\left\langle\left(\left[\min_j\left(T_j^L\right),\min_j\left(T_j^U\right)\right],\left[\max_j\left(I_j^L\right),\max_j\left(I_j^U\right)\right],\left[\max_j\left(F_j^L\right),\max_j\left(F_j^U\right)\right]\right)\right\rangle \text{and}$$

$$x^{+}=\left\langle\left(\left[\max_j\left(T_j^L\right),\max_j\left(T_j^U\right)\right],\left[\min_j\left(I_j^L\right),\min_j\left(I_j^U\right)\right],\left[\min_j\left(F_j^L\right),\min_j\left(F_j^U\right)\right]\right)\right\rangle.$$

Then, $x^{-}\le INSSWG_{\upsilon}\left(x_1,x_2,...,x_n\right)\le x^{+}$

Proof:

Since, $\min_j\left(T_j^L\right)\le T_j^L\le\max_j\left(T_j^L\right)$, $\min_j\left(T_j^U\right)\le T_j^U\le\max_j\left(T_j^U\right)$

$$\min_j\left(I_j^L\right)\le I_j^L\le\max_j\left(I_j^L\right),\ \min_j\left(I_j^U\right)\le I_j^U\le\max_j\left(I_j^U\right)$$

$$\min_j\left(F_j^L\right)\le F_j^L\le\max_j\left(F_j^L\right),\ \min_j\left(F_j^U\right)\le F_j^U\le\max_j\left(F_j^U\right)$$

the following inequalities are holding good.

$$1-\left(\mathrm{S}\left(\upsilon_j \min\left(1-T_j^L\right)^{\delta}\right)-\wp\left(\upsilon_j \min\left(1-T_j^L\right)^{\delta}\right)\right)^{1/\delta}$$

$$\leq 1-\left(\mathrm{S}\left(\upsilon_j \left(1-T_j^L\right)^{\delta}\right)-\wp\left(\upsilon_j \left(1-T_j^L\right)^{\delta}\right)\right)^{1/\delta}$$

$$\leq 1-\left(\mathrm{S}\left(\upsilon_j \max\left(1-T_j^L\right)^{\delta}\right)-\wp\left(\upsilon_j \max\left(1-T_j^L\right)^{\delta}\right)\right)^{1/\delta},$$

$$1-\left(\mathrm{S}\left(\upsilon_j \min\left(1-T_j^U\right)^{\delta}\right)-\wp\left(\upsilon_j \min\left(1-T_j^U\right)^{\delta}\right)\right)^{1/\delta}$$

$$\leq 1-\left(\mathrm{S}\left(\upsilon_j \left(1-T_j^U\right)^{\delta}\right)-\wp\left(\upsilon_j \left(1-T_j^U\right)^{\delta}\right)\right)^{1/\delta}$$

$$\leq 1-\left(\mathrm{S}\left(\upsilon_j \max\left(1-T_j^U\right)^{\delta}\right)-\wp\left(\upsilon_j \max\left(1-T_j^U\right)^{\delta}\right)\right)^{1/\delta},$$

$$\left(\mathrm{S}\left(\upsilon_{j\mathrm{i}} \min\left(I_j^L\right)^{\delta}\right)-\wp\left(\upsilon_j \min\left(I_j^L\right)^{\delta}\right)\right)^{1/\delta} \leq \left(\mathrm{S}\left(\upsilon_{j\mathrm{i}}\left(I_j^L\right)^{\delta}\right)-\wp\left(\upsilon_j \left(I_j^L\right)^{\delta}\right)\right)^{1/\delta}$$

$$\leq \left(\mathrm{S}\left(\upsilon_{j\mathrm{i}} \max\left(I_j^L\right)^{\delta}\right)-\wp\left(\upsilon_j \max\left(I_j^L\right)^{\delta}\right)\right)^{1/\delta},$$

$$\left(\mathrm{S}\left(\upsilon_{j\mathrm{i}} \min\left(I_j^U\right)^{\delta}\right)-\wp\left(\upsilon_j \min\left(I_j^U\right)^{\delta}\right)\right)^{1/\delta} \leq \left(\mathrm{S}\left(\upsilon_{j\mathrm{i}}\left(I_j^U\right)^{\delta}\right)-\wp\left(\upsilon_j \left(I_j^U\right)^{\delta}\right)\right)^{1/\delta}$$

$$\leq \left(\mathrm{S}\left(\upsilon_{j\mathrm{i}} \max\left(I_j^U\right)^{\delta}\right)-\wp\left(\upsilon_j \max\left(I_j^U\right)^{\delta}\right)\right)^{1/\delta},$$

$$\left(\mathrm{S}\left(\upsilon_{j\mathrm{i}} \min\left(F_j^L\right)^{\delta}\right)-\wp\left(\upsilon_j \min\left(F_j^L\right)^{\delta}\right)\right)^{1/\delta} \leq \left(\mathrm{S}\left(\upsilon_{j\mathrm{i}}\left(F_j^L\right)^{\delta}\right)-\wp\left(\upsilon_j \left(F_j^L\right)^{\delta}\right)\right)^{1/\delta}$$

$$\leq \left(\mathrm{S}\left(\upsilon_{j\mathrm{i}} \max\left(F_j^L\right)^{\delta}\right)-\wp\left(\upsilon_j \max\left(F_j^L\right)^{\delta}\right)\right)^{1/\delta},$$

$$\left(\mathrm{S}\left(\upsilon_{j\mathrm{i}} \min\left(F_j^U\right)^{\delta}\right)-\wp\left(\upsilon_j \min\left(F_j^U\right)^{\delta}\right)\right)^{1/\delta} \leq \left(\mathrm{S}\left(\upsilon_{j\mathrm{i}}\left(F_j^U\right)^{\delta}\right)-\wp\left(\upsilon_j \left(F_j^U\right)^{\delta}\right)\right)^{1/\delta}$$

$$\leq \left(\mathrm{S}\left(\upsilon_{j\mathrm{i}} \max\left(F_j^U \right)^{\delta} \right) - \wp\left(\upsilon_j \max\left(F_j^U \right)^{\delta} \right) \right)^{1/\delta}$$

Therefore, $x^{-} \leq INSSWG_{\upsilon}\left(x_1, x_2, ..., x_n\right) \leq x^{+}$.

Hence, the result.

Numerical computation can be performed in a similar way (Theorem 4.1b.6 (i)).

4.1b.6(vi) Theorem (Stability)

If $x_{n+1} = \left\langle \left[T_{n+1}^L, T_{n+1}^U\right], \left[I_{n+1}^L, I_{n+1}^U\right], \left[F_{n+1}^L, F_{n+1}^U\right] \right\rangle$ is also an INN and $k > 0$ then

$$INSSWG_{\upsilon}\left(x_1^{\ k} \otimes x_{n+1}, x_2^{\ k} \otimes x_{n+1}, ..., x_n^{\ k} \otimes x_{n+1}\right) = \left(INSSWG_{\upsilon}\left(x_1, x_2, ..., x_n\right)\right)^k \otimes x_{n+1}$$

Proof:

Based on the operational laws and above results, the following results are true for INNs.

$$INSSWG_{\upsilon}\left(x_1 \otimes x_{n+1}, x_2 \otimes x_{n+1}, ..., x_n \otimes x_{n+1}\right) = INSSWG_{\upsilon}\left(x_1, x_2, ..., x_n\right) \otimes x_{n+1} \quad (4.14)$$

$$INSSWG_{\upsilon}\left(x_1^{\ k}, x_2^{\ k}, ..., x_n^{\ k}\right) = \left(INSSWG_{\upsilon}\left(x_1, x_2, ..., x_n\right)\right)^k \quad (4.15)$$

From Equation (4.14) and Equation (4.15), it is obvious that,

$$INSSWG_{\upsilon}\left(x_1^{\ k} \otimes x_{n+1}, x_2^{\ k} \otimes x_{n+1}, ..., x_n^{\ k} \otimes x_{n+1}\right) = \left(INSSWG_{\upsilon}\left(x_1, x_2, ..., x_n\right)\right)^k \otimes x_{n+1}$$

Numerical computation can be performed in a similar way (Theorem 4.1b.6 (i)).

4.1b.6(vii) Theorem

Let $x_j = \left\langle \left[T_j^L, T_j^U\right], \left[I_j^L, I_j^U\right], \left[F_j^L, F_j^U\right] \right\rangle$, $j = 1,2,3,...,n$ be a collection of INNs and their weight vector is $\upsilon = (\upsilon_1, \upsilon_2, ..., \upsilon_n)$, $\upsilon_i \in [0,1]$ and $\sum_{j=1}^{n} \upsilon_j = 1$. Then, the aggregated value of the interval neutrosophic Schweizer and Sklar weighted averaging (INSSWA) operator is still an INN and

$$INSSWA_{\upsilon}(x_1, x_2, ..., x_n)$$

$$= \left\{ \left[\left(\left(\mathrm{S}\left(\upsilon_j \left(T_j^L\right)^{\delta}\right) - \wp\left(\upsilon_j \left(T_j^L\right)^{\delta}\right)\right)^{1/\delta}, \left(\mathrm{S}\left(\upsilon_j \left(T_j^U\right)^{\delta}\right) - \wp\left(\upsilon_j \left(T_j^U\right)^{\delta}\right)\right)^{1/\delta} \right) \right], \right.$$

$$\left[1 - \left(\mathrm{S}\left(\upsilon_{ji}\left(1 - I_j^L\right)^{\delta}\right) - \wp\left(\upsilon_j \left(1 - I_j^L\right)^{\delta}\right)\right)^{1/\delta}, 1 - \left(\mathrm{S}\left(\upsilon_{ji}\left(1 - I_j^U\right)^{\delta}\right) - \wp\left(\upsilon_j \left(1 - I_j^U\right)^{\delta}\right)\right)^{1/\delta} \right],$$

$$\left. \left[1 - \left(\mathrm{S}\left(\upsilon_j \left(1 - F_j^L\right)^{\delta}\right) - \wp\left(\upsilon_j \left(1 - F_j^L\right)^{\delta}\right)\right)^{1/\delta}, 1 - \left(\mathrm{S}\left(\upsilon_j \left(1 - F_j^U\right)^{\delta}\right) - \wp\left(\upsilon_j \left(1 - F_j^U\right)^{\delta}\right)\right)^{1/\delta} \right] \right\}$$

4.1b.6(viii) Theorem

If all x_i, $i = 1,2,...,n$ are equal, i.e., $x_i = x$ then $INSSWA_{\varpi}(x_1, x_2, ..., x_n) = x$.

4.1b.6(ix) Theorem

$$INSSWA_{\upsilon}(x_1 \oplus x_{n+1}, x_2 \oplus x_{n+1}, ..., x_n \oplus x_{n+1}) = INSSWA_{\upsilon}(x_1, x_2, ..., x_n) \oplus x_{n+1}$$

4.1b.6(x) Theorem

$$INSSWA_{\varpi}(kx_1, kx_2, ..., kx_n) = kINSSWA_{\varpi}(x_1, x_2, ..., x_n)$$

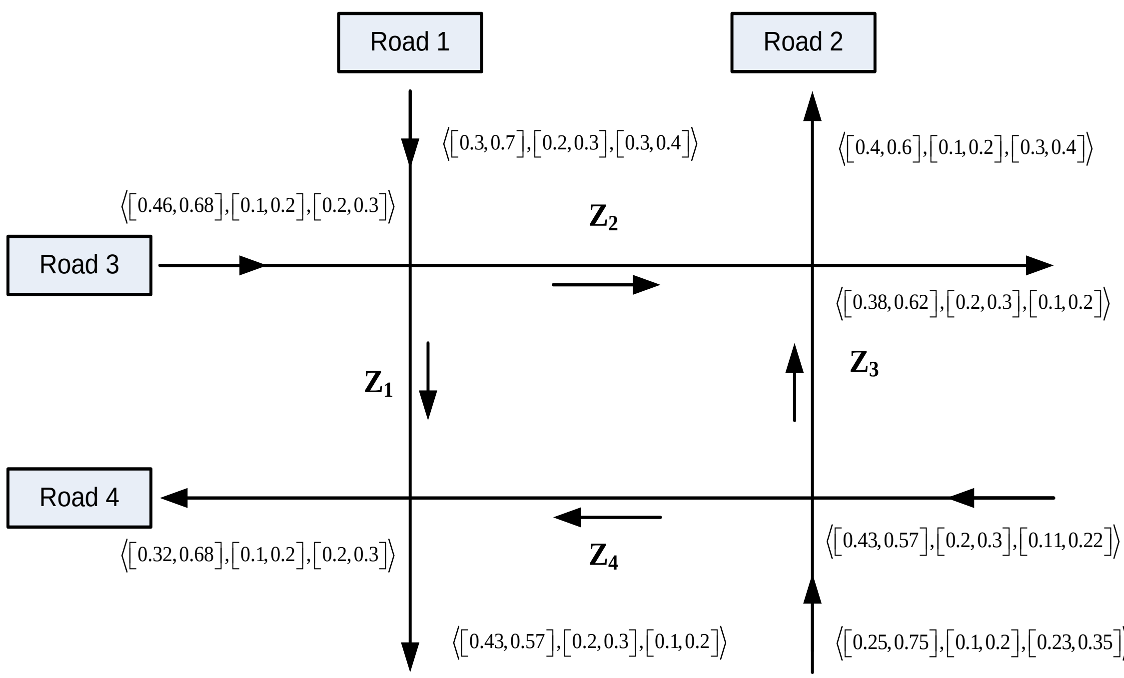
4.1b.6(xi)Theorem $TIT2SSWA_{\varpi}\left(k\overline{F_1} \oplus \overline{F_{n+1}}, k\overline{F_2} \oplus \overline{F_{n+1}}, ..., k\overline{F_n} \oplus \overline{F_{n+1}}\right)$

$$= kTIT2SSWA_{\varpi}\left(\overline{F_1}, \overline{F_2}, ..., \overline{F_n}\right) \oplus \overline{F_{n+1}}$$

4.1b.7 PROPOSED METHOD FOR TRAFFIC FLOW CONTROL USING INSSWA OPERATOR

For the same experiment as in the previous case, the average number of vehicles per hour coming in and departing at each intersecting point is taken as INNs instead of crisp numbers (Figure 4.2). The aim of this work is to identify the junction that has more vehicles (traffic), which need to be cleared first using the score value of the INN and the higher score value represents the junction that has more traffic.

Figure 4.2: Traffic flow on the road with four junctions using interval neutrosophic numbers



Using Theorem 4.1b.6(vii),

$$INSSWA_{\varpi}(Z_1, Z_2) = (0.45)Z_1 \oplus (0.55)Z_2$$

$$
\begin{aligned}
=\Bigg\langle &\Bigg[\left(0.45(0.3)^2+0.55(0.4)^2-\left(0.45(0.3)^2\right)\left(0.55(0.4)^2\right)\right)^{1/2},\\
&\left(0.45(0.7)^2+0.55(0.6)^2-\left(0.45(0.7)^2\right)\left(0.55(0.6)^2\right)\right)^{1/2}\Bigg],\\
&\Bigg[1-\left(0.45(1-0.2)^2+0.55(1-0.1)^2-\left(0.45(1-0.2)^2\right)\left(0.55(1-0.1)^2\right)\right)^{1/2},\\
&1-\left(0.45(1-0.3)^2+0.55(1-0.2)^2-\left(0.45(1-0.3)^2\right)\left(0.55(1-0.2)^2\right)\right)^{1/2}\Bigg],\\
&\Bigg[1-\left(0.45(1-0.3)^2+0.55(1-0.2)^2-\left(0.45(1-0.3)^2\right)\left(0.55(1-0.2)^2\right)\right)^{1/2},\\
&1-\left(0.45(1-0.4)^2+0.55(1-0.3)^2-\left(0.45(1-0.4)^2\right)\left(0.55(1-0.3)^2\right)\right)^{1/2}\Bigg]\Bigg\rangle
\end{aligned}
$$

$$=\langle[0.35,0.61],[0.22,0.29],[0.29,0.37]\rangle$$

Similarly,

$$INSSWA_{\varpi}(Z_2,Z_3)=(0.45)Z_2\oplus(0.55)Z_3=\langle[0.38,0.58],[0.22,0.30],[0.26,0.34]\rangle$$

$$INSSWA_{\varpi}(Z_3,Z_4)=(0.45)Z_3\oplus(0.55)Z_4=\langle[0.35,0.62],[0.23,0.31],[0.23,0.32]\rangle$$

$$INSSWA_{\varpi}(Z_4,Z_1)=(0.45)Z_4\oplus(0.55)Z_1=\langle[0.4,0.57],[0.23,0.31],[0.22,0.29]\rangle$$

The score values of the entire four junctions are obtained using Equation (4.13) such as

$$SV(Z_1,Z_2)=\frac{1}{2}\left[(0.35+0.61)-(0.22\times0.29)+(0.29-1)^2+0.37\right]=0.89$$

Similarly, $SV(Z_2, Z_3) = 0.85$, $SV(Z_3, Z_4) = 0.84$, $SV(Z_4, Z_1) = 0.83$

Based on the score values, the junction between Z_1 and Z_2 has higher value, and therefore it is recommended that this junction may be cleared first as it has more traffic.

4.1b.8 TRAFFIC FLOW USING PROPOSED OPERATORS

The proposed operators under interval type-2 fuzzy environment and interval neutrosophic environment are listed (Table 4.1). Controlling traffic flow has been handled using TIT2SSWA and INSSWA operators. There is a similar procedure for the geometric case.

Junction (Z_1, Z_2) has the higher score value, as determined using both the proposed methods, and therefore the traffic may be cleared in that junction first (Table 4.1).

Table 4.1: Aggregated traffic flow and score value

Junction	TIT2SSWA	SV	INSSWA	SV
(Z_1, Z_2)	$\langle[0.62, 0.73], 0.84, [0.92, 1]\rangle$	**1.5**	$\langle[0.35, 0.61], [0.22, 0.29], [0.29, 0.37]\rangle$	**0.89**
(Z_2, Z_3)	$\langle[0.56, 0.68], 0.78, [0.87, 0.9]\rangle$	1.33	$\langle[0.38, 0.58], [0.22, 0.30], [0.26, 0.34]\rangle$	0.85
(Z_3, Z_4)	$\langle[0.41, 0.53], 0.65, [0.75, 0.85]\rangle$	1.05	$\langle[0.35, 0.62], [0.23, 0.31], [0.23, 0.32]\rangle$	0.84
(Z_4, Z_1)	$\langle[0.56, 0.68], 0.79, [0.88, 0.95]\rangle$	1.37	$\langle[0.4, 0.57], [0.23, 0.31], [0.22, 0.29]\rangle$	0.83

4.1c QUALITATIVE COMPARISON OF TRAFFIC CONTROL MANAGEMENT USING CRISP SETS, FUZZY SETS, TYPE-2 FUZZY SETS, NEUTROSOPHIC SET AND INTERVAL NEUTROSOPHIC SETS

A comparative analysis has been done with advantages and limitations of different types of sets such as crisp, fuzzy, type-2 fuzzy, neutrosophic and interval valued neutrosophic sets in traffic control management. This analysis will be helpful in understanding the role of all types of sets mentioned and will provide the motivation for conducting research on these areas and applying them in real world problems according to the capacity of the type of sets. From the analysis, it is found that interval-based fuzzy and neutrosophic sets can handle more uncertainties than the single valued type of sets. This point will give a different perspective to new researchers.

Traffic Control Management	Advantages	Limitations
Using Crisp Sets	Fixed time period for all traffic densities Achieved to characterize the real situation appropriately	Cannot act while there is a fluctuation in traffic density Unable to react immediately to unpredictable changes such as a driver's behaviour. Unable to handle rapid momentous changes that

		disturb the continuity of the traffic.
Using Fuzzy Sets	Various time durations can be considered according to the traffic density Follow a rule-based approach that accepts uncertainties Able to model the reasoning of an experienced human being Adaptive and intelligent Able to apply and handle real-life rules identical to human thinking Admits fuzzy terms and conditions Has the best security Makes it simpler to convert knowledge beyond the domain	Adaptiveness is missing while computing the connectedness of the interval-based input Cannot be used to show uncertainty as it applies crisp and accurate functions Cannot handle uncertainties such as stability, flexibility and on-line planning completely as consequences can be uncertain.
Using Type-2 Fuzzy Sets	Rule-based approach that accepts uncertainties completely Adaptiveness (Fixed type-1 fuzzy sets are used to calculate the bounds of the type-reduced interval change as input changes) Novelty (the upper and lower membership functions may be used concurrently in calculating every bound of the type-reduced interval)	Computational complexity is high as the membership functions are themselves fuzzy.

Neutrosophic Set	Deals not only with uncertainty but also indeterminacy owing to unpredictable environmental disturbances	Unable to round up and down errors of calculations
Interval Neutrosophic Set	Deals with more uncertainties and indeterminacy Flexible and adaptable Able to address issues with a set of numbers in the real unit interval, not just a particular number. Able to round up and down errors of calculations	Unable to deal with criterion incomplete weight information.

4.2 DOMBI NEUTROSOPHIC GRAPH

Using Dombi and Hamacher triangular norms, Dombi single valued neutrosophic (DSVNG) and Dombi interval valued neutrosophic graph (DIVNG) have been proposed and derived the Cartesian and composite products of DIVNG along with the numerical validation.

4.2.1 DOMBI SINGLE VALUED NEUTROSOPHIC GRAPH (DSVNG)

A single Valued Neutrosophic Graph is a DSVNG on V is defined to be a pair $\mathrm{G_N} = (P, Q)$, where:

The functions $T_P : \mathrm{V} \to [0,1]$, $I_P : \mathrm{V} \to [0,1]$ and $F_P : \mathrm{V} \to [0,1]$ are the degree of truth membership. Indeterminacy membership and falsity membership of the element $x_i \in \mathrm{V}$ respectively and $0 \le T_P(x_i) + I_P(x_i) + F_P(x_i) \le 3, \forall x_i \in \mathbf{V}, i = 1,2,3,...,n$

The functions $T_Q : \mathrm{E} \subseteq \mathrm{V} \times \mathrm{V} \to [0,1]$, $I_Q : \mathrm{E} \subseteq \mathrm{V} \times \mathrm{V} \to [0,1]$ and $F_Q : \mathrm{E} \subseteq \mathrm{V} \times \mathrm{V} \to [0,1]$ are defined by

$$T_Q(\{x_i, y_j\}) \le \frac{T_P(x_i)T_P(y_j)}{T_P(x_i) + T_P(y_j) - T_P(x_i)T_P(y_j)} \tag{4.16}$$

$$I_Q(\{x_i, y_j\}) \ge \frac{I_P(x_i) + I_P(y_j) - 2I_P(x_i)I_P(y_j)}{1 - I_P(x_i)I_P(y_j)} \tag{4.17}$$

$$F_Q(\{x_i, y_j\}) \ge \frac{F_P(x_i) + F_P(y_j) - 2F_P(x_i)F_P(y_j)}{1 - F_P(x_i)F_P(y_j)} \tag{4.18}$$

4.2.1(i) NUMERICAL VALIDATION

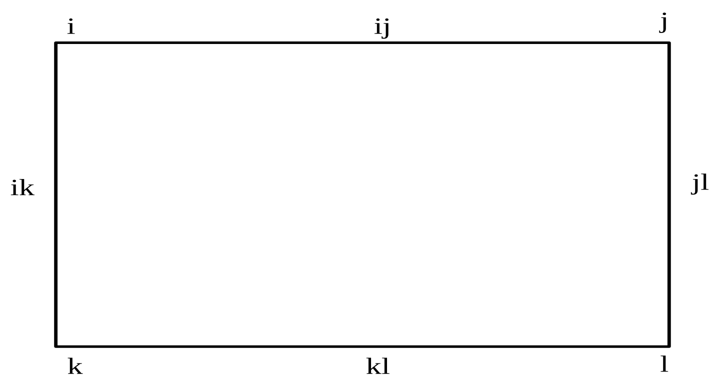
The DSVNG $\mathrm{G_N} = (P, Q)$ (Figure 4.3) of the graph $G = (V, E)$ such that the vertex set is

$$P = \{\langle i, (0.5, 0.1, 0.4)\rangle, \langle j, (0.6, 0.3, 0.2)\rangle, \langle k, (0.2, 0.3, 0.4)\rangle, \langle l, (0.4, 0.2, 0.5)\rangle\}$$

and the edge set is given by

$$Q = \{\langle ij, (0.4, 0.3, 0.5)\rangle, \langle jk, (0.2, 0.4, 0.5)\rangle\}, \langle kl, (0.2, 0.4, 0.6)\rangle, \langle il, (0.3, 0.4, 0.6)\rangle$$

Figure 4.3: Dombi Single Valued Neutrosophic Graph



To find $T_Q(ij)$:

$$T_Q(ij) \le \frac{(0.5)(0.6)}{(0.5)+(0.6)-(0.5)(0.6)} \le 0.4, \text{ hence } T_Q(ij) = 0.4$$

To find $I_Q(ij)$:

$$I_Q(ij) \ge \frac{(0.1)+(0.3)-2(0.1)(0.3)}{1-(0.1)(0.3)} \ge 0.3, \text{ hence } I_Q(ij) = 0.3$$

To find $F_Q(ij)$:

$$F_Q(ij) \ge \frac{(0.4)+(0.2)-2(0.4)(0.2)}{1-(0.4)(0.2)} \ge 0.5, \text{ hence } I_Q(ij) = 0.5$$

Similarly other values can be found.

While the membership values of truth, indeterminacy and falsity are not in the interval form then the DSVNG can be used to get the solution or output of the system for the linguistic terms. Hence DSVNG is a special case of interval based neutrosophic graph.

4.2.2 DOMBI INTERVAL VALUED NEUTROSOPHIC GRAPH (DIVNG)

A pair $\mathrm{G_N} = (P,Q)$ is DIVNG, where $P = \left\langle \left[T_P^L, T_P^U\right], \left[I_P^L, I_P^U\right], \left[F_P^L, F_P^U\right] \right\rangle$, is an IVN set on V and $Q = \left\langle \left[T_Q^L, T_Q^U\right], \left[I_Q^L, I_Q^U\right], \left[F_Q^L, F_Q^U\right] \right\rangle$ is an IVN edge set on $\mathbf{E}$ satisfying the following conditions:

Degree of truth membership, indeterminacy membership and falsity membership of the element $x_i \in \mathbf{V}$ are defined by $T_P^L : \mathrm{V} \to [0,1]$, $T_P^U : \mathrm{V} \to [0,1]$, $I_P^L : \mathrm{V} \to [0,1]$, $I_P^L : \mathrm{V} \to [0,1]$, $I_P^U : \mathrm{V} \to [0,1]$ and $F_P^L : \mathrm{V} \to [0,1]$, $F_P^U : \mathrm{V} \to [0,1]$ respectively and $0 \le T_P(x_i) + I_P(x_i) + F_P(x_i) \le 3, \forall x_i \in \mathbf{V}, i = 1,2,3,...,n$.

Degree of truth membership, indeterminacy membership and falsity membership of the edge $(x_i, y_j) \in \mathbf{E}$ are denoted by the functions, $T_Q^L : \mathrm{V} \times \mathrm{V} \to [0,1]$, $T_Q^U : \mathrm{V} \times \mathrm{V} \to [0,1]$ $I_Q^L : \mathrm{V} \times \mathrm{V} \to [0,1]$, $I_Q^U : \mathrm{V} \times \mathrm{V} \to [0,1]$ and $F_Q^L : \mathrm{V} \times \mathrm{V} \to [0,1]$, $F_Q^U : \mathrm{V} \times \mathrm{V} \to [0,1]$ respectively and are defined by

$$T_Q^L\left(\left\{x_i, y_j\right\}\right) \le \frac{T_P^L(x_i) T_P^L(y_j)}{T_P^L(x_i) + T_P^L(y_j) - T_P^L(x_i) T_P^L(y_j)} \tag{4.19}$$

$$T_Q^U\left(\left\{x_i, y_j\right\}\right) \le \frac{T_P^U(x_i) T_P^U(y_j)}{T_P^U(x_i) + T_P^U(y_j) - T_P^U(x_i) T_P^U(y_j)} \tag{4.20}$$

$$I_Q^L\left(\left\{x_i, y_j\right\}\right) \ge \frac{I_P^L(x_i) + I_P^L(y_j) - 2 I_P^L(x_i) I_P^L(y_j)}{1 - I_P^L(x_i) I_P^L(y_j)} \tag{4.21}$$

$$I_Q^U\left(\left\{x_i, y_j\right\}\right) \geq \frac{I_P^U\left(x_i\right)+I_P^U\left(y_j\right)-2 I_P^U\left(x_i\right) I_P^U\left(y_j\right)}{1-I_P^U\left(x_i\right) I_P^U\left(y_j\right)} \tag{4.22}$$

$$F_Q^L\left(\left\{x_i, y_j\right\}\right) \geq \frac{F_P^L\left(x_i\right)+F_P^L\left(y_j\right)-2 F_P^L\left(x_i\right) F_P^L\left(y_j\right)}{1-F_P^L\left(x_i\right) F_P^L\left(y_j\right)} \tag{4.23}$$

$$F_Q^U\left(\left\{x_i, y_j\right\}\right) \geq \frac{F_P^U\left(x_i\right)+F_P^U\left(y_j\right)-2 F_P^U\left(x_i\right) F_P^U\left(y_j\right)}{1-F_P^U\left(x_i\right) F_P^U\left(y_j\right)} \tag{4.24}$$

where, $0 \leq T_Q\left(\left\{x_i, y_j\right\}\right)+I_Q\left(\left\{x_i, y_j\right\}\right)+F_Q\left(\left\{x_i, y_j\right\}\right) \leq 3, \forall\left\{x_i y_j\right\} \in \mathbf{E}\ \ (i, j=1,2, \ldots, n)$.

4.2.2(i) NUMERICAL VALIDATION

For the Dombi IVNG (Figure 4.4), the vertex and edge sets are

$$P=\{\langle i,[0.5,0.7],[0.2,0.3],[0.1,0.3]\rangle,\langle j,[0.6,0.7],[0.2,0.4],[0.1,0.3]\rangle$$
$$\langle k,[0.4,0.6],[0.1,0.3],[0.2,0.4]\rangle\}$$

$$Q=\{\langle ij,[0.4,0.5],[0.4,0.5],[0.3,0.5]\rangle,\langle jk,[0.3,0.5],[0.3,0.5],[0.3,0.5]\rangle$$
$$\langle ik,[0.3,0.5],[0.3,0.5],[0.3,0.5]\rangle\}$$

To find $T_Q^L(ij)$:

$T_Q^L(ij) \leq \dfrac{(0.5)(0.4)}{(0.5)+(0.4)-(0.5)(0.4)} \leq 0.4$, hence $T_Q^L(ij)=0.4$

To find $I_Q^L(ij)$:

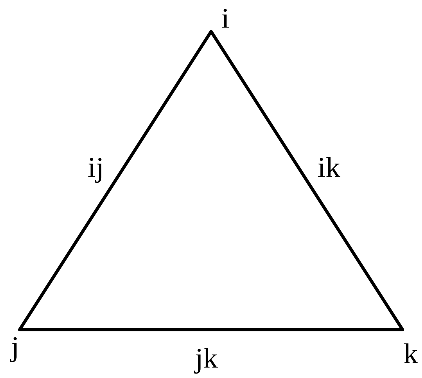
$I_Q^L(ij) \geq \dfrac{(0.2)+(0.2)-2(0.2)(0.2)}{1-(0.2)(0.2)} \geq 0.3$, hence $I_Q^L(ij)=0.4$

To find $F_Q^L(ij)$:

$F_Q^L(ij) \geq \dfrac{(0.1)+(0.1)-2(0.1)(0.1)}{1-(0.1)(0.1)} \geq 0.2$, hence $I_Q^L(ij)=0.3$

Similarly other values can be found.

Figure 4.4: Dombi Interval Valued Neutrosophic Graph



From the definition and numerical validation of DSVNG and DIVNG, it is found that Dombi FG is a special case of DSVNG and DIVNG.

4.2.3 CARTESIAN PRODUCT OF DOMBI INTERVAL VALUED NEUTROSOPHIC GRAPHS

Let λ_i be a Neutrosophic fuzzy subset of $\mathbf{V}_i$ and δ_i, a fuzzy subset of $\mathbf{E}_i, i=1,2$. Let $\mathbf{G}_{\mathbf{N}1}(\lambda_1,\delta_1)$ and $\mathbf{G}_{\mathbf{N}2}(\lambda_2,\delta_2)$ be two Dombi NGs of the crisp graphs $G_1^*(V_1,E_1)$ and $G_2^*(V_2,E_2)$ respectively and are defined by,

for all $(x_1,x_2)\in \mathrm{V}_1\times\mathrm{V}_2$,

$$\left(\lambda_1^L\times\lambda_2^L\right)(x_1,x_2)=\frac{\lambda_1^L(x_1)\lambda_2^L(x_2)}{\lambda_1^L(x_1)+\lambda_2^L(x_2)-\lambda_1^L(x_1)\lambda_2^L(x_2)} \tag{4.25}$$

for all $x\in\mathrm{V}_1,\ x_2y_2\in\mathbf{E}_2$,

$$\left(\delta_1^L\times\delta_2^L\right)\left((x,x_2)(x,y_2)\right)=\frac{\lambda_1^L(x)\lambda_2^L(x_2y_2)}{\lambda_1^L(x)+\lambda_2^L(x_2y_2)-\lambda_1^L(x)\lambda_2^L(x_2y_2)} \tag{4.26}$$

for all $z\in\mathbf{V}_2,\ x_1y_1\in\mathrm{E}_1$

$$\left(\delta_1^L\times\delta_2^L\right)\left((x_1,z)(y_1,z)\right)=\frac{\lambda_2^L(z)\delta_1^L(x_1y_1)}{\lambda_2^L(z)+\delta_1^L(x_1y_1)-\lambda_2^L(z)\delta_1^L(x_1y_1)} \tag{4.27}$$

Similarly for indeterminacy and falsity memberships with upper and lower membership values.

4.2.4 PROPOSITION

Let $\mathbf{G}_{\mathrm{N1}}$ and $\mathbf{G}_{\mathrm{N2}}$ be the Dombi IVN edge graphs (DIVNEGs) of G_1 and G_2 respectively. Then cartesian product of two DIVNEGs is not necessarily being a DIVNEG.

Proof:

Let $\mathbf{E} = \left\{ \left(x, x_2\right)\left(x, y_2\right) / x \in \mathrm{V}_1, x_2 y_2 \in \mathbf{E}_2 \right\} \cup \left\{ \left(x_1, z\right)\left(y_1, z\right) / z \in \mathrm{V}_2, x_1 y_1 \in \mathbf{E}_1 \right\}$

Consider $x \in \mathrm{V}_1,\ x_2 y_2 \in \mathrm{E}_2$

By the definition of Cartesian product of IVNG

$$\left(T_{Q_1}^L \times T_{Q_2}^L\right)\left(\left(x, x_2\right)\left(x, y_2\right)\right)$$

$$= \min\left(T_{P_1}^L\left(x\right), T_{Q_2}^L\left(x_2 y_2\right)\right) = TN\left(1, T_{Q_2}^L\left(x_2 y_2\right)\right) \le \frac{T_{P_2}^L\left(x_2\right) T_{P_2}^L\left(y_2\right)}{T_{P_2}^L\left(x_2\right) + T_{P_2}^L\left(y_2\right) - T_{P_2}^L\left(x_2\right) T_{P_2}^L\left(y_2\right)}$$

$$\le \min\left(T_{P_1}^L\left(x\right), \min\left(T_{P_2}^L\left(x_2\right), T_{P_2}^L\left(y_2\right)\right)\right)$$

$$= \min\left(\min\left(T_{P_1}^L\left(x\right), T_{P_2}^L\left(x_2\right)\right), \min\left(T_{P_1}^L\left(x\right), T_{A_2}^L\left(y_2\right)\right)\right)$$

$$= \min\left(\left(T_{P_1}^L \times T_{P_2}^L\right)\left(x, x_2\right), \left(T_{P_1}^L \times T_{P_2}^L\right)\left(x, y_2\right)\right)$$

$$\le \frac{\left(T_{P_1}^L \times T_{P_2}^L\right)\left(\left(x, x_2\right)\right)\left(T_{P_1}^L \times T_{P_2}^L\right)\left(\left(x, y_2\right)\right)}{\left(T_{P_1}^L \times T_{P_2}^L\right)\left(\left(x, x_2\right)\right) + \left(T_{P_1}^L \times T_{P_2}^L\right)\left(\left(x, y_2\right)\right) - \left(T_{P_1}^L \times T_{P_2}^L\right)\left(\left(x, x_2\right)\right)\left(T_{P_1}^L \times T_{P_2}^L\right)\left(\left(x, y_2\right)\right)}$$

$$\left(T_{Q_1}^U \times T_{Q_2}^U\right)\left(\left(x, x_2\right)\left(x, y_2\right)\right)$$

$$= \min\left(T_{P_1}^U\left(x\right), T_{Q_2}^U\left(x_2 y_2\right)\right) = TN\left(1, T_{Q_2}^U\left(x_2 y_2\right)\right) \le \frac{T_{P_2}^U\left(x_2\right) T_{P_2}^U\left(y_2\right)}{T_{P_2}^U\left(x_2\right) + T_{P_2}^U\left(y_2\right) - T_{P_2}^U\left(x_2\right) T_{P_2}^U\left(y_2\right)}$$

$$\leq \min\left(T_{P_1}^U(x), \min\left(T_{P_2}^U(x_2), T_{P_2}^U(y_2)\right)\right)$$

$$= \min\left(\min\left(T_{P_1}^U(x), T_{P_2}^U(x_2)\right), \min\left(T_{P_1}^U(x), T_{A_2}^U(y_2)\right)\right)$$

$$= \min\left(\left(T_{P_1}^U \times T_{P_2}^U\right)(x, x_2), \left(T_{P_1}^U \times T_{P_2}^U\right)(x, y_2)\right)$$

$$= \frac{\left(T_{P_1}^U \times T_{P_2}^U\right)\left((x, x_2)\right)\left(T_{P_1}^U \times T_{P_2}^U\right)\left((x, y_2)\right)}{\left(T_{P_1}^U \times T_{P_2}^U\right)\left((x, x_2)\right) + \left(T_{P_1}^U \times T_{P_2}^U\right)\left((x, y_2)\right) - \left(T_{P_1}^U \times T_{P_2}^U\right)\left((x, x_2)\right)\left(T_{P_1}^U \times T_{P_2}^U\right)\left((x, y_2)\right)}$$

$$\left(I_{Q_1}^L \times I_{Q_2}^L\right)\left((x, x_2)(x, y_2)\right)$$

$$= \max\left(I_{P_1}^L(x), I_{Q_2}^L(x_2 y_2)\right)$$

$$\geq TCN\left(I_{P_1}^L(x), I_{Q_2}^L(x_2 y_2)\right)$$

$$= \max\left(\max\left(I_{P_1}^L(x), I_{P_2}^L(x_2)\right), \max\left(I_{P_1}^L(x), T_{P_2}^L(y_2)\right)\right)$$

$$= \max\left(\left(I_{P_1}^L \times I_{P_2}^L\right)\left((x, x_2)\right), \left(I_{P_1}^L \times I_{P_2}^L\right)\left((x, y_2)\right)\right)$$

$$= \frac{\left(I_{P_1}^L \times I_{P_2}^L\right)\left((x, x_2)\right) + \left(I_{P_1}^L \times I_{P_2}^L\right)\left((x, y_2)\right) - 2\left(I_{P_1}^L \times I_{P_2}^L\right)\left((x, x_2)\right)\left(I_{P_1}^L \times I_{P_2}^L\right)\left((x, y_2)\right)}{1 - \left(I_{P_1}^L \times I_{P_2}^L\right)\left((x, x_2)\right).\left(I_{P_1}^L \times I_{P_2}^L\right)\left((x, y_2)\right)}$$

$$\left(I_{Q_1}^U \times I_{Q_2}^U\right)\left((x, x_2)(x, y_2)\right)$$

$$= \max\left(I_{P_1}^U(x), I_{Q_2}^U(x_2 y_2)\right)$$

$$\geq TCN\left(I_{P_1}^U(x), I_{Q_2}^U(x_2 y_2)\right)$$

$$= \max\left(\max\left(I_{P_1}^U(x), I_{P_2}^U(x_2)\right), \max\left(I_{P_1}^U(x), T_{P_2}^U(y_2)\right)\right)$$

$$= \max\left(\left(I_{P_1}^U \times I_{P_2}^U\right)\left((x, x_2)\right), \left(I_{P_1}^U \times I_{P_2}^U\right)\left((x, y_2)\right)\right)$$

$$= \frac{\left(I_{P_1}^U \times I_{P_2}^U\right)\left((x, x_2)\right) + \left(I_{P_1}^U \times I_{P_2}^U\right)\left((x, y_2)\right) - 2\left(I_{P_1}^U \times I_{P_2}^U\right)\left((x, x_2)\right)\left(I_{P_1}^U \times I_{P_2}^U\right)\left((x, y_2)\right)}{1 - \left(I_{P_1}^U \times I_{P_2}^U\right)\left((x, x_2)\right).\left(I_{P_1}^U \times I_{P_2}^U\right)\left((x, y_2)\right)}$$

$$\left(F_{Q_1}^L \times F_{Q_2}^L\right)\left(\left(x, x_2\right)\left(x, y_2\right)\right)$$

$$= \max\left(F_{P_1}^L\left(x\right), F_{Q_2}^L\left(x_2 y_2\right)\right)$$

$$\geq TCN\left(F_{P_1}^L\left(x\right), F_{Q_2}^L\left(x_2 y_2\right)\right)$$

$$= \max\left(\max\left(F_{P_1}^L\left(x\right), F_{P_2}^L\left(x_2\right)\right), \max\left(F_{P_1}^L\left(x\right), F_{P_2}^L\left(y_2\right)\right)\right)$$

$$= \max\left(\left(F_{P_1}^L \times F_{P_2}^L\right)\left(\left(x, x_2\right)\right), \left(F_{P_1}^L \times F_{P_2}^L\right)\left(\left(x, y_2\right)\right)\right)$$

$$\geq \frac{\left(F_{P_1}^L \times F_{P_2}^L\right)\left(\left(x, x_2\right)\right) + \left(F_{P_1}^L \times F_{P_2}^L\right)\left(\left(x, y_2\right)\right) - 2\left(F_{P_1}^L \times F_{P_2}^L\right)\left(\left(x, x_2\right)\right)\left(F_{P_1}^L \times F_{P_2}^L\right)\left(\left(x, y_2\right)\right)}{1 - \left(F_{P_1}^L \times F_{P_2}^L\right)\left(\left(x, x_2\right)\right).\left(F_{P_1}^L \times F_{P_2}^L\right)\left(\left(x, y_2\right)\right)}$$

$$\left(F_{Q_1}^U \times F_{Q_2}^U\right)\left(\left(x, x_2\right)\left(x, y_2\right)\right)$$

$$= \max\left(F_{P_1}^U\left(x\right), F_{Q_2}^U\left(x_2 y_2\right)\right)$$

$$\geq TCN\left(F_{P_1}^U\left(x\right), F_{Q_2}^U\left(x_2 y_2\right)\right)$$

$$= \max\left(\max\left(F_{P_1}^U\left(x\right), F_{P_2}^U\left(x_2\right)\right), \max\left(F_{P_1}^U\left(x\right), F_{P_2}^U\left(y_2\right)\right)\right)$$

$$= \max\left(\left(F_{P_1}^U \times F_{P_2}^U\right)\left(\left(x, x_2\right)\right), \left(F_{P_1}^U \times F_{P_2}^U\right)\left(\left(x, y_2\right)\right)\right)$$

$$\geq \frac{\left(F_{P_1}^U \times F_{P_2}^U\right)\left(\left(x, x_2\right)\right) + \left(F_{P_1}^U \times F_{P_2}^U\right)\left(\left(x, y_2\right)\right) - 2\left(F_{P_1}^U \times F_{P_2}^U\right)\left(\left(x, x_2\right)\right)\left(F_{P_1}^U \times F_{P_2}^U\right)\left(\left(x, y_2\right)\right)}{1 - \left(F_{P_1}^U \times F_{P_2}^U\right)\left(\left(x, x_2\right)\right).\left(F_{P_1}^U \times F_{P_2}^U\right)\left(\left(x, y_2\right)\right)}$$

Consider $z \in \mathrm{V}_2$, $x_1 y_1 \in \mathrm{E}_1$

$$\left(T_{Q_1}^L \times T_{Q_2}^L\right)\left(\left(x_1, z\right)\left(y_1, z\right)\right)$$

$$= \min\left(T_{Q_1}^L\left(x_1 y_1\right), T_{P_2}^L\left(z\right)\right) = TN\left(T_{Q_1}^L\left(x_1 y_1\right), 1\right) = T_{Q_1}^L\left(x_1 y_1\right)$$

$$\leq \frac{T_{P_1}^L\left(x_1\right) T_{P_1}^L\left(y_1\right)}{T_{P_1}^L\left(x_1\right) + T_{P_1}^L\left(y_1\right) - T_{P_1}^L\left(x_1\right).T_{P_1}^L\left(y_1\right)}$$

$$\leq \min\left(\min\left(T_{P_1}^L\left(x_1\right), T_{P_1}^L\left(y_1\right)\right), T_{P_2}^L\left(z\right)\right)$$

$$= \min\left(\min\left(T_{P_1}^{L}(x_1), T_{P_2}^{L}(z)\right), \min\left(T_{P_1}^{L}(y_1), T_{P_2}^{L}(z)\right)\right)$$

$$= \min\left(\left(T_{P_1}^{L} \times T_{P_2}^{L}\right)(x_1, z), \left(T_{P_1}^{L} \times T_{P_2}^{L}\right)(y_1, z)\right)$$

$$\le \frac{\left(T_{P_1}^{L} \times T_{P_2}^{L}\right)(x_1, z)\left(T_{P_1}^{L} \times T_{P_2}^{L}\right)(y_1, z)}{\left(T_{P_1}^{L} \times T_{P_2}^{L}\right)(x_1, z) + \left(T_{P_1}^{L} \times T_{P_2}^{L}\right)(y_1, z) - \left(T_{P_1}^{L} \times T_{P_2}^{L}\right)(x_1, z)\left(T_{P_1}^{L} \times T_{P_2}^{L}\right)(y_1, z)}$$

$$\left(T_{Q_1}^{U} \times T_{Q_2}^{U}\right)\left((x_1, z)(y_1, z)\right)$$

$$= \min\left(T_{Q_1}^{U}(x_1 y_1), T_{P_2}^{U}(z)\right) = TN\left(T_{Q_1}^{U}(x_1 y_1), 1\right) = T_{Q_1}^{U}(x_1 y_1)$$

$$\le \frac{T_{P_1}^{U}(x_1) T_{P_1}^{U}(y_1)}{T_{P_1}^{U}(x_1) + T_{P_1}^{U}(y_1) - T_{P_1}^{U}(x_1) . T_{P_1}^{U}(y_1)}$$

$$\le \min\left(\min\left(T_{P_1}^{U}(x_1), T_{P_1}^{U}(y_1)\right), T_{P_2}^{U}(z)\right)$$

$$= \min\left(\min\left(T_{P_1}^{U}(x_1), T_{P_2}^{U}(z)\right), \min\left(T_{P_1}^{U}(y_1), T_{P_2}^{U}(z)\right)\right)$$

$$= \min\left(\left(T_{P_1}^{U} \times T_{P_2}^{U}\right)(x_1, z), \left(T_{P_1}^{U} \times T_{P_2}^{U}\right)(y_1, z)\right)$$

$$\le \frac{\left(T_{P_1}^{U} \times T_{P_2}^{U}\right)(x_1, z)\left(T_{P_1}^{U} \times T_{P_2}^{U}\right)(y_1, z)}{\left(T_{P_1}^{U} \times T_{P_2}^{U}\right)(x_1, z) + \left(T_{P_1}^{U} \times T_{P_2}^{U}\right)(y_1, z) - \left(T_{P_1}^{U} \times T_{P_2}^{U}\right)(x_1, z)\left(T_{P_1}^{U} \times T_{P_2}^{U}\right)(y_1, z)}$$

$$\left(I_{Q_1}^{L} \times I_{Q_2}^{L}\right)\left((x_1, z)(y_1, z)\right)$$

$$= \max\left(T_{Q_1}^{L}(x_1 y_1), T_{P_2}^{L}(z)\right)$$

$$\ge \max\left(\max\left(I_{P_1}^{L}(x_1), I_{P_1}^{L}(y_1)\right), I_{P_2}^{L}(z)\right)$$

$$= \max\left(\max\left(I_{P_1}^{L}(x_1), I_{P_2}^{L}(z)\right), \max\left(T_{P_1}^{L}(y_1), T_{P_2}^{L}(z)\right)\right)$$

$$= \max\left(\left(I_{P_1}^{L} \times I_{P_2}^{L}\right)(x_1, z), \left(I_{P_1}^{L} \times I_{P_2}^{L}\right)(y_1, z)\right)$$

$$\geq \frac{\left(I_{P_1}^L \times I_{P_2}^L\right)(x_1,z)+\left(I_{P_1}^L \times I_{P_2}^L\right)(y_1,z)-2\left(I_{P_1}^L \times I_{P_2}^L\right)(x_1,z)\left(I_{P_1}^L \times I_{P_2}^L\right)(y_1,z)}{1-\left(I_{P_1}^L \times I_{P_2}^L\right)(x_1,z)\left(I_{P_1}^L \times I_{P_2}^L\right)(y_1,z)}$$

$$\left(I_{Q_1}^U \times I_{Q_2}^U\right)\left((x_1,z)(y_1,z)\right)$$

$$=\max\left(T_{Q_1}^U(x_1y_1),T_{P_2}^U(z)\right)$$

$$\geq \max\left(\max\left(I_{P_1}^U(x_1),I_{P_1}^U(y_1)\right),I_{P_2}^U(z)\right)$$

$$=\max\left(\max\left(I_{P_1}^U(x_1),I_{P_2}^U(z)\right),\max\left(T_{P_1}^U(y_1),T_{P_2}^U(z)\right)\right)$$

$$=\max\left(\left(I_{P_1}^U \times I_{P_2}^U\right)(x_1,z),\left(I_{P_1}^U \times I_{P_2}^U\right)(y_1,z)\right)$$

$$\geq \frac{\left(I_{P_1}^U \times I_{P_2}^U\right)(x_1,z)+\left(I_{P_1}^U \times I_{P_2}^U\right)(y_1,z)-2\left(I_{P_1}^U \times I_{P_2}^U\right)(x_1,z)\left(I_{P_1}^U \times I_{P_2}^U\right)(y_1,z)}{1-\left(I_{P_1}^U \times I_{P_2}^U\right)(x_1,z)\left(I_{P_1}^U \times I_{P_2}^U\right)(y_1,z)}$$

$$\left(F_{Q_1}^L \times F_{Q_2}^L\right)\left((x_1,z)(y_1,z)\right)$$

$$=\max\left(F_{Q_1}^L(x_1y_1),F_{P_2}^L(z)\right)$$

$$\geq \max\left(\max\left(F_{P_1}^L(x_1),F_{P_1}^L(y_1)\right),F_{P_2}^L(z)\right)$$

$$=\max\left(\max\left(F_{P_1}^L(x_1),F_{P_2}^L(z)\right),\max\left(F_{P_1}^L(y_1),F_{P_2}^L(z)\right)\right)$$

$$=\max\left(\left(F_{P_1}^L \times F_{P_2}^L\right)(x_1,z),\left(F_{P_1}^L \times F_{P_2}^L\right)(y_1,z)\right)$$

$$\geq \frac{\left(F_{P_1}^L \times F_{P_2}^L\right)(x_1,z)+\left(F_{P_1}^L \times F_{P_2}^L\right)(y_1,z)-2\left(F_{P_1}^L \times F_{P_2}^L\right)(x_1,z)\left(F_{P_1}^L \times F_{P_2}^L\right)(y_1,z)}{1-\left(F_{P_1}^L \times F_{P_2}^L\right)(x_1,z)\left(F_{P_1}^L \times F_{P_2}^L\right)(y_1,z)}$$

$$\left(F_{Q_1}^U \times F_{Q_2}^U\right)\left((x_1,z)(y_1,z)\right)$$

$$=\max\left(F_{Q_1}^U(x_1y_1),F_{P_2}^U(z)\right)$$

$$\geq \max\left(\max\left(F_{P_1}^U(x_1),F_{P_1}^U(y_1)\right),F_{P_2}^U(z)\right)$$

$$=\max\left(\max\left(F_{P_1}^U(x_1),F_{P_2}^U(z)\right),\max\left(F_{P_1}^U(y_1),F_{P_2}^U(z)\right)\right)$$

$$= \max\left(\left(F_{P_1}^U \times F_{P_2}^U\right)(x_1, z), \left(F_{P_1}^U \times F_{P_2}^U\right)(y_1, z)\right)$$

$$\geq \frac{\left(F_{P_1}^U \times F_{P_2}^U\right)(x_1, z) + \left(F_{P_1}^U \times F_{P_2}^U\right)(y_1, z) - 2\left(F_{P_1}^U \times F_{P_2}^U\right)(x_1, z)\left(F_{P_1}^U \times F_{P_2}^U\right)(y_1, z)}{1 - \left(F_{P_1}^U \times F_{P_2}^U\right)(x_1, z)\left(F_{P_1}^U \times F_{P_2}^U\right)(y_1, z)}$$

Note: Cartesian product of two IVNEGs need not be an IVNEG.

4.2.4(i) NUMERICAL VALIDATION

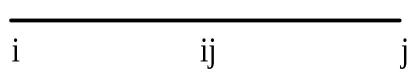
Consider two crisp graphs $G_1 = (A_1, B_1)$ and $G_2 = (A_2, B_2)$, where $V_1 = \{a, b\}$, $V_2 = \{c, d\}$, $E_1 = \{a, b\}$ and $E_2 = \{c, d\}$. Consider two IIVNGs $\mathbf{G}_{\mathbf{N}1} = (P_1, Q_1)$ and $\mathbf{G}_{\mathbf{N}2} = (P_2, Q_2)$ (Figure 4.5 and Figure 4.6).

$P_1 = \{\langle i, [0.5, 0.7], [0.2, 0.3], [0.1, 0.3]\rangle, \langle j, [0.6, 0.7], [0.2, 0.4], [0.1, 0.3]\rangle\}$

$Q_1 = \{\langle ij, [0.3, 0.6], [0.2, 0.4], [0.2, 0.4]\rangle\}$

where the vertex and edge sets have been taken from Broumi *et al.* (2016)

Figure 4.5: IVNG 1 $\mathbf{G}_{\mathbf{N}1}$



$P_2 = \{\langle k, [0.4, 0.6], [0.2, 0.3], [0.1, 0.3]\rangle, \langle l, [0.4, 0.7], [0.2, 0.4], [0.1, 0.3]\rangle\}$

$Q_2 = \{\langle kl, [0.3, 0.5], [0.4, 0.5], [0.3, 0.5]\rangle\}$

Figure 4.6: IVNG 2 $\mathbf{G}_{\mathbf{N}2}$

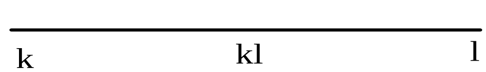
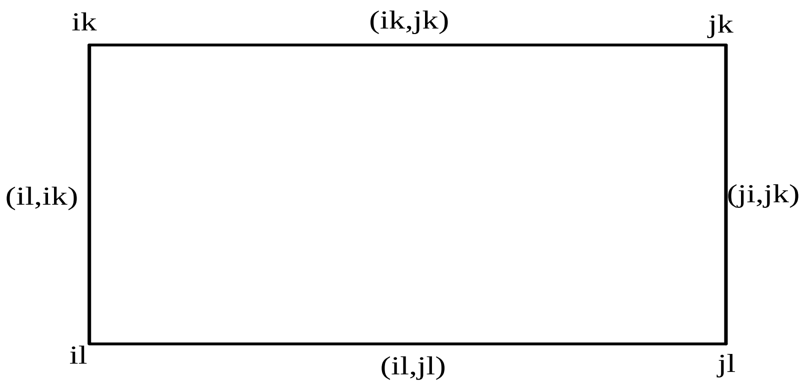


Figure 4.7: Cartesian product of DIVNGs$\left(\mathbf{G}_{N1} \times \mathbf{G}_{N2}\right)$



The vertices and edges are taken from Broumi *et al.* (2016).

$ik = \langle [0.4, 0.6], [0.2, 0.3], [0.1, 0.3] \rangle$, $jk = \langle [0.4, 0.6], [0.2, 0.4], [0.1, 0.3] \rangle$

$il = \langle [0.4, 0.7], [0.2, 0.4], [0.1, 0.3] \rangle$, $jl = \langle [0.4, 0.7], [0.2, 0.4], [0.1, 0.3] \rangle$

$(ik, jk) = \langle [0.3, 0.6], [0.2, 0.4], [0.2, 0.4] \rangle$, $(jl, jk) = \langle [0.3, 0.5], [0.4, 0.5], [0.3, 0.5] \rangle$

$(il, jl) = \langle [0.3, 0.6], [0.2, 0.4], [0.2, 0.4] \rangle$, $(il, ik) = \langle [0.3, 0.5], [0.4, 0.5], [0.3, 0.5] \rangle$

Cartesian product of interval neutrosophic vertex is obtained (Figure 4.7) and the computation has been done such as,

$$\left(T_{P_1}^L \times T_{P_2}^L\right)(i,k) = \min\left(T_{P_1}^L(i), T_{P_2}^L(k)\right) = \min(0.5, 0.4) = 0.4$$

$$\left(I_{P_1}^L \times I_{P_2}^L\right)(i,k) = \max\left(I_{P_1}^L(i), I_{P_2}^L(k)\right) = \max(0.2, 0.2) = 0.2$$

$$\left(F_{P_1}^L \times F_{P_2}^L\right)(i,k) = \min\left(F_{P_1}^L(i), F_{P_2}^L(k)\right) = \max(0.1, 0.1) = 0.1$$

Similarly for other vertices.

And interval neutrosophic edges are,

$$\left(T_{Q_1}^L \times T_{Q_2}^L\right)\left((i,k)(j,k)\right) = \min\left(T_{Q_1}^L(ij), T_{P_2}^L(k)\right) = \min(0.3, 0.4) = 0.3$$

$$\left(I_{Q_1}^L \times I_{Q_2}^L\right)\left((i,k)(j,k)\right) = \max\left(I_{Q_1}^L(ij), I_{P_2}^L(k)\right) = \max(0.2, 0.2) = 0.2$$

$$\left(F_{Q_1}^L \times F_{Q_2}^L\right)\left((i,k)(j,k)\right) = \max\left(F_{Q_1}^L(ij), F_{P_2}^L(k)\right) = \max(0.2, 0.1) = 0.2$$

Similar calculation can be done for other edges.

Further,

$$\left(T_{Q_1}^L \times T_{Q_2}^L\right)\left((i,k)(j,k)\right)$$

$$\leq \frac{\left(T_{P_1}^L \times T_{P_2}^L\right)\left((i,k)\right)\left(T_{P_1}^L \times T_{P_2}^L\right)\left((j,k)\right)}{\left(T_{P_1}^L \times T_{P_2}^L\right)\left((i,k)\right) + \left(T_{P_1}^L \times T_{P_2}^L\right)\left((j,k)\right) - \left(T_{P_1}^L \times T_{P_2}^L\right)\left((i,k)\right)\left(T_{P_1}^L \times T_{P_2}^L\right)\left((j,k)\right)}$$

$0.3 \leq \dfrac{(0.4)(0.4)}{(0.4)+(0.4)-(0.4)(0.4)} = 0.3$, hence satisfied.

$$\left(I_{Q_1}^L \times I_{Q_2}^L\right)\left((i,k)(j,k)\right)$$

$$\geq \frac{\left(I_{P_1}^L \times I_{P_2}^L\right)\left((i,k)\right) + \left(I_{P_1}^L \times I_{P_2}^L\right)\left((j,k)\right) - 2\left(I_{P_1}^L \times I_{P_2}^L\right)\left((i,k)\right)\left(I_{P_1}^L \times I_{P_2}^L\right)\left((j,k)\right)}{1 - \left(I_{P_1}^L \times I_{P_2}^L\right)\left((i,k)\right)\left(I_{P_1}^L \times I_{P_2}^L\right)\left((j,k)\right)}$$

$0.2 \geq \dfrac{(0.2)+(0.2)-2(0.2)(0.2)}{1-(0.2)(0.2)} = 0.33$, hence not satisfied

$$\left(F_{Q_1}^L \times F_{Q_2}^L\right)\left((i,k)(j,k)\right)$$

$$\geq \frac{\left(F_{P_1}^L \times F_{P_2}^L\right)\left((i,k)\right) + \left(F_{P_1}^L \times F_{P_2}^L\right)\left((j,k)\right) - 2\left(F_{P_1}^L \times F_{P_2}^L\right)\left((i,k)\right)\left(F_{P_1}^L \times F_{P_2}^L\right)\left((j,k)\right)}{1 - \left(F_{P_1}^L \times F_{P_2}^L\right)\left((i,k)\right)\left(F_{P_1}^L \times F_{P_2}^L\right)\left((j,k)\right)}$$

$0.2 \geq \dfrac{(0.1)+(0.1)-2(0.1)(0.1)}{1-(0.1)(0.1)} = 0.2$, hence satisfied.

Similarly for other edges.

Hence Cartesian product of two DIVNEGs need not be a DIVNEG.

4.2.5 COMPOSITE PRODUCT OF DOMBI INTERVAL VALUED NEUTROSOPHIC GRAPHS

Consider λ_i, a neutrosophic fuzzy subset of $\mathbf{V}_i$ and δ_i, a fuzzy subset of $\mathbf{E}_i, i=1,2$

Let $\mathbf{G}_{\mathbf{N1}}(\lambda_1,\delta_1)$ and $\mathbf{G}_{\mathbf{N2}}(\lambda_2,\delta_2)$ be two Dombi Interval Valued neutrosophic graphs of the crisp graphs $G_1(V_1,E_1)$ and $G_2(V_2,E_2)$ respectively and are defined by,

for all $(x_1,x_2)\in \mathrm{V}_1\times \mathrm{V}_2$,

$$\left(\lambda_1^L\circ\lambda_2^L\right)(x_1,x_2)=\frac{\lambda_1^L(x_1)\lambda_2^L(x_2)}{\lambda_1^L(x_1)+\lambda_2^L(x_2)-\lambda_1^L(x_1)\lambda_2^L(x_2)} \tag{4.28}$$

for all $x\in \mathrm{V}_1,\ x_2y_2\in \mathbf{E}_2$,

$$\left(\delta_1^L\,\mathrm{o}\,\delta_2^L\right)\left((x,x_2)(x,y_2)\right)=\frac{\lambda_1^L(x)\lambda_2^L(x_2y_2)}{\lambda_1^L(x)+\lambda_2^L(x_2y_2)-\lambda_1^L(x)\lambda_2^L(x_2y_2)} \tag{4.29}$$

for all $z\in \mathbf{V}_2,\ x_1y_1\in \mathrm{E}_1$,

$$\left(\delta_1^L\,\mathrm{o}\,\delta_2^L\right)\left((x_1,z)(y_1,z)\right)=\frac{\lambda_2^L(z)\delta_1^L(x_1y_1)}{\lambda_2^L(z)+\delta_1^L(x_1y_1)-\lambda_2^L(z)\delta_1^L(x_1y_1)} \tag{4.30}$$

for all $x_1y_1\in \mathrm{E}_1$ and $x_2\neq y_2$,

$$\left(\delta_1^L\,\mathrm{o}\,\delta_2^L\right)\left((x_1,x_2)(y_1,y_2)\right) \tag{4.31}$$

$$=\frac{\lambda_2^L(x_2)\lambda_2^L(y_2)\delta_1^L(x_1y_1)}{\lambda_2^L(x_2)\lambda_2^L(y_2)+\lambda_2^L(y_2)\delta_1^L(x_1y_1)+\lambda_2^L(x_2)\delta_1^L(x_1y_1)-2\lambda_2^L(x_2)\lambda_2^L(y_2)\delta_1^L(x_1y_1)}$$

Similarly, for indeterminacy and falsity memberships with upper and lower membership values.

4.2.6 PROPOSITION

The composite product of two Dombi interval valued neutrosophic edge graphs (DIVNEGs) of G_1 and G_2 is the DIVNEG.

Proof:

Since, $\left(T_{Q_1}^L \circ T_{Q_2}^L\right)\left((x,x_2)(x,y_2)\right)$

$$\leq \frac{\left(T_{P_1}^L \circ T_{P_2}^L\right)\left((x,x_2)\right)\left(T_{P_1}^L \circ T_{P_2}^L\right)\left((x,y_2)\right)}{\left(T_{P_1}^L \circ T_{P_2}^L\right)\left((x,x_2)\right)+\left(T_{P_1}^L \circ T_{P_2}^L\right)\left((x,y_2)\right)-\left(T_{P_1}^L \circ T_{P_2}^L\right)\left((x,x_2)\right)\left(T_{P_1}^L \circ T_{P_2}^L\right)\left((x,y_2)\right)}$$

$\left(T_{Q_1}^U \circ T_{Q_2}^U\right)\left((x,x_2)(x,y_2)\right)$

$$\leq \frac{\left(T_{P_1}^U \times T_{P_2}^U\right)\left((x,x_2)\right)\left(T_{P_1}^U \times T_{P_2}^U\right)\left((x,y_2)\right)}{\left(T_{P_1}^U \times T_{P_2}^U\right)\left((x,x_2)\right)+\left(T_{P_1}^U \times T_{P_2}^U\right)\left((x,y_2)\right)-\left(T_{P_1}^U \times T_{P_2}^U\right)\left((x,x_2)\right)\left(T_{P_1}^U \times T_{P_2}^U\right)\left((x,y_2)\right)}$$

$\left(I_{Q_1}^L \circ I_{Q_2}^L\right)\left((x,x_2)(x,y_2)\right)$

$$\geq \frac{\left(I_{P_1}^L \circ I_{P_2}^L\right)\left((x,x_2)\right)+\left(I_{P_1}^L \circ I_{P_2}^L\right)\left((x,y_2)\right)-2\left(I_{P_1}^L \circ I_{P_2}^L\right)\left((x,x_2)\right)\left(I_{P_1}^L \circ I_{P_2}^L\right)\left((x,y_2)\right)}{1-\left(I_{P_1}^L \circ I_{P_2}^L\right)\left((x,x_2)\right).\left(I_{P_1}^L \circ I_{P_2}^L\right)\left((x,y_2)\right)}$$

$\left(I_{Q_1}^U \circ I_{Q_2}^U\right)\left((x,x_2)(x,y_2)\right)$

$$\geq \frac{\left(I_{P_1}^U \circ I_{P_2}^U\right)\left((x,x_2)\right)+\left(I_{P_1}^U \circ I_{P_2}^U\right)\left((x,y_2)\right)-2\left(I_{P_1}^U \circ I_{P_2}^U\right)\left((x,x_2)\right)\left(I_{P_1}^U \circ I_{P_2}^U\right)\left((x,y_2)\right)}{1-\left(I_{P_1}^U \circ I_{P_2}^U\right)\left((x,x_2)\right).\left(I_{P_1}^U \circ I_{P_2}^U\right)\left((x,y_2)\right)}$$

$\left(F_{Q_1}^L \circ F_{Q_2}^L\right)\left((x,x_2)(x,y_2)\right)$

$$\geq \frac{\left(F_{P_1}^L \circ F_{P_2}^L\right)\left((x,x_2)\right)+\left(F_{P_1}^L \circ F_{P_2}^L\right)\left((x,y_2)\right)-2\left(F_{P_1}^L \circ F_{P_2}^L\right)\left((x,x_2)\right)\left(F_{P_1}^L \circ F_{P_2}^L\right)\left((x,y_2)\right)}{1-\left(F_{P_1}^L \circ F_{P_2}^L\right)\left((x,x_2)\right).\left(F_{P_1}^L \circ F_{P_2}^L\right)\left((x,y_2)\right)}$$

$\left(F_{Q_1}^U \circ F_{Q_2}^U\right)\left((x,x_2)(x,y_2)\right)$

$$\geq \frac{\left(F_{P_1}^U \circ F_{P_2}^U\right)\left((x,x_2)\right)+\left(F_{P_1}^U \circ F_{P_2}^U\right)\left((x,y_2)\right)-2\left(F_{P_1}^U \circ F_{P_2}^U\right)\left((x,x_2)\right)\left(F_{P_1}^U \circ F_{P_2}^U\right)\left((x,y_2)\right)}{1-\left(F_{P_1}^U \circ F_{P_2}^U\right)\left((x,x_2)\right).\left(F_{P_1}^U \circ F_{P_2}^U\right)\left((x,y_2)\right)}$$

Similarly for $z \in \mathrm{V}_2$, $x_1 y_1 \in \mathrm{E}_1$.

Now, consider $x_1y_1 \in E_1$, $x_2 \neq y_2$

$$\left(T_{Q_1}^L \circ T_{Q_2}^L\right)\left(\left(x_1,x_2\right)\left(y_1,y_2\right)\right)$$

$$\leq \min\left(T_{P_2}^L\left(x_2\right),T_{P_2}^L\left(y_2\right),T_{Q_1}^L\left(x_1y_1\right)\right)$$

$$\leq \min\left(T_{P_2}^L\left(x_2\right),T_{P_2}^L\left(y_2\right),\min\left(T_{P_1}^L\left(x_1\right),T_{P_1}^L\left(y_1\right)\right)\right)$$
$$=\min\left(\min\left(T_{P_1}^L\left(x_1\right),T_{P_2}^L\left(x_2\right)\right),\min\left(T_{P_1}^L\left(y_1\right),T_{P_1}^L\left(y_2\right)\right)\right)$$

$$=\min\left(\left(T_{P_1}^L \circ T_{P_2}^L\right)\left(x_1,x_2\right),\left(T_{P_1}^L \circ T_{P_2}^L\right)\left(y_1,y_2\right)\right)$$

$$=TN\left(\left(T_{P_1}^L \circ T_{P_2}^L\right)\left(\left(x_1,x_2\right)\right),\left(T_{P_1}^L \circ T_{P_2}^L\right)\left(\left(y_1,y_2\right)\right)\right)$$

$$\leq \frac{\left(T_{P_1}^L \circ T_{P_2}^L\right)\left(\left(x_1,x_2\right)\right)\left(T_{P_1}^L \circ T_{P_2}^L\right)\left(\left(y_1,y_2\right)\right)}{\left(T_{P_1}^L \circ T_{P_2}^L\right)\left(\left(x_1,x_2\right)\right)+\left(T_{P_1}^L \circ T_{P_2}^L\right)\left(\left(y_1,y_2\right)\right)-\left(T_{P_1}^L \circ T_{P_2}^L\right)\left(\left(x_1,x_2\right)\right)\left(T_{P_1}^L \circ T_{P_2}^L\right)\left(\left(y_1,y_2\right)\right)}$$

$$\left(T_{Q_1}^U \circ T_{Q_2}^U\right)\left(\left(x_1,x_2\right)\left(y_1,y_2\right)\right)$$

$$=\min\left(T_{P_2}^U\left(x_2\right),T_{P_2}^U\left(y_2\right),T_{Q_1}^U\left(x_1y_1\right)\right)$$

$$\leq \min\left(T_{P_2}^U\left(x_2\right),T_{P_2}^U\left(y_2\right),\min\left(T_{P_1}^U\left(x_1\right),T_{P_1}^U\left(y_1\right)\right)\right)$$
$$=\min\left(\min\left(T_{P_1}^U\left(x_1\right),T_{P_2}^U\left(x_2\right)\right),\min\left(T_{P_1}^U\left(y_1\right),T_{P_1}^U\left(y_2\right)\right)\right)$$

$$=\min\left(\left(T_{P_1}^U \circ T_{P_2}^U\right)\left(x_1,x_2\right),\left(T_{P_1}^U \circ T_{P_2}^U\right)\left(y_1,y_2\right)\right)$$

$$=TN\left(\left(T_{P_1}^U \circ T_{P_2}^U\right)\left(\left(x_1,x_2\right)\right),\left(T_{P_1}^U \circ T_{P_2}^U\right)\left(\left(y_1,y_2\right)\right)\right)$$

$$\leq \frac{\left(T_{P_1}^U \circ T_{P_2}^U\right)\left(\left(x_1,x_2\right)\right)\left(T_{P_1}^U \circ T_{P_2}^U\right)\left(\left(y_1,y_2\right)\right)}{\left(T_{P_1}^U \circ T_{P_2}^U\right)\left(\left(x_1,x_2\right)\right)+\left(T_{P_1}^U \circ T_{P_2}^U\right)\left(\left(y_1,y_2\right)\right)-\left(T_{P_1}^U \circ T_{P_2}^U\right)\left(\left(x_1,x_2\right)\right)\left(T_{P_1}^U \circ T_{P_2}^U\right)\left(\left(y_1,y_2\right)\right)}$$

$$\left(I_{Q_1}^L \circ I_{Q_2}^L\right)\left(\left(x_1,x_2\right)\left(y_1,y_2\right)\right)$$

$$=\max\left(I_{P_2}^L\left(x_2\right),I_{P_2}^L\left(y_2\right),I_{Q_1}^L\left(x_1y_1\right)\right)$$

$$\geq \max\left(I_{P_2}^L\left(x_2\right),I_{P_2}^L\left(y_2\right),\max\left(I_{P_1}^L\left(x_1\right),I_{P_1}^L\left(y_1\right)\right)\right)$$
$$=\max\left(\max\left(I_{P_1}^L\left(x_1\right),I_{P_2}^L\left(x_2\right)\right),\max\left(I_{P_1}^L\left(y_1\right),I_{P_1}^L\left(y_2\right)\right)\right)$$

$$=\max\left(\left(I_{P_1}^L \circ I_{P_2}^L\right)\left(x_1,x_2\right),\left(I_{P_1}^L \circ I_{P_2}^L\right)\left(y_1,y_2\right)\right)$$

$$= TCN\left(\left(I_{P_1}^{L} \circ I_{P_2}^{L}\right)\left(\left(x_1, x_2\right)\right), \left(I_{P_1}^{L} \circ I_{P_2}^{L}\right)\left(\left(y_1, y_2\right)\right)\right)$$

$$\geq \frac{\left(I_{P_1}^{L} \circ I_{P_2}^{L}\right)\left(\left(x_1, x_2\right)\right) + \left(I_{P_1}^{L} \circ I_{P_2}^{L}\right)\left(\left(y_1, y_2\right)\right) - 2\left(I_{P_1}^{L} \circ I_{P_2}^{L}\right)\left(\left(x_1, x_2\right)\right).\left(I_{P_1}^{L} \circ I_{P_2}^{L}\right)\left(\left(y_1, y_2\right)\right)}{1 - \left(I_{P_1}^{L} \circ I_{P_2}^{L}\right)\left(\left(x_1, x_2\right)\right).\left(I_{P_1}^{L} \circ I_{P_2}^{L}\right)\left(\left(y_1, y_2\right)\right)}$$

$$\left(I_{Q_1}^{U} \circ I_{Q_2}^{U}\right)\left(\left(x_1, x_2\right)\left(y_1, y_2\right)\right)$$

$$= \frac{I_{P_2}^{U}\left(x_2\right) I_{P_2}^{U}\left(y_2\right) I_{Q_1}^{U}\left(x_1 y_1\right)}{I_{P_2}^{U}\left(x_2\right) I_{P_2}^{U}\left(y_2\right) + I_{P_2}^{U}\left(y_2\right) I_{Q_1}^{U}\left(x_1 y_1\right) + I_{P_2}^{U}\left(x_2\right) I_{Q_1}^{U}\left(x_1 y_1\right) - 2 I_{P_2}^{U}\left(x_2\right) I_{P_2}^{U}\left(y_2\right) I_{Q_1}^{U}\left(x_1 y_1\right)},$$

$$= \max\left(I_{P_2}^{U}\left(x_2\right), I_{P_2}^{U}\left(y_2\right), I_{Q_1}^{U}\left(x_1 y_1\right)\right)$$

$$= TCN\left(\left(I_{P_1}^{U} \circ I_{P_2}^{U}\right)\left(\left(x_1, x_2\right)\right), \left(I_{P_1}^{U} \circ I_{P_2}^{U}\right)\left(\left(y_1, y_2\right)\right)\right)$$

$$\geq \frac{\left(I_{P_1}^{U} \circ I_{P_2}^{U}\right)\left(\left(x_1, x_2\right)\right) + \left(I_{P_1}^{U} \circ I_{P_2}^{U}\right)\left(\left(y_1, y_2\right)\right) - 2\left(I_{P_1}^{U} \circ I_{P_2}^{U}\right)\left(\left(x_1, x_2\right)\right).\left(I_{P_1}^{U} \circ I_{P_2}^{U}\right)\left(\left(y_1, y_2\right)\right)}{1 - \left(I_{P_1}^{U} \circ I_{P_2}^{U}\right)\left(\left(x_1, x_2\right)\right).\left(I_{P_1}^{U} \circ I_{P_2}^{U}\right)\left(\left(y_1, y_2\right)\right)}$$

$$\left(F_{Q_1}^{L} \circ F_{Q_2}^{L}\right)\left(\left(x_1, x_2\right)\left(y_1, y_2\right)\right)$$

$$= \max\left(F_{P_2}^{L}\left(x_2\right), F_{P_2}^{L}\left(y_2\right), F_{Q_1}^{L}\left(x_1 y_1\right)\right)$$

$$\geq \max\left(F_{P_2}^{L}\left(x_2\right), F_{P_2}^{L}\left(y_2\right), \max\left(F_{P_1}^{L}\left(x_1\right), F_{P_1}^{L}\left(y_1\right)\right)\right)$$
$$= \max\left(\max\left(F_{P_1}^{L}\left(x_1\right), F_{P_2}^{L}\left(x_2\right)\right), \max\left(F_{P_1}^{L}\left(y_1\right), F_{P_1}^{L}\left(y_2\right)\right)\right)$$
$$= \max\left(\left(F_{P_1}^{L} \circ F_{P_2}^{L}\right)\left(x_1, x_2\right), \left(F_{P_1}^{L} \circ F_{P_2}^{L}\right)\left(y_1, y_2\right)\right)$$

$$= TCN\left(\left(F_{P_1}^{L} \circ F_{P_2}^{L}\right)\left(\left(x_1, x_2\right)\right), \left(F_{P_1}^{L} \circ F_{P_2}^{L}\right)\left(\left(y_1, y_2\right)\right)\right)$$

$$\geq \frac{\left(F_{P_1}^{L} \circ F_{P_2}^{L}\right)\left(\left(x_1, x_2\right)\right) + \left(F_{P_1}^{L} \circ F_{P_2}^{L}\right)\left(\left(y_1, y_2\right)\right) - 2\left(F_{P_1}^{L} \circ F_{P_2}^{L}\right)\left(\left(x_1, x_2\right)\right).\left(F_{P_1}^{L} \circ F_{P_2}^{L}\right)\left(\left(y_1, y_2\right)\right)}{1 - \left(F_{P_1}^{L} \circ F_{P_2}^{L}\right)\left(\left(x_1, x_2\right)\right).\left(F_{P_1}^{L} \circ F_{P_2}^{L}\right)\left(\left(y_1, y_2\right)\right)}$$

$$\left(F_{Q_1}^{U} \circ F_{Q_2}^{U}\right)\left(\left(x_1, x_2\right)\left(y_1, y_2\right)\right)$$

$$= \max\left(F_{P_2}^{U}\left(x_2\right), F_{P_2}^{U}\left(y_2\right), F_{Q_1}^{U}\left(x_1 y_1\right)\right)$$

$$= TCN\left(\left(F_{P_1}^{U} \circ F_{P_2}^{U}\right)\left(\left(x_1, x_2\right)\right), \left(F_{P_1}^{U} \circ F_{P_2}^{U}\right)\left(\left(y_1, y_2\right)\right)\right)$$

$$\geq \frac{\left(F_{P_1}^U \circ F_{P_2}^U\right)\left((x_1,x_2)\right)+\left(F_{P_1}^U \circ F_{P_2}^U\right)\left((y_1,y_2)\right)-2\left(F_{P_1}^U \circ F_{P_2}^U\right)\left((x_1,x_2)\right).\left(F_{P_1}^U \circ F_{P_2}^U\right)\left((y_1,y_2)\right)}{1-\left(F_{P_1}^U \circ F_{P_2}^U\right)\left((x_1,x_2)\right).\left(F_{P_1}^U \circ F_{P_2}^U\right)\left((y_1,y_2)\right)}$$

Hence the proposition.

4.2.6(i) NUMERICAL VALIDATION

Consider the same example considered for cartesian product.

$P_1 = \left\{\left\langle i,[0.5,0.7],[0.2,0.5],[0.1,0.3]\right\rangle,\left\langle j,[0.6,0.7],[0.2,0.4],[0.1,0.3]\right\rangle\right\}$

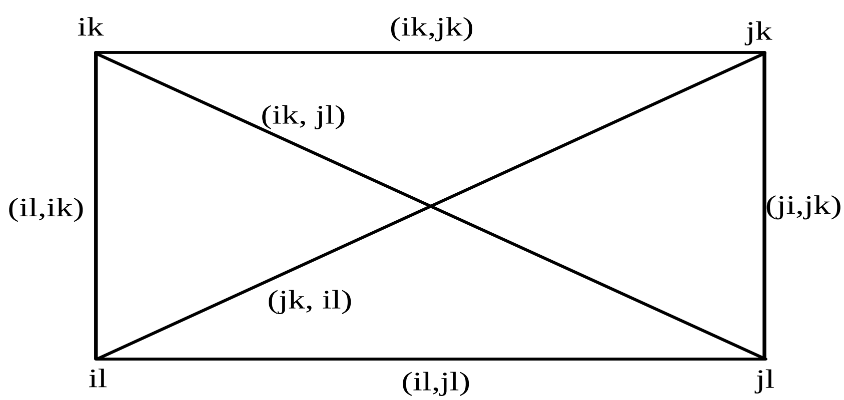
$Q_1 = \left\{\left\langle ij,[0.3,0.6],[0.2,0.4],[0.2,0.4]\right\rangle\right\}$

$P_2 = \left\{\left\langle k,[0.4,0.6],[0.3,0.4],[0.1,0.3]\right\rangle,\left\langle l,[0.4,0.7],[0.2,0.4],[0.1,0.3]\right\rangle\right\}$

$Q_2 = \left\{\left\langle kl,[0.3,0.5],[0.4,0.5],[0.3,0.5]\right\rangle\right\}$

Then composition of DIVNEGs $\left(\mathbf{G}_{\mathbf{N}1} \circ \mathbf{G}_{\mathbf{N}2}\right)$(Figure 4.8) can be calculated.

Figure 4.8: Composition of DIVNEGs $\mathbf{G}_{\mathbf{N}1}$ and $\mathbf{G}_{\mathrm{N}2}$



The edge are,

$(ik, jk)=\left\langle[0.3,0.6],[0.3,0.4],[0.2,0.4]\right\rangle$, $(jl, jk)=\left\langle[0.3,0.5],[0.4,0.5],[0.3,0.5]\right\rangle$

$(il, jl)=\left\langle[0.3,0.6],[0.2,0.4],[0.2,0.4]\right\rangle$, $(il,ik)=\left\langle[0.3,0.5],[0.4,0.5],[0.3,0.5]\right\rangle$

$(ik, jl)=\left\langle[0.3,0.6],[0.3,0.4],[0.2,0.4]\right\rangle$, $(jk,il)=\left\langle[0.3,0.6],[0.3,0.4],[0.2,0.4]\right\rangle$

To check for Dombi Composition of two IVNEGs is an IVNEG.

Consider $ij \in \mathrm{E}_1$,

$$\left(T_{Q_1}^L \circ T_{Q_2}^L\right)\left((i,k)(j,l)\right) = \min\left[T_{P_2}^L(k), T_{P_2}^L(l), T_{Q_1}^L(ij)\right] = \min[0.4, 0.4, 0.3] = 0.3$$

$$\leq \frac{\left(T_{P_1}^L \circ T_{P_1}^L\right)\left((i,k)\right)\left(T_{P_1}^L \circ T_{P_1}^L\right)\left((j,l)\right)}{\left(T_{P_1}^L \circ T_{P_1}^L\right)\left((i,k)\right) + \left(T_{P_1}^L \circ T_{P_1}^L\right)\left((j,l)\right) - \left(T_{P_1}^L \circ T_{P_1}^L\right)\left((i,k)\right)\left(T_{P_1}^L \circ T_{P_1}^L\right)\left((j,l)\right)}$$

$$= \frac{(0.4)(0.4)}{(0.4)+(0.4)-(0.4)(0.4)} = 0.3$$

Hence satisfied.

$$\left(I_{Q_1}^L \circ I_{Q_2}^L\right)\left((i,k)(j,l)\right) = \max\left[I_{P_2}^L(k), I_{P_2}^L(l), I_{Q_1}^L(ij)\right] = \max[0.3, 0.2, 0.2] = 0.3$$

$$\geq \frac{\left(I_{P_1}^L \circ I_{P_1}^L\right)\left((i,k)\right) + \left(I_{P_1}^L \circ I_{P_1}^L\right)\left((j,l)\right) - 2\left(I_{P_1}^L \circ I_{P_1}^L\right)\left((i,k)\right)\left(I_{P_1}^L \circ I_{P_1}^L\right)\left((j,l)\right)}{1 - \left(I_{P_1}^L \circ I_{P_1}^L\right)\left((i,k)\right)\left(I_{P_1}^L \circ I_{P_1}^L\right)\left((j,l)\right)}$$

$= \frac{(0.2)+(0.2)-2(0.2)(0.2)}{1-(0.2)(0.2)} = 0.3$, hence satisfied

$$\left(F_{Q_1}^L \circ F_{Q_2}^L\right)\left((i,k)(j,l)\right) = \max\left[F_{P_2}^L(k), F_{P_2}^L(l), F_{Q_1}^L(ij)\right] = \max[0.1, 0.1, 0.2] = 0.2$$

$$\geq \frac{\left(F_{P_1}^L \circ F_{P_1}^L\right)\left((i,k)\right) + \left(F_{P_1}^L \circ F_{P_1}^L\right)\left((j,l)\right) - 2\left(F_{P_1}^L \circ F_{P_1}^L\right)\left((i,k)\right)\left(F_{P_1}^L \circ F_{P_1}^L\right)\left((j,l)\right)}{1 - \left(F_{P_1}^L \circ F_{P_1}^L\right)\left((i,k)\right)\left(F_{P_1}^L \circ F_{P_1}^L\right)\left((j,l)\right)}$$

$= \frac{(0.1)+(0.1)-2(0.1)(0.1)}{1-(0.1)(0.1)} = 0.2$, hence satisfied

Similarly for other edges.

Hence the proposition.

4.3 BLOCKCHAIN SINGLE AND INTERVAL VALUED NEUTROSOPHIC GRAPH

In this section, Blockchain neutrosophic graphs have been introduced under single and interval valued neutrosophic environments.

4.3.1BLOCKCHAIN SINGLE VALUED NEUTROSOPHIC GRAPH (BCSVNG)

A pair $\mathrm{G} = (R, S)$ is BCSVNG with elemental set V . Where:

1. $T_R : \mathrm{V} \to [0,1]$, $I_R : \mathrm{V} \to [0,1]$ and $F_R : \mathrm{V} \to [0,1]$ and

   $0 \le T_R(x_i) + I_R(x_i) + F_R(x_i) \le 3, \forall x_i \in \mathbf{V}, i = 1, 2, 3, ..., n$

2. $T_S : \mathrm{E} \subseteq \mathrm{V} \times \mathrm{V} \to [0,1]$, $I_S : \mathrm{E} \subseteq \mathrm{V} \times \mathrm{V} \to [0,1]$ and $F_S : \mathrm{E} \subseteq \mathrm{V} \times \mathrm{V} \to [0,1]$ are defined by

Case (i): If $i \ne j$ then

$$\sum \left[T_S(x_i, y_j) \le \min \left[T_R(x_i), T_R(y_j) \right] \right] = 1 \qquad (4.32)$$

$$\sum \left[I_S(x_i, y_j) \ge \max \left[I_R(x_i), I_R(y_j) \right] \right] = 1 \qquad (4.33)$$

$$\sum \left[F_S(x_i, y_j) \ge \max \left[F_R(x_i), F_R(y_j) \right] \right] = 1 \qquad (4.34)$$

Case (ii): If $i = j$ then the above values are 0.

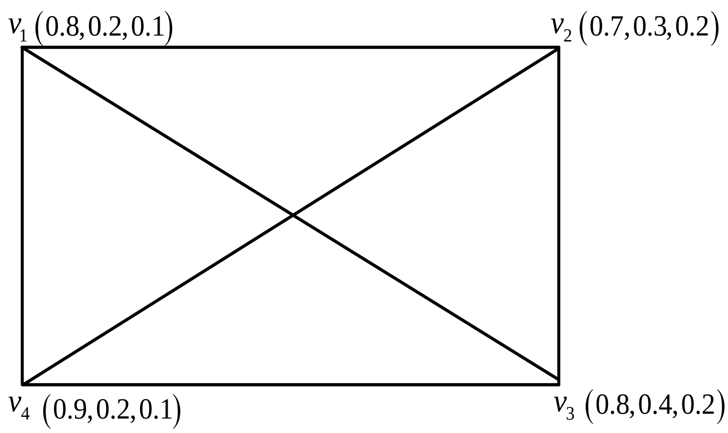
Where, $0 \le T_S(\{x_i, y_j\}) + I_S(\{x_i, y_j\}) + F_S(\{x_i, y_j\}) \le 3, \forall \{x_i, y_j\} \in \mathbf{E}(i, j = 1, 2, ..., n)$

Also R is a SVN vertex of V and S is a SVN edge set of $\mathbf{E}$. S is a symmetric SVN relation on R.

4.3.2 BCSVNG IN BLOCKCHAIN TECHNOLOGY

Four persons are connected (Figure 4.9) in the Blockchain and everyone is doing a transaction using Bitcoin and they are saving 40% and investing the remaining 60% in Bitcoin.

Figure 4.9: Blockchain Single Valued Neutrosophic Graph



Party 1: investing 20 lakhs and doing 3 transactions

Party 2: investing 15 lakhs and doing 3 transactions

Party 3: investing 10 lakhs and doing 3 transactions

Party 4: investing 5.5 lakhs and doing 3 transactions

Assume that the party-1 (v_1) has the total amount of 20 lakhs, from this he is saving 40% and investing the remaining 60% as Bitcoins for his crypto currencies and have the transactions with other parties (Table 4.2).

The transactions of Party-1:

Transaction 1: Party-1 to Party-2 : (v_1 to v_2)

$(0.7, 0.3, 0.2) \times 12,00,000$

$= \left\langle \left(1-(1-T_R)^k\right), \left(1-(1-T_R)^k\right), \left(1-(1-T_R)^k\right) \right\rangle$, $k > 0$ (any arbitrary number)

$= \left\langle \left(1-(1-0.7)^{12,00,000}\right), \left(1-(1-0.3)^{12,00,000}\right), \left(1-(1-0.2)^{12,00,000}\right) \right\rangle$

$= \left\langle \left(1-(1-0.7)^{12,00,000}\right), \left(1-(1-0.3)^{12,00,000}\right), \left(1-(1-0.2)^{12,00,000}\right) \right\rangle$

$= \langle 1,1,1 \rangle$

Similarly for other transactions namely

Transaction 2: Party-1 to Party-3 : (v_1 to v_3)

Transaction 3: Party-1 to Party-4 : (v_1 to v_4)

Table 4.2: Transaction Table for BCSVNG

		$(0.8, 0.2, 0.1)$	$(0.7, 0.3, 0.2)$	$(0.8, 0.4, 0.2)$	$(0.9, 0.2, 0.1)$	Sum
		v_1	v_2	v_3	v_4	
$(0.8, 0.2, 0.1)$	v_1	0	$(0.4, 0.38, 0.3)$	$(0.3, 0.41, 0.4)$	$(0.3, 0.21, 0.3)$	$(1,1,1)$
$(0.7, 0.3, 0.2)$	v_2	$(0.4, 0.38, 0.3)$	0	$(0.4, 0.37, 0.3)$	$(0.2, 0.25, 0.4)$	$(1,1,1)$
$(0.8, 0.4, 0.2)$	v_3	$(0.3, 0.41, 0.4)$	$(0.4, 0.37, 0.3)$	0	$(0.3, 0.54, 0.3)$	$(1,1,1)$
$(0.9, 0.2, 0.1)$	v_4	$(0.3, 0.21, 0.3)$	$(0.2, 0.25, 0.4)$	$(0.3, 0.54, 0.3)$	0	$(1,1,1)$
Sum		$(1,1,1)$	$(1,1,1)$	$(1,1,1)$	$(1,1,1)$	

Where, Sum= $\sum(\lambda_i, \lambda_j)$

4.3.2 DEGREE OF SINGLE VALUED NEUTROSOPHIC GRAPH (SVNG)

The definitions for finding the degrees have been taken from Broumi *et al.* (2016)

$d(v_1) = (d_T(v_1), d_I(v_1), d_F(v_1)) = (1,1,1)$

Where, $d_T(v_1) = T_S(v_1,v_2) + T_S(v_1,v_3) + T_S(v_1,v_4) = 0.4 + 0.3 + 0.3 = 1$

$d_I(v_1) = I_S(v_1,v_2) + I_S(v_1,v_3) + I_S(v_1,v_4) = 0.38 + 0.41 + 0.21 = 1$

$d_F(v_1) = F_S(v_1,v_2) + F_S(v_1,v_3) + F_S(v_1,v_4) = 0.3 + 0.4 + 0.3 = 1$

Similarly $d(v_2) = (d_T(v_2), d_I(v_2), d_F(v_2)) = (1,1,1)$

$d(v_3) = (d_T(v_3), d_I(v_3), d_F(v_3)) = (1,1,1)$

$d(v_4) = (d_T(v_4), d_4(v_4), d_F(v_4)) = (1,1,1)$

and $\sum d(v_1) = \left(2\sum_{v_1 \neq v_j} T_S(v_1,v_j), 2\sum_{v_1 \neq v_j} I_S(v_1,v_j), 2\sum_{v_1 \neq v_j} F_S(v_1,v_j) \right)$

$= (2(1), 2(1), 2(1)) = (2,2,2)$

4.3.3 Total Degree of SVNG

$td(v_i) = (td_T(v_i), td_I(v_i), td_F(v_i))$

Where $td_T(v_i) = \sum T_S(v_i,v_j) + T_R(v_i)$

$td_T(v_1) = \sum T_S(v_1,v_j) + T_R(v_1) = 1 + 0.8 = 1.8$

$td_I(v_1) = \sum I_S(v_1,v_j) + I_R(v_1) = 1 + 0.2 = 1.2$

$td_F(v_1) = \sum F_S(v_1,v_j) + F_R(v_1) = 1 + 0.1 = 1.1$

Therefore, $td(v_1) = (td_T(v_1), td_I(v_1), td_F(v_1)) = (1.8, 1.2, 1.1)$

Similarly, $td(v_2) = (td_T(v_2), td_I(v_2), td_F(v_2)) = (1.7, 1.3, 1.2)$

$td(v_3) = (td_T(v_3), td_I(v_3), td_F(v_3)) = (1.8, 1.4, 1.2)$

$$td(v_4)=(td_T(v_4),td_I(v_4),td_F(v_4))=(1.9,1.2,1.1)$$

4.3.4 Minimum degree of SVNG

It is $\xi(\mathrm{G})=(\xi_T(\mathrm{G}),\xi_I(\mathrm{G}),\xi_F(\mathrm{G}))$, where

$\xi_T(\mathrm{G})=\min\{d_T(v)/v\in \mathrm{V}\}$,

$\xi_I(\mathrm{G})=\min\{d_I(v)/v\in \mathrm{V}\}$ and $\xi_F(\mathrm{G})=\min\{d_F(v)/v\in \mathrm{V}\}$

For the considered SVNG,

$$\xi_T(\mathrm{G})=\min\{d_T(v)/v\in \mathrm{V}\}=1$$

$$\xi_I(\mathrm{G})=\min\{d_I(v)/v\in \mathrm{V}\}=1$$

$$\xi_F(\mathrm{G})=\min\{d_F(v)/v\in \mathrm{V}\}=1$$

4.3.5 Maximum degree of SVNG

It is defined by $\eta(\mathrm{G})=(\eta_T(\mathrm{G}),\eta_I(\mathrm{G}),\eta_F(\mathrm{G}))$, where

$\eta_T(\mathrm{G})=\max\{d_T(v)/v\in \mathrm{V}\},\eta_T(\mathrm{G})=\max\{d_T(v)/v\in \mathrm{V}\}$,

$\eta_F(\mathrm{G})=\max\{d_F(v)/v\in \mathrm{V}\}$

Therefore,

$$\eta_T(\mathrm{G})=\max\{d_T(v)/v\in \mathrm{V}\}=1$$

$$\eta_I(\mathrm{G})=\max\{d_I(v)/v\in \mathrm{V}\}=1$$

$$\eta_F(\mathrm{G})=\max\{d_F(v)/v\in \mathrm{V}\}=1$$

For the considered SVNG,

$$\eta_T(\mathrm{G})=\max\{d_T(v)/v\in \mathrm{V}\}=\eta_T(\mathrm{G})=\max\{d_T(v)/v\in \mathrm{V}\}$$

$$=\eta_F(\mathrm{G})=\max\{d_F(v)/v\in \mathrm{V}\}=1$$

4.3.6 BLOCKCHAIN INTERVAL VALUED NEUTROSOPHIC GRAPH (BCIVNG)

A pair $\mathrm{G}=(R,S)$ is BCIVNG, where $R=\left\langle\left[T_R^L,T_R^U\right],\left[I_R^L,I_R^U\right],\left[F_R^L,F_R^U\right]\right\rangle$, is an IVN set on V and $S=\left\langle\left[T_S^L,T_S^U\right],\left[I_S^L,I_S^U\right],\left[F_S^L,F_S^U\right]\right\rangle$ is an IVN edge set on $\mathbf{E}$ satisfying conditions of IVNG and with the following criterions.

Case (i): If $i\neq j$ then

$$\sum\left[T_S^L\left(x_i,y_j\right)\leq\min\left[T_R^L\left(x_i\right),T_R^L\left(y_j\right)\right]\right]=0.5 \tag{4.35}$$

$$\sum\left[T_S^U\left(x_i,y_j\right)\leq\min\left[T_R^U\left(x_i\right),T_R^U\left(y_j\right)\right]\right]=0.5 \tag{4.36}$$

$$\sum\left[I_S^L\left(x_i,y_j\right)\geq\max\left[I_R^L\left(x_i\right),I_R^L\left(y_j\right)\right]\right]=0.5 \tag{4.37}$$

$$\sum\left[I_S^U\left(x_i,y_j\right)\geq\max\left[I_R^U\left(x_i\right),I_R^U\left(y_j\right)\right]\right]=0.5 \tag{4.38}$$

$$\sum\left[F_S^L\left(x_i,y_j\right)\geq\max\left[F_R^L\left(x_i\right),F_R^L\left(y_j\right)\right]\right]=0.5 \tag{4.39}$$

$$\sum\left[F_S^U\left(x_i,y_j\right)\geq\max\left[F_R^U\left(x_i\right),F_R^U\left(y_j\right)\right]\right]=0.5 \tag{4.40}$$

Case (ii): If $i=j$ then the above six values are 0.

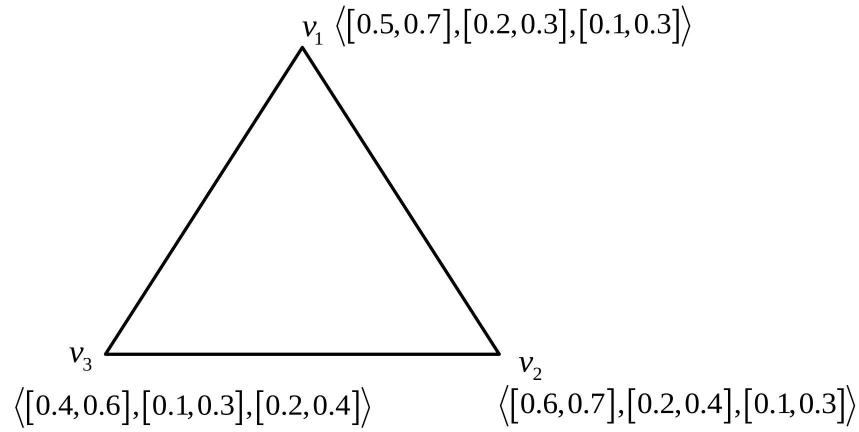
Where $0\leq T_S\left(\left\{x_i,y_j\right\}\right)+I_S\left(\left\{x_i,y_j\right\}\right)+F_S\left(\left\{x_i,y_j\right\}\right)\leq 3,\forall\left\{x_i,y_j\right\}\in\mathbf{E}\ (i,j=1,2,...,n)$

Also R is an interval valued neutrosophic vertex of V and S is an interval valued neutrosophic edge set of $\mathbf{E}$. S is a symmetric interval valued neutrosophic relation on R.

4.3.7 BCIVNG IN BLOCKCHAIN TECHNOLGY

Three persons are connected in the Blockchain and everyone is doing a transaction using Bitcoin and they are saving 40% and investing the remaining 60% in Bitcoin. Here the vertices are the three parties (Figure 4.10).

Figure 4.10: Blockchain Interval Valued Neutrosophic Graph



Party 1: investing 20 lakhs and doing 2 transactions

Party 2: investing 15 lakhs and doing 2 transactions

Party 3: investing 10 lakhs and doing 2 transactions

Assume that the party-1 (v_1) has the total amount of 20 lakhs, from this he is saving 40% and investing the remaining 60% as Bitcoins for his crypto currencies.

The transactions of Party-1:

Transaction 1: Party-1 to Party-2 : (v_1 to v_2)

$$\langle[0.6,0.7],[0.2,0.4],[0.1,0.3]\rangle\times 12,00,000$$

$$=\left\{\left[1-\left(1-T_R^L\right)^k,1-\left(1-T_R^U\right)^k\right],\left[\left(I_R^L\right)^k,\left(I_R^U\right)^k\right],\left[\left(F_R^L\right)^k,\left(F_R^U\right)^k\right]\right\}$$

$$=\left\{\left[1-\left(1-0.6\right)^{12,00,000},1-\left(1-0.7\right)^{12,00,000}\right],\left[\left(0.2\right)^{12,00,000},\left(0.4\right)^{12,00,000}\right],\right.$$
$$\left.\left[\left(0.1\right)^{12,00,000},\left(0.3\right)^{12,00,000}\right]\right\}$$

$$=\left\{\left[1-(0.4)^{12,00,000},1-(0.3)^{12,00,000}\right],\left[(0.2)^{12,00,000},(0.4)^{12,00,000}\right],\right.$$

$$\left.\left[(0.1)^{12,00,000},(0.3)^{12,00,000}\right]\right\}$$

$$=\left\{[1-0,1-0],[0,0],[0,0]\right\}=\{[1,1],[0,0],[0,0]\}$$

Transaction 2: Party-1 to Party-3: $(v_1 \text{ to } v_3)=\{[1,1],[0,0],[0,0]\}$

Table 4.3: Transaction Table for BCIVNG

		$\langle[0.5,0.7],$ $[0.2,0.3],$ $[0.1,0.3]\rangle$	$\langle[0.6,0.7],$ $[0.2,0.4],$ $[0.1,0.3]\rangle$	$\langle[0.4,0.6],$ $[0.1,0.3],$ $[0.2,0.4]\rangle$	Sum
		v_1	v_2	v_3	
$\langle[0.5,0.7],$ $[0.2,0.3],$ $[0.1,0.3]\rangle$	v_1	0	$\langle[0.217,0.283],$ $[0.211,0.289],$ $[0.302,0.313]\rangle$	$\langle[0.281,0.282],$ $[0.198,0.199],$ $[0.208,0.209]\rangle$	$(1,1,1)$
$\langle[0.6,0.7],$ $[0.2,0.4],$ $[0.1,0.3]\rangle$	v_2	$\langle[0.217,0.283],$ $[0.211,0.289],$ $[0.302,0.313]\rangle$	0	$\langle[0.28,0.283],$ $[0.197,0.198],$ $[0.208,0.209]\rangle$	$(1,1,1)$
$\langle[0.4,0.6],$ $[0.1,0.3],$ $[0.2,0.4]\rangle$	v_3	$\langle[0.281,0.282],$ $[0.198,0.199],$ $[0.208,0.209]\rangle$	$\langle[0.217,0.283],$ $[0.302,0.313],$ $[0.292,0.302]\rangle$	0	$(1,1,1)$
Sum		$(1,1,1)$	$(1,1,1)$	$(1,1,1)$	

where Sum= $\sum(\lambda_i,\lambda_j)$. It is observed that sum of all single and interval valued neutrosophic edges of a particular neutrosophic vertex is equal to (1, 1, 1) (Table 4.2 and Table 4.3). Hence the proposed method is an optimized one to deal indeterminacy of the data.

4.4 EDGE DETECTION ON DICOM IMAGE USING INTERVAL TYPE-2 FUZZY

In this chapter, operational laws for triangular interval type-2 fuzzy numbers, aggregation operators using Yager triangular norms with their desirable properties have been derived and applied the proposed aggregation operators in the process of edge detection on DICOM image has been done.

4.4.1 PROPOSED OPERATIONAL LAWS

Let $\overline{M}, \overline{M}_1, \overline{M}_2$ be three TIT2FNs and $\eta > 0$. Here, the operational laws have been defined.

4.4.1(i) ADDITION

Consider,

$$A_1 = \underset{p=1}{\overset{2}{sum}}\left(\underline{l_{M_p}}\right), B_1 = \underset{p=1}{\overset{2}{sum}}\left(\overline{l_{M_p}}\right), C_1 = \underset{p=1}{\overset{2}{sum}}\left(c_{M_p}\right), D_1 = \underset{p=1}{\overset{2}{sum}}\left(\underline{r_{M_p}}\right), E_1 = \underset{p=1}{\overset{2}{sum}}\left(\overline{r_{M_p}}\right).$$

$$\overline{M}_1 \underset{Y}{\oplus} \overline{M}_2 = \left(\left[\min\left(A_1^{\frac{1}{\eta}}, 1\right), \min\left(B_1^{\frac{1}{\eta}}, 1\right)\right], \min\left(C_1^{\frac{1}{\eta}}, 1\right), \left[\min\left(D_1^{\frac{1}{\eta}}, 1\right), \min\left(E_1^{\frac{1}{\eta}}, 1\right)\right]\right). \tag{4.41}$$

4.4.1(ii) MULTIPLICATION

Let, $A_2 = \underset{p=1}{\overset{2}{sum}}\left(1-\underline{l_{M_p}}\right)^{\eta}, B_2 = \underset{p=1}{\overset{2}{sum}}\left(1-\overline{l_{M_p}}\right)^{\eta}, \; C_2 = \underset{p=1}{\overset{2}{sum}}\left(1-c_{M_p}\right)^{\eta},$

$$D_2 = \underset{p=1}{\overset{2}{sum}}\left(1-\underline{r_{M_p}}\right)^{\eta}, E_2 = \underset{p=1}{\overset{2}{sum}}\left(1-\overline{r_{M_p}}\right)^{\eta}.$$

$$\overline{M}_1 \underset{Y}{\otimes} \overline{M}_2 = \left\{\left[\max\left(1-A_2^{\frac{1}{\eta}}, 0\right), \max\left(1-B_2^{\frac{1}{\eta}}, 0\right)\right], \max\left(1-C_2^{\frac{1}{\eta}}, 0\right), \left[\max\left(1-D_2^{\frac{1}{\eta}}, 0\right), \max\left(1-E_2^{\frac{1}{\eta}}, 0\right)\right]\right\} \tag{4.42}$$

4.4.1(iii) MULTIPLICATION BY AN ORDINARY NUMBER

Consider, $A=\underline{l_M}, B=\overline{l_M}, C=c_M, D=\underline{r_M}, E=\overline{r_M}$

$$k\underset{Y}{\oplus}\overline{M}=\left\{\left[\min\left\langle A^{\frac{k}{\eta}},1\right\rangle,\min\left\langle B^{\frac{k}{\eta}},1\right\rangle\right],\min\left\langle C^{\frac{k}{\eta}},1\right\rangle,\left[\min\left\langle D^{\frac{k}{\eta}},1\right\rangle,\min\left\langle E^{\frac{k}{\eta}},1\right\rangle\right]\right\}. \tag{4.43}$$

4.4.1(iv) POWER

Consider $A_3=1-\underline{l_M}, B_3=1-\overline{l_M}, C_3=1-c_M,\ D_3=1-\underline{r_M}, E_3=1-\overline{r_M}$

$$\overline{M}^{\hat{Y}k}=\left\{\left[\max\left(1-\left[A_3^{\eta}\right]^{k/\eta},0\right),\max\left(1-\left[B_3^{\eta}\right]^{k/\eta},0\right)\right],\max\left(1-\left[C_3^{\eta}\right]^{k/\eta},0\right),\right.$$
$$\left.\left[\max\left(1-\left[D_3^{\eta}\right]^{k/\eta},0\right),\max\left(1-\left[E_3^{\eta}\right]^{k/\eta},0\right)\right]\right\} \tag{4.44}$$

4.4.2 PROPOSED THEOREMS

Here the mathematical properties of aggregation properties for TIT2FN using triangular interval type-2 fuzzy Yager weighted geometric (TIT2FYWG) and triangular interval type-2 fuzzy Yager weighted arithmetic (TIT2FYWA) operators are proved and they are playing an important role in image processing.

Consider a collection of TIT2FNs, $\overline{M}=\left([\underline{l_{M_p}},\overline{l_{M_p}}],c_{M_p},[\underline{r_{M_p}},\overline{r_{M_p}}]\right), p=1,2,...,n$,

where $0\le\underline{l_M}\le\overline{l_M}\le c_M\le\underline{r_M}\le\overline{r_M}\le1$.

4.4.2(i) Theorem

The aggregation value of these fuzzy numbers using TIT2FYWG operator is again a TIT2FN and $TIT2FYWG_{\varepsilon}\left\langle\overline{M_1},\overline{M_2},...,\overline{M_n}\right\rangle=\left\{\left[\max\left(1-\left[A_n^{\eta}\right]^{\varepsilon_p/\eta},0\right),\right.\right.$

$\left.\max\left(1-\left[B_n^{\eta}\right]^{\varepsilon_p/\eta},0\right)\right],\ \max\left(1-\left[C_n^{\eta}\right]^{\varepsilon_p/\eta},0\right),\left[\max\left(1-\left[D_n^{\eta}\right]^{\varepsilon_p/\eta},0\right),\ \max\left(1-\left[E_n^{\eta}\right]^{\varepsilon_p/\eta},0\right)\right]\Big\}$, where the weight vector is $\varepsilon = \left(\varepsilon_1, \varepsilon_2, ..., \varepsilon_n\right)^T, \varepsilon_n \geq 0$, the sum of the weight vectors is equal to 1.

Proof:

Here use mathematical induction method.

Case (i): For $n = 2$.

Consider, $P_1 = \left(1 - \underline{l_{M_1}}\right)^\eta, Q_1 = \left(1 - \overline{l_{M_1}}\right)^\eta, R_1 = \left(1 - c_{M_1}\right)^\eta, S_1 = \left(1 - \underline{r_{M_1}}\right)^\eta, T_1 = \left(1 - \overline{r_{M_1}}\right)^\eta$

Using Yager power operation

$$\overline{M_1}^{\overset{\bullet}{Y}k} = \left\{\left[\max\left(1 - P_1^{\frac{k}{\eta}}, 0\right), \max\left(1 - Q_1^{\frac{k}{\eta}}, 0\right)\right], \max\left(1 - R_1^{\frac{k}{\eta}}, 0\right), \left[\max\left(1 - S_1^{\frac{k}{\eta}}, 0\right), \max\left(1 - T_1^{\frac{k}{\eta}}, 0\right)\right]\right\}$$

Consider, $P_2 = \left(1 - \underline{l_{M_1}}\right)^\eta$, $Q_2 = \left(1 - \overline{l_{M_1}}\right)^\eta, R_2 = \left(1 - c_{M_1}\right)^\eta, S_2 = \left(1 - \underline{r_{M_1}}\right)^\eta$, $T_2 = \left(1 - \overline{r_{M_1}}\right)^\eta$

$$\overline{M_2}^{\overset{\bullet}{Y}k} = \left\{\left[\max\left(1 - P_2^{\frac{k}{\eta}}, 0\right), \max\left(1 - Q_2^{\frac{k}{\eta}}, 0\right)\right], \max\left(1 - R_2^{\frac{k}{\eta}}, 0\right),\right.$$

$$\left.\left[\max\left(1 - S_2^{\frac{k}{\eta}}, 0\right), \max\left(1 - T_2^{\frac{k}{\eta}}, 0\right)\right]\right\}$$

$$TIT2FYWG_\varepsilon\left(\overline{M_1}, \overline{M_2}\right) = \overline{M_1}^{\overset{\bullet}{Y}\varepsilon_1} \underset{Y}{\otimes} \overline{M_2}^{\overset{\bullet}{Y}\varepsilon_2}$$

$$= \left\{\left[\max\left(1 - \left[\underset{p=1}{\overset{2}{sum}}\left(1 - \max\left(1 - \left(1 - A_2\right)^{\frac{\varepsilon_p}{\eta}}\right), 0\right)\right], 0\right),\right.\right.$$

$$\left.\max\left(1 - \left[\underset{p=1}{\overset{2}{sum}}\left(1 - \max\left(1 - \left(1 - B_2\right)^{\frac{\varepsilon_p}{\eta}}\right), 0\right)\right], 0\right)\right],$$

$$\max\left(1 - \left[\underset{p=1}{\overset{2}{sum}}\left(1 - \max\left(1 - \left(1 - C_2\right)^{\frac{\varepsilon_p}{\eta}}\right), 0\right)\right], 0\right),$$

$$\left[\max\left(1-\left[\underset{p=1}{\overset{2}{sum}}\left(1-\max\left(1-\left(1-D_2\right)^{\frac{\varepsilon_p}{\eta}}\right),0\right)\right],0\right),\right.$$

$$\left.\left.\max\left(1-\left[\underset{p=1}{\overset{2}{sum}}\left(1-\max\left(1-\left(1-E_2\right)^{\frac{\varepsilon_p}{\eta}}\right),0\right)\right],0\right)\right]\right\}$$

Consider, $A_4=\left(1-\underline{l_{M_p}}\right)^{\eta}, B_4=\left(1-\overline{l_{M_p}}\right)^{\eta}, C_4=\left(1-c_{M_p}\right)^{\eta}, D_4=\left(1-\underline{r_{M_p}}\right)^{\eta}, E_4=\left(1-\overline{r_{M_p}}\right)^{\eta}$.

$$=\left\{\left[\max\left(1-\left(\underset{p=1}{\overset{2}{sum}}\left(1-\max\left(1-A_4^{\frac{\varepsilon_p}{\eta}},0\right)\right)\right),0\right)\right),\max\left(1-\left(\underset{p=1}{\overset{2}{sum}}\left(1-\max\left(1-B_4^{\frac{\varepsilon_p}{\eta}},0\right)\right)\right),0\right)\right)\right],\right.$$

$$\max\left(1-\left(\underset{p=1}{\overset{2}{sum}}\left(1-\max\left(1-C_4^{\frac{\varepsilon_p}{\eta}},0\right)\right)\right),0\right)\right),$$

$$\left.\left[\max\left(1-\left(\underset{p=1}{\overset{2}{sum}}\left(1-\max\left(1-D_4^{\frac{\varepsilon_p}{\eta}},0\right)\right)\right),0\right)\right),\max\left(1-\left(\underset{p=1}{\overset{2}{sum}}\left(1-\max\left(1-E_4^{\frac{\varepsilon_p}{\eta}},0\right)\right)\right),0\right)\right)\right]\right\}.$$

$$=\left\{\left[\max\left[1-A_2^{\frac{\varepsilon_p}{\eta}},0\right],\max\left[1-B_2^{\frac{\varepsilon_p}{\eta}},0\right]\right],\max\left[1-C_2^{\frac{\varepsilon_p}{\eta}},0\right],\left[\max\left[1-D_2^{\frac{\varepsilon_p}{\eta}},0\right],\max\left[1-E_2^{\frac{\varepsilon_p}{\eta}},0\right]\right]\right\}.$$

For $n=k$,

$A_k=\underset{p=1}{\overset{k}{sum}}\left(1-\underline{l_{M_p}}\right)^{\eta}, B_k=\underset{p=1}{\overset{k}{sum}}\left(1-\overline{l_{M_p}}\right)^{\eta}, C_k=\underset{p=1}{\overset{k}{sum}}\left(1-c_{M_p}\right)^{\eta}, D_k=\underset{p=1}{\overset{k}{sum}}\left(1-\underline{r_{M_p}}\right)^{\eta}, E_k=\underset{p=1}{\overset{k}{sum}}\left(1-\overline{r_{M_p}}\right)^{\eta}$.

$TIT2FYWG_{\varepsilon}\left\langle\overline{M_1},\overline{M_2},...,\overline{M_k}\right\rangle$

$$=\left\{\left[\max\left(1-\left[\underset{p=1}{\overset{k}{sum}}\left(1-\max\left(1-\left[A_k\right]^{\frac{\omega_p}{\eta}},0\right)\right],0\right),\max\left(1-\left[\underset{p=1}{\overset{k}{sum}}\left(1-\max\left(1-B_k^{\frac{\omega_p}{\eta}},0\right)\right],0\right)\right],\right.$$

$$\max\left(1-\left[\underset{p=1}{\overset{k}{sum}}\left(1-\max\left(1-C_k^{\frac{\omega_p}{\eta}},0\right)\right)\right],0\right),$$

$$\left[\max\left(1-\left[\underset{p=1}{\overset{k}{sum}}\left(1-\max\left(1-D_k^{\frac{\varepsilon_p}{\eta}},0\right)\right)\right],0\right),\ \max\left(1-\left[\underset{p=1}{\overset{k}{sum}}\left(1-\max\left(1-E_k^{\frac{\varepsilon_p}{\eta}},0\right)\right)\right],0\right)\right]\Bigg\}.$$

$$=\left\{\left[\max\left[1-A_k^{\frac{\varepsilon_p}{\eta}},0\right],\max\left[1-B_k^{\frac{\varepsilon_p}{\eta}},0\right]\right],\ \max\left[1-C_k^{\frac{\varepsilon_p}{\eta}},0\right],\left[\max\left[1-D_k^{\frac{\varepsilon_p}{\eta}},0\right],\max\left[1-E_k^{\frac{\varepsilon_p}{\eta}},0\right]\right]\right\}.$$

For $n=k+1$,

$$TIT2FYWG_{\varepsilon}\left\langle\overline{M_1},\overline{M_2},...,\overline{M_k}\right\rangle\underset{Y}{\otimes}TIT2FYWG_{\varepsilon}\left\langle\overline{M_{k+1}}\right\rangle$$

$$=\left\{\left[\max\left(1-\left[\underset{p=1}{\overset{k}{sum}}\left(1-\max\left(1-A_k^{\frac{\varepsilon_p}{\eta}},0\right)\right)\right],0\right),\max\left(1-\left[\underset{p=1}{\overset{k}{sum}}\left(1-\max\left(1-B_k^{\frac{\varepsilon_p}{\eta}},0\right)\right)\right],0\right)\right],\right.$$

$$\max\left(1-\left[\underset{p=1}{\overset{k}{sum}}\left(1-\max\left(1-C_k^{\frac{\varepsilon_p}{\eta}},0\right)\right)\right],0\right),$$

$$\left[\max\left(1-\left[\underset{p=1}{\overset{k}{sum}}\left(1-\max\left(1-D_k^{\frac{\varepsilon_p}{\eta}},0\right)\right)\right],0\right),\quad \max\left(1-\left[\underset{p=1}{\overset{k}{sum}}\left(1-\max\left(1-E_k^{\frac{\varepsilon_p}{\eta}},0\right)\right)\right],0\right)\right]\Bigg\}.$$

$$\underset{Y}{\otimes}\left\{\left[\max\left(1-P_{k+1}^{\frac{\omega_{k+1}}{\eta}},0\right),\max\left(1-Q_{k+1}^{\frac{\varepsilon_{k+1}}{\eta}},0\right)\right],\max\left(1-R_{k+1}^{\frac{\varepsilon_{k+1}}{\eta}},0\right),\right.$$

$$\left.\left[\max\left(1-S_{k+1}^{\frac{\varepsilon_{k+1}}{\eta}},0\right),\max\left(1-T_{k+1}^{\frac{\varepsilon_{k+1}}{\eta}},0\right)\right]\right\}$$

$$=\left\{\left[\max\left(1-\left[\underset{p=1}{\overset{k+1}{sum}}\left(1-\max\left(1-A_k^{\frac{\varepsilon_p}{\eta}}\right),0\right)\right],0\right),\right.\right.$$

$$\max\left(1-\left[\underset{p=1}{\overset{k+1}{sum}}\left(1-\max\left(1-B_k^{\frac{\varepsilon_p}{\eta}}\right),0\right)\right],0\right)\Bigg],$$

$$\max\left(1-\left[\underset{p=1}{\overset{k+1}{sum}}\left(1-\max\left(1-C_k^{\frac{\varepsilon_p}{\eta}}\right),0\right)\right],0\right),$$

$$\left[\max\left(1-\left[\underset{p=1}{\overset{k+1}{sum}}\left(1-\max\left(1-D_k^{\frac{\varepsilon_p}{\eta}}\right),0\right)\right],0\right),\right.$$

$$\left.\max\left(1-\left[\underset{p=1}{\overset{k+1}{sum}}\left(1-\max\left(1-E_k^{\frac{\varepsilon_p}{\eta}}\right),0\right)\right],0\right)\right]\Bigg\}.$$

$$=\left\{\left[\max\left[1-A_{k+1}^{\frac{\varepsilon_p}{\eta}},0\right],\max\left[1-B_{k+1}^{\frac{\varepsilon_p}{\eta}},0\right]\right],\ \max\left[1-C_{k+1}^{\frac{\varepsilon_p}{\eta}},0\right],\right.$$

$$\left.\left[\max\left[1-D_{k+1}^{\frac{\omega_p}{\eta}},0\right],\max\left[1-E_{k+1}^{\frac{\varepsilon_p}{\eta}},0\right]\right]\right\}$$

Hence the result holds for all the values of n.

4.4.2(ii) Theorem (Idempotency)

If $\overline{M_p}=\overline{M}$ for all the values of p then $TIT2FYWG_{\varepsilon}\left\langle\overline{M_1},\overline{M_2},...,\overline{M_n}\right\rangle=\overline{M}$.

Proof:

Since, $TIT2FYWG_{\varepsilon}\left\langle\overline{M_1},\overline{M_2},...,\overline{M_n}\right\rangle$

$$=\left\{\left[\max\left(1-A_n^{\frac{\varepsilon_p}{\eta}},0\right),\max\left(1-B_n^{\frac{\omega_p}{\eta}},0\right)\right],\ \max\left(1-C_n^{\frac{\varepsilon_p}{\eta}},0\right),\left[\max\left(1-D_n^{\frac{\varepsilon_p}{\eta}},0\right),\max\left(1-E_n^{\frac{\varepsilon_p}{\eta}},0\right)\right]\right\}.$$

$$TIT2FYWG_{\varepsilon}\left\langle \overline{M_1}, \overline{M_2}, ..., \overline{M_n} \right\rangle$$

$$=\left\{\left[\max\left(1-A_4^{\underset{p=1}{\overset{k+1}{sum}}\left(\frac{\varepsilon_p}{\eta}\right)},0\right),\max\left(1-B_4^{\underset{p=1}{\overset{k+1}{sum}}\left(\frac{\omega_p}{\eta}\right)},0\right)\right],\ \max\left(1-C_4^{\underset{p=1}{\overset{k+1}{sum}}\left(\frac{\varepsilon_p}{\eta}\right)},0\right),\right.$$

$$\left.\left[\max\left(1-D_4^{\underset{p=1}{\overset{k+1}{sum}}\left(\frac{\varepsilon_p}{\eta}\right)},0\right),\max\left(1-E_4^{\underset{p=1}{\overset{k+1}{sum}}\left(\frac{\varepsilon_p}{\eta}\right)},0\right)\right]\right\}$$

$$=\left\{\left[\max\left(1-A_5^{\frac{1}{\eta}},0\right),\max\left(1-B_5^{\frac{1}{\eta}},0\right)\right],\max\left(1-C_5^{\frac{1}{\eta}},0\right),\right.$$

$$\left.\left[\max\left(1-D_5^{\frac{1}{\eta}},0\right),\max\left(1-E_5^{\frac{1}{\eta}},0\right)\right]\right\}.$$

$$=\left\{\ \left[A_3, B_3, C_3, D_3, E_3\right]\right\}=\left\{\ \left[A,B\right],C,\left[D,E\right]\right\}=\overline{M}$$

4.4.2(iii) Theorem (Boundary)

Let $\overline{M}^{+}=\left\{\left[\max\limits_{p=1}^{n}\left(\underline{l_{M_p}}\right),\max\limits_{p=1}^{n}\left(\overline{l_{M_p}}\right)\right],\max\limits_{p=1}^{n}c_{M_p},\left[\max\limits_{p=1}^{n}\left(\underline{r_{M_p}}\right),\max\limits_{p=1}^{n}\left(\overline{r_{M_p}}\right)\right]\right\}$

$\overline{M}^{-}=\left\{\left[\min\limits_{p=1}^{n}\left(\underline{l_{M_p}}\right),\min\limits_{p=1}^{n}\left(\overline{l_{M_p}}\right)\right],\min\limits_{p=1}^{n}c_{M_p},\left[\min\limits_{p=1}^{n}\left(\underline{r_{M_p}}\right),\min\limits_{j=p}^{n}\left(\overline{r_{M_p}}\right)\right]\right\}$

Then $\overline{M}^{-}\le TIT2FYWG_{\varepsilon}\left\langle \overline{M_1}, \overline{M_2}, ..., \overline{M_n} \right\rangle \le \overline{M}^{+}$

Proof:

Since, $\min\limits_{p=1}^{n}\left(\underline{l_{M_p}}\right)\le \underline{l_{M_p}}\le \max\limits_{p=1}^{n}\left(\underline{l_{M_p}}\right)$, $\min\limits_{p=1}^{n}\left(\overline{l_{M_p}}\right)\le \overline{l_{M_p}}\le \max\limits_{p=1}^{n}\left(\overline{l_{M_p}}\right)$,

$\min\limits_{p=1}^{n}\left(c_{M_p}\right)\le c_{M_p}\le \max\limits_{p=1}^{n}\left(c_{M_p}\right)$, $\min\limits_{p=1}^{n}\left(\underline{r_{M_p}}\right)\le \underline{r_{M_p}}\le \max\limits_{p=1}^{n}\left(\underline{r_{M_p}}\right)$,

$\min\limits_{p=1}^{n}\left(\overline{r_{M_p}}\right)\le \overline{r_{M_p}}\le \max\limits_{j=1}^{n}\left(\overline{r_{M_p}}\right)$.

we have, $1-\max\limits_{p=1}^{n}\left(\underline{l_{M_p}}\right)\le \underline{l_{M_p}}\le 1-\min\limits_{p=1}^{n}\left(\underline{l_{M_p}}\right)$

$$\Rightarrow \min\left(\left[\operatorname*{sum}_{p=1}^{n}\left(1-\max\left(\underline{l_{M_p}}\right)\right)^{\eta}\right]^{\frac{1}{\eta}},1\right) \le \min\left(\left[\operatorname*{sum}_{p=1}^{n}\left(1-\left(\underline{l_{M_p}}\right)\right)^{\eta}\right]^{\frac{1}{\eta}},1\right)$$

$$\le \min\left(\left[\operatorname*{sum}_{p=1}^{n}\left(1-\min\left(\underline{l_{M_p}}\right)\right)^{\eta}\right]^{\frac{1}{\eta}},1\right)$$

$$\Rightarrow \min\left(\left[\left(1-\max\left(\underline{l_{Mp}}\right)\right)^{\eta}\right]^{\operatorname*{prod}_{p=1}^{n}\left(\frac{1}{\eta}\right)},1\right) \le \min\left(\left[\left(1-\underline{l_{M_p}}\right)^{\gamma}\right]^{\operatorname*{prod}_{p=1}^{n}\left(\frac{1}{\eta}\right)},1\right)$$

$$\le \min\left(\left[\left(1-\min\left(\underline{l_{M_p}}\right)\right)^{\eta}\right]^{\operatorname*{prod}_{p=1}^{n}\left(\frac{1}{\eta}\right)},1\right)$$

$$\Rightarrow \min\left(\left[\left(1-\max\left(\underline{l_{M_p}}\right)\right)\right],1\right) \le \min\left(\left[\left(1-\underline{l_{M_p}}\right)\right],1\right)$$

$$\le \min\left(\left[\left(1-\min\left(\underline{l_{M_p}}\right)\right)\right],1\right).$$

$$\Rightarrow \min\left(\min\left(\underline{l_{M_p}}\right),1\right) \le \min\left(1-\operatorname*{sum}_{p=1}^{n}\left(\underline{l_{M_p}}\right),1\right) \le \min\left(\max\left(\underline{l_{M_p}}\right),1\right)$$

$$\Rightarrow \min\left(\underline{l_{M_p}}\right) \le \min\left(1-\operatorname*{sum}_{p=1}^{n}\left(\underline{l_{M_p}}\right),1\right) \le \max\left(\underline{l_{M_p}}\right)$$

Similarly we have,

$$\min\left(\overline{l_{M_p}}\right) \le \min\left(1-\operatorname*{sum}_{p=1}^{n}\left(\overline{l_{M_p}}\right),1\right) \le \max\left(\overline{l_{M_p}}\right), \min\left(c_{M_p}\right)$$

$$\le \min\left(1-\operatorname*{sum}_{p=1}^{n}\left(c_{M_p}\right),1\right) \le \max\left(c_{M_p}\right)$$

$$\min\left(\underline{r_{M_p}}\right) \le \min\left(1-\operatorname*{sum}_{p=1}^{n}\left(\underline{r_{M_p}}\right),1\right) \le \max\left(\underline{r_{M_p}}\right),$$

$$\min\left(\overline{r_{M_p}}\right) \le \min\left(1-\operatorname*{sum}_{p=1}^{n}\left(\overline{l_{M_p}}\right),1\right) \le \max\left(\overline{r_{M_p}}\right)$$

By using the ranking value formula for TIT2FN and using the arithmetic average ranking value,

$$R\left(\overline{M}\right)=\left(\frac{\underline{l_{M_p}}+\overline{r_{M_p}}}{2}+1\right)\times\frac{\underline{l_{M_p}}+\overline{l_{M_p}}+\underline{r_{M_p}}+\overline{r_{M_p}}+4c_M}{8}$$

$$\leq\left(\frac{\max\left(\underline{l_{M_p}}\right)+\max\left(\overline{r_{M_p}}\right)}{2}+1\right)\times\left(\max_{p=1}^{n}\left(\underline{l_{M_p}}\right)+\max_{p=1}^{n}\left(\overline{l_{M_p}}\right)+\max_{p=1}^{n}\left(\underline{r_{M_p}}\right)\right.$$

$$\left.+\max_{p=1}^{n}\left(\overline{r_{M_p}}\right)+4\max_{p=1}^{n}\left(c_{M_p}\right)\right)\times 8^{-1}=R\left(\overline{M}^{+}\right)$$

Hence the result.

4.4.2(iv) Theorem

If $t>0$ for all the values of p then $TIT2FYWG_{\varepsilon}\left(\overline{M_1}^{\bullet_Y t},\overline{M_2}^{\bullet_Y t},...,\overline{M_n}^{\bullet_Y t}\right)$

$=TIT2FYWG_{\varepsilon}\left(\overline{M_1},\overline{M_2},...,\overline{M_n}\right)^{\bullet_Y t}$

Proof:

$$\overline{M}^{\bullet_Y t}=\left\{\left[\max\left(1-A_5^{\frac{t}{\eta}},0\right),\max\left(1-B_5^{\frac{t}{\eta}},0\right)\right],\max\left(1-C_5^{\frac{t}{\eta}},0\right),.\right.$$

$$\left.\left[\max\left(1-D_5^{\frac{t}{\eta}},0\right),\max\left(1-E_5^{\frac{t}{\eta}},0\right)\right]\right\}$$

$$=\left\{\left[\max\left(1-\left[\underset{p=1}{\overset{k}{sum}}\left(1-\max\left(1-A_k^{\frac{\varepsilon_p}{\eta}},0\right)\right)\right],0\right),\right.\right.$$

$$\max\left(1-\left[\underset{p=1}{\overset{k}{sum}}\left(1-\max\left(1-B_k^{\frac{\varepsilon_p}{\eta}},0\right)\right)\right],0\right)\Bigg],$$

$$\max\left(1-\left[\underset{p=1}{\overset{k}{sum}}\left(1-\max\left(1-C_k^{\frac{\varepsilon_p}{\eta}},0\right)\right)\right],0\right),$$

$$\Bigg[\max\left(1-\left[\underset{p=1}{\overset{k}{sum}}\left(1-\max\left(1-D_k^{\frac{\varepsilon_p}{\eta}},0\right)\right)\right],0\right),$$

$$\max\left(1-\left[\underset{p=1}{\overset{k}{sum}}\left(1-\max\left(1-E_k^{\frac{\varepsilon_p}{\eta}},0\right)\right)\right],0\right)\Bigg]\Bigg\}.$$

$$TIT2FYWG_{\varepsilon}\left\langle\overline{M_1},\overline{M_2},\ldots,\overline{M_n}\right\rangle$$

$$=\overline{M_1}^{\bullet_Y \varepsilon_1}\underset{Y}{\otimes}\overline{M_2}^{\bullet_Y \varepsilon_2}\underset{Y}{\otimes}\ldots\underset{Y}{\otimes}\overline{M_n}^{\bullet_Y \varepsilon_n}$$

$$=\Bigg\{\Bigg[\max\left(1-\left[\underset{p=1}{\overset{n}{sum}}\left(1-\max\left(1-\left(1-A_2\right)^{\frac{\varepsilon_p}{\eta}}\right),0\right)\right],0\right),$$

$$\max\left(1-\left[\underset{p=1}{\overset{n}{sum}}\left(1-\max\left(1-\left(1-B_2\right)^{\frac{\varepsilon_p}{\eta}}\right),0\right)\right],0\right)\Bigg]$$

$$\max\left(1-\left[\underset{p=1}{\overset{n}{sum}}\left(1-\max\left(1-\left(1-C_2\right)^{\frac{\varepsilon_p}{\eta}}\right),0\right)\right],0\right),$$

$$\Bigg[\max\left(1-\left[\underset{p=1}{\overset{n}{sum}}\left(1-\max\left(1-\left(1-D_2\right)^{\frac{\varepsilon_p}{\eta}}\right),0\right)\right],0\right),$$

$$\max\left(1-\left[\underset{p=1}{\overset{n}{sum}}\left(1-\max\left(1-\left(1-E_2\right)^{\frac{\varepsilon_p}{\eta}}\right),0\right)\right],0\right)\Bigg]\Bigg\}$$

$$=\left\{\left[\max\left(1-A_n^{\frac{t\varepsilon_p}{\eta}},0\right),\ \max\left(1-B_n^{\frac{t\varepsilon_p}{\eta}},0\right)\right],\ \max\left(1-C_n^{\frac{t\varepsilon_p}{\eta}},0\right),\right.$$

$$\left.\left[\max\left(1-D_n^{\frac{t\varepsilon_p}{\eta}},0\right),\max\left(1-E_n^{\frac{t\varepsilon_p}{\eta}},0\right)\right]\right\} \tag{4.45}$$

Also since, $TIT2FYWG_{\varepsilon}\left(\overline{M_1},\overline{M_2},...,\overline{M_n}\right)^{\bullet_Y t}$

$$=\left\{\left[\max\left(1-A_n^{\frac{t\varepsilon_p}{\eta}},0\right),\ \max\left(1-B_n^{\frac{t\varepsilon_p}{\eta}},0\right)\right],\max\left(1-C_n^{\frac{t\varepsilon_p}{\eta}},0\right),\right.$$

$$\left.\left[\max\left(1-D_n^{\frac{t\varepsilon_p}{\eta}},0\right),\max\left(1-E_n^{\frac{t\varepsilon_p}{\eta}},0\right)\right]\right\}^{\bullet_Y t}.$$

$$=\left\{\left[\max\left(\left(1-A_n^{\frac{t}{\eta}}\right)^{\varepsilon_p},0\right),\ \max\left(\left(1-B_n^{\frac{t}{\eta}}\right)^{\varepsilon_p},0\right)\right],\max\left(\left(1-C_n^{\frac{t}{\eta}}\right)^{\varepsilon_p},0\right),\right.$$

$$\left.\left[\max\left(\left(1-D_n^{\frac{t}{\eta}}\right)^{\varepsilon_p},0\right),\max\left(\left(1-E_n^{\frac{t}{\eta}}\right)^{\varepsilon_p},0\right)\right]\right\}$$

$$=\left\{\left[\max\left(\left(1-A_n^{\frac{t\varepsilon_p}{\eta}}\right),0\right),\ \max\left(\left(1-B_n^{\frac{t\varepsilon_p}{\eta}}\right),0\right)\right],\max\left(\left(1-C_n^{\frac{t\varepsilon_p}{\eta}}\right),0\right),\right.$$

$$\left.\left[\max\left(\left(1-D_n^{\frac{t\varepsilon_p}{\eta}}\right),0\right),\max\left(\left(1-E_n^{\frac{t\varepsilon_p}{\eta}}\right),0\right)\right]\right\} \tag{4.46}$$

Equation (4.45) and Equation (4.46) are equal.

Hence the result.

4.4.2(v) Theorem (Stability)

If $t>0$, $\overline{M_{n+1}}=\left(\left[\underline{l_{M_{n+1}}},\overline{l_{M_{n+1}}}\right],c_{M_{n+1}},\left[\underline{r_{M_{n+1}}},\overline{r_{M_{n+1}}}\right]\right)$ then

$$TIT2FYWG_{\varepsilon}\left(\overline{M_1}^{\bullet_Y t}\underset{Y}{\otimes}\overline{M_{n+1}},\overline{M_2}^{\bullet_Y t}\underset{Y}{\otimes}\overline{M_{n+1}},...,\overline{M_n}^{\bullet_Y t}\underset{Y}{\otimes}\overline{M_{n+1}}\right)$$
$$=TIT2FYWG_{\varepsilon}\left(\overline{M_1},\overline{M_2},...,\overline{M_n}\right)^{\bullet_Y t}\underset{Y}{\otimes}\overline{M_{n+1}}$$

Proof:

$$TIT2FYWG_{\varepsilon}\left(\overline{M_1}^{\bullet_Y t}\underset{Y}{\otimes}\overline{M_{n+1}},\overline{M_2}^{\bullet_Y t}\underset{Y}{\otimes}\overline{M_{n+1}},...,\overline{M_n}^{\bullet_Y t}\underset{Y}{\otimes}\overline{M_{n+1}}\right)$$

$$=\left\{\left[\max\left(1-A_n^{\frac{t\varepsilon_p}{\eta}},0\right)\underset{Y}{\otimes}\overline{M_{n+1}},\ \max\left(1-B_n^{\frac{t\varepsilon_p}{\eta}},0\right)\underset{Y}{\otimes}\overline{M_{n+1}}\right],\max\left(1-C_n^{\frac{t\varepsilon_p}{\eta}},0\right)\underset{Y}{\otimes}\overline{M_{n+1}},\right.$$

$$\left.\left[\max\left(1-D_n^{\frac{t\varepsilon_p}{\eta}},0\right)\underset{Y}{\otimes}\overline{M_{n+1}},\max\left(1-E_n^{\frac{t\varepsilon_p}{\eta}},0\right)\underset{Y}{\otimes}\overline{M_{n+1}}\right]\right\}$$

$$=\left\{\left[\max\left(1-\left[\underset{p=1}{\overset{n}{sum}}\left(1-\max\left(1-\left[\underset{q=\{p,n+1\}}{sum}\left(1-\underline{l_{M_q}}\right)^{\eta}\right]^{\frac{t}{\eta}},0\right)^{\varepsilon_p}\right)\right],0\right),\right.\right.$$

$$\left.\max\left(1-\left[\underset{p=1}{\overset{n}{sum}}\left(1-\max\left(1-\left[\underset{q=\{p,n+1\}}{sum}\left(1-\overline{l_{M_q}}\right)^{\eta}\right]^{\frac{t}{\eta}},0\right)^{\varepsilon_p}\right)\right],0\right)\right]$$

$$\max\left(1-\left[\underset{p=1}{\overset{n}{sum}}\left(1-\max\left(1-\left[\underset{q=\{p,n+1\}}{sum}\left(1-c_{M_q}\right)^{\eta}\right]^{\frac{t}{\eta}},0\right)^{\varepsilon_p}\right)\right],0\right),$$

$$\left[\max\left(1-\left[\underset{p=1}{\overset{n}{sum}}\left(1-\max\left(1-\left[\underset{q=\{p,n+1\}}{sum}\left(1-\underline{r_{M_q}}\right)^{\eta}\right]^{\frac{t}{\eta}},0\right)^{\varepsilon_p}\right)\right],0\right),\right.$$

$$\left.\left.\max\left(1-\left[\underset{p=1}{\overset{n}{sum}}\left(1-\max\left(1-\left[\underset{q=\{p,n+1\}}{sum}\left(1-\overline{r_{M_q}}\right)^{\eta}\right]^{\frac{t}{\eta}},0\right)^{\varepsilon_p}\right)\right],0\right)\right]\right\}$$

$$=\left\{\left[\max\left(\left(1-A_n^{\frac{\varepsilon_p}{\eta}}\right)+\left(\left[\left(1-\underline{l_{M+1}}\right)^{\eta}\right]^{\frac{1}{\eta}}\right)^{\underset{p=1}{\overset{n}{sum}}\varepsilon_p},0\right),\right.\right.$$

$$\left.\max\left(\left(1-B_n^{\frac{\varepsilon_p}{\eta}}\right)+\left(\left[\left(1-\overline{l_{M+1}}\right)^{\eta}\right]^{\frac{1}{\eta}}\right)^{\underset{p=1}{\overset{n}{sum}}\varepsilon_p},0\right)\right],$$

$$\max\left(\left(1-C_n^{\frac{\varepsilon_p}{\eta}}\right)+\left(\left[\left(1-c_{M+1}\right)^{\eta}\right]^{\frac{1}{\eta}}\right)^{\underset{p=1}{\overset{n}{sum}}\varepsilon_p},0\right),$$

$$\left[\max\left(\left(1-D_n^{\frac{\varepsilon_p}{\eta}}\right)+\left(\left[\left(1-\underline{r_{M+1}}\right)^{\eta}\right]^{\frac{1}{\eta}}\right)^{\underset{p=1}{\overset{n}{sum}}\varepsilon_p},0\right),\right.$$

$$\left.\left.\max\left(\left(1-E_n^{\frac{\varepsilon_p}{\eta}}\right)+\left(\left[\left(1-\overline{r_{M+1}}\right)^{\eta}\right]^{\frac{1}{\eta}}\right)^{\underset{p=1}{\overset{n}{sum}}\varepsilon_p},0\right)\right]\right\} \tag{4.47}$$

Based on the Theorem 4.4.2(i) and Equation 4.44,

$$\text{TIT2FYWG}_{\varepsilon}\left\langle \overline{M_1}, \overline{M_2}, ..., \overline{M_n} \right\rangle_Y^{\bullet t} \underset{Y}{\otimes} \overline{M_{n+1}}$$

$$= \left\{ \left[\max\left(1 - A_n^{\frac{t\varepsilon_p}{\eta}}, 0\right), \max\left(1 - B_n^{\frac{t\varepsilon_p}{\eta}}, 0\right) \right], \max\left(1 - C_n^{\frac{t\varepsilon_p}{\eta}}, 0\right), \right.$$

$$\left. \left[\max\left(1 - D_n^{\frac{t\varepsilon_p}{\eta}}, 0\right),\ \max\left(1 - E_n^{\frac{t\varepsilon_p}{\eta}}, 0\right) \right] \right\}$$

$$\underset{Y}{\otimes} \left\langle \left[\underline{l_{M_{n+1}}}, \overline{l_{M_{n+1}}} \right], c_{M_{n+1}}, \left[\underline{r_{M_{n+1}}}, \overline{r_{M_{n+1}}} \right] \right\rangle$$

$$= \left\{ \left[\max\left(\left(1 - A_n^{\frac{\varepsilon_p}{\eta}}\right) + \left(\left[\left(1 - \underline{l_{M+1}}\right)^{\eta} \right]^{\frac{1}{\eta}} \right), 0 \right),\ \max\left(\left(1 - B_n^{\frac{\varepsilon_p}{\eta}}\right) + \left(\left[\left(1 - \overline{l_{M+1}}\right)^{\eta} \right]^{\frac{1}{\eta}} \right), 0 \right) \right], \right.$$

$$\max\left(\left(1 - C_n^{\frac{\varepsilon_p}{\eta}}\right) + \left(\left[\left(1 - c_{M+1}\right)^{\eta} \right]^{\frac{1}{\eta}} \right), 0 \right), \tag{4.48}$$

$$\left. \left[\max\left(\left(1 - D_n^{\frac{\varepsilon_p}{\eta}}\right) + \left(\left[\left(1 - \underline{r_{M+1}}\right)^{\eta} \right]^{\frac{1}{\eta}} \right), 0 \right),\ \max\left(\left(1 - E_n^{\frac{\varepsilon_p}{\eta}}\right) + \left(\left[\left(1 - \overline{r_{M+1}}\right)^{\eta} \right]^{\frac{1}{\eta}} \right), 0 \right) \right] \right\}$$

Therefore, Equation (4.47) and Equation (4.48) are equal. Hence the result.

4.4.2(vi) Theorem (Image Contrast)

For given arguments $\overline{M}_p, p = 1, 2, ..., n$ and the parameter $\eta \in (1, +\infty)$ then TIT2FYWG operator is monotonically non-decreasing (MND) with respect to the parameter.

Proof:

To prove the operator is MND with respect to the parameter, we have to prove the same for every reference point function is MND w.r.t the parameter.

Since $0 \leq \underline{l_M} \leq \overline{l_M} \leq c_M \leq \underline{r_M} \leq \overline{r_M} \leq 1$, $\max\left(1 - A_5^{\frac{k}{\eta}}, 0\right) > 0.$

And it is true for all the reference points. Hence the result.

Note: The above theorems also can be proved by using TIT2FYWA operator.

4.4.4 APPLICATION OF IMAGE PROCESSING

Using MATLAB 2015a, triangular norms have been applied in medical image processing from a patient DICOM image. In this case three dimensional image is converted to two dimensional image. The image is collected from our experimental data set from a patient DICOM image (Plate 4.1).

Size of the image	= 512 x 517.
Mean of the image	= 28.83.
Standard deviation	= 60.79
Mean absolute deviation	= 40.03.

4.4.5 Algorithm for the proposed edge detection method

Step 1: Input the DICOM image.

Step 2: Apply discrete wavelet transform on the image until desired level

Step 3: At every level aggregate the detailed coefficients by choosing the maximum value

Step 4: Find the gradient through x axis and y axis using t-norm and t-conorm respectively.

Step 5: Detect and save the common regions of the image and output

Step 6: Arrange the remaining region of the image into column vectors

Step 7: Design the fuzzy logic system as follows.

(i). choose the fuzzifier

(ii). choose the membership functions for input and output (iii). Design the rules for edge detection

(iv). Perform the fuzzy inference using lower and upper Membership functions of the antecedent operation and the fuzzified output is calculated by IT2TFYWG operator

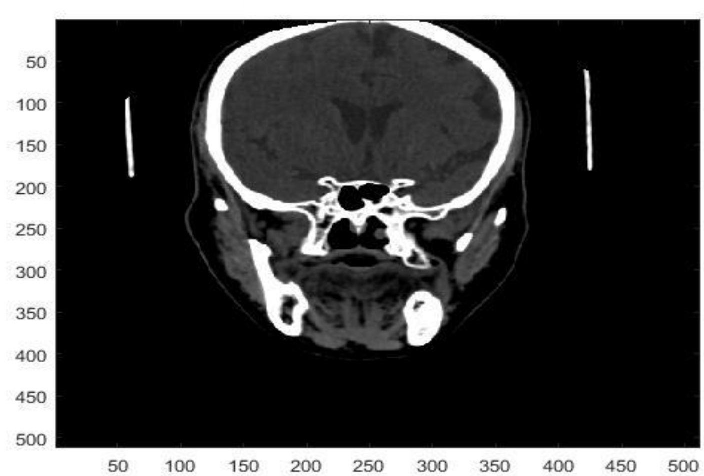
(v). Apply centroid formula to transform the output into type-1 fuzzy set to find the lower and upper membership functions.

(vi). The crisp output is obtained by ranking formula

Step 8: Covert the output column into edge detected image

To identify the gradient of the image dilation-erosion, triangular norms have been used.

Plate 4.1: Original gray scale DICOM image



Structuring elements are used in gradient value.

Image Erosion is

0.9961	1.0000	0
1.0000	0.9961	0
1.0000	1.0000	1.0000

The output of the image processing application in edge detection through triangular norms by MATLAB 2015 a.

The gradient through x axis and the gradient through the y axis using have been done using triangular norms (Plate 4.2 and Plate 4.3).

The plates reveal that the image gradient to identify the region uniformly.

Plate 4.2: Gradient through X axis

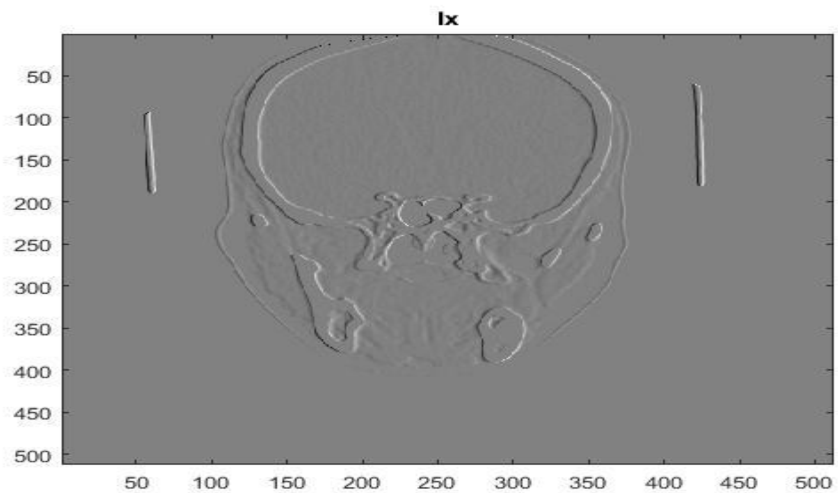
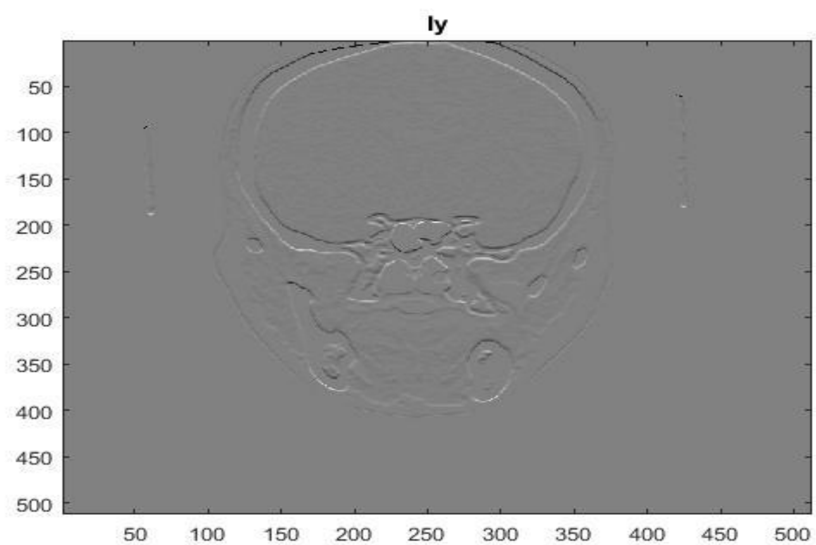
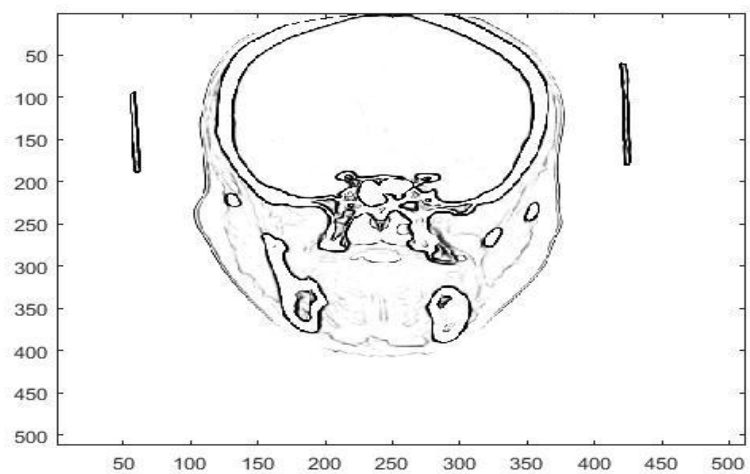


Plate 4.3: Gradient through Y axis



The output of the edge detection using type-2 fuzzy logic from our experimental data using MATLAB 2015a (Plate 4.4).

Plate 4.4: Image after Edge detection



The edges of the object through FIS and equating the pixel on both directions. If the edge is block then pixel is not 0 (Plate 4.4).

Edge detection plays a vital role in image identification. It is observed that, it helps in reducing the memory for saving medical images.

4.5 AUTOMATIC SELECTION OF DETERGENT QUANTITY USING INTERVAL TYPE-2 FUZZY LOGIC CONTROLLER

In this chapter, automatic detergent intake according to the amount of cloths and their level of dirtiness has been analyzed using interval type-2 fuzzy logic controller in a washing machine. Also transporation delay and stability analysis have been analyzed by using four different defuzzification methods.

4.5.1 PROBLEM FORMATION

While using washing machine, the amount of washing powder or detergent is used manually on the user's knowledge about the parameters such as the amount of cloths need to be washed, type of the material and degree of dirt of the cloths. Depending on the decision of the user for the selection of the detergent quantity, as a result the uncertainty exists. Hence to automate this process, we address the problem by using IT2FLC. The reason of using this type of fuzzy logic controller is that, the controller provides the exact amount of detergent even though an exact model of the input/output is not available. Here the parameters are considered as interval data instead of crisp one.

4.5.2 DETAILS OF THE PROBLEM

Consider two input variables (IVs) to make the study comprehensible and easy and the inputs are,

1. Weight of the cloths (WC)
2. Type of dirt (TD).

IT2FLC takes these two inputs in the form of interval data, measures the information and the output of the quantity of the detergent. At the same time, getting the sensorial inputs of the WM and their working process (electrical, optical or any other type) is not the concern of this study.

The introduction of the inputs is the weight of the cloths decided by the knowledge of the person who is using the washing machine. The more amounts of the cloths necessitates the more consumption of the detergent. The type of dirt is inclined by the saturation time. The point at which no further detectable changes in the water color is called saturation point. The dirt type determines the nature of dirt such as the excess level of grease, the medium level of grease or no content of grease at all.

Greasy cloths take large amount of detergent according to clearness and the purity level of the water since greasy type needs more amount of detergent for an effective washing process than the dirt of other types. Hence, the straight forward sensor system can supply us the necessary input for the IT2FLC.

4.5.3 PLAN OF THE SET APPLIED

The input and output variables (OVs) and their ranges are needed to be determined before dealing with IT2FLC. MF connects the real world calculations with fuzzy rules. The labels of IVs and OVs and their corresponding MFs are defined as follows. The values of the IVs assigned (-1 to 100) in optical sensor's domain (Plate 4.7). The determination of the IT2FLC is derived from the rules which are saved in the database.

Generally, the rules are supposedly IF-THEN statements which are spontaneous and understandable (common English statements). Here the rules used are derived from the knowledge of an individual in the form of IF-THEN statement like IF (linguistic term) and (another linguistic term) THEN (linguistic term). Here the linguistic term is the result of the natural language with inherently uncertainty exists.

The set of rules are:

1. If the WC is more and TD is greasy then the detergent intake is very high (VH)
2. If the WC is medium and TD is greasy then the detergent intake is high (H)
3. If the WC is low and TD then the detergent intake is low (H)
4. If the WC is high and TD is medium then the detergent intake is high (H)
5. If the WC and TD then the detergent intake is medium (M)
6. If the WC is low and TD is medium then the detergent intake is low (L)
7. If the WC is high and TD is not greasy then the detergent intake is medium (M)
8. If the WC is medium and TD is not greasy then the detergent intake is low (L)
9. If the WC is low and TD is not greasy then the detergent intake is very low (VL)

Here the level of greasiness is the reason for consuming the higher quantity of detergent. The case, not greasy which is mentioned in the above fuzzy rules has been considered as low and medium level of dirtiness of the cloths respectively for the calculation.

4.5.4 ALGORITHM FOR RULE CALCULATION

Step 1: Assume the range of the inputs in terms of IT2FNs

Step 2: Find the score values of the inputs

Step 3: Assume the values lie in [1.5, 2] for high and maximum value 2 for very high, values lies in [1, 1.5] for medium and [0, 1] for low and the values lies in [0,0.2] for very low. The resultant value is the level of detergent intake.

Step 4: End

4.5.5 APPLICATION OF THE PROPOSED ALGORITHM

The weight and type of dirtiness of the cloths are considered as TIT2FNs for the levels of low, medium and high (Table 4.4). Detergent intake is calculated by the score function formula for TIT2FN.

Table 4.4: Range of the input assumed for weight, dirt

Weight of cloths	Type of dirt
L: ([0.2,0.3],0.4,[0.5,0.6])	L: ([0.24,0.36], 0.47, [0.58,0.67])
M: ([0.4,0.5],0.6,[0.7,0.8])	M: ([0.45,0.55], 0.66, [0.73,0.84])
H: ([0.6,0.7],0.8,[0.9,0.95])	H:[0.67,0.79], 0.85, [0.92,0.97])

The maximum value 2 denotes the very high intake and the minimum value 0.1 denotes very low intake of detergent (Table 4.5).

Table 4.5: Interval Type-2 Fuzzy Rule Set

Rule No.	Inputs		Output
	Weight of cloths	Type of dirt	Detergent Intake
1	H: ([0.6,0.7],0.8, [0.9,0.95])	H: ([0.67,0.79], 0.85, [0.92,0.97])	VH:2
2	M: ([0.4,0.5],0.6, [0.7,0.8])	H: ([0.67,0.79], 0.85, [0.92,0.97])	H:1.5
3	L: ([0.2,0.3],0.4, [0.5,0.6])	H: ([0.67,0.79], 0.85, [0.92,0.97])	H:0.44

4	H: ([0.6,0.7],0.8, [0.9,0.95])	M: ([0.45,0.55], 0.66, [0.73,0.84])	H:1.51
5	M: ([0.4,0.5],0.6, [0.7,0.8])	M: ([0.45,0.55], 0.66, [0.73,0.84])	M:1.03
6	L: ([0.2,0.3],0.4, [0.5,0.6])	M: ([0.45,0.55], 0.66, [0.73,0.84])	L:0.31
7	H: ([0.6,0.7],0.8, [0.9,0.95])	M (not greasy): ([0.45,0.55], 0.66, [0.73,0.84])	M:1.51
8	M: ([0.4,0.5],0.6, [0.7,0.8])	L(not greasy): ([0.24,0.36], 0.47, [0.58,0.67])	L: 0.65
9	L: ([0.2,0.3],0.4, [0.5,0.6])	M(not greasy): ([0.24,0.36], 0.47, [0.58,0.67])	L: 0.20

4.5.6 OUTPUT FOR THE RULES

The crisp output of the rules has been obtained by the score value (SV) using score function (SF) of the TIT2FN for each input using Equation 4.49.

$$S(F)=\left(\frac{A_l^L+A_r^U}{2}+1\right)\times\left(\frac{A_l^L+A_l^U+A_r^L+A_r^U+4A_c}{8}\right) \quad (4.49)$$

$$SV\left(\left\langle [0.2,0.3],0.4,[0.5,0.6]\right\rangle\right)$$

$$=\left(\frac{0.2+0.6}{2}+1\right)\times\left(\frac{0.2+0.3+0.5+0.6+4(0.4)}{8}\right)=0.29$$

$$SV\left(\left\langle [0.24,0.36],0.47,[0.58,0.67]\right\rangle\right)$$

$$=\left(\frac{0.24+0.67}{2}+1\right)\times\left(\frac{0.24+0.36+0.58+0.67+4(0.47)}{8}\right)=0.68$$

$$SV\left(\left\langle [0.4,0.5],0.6,[0.7,0.8]\right\rangle\right)$$

$$=\left(\frac{0.4+0.8}{2}+1\right)\times\left(\frac{0.4+0.5+0.7+0.8+4(0.6)}{8}\right)=0.96$$

$$SV\left(\left\langle [0.45,0.55],0.66,[0.73,0.84]\right\rangle\right)$$

$$=\left(\frac{0.45+0.84}{2}+1\right)\times\left(\frac{0.45+0.55+0.73+0.84+4(0.66)}{8}\right)=1.07$$

$$SV\left(\left\langle [0.6,0.7],0.8,[0.9,0.95]\right\rangle\right)$$

$$=\left(\frac{0.6+0.96}{2}+1\right)\times\left(\frac{0.6+0.7+0.9+0.95+4(0.8)}{8}\right)=1.42$$

$$SV\left(\left\langle [0.67,0.79],0.85,[0.92,0.97]\right\rangle\right)$$

$$=\left(\frac{0.67+0.97}{2}+1\right)\times\left(\frac{0.67+0.79+0.92+0.97+4(0.85)}{8}\right)=1.52$$

$$SV\left(\left\langle [0.65,0.75],0.84,[0.95,1]\right\rangle\right)$$

$$=\left(\frac{0.65+1}{2}+1\right)\times\left(\frac{0.65+0.75+0.95+1+4(0.84)}{8}\right)=1.53$$

$$SV\left(\left\langle [0.68,0.8],0.86,[0.96,1]\right\rangle\right)$$

$$=\left(\frac{0.68+1}{2}+1\right)\times\left(\frac{0.68+0.8+0.96+1+4(0.86)}{8}\right)=1.86$$

$$SV\left(\langle[0.18,0.25],0.3,[0.4,0.46]\rangle\right)$$

$$=\left(\frac{0.18+0.46}{2}+1\right)\times\left(\frac{0.18+0.25+0.4+0.46+4(0.3)}{8}\right)=0.41$$

$$SV\left(\langle[0.19,0.2],0.35,[0.45,0.5]\rangle\right)$$

$$=\left(\frac{0.19+0.5}{2}+1\right)\times\left(\frac{0.19+0.2+0.5+0.46+4(0.35)}{8}\right)=0.46$$

Therefore, maximum detergent intake = 2

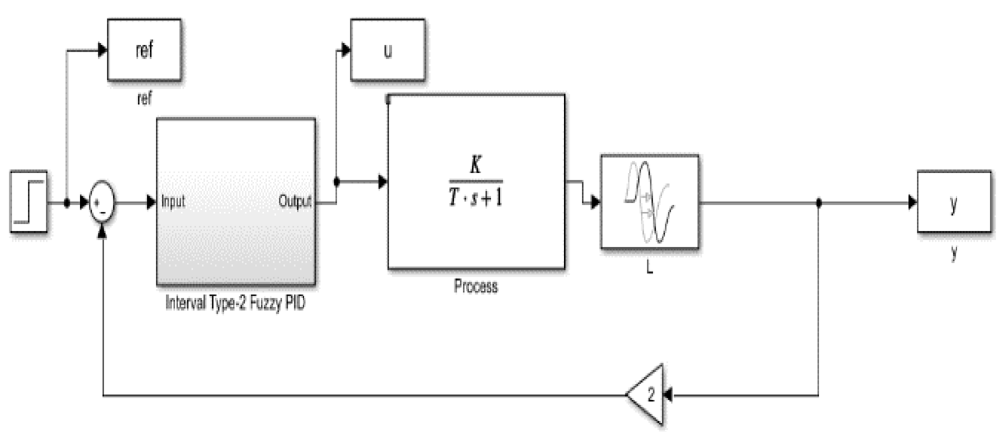
4.5.7 DESCRIPTION OF THE SYSTEM

Two inputs dirt, weight and one output designed in FIS. The dirt level and weight of the cloth are divided into three stages such as low, medium and high and are represented by TIT2FNs.

The fuzzy inference system was developed for interval type 2 fuzzy and named as IT2 FPID WM and nine rules were framed.

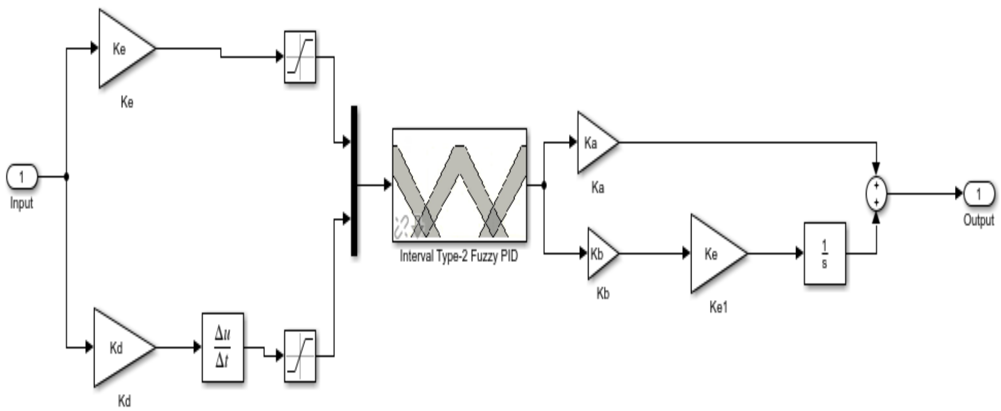
Fuzzy inference system (FIS) is named as interval type-2 fuzzy PID, output response is y and time delay denoted by block L (Plate 4.5). The time delay can be varied in the program using variable L.

Plate 4.5: Block diagram of IT2F WM contol system



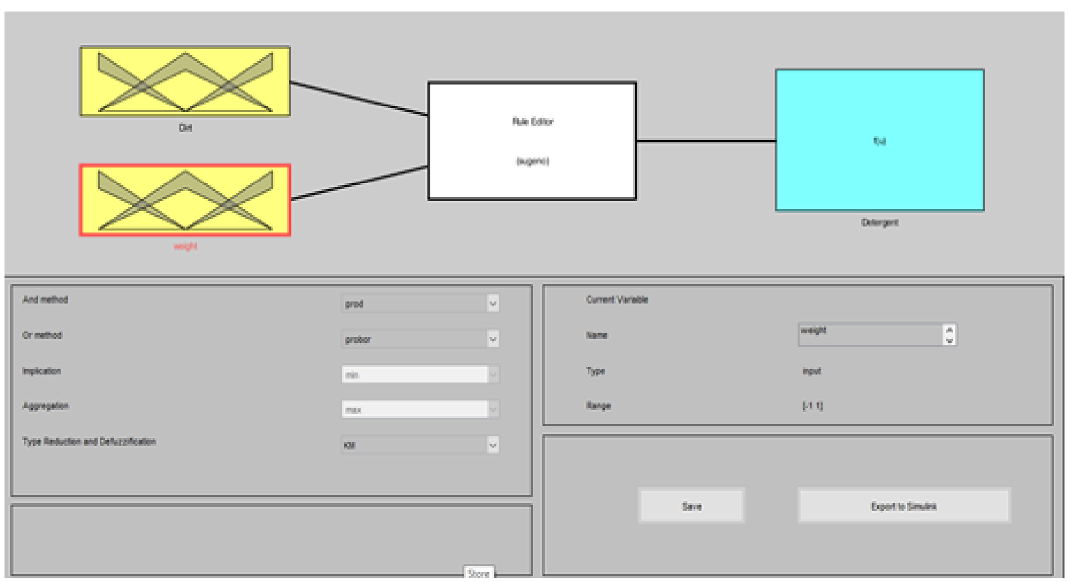
The sub system (Plate 4.6) has fuzzy inference system and other blocks with variables which are represented in program scripts.

Plate 4.6: Block diagram of IT2FC Sub system



Sugeno FIS (Plate 4.7) with membership functions where the editing of fuzzy rule is possible. Here two inputs give nine potential considerations.

Plate 4.7: IT2FC parameters



The overall input membership and output memberships are given in the following table:

The rule base of IT2FLC in terms of linguistic terms for weight of cloths, level of dirt and necessary detergent intake (Table 4.6).

Table 4.6: Rule base of IT2FLC

Weight of the Cloths	Dirt	Detergent Intake
H	H	VH
M	H	H
L	H	H
H	M	H
M	M	M
L	M	M
H	M (Not greasy)	M
M	L (Not greasy)	L
L	M (Not greasy)	VL

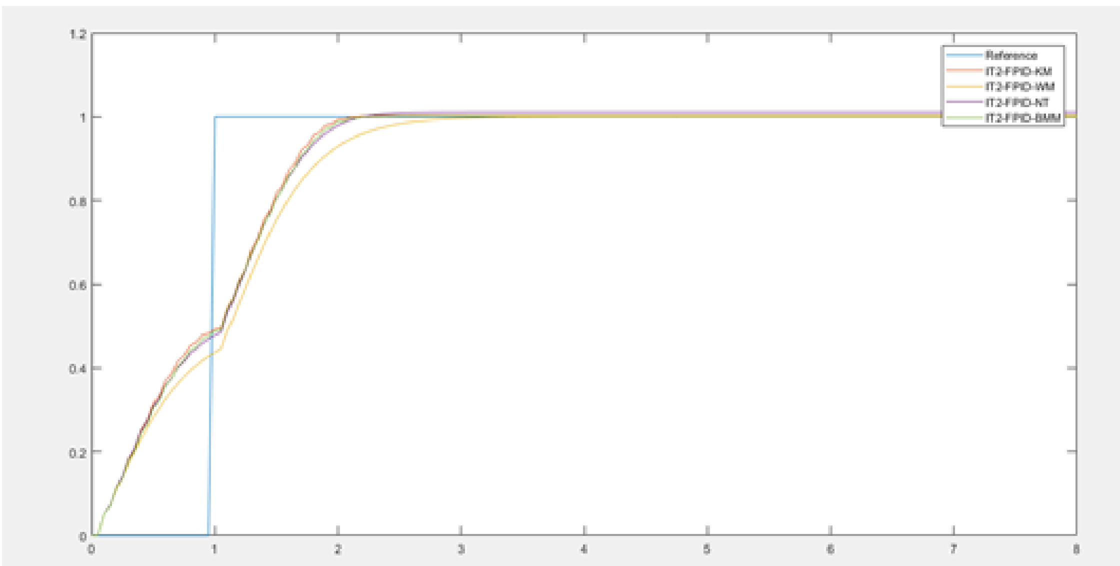
Hence, the intake of the detergent varies according to the weight of the cloths and dirtiness of the cloths.

4.5.8 STABILIZE THE OUTPUT WITH REFERENCE

To increase the stability of the output, there are four defuzzification methods used namely: **k**arnik Mendal Algorithm (KM), Wu Mendal Uncertainty Bound Method (WM), Begain Melak Mendal method (BMM) and Nie-tan method (NT) through type of road (TR) method to analyze delay in transportation.

The stabilize value of Ke=1, Kd=0.5141, Ka=0.077, Kb =17.336, sampling Time=0.05, K=1, T=2.1, and L=0.01.

Plate 4.8: Settling point of the overall system response



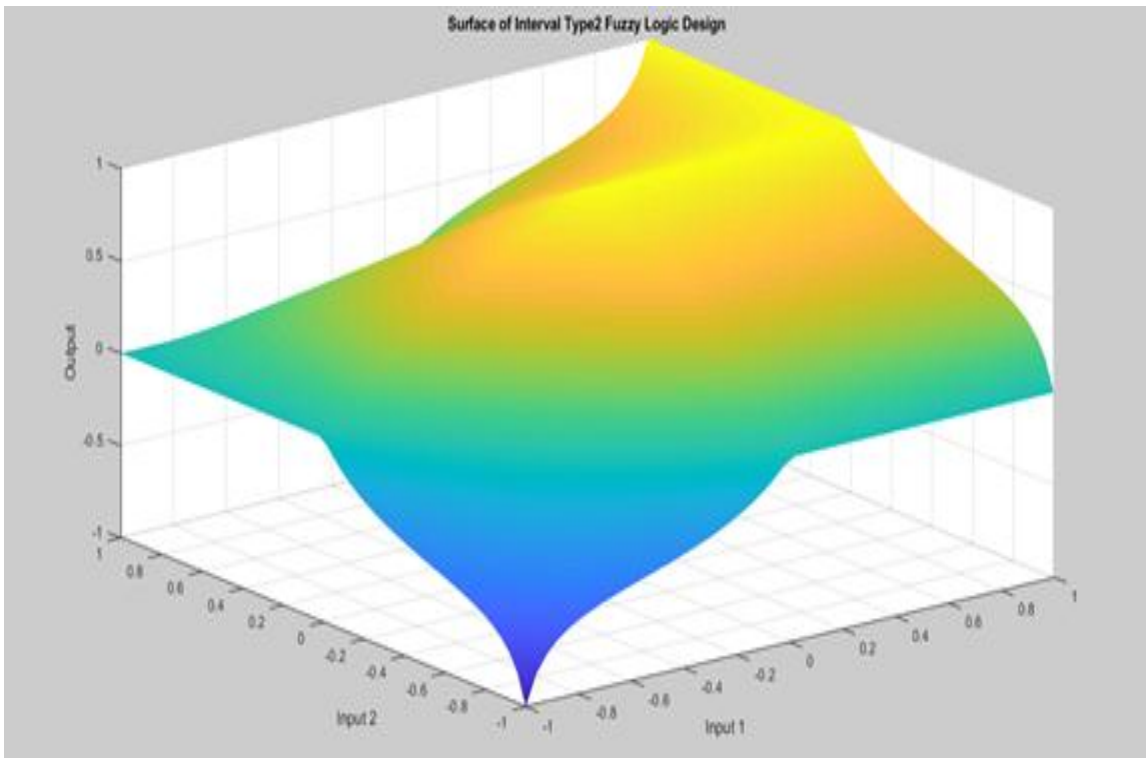
The different type of responses using karnik Mendal Algorithm (KM) , Wu Mendal Uncertainty Bound Method (WM), Nie tan method (NT) and Begain-Melak-Mendal method (BMM) through method time with reference value (Plate 4.8).

From the observation made, it is found that Wu-Mendal Uncertainty Bound Method (WM) gives better performance for automatic detergent intake.

4.5.9 PRECISION

The Fuzzy rule based subsets are modified and reshaping the sub degree within the subset (Plate 4.9). Fuzzy knowledge based on Fuzzy logic gives much better outcome. The output of the model shows that the detergent intake is varying according to the inputs.

Plate 4.9: Surface of Interval Type-2 Fuzzy Logic



CHAPTER-5

DISCUSSION

5.0 DISCUSSION

It is proved that, aggregation and fusion of information are essential concerns for all types of knowledge based systems, from image processing to decision making and from pattern recognition to machine learning. From a general perspective, one can say that, aggregation is a process getting a conclusion or decision by utilizing the information provided by the various sources simultaneously.

Many research groups, namely, the communities of, multi-criteria, sensor fusion, decision making and data mining are exceptionally interested to find the solution of the problem. All these communities are using some kind of techniques or methodologies to aggregate the provided information by applying some suitable rules, probability theory, neural network and theory of FS where the procedures are established on some numerical AO. Hence it plays a vital role in fusing the information in all the areas of real world problems.

From the mathematical perspective, mathematical objects that have the function of reduction of set of numbers into a single number. Hence developing aggregation operators is an important task to solve real world problems under various sets. There are many AOs have been developed under fuzzy and neutrosophic environments. Also there are some AOs that have been developed under interval valued fuzzy environment.

Qin and Liu (2014) have developed triangular interval type-2 Frank AOs and applied them in a decision making process by considering the data as a TIT2FNs. Liu and Wang (2018) have proposed the interval valued intuitionistic fuzzy Schweizer- Sklar power aggregation operators and applied them in a decision making problem for supplier selection by considering the data as an interval valued intuitionistic FNs. But there is no contribution of TCM using interval valued fuzzy and intuitionistic setting.

Ye (2014) proposed AOs under simplified neutrosophic set environment and utilized them in a decision making process. But it is yet to develop aggregation operators under single and interval neutrosophic environments with parameterized aggregation operators. In such a way, the present study gives a new perspective. Here we have developed interval valued Schweizer- Sklar AOs and applied them in traffic flow management. Hence the proposed methods are superior methods.

Though there are many findings related to graph, FG, NG, however the NGs based on AOs are yet to be developed. Especially Broumi *et al.* (2016) applied interval valued neutrosophic in DMP and introduced the properties of different types of degrees size and order of single valued neutrosophic graphs (SNGSs). Further they have proposed the definition of regular SVNG. Ashraf *et al.* (2018) proposed DFG and proved the standard operations on DFGs. There is no finding on neutrosophic graph using triangular norms and hence this present study superseding the previous findings.

One could clearly understand and study the process of Blockchain technology with crisp numbers. As the real world problems have uncertainty in nature, FL is needed to be applied for getting the acceptable result. It is found that, in the article, Arockiaraj and Charumathi (2018), the Blockchain FG has been proposed for crypto currency transaction such as Bitcoin and Ethereum. It is also consider only truth membership grades. It is unable to handle indeterminacy of the transactions. But the proposed study can able to handle falsity and indeterminacy of the transaction which could give an optimized result of the transaction, as the real life situation has indeterminacy also. Hence in this way, the result of the current study superseding the existing results.

It is found that, Myna and Prakash (2018) have used interval type-2 fuzzy logic for the fusion of multifocus image. Here upper and lower membership functions of the Gaussian membership functions to measure the quality of the image and the operation used was ordinary triangular norms operations. But in this present study Yager triangular norms have been used under triangular interval type-2 fuzzy environment. The data set is taken from our experimental data set.

A patient MRI has been considered which is in the three dimensional form and is converted to two dimensional form (DICOM) using MATLAB 2015a. The 3D format consists of 25 DICOM file formats; the montage of the images is obtained as a single frame. Out of these 25 DICOM images a clear full image is chosen. Using Dilation corrosion method, the gradient is identified. The edge detection is performed through triangular norms using MATLAB 2015a. Since it is very useful for the medical field to save their data with less memory, our method is superseded than the existing methods mentioned in the literature and particularly the method compared.

It is found that, Roseline (2016) designed a smart washing machine and developed it using type-1 fuzzy logic controller, which is capable of automatic inputs and desired output using type-1 fuzzy logic controller. Masood (2017) analyzed T1FLC of washing machine, where the procedure has been used for fixing washing time according to the type of cloths. Stammingera (2018) developed rule based fuzzy logic, where parameters are crisp numbers and the desired output is obtained in terms of percentage and the standard units like temperature. But in the present study, IT2FLC has been used to automate the detergent intake as it handles more uncertainty than the T1FLCs and stability has been analyzed using four different defuzzification methods and concluded the better method. Also the parameters have been considered as IT2FNs and the desired output is obtained using the score of the inputs

5.1 CONCLUSION

Controlling traffic flow on roads is an important traffic management task necessary to ensure a peaceful and safe environment for people. The number of cars on roads at any given time is always unknown. Type-2 fuzzy sets and neutrosophic sets play a vital role in dealing efficiently with such uncertainty.

A triangular interval type-2 Schweizer and Sklar weighted arithmetic (TIT2SSWA) operator and a triangular interval type-2 Schweizer and Sklar weighted geometric (TIT2SSWG) operator based on Schweizer and Sklar triangular norms have been studied, and the validity of these operators has been checked using a numerical validation and extended to an interval neutrosophic environment by proposing interval neutrosophic Schweizer and Sklar weighted arithmetic (INSSWA) and interval neutrosophic Schweizer and Sklar weighted geometric (INSSWG) operators.

Furthermore, their properties have been examined; some of the more important properties are examined in detail. Moreover, an improved score function for interval neutrosophic numbers (INNs) has been proposed. Moreover traffic flow control has been analyzed by identifying the junction that has more vehicles.

Dealing indeterminacy is an essential work to get an optimized output in any problem and control system as well. It is possible only with neutrosophic logic and that too effectively by interval valued neutrosophic setting as it has lower and upper membership function for three independent membership functions namely truth, indeterminacy and falsity.

DSVNG and DIVNG have been proposed. Subsequently, it has been proved that Cartesian product and composition of two DIVNGs are DIVNG with numerical validation.

Reliability and assurance of the dealing are very important for any business transaction. Blockchain technology is such technology and recently it is widely applied in many fields. In any field uncertainty is unavoidable one as the human behavior always uncertainty in nature. The indeterminacy also does not deal in any area field of mathematics whereas neutrosophic set deals indeterminacy and hence an optimized solution can be obtained for any problem.

Blockchain network has been used for Bitcoin transaction where the vertex and edges have been considered as single and interval valued Neutrosophic sets. Moreover, the degree, total degree, minimum and maximum degree have been found for the proposed BCSVNG. In addition to this, the contingent study has been done for various types of Blockchain graphs.

Operational laws of addition, multiplication, power and multiplication by an arbitrary number using Yager triangular norms for TIT2FN have been derived. Also some properties of aggregation operation using fuzzy Yager weighted geometric operators have been proved. Since Yager aggregation operator contains minimum and maximum operator, it will be acted as a morphological filters in medical image processing.

Detailed representation of the mathematical properties in image processing is presented. Also, the gradient of the DICOM image of MRI scan of a patient using triangular norms have been found and edge detection of the Brain has been done using MATLAB in type-2 fuzzy logic.

The system which has uncertain parameters will have the problem of maintaining the system's stability. This issue can be solved by fuzzy logic. Interval T2FL handles more uncertainty to keep the system stabilized as it can deal imprecise parameters for interval data. It has an impressive capability to conclude the output and it can be characterized by aggressive growth and decomposition with respect to time for the system having uncertain criterion. In this work, IT2FLC is used for automatic detergent intake according to weight and dirt of the cloths automatically. Four defuzzification methods were executed to analyze the transportation delay and among which Wu-Mendal Uncertainty Bound Method (WM) gives better result for maintaining the system the system stabilized.

CHAPTER-6

SUMMARY

6.0 SUMMARY

Controlling and clearing traffic is an essential daily traffic management task. In this chapter, operational laws, and aggregation operators have been proposed under triangular interval type-2 fuzzy and interval neutrosophic environments. The validity of the proposed concepts has been verified using a numerical validation.

Moreover, a novel traffic flow control method using the proposed operators is proposed. An improved score function has also been proposed for INNs. Using TIT2SSWA and INSSWA operators, the traffic flow is analyzed with the score values using the score functions and the same can be derived using TIT2SSWG and INSSWG operators. The junction identified as having more traffic is the same for both the methods applied.

The advantage of dealing indeterminacy is possible only with Neutrosophic Sets. Graph theory plays a vital role in the field of networking. If uncertainty exists in the set of vertices and edge then that it can be dealt by fuzzy graphs in any application and by using neutrosophic graph, the uncertainty of the problems can be completely dealt with the concept of indeterminacy.

Dombi interval valued neutrosophic graph has been proposed and Cartesian product and composition of the proposed graphs have been derived. The validity of the derived results has been proved with the numerical example. This study exposes the use of Dombi triangular norms in the area of neutrosophic graph theory. The advantages and limitations have been discussed for crisp, type-1fuzzy, type-2 fuzzy, neutrosophic set, interval neutrosophic set, neutrosophic graph and interval neutrosophic graph.

Blockchain Technology (BCT) is a growing and reliable technology in various fields such as developing business deals, economic environments, social and politics as well. Without having a trusted central party this technology, gives the guarantee for safe and reliable transactions using Bitcoin or Ethereum. In this chapter, BCT has been considered using Bitcoins.

Further, the Blockchain single and interval valued neutrosophic graphs have been proposed and applied in a transaction of Bitcoins. Moreover, degree, total degree, minimum and maximum degree have been found for the proposed graphs. Further, comparative analysis is done with the advantages and limitations of different types of Blockchain graphs.

In image processing, edge detection is an important venture. Fuzzy logic plays a vital role in image processing to deal with lacking in quality of an image or imprecise in nature. This present study contributes an authentic method of fuzzy edge detection through image segmentation. Gradient of the image is done by triangular norms to extract the information. Triangular norms (t-norms) and triangular conorms (t-conorms) are specialized in dealing uncertainty. Therefore triangular norms are chosen with minimum and maximum operators for the purpose of morphological operations.

Also, mathematical properties of aggregation operator to represent the role of morphological operations using triangular interval type-2 fuzzy Yager weighted geometric (TIT2FYWG) and triangular interval type-2 fuzzy Yager weighted arithmetic (TIT2FYWA) operators are derived. These properties represent the qualities of image processing. Here edge detection is done for DICOM image by converting into 2D gray scale image, using MATLAB 2015a under type-2 fuzzy environment and which is the novelty of this work.

By using fuzzy logic controller, one can control uncertain system where uncertainty exists on the input parameters. Generally, the delay in the system may cause time shift which disturbs the stability of the system. Therefore, the time delay systems with uncertainty can be represented by type -2 fuzzy logic controllers.

The reference values are considered as intervals and automatic selection of detergent intake needed for good washing is proposed by using Interval type-2 fuzzy controller for the washing machine. Also transportation delay has been analyzed with the support of four different defuzzification methods and it is observed that Wu-Mendal Uncertainty Bound Method gives better result to keep the system stabilized than other methods.

CHAPTER-7

FUTURE DIRECTION

7.0 FUTURE DIRECTION

The novel concepts such as traffic flow management using aggregation operators under the following environments such as interval type-2 soft sets, interval type-2 Pythagorean sets and single and interval valued neutrosophic soft sets and cubic sets can be developed.

The new operations of Dombi IVNG such as, Strong product and Lexicographic Product can be proposed, studied, and applied in network system.

Blockchain technology using, neutrosophic soft sets and neutrosophic cubic sets can be analyzed.

Feature extraction and edge detection using, neutrosophic set and interval neutrosophic set can be studied.

Smart washing machine using, neutrosophic controller can be designed and studied.

CHAPTER-8

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8.0 REFERENCES

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ANNEXURE - I

(i). LIST OF PUBLICATIONS

LIST OF PUBLICATIONS

1. Nagarajan, D., **Lathamaheswari, M.,** Broumi, S. and Kavikumar, J. A. (2019). New Perspective of Traffic Control Management using Type 2 Fuzzy and Interval Neutrosophic Sets. *Operations Research Perspectives.*, DOI: https://doi. Org/10.1016/j.orp.2019.100099.
2. Nagarajan, D., **Lathamaheswari, M.,** Kavikumar, J. (2019). Type-2 Fuzzy Controller for Stability of a System. *Cybernetics and Automation Control Theory Methods in Intelligent Algorithms.*, 986: 197-213.
3. Broumi, S., Nagarajan, D., Talea, M., Bakali, A., Smarandache, F. and **Lathamaheswari, M.** (2019). The Shortest Path Problem In Interval Valued Trapezoidal and Triangular Neutrosophic Environment. *Complex and Intelligent Systems.*, https://doi.org/10.1007/s40747-019-0092-5.
4. Broumi, S., Dey, A., Talea, M., Bakali, A., Smarandache, F., Nagarajan, D., Lathamaheswari, M. and Kumar, R. (2019). Shortest path problem using Bellman algorithm under neutrosophic environment. *Complex and Intelligent Systems.*, https://doi.org/10.1007/s40747-019-0101-8.
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6. Nagarajan, D., Tamizhi, T., **Lathamaheswari, M**. and Kavikumar, J. (2019). Traffic Control Management using Gauss Jordan Method under Neutrosophic Environment. *Proceedings in American Institute of Physics.*, 2112, 020060, https://doi.org/10.1063/1.5112245.

7. Nagarajan, D., **Lathamaheswari**, M., Broumi, S. and Kavikumar, J. (2019). Blockchain Single and Interval Valued Neutrosophic Graph. *Neutrosophic Sets and Systems.*, DOI: 10.5281/zenodo.2593909.

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11. Nagarajan, D., **Lathamaheswari, M.,** Sujatha, R. and Kavikumar, J. (2018). Edge Detection on DIOM Image using Triangular Norms in Type-2 Fuzzy. *International Journal of Advanced Computer Science and Applications.*, 9(11):462-475.

12. Nagarajan, D., **Lathamaheswari**, M., Kavikumar, J. and Hamzha. (2018). A Type-2 Fuzzy in Image Extraction for DICOM Image. *International Journal of Advanced Computer Science and Applications.*, 9(12):352-362.

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ANNEXURE - I

(ii). REPRINTS OF PUBLICATIONS

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A new perspective on traffic control management using triangular interval type-2 fuzzy sets and interval neutrosophic sets

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ABSTRACT

Controlling traffic flow on roads is an important traffic management task necessary to ensure a peaceful and safe environment for people. The number of cars on roads at any given time is always unknown. Type-2 fuzzy sets and neutrosophic sets play a vital role in dealing efficiently with such uncertainty. In this paper, a triangular interval type-2 Schweizer and Sklar weighted arithmetic (TIT2SSWA) operator and a triangular interval type-2 Schweizer and Sklar weighted geometric (TIT2SSWG) operator based on Schweizer and Sklar triangular norms have been studied, and the validity of these operators has been checked using a numerical example and extended to an interval neutrosophic environment by proposing interval neutrosophic Schweizer and Sklar weighted arithmetic (INSSWA) and interval neutrosophic Schweizer and Sklar weighted geometric (INSSWG) operators. Furthermore, their properties have been examined; some of the more important properties are examined in detail. Moreover, we proposed an improved score function for interval neutrosophic numbers (INNs) to control traffic flow that has been analyzed by identifying the junction that has more vehicles. This improved score function uses score values of triangular interval type-2 fuzzy numbers (TIT2FNs) and interval neutrosophic numbers.

1. Introduction

Issues related to traffic congestion is regularly experienced in daily life. Controlling traffic signals is one of the areas in which fuzzy logic is most popularly employed in transportation engineering. Traffic congestion affects the safety of the people, disrupts routine (daily/everyday) activities and the quality of lifestyle and leads to a commercial, natural and health burden for the government and related organizations. Traffic control aims to reduce the negative effects of traffic by establishing intelligent models to correct state calculation, control and forecasting. The theory of triangular norms provides the mathematical properties, and these properties represent the crucial qualities of the control system, such as stability. Control problems have attracted considerable attention in the control community [1,4,11].

As real-world problems in nature often involve uncertainty, fuzzy logic has been applied successfully to deal with impreciseness. This theory is based on the concepts of degree to deal with uncertainties in a field of knowledge. This logic agrees with linguistic and imprecise traffic data as well as in modeling signal timings. Modeling the control is the basic principle of fuzzy signal control with respect to a human expert's knowledge. The model of the fuzzy controller needs an expert's knowledge and experience in the traffic control field in developing the linguistic protocol that produces the input of the control in the system. As fuzzy logic exploits linguistic information, reproduces human thinking and captures the uncertainty of the real-world problems, it is successful in producing good performance for various practical problems [2].

A fuzzy logic system works with the use of IF-THEN rules, where the knowledge will be often uncertain. It is very useful for decoy approximation. If the antecedent and consequent parts are type-1 fuzzy sets (T1FSs), then the system is called type-1 fuzzy logic system (T1FLS), whereas in type-2 fuzzy logic system (T2FLS), the antecedent or consequent set will be of type-2 fuzzy sets (T2FSs). The membership function of T2FS is a three-dimensional one, which includes upper and lower membership functions, and the area between them is called the footprint of uncertainty (FOU) [3,6,13,18]. For a set of regional linear models, the Takagi Sugeno model will be used for an optimized output [14,30–31].

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As noise is nonlinear, systems that use traditional logic and electronics fail to deal with the complex nature of the signal in terms of algorithms and circuits. Fuzzy logic is the most appropriate method to describe imprecise characteristics accurately [12]. Traffic congestion problems may arise owing to different conditions such as insufficient number of lanes, broken road surface, high volume of vehicles, irrational allocation of signaling system and poor visibility of the road. Furthermore, traffic congestion increases the level of pollution, as in most of the cases, the engines of the vehicles are left running.

To mitigate these problems, a new methodology has been implemented by accompanying the automated sensor approach in the system of traffic signaling. In early years, and at present, in some places, traffic is controlled by the usage of hand signs by the traffic police, signals and markings. This impreciseness cannot be dealt with by type-1 fuzzy as it is precise in nature, whereas type-2 fuzzy, an extension of type-1 fuzzy, can handle a high level of uncertainty [17]. The approach of type-2 fuzzy will overcome the consequence of time delay in control systems [19,20]. Hence, fuzzy logic controllers have been applied for controlling several physical processes successfully [15,22].

Type-1 fuzzy set induces a unique membership value to every set element between 0 and 1 and is useful to model knowledge, but it fails to deal with special uncertainties such as different opinions of experts on the same concept. It also cannot deal with only degree of truth and is not able to minimize the noise. In spite of these shortcomings, different opinions may produce different membership functions, therefore a model can be designed using T2FS [23].

T-norms and *t*-conorms are the triangular norms and preferable operators for controlling the system as it satisfies Commutativity, Idempotency, monotonicity and boundary conditions, which represent the qualities of a good system and generalize conjunction and disjunction respectively. Fuzzy conjunction is used for the system to decide the particular decision in a given period of time. Additionally, it is an operation between two membership degrees, which describes two fuzzy sets treated as a premise in an inference system. *T*-norms use some parameters of the system to control the inference, and may have different behavior based on the parameters [24].

Interval-valued neutrosophic sets (IVNSs) are determined by an interval membership grade, interval indeterminacy grade and interval nonmembership grade [32,33]. The generalization of intuitionistic fuzzy is the neutrosophic set with the indeterminate reasoning, and interval-valued neutrosophic set is the general case of single-valued neutrosophic environment. Using these concepts, the uncertainty of the problem can be dealt with effectively, as the neutrosophic concept deals with indeterminacy also. Interval-valued sets, especially neutrosophic sets, handle indeterminacy with the lower and upper membership functions, and hence uncertainty in a real- world problem can be solved in an optimized way [26,35–37]. Many aggregation operators have been proposed under single-valued and interval-valued neutrosophic environments and applied in decision-making problems to choose the best option [25,27,34,38,39].

By considering the determinate part and indeterminacy, a neutrosophic number can be formed and interchanged in the form of an interval number. Using this concept and the operational laws of neutrosophic matrices, traffic flow can be identified in each intersection by considering neutrosophic linear equations, and is an effective way of finding traffic flow [43]. In addition, some models have been designed to avoid accidents, and unwanted situations while inspecting and collecting information about individuals [49–51].

The rest of the paper is arranged as follows. In, Section 2, a literature review is provided for the proposed concept, and this will show the novelty of the methods proposed in this paper. In Section 3 and Section 5.1, basic concepts have been given. In Section 4, operational laws, aggregation operators and their properties are examined with a numerical example and applied in traffic flow control under triangular interval type-2 environment. In Section 5, the proposed concepts in Section 4 have been extended to an interval neutrosophic environment. In Section 6, traffic flow analysis using the proposed operators are listed. In Section 7, qualitative analysis is provided for different fuzzy environments and crisp set as well. In Section 8, conclusion of the present work is given.

2. Review of literature

The authors of Gupta and Qi [1] proposed the theoretical concepts of *t*-norms and methods of fuzzy reasoning. Castro [2] proved that fuzzy logic controllers (FLCs) are global approximations. Karnik et al. [3] presented type-2 fuzzy logic systems, which can deal with more uncertainties. Niittymaki and Pursula [4] proved that signal control using fuzzy logic can be an efficient controlling method for signalized intersections. Wei et al. [5] presented traffic signal control management using fuzzy logic and MOGA. Wu and Mendel [6] applied imprecise bounds in the model of interval type-2 fuzzy logic systems (IT2FLSs).

Wang et al. [7] introduced interval neutrosophic sets based on truth value. Aguero and Vargas [8] concluded the dynamic structure of distribution networks using T2FLSs. Wang et al. [9] presented, in detail, the theoretical concepts about interval based neutrosophic set and its application in computing. Smarandache [10] proved that a neutrosophic set is the logical reasoning and generalization of intuitionistic fuzzy set. Li et al. [11] proposed a different method for predicting traffic using type-2 fuzzy logic. Jarrah and Shaout [12] proposed volume control of motor vehicles using fuzzy logic. Ozek and Akpolat [13] proposed an operating system for type-2 fuzzy logic tool box.

Petrescu et al. [14] proposed a fuzzy control design for an independent vehicle governing system where the design is replaced by some local linear systems, which are defined over the given points and the union of these systems inclined a Takagi Sugeno model. Algreer and Ali [15] accomplished position control using fuzzy logic. Wang et al. [16] introduced single-valued neutrosophic sets (SVNSs). Almaraaashi et al. [17] constructed generalized T2FLSs using interval type-2 setting and artificial strengthening. Tellez et al. [18] proposed T2FLSs using parametric representation. Li et al. [19] described mathematical properties such as monotonicity of IT2FLSs.

Blaho et al. [20] used type-2 fuzzy logic in diminishing the collision of impreciseness in a chain of control systems. Singhala et al. [21] developed a temperature control system using fuzzy logic. Patel [22] explained the situations and methods in which fuzzy logic can be applied. Comas et al. [23] defined measures to determine the degree of truth and the theoretical background of the decision support system. Qin and Liu [24] proposed Frank triangular norms for triangular interval type-2 fuzzy set and applied it in a decision-making process.

Ye [25] improved the correlation coefficient of SVNSs, examined their properties and extended the concept to interval neutrosophic sets (INSs). They also applied the proposed concepts in decision-making problems. Ye [26] generalized Jaccard, Dice and cosine similarity measures in vector space and presented three vector similarity measures between simplified NSs (SNSs). They also used it in a decision-making problem. Ye [27] proposed the concept of SNS, its operational laws and two aggregation operators, namely, simplified neutrosophic weighted (SNW) arithmetic average operator and SNW geometric averaging operator and applied them in a decision-making problem.

Singh et al. [28] used comparative analysis of neural network and fuzzy algorithm in implementing ACC. Shafeek [29] designed an autopilot to control the header of an aircraft using PD-like T1 and T2 fuzzy logic controllers. Wen et al. [30] proposed an intelligent signal controller using T2FL and NSGAII. Lafta and Hassan [31] introduced mobile automation control using fuzzy logic. Broumi and Smarandache [32] proposed interval-valued neutrosophic soft rough sets. Smarandache [33] explained very clearly about neutrosophic theory symbolically. Ye [34] proposed the ranking method on possibility degree for INNs from the probability aspect. Poyen et al. [35] designed a dynamic traffic signal system based on density where the signal timing changes automatically on sensing traffic. Sharma and Sahu [36] reviewed fuzzy logic-based traffic signal control.

Singh et al. [37] analyzed an uncertainty for the provided many-valued context. Ye [38] proposed new exponential laws of INSs, interval neutrosophic weighted exponential aggregation operator and its dual operator and applied them in a decision-making problem for global supplier selection. Ye [39] introduced a credibility induced INWA averaging operator and a credibility induced INWG averaging operator and examined their properties. They also presented a measure of projection between INNs and its ranking method and applied it in a decision-making problem. Bouyahia et al. [40] used fuzzy switching linear models to present real-time traffic smoothing from GPS spare measures. Chen and Ye [41] derived the mathematical properties of Dombi triangular norms based on a SVNS and applied them in a decision- making method.

Singh [42] discovered some of the important hidden patterns in the interval-valued neutrosophic context. Ye [43] presented the concepts of neutrosophic linear equations (NLEs) and, neutrosophic matrix (NM), and proposed NM operations for the first time. They also introduced some solving methods on NMs. Laxmi et al. [44] proposed an intelligent system for traffic control to enable emergency vehicles to pass without any disruptions. Noormohammadpour and Raghavendra [45] have brought out the important characteristic of traffic control in data centers. Shi and Ye [46] derived Dombi aggregation operators of neutrosophic cubic sets and applied them in a decision-making process. Liu and Wang [47] proposed interval-valued intuitionist fuzzy Schweizer–Sklar power aggregation operators and applied them in a decision-making problem for supplier selection.

Broumi et al. [48] discussed the lack of knowledge partially for [0, 1] using IVNSs. Nagarajan et al. [49] applied triangular norms under a type-2 fuzzy environment for edge detection on a DICOM image. Mayouf et al. [50] developed an accident management system applicable for cellular technology in public transportation. Sumia and Ranga [51] proposed a new intelligent traffic management system (TMS). Ankam et al. [52] designed a new TMS for the benefit of vehicle owners to carry the documents such as license and insurance during investigation by the authorities.

This review is the motivation of the present study, as the use of aggregation operators for controlling traffic flow has not yet been studied.

3. Fundamental perception

In this section, basic concepts of a traffic control system, role of fuzzy logic, output methods from fuzzy linguistic terms and structure of the fuzzy control system have been given for better understanding.

3.1. Traffic signal control [33]

This is a pretimed or induced or flexible control and is described as follows.

Pretimed Controllers - Such controllers decide the signal timings in advance, which are collected from earlier pattern of traffic, and repeat the same.

Actuated Controllers – These will identify the moving and interrupted traffic on each lane towards cross-roads and estimate the duration of the signal phase.

Adaptive Controllers – These identify the entire cross-roads and modify the signal phase and response timings to real-time traffic.

3.2. Levels of signal control [33]

The fuzziness of signal control can be classified into three levels, namely, input, control and output levels, and are described as follows.

Input Level – Here, a partial picture of the succeeding traffic environment will be drawn using measurements.

Control Level – At this level, there will be various possibilities and it is difficult to decide the right or the best possibility because the relationship of source and reaction of the signal control cannot be explained.

Output Level – Here, the exact criteria of the control are not known, such as extension gap.

3.3. Fuzzy logic in traffic control system [22]

As fuzzy logic is theoretically easy to understand, adaptable, lenient with uncertain data, can design nonlinear functions of inconsistent complexity, can be built with the knowledge of the experts, and flexible with traditional control approach, a fuzzy logic-based control system has been a successful pursuit to implement intelligence in a traffic control system.

A nonlinear mapping of an input data set to scalar output is called a fuzzy logic system. It consists of four parts i.e., fuzzifier, fuzzy rules, inference engine and defuzzifier. The fuzzy system converts the input to the output. Here, the linguistic values are divided into fuzzy sets, e.g., traffic flow can be defined as low, high and medium. The degree of addiction to every fuzzy set is shown by membership functions. The input value of the fuzzy system may exist in more than one fuzzy set. The corresponding numeric values to fuzzy set are called fuzzification and the reverse is called defuzzification.

Fuzzy IF-THEN rules are the main logic of the inference system and involve vague reasoning. Fuzzy rules are well defined using an expert's knowledge, and hence a mathematical model is not necessary for the objects and the system is very flexible. The parameters of the membership functions and its values, operators, fuzzy rules, defuzzification and other parameters can be modified according to the desired result.

Interval type-2 fuzzy logic systems are used to recognize control laws to minimize control errors. The output of this system, called the control signal, is supposed to be monotonic with respect to the error and/or variation of the error called inputs of the system.

The fuzzy control is found to be preferable in complicated problems with multi-objective decisions. Various traffic flows contest for the same time and space and various preferences are frequently set to different flows or vehicle groups.

3.4. Defuzzification methods [19]

The following methods are often applied in a control system to get the precise output from the fuzzy inputs: Karnik Mendel, Du Ying, Begian Melek Mendel, Wu Tan and Nie Tan methods.

3.5. Role of membership functions [21]

Application of membership function is an essential role in the stage of fuzzification and defuzzification of the fuzzy logic system to calculate the nonfuzzy input values to fuzzy linguistic terms and for the converse. It is used to measure the linguistic term. An amazing characteristic of the fuzzy logic lying in the fuzzification of the numerical value is that it need not be fuzzified using only one membership function, and hence the value can be described by different sets at a particular time.

3.6. Algorithm of fuzzy logic [5]

The following figure represents the algorithm of traffic control system using fuzzy logic.

3.7. Triangular norms considered

In this paper, Schweizer and Sklar (SS) triangular norms have been considered and defined as follows. [38]

T − norm: $TN(p,\, q) = p \otimes\ q$

$$= 1 - [(1-p)^{\varphi} + (1-q)^{\varphi} - (1-p)^{\varphi}(1-q)^{\varphi}]^{\frac{1}{\varphi}},\ p,\, q \in [0,\, 1]^2 \quad (1)$$

T − conorm: $TCN(p,\, q) = p \oplus q = (p^{\varphi} + q^{\varphi} - p^{\varphi}q^{\varphi})^{\frac{1}{\varphi}},\ p,\, q \in [0,\, 1]^2$ (2)

whereφ > 0is the parameter.

3.8. Triangular interval type-2 weighted arithmetic/geometric operator [24]

Let $\overline{F_i} = ([\underline{l_{F_i}}, \overline{l_{F_i}}], c_{F_i}, [\underline{r_{F_i}}, \overline{r_{F_i}}])$,$i = 1, 2, ...,n$ be a set of TIT2FNs of the triangular interval type-2 fuzzy Set X. Let $TIT2WA_{\varpi}/TIT2WG_{\varpi}$: $\Omega^n \to \Omega$ if,

$$TIT2WA_{\varpi}(\overline{F_1}, \overline{F_2}, ...,\overline{F_n}) = \varpi_1{\cdot}\overline{F_1} \otimes\ \varpi_2{\cdot}\overline{F_2} \otimes\ .....\otimes\varpi_n{\cdot}\overline{F_n} \quad (3)$$

$$TIT2WG_{\varpi}(\overline{F_1}, \overline{F_2}, ...,\overline{F_n}) = \overline{F_1}^{\varpi_1} \otimes\ \overline{F_2}^{\varpi_2} \otimes\ .....\otimes\overline{F_n}^{\varpi_n} \quad (4)$$

then the function TIT2WA/TIT2WG are called triangular interval type-2 weighted arithmetic and geometric operators respectively, and $\varpi = (\varpi_1, \varpi_2, ...,\varpi_n)^T$ is the weight vector of$\overline{F_i} = ,i = 1, 2, ...,n$, $\varpi_i \geq 0$ and$\sum_{i=1}^{n} \varpi_i = 1$. If $\varpi = (\frac{1}{n}, \frac{1}{n}, ...,\frac{1}{n})^T$then the TIT2WG operator is reduced to a triangular interval type-2 geometric averaging operator of dimension$n$.

3.9. Score function of TIT2FN [24]

To rank two TIT2FNs the following score function is used, defined by

$$SF(\overline{F}) = \left(\frac{\underline{l_F} + \overline{r_F}}{2} + 1\right) \times \frac{\underline{l_F} + \overline{l_F}\ +\ \underline{r_F} + \overline{r_F} + 4c_F}{8} \quad (5)$$

4. Proposed operational laws

The following operational laws can be defined using triangular interval type-2 fuzzy numbers for SS triangular norms. Consider three triangular interval type-2 fuzzy numbers $\overline{F}$, $\overline{F_1}$, $\overline{F_2}$ and$\varphi > 0$.

Addition

$$\overline{F_1} \oplus \overline{F_2} = \left\langle \left[(sum(\underline{l_{F_i}})^{\varphi} - prod(\underline{l_{F_i}})^{\varphi})^{\frac{1}{\varphi}}, (sum(\overline{l_{F_i}})^{\varphi} - prod(\overline{l_{F_i}})^{\varphi})^{\frac{1}{\varphi}}\right], (sum(c_{F_i})^{\varphi} - prod(c_{F_i})^{\varphi})^{\frac{1}{\varphi}}, \left[(sum(\underline{r_{F_i}})^{\varphi} - prod(\underline{r_{F_i}})^{\varphi})^{\frac{1}{\varphi}}, (sum(\overline{r_{F_i}})^{\varphi} - prod(\overline{r_{F_i}})^{\varphi})^{\frac{1}{\varphi}}\right] \right\rangle \quad (6)$$

where $\sum_{i=1}^{2} =$, $\prod_{i=1}^{2} = \wp$

Numerical Example:

If $\overline{F_1} = \langle[0.4, 0.5], 0.6, [0.7, 0.8]\rangle$and $\overline{F_2} = \langle[0.5, 0.6], 0.7, [0.8, 0.9]\rangle$be any two TIT2FNs and if $\varphi = 2$then

$$\overline{F_1} \oplus \overline{F_2} = \langle[((0.4)^2 + (0.5)^2 - (0.4)^2.\ (0.5)^2)^{1/2}, ((0.5)^2 + (0.6)^2 - (0.5)^2.\ (0.6)^2)^{1/2}], ((0.6)^2 + (0.7)^2 - (0.6)^2.\ (0.7)^2)^{1/2}, [((0.7)^2 + (0.8)^2 - (0.7)^2.\ (0.8)^2)^{1/2}, ((0.8)^2 + (0.9)^2 - (0.8)^2.\ (0.9)^2)^{1/2}]\rangle = \langle[0.6096, 0.7211], 0.8207, [0.9035, 0.9652]\rangle$$

Multiplication

$$\overline{F_1} \otimes\ \overline{F_2} = \left\langle \left[1 - ((1 - \underline{l_{F_i}})^{\varphi} - \wp(1 - \underline{l_{F_i}})^{\varphi})^{\frac{1}{\varphi}}, 1 - ((1 - \overline{l_{F_i}})^{\varphi} - \wp(1 - \overline{l_{F_i}})^{\varphi})^{\frac{1}{\varphi}}\right], 1 - ((1 - c_{F_i})^{\varphi} - \wp(1 - c_{F_i})^{\varphi})^{\frac{1}{\varphi}}, \left[1 - ((1 - \underline{r_{F_i}})^{\varphi} - \wp(1 - \underline{r_{F_i}})^{\varphi})^{\frac{1}{\varphi}}, 1 - ((1 - \overline{r_{F_i}})^{\varphi} - \wp(1 - \overline{r_{F_i}})^{\varphi})^{\frac{1}{\varphi}}\right] \right\rangle, \text{ where } \sum_{i=1}^{2} =,\ \prod_{i=1}^{2} = \wp \quad (7)$$

Numerical Example:

$$\overline{F_1} \otimes\ \overline{F_2} = \langle[1 - ((1-0.4)^2 + (1-0.5)^2 - (1-0.4)^2.\ (1-0.5)^2)^{1/2}, 1 - ((1-0.5)^2 + (1-0.6)^2 - (1-0.5)^2.\ (1-0.6)^2)^{1/2}] \; 1 - ((1-0.6)^2 + (1-0.7)^2 - (1-0.6)^2.\ (1-0.7)^2)^{1/2}, [1 - ((1-0.7)^2 + (1-0.8)^2 - (1-0.7)^2.\ (1-0.8)^2)^{1/2}, 1 - ((1-0.8)^2 + (1-0.9)^2 - (1-0.8)^2.\ (1-0.9)^2)^{1/2}]\rangle = \langle[0.2789, 0.3917], 0.5146, [0.6445, 0.777]\rangle = \text{TIT2FN}$$

Multiplication by an ordinary number

$$v.\ \overline{F_1} = \left\langle \left[((\underline{l_{F_1}})^{\varphi})^{\frac{v}{\varphi}}, ((\overline{l_{F_1}})^{\varphi})^{\frac{v}{\varphi}}\right], ((c_{F_1})^{\varphi})^{\frac{v}{\varphi}}, \left[((\underline{r_{F_1}})^{\varphi})^{\frac{v}{\varphi}}, ((\overline{r_{F_1}})^{\varphi})^{\frac{v}{\varphi}}\right] \right\rangle \quad (8)$$

Here, $v$is an ordinary number.

Numerical Example:

Consider $\overline{F_1} = \langle[0.4, 0.5], 0.6, [0.7, 0.8]\rangle$ and $v = 0.3$

$$0.3.\ \overline{F_1} = \left\langle \left[((0.4)^2)^{\frac{0.3}{2}}, ((0.5)^2)^{\frac{0.3}{2}}\right], ((0.6)^2)^{\frac{0.3}{2}}, \left[((0.7)^2)^{\frac{0.3}{2}}, ((0.8)^2)^{\frac{0.3}{2}}\right] \right\rangle = \langle[0.2789, 0.3917], 0.5146, [0.6445, 0.777]\rangle = \text{TIT2FN}$$

Power Operation

$$\overline{F_1}^{v} = \left\langle \left[1 - ((1 - \underline{l_{F_1}})^{\varphi})^{\frac{v}{\varphi}}, 1 - ((1 - \overline{l_{F_1}})^{\varphi})^{\frac{v}{\varphi}}\right], 1 - ((1 - c_{F_1})^{\varphi})^{\frac{v}{\varphi}}, \left[1 - ((1 - \underline{r_{F_1}})^{\varphi})^{\frac{v}{\varphi}}, 1 - ((1 - \overline{r_{F_1}})^{\varphi})^{\frac{v}{\varphi}}\right] \right\rangle \quad (9)$$

Numerical Example: Consider $v = 0.3$and $\overline{F_1}$

$$\overline{F_1}^{0.3} = \left\langle \left[1 - ((1-0.4)^2)^{\frac{0.3}{2}}, 1 - ((1-0.5)^2)^{\frac{0.3}{2}}\right], 1 - ((1-0.6)^2)^{\frac{0.3}{2}}, \left[1 - ((1-0.7)^2)^{\frac{0.3}{2}}, 1 - ((1-0.8)^2)^{\frac{0.3}{2}}\right] \right\rangle = \langle[0.1421, 0.1877], 0.2403, [0.3032, 0.383]\rangle = \text{TIT2FN}$$

4.1. Proposed theorems using TIT2SSWG operator

Here, the SS operator under triangular interval type-2 setting has been developed and proposed as a triangular interval type-2 Schweizer and Sklar weighted geometric (TIT2SSWG) operator based on SS triangular norms.

4.1.1. Theorem

Let $\overline{F_i} = ([\underline{l_{F_i}}, \overline{l_{F_i}}], c_{F_i}, [\underline{r_{F_i}}, \overline{r_{F_i}}])$, $i = 1, 2, ...,n$ be a set of TIT2FNs; then their aggregated value using TIT2SSWG operator is still a TIT2FN, $0 \leq \underline{l_{F_i}} \leq \overline{l_{F_i}} \leq c_{F_i} \leq \underline{r_{F_i}} \leq \overline{r_{F_i}} \leq 1$, $i = 1, 2, ...,n$ and

$$TIT2SSWG_{\varpi}(\overline{F_1}, \overline{F_2}, ...,\overline{F_n}) = \left\langle \left[1 - ((1 - \underline{l_{F_i}})^{\varphi} - \wp(1 - \underline{l_{F_i}})^{\varphi})^{\frac{\varpi_i}{\varphi}}, 1 - ((1 - \overline{l_{F_i}})^{\varphi} - \wp(1 - \overline{l_{F_i}})^{\varphi})^{\frac{\varpi_i}{\varphi}}\right], 1 - ((1 - c_{F_i})^{\varphi} - \wp(1 - c_{F_i})^{\varphi})^{\frac{\varpi_i}{\varphi}}, \left[1 - ((1 - \underline{r_{F_i}})^{\varphi} - \wp(1 - \underline{r_{F_i}})^{\varphi})^{\frac{\varpi_i}{\varphi}}, 1 - ((1 - \overline{r_{F_i}})^{\varphi} - \wp(1 - \overline{r_{F_i}})^{\varphi})^{\frac{\varpi_i}{\varphi}}\right] \right\rangle \quad (10)$$

where $\sum_{i=1}^{n} =$, $\prod_{i=1}^{n} = \wp$

Proof. :

By mathematical induction method, we prove this theorem.

For $n = 2$

Consider the power operation

$$\overline{F}^{v_1} = \left\langle \left[1 - ((1 - \underline{l}_F)^{\varphi})^{\frac{v_1}{\varphi}}, 1 - ((1 - \overline{l_F})^{\varphi})^{\frac{v_1}{\varphi}}\right], 1 - ((1 - c_F)^{\varphi})^{\frac{v_1}{\varphi}}, \left[1 - ((1 - \underline{r}_F)^{\varphi})^{\frac{v_1}{\varphi}}, 1 - ((1 - \overline{r_F})^{\varphi})^{\frac{v_1}{\varphi}}\right] \right\rangle$$

$$\overline{F}^{v_2} = \left\langle \left[1 - ((1 - \underline{l}_F)^{\varphi})^{\frac{v_2}{\varphi}}, 1 - ((1 - \overline{l_F})^{\varphi})^{\frac{v_2}{\varphi}}\right], 1 - ((1 - c_F)^{\varphi})^{\frac{v_2}{\varphi}}, \left[1 - ((1 - \underline{r}_F)^{\varphi})^{\frac{v_2}{\varphi}}, 1 - ((1 - \overline{r_F})^{\varphi})^{\frac{v_2}{\varphi}}\right] \right\rangle$$

$$TIT2WG_{\varpi}(\overline{F_1}, \overline{F_2}) = \overline{F_1}^{\varpi_1} \otimes \overline{F_1}^{\varpi_2}$$
$$= \left\langle \left[1 - ((1 - \underline{l}_{F_i})^{\varphi} - \wp(1 - \underline{l}_{F_i})^{\varphi})^{\frac{\varpi_i}{\varphi}}, 1 - ((1 - \overline{l_{F_i}})^{\varphi} - \wp(1 - \overline{l_{F_i}})^{\varphi})^{\frac{\varpi_i}{\varphi}}\right], 1 - ((1 - c_{F_i})^{\varphi} - \wp(1 - c_{F_i})^{\varphi})^{\frac{\varpi_i}{\varphi}}, \left[1 - ((1 - \underline{r}_{F_i})^{\varphi} - \wp(1 - \underline{r}_{F_i})^{\varphi})^{\frac{\varpi_i}{\varphi}}, 1 - ((1 - \overline{r_{F_i}})^{\varphi} - \wp(1 - \overline{r_{F_i}})^{\varphi})^{\frac{\varpi_i}{\varphi}}\right] \right\rangle$$

where $\sum_{i=1}^{2} =$, $\prod_{i=1}^{2} = \wp$

For $n = k$

$$TIT2SSWG_{\varpi}(\overline{F_1}, \overline{F_2}, \ldots, \overline{F_k}) = \overline{F_1}^{\varpi_1} \otimes \overline{F_2}^{\varpi_2} \otimes \ldots \otimes \overline{F_k}^{\varpi_k}$$
$$= \left\langle \left[1 - ((\underline{l}_{F_i})^{\varphi} - \wp(\underline{l}_{F_i})^{\varphi})^{\frac{\varpi_i}{\varphi}}, 1 - ((\overline{l_{F_i}})^{\varphi} - \wp(\overline{l_{F_i}})^{\varphi})^{\frac{\varpi_i}{\varphi}}\right], 1 - ((1 - c_{F_i})^{\varphi} - \wp(1 - c_{F_i})^{\varphi})^{\frac{\varpi_i}{\varphi}}, \left[1 - ((1 - \underline{r}_{F_i})^{\varphi} - \wp(1 - \underline{r}_{F_i})^{\varphi})^{\frac{\varpi_i}{\varphi}}, 1 - ((1 - \overline{r_{F_i}})^{\varphi} - \wp(1 - \overline{r_{F_i}})^{\varphi})^{\frac{\varpi_i}{\varphi}}\right] \right\rangle$$

where $\sum_{i=1}^{k} =$and $\prod_{i=1}^{k} = \wp$

For $n = k + 1$

$$TIT2SSWG_{\varpi}(\overline{F_1}, \overline{F_2}, \ldots, \overline{F_{k+1}}) = (\overline{F_1}^{\varpi_1} \otimes \overline{F_2}^{\varpi_2} \otimes \ldots \otimes \overline{F_k}^{\varpi_k}) \otimes \overline{F_{k+1}}^{\varpi_{k+1}}$$
$$= \left\langle \left[1 - ((\underline{l}_{F_i})^{\varphi} - \wp(\underline{l}_{F_i})^{\varphi})^{\frac{\varpi_i}{\varphi}}, 1 - ((\overline{l_{F_i}})^{\varphi} - \wp(\overline{l_{F_i}})^{\varphi})^{\frac{\varpi_i}{\varphi}}\right], 1 - ((1 - c_{F_i})^{\varphi} - \wp(1 - c_{F_i})^{\varphi})^{\frac{\varpi_i}{\varphi}}, \left[1 - ((1 - \underline{r}_{F_i})^{\varphi} - \wp(1 - \underline{r}_{F_i})^{\varphi})^{\frac{\varpi_i}{\varphi}}, 1 - ((1 - \overline{r_{F_i}})^{\varphi} - \wp(1 - \overline{r_{F_i}})^{\varphi})^{\frac{\varpi_i}{\varphi}}\right] \right\rangle$$
$$\otimes \left\langle \left[1 - ((1 - \underline{l}_{F_{k+1}})^{\varphi})^{\frac{\varpi_{k+1}}{\varphi}}, 1 - ((1 - \overline{l_{S_{k+1}}})^{\varphi})^{\frac{\varpi_{k+1}}{\varphi}}\right], 1 - ((1 - c_{F_{k+1}})^{\varphi})^{\frac{\varpi_{k+1}}{\varphi}}, \left[1 - ((1 - \underline{r}_{F_{k+1}})^{\varphi})^{\frac{\varpi_{k+1}}{\varphi}}, 1 - ((1 - \overline{r_{F_{k+1}}})^{\varphi})^{\frac{\varpi_{k+1}}{\varphi}}\right] \right\rangle$$
$$= \left\langle \left[1 - ((1 - \underline{l}_{F_i})^{\varphi} - \wp(1 - \underline{l}_{F_i})^{\varphi})^{\frac{\varpi_i}{\varphi}}, 1 - ((1 - \overline{l_{F_i}})^{\varphi} - \wp(1 - \overline{l_{F_i}})^{\varphi})^{\frac{\varpi_i}{\varphi}}\right], 1 - ((1 - c_{F_i})^{\varphi} - \wp(1 - c_{F_i})^{\varphi})^{\frac{\varpi_i}{\varphi}}, \left[1 - ((1 - \underline{r}_{F_i})^{\varphi} - \wp(1 - \underline{r}_{F_i})^{\varphi})^{\frac{\varpi_i}{\varphi}}, 1 - ((1 - \overline{r_{F_i}})^{\varphi} - \wp(1 - \overline{r_{F_i}})^{\varphi})^{\frac{\varpi_i}{\varphi}}\right] \right\rangle$$

where $\sum_{i=1}^{k+1} =$and $\prod_{i=1}^{k+1} = \wp$

Hence, the result is true for all values of n.

Numerical Example:

For $n = 2$ the calculation has been given and the computation is similar for all the values of n. In addition, consider the weight vector $\varpi_1 = 0.55$and$\varpi_2 = 0.45$. Without loss of generality, take $\varphi = 2$throughout the paper.

$$TIT2SSWG_{\varpi}(\overline{F_1}, \overline{F_2}) = \overline{F_1}^{0.55} \otimes \overline{F_2}^{0.45}$$
$$= \left\langle \left[1 - ((1 - 0.4)^2 + (1 - 0.5)^2 - (1 - 0.4)^2.\ (1 - 0.5)^2)^{\frac{0.55+0.45}{2}}, 1 - ((1 - 0.5)^2 + (1 - 0.6)^2 - (1 - 0.5)^2.\ (1 - 0.6)^2)^{\frac{0.55+0.45}{2}}\right], 1 - ((1 - 0.6)^2 + (1 - 0.7)^2 - (1 - 0.6)^2.\ (1 - 0.7)^2)^{\frac{0.55+0.45}{2}}, \left[1 - ((1 - 0.7)^2 + (1 - 0.8)^2 - (1 - 0.7)^2.\ (1 - 0.8)^2)^{\frac{0.55+0.45}{2}}, 1 - ((1 - 0.8)^2 + (1 - 0.9)^2 - (1 - 0.8)^2.\ (1 - 0.9)^2)^{\frac{0.55+0.45}{2}}\right] \right\rangle$$
$$= \langle [0.48, 0.63], 0.7644, [0.8736, 0.9504] \rangle = \text{TIT2FN}$$

4.1.2. Theorem (Idempotency)

$$\text{Let } \overline{F_i} = ([\underline{l}_{F_i}, \overline{l_{F_i}}], c_{F_i}, [\underline{r}_{F_i}, \overline{r_{F_i}}]),\ i = 1, 2, \ldots, n \text{ be a set of TIT2FNs}, 0 \le \underline{l}_{F_i} \le \overline{l_{F_i}} \le c_{F_i} \le \underline{r}_{F_i} \le \overline{r_{F_i}} \le 1,\ i = 1, 2, \ldots, n.$$
$$\text{If all } \overline{F_i},\ i = 1, 2, \ldots, n \text{ are equal, i.e.,} \overline{F_i} = \overline{F} \text{ then } TIT2SSWG_{\varpi}(\overline{F_1}, \overline{F_2}, \ldots, \overline{F_n}) = \overline{F} \tag{11}$$

Proof. :

$$TIT2SSWG_{\varpi}(\overline{F_1}, \overline{F_2}, \ldots, \overline{F_n})$$
$$= \left\langle \left[1 - ((1 - \underline{l}_{F_i})^{\varphi} - \wp(1 - \underline{l}_{S_t})^{\varphi})^{\frac{\varpi_i}{\varphi}}, 1 - ((1 - \overline{l_{S_t}})^{\varphi} - \wp(1 - \overline{l_{S_t}})^{\varphi})^{\frac{\varpi_i}{\varphi}}\right], 1 - ((1 - m_{S_t})^{\varphi} - \wp(1 - m_{S_t})^{\varphi})^{\frac{\varpi_i}{\varphi}}, \left[1 - ((1 - \underline{r}_{F_i})^{\varphi} - \wp(1 - \underline{r}_{F_i})^{\varphi})^{\frac{\varpi_i}{\varphi}}, 1 - ((1 - \overline{r_{F_i}})^{\varphi} - \wp(1 - \overline{r_{F_i}})^{\varphi})^{\frac{\varpi_i}{\varphi}}\right] \right\rangle$$
$$= \left\langle \left[1 - ((1 - \underline{l}_{F_i})^{\varphi})^{\frac{\sum_{i=1}^{n} \varpi_i}{\varphi}}, 1 - ((1 - \overline{l_{F_i}})^{\varphi})^{\frac{\sum_{i=1}^{n} \varpi_i}{\varphi}}\right], (1 - (1 - c_{F_i})^{\varphi})^{\frac{\sum_{i=1}^{n} \varpi_i}{\varphi}}, \left[1 - ((1 - \underline{r}_{F_i})^{\varphi})^{\frac{\sum_{i=1}^{n} \varpi_i}{\varphi}}, 1 - ((1 - \overline{r_{F_i}})^{\varphi})^{\frac{\sum_{i=1}^{n} \varpi_i}{\varphi}}\right] \right\rangle$$
$$= \left\langle \left[1 - ((1 - \underline{l}_{F_i})^{\varphi})^{\frac{1}{\varphi}}, 1 - ((1 - \overline{l_{F_i}})^{\varphi})^{\frac{1}{\varphi}}\right], 1 - ((1 - m_{S_t})^{\varphi})^{\frac{1}{\varphi}}, \left[1 - ((1 - \underline{r}_{F_i})^{\varphi})^{\frac{1}{\varphi}}, 1 - ((1 - \overline{r_{F_i}})^{\varphi})^{\frac{1}{\varphi}}\right] \right\rangle$$
$$= \langle [(1 - \underline{l}_{F_i})^{\varphi}, (1 - \overline{l_{F_i}})^{\varphi}],\ (1 - c_{F_i})^{\varphi},\ [(1 - \underline{r}_{F_i})^{\varphi}, (1 - \overline{r_{F_i}})^{\varphi}] \rangle$$
$$= \langle [(1 - \underline{l}_{F_i}), (1 - \overline{l_{F_i}})],\ (1 - c_{F_i}),\ [(1 - \underline{r}_{F_i}), (1 - \overline{r_{F_i}})] \rangle$$
$$= \langle [\underline{l}_{F_i}, \overline{l_{F_i}}], c_{F_i}, [\underline{r}_{F_i}, \overline{r_{F_i}}] \rangle$$
$$= \langle [\underline{l}_F, \overline{l_F}], c_F, [\underline{r}_F, \overline{r_F}] \rangle$$
$$= \overline{F},\ \text{where } \sum_{i=1}^{n} = \text{ and } \prod_{i=1}^{n} = \wp$$

4.1.3. Theorem

Let $\overline{F_i} = ([\underline{l_{F_i}}, \overline{l_{F_i}}], c_{F_i}, [\underline{r_{F_i}}, \overline{r_{F_i}}]),\ i = 1, 2, ..., n$ be a set of TIT2FNs, $0 \le \underline{l_{F_i}} \le \overline{l_{F_i}} \le c_{F_i} \le \underline{r_{F_i}} \le \overline{r_{F_i}} \le 1,\ i = 1, 2, ..., n.$ if $\overline{S}_{n+1} = ([\underline{l_{S_{n+1}}}, \overline{l_{S_{n+1}}}], m_{S_{n+1}}, [\underline{r_{S_{n+1}}}, \overline{r_{S_{n+1}}}])$ is also a TIT2FN on X then,

$$TIT2SSWG_{\varpi}(\overline{F_1} \otimes \overline{S_{n+1}}, \overline{F_2} \otimes \overline{F_{n+1}}, ..., \overline{F_n} \otimes \overline{F_{n+1}}) = TIT2SSWG_{\varpi}(\overline{F_1}, \overline{F_2}, ..., \overline{F_n}) \otimes \overline{F_{n+1}} \tag{12}$$

Proof. :

$$\text{Since, } \overline{F_i} \otimes \overline{F_{n+1}} = \Big\langle \Big[1 - (*(1 - \underline{l_{F_t}})^{\varphi} - \wp^*(1 - \underline{l_{F_t}})^{\varphi})^{\frac{1}{\varphi}}, 1 - (*(1 - \overline{l_{F_t}})^{\varphi} - \wp^*(1 - \overline{l_{F_t}})^{\varphi})^{\frac{1}{\varphi}}\Big],$$
$$1 - (*(1 - c_{F_t})^{\varphi} - \wp^*(1 - c_{F_t})^{\varphi})^{\frac{1}{\varphi}},$$
$$\Big[1 - (*(1 - \underline{r_{F_t}})^{\varphi} - \wp^*(1 - \underline{r_{F_t}})^{\varphi})^{\frac{1}{\varphi}}, 1 - (*(1 - \overline{r_{F_t}})^{\varphi} - \wp^*(1 - \overline{r_{F_t}})^{\varphi})^{\frac{1}{\varphi}}\Big]\Big\rangle$$

where $* = \sum_{t=i,n+1}$, $\wp^* = \prod_{t=i,n+1}$

Consider LHS,

$$TIT2SSWG_{\varpi}(\overline{F_1} \otimes \overline{S_{n+1}}, \overline{F_2} \otimes \overline{F_{n+1}}, ..., \overline{F_n} \otimes \overline{F_{n+1}})$$
$$= \Big\langle \Big[1 - ((*(1 - \underline{l_{F_i}})^{\varphi}) - \wp(\wp^*(1 - \underline{l_{F_i}})^{\varphi}))^{\frac{\varpi_i}{\varphi}}, 1 - ((*(1 - \overline{l_{F_i}}))^{\xi} - \wp(\wp^*(1 - \overline{l_{F_i}}))^{\varphi})^{\frac{\varpi_i}{\varphi}}\Big],$$
$$1 - ((*(1 - c_{F_i})^{\varphi}) - \wp(\wp^*(1 - c_{F_i})^{\varphi}))^{\frac{\varpi_i}{\varphi}},$$
$$\Big[1 - ((*(1 - \underline{r_{F_i}})^{\varphi}) - \wp(\wp^*(1 - \underline{r_{F_i}})^{\varphi}))^{\frac{\varpi_i}{\varphi}}, 1 - ((*(1 - \overline{r_{F_i}}))^{\xi} - \wp(\wp^*(1 - \overline{r_{F_i}}))^{\varphi})^{\frac{\varpi_i}{\varphi}}\Big]\Big\rangle$$
$$= \Big\langle \Big[1 - ((1 - \underline{l_{F_i}})^{\varphi} + (1 - \underline{l_{F_{n+1}}})^{\varphi} - \wp(1 - \underline{l_{F_i}})^{\varphi}(1 - \underline{l_{F_{n+1}}})^{\varphi})^{\frac{\varpi_i}{\varphi}}, 1 - ((1 - \overline{l_{F_i}})^{\varphi} + (1 - \overline{l_{F_{n+1}}})^{\varphi} - \wp(1 - \overline{l_{F_i}})^{\varphi}(1 - \overline{l_{F_{n+1}}})^{\varphi})^{\frac{\varpi_i}{\varphi}}\Big],$$
$$1 - ((1 - c_{F_i})^{\varphi} + (1 - c_{F_{n+1}})^{\varphi} - \wp(1 - c_{F_i})^{\varphi}(1 - c_{F_{n+1}})^{\varphi})^{\frac{\varpi_i}{\varphi}},$$
$$\Big[1 - ((1 - \underline{r_{F_i}})^{\varphi} + (1 - \underline{r_{F_{n+1}}})^{\varphi} - \wp(1 - \underline{r_{F_t}})^{\varphi}(1 - \underline{r_{F_{n+1}}})^{\varphi})^{\frac{\varpi_i}{\varphi}},$$
$$1 - ((1 - \overline{r_{F_i}})^{\varphi} + (1 - \overline{r_{F_{n+1}}})^{\varphi} - \wp(1 - \overline{r_{F_i}})^{\varphi}(1 - \overline{r_{F_{n+1}}})^{\varphi})^{\frac{\varpi_i}{\varphi}}\Big]\Big\rangle$$
$$= \Big\langle \Big[1 - (((1 - \underline{l_{F_i}})^{\varphi} + (1 - \underline{l_{F_{n+1}}})^{\varphi}) - \wp((1 - \underline{l_{F_i}})^{\varphi} + (1 - \underline{l_{F_{n+1}}})^{\varphi}))^{\frac{\varpi_i}{\varphi}},$$
$$1 - (((1 - \overline{l_{F_i}})^{\varphi} + (1 - \overline{l_{F_{n+1}}})^{\varphi}) - \wp((1 - \overline{l_{F_i}})^{\varphi} + (1 - \overline{l_{F_{n+1}}})^{\varphi}))^{\frac{\varpi_i}{\varphi}}\Big],$$
$$1 - (((1 - c_{F_i})^{\varphi} + (1 - c_{F_{n+1}})^{\varphi}) - \wp((1 - c_{F_i})^{\varphi} + (1 - c_{F_{n+1}})^{\varphi}))^{\frac{\varpi_i}{\varphi}},$$
$$\Big[1 - (((1 - \underline{r_{F_i}})^{\varphi} + (1 - \underline{r_{F_{n+1}}})^{\varphi}) - \wp((1 - \underline{r_{F_i}})^{\varphi} + (1 - \underline{r_{F_{n+1}}})^{\varphi}))^{\frac{\varpi_i}{\varphi}},$$
$$1 - (((1 - \overline{r_{F_i}})^{\varphi} + (1 - \overline{r_{F_{n+1}}})^{\varphi}) - \wp((1 - \overline{r_{F_i}})^{\varphi} + (1 - \overline{r_{F_{n+1}}})^{\varphi}))^{\frac{\varpi_i}{\varphi}}\Big]\Big\rangle \tag{13}$$

$$\text{RHS} = TIT2SSWG_{\varpi}(\overline{F_1}, \overline{F_2}, ..., \overline{F_n}) \otimes \overline{F_{n+1}}$$
$$= \Big\langle \Big[1 - ((1 - \underline{l_{F_i}})^{\varphi} - \wp(1 - \underline{l_{F_i}})^{\varphi})^{\frac{\varpi_i}{\varphi}}, ((1 - \overline{l_{F_i}})^{\varphi} - \wp(1 - \overline{l_{F_i}})^{\varphi})^{\frac{\varpi_i}{\varphi}}\Big],$$
$$1 - ((1 - c_{F_i})^{\varphi} - \wp(1 - c_{F_i})^{\varphi})^{\frac{\varpi_i}{\varphi}},$$
$$\Big[1 - ((1 - \underline{r_{F_i}})^{\varphi} - \wp(1 - \underline{r_{F_i}})^{\varphi})^{\frac{\varpi_i}{\varphi}}, ((1 - \overline{r_{F_i}})^{\varphi} - \wp(1 - \overline{r_{F_i}})^{\varphi})^{\frac{\varpi_i}{\varphi}}\Big]\Big\rangle \oplus$$
$$\Big\langle \Big[1 - ((1 - \underline{l_{F_{n+1}}})^{\varphi})^{\frac{\varpi_i}{\varphi}}, 1 - ((1 - \overline{l_{F_{n+1}}})^{\varphi})^{\frac{\varpi_i}{\varphi}}\Big], (1 - (1 - c_{F_{n+1}})^{\varphi})^{\frac{\varpi_i}{\varphi}},$$
$$\Big[1 - ((1 - \underline{r_{F_{n+1}}})^{\varphi})^{\frac{\varpi_i}{\varphi}}, 1 - ((1 - \overline{r_{F_{n+1}}})^{\varphi})^{\frac{\varpi_i}{\varphi}}\Big]\Big\rangle$$
$$= \Big\langle \Big[1 - (((1 - \underline{l_{F_i}})^{\varphi} + (1 - \underline{l_{F_{n+1}}})^{\varphi}) - \wp((1 - \underline{l_{F_i}})^{\varphi} + (1 - \underline{l_{F_{n+1}}})^{\varphi}))^{\frac{\varpi_i}{\varphi}},$$
$$1 - (((1 - \overline{l_{F_i}})^{\varphi} + (1 - \overline{l_{F_{n+1}}})^{\varphi}) - \wp((1 - \overline{l_{F_i}})^{\varphi} + (1 - \overline{l_{F_{n+1}}})^{\varphi}))^{\frac{\varpi_i}{\xi}}\Big],$$
$$1 - (((1 - c_{F_i})^{\varphi} + (1 - c_{F_{n+1}})^{\varphi}) - \wp((1 - c_{F_i})^{\varphi} + (1 - c_{F_{n+1}})^{\varphi}))^{\frac{\varpi_i}{\varphi}}$$
$$\Big[1 - (((1 - \underline{r_{F_i}})^{\varphi} + (1 - \underline{r_{F_{n+1}}})^{\varphi}) - \wp((1 - \underline{r_{F_i}})^{\varphi} + (1 - \underline{r_{F_{n+1}}})^{\varphi}))^{\frac{\varpi_i}{\varphi}},$$
$$1 - (((1 - \overline{r_{F_i}})^{\varphi} + (1 - \overline{r_{F_{n+1}}})^{\varphi}) - \wp((1 - \overline{r_{F_i}})^{\varphi} + (1 - \overline{r_{F_{n+1}}})^{\varphi}))^{\frac{\varpi_i}{\varphi}}\Big]\Big\rangle \tag{14}$$

From (13) and (14), the theorem holds.

4.1.4. Theorem

Let $\overline{F_i} = ([\underline{l_{F_i}}, \overline{l_{F_i}}], c_{F_i}, [\underline{r_{F_i}}, \overline{r_{F_i}}]),\ i = 1, 2, ..., n$ be a set of TIT2FNs, $0 \le \underline{l_{F_i}} \le \overline{l_{F_i}} \le c_{F_i} \le \underline{r_{F_i}} \le \overline{r_{F_i}} \le 1,\ i = 1, 2, ..., n.$ If $k > 0,\ \overline{F}_{n+1} = ([\underline{l_{F_{n+1}}}, \overline{l_{F_{n+1}}}], c_{F_{n+1}}, [\underline{r_{F_{n+1}}}, \overline{r_{F_{n+1}}}])$ is a TIT2FN on X then,

$$TIT2SSWG_{\varpi}(\overline{F_1}^k, \overline{F_2}^k, ..., \overline{F_n}^k) = (TIT2SSWG_{\varpi}(\overline{F_1}, \overline{F_2}, ..., \overline{F_n}))^k \tag{15}$$

Proof. :

$$k.\ \overline{S_i} = \Big\langle \Big[1 - ((1 - \underline{l_{F_i}})^{\varphi})^{\frac{k}{\varphi}}, 1 - (\wp(1 - \overline{l_{F_i}})^{\varphi})^{\frac{k}{\varphi}}\Big], 1 - ((1 - c_{F_i})^{\varphi})^{\frac{k}{\varphi}}, \Big[1 - ((1 - \underline{r_{F_i}})^{\varphi})^{\frac{k}{\varphi}}, 1 - (\wp(1 - \overline{r_{F_i}})^{\varphi})^{\frac{k}{\varphi}}\Big]\Big\rangle$$

$$\text{LHS} = TIT2SSWG_{\varpi}(\overline{F_1}^k, \overline{F_2}^k, ..., \overline{F_n}^k)$$
$$= \Big\langle \Big[1 - ([((1 - \underline{l_{F_i}})^{\varphi}) - \wp(\wp(1 - \underline{l_{F_i}})^{\varphi})]^k)^{\frac{\varpi_i}{\varphi}}, 1 - ([((1 - \overline{l_{F_i}})^{\xi}) - \wp(\wp(1 - \overline{l_{F_i}})^{\varphi})]^k)^{\frac{\varpi_i}{\varphi}}\Big],$$
$$1 - [([((1 - c_{F_i})^{\varphi})])^k - ([\wp(\wp(1 - c_{F_i})^{\varphi})])^k]^{\frac{\varpi_i}{\varphi}},$$
$$\Big[1 - ([((1 - \underline{r_{F_i}})^{\varphi}) - \wp(\wp(1 - \underline{r_{F_i}})^{\varphi})]^k)^{\frac{\varpi_i}{\varphi}}, 1 - ([((1 - \overline{r_{F_i}})^{\xi}) - \wp(\wp(1 - \overline{r_{F_i}})^{\varphi})]^k)^{\frac{\varpi_i}{\varphi}}\Big]\Big\rangle$$
$$= \Big\langle \Big[1 - ((1 - \underline{l_{F_i}})^{\varphi} - \wp(1 - \underline{l_{F_i}})^{\varphi})^{\frac{k\varpi_i}{\varphi}}, 1 - ((1 - \overline{l_{F_i}})^{\varphi} - \wp(1 - \overline{l_{F_i}})^{\varphi})^{\frac{k\varpi_i}{\varphi}}\Big],$$
$$1 - ((1 - c_{F_i})^{\varphi} - \wp(1 - c_{F_i})^{\varphi})^{\frac{k\varpi_i}{\varphi}},$$
$$\Big[1 - ((1 - \underline{r_{F_i}})^{\varphi} - \wp(1 - \underline{r_{F_i}})^{\varphi})^{\frac{k\varpi_i}{\varphi}}, 1 - ((1 - \overline{r_{F_i}})^{\varphi} - \wp(1 - \overline{r_{F_i}})^{\varphi})^{\frac{k\varpi_i}{\varphi}}\Big]\Big\rangle \tag{16}$$

RHS= $(TIT2SSWG_{\varpi}(\overline{F_1}, \overline{F_2}, ...,\overline{F_n}))^k$

$$= \left\langle \left[1 - ((1 - \underline{l_{F_i}})^{\varphi} - \wp(1 - \underline{l_{F_i}})^{\varphi})^{\frac{k\varpi_i}{\varphi}}, 1 - ((1 - \overline{l_{F_i}})^{\varphi} - \wp(1 - \overline{l_{F_i}})^{\varphi})^{\frac{k\varpi_i}{\varphi}}\right], 1 - ((1 - c_{F_i})^{\varphi} - \wp(1 - c_{F_i})^{\varphi})^{\frac{k\varpi_i}{\varphi}}, \left[1 - ((1 - \underline{r_{F_i}})^{\varphi} - \wp(1 - \underline{r_{F_i}})^{\varphi})^{\frac{k\varpi_i}{\varphi}}, 1 - ((1 - \overline{r_{F_i}})^{\varphi} - \wp(1 - \overline{r_{F_i}})^{\varphi})^{\frac{k\varpi_i}{\varphi}}\right] \right\rangle$$

$$= \left\langle \left[1 - ((1 - \underline{l_{F_i}})^{\varphi} - \wp(1 - \underline{l_{F_i}})^{\varphi})^{\frac{k\varpi_i}{\varphi}}, 1 - ((1 - \overline{l_{F_i}})^{\varphi} - \wp(1 - \overline{l_{F_i}})^{\varphi})^{\frac{k\varpi_i}{\varphi}}\right], 1 - ((1 - c_{F_i})^{\varphi} - \wp(1 - c_{F_i})^{\varphi})^{\frac{k\varpi_i}{\varphi}}, \left[1 - ((1 - \underline{r_{F_i}})^{\varphi} - \wp(1 - \underline{r_{F_i}})^{\varphi})^{\frac{k\varpi_i}{\varphi}}, 1 - ((1 - \overline{r_{F_i}})^{\varphi} - \wp(1 - \overline{r_{F_i}})^{\varphi})^{\frac{k\varpi_i}{\varphi}}\right] \right\rangle \quad (17)$$

From (16) and (17),

$TIT2SSWG_{\varpi}(\overline{F_1}^k, \overline{F_2}^k, ...,\overline{F_n}^k) = (TIT2SSWG_{\varpi}(\overline{F_1}, \overline{F_2}, ...,\overline{F_n}))^k$

4.1.5. Theorem (Stability)

Let $\overline{F_i} = ([\underline{l_{F_i}}, \overline{l_{F_i}}], c_{F_i}, [\underline{r_{F_i}}, \overline{r_{F_i}}])$, $i = 1, 2, ...,n$ be a set of TIT2FNs, $0 \le \underline{l_{F_i}} \le \overline{l_{F_i}} \le c_{F_i} \le \underline{r_{F_i}} \le \overline{r_{F_i}} \le 1$, $i = 1, 2, ...,n$. If $\overline{F_{n+1}} = ([\underline{l_{F_{n+1}}}, \overline{l_{F_{n+1}}}], c_{F_{n+1}}, [\underline{r_{F_{n+1}}}, \overline{r_{F_{n+1}}}])$ is also a TIT2FN on X. If $k > 0$ then,

$$TIT2SSWG_{\varpi}(\overline{F_1}^k \otimes \overline{F_{n+1}}, \overline{F_2}^k \otimes \overline{F_{n+1}}, ...,\overline{F_n}^k \otimes \overline{F_{n+1}}) = (TIT2SSWG_{\varpi}(\overline{F_1}, \overline{F_2}, ...,\overline{F_n}))^k \otimes \overline{F_{n+1}}. \quad (18)$$

Proof. :

From theorems 4.1.4. and 4.1.5. $TIT2SSWG_{\varpi}(\overline{F_1}^k \otimes \overline{F_{n+1}}, \overline{F_2}^k \otimes \overline{F_{n+1}}, ...,\overline{F_n}^k \otimes \overline{F_{n+1}})$ is true.

$= (TIT2SSWG_{\varpi}(\overline{F_1}, \overline{F_2}, ...,\overline{F_n}))^k \otimes \overline{F_{n+1}}$

4.2. Proposed theorems using TIT2SSWA operator

In this section, the same theorems are listed out and the proof is similar

4.2.1. Let $\overline{F_i} = ([\underline{l_{F_i}}, \overline{l_{F_i}}], c_{F_i}, [\underline{r_{F_i}}, \overline{r_{F_i}}])$, $i = 1, 2, ...,n$ be a set of TIT2FNs; then their aggregated value using TIT2SSWA operator is still a TIT2FN, $0 \le \underline{l_{F_i}} \le \overline{l_{F_i}} \le c_{F_i} \le \underline{r_{F_i}} \le \overline{r_{F_i}} \le 1$, $i = 1, 2, ...,n$ and

$$TIT2SSWA_{\varpi}(\overline{F_1}, \overline{F_2}, ...,\overline{F_n}) = \left\langle \left[((\underline{l_{F_i}})^{\varphi} - \wp(\underline{l_{F_i}})^{\varphi})^{\frac{\varpi_i}{\varphi}}, ((\overline{l_{F_i}})^{\varphi} - \wp(\overline{l_{F_i}})^{\varphi})^{\frac{\varpi_i}{\varphi}}\right], ((c_{F_i})^{\varphi} - \wp(c_{F_i})^{\varphi})^{\frac{\varpi_i}{\varphi}}, \left[((1 - \underline{r_{F_i}})^{\varphi} - \wp(\underline{r_{F_i}})^{\varphi})^{\frac{\varpi_i}{\varphi}}, ((\overline{r_{F_i}})^{\varphi} - \wp(\overline{r_{F_i}})^{\varphi})^{\frac{\varpi_i}{\varphi}}\right] \right\rangle \quad (19)$$

where $\sum_{i=1}^{n} =, \prod_{i=1}^{n} = \wp$

If all $\overline{F_i}$, $i = 1, 2, ...,n$ are equal, i.e., $\overline{F_i} = \overline{F}$ then $TIT2SSWA_{\varpi}(\overline{F_1}, \overline{F_2}, ...,\overline{F_n}) = \overline{F}$. (20)

$$TIT2SSWA_{\varpi}(\overline{F_1} \oplus \overline{F_{n+1}}, \overline{F_2} \oplus \overline{F_{n+1}}, ...,\overline{F_n} \oplus \overline{F_{n+1}}) = TIT2SSWA_{\varpi}(\overline{F_1}, \overline{F_2}, ...,\overline{F_n}) \oplus \overline{F_{n+1}} \quad (21)$$

$$TIT2SSWA_{\varpi}(k{\cdot}\overline{F_1}, k{\cdot}\overline{F_2}, ...,k{\cdot}\overline{F_n}) = k{\cdot}TIT2SSWA_{\varpi}(\overline{F_1}, \overline{F_2}, ...,\overline{F_n}) \quad (22)$$

$$TIT2SSWA_{\varpi}(k{\cdot}\overline{F_1} \oplus \overline{F_{n+1}}, k{\cdot}\overline{F_2} \oplus \overline{F_{n+1}}, ...,k{\cdot}\overline{F_n} \oplus \overline{F_{n+1}}) = k{\cdot}TIT2SSWA_{\varpi}(\overline{F_1}, \overline{F_2}, ...,\overline{F_n}) \oplus \overline{F_{n+1}} \quad (23)$$

4.3. Proposed method for traffic flow control using TIT2SSWA operator

The traffic flow of the junction is considered during rush hour on a working day inFig. 2. The arrow marks represent the direction of the flow in each direction. The average number of vehicles per hour coming in and departing at each intersecting point are taken as triangular interval type-2 fuzzy numbers instead of crisp numbers. The aim of this work is to identify the junction that has a higher number of vehicles (traffic) that need to be cleared first using the score value of the IT2FNs, and the result can be concluded based on the greater score value.

Using Eq. (19),

$$TIT2SSWA_{\varpi}(Z_1, Z_2) = (0.45)Z_1 \oplus (0.55)Z_2$$

$$= \left\langle \left[((0.6)^2 + (0.2)^2 - (0.6)^2.\ (0.2)^2)^{\frac{0.45+0.55}{2}}, ((0.7)^2 + (0.3)^2 - (0.7)^2.\ (0.3)^2)^{\frac{0.45+0.55}{2}}\right], ((0.8)^2 + (0.4)^2 - (0.8)^2.\ (0.4)^2)^{\frac{0.45+0.55}{2}}, \left[((0.9)^2 + (0.5)^2 - (0.9)^2.\ (0.5)^2)^{\frac{0.45+0.55}{2}}, ((1)^2 + (0.6)^2 - (1)^2.\ (0.6)^2)^{\frac{0.45+0.55}{2}}\right] \right\rangle$$

$$= \langle [0.62, 0.73], 0.84, [0.92, 1] \rangle$$

Similarly,

$$TIT2SSWA_{\varpi}(Z_2, Z_3) = (0.45)Z_2 \oplus (0.55)Z_3 = \langle [0.56, 0.68], 0.78, [0.87, 0.9] \rangle$$

$$TIT2SSWA_{\varpi}(Z_3, Z_4) = (0.45)Z_3 \oplus (0.55)Z_4 = \langle [0.41, 0.53], 0.65, [0.75, 0.85] \rangle$$

$$TIT2SSWA_{\varpi}(Z_4, Z_1) = (0.45)Z_4 \oplus (0.55)Z_1 = \langle [0.56, 0.68], 0.79, [0.88, 0.95] \rangle$$

Finding the score values (SVs)

Using Eq. (5)

$$SV(Z_1, Z_2) = \left(\frac{0.62 + 1}{2} + 1\right) \times \frac{0.62 + 0.73 + 0.92 + 1 + 4(0.84)}{8} = 1.5$$

Similarly,

$$SV(Z_2, Z_3) = 1.33,\ SV(Z_3, Z_4) = 1.05,\ SV(Z_4, Z_1) = 1.37$$

From the score values, the junction between $Z_1$and Z_2 has a higher value, and therefore it is recommended that this junction has more traffic and may be cleared first.

5. Neutrosophic perspective

The concept of interval type-2 fuzzy sets can be extended to interval neutrosophic sets. As fuzzy sets handle only truth and false membership grades whereas neutrosophic sets handle not only truth and false membership grades but also indeterminacy grade, extension of the above results would provide an efficient way of handling uncertainties existing in the real-world problems.

The above theorems have been extended to an interval neutrosophic setting. The following are the basic concepts related to interval neutrosophic sets.

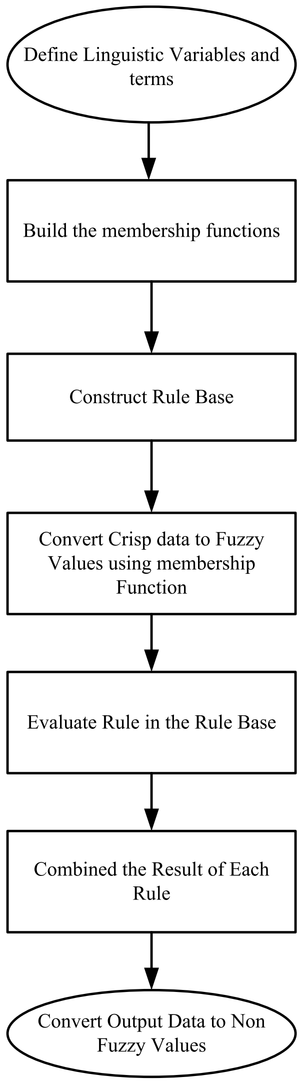

Fig. 1. Traffic control system using fuzzy logic.

5.1. Basic concepts

In this section, essential definitions of interval neutrosophic set and numbers are given, and based on these definitions, Schweizer and Sklar operations of INNs have been proposed.

5.1.1. Interval Neutrosophic Set (INS) [46]

Let U be a nonempty set. An interval neutrosophic set $B$is defined as follows.

$B = \{x, \langle T(x),\ I(x),\ F(x)\rangle | x \in B\}$, where the intervals $T(x) = [T^L(x),\ T^U(x)] \subseteq [0, 1]$, $I(x) = [I^L(x),\ I^U(x)] \subseteq [0, 1]$, $F(x) = [F^L(x),\ F^U(x)] \subseteq [0, 1]$ for $x \in U$are the grades of the truth-membership, indeterminacy-membership and false-membership respectively.

5.1.2. Interval neutrosophic numbers [32]

Let $X = \{x_1, x_2, \ldots, x_n\}$be an INS, where $x_j = \langle [T_j^L,\ T_j^U],\ [I_j^L,\ I_j^U],\ [F_j^L,\ F_j^U]\rangle$for $j = 1, 2, 3, \ldots, n$ is a collection of INNs and$T_j^L, T_j^U, I_j^L, I_j^U, F_j^L, F_j^U \in (0, 1)$, $\upsilon > 0$and$\delta > 0$. Then, the SS T-norm and T-conorm operations on INNs, are defined as follows.

5.2. Proposed schweizer and sklar operations of interval neutrosophic numbers

Addition:

$$x_1 \oplus x_2 = \{[((T_1^L)^\delta + (T_2^L)^\delta - (T_1^L)^\delta (T_2^L)^\delta)^{1/\delta},\ ((T_1^U)^\delta + (T_2^U)^\delta - (T_1^U)^\delta (T_2^U)^\delta)^{1/\delta}], \tag{24}$$

$$[1 - ((1 - I_1^L)^\delta + (1 - I_2^L)^\delta - (1 - I_1^L)^\delta (1 - I_2^L)^\delta)^{1/\delta},\ 1 - ((1 - I_1^U)^\delta + (1 - I_2^U)^\delta - (1 - I_1^U)^\delta (1 - I_2^U)^\delta)^{1/\delta}],$$
$$[1 - ((1 - F_1^L)^\delta + (1 - F_2^L)^\delta - (1 - F_1^L)^\delta (1 - I_2^L)^\delta)^{1/\delta},\ 1 - ((1 - F_1^U)^\delta + (1 - F_2^U)^\delta - (1 - F_1^U)^\delta (1 - F_2^U)^\delta)^{1/\delta}]\}$$

Numerical Example:

If $x_1 = \langle [0.7, 0.8], [0.0, 0.1], [0.1, 0.2]\rangle$ and $x_2 = \langle [0.4, 0.5], [0.2, 0.3], [0.3, 0.4]\rangle$are the two INNs and $\delta = 2$then

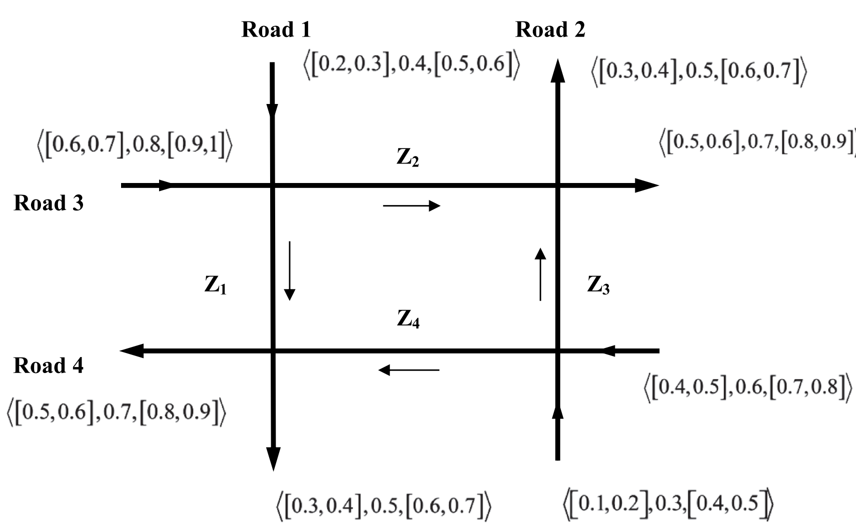

Fig. 2. Traffic flow on the road with four junctions using triangular interval type-2 fuzzy numbers.

$$x_1 \oplus x_2 = \{[((0.7)^2+(0.4)^2-(0.7)^2.(0.4)^2)^{1/2}, ((0.8)^2+(0.5)^2-(0.8)^2.(0.5)^2)^{1/2}],$$

$$[1-((1-0)^2+(1-0.2)^2-(1-0)^2.(1-0.2)^2)^{1/2}, 1-((1-0.1)^2+(1-0.3)^2-(1-0.1)^2.(1-0.3)^2)^{1/2}],$$

$$[1-((1-0.1)^2+(1-0.3)^2-(1-0.1)^2.(1-0.3)^2)^{1/2}, 1-((1-0.2)^2+(1-0.4)^2-(1-0.2)^2.(1-0.4)^2)^{1/2}]\}$$

$x_1 \oplus x_2 = \{[0.76, 0.85], [0.11, 0.13], [0.05, 0.12]\}$= INN

Multiplication:

$$x_1 \otimes x_2 = \{[1-((1-T_1^L)^\delta+(1-T_2^L)^\delta-(1-T_1^L)^\delta(1-T_2^L)^\delta)^{1/\delta}, 1-((1-T_1^U)^\delta+(1-T_2^U)^\delta-(1-T_1^U)^\delta(1-T_2^U)^\delta)^{1/\delta}],$$
$$[((I_1^L)^\delta+(I_2^L)^\delta-(I_1^L)^\delta(I_2^L)^\delta)^{1/\delta}, ((I_1^U)^\delta+(I_2^U)^\delta-(I_1^U)^\delta(I_2^U)^\delta)^{1/\delta}],$$
$$[((F_1^L)^\delta+(F_2^L)^\delta-(F_1^L)^\delta(I_2^L)^\delta)^{1/\delta}, ((F_1^U)^\delta+(F_2^U)^\delta-(F_1^U)^\delta(F_2^U)^\delta)^{1/\delta}]\} \quad (25)$$

Numerical Example:

$$x_1 \otimes x_2 = \{[1-((1-0.7)^2+(1-0.4)^2-(1-0.7)^2.(1-0.4)^2)^{1/2}, 1-((1-0.8)^2+(1-0.5)^2-(1-0.8)^2.(1-0.5)^2)^{1/2}],$$
$$[((0)^2+(0.2)^2-(0)^2.(0.2)^2)^{1/2}, ((0.1)^2+(0.3)^2-(0.1)^2.(0.3)^2)^{1/2}],$$
$$[((0.1)^2+(0.3)^2-(0.1)^2.(0.3)^2)^{1/2}, ((0.2)^2+(0.4)^2-(0.2)^2.(0.4)^2)^{1/2}]\}$$

$x_1 \otimes x_2 = \{[0.35, 0.47], [0.2, 0.31], [0.3148, 0.44]\}$ = INN

Multiplication by an ordinary Numbers:

$$g.\,x_1 = \{[(g(T_1^L)^\delta)^{1/\delta}, (g(T_1^U)^\delta)^{1/\delta}], [1-(g(1-I_1^L)^\delta)^{1/\delta}, 1-(g(1-I_1^U)^\delta)^{1/\delta}], [1-(g(1-F_1^L)^\delta)^{1/\delta}, 1-(g(1-F_1^U)^\delta)^{1/\delta}]\} \quad (26)$$

Numerical Example: Consider $g = 0.2$

$$(0.2).\,x_1 = \{[(0.2(0.7)^2)^{1/2}, (0.2(0.8)^2)^{1/2}], [1-(0.2(1-0.0)^2)^{1/2}, 1-(0.2(1-0.1)^2)^{1/2}], [1-(0.2(1-0.1)^2)^{1/2}, 1-(0.2(1-0.2)^2)^{1/2}]\}$$
$= \{[0.3130, 357], [0.55, 0.5976], [0.5976, 0.6422]\}$ = INN

Power Operation:

$$x_1^g = \{[1-(g(1-T_1^L)^\delta)^{1/\delta}, 1-(g(1-T_1^U)^\delta)^{1/\delta}], [(g(I_1^L)^\delta)^{1/\delta}, (g(I_1^U)^\delta)^{1/\delta}], [(g(F_1^L)^\delta)^{1/\delta}, (g(F_1^U)^\delta)^{1/\delta}]\} \quad (27)$$

Numerical Example: Consider $g = 0.2$

$$x_1^{0.2} = \{[1-(0.2(1-0.7)^2)^{1/2}, 1-(0.2(1-0.8)^2)^{1/2}], [(0.2(0.0)^2)^{1/2}, (0.2(0.1)^2)^{1/2}], [(0.2(0.1)^2)^{1/2}, (0.2(0.2)^2)^{1/2}]\}$$
$= \{[0.8658, 0.9106], [0.0, 0.0447], [0.0447, 0.0894]\}$=INN

5.2.1. Proposed score function

For ranking INNs, a new score function is proposed in this section, defined by

$$SF(\overline{F}) = \frac{1}{2}[(T_F^L + T_F^U) - (I_F^L I_F^U) + (I_F^U - 1)^2 + F_F^U] \quad (28)$$

5.3. Proposed theorems using INSSWG operator

The following theorems are proved using the proposed aggregation operator

5.3.1. Theorem

Let $x_j = \langle [T_j^L, T_j^U], [I_j^L, I_j^U], [F_j^L, F_j^U] \rangle$, $j = 1, 2, 3, \ldots, n$ be a collection of INNs and their weight vector is $\upsilon = (\upsilon_1, \upsilon_2, \ldots, \upsilon_n)$, $\upsilon_i \in [0, 1]$ and $\sum_{j=1}^n \upsilon_j = 1$. Then, the aggregated value of the interval neutrosophic Schweizer and Sklar weighted geometric (INSSWG) operator is still an INN, i.e.,

$$INSSWG_\upsilon(x_1, x_2, \ldots, x_n) = \{[1-(((\upsilon_j(1-T_j^L)^\delta) - \wp(\upsilon_j(1-T_j^L)^\delta))^{1/\delta}, 1-((\upsilon_j(1-T_j^U)^\delta) - \wp(\upsilon_j(1-T_j^U)^\delta))^{1/\delta})],$$
$$\left[\left(\left(\upsilon_{j_{(I_j^L)^\delta}}\right) - \wp(\upsilon_j(I_j^L)^\delta)\right)^{1/\delta}, \left(\left(\upsilon_{j_{(I_j^U)^\delta}}\right) - \wp(\upsilon_j(I_j^U)^\delta)\right)^{1/\delta}\right],$$
$$[((\upsilon_j(F_j^L)^\delta) - \wp(\upsilon_j(F_j^L)^\delta))^{1/\delta}, ((\upsilon_j(F_j^U)^\delta) - \wp(\upsilon_j(F_j^U)^\delta))^{1/\delta}]\} \quad (29)$$

where $\sum_{i=1}^n$ =, $\prod_{i=1}^n$ =$\wp$.

Proof. :

Mathematical induction is used to prove this theorem.

When $n = 2$, using SS triangular norms, we get,

$$INSSWG_\upsilon(x_1, x_2) = x_1^{\upsilon_1} \otimes x_2^{\upsilon_2}$$
$$= \{[1-(\upsilon_1(1-T_1^L)^\delta + \upsilon_2(1-T_2^L)^\delta - \upsilon_1(1-T_1^L)^\delta\upsilon_2(1-T_2^L)^\delta)^{1/\delta}, 1-(\upsilon_1(1-T_1^U)^\delta + \upsilon_2(1-T_2^U)^\delta - \upsilon_1(1-T_1^U)^\delta\upsilon_2(1-T_2^U)^\delta)^{1/\delta}]$$
$$[(\upsilon_1(I_1^L)^\delta + \upsilon_2(I_2^L)^\delta - \upsilon_1(I_1^L)^\delta\upsilon_2(I_2^L)^\delta)^{1/\delta}, (\upsilon_1(I_1^U)^\delta + \upsilon_2(I_2^U)^\delta - \upsilon_1(I_1^U)^\delta\upsilon_2(I_2^U)^\delta)^{1/\delta}],$$
$$[(\upsilon_1(F_1^L)^\delta + \upsilon_2(F_2^L)^\delta - \upsilon_1(F_1^L)^\delta\upsilon_2(I_2^L)^\delta)^{1/\delta}, (\upsilon_1(F_1^U)^\delta + \upsilon_2(F_2^U)^\delta - \upsilon_1(F_1^U)^\delta\upsilon_2(F_2^U)^\delta)^{1/\delta}]\}$$
$$= \{[1-(((\upsilon_j(1-T_j^L)^\delta) - \wp(\upsilon_j(1-T_j^L)^\delta))^{1/\delta}, 1-((\upsilon_j(1-T_j^U)^\delta) - \wp(\upsilon_j(1-T_j^U)^\delta))^{1/\delta})],$$
$$\left[\left(\left(\upsilon_{j_{(I_j^L)^\delta}}\right) - \wp(\upsilon_j(I_j^L)^\delta)\right)^{1/\delta}, \left(\left(\upsilon_{j_{(I_j^U)^\delta}}\right) - \wp(\upsilon_j(I_j^U)^\delta)\right)^{1/\delta}\right],$$
$$[((\upsilon_j(F_j^L)^\delta) - \wp(\upsilon_j(F_j^L)^\delta))^{1/\delta}, ((\upsilon_j(F_j^U)^\delta) - \wp(\upsilon_j(F_j^U)^\delta))^{1/\delta}]\}$$

where $\sum_{j=1}^2$ =, $\prod_{j=1}^2$ =$\wp$.

When $n = k$,

$$INSSWG_\upsilon(x_1, x_2, \ldots, x_k) = x_1^{\upsilon_1} \otimes x_2^{\upsilon_2} \otimes \ldots \otimes x_k^{\upsilon_k}$$
$$= \{[1-(((\upsilon_j(1-T_j^L)^\delta) - \wp(\upsilon_j(1-T_j^L)^\delta))^{1/\delta}, 1-((\upsilon_j(1-T_j^U)^\delta) - \wp(\upsilon_j(1-T_j^U)^\delta))^{1/\delta})],$$
$$\left[\left(\left(\upsilon_{j_{(I_j^L)^\delta}}\right) - \wp(\upsilon_j(I_j^L)^\delta)\right)^{1/\delta}, \left(\left(\upsilon_{j_{(I_j^U)^\delta}}\right) - \wp(\upsilon_j(I_j^U)^\delta)\right)^{1/\delta}\right],$$
$$[((\upsilon_j(F_j^L)^\delta) - \wp(\upsilon_j(F_j^L)^\delta))^{1/\delta}, ((\upsilon_j(F_j^U)^\delta) - \wp(\upsilon_j(F_j^U)^\delta))^{1/\delta}]\}$$

where$\sum_{j=1}^{k} =$, $\prod_{j=1}^{k} =\wp$.

When$n = k + 1$,

$$INSSWG_{\upsilon}(x_1, x_2, ..., x_k, x_{k+1}) = (x_1^{\upsilon_1} \otimes x_2^{\upsilon_2} \otimes ...\otimes x_k^{\upsilon_k}) \otimes x_{k+1}^{\upsilon_{k+1}}$$
$$= \langle\{[1 - (((\upsilon_i(1 - T_i^L)^{\delta}) - \wp(\upsilon_i(1 - T_i^L)^{\delta}))^{1/\delta}, 1 - ((\upsilon_i(1 - T_i^U)^{\delta}) - \wp(\upsilon_i(1 - T_i^U)^{\delta}))^{1/\delta})],$$
$$\left[\left(\left(\upsilon_{i_{(I_i^L)^{\delta}}}\right) - \wp(\upsilon_i(I_i^L)^{\delta})\right)^{1/\delta}, \left(\left(\upsilon_{i_{(I_i^U)^{\delta}}}\right) - \wp(\upsilon_i(I_i^U)^{\delta})\right)^{1/\delta}\right],$$
$$[((\upsilon_j(F_j^L)^{\delta}) - \wp(\upsilon_j(F_j^L)^{\delta}))^{1/\delta}, ((\upsilon_j(F_j^U)^{\delta}) - \wp(\upsilon_j(F_j^U)^{\delta}))^{1/\delta}]\}\rangle \otimes \upsilon_{k+1}x_{k+1}$$
$$= \left\{\left[1 - \left(\sum_{j=1}^{k+1}(\upsilon_j(1 - T_j^L)^{\delta}) - \prod_{j=1}^{k+1}(\upsilon_j(1 - T_j^L)^{\delta})\right)^{1/\delta}, 1 - \left(\sum_{j=1}^{k+1}(\upsilon_j(1 - T_j^U)^{\delta}) - \prod_{j=1}^{k+1}(\upsilon_j(1 - T_j^U)^{\delta})\right)^{1/\delta}\right],\right.$$
$$\left[\left(\sum_{j=1}^{k+1}\left(\upsilon_{j_{(I_j^L)^{\delta}}}\right) - \prod_{j=1}^{k+1}(\upsilon_j(I_j^L)^{\delta})\right)^{1/\delta}, \left(\sum_{j=1}^{k+1}\left(\upsilon_{j_{(I_j^U)^{\delta}}}\right) - \prod_{j=1}^{k+1}(\upsilon_j(I_j^U)^{\delta})\right)^{1/\delta}\right],$$
$$\left.\left[\left(\sum_{j=1}^{k+1}(\upsilon_j(F_j^L)^{\delta}) - \prod_{j=1}^{k+1}(\upsilon_j(F_j^L)^{\delta})\right)^{1/\delta}, \left(\sum_{j=1}^{k+1}(\upsilon_j(F_j^U)^{\delta}) - \prod_{j=1}^{k+1}(\upsilon_j(F_j^U)^{\delta})\right)^{1/\delta}\right]\right\}$$

Hence, the theorem is true for all values of n.

Numerical Example: For $n = 2$

Consider the same $x_1$and x_2, and consider the weight vectors $\upsilon_1 = 0.55$and $\upsilon_1 = 0.45$

$$INSSWG_{\upsilon}(x_1, x_2) = x_1^{0.55} \otimes x_2^{0.45}$$
$$= \{[1 - (0.55(1 - 0.7)^2 + 0.45(1 - 0.4)^2 - (0.55(1 - 0.7)^2)(0.45(1 - 0.4)^2))^{1/2},$$
$$1 - (0.55(1 - 0.8)^2 + 0.45(1 - 0.5)^2 - (0.55(1 - 0.8)^2)(0.45(1 - 0.5)^2))^{1/2}]$$
$$[(0.55(0)^2 + 0.45(0.1)^2 - (0.55(0)^2)(0.45(0.1)^2))^{1/2},$$
$$(0.55(0.1)^2 + 0.45(0.3)^2 - (0.55(0.1)^2)(0.45(0.3)^2))^{1/2}]$$
$$[(0.55(0.1)^2 + 0.45(0.3)^2 - (0.55(0.1)^2)(0.45(0.3)^2))^{1/2},$$
$$(0.55(0.2)^2 + 0.45(0.4)^2 - (0.55(0.2)^2)(0.45(0.4)^2))^{1/2}]\}$$
$$= \{[0.5489, 0.6367], [0.0671, 0.2140], [0.2140, 0.3040]\}.$$

5.3.2. Theorem

If $\upsilon = (1/n, 1/n, ..., 1/n)$then,

$$INSSWG_{\upsilon}(x_1, x_2, ..., x_n)$$
$$= \left\{\left[1 - \left(\left(\left(\tfrac{1}{n}(1 - T_j^L)^{\delta}\right) - \wp\left(\tfrac{1}{n}(1 - T_j^L)^{\delta}\right)\right)^{1/\delta}, 1 - \left(\left(\tfrac{1}{n}(1 - T_j^U)^{\delta}\right) - \wp\left(\tfrac{1}{n}(1 - T_j^U)^{\delta}\right)\right)^{1/\delta}\right)\right],\right.$$
$$\left[\left(\left(\tfrac{1}{n}(I_j^L)^{\delta}\right) - \wp\left(\tfrac{1}{n}(I_j^L)^{\delta}\right)\right)^{1/\delta}, \left(\left(\tfrac{1}{n}(I_j^U)^{\delta}\right) - \wp\left(\tfrac{1}{n}(I_j^U)^{\delta}\right)\right)^{1/\delta}\right],$$
$$\left.\left[\left(\left(\tfrac{1}{n}(F_j^L)^{\delta}\right) - \wp\left(\tfrac{1}{n}(F_j^L)^{\delta}\right)\right)^{1/\delta}, \left(\left(\tfrac{1}{n}(F_j^U)^{\delta}\right) - \wp\left(\tfrac{1}{n}(F_j^U)^{\delta}\right)\right)^{1/\delta}\right]\right\} \quad (30)$$

where $\sum_{i=1}^{n} =$, $\prod_{i=1}^{n} =\wp$. Therefore, INSSWG operator reduces into an interval neutrosophic Schweizer and Sklar weighted arithmetic (INSSWA) operator when the weight vector$\upsilon = (1/n, 1/n, ..., 1/n)$.

5.3.3. Theorem (Idempotency)

Let $x_j = \langle[T_j^L, T_j^U], [I_j^L, I_j^U], [F_j^L, F_j^U]\rangle$,

$j = 1, 2, 3, ..., n$ be a collection of INNs and if $x_j = x$, then

$$INSSWG_{\upsilon}(x_1, x_2, ..., x_n) = x. \quad (31)$$

Proof. :

$$INSSWG_{\upsilon}(x_1, x_2, ..., x_n)$$
$$= \{[1 - (((\upsilon_i(1 - T_i^L)^{\delta}) - \wp(\upsilon_i(1 - T_i^L)^{\delta}))^{1/\delta}, 1 - ((\upsilon_i(1 - T_i^U)^{\delta}) - \wp(\upsilon_i(1 - T_i^U)^{\delta}))^{1/\delta})],$$
$$\left[\left(\left(\upsilon_{i_{(I_i^L)^{\delta}}}\right) - \wp(\upsilon_i(I_i^L)^{\delta})\right)^{1/\delta}, \left(\left(\upsilon_{i_{(I_i^U)^{\delta}}}\right) - \wp(\upsilon_i(I_i^U)^{\delta})\right)^{1/\delta}\right],$$
$$[((\upsilon_j(F_j^L)^{\delta}) - \wp(\upsilon_j(F_j^L)^{\delta}))^{1/\delta}, ((\upsilon_j(F_j^U)^{\delta}) - \wp(\upsilon_j(F_j^U)^{\delta}))^{1/\delta}]\}$$

$$= \{[1 - ((1 - T^L)^{\delta})^{1/\delta}, 1 - ((1 - T^U)^{\delta})^{1/\delta}],$$
$$\left[\left(_{(F^L)^{\delta}}\right)^{1/\delta}, \left(_{(F^U)^{\delta}}\right)^{1/\delta}\right]\}\left[\left(_{(I^L)^{\delta}}\right)^{1/\delta}, \left(_{(I^U)^{\delta}}\right)^{1/\delta}\right],$$

$$= \{[1 - ((1 - T^L)), 1 - ((1 - T^U))], [(_{(I^L)}), (_{(I^U)})], [(_{(F^L)}), (_{(F^U)})]\}$$
$$= \{[1 - T^L, 1 - T^U], [I^L, I^U], [F^L, F^U]\}$$
$$= \{[T^L, T^U], [I^L, I^U], [F^L, F^U]\}$$
$$= x$$

Hence, the theorem is proved.

Numerical computation can be performed as in theorem 5.3.1.

5.3.4. Theorem (Boundedness)

Let x_j, $j = 1, 2, ..., n$be a collection of INNs and let

$$x^- = \left\langle\left(\left[\min_j(T_j^L), \min_j(T_j^U)\right], \left[\max_j(I_j^L), \max_j(I_j^U)\right], \left[\max_j(F_j^L), \max_j(F_j^U)\right]\right)\right\rangle \text{ and}$$

$$x^+ = \left\langle\left(\left[\max_j(T_j^L), \max_j(T_j^U)\right], \left[\min_j(I_j^L), \min_j(I_j^U)\right], \left[\min_j(F_j^L), \min_j(F_j^U)\right]\right)\right\rangle. \text{ then,}$$

$$x^- \le INSSWG_{\upsilon}(x_1, x_2, ..., x_n) \le x^+ \quad (32)$$

Proof. :

Since, $\min_j(T_j^L) \le T_j^L \le \max_j(T_j^L)$, $\min_j(T_j^U) \le T_j^U \le \max_j(T_j^U)$

$\min_j(I_j^L) \le I_j^L \le \max_j(I_j^L)$, $\min_j(I_j^U) \le I_j^U \le \max_j(I_j^U)$

$\min_j(F_j^L) \le F_j^L \le \max_j(F_j^L)$, $\min_j(F_j^U) \le F_j^U \le \max_j(F_j^U)$

the following inequalities are holding good.

$1-((\upsilon_j \min(1-T_j^L)^\delta)-\wp(\upsilon_j \min(1-T_j^L)^\delta))^{1/\delta}$
$\le 1-((\upsilon_j(1-T_j^L)^\delta)-\wp(\upsilon_j(1-T_j^L)^\delta))^{1/\delta}$
$\le 1-((\upsilon_j \max(1-T_j^L)^\delta)-\wp(\upsilon_j \max(1-T_j^L)^\delta))^{1/\delta}$
$1-((\upsilon_j \min(1-T_j^U)^\delta)-\wp(\upsilon_j \min(1-T_j^U)^\delta))^{1/\delta}$
$\le 1-((\upsilon_j(1-T_j^U)^\delta)-\wp(\upsilon_j(1-T_j^U)^\delta))^{1/\delta}$
$\le 1-((\upsilon_j \max(1-T_j^U)^\delta)-\wp(\upsilon_j \max(1-T_j^U)^\delta))^{1/\delta}$,
$((\upsilon_j \min(I_j^L)^\delta)-\wp(\upsilon_j \min(I_j^L)^\delta))^{1/\delta} \le ((\upsilon_j(I_j^L)^\delta)-\wp(\upsilon_j(I_j^L)^\delta))^{1/\delta}$
$\le ((\upsilon_j \max(I_j^L)^\delta)-\wp(\upsilon_j \max(I_j^L)^\delta))^{1/\delta}$,
$((\upsilon_j \min(I_j^U)^\delta)-\wp(\upsilon_j \min(I_j^U)^\delta))^{1/\delta} \le ((\upsilon_j(I_j^U)^\delta)-\wp(\upsilon_j(I_j^U)^\delta))^{1/\delta}$
$\le ((\upsilon_j \max(I_j^U)^\delta)-\wp(\upsilon_j \max(I_j^U)^\delta))^{1/\delta}$,
$((\upsilon_j \min(F_j^L)^\delta)-\wp(\upsilon_j \min(F_j^L)^\delta))^{1/\delta} \le ((\upsilon_j(F_j^L)^\delta)-\wp(\upsilon_j(F_j^L)^\delta))^{1/\delta}$
$\le ((\upsilon_j \max(F_j^L)^\delta)-\wp(\upsilon_j \max(F_j^L)^\delta))^{1/\delta}$,
$((\upsilon_j \min(F_j^U)^\delta)-\wp(\upsilon_j \min(F_j^U)^\delta))^{1/\delta} \le ((\upsilon_j(F_j^U)^\delta)-\wp(\upsilon_j(F_j^U)^\delta))^{1/\delta}$
$\le ((\upsilon_j \max(F_j^U)^\delta)-\wp(\upsilon_j \max(F_j^U)^\delta))^{1/\delta}$

Therefore, $x^- \le INSSWG_\upsilon(x_1, x_2, ..., x_n) \le x^+$. Hence, the result.Numerical computation can be performed as in theorem 5.3.1.

5.3.5. Theorem (Stability)

Let$x_j = \langle [T_j^L, T_j^U], [I_j^L, I_j^U], [F_j^L, F_j^U] \rangle$, $j = 1, 2, 3, ..., n$be a collection of INNs and their weight vector is$\upsilon = (\upsilon_1, \upsilon_2, ..., \upsilon_n)$, $\upsilon_i \in [0, 1]$ and$\sum_{j=1}^n \upsilon_j = 1$. If $x_{n+1} = \langle [T_{n+1}^L, T_{n+1}^U], [I_{n+1}^L, I_{n+1}^U], [F_{n+1}^L, F_{n+1}^U] \rangle$ is also an INN and $k > 0$ then

$$INSSWG_\upsilon(x_1{}^k \otimes x_{n+1}, x_2{}^k \otimes x_{n+1}, ..., x_n{}^k \otimes x_{n+1}) = (INSSWG_\upsilon(x_1, x_2, ..., x_n))^k \otimes x_{n+1} \quad (33)$$

Proof. :

Based on the operational laws and above results, the following results are true for INNs.

$$INSSWG_\upsilon(x_1 \otimes x_{n+1}, x_2 \otimes x_{n+1}, ..., x_n \otimes x_{n+1}) = INSSWG_\upsilon(x_1, x_2, ..., x_n) \otimes x_{n+1} \quad (34)$$

$$INSSWG_\upsilon(x_1{}^k, x_2{}^k, ..., x_n{}^k) = (INSSWG_\upsilon(x_1, x_2, ..., x_n))^k \quad (35)$$

From (34) and (35), it is obvious that,

$$INSSWG_\upsilon(x_1{}^k \otimes x_{n+1}, x_2{}^k \otimes x_{n+1}, ..., x_n{}^k \otimes x_{n+1}) = (INSSWG_\upsilon(x_1, x_2, ..., x_n))^k \otimes x_{n+1}$$

Numerical computation can be performed as in theorem 5.3.1.

5.4. Proposed theorems using INSSWA operator

Here, statements of the above theorems are given, and the proof is similar.

5.4.1. Theorem

Let$x_j = \langle [T_j^L, T_j^U], [I_j^L, I_j^U], [F_j^L, F_j^U] \rangle$, $j = 1, 2, 3, ..., n$be a collection of INNs and their weight vector is$\upsilon = (\upsilon_1, \upsilon_2, ..., \upsilon_n)$, $\upsilon_i \in [0, 1]$ and$\sum_{j=1}^n \upsilon_j = 1$. Then, the aggregated value of the interval neutrosophic Schweizer and Sklar weighted averaging (INSSWA) operator is still an INN, i.e.,

$$\begin{aligned} &INSSWA_\upsilon(x_1, x_2, ..., x_n)\\ &= \{[(((\upsilon_j(T_j^L)^\delta)-\wp(\upsilon_j(T_j^L)^\delta))^{1/\delta}, ((\upsilon_j(T_j^U)^\delta)-\wp(\upsilon_j(T_j^U)^\delta))^{1/\delta})],\\ &\left[1-\left(\left(\upsilon_{j_{(1-I_j^L)^\delta}}\right)-\wp(\upsilon_j(1-I_j^L)^\delta)\right)^{1/\delta}, 1 - ((\upsilon_j(1-I_j^U)^\delta)-\wp(\upsilon_j(1-I_j^U)^\delta))^{1/\delta}\right],\\ &[1-((\upsilon_j(1-F_j^L)^\delta)-\wp(\upsilon_j(1-F_j^L)^\delta))^{1/\delta}, 1 - ((\upsilon_j(1-F_j^U)^\delta)-\wp(\upsilon_j(1-F_j^U)^\delta))^{1/\delta}]\} \end{aligned} \quad (36)$$

$$\text{If all } x_i,\ i = 1, 2, ..., n \text{ are equal, i.e.,} x_i = x \text{ then } INSSWA_\varpi(x_1, x_2, ..., x_n) = x.. \quad (37)$$

$$INSSWA_\upsilon(x_1 \oplus x_{n+1}, x_2 \oplus x_{n+1}, ..., x_n \oplus x_{n+1}) = INSSWA_\upsilon(x_1, x_2, ..., x_n) \oplus x_{n+1} \quad (38)$$

$$INSSWA_\varpi(k \cdot x_1, k \cdot x_2, ..., k \cdot x_n) = k \cdot INSSWA_\varpi(x_1, x_2, ..., x_n) \quad (39)$$

$$TIT2SSWA_\varpi(k \cdot \overline{F_1} \oplus \overline{F_{n+1}}, k \cdot \overline{F_2} \oplus \overline{F_{n+1}}, ..., k \cdot \overline{F_n} \oplus \overline{F_{n+1}}) = k \cdot TIT2SSWA_\varpi(\overline{F_1}, \overline{F_2}, ..., \overline{F_n}) \oplus \overline{F_{n+1}} \quad (40)$$

5.5. Proposed method for traffic flow control using INSSWA operator

For the same experiment as in the previous case, the average number of vehicles per hour coming in and departing at each intersecting point is taken as INNs instead of crisp numbers as in Fig. 3. The aim of this work is to identify the junction that has more vehicles (traffic), which need to be cleared first using the score value of the INN and the higher score value represents the junction that has more traffic.

Using Eq. (36),

$$\begin{aligned} &INSSWA_\varpi(Z_1, Z_2) = (0.45)Z_1 \oplus (0.55)Z_2\\ &= \langle [(0.45(0.3)^2 + 0.55(0.4)^2 - (0.45(0.3)^2)(0.55(0.4)^2))^{1/2},\\ &(0.45(0.7)^2 + 0.55(0.6)^2 - (0.45(0.7)^2)(0.55(0.6)^2))^{1/2}],\\ &[1-(0.45(1-0.2)^2 + 0.55(1-0.1)^2 - (0.45(1-0.2)^2)(0.55(1-0.1)^2))^{1/2},\\ &1-(0.45(1-0.3)^2 + 0.55(1-0.2)^2 - (0.45(1-0.3)^2)(0.55(1-0.2)^2))^{1/2}],\\ &[1-(0.45(1-0.3)^2 + 0.55(1-0.2)^2 - (0.45(1-0.3)^2)(0.55(1-0.2)^2))^{1/2},\\ &1-(0.45(1-0.4)^2 + 0.55(1-0.3)^2 - (0.45(1-0.4)^2)(0.55(1-0.3)^2))^{1/2}]\rangle\\ &= \langle [0.35, 0.61], [0.22, 0.29], [0.29, 0.37]\rangle \end{aligned}$$

Similarly,

$$\begin{aligned} INSSWA_\varpi(Z_2, Z_3) &= (0.45)Z_2 \oplus (0.55)Z_3\\ &= \langle [0.38, 0.58], [0.22, 0.30], [0.26, 0.34]\rangle\\ INSSWA_\varpi(Z_3, Z_4) &= (0.45)Z_3 \oplus (0.55)Z_4\\ &= \langle [0.35, 0.62], [0.23, 0.31], [0.23, 0.32]\rangle\\ INSSWA_\varpi(Z_4, Z_1) &= (0.45)Z_4 \oplus (0.55)Z_1\\ &= \langle [0.4, 0.57], [0.23, 0.31], [0.22, 0.29]\rangle \end{aligned}$$

Finding the score values (SVs)

Using Eq. (28),

$$SV(Z_1, Z_2) = \frac{1}{2}[(0.35 + 0.61) - (0.22 \times 0.29) + (0.29 - 1)^2 + 0.37] = 0.89$$

Similarly, $SV(Z_2, Z_3) = 0.85$, $SV(Z_3, Z_4) = 0.84$,$SV(Z_4, Z_1) = 0.83$

Based on the score values, the junction between $Z_1$and Z_2 has higher value, and therefore it is recommended that this junction may be cleared first as it has more traffic.

6. Traffic flow using proposed operators

The proposed operators under interval type-2 fuzzy environment

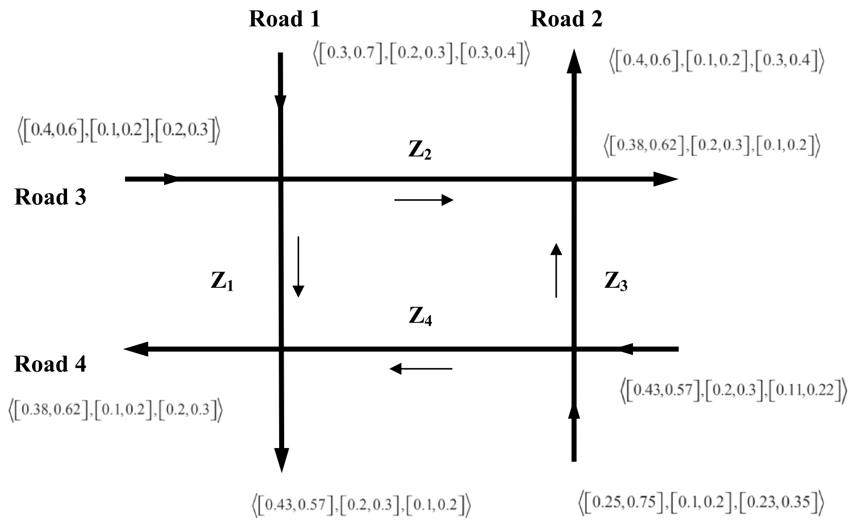

Fig. 3. Traffic flow on the road with four junctions using interval neutrosophic numbers.

Table 1
Aggregated traffic flow and score value.

Junction	TIT2SSWA	SV	INSSWA	SV
(Z_1, Z_2)	⟨[0.62, 0.73], 0.84, [0.92, 1]⟩	**1.5**	⟨[0.35, 0.61], [0.22, 0.29], [0.29, 0.37]⟩	**0.89**
(Z_2, Z_3)	⟨[0.56, 0.68], 0.78, [0.87, 0.9]⟩	**1.33**	⟨[0.38, 0.58], [0.22, 0.30], [0.26, 0.34]⟩	**0.85**
(Z_3, Z_4)	⟨[0.41, 0.53], 0.65, [0.75, 0.85]⟩	**1.05**	⟨[0.35, 0.62], [0.23, 0.31], [0.23, 0.32]⟩	**0.84**
(Z_4, Z_1)	⟨[0.56, 0.68], 0.79, [0.88, 0.95]⟩	**1.37**	⟨[0.4, 0.57], [0.23, 0.31], [0.22, 0.29]⟩	**0.83**

and interval neutrosophic environment are listed in Table 1. Controlling traffic flow has been handled using TIT2SSWA and INSSWA operators. There is a similar procedure for the geometric case.

In Table 1, junction (Z_1, Z_2)has the higher score value, as determined using both the proposed methods, and therefore the traffic may be cleared in that junction first.

7. Qualitative comparison of traffic control management using crisp sets, fuzzy sets, type-2 fuzzy sets, neutrosophic set and interval neutrosophic sets

In this section, a comparative analysis has been done with advantages and limitations of different types of sets such as crisp, fuzzy, type-2 fuzzy, neutrosophic and interval valued neutrosophic sets in traffic control management. This analysis will be helpful in understanding the role of all types of sets mentioned and will provide the motivation for conducting research on these areas and applying them in real-world problems according to the capacity of the type of sets. From the analysis, it is found that interval-based fuzzy and neutrosophic sets can handle more uncertainties than the single-valued type of sets. This point will give a different perspective to new researchers.

Traffic control management	Advantages	Limitations
Using crisp sets	• Fixed time period for all traffic densities • Achieved to characterize the real situation appropriately	• Cannot act while there is a fluctuation in traffic density • Unable to react immediately to unpredictable changes such as a driver's behavior • Unable to handle rapid momentous changes that disturb the continuity of the traffic
Using fuzzy sets	• Various time durations can be considered according to the traffic density • Follow a rule-based approach that accepts uncertainties • Able to model the reasoning of an experienced human being • Adaptive and intelligent • Able to apply and handle real-life rules identical to human thinking • Admits fuzzy terms and conditions • Has the best security • Makes it simpler to convert knowledge beyond the domain	• Adaptiveness is missing while computing the connectedness of the interval-based input • Cannot be used to show uncertainty as it applies crisp and accurate functions • Cannot handle uncertainties such as stability, flexibility and on-line planning completely as consequences can be uncertain
Using type-2 fuzzy sets	• Rule-based approach that accepts uncertainties completely • Adaptiveness (Fixed type-1 fuzzy sets are used to calculate the bounds of the type-reduced interval change as input changes) • Novelty (the upper and lower membership functions may be used concurrently in calculating every bound of the type-reduced interval)	• Computational complexity is high as the membership functions are themselves fuzzy
Neutrosophic set	• Deals not only with uncertainty but also indeterminacy owing to unpredictable environmental disturbances	• Unable to round up and down errors of calculations
Interval neutrosophic set	• Deals with more uncertainties and indeterminacy • Flexible and adaptable • Able to address issues with a set of numbers in the real unit interval, not just a particular number • Able to round up and down errors of calculations	• Unable to deal with criterion incomplete weight information

8. Conclusion

Controlling and clearing traffic is an essential daily traffic management task. In this paper, operational laws, and aggregation operators have been proposed under triangular interval type-2 fuzzy and interval neutrosophic environments. The validity of the proposed concepts has been verified using a numerical example. Furthermore, a novel traffic flow control method using the proposed operators is proposed. An improved score function is also proposed. Using TIT2SSWA and INSSWA operators, the traffic flow is analyzed with the score values using the score functions and the same can be derived using TIT2SSWG and INSSWG operators. The junction identified as having more traffic is the same for both the methods applied.

Notes

Compliance with ethical standards

Conflicts of interest

The authors declare that they have no conflicts of interest.

Ethical approval

The article does not involve any studies with human participants or animals. All activities have been performed by one or more of the authors.

Informed consent

Informed consent was obtained from all individual participants included in the study.

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Type-2 Fuzzy Controller for Stability of a System

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Abstract. Type-2 Fuzzy sets (T2FSs) handle a greater modeling and uncertainties which exist in the real world applications especially control systems. To avoid mathematical complexity, interval T2FSs (IT2FSs) have been pertained in majority of the fields. One of the important components which influence the fuzzy controller is the t-norm. For obtaining the stability of a control system, t-norm operator can be preferred for better results and in this paper the minimum and maximum operations have been used to simplify the work of the system with Gaussian interval type-2 membership function (GIT2MF). Also proposed Gaussian interval type-2 weighted geometric (GIT2WG) operator and mathematical properties of aggregation operator have been proved using the proposed operator. The goal of this work is to analyze the stability of an inverted pendulum using interval type-2 fuzzy logic controller (IT2FLC) and the results are compared with Proportional Integrated Derivative (PID) Controller. It is observed that IT2FL controller gives the better stability.

Keywords: Aggregation properties · Control systems · Gaussian membership function · T Norm · Type-2 fuzzy sets · PID controller · Interval type-2 fuzzy logic controller · Inverted pendulum

1 Introduction

Fuzzy Control System (FCS) is reflects the human maturity for using linguistic rules with vague implication in order to develop control behavior. Triangular norms play an important role in control systems. Fuzzy logic controller (FLC) consists of linguistic IF-THEN rules. In the comparison of type-1 (T1) and type-2 (T2) categories of fuzzy logic, T1FL system has the difficulties in imitate and decrease the effect of uncertainties. Whereas in the case of T2 Fuzzy logic system, at least one T2FS must be taken and should be characterized by fuzzy membership grades.

An important part of T2FS is that Foot Print of Uncertainty (FOU), which constitutes the uncertainties in nature and posture of T1FS. Also it is an extra mathematical aspect provided by the focusable area and this area contains T1 membership function

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(T1MF) and has crisp membership grades. Simply FOU is the area between upper and lower MFs and which tells about the level of uncertainty of the information. Also it has the possibility of outperforming their T1 counterparts. Type reducer will convert the type-2 fuzzy outputs into crisp outputs. Moreover in IT2FSs, every element of FOU has a secondary membership grade as unity.

Under fuzzy based Control design, membership functions and rule base are the essential things and usually it is difficult to determine. In this work, the antecedent part of the rule base has been designed using IT2FS, whereas for consequent part T1FS has been applied. Type reduction process is differentiate T2 from T1, since for each fired rules the outputs are T2FS and this should be done prior to the defuzzifier is manipulate to provoke a crisp output. One of the type reducer is center of sets. This will incorporate all the type-2 fuzzy outputs and produce type-1 fuzzy set, which is the type reduced set.

FL controller is usually designed by T1FS which is known as T1FL controller and it has been applied in many of the fields, specifically in controlling complex non-linear systems where the researchers faced difficulties in designing and handling uncertainties. The disadvantage is failing to catch all the feature of a certain plant. Generally, controller which handles more uncertainties is preferable. It has been noted that interval type-2 fuzzy logic controllers have been applied for controlling the stability of mobile robot quality, sound speakers and admission in ATM networks [13]. The most applicable membership functions in control system are triangular and trapezoidal are the most applied membership function in control system. Due to their poor approximation, Gaussian membership function (GMF) is chosen as it gives actual representation at each point. Since MF virtually expresses the fuzziness, its characterization is the main aspect of the fuzzy operation [8].

In fuzzy inference theory, MFs, triangular norm operators, defuzzification methods and types of input to the controller are the main components. Though there are many MFs to represent a linguistic value, researchers found that GMF is the ideal one. There is a possibility of getting negative effects while getting the information from sensors corrupted by noise and this is the reason why fuzzy inputs are introduced with the computational complexity. For an efficient system, an influence of the operators must be taken care before implementation to get an enforced achievement level. Therefore, the selection of T Norm, defuzzifier and GMF has the greatest influence of the fuzzy controller [10].

Classical control designs are based on point to point whereas FL controller is either range to point or range to range i.e. FL controller is a function from an input data vector to a scalar output [1, 6]. Using MFs, outputs of the FL controller by fuzzifying both input and outputs are obtained. T2FLSs distinguished by fuzzy IF-THEN rules and T2 fuzzy values are used for antecedent and consequent parts as the parameters. In GIT2FSs, uncertainties may be incorporated with the mean or standard deviation and therefore consider either GIT2FS with uncertain mean or uncertain standard deviation. In this work, GIT2MF with uncertain mean and standard deviation is considered. In the case of T2FS, the antecedent and consequent parts are T2 or any one of the two. Usually consequent part is taken as T1 [2, 3, 7, 11, 12].

In this work the performance of PID controller and IT2FL controller is compared. PID controller has been used in many applications but it gives poor performance as it has poor knowledge of input and output parameters and therefore tuning is very difficult in this case. Neuro fuzzy controller works better than PID controllers as it performs better with reduced fluctuations and settling time faster. The performance of the controllers can be improved by increasing the number of combinations of input and output. From these observations it is observed that FL controller, which handles imprecision and uncertainty, is a good replacement to PID controller.

The lower and upper MFs can be captured effectively in IT2FL which is the collection of T1 fuzzy models. The FOU processes the stability analysis of the system [4, 5] and [9]. The operators may be chosen according to the characteristic properties and then the operations for minimum and maximum can be applied [13, 14]. Norms play as the synthesize operators for which these maximum and minimum operators are just an exclusive choice [15–17]. In many of the real world problems it is necessary to have a MV itself fuzzy instead of crisp value which is called T2FS [18, 19]. The parameter η in Yager triangular norms, accepts for tuning the norm between the other norms [20–22]. Image processing based on T2 Fuzzy system has been considered [23, 24]. Every pixel has some number of bits that determines available number of various gray levels [25–30]. Interval type-2 fuzzy set has been applied in image processing (edge detection and feature extraction), traffic control management to handle more uncertainties exists in the image and in the data successfully [31–36].

In this paper, the rest of the part is organized as follows. In Sect. 2, basic concepts have been given for better understanding of the paper. In Sect. 3, operational laws have been derived using Gaussian interval type-2 fuzzy numbers. In Sect. 4, the mathematical properties have been proved using proposed operator namely Gaussian interval type-2 weighted geometric operator. In Sect. 5, basic concepts of control system is given. In Sect. 6, stability analysis has been done for inverted pendulum using interval type-2 fuzzy logic controller and the result is compared with PID controller. In Sect. 7, conclusion of the work is given with future direction.

2 Basic Concepts

2.1 Gaussian Membership Function

Gaussian membership function for a Fuzzy set is defined by

$\psi_{\overline{D}}(x) = \exp\left[-\frac{1}{2}\left(\frac{x-m}{\sigma}\right)\right],\ -\infty < x < \infty$, where m is the mean and σ is the standard deviation.

2.2 GMF with Type-2 Fuzzy Set

Here two different cases are considered for GITMF, according to the nature of the parameters namely mean (m) and standard deviation (σ) namely GIT2MF with fixed mean and uncertain standard deviation (FM&USD) and fixed standard deviation and uncertain mean (FSD&UM) as follows (Figs. 1 and 2).

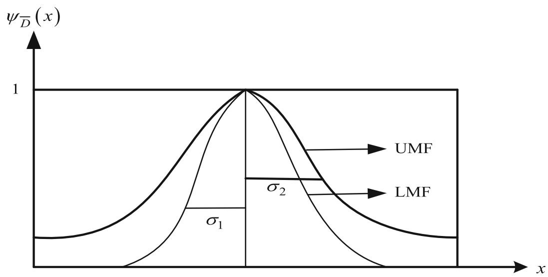

Fig. 1. GIT2MF with FM&USD

$$\text{and is defined by,} \quad \psi_{\overline{D}}(x) = \exp\left[-\frac{1}{2}\left(\frac{x-m}{\sigma}\right)^2\right], \ \sigma \in [\sigma_1, \sigma_2] \tag{1}$$

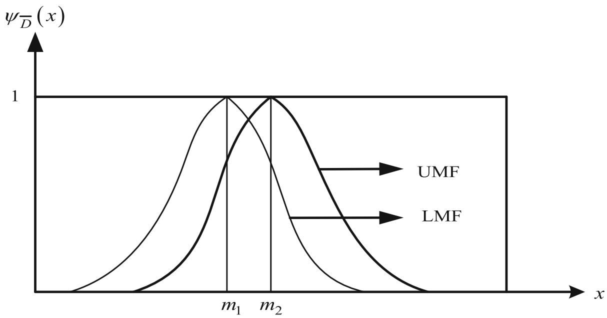

Fig. 2. GIT2MF with FSD&UM

$$\text{and is defined by } \psi_{\overline{D}}(x) = \exp\left[-\frac{1}{2}\left(\frac{x-m}{\sigma}\right)^2\right], \ m \in [m_1, m_2] \tag{2}$$

Where LMF is the lower membership function and UMF is the upper membership function in the pictorial representations.

2.3 Triangular Norms Used

Consider Dubois Prade (DP) triangular norms as defined below.

$$\text{DP T Norm:} \ T(x,y) = \frac{xy}{\max(x,y,\upsilon)} \tag{3}$$

$$\text{DP T Conorm:} \ TC(x,y) = 1 - \frac{(1-x)(1-y)}{\max[(1-x),(1-y),(1-\upsilon)]} \tag{4}$$

In this paper T Norm is used as it is preferable for control systems with min and max operations and T conorm will be used in the stage of defuzzification in the control system with uncertain parameters.

3 Operational Laws

Let $\overline{D}_1, \overline{D}_2, \ldots, \overline{D}_n,\ n = 1, 2, 3, \ldots, n$ be three Gaussian interval type-2 fuzzy numbers and the parameter $\vartheta \in [0, 1]$, then the following operations are hold.

3.1 Addition Operation

$$\overline{D_1} \oplus \overline{D_2} = 1 - \left[\frac{\left(1 - \exp\left[-\frac{1}{2}\left(\frac{x_1 - m_1}{\sigma_1}\right)^2\right]\right)\left(1 - \exp\left[-\frac{1}{2}\left(\frac{x_2 - m_2}{\sigma_2}\right)^2\right]\right)}{\max\left(\left(1 - \exp\left[-\frac{1}{2}\left(\frac{x_1 - m_1}{\sigma_1}\right)^2\right]\right),\ \left(1 - \exp\left[-\frac{1}{2}\left(\frac{x_1 - m_1}{\sigma_1}\right)^2\right]\right),\ (1 - \vartheta)\right)} \right] \tag{5}$$

3.2 Multiplication Operation

$$\overline{D_1} \otimes \overline{D_2} = \frac{\left(\exp\left[-\frac{1}{2}\left(\frac{x_1 - m_1}{\sigma_1}\right)^2\right]\right)\left(\exp\left[-\frac{1}{2}\left(\frac{x_2 - m_2}{\sigma_2}\right)^2\right]\right)}{\max\left(\left(\exp\left[-\frac{1}{2}\left(\frac{x_1 - m_1}{\sigma_1}\right)^2\right]\right),\ \left(\exp\left[-\frac{1}{2}\left(\frac{x_1 - m_1}{\sigma_1}\right)^2\right]\right),\ \vartheta\right)} \tag{6}$$

3.3 Multiplication by an Ordinary Number and Power

$$p.\overline{D} = 1 - \left[\frac{\left(1 - \exp\left[-\frac{1}{2}\left(\frac{x - m}{\sigma}\right)^2\right]^p\right)}{\max\left(\left(1 - \exp\left[-\frac{1}{2}\left(\frac{x - m}{\sigma}\right)^2\right]^p\right),\ (1 - \vartheta)\right)} \right] \tag{7}$$

$$\text{and} \quad \overline{D}^p = \frac{\left(\exp\left[-\frac{1}{2}\left(\frac{x - m}{\sigma}\right)^2\right]^p\right)}{\max\left(\left(\exp\left[-\frac{1}{2}\left(\frac{x - m}{\sigma}\right)^2\right]^p\right),\ \vartheta\right)} \tag{8}$$

4 Proposed Theorems

The below theorems are constituting the mathematical properties of aggregation operator (AO) namely triangular norms and shows the role of their properties in control system. Here the theorems of first, Idempotency, associativity and stability represents the facts that a control system can have any number of inputs (finite), unanimity of the system, the system can extend the process without ambiguity and the strength of the system respectively.

4.1 Theorem

Let $\overline{D}_i = \left(\exp\left[-\frac{1}{2}\left(\frac{x_i - m_i}{\sigma_i}\right)^2\right]\right)$, $i = 1, 2, 3, \ldots,$ n be a collection of GIT2FNs then their aggregated value by

GIT2WG operator is still a GIT2FN and

$$GIT2WG_{\varpi}\left(\overline{D}_1, \overline{D}_2, \ldots, \overline{D}_n\right) = \frac{MOT\left(\exp\left[-\frac{1}{2}\left(\frac{x_i - m_i}{\sigma_i}\right)^2\right]\right)^{\varsigma_i}}{\max\left(\left(\exp\left[-\frac{1}{2}\left(\frac{x_1 - m_1}{\sigma_1}\right)^2\right]\right)^{\varsigma_1}, \left(\exp\left[-\frac{1}{2}\left(\frac{x_2 - m_2}{\sigma_2}\right)^2\right]\right)^{\varsigma_2}, \ldots, \left(\exp\left[-\frac{1}{2}\left(\frac{x_n - m_n}{\sigma_n}\right)^2\right]\right)^{\varsigma_n}, \vartheta\right)} \quad (9)$$

Proof:
By the method of mathematical induction. For $n = 2$, using Power operation

$$\overline{D}^{\varsigma_1} = \frac{\left(\exp\left[-\frac{1}{2}\left(\frac{x-m}{\sigma}\right)^2\right]\right)^{\varsigma_1}}{\max\left(\left(\exp\left[-\frac{1}{2}\left(\frac{x-m}{\sigma}\right)^2\right]\right)^{\varsigma_1}, \vartheta\right)} \quad \text{and} \quad \overline{D}^{\varsigma_2} = \frac{\left(\exp\left[-\frac{1}{2}\left(\frac{x-m}{\sigma}\right)^2\right]\right)^{\varsigma_2}}{\max\left(\left(\exp\left[-\frac{1}{2}\left(\frac{x-m}{\sigma}\right)^2\right]\right)^{\varsigma_2}, \vartheta\right)}$$

Here $i = 1, 2$, MOT = Multiplication Of Terms

$$GIT2WG_{\varpi}\left(\overline{D}_1, \overline{D}_2\right) = \frac{MOT\left(\exp\left[-\frac{1}{2}\left(\frac{x_i - m_i}{\sigma_i}\right)^2\right]\right)^{\varsigma_i}}{\max\left(\left(\exp\left[-\frac{1}{2}\left(\frac{x_1 - m_1}{\sigma_1}\right)^2\right]\right)^{\varsigma_1}, \left(\exp\left[-\frac{1}{2}\left(\frac{x_2 - m_2}{\sigma_2}\right)^2\right]\right)^{\varsigma_2}, \vartheta\right)}$$

$$= \frac{MOT\left(\exp\left[-\frac{1}{2}\left(\frac{x_i - m_i}{\sigma_i}\right)^{2\varsigma_i}\right]\right)}{\max\left(\left(\exp\left[-\frac{1}{2}\left(\frac{x_1 - m_1}{\sigma_1}\right)^{2\varsigma_1}\right]\right), \left(\exp\left[-\frac{1}{2}\left(\frac{x_2 - m_2}{\sigma_2}\right)^{2\varsigma_2}\right]\right), \vartheta\right)}$$

For $n = k$,

$$GIT2WG_{\varpi}\left(\overline{D}_1, \overline{D}_2, \ldots, \overline{D}_k\right) = \frac{MOT\left(\exp\left[-\frac{1}{2}\left(\frac{x_i - m_i}{\sigma_i}\right)^2\right]\right)^{\varsigma_i}}{\max\left(\left(\exp\left[-\frac{1}{2}\left(\frac{x_1 - m_1}{\sigma_1}\right)^2\right]\right)^{\varsigma_1}, \left(\exp\left[-\frac{1}{2}\left(\frac{x_2 - m_2}{\sigma_2}\right)^2\right]\right)^{\varsigma_2}, \ldots, \left(\exp\left[-\frac{1}{2}\left(\frac{x_k - m_k}{\sigma_k}\right)^2\right]\right)^{\varsigma_k}, \vartheta\right)}$$

For $n = k + 1$, $GIT2WG_{\varpi}\left(\overline{D}_1, \overline{D}_2, \ldots, \overline{D}_k, \overline{D}_{k+1}\right)$

$$= \frac{MOT\left(\exp\left[-\frac{1}{2}\left(\frac{x_i-m_i}{\sigma_i}\right)^2\right]\right)^{\varsigma_i}}{\max\left(\left(\exp\left[-\frac{1}{2}\left(\frac{x_1-m_1}{\sigma_1}\right)^2\right]\right)^{\varsigma_1}, \left(\exp\left[-\frac{1}{2}\left(\frac{x_2-m_2}{\sigma_2}\right)^2\right]\right)^{\varsigma_2}, \ldots, \left(\exp\left[-\frac{1}{2}\left(\frac{x_k-m_k}{\sigma_k}\right)^2\right]\right)^{\varsigma_k}, \vartheta\right)}$$
$$\otimes \frac{\left(\exp\left[-\frac{1}{2}\left(\frac{x_{k+1}-m_{k+1}}{\sigma_{k+1}}\right)^2\right]\right)^{\varsigma_{k+1}}}{\max\left(\left(\exp\left[-\frac{1}{2}\left(\frac{x_{k+1}-m_{k+1}}{\sigma_{k+1}}\right)^2\right]\right)^{\varsigma_{k+1}}, \vartheta\right)}$$
$$= \frac{MOT\left(\exp\left[-\frac{1}{2}\left(\frac{x_i-m_i}{\sigma_i}\right)^2\right]\right)^{\varsigma_i}}{\max\left(\left(\exp\left[-\frac{1}{2}\left(\frac{x_1-m_1}{\sigma_1}\right)^2\right]\right)^{\varsigma_1}, \left(\exp\left[-\frac{1}{2}\left(\frac{x_2-m_2}{\sigma_2}\right)^2\right]\right)^{\varsigma_2}, \ldots, \left(\exp\left[-\frac{1}{2}\left(\frac{x_{k+1}-m_{k+1}}{\sigma_{k+1}}\right)^2\right]\right)^{\varsigma_{k+1}}, \vartheta\right)}$$

Hence, the result is true for all the values of n.

4.2 Theorem (Idempotency)

Let $\overline{D}_i = \left(\exp\left[-\frac{1}{2}\left(\frac{x_i-m_i}{\sigma_i}\right)^2\right]\right),\ i = 1, 2, 3, \ldots,$ n be a collection of GIT2FNs. If be a collection of GIT2FNs. If $\forall \overline{D}_i,\ i = 1, 2, 3, \ldots, n$ are equal i.e., $\overline{D}_i = \overline{D}$ then

$$GIT2WG_{\varpi}\left(\overline{D}_1, \overline{D}_2, \ldots, \overline{D}_n\right) = \overline{D} \tag{10}$$

Proof:
Using Theorem 4.1, $GIT2WG_{\varsigma}\left(\overline{D}_1, \overline{D}_2, \ldots, \overline{D}_n\right)$

$$= \frac{MOT\left(\exp\left[-\frac{1}{2}\left(\frac{x_i-m_i}{\sigma_i}\right)^2\right]\right)^{\varsigma_i}}{\max\left(\left(\exp\left[-\frac{1}{2}\left(\frac{x_1-m_1}{\sigma_1}\right)^2\right]\right)^{\varsigma_1}, \left(\exp\left[-\frac{1}{2}\left(\frac{x_2-m_2}{\sigma_2}\right)^2\right]\right)^{\varsigma_2}, \ldots, \left(\exp\left[-\frac{1}{2}\left(\frac{x_n-m_n}{\sigma_n}\right)^2\right]\right)^{\varsigma_n}, \vartheta\right)}$$
$$= \frac{\left(\exp\left[-\frac{1}{2}\left(\frac{x_i-m_i}{\sigma_i}\right)^2\right]\right)^{\sum_{i=1}^{n}\varsigma_i}}{\max\left(\left(\exp\left[-\frac{1}{2}\left(\frac{x_1-m_1}{\sigma_1}\right)^2\right]\right)^{\varsigma_1}, \left(\exp\left[-\frac{1}{2}\left(\frac{x_2-m_2}{\sigma_2}\right)^2\right]\right)^{\varsigma_2}, \ldots, \left(\exp\left[-\frac{1}{2}\left(\frac{x_n-m_n}{\sigma_n}\right)^2\right]\right)^{\varsigma_n}, \vartheta\right)}$$
$$= \frac{\left(\exp\left[-\frac{1}{2}\left(\frac{x_i-m_i}{\sigma_i}\right)^2\right]\right)}{\max\left(\left(\exp\left[-\frac{1}{2}\left(\frac{x_1-m_1}{\sigma_1}\right)^2\right]\right), \left(\exp\left[-\frac{1}{2}\left(\frac{x_2-m_2}{\sigma_2}\right)^2\right]\right), \ldots, \left(\exp\left[-\frac{1}{2}\left(\frac{x_n-m_n}{\sigma_n}\right)^2\right]\right), \vartheta\right)}$$
$$= \overline{D}$$

Hence, the theorem holds.

4.3 Theorem (Associativity)

If $\overline{D}_1, \overline{D}_2$ and $\overline{D}_3$ are the three GIT2FNs then the following result

$$\left(\overline{D}_1 \otimes \overline{D}_2 \otimes \overline{D}_3\right) = \left(\overline{D}_1 \otimes \overline{D}_2\right) \otimes \overline{D}_3 \text{ is hold.} \tag{11}$$

Proof:
Using associativity property we have,

$$\left(\overline{D}_1 \otimes \overline{D}_2 \otimes \overline{D}_3\right) = \left(\overline{D}_1 \otimes \overline{D}_2\right) \otimes \overline{D}_3.$$

Consider, $\left(\overline{D}_1 \otimes \overline{D}_2\right) \otimes \overline{D}_3$

$$= \left[\left(\exp\left[-\frac{1}{2}\left(\frac{x_1 - m_1}{\sigma_1}\right)^2\right]\right) \otimes \left(\exp\left[-\frac{1}{2}\left(\frac{x_2 - m_2}{\sigma_2}\right)^2\right]\right)\right] \otimes \left(\exp\left[-\frac{1}{2}\left(\frac{x_3 - m_3}{\sigma_3}\right)^2\right]\right)$$

$$= \frac{\left[\left(\exp\left[-\frac{1}{2}\left(\frac{x_1-m_1}{\sigma_1}\right)^2\right]\right) \otimes \left(\exp\left[-\frac{1}{2}\left(\frac{x_2-m_2}{\sigma_2}\right)^2\right]\right)\right] \cdot \left(\exp\left[-\frac{1}{2}\left(\frac{x_3-m_3}{\sigma_3}\right)^2\right]\right)}{\max\left[\left(\exp\left[-\frac{1}{2}\left(\frac{x_1-m_1}{\sigma_1}\right)^2\right]\right) \otimes \left(\exp\left[-\frac{1}{2}\left(\frac{x_2-m_2}{\sigma_2}\right)^2\right]\right),\ \left(\exp\left[-\frac{1}{2}\left(\frac{x_3-m_3}{\sigma_3}\right)^2\right]\right),\ \upsilon\right]}$$

$$= \frac{\dfrac{\left(\exp\left[-\frac{1}{2}\left(\frac{x_1-m_1}{\sigma_1}\right)^2\right]\right) \cdot \left(\exp\left[-\frac{1}{2}\left(\frac{x_2-m_2}{\sigma_2}\right)^2\right]\right)}{\max\left[\left(\exp\left[-\frac{1}{2}\left(\frac{x_1-m_1}{\sigma_1}\right)^2\right]\right), \left(\exp\left[-\frac{1}{2}\left(\frac{x_2-m_2}{\sigma_2}\right)^2\right]\right), \upsilon\right]} \cdot \left(\exp\left[-\frac{1}{2}\left(\frac{x_3-m_3}{\sigma_3}\right)^2\right]\right)}{\max\left[\dfrac{\left(\exp\left[-\frac{1}{2}\left(\frac{x_1-m_1}{\sigma_1}\right)^2\right]\right) \cdot \left(\exp\left[-\frac{1}{2}\left(\frac{x_2-m_2}{\sigma_2}\right)^2\right]\right)}{\max\left[\left(\exp\left[-\frac{1}{2}\left(\frac{x_1-m_1}{\sigma_1}\right)^2\right]\right), \left(\exp\left[-\frac{1}{2}\left(\frac{x_2-m_2}{\sigma_2}\right)^2\right]\right), \upsilon\right]}, \left(\exp\left[-\frac{1}{2}\left(\frac{x_3-m_3}{\sigma_3}\right)^2\right]\right),\ \upsilon\right]}$$

$$= \frac{\overline{D}_1.\overline{D}_2.\overline{D}_3}{\max\left[\overline{D}_1, \overline{D}_2, \upsilon\right] \max\left[\frac{\overline{D}_1.\overline{D}_2}{\max\left[\overline{D}_1, \overline{D}_2, \upsilon\right]}, \overline{D}_3, \upsilon\right]}$$

$$= \frac{\left(\exp\left[-\frac{1}{2}\left(\frac{x_1-m_1}{\sigma_1}\right)^2\right]\right) \cdot \left(\exp\left[-\frac{1}{2}\left(\frac{x_2-m_2}{\sigma_2}\right)^2\right]\right) \cdot \left(\exp\left[-\frac{1}{2}\left(\frac{x_3-m_3}{\sigma_3}\right)^2\right]\right)}{\max\left[\left(\exp\left[-\frac{1}{2}\left(\frac{x_1-m_1}{\sigma_1}\right)^2\right]\right),\ \left(\exp\left[-\frac{1}{2}\left(\frac{x_2-m_2}{\sigma_2}\right)^2\right]\right),\ \left(\exp\left[-\frac{1}{2}\left(\frac{x_3-m_3}{\sigma_3}\right)^2\right]\right),\ \upsilon\right]}$$

$$= \frac{\overline{D}_1.\overline{D}_2.\overline{D}_3}{\max\left[\overline{D}_1, \overline{D}_2, \overline{D}_3, \upsilon\right]} = \left(\overline{D}_1 \otimes \overline{D}_2 \otimes \overline{D}_3\right)$$

Hence, this result also holds for all the values of n.

4.4 Theorem (Stability)

Let $\overline{D}_i = \left(\exp\left[-\frac{1}{2}\left(\frac{x_i - m_i}{\sigma_i}\right)^2\right]\right),\ i = 1, 2, 3, \ldots,$ n be a collection of GIT2FNs. If $p > 0$ and $\overline{D}_{n+1} = \left(\exp\left[-\frac{1}{2}\left(\frac{x_{n+1} - m_{n+1}}{\sigma_{n+1}}\right)^2\right]\right)$ is a GIT2FN on the set X then,

$$\begin{aligned} &GIT2WG_{\varpi}\left(\overline{D}_1^p \otimes \overline{D}_{n+1}, \overline{D}_2^p \otimes \overline{D}_{n+1}, \ldots, \overline{D}_n^p \otimes \overline{D}_{n+1}\right) \\ &= \left[GIT2WG_{\varpi}\left(\overline{D}_1, \overline{D}_2, \ldots, \overline{D}_n\right)\right]^p \otimes \overline{D}_{n+1} \end{aligned} \tag{12}$$

Proof:
Using Power operation of GIT2FN,
We know that,

$$\begin{aligned} &GIT2WG_{\varpi}\left(\overline{D}_1^p \otimes \overline{D}_{n+1}, \overline{D}_2^p \otimes \overline{D}_{n+1}, \ldots, \overline{D}_n^p \otimes \overline{D}_{n+1}\right) \\ &= GIT2WG_{\varpi}\left(\overline{D}_1, \overline{D}_2, \ldots, \overline{D}_n\right) \otimes \overline{D}_{n+1} \end{aligned} \tag{13}$$

$$\begin{aligned} &\overline{D}_i \otimes \overline{D}_{n+1} = \frac{\underset{j=\{i,n+1\}}{MOT}\left[\left(\exp\left[-\frac{1}{2}\left(\frac{x_j - m_j}{\sigma_j}\right)^2\right]\right)\right]}{\max\left[\left(\exp\left[-\frac{1}{2}\left(\frac{x_j - m_j}{\sigma_j}\right)^2\right]\right),\ \upsilon\right]} \\ &GIT2WG_{\varpi}\left(\overline{D}_1^p \otimes \overline{D}_{n+1}, \overline{D}_2^p \otimes \overline{D}_{n+1}, \ldots, \overline{D}_n^p \otimes \overline{D}_{n+1}\right) \\ &= \frac{\overset{n}{\underset{j=1}{MOT}}\left[\exp\left[-\frac{1}{2}\left(\frac{x_j - m_j}{\sigma_j}\right)^2\right]\right]\left(\exp\left[-\frac{1}{2}\left(\frac{x_{n+1} - m_{n+1}}{\sigma_{n+1}}\right)^2\right]\right)}{\max\left[\exp\left[-\frac{1}{2}\left(\frac{x_j - m_j}{\sigma_j}\right)^2\right],\ \exp\left[-\frac{1}{2}\left(\frac{x_{n+1} - m_{n+1}}{\sigma_{n+1}}\right)^2\right],\ \upsilon\right]} \end{aligned} \tag{14}$$

$$\begin{aligned} &GIT2WG_{\varpi}\left(\overline{D}_1, \overline{D}_2, \ldots, \overline{D}_n\right) \otimes \overline{D}_{n+1} \\ &= \frac{\overset{n}{\underset{j=1}{MOT}}\left[\exp\left[-\frac{1}{2}\left(\frac{x_j - m_j}{\sigma_j}\right)^2\right]^{\varpi_j}\right]}{\max\left[\exp\left[-\frac{1}{2}\left(\frac{x_1 - m_1}{\sigma_1}\right)^2\right],\ \exp\left[-\frac{1}{2}\left(\frac{x_2 - m_2}{\sigma_2}\right)^2\right], \ldots, \exp\left[-\frac{1}{2}\left(\frac{x_n - m_n}{\sigma_n}\right)^2\right],\ \upsilon\right]} \otimes \exp\left[-\frac{1}{2}\left(\frac{x_{n+1} - m_{n+1}}{\sigma_{n+1}}\right)^2\right] \\ &= \frac{\overset{n}{\underset{j=1}{MOT}}\left[\exp\left[-\frac{1}{2}\left(\frac{x_j - m_j}{\sigma_j}\right)^2\right]^{\varpi_j}\right] . \exp\left[-\frac{1}{2}\left(\frac{x_{n+1} - m_{n+1}}{\sigma_{n+1}}\right)^2\right]}{\max\left[\overset{n}{\underset{j=1}{MOT}}\left[\exp\left[-\frac{1}{2}\left(\frac{x_j - m_j}{\sigma_j}\right)^2\right]^{\varpi_j}\right],\ \exp\left[-\frac{1}{2}\left(\frac{x_{n+1} - m_{n+1}}{\sigma_{n+1}}\right)^2\right],\ \upsilon\right]} \end{aligned} \tag{15}$$

From (14) and (15), $GIT2WG_{\varpi}\left(\overline{D}_1^p \otimes \overline{D}_{n+1}, \overline{D}_2^p \otimes \overline{D}_{n+1}, \ldots, \overline{D}_n^p \otimes \overline{D}_{n+1}\right) = GIT2WG_{\varpi}\left(\overline{D}_1, \overline{D}_2, \ldots, \overline{D}_n\right) \otimes \overline{D}_{n+1}$.

Also we have, $GIT2WG_{\varpi}\left(\overline{D}_1^p, \overline{D}_2^p, \ldots, \overline{D}_n^p\right) = \left(GIT2WG_{\varpi}\left(\overline{D}_1, \overline{D}_2, \ldots, \overline{D}_n\right)\right)^p$ (16)

$$GIT2WG_{\varpi}\left(\overline{D}_1^p, \overline{D}_2^p, \ldots, \overline{D}_n^p\right)$$
$$= \frac{\underset{j=1}{\overset{n}{MOT}}\left[\exp\left(\left[-\frac{1}{2}\left(\frac{x_j - m_j}{\sigma_j}\right)^2\right]^p\right)^{\varpi_j}\right]}{\max\left[\exp\left(\left[-\frac{1}{2}\left(\frac{x_1 - m_1}{\sigma_1}\right)^2\right]^p\right)^{\varpi_1}, \exp\left(\left[-\frac{1}{2}\left(\frac{x_2 - m_2}{\sigma_2}\right)^2\right]^p\right)^{\varpi_2}, \ldots \exp\left(\left[-\frac{1}{2}\left(\frac{x_n - m_n}{\sigma_n}\right)^2\right]^p\right)^{\varpi_n}, \upsilon\right]}$$
$$= \frac{\underset{j=1}{\overset{n}{MOT}}\left[\exp\left(\left[-\frac{1}{2}\left(\frac{x_j - m_j}{\sigma_j}\right)^2\right]^{p\varpi_j}\right)\right]}{\max\left[\exp\left(\left[-\frac{1}{2}\left(\frac{x_1 - m_1}{\sigma_1}\right)^2\right]^{p\varpi_1}\right), \exp\left(\left[-\frac{1}{2}\left(\frac{x_2 - m_2}{\sigma_2}\right)^2\right]^{p\varpi_2}\right), \ldots \exp\left(\left[-\frac{1}{2}\left(\frac{x_n - m_n}{\sigma_n}\right)^2\right]^{p\varpi_n}\right), \upsilon\right]} \tag{17}$$

Also since, $\left(GIT2WG_{\varpi}\left(\overline{D}_1, \overline{D}_2, \ldots, \overline{D}_n\right)\right)^p$

$$= \frac{\underset{j=1}{\overset{n}{MOT}}\left[\exp\left(\left[-\frac{1}{2}\left(\frac{x_j - m_j}{\sigma_j}\right)^2\right]^{\varpi_j}\right)\right]^p}{\max\left[\exp\left(\left[-\frac{1}{2}\left(\frac{x_1 - m_1}{\sigma_1}\right)^2\right]^{\varpi_1}\right)^p, \exp\left(\left[-\frac{1}{2}\left(\frac{x_2 - m_2}{\sigma_2}\right)^2\right]^{\varpi_2}\right)^p, \ldots, \exp\left(\left[-\frac{1}{2}\left(\frac{x_n - m_n}{\sigma_n}\right)^2\right]^{\varpi_n}\right)^p, \upsilon\right]}$$
$$= \frac{\underset{j=1}{\overset{n}{MOT}}\left[\exp\left(\left[-\frac{1}{2}\left(\frac{x_j - m_j}{\sigma_j}\right)^2\right]^{p\varpi_j}\right)\right]}{\max\left[\exp\left(\left[-\frac{1}{2}\left(\frac{x_1 - m_1}{\sigma_1}\right)^2\right]^{p\varpi_1}\right), \exp\left(\left[-\frac{1}{2}\left(\frac{x_2 - m_2}{\sigma_2}\right)^2\right]^{p\varpi_2}\right), \ldots \exp\left(\left[-\frac{1}{2}\left(\frac{x_n - m_n}{\sigma_n}\right)^2\right]^{p\varpi_n}\right), \upsilon\right]} \tag{18}$$

From (17) and (18), we get $GIT2WG_{\varpi}\left(\overline{D}_1^p \otimes \overline{D}_{n+1}, \overline{D}_2^p \otimes \overline{D}_{n+1}, \ldots, \overline{D}_n^p \otimes \overline{D}_{n+1}\right) = \left[GIT2WG_{\varpi}\left(\overline{D}_1, \overline{D}_2, \ldots, \overline{D}_n\right)\right]^p \otimes \overline{D}_{n+1}$.
Hence the theorem.

5 Basics of Control System

5.1 Components of Fuzzy Inference System (FIS)

Rule Base, database and reasoning Mechanism are the components of FIS used for selecting fuzzy rules, to define membership function and to derive sensible conclusion based on the rule of fuzzy reasoning respectively.

5.2 Components of Fuzzy Logic System

Rule base, fuzzy inference engine (FIE), fuzzifier and defuzzifier are the four components used to choose Fuzzy rule which shows the human thinking, judgment and perception, to combine rules for developing a scaling from crisp inputs to type-2 fuzzy outputs, Gaussian fuzzifier to simplify the computation in the FIE when the MFs in the IF-THEN rules are Gaussian and a mapping from fuzzy set to crisp point and calculates the crisp Output respectively.

5.3 Role of T-Norm in Control System

The role of triangular norms plays a key role in fuzzy control system, especially in getting an output. The T Norms are expresses differently and come out with different properties as proved by the theorems.

6 Application

The pendulum moves vertically, the force F is the control input of the cart which moves horizontally and the angular position of the pendulum θ and the horizontal position of the cart x are the outputs. Also N is the reaction force [5] (Fig. 3).

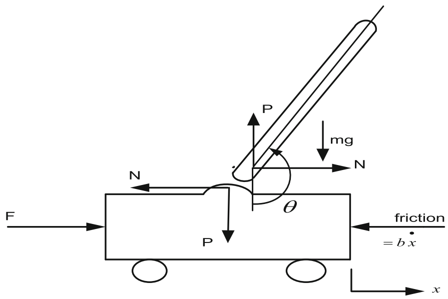

Fig. 3. Inverted Pendulum

The motion in the cart is defined by

$$M\ddot{x} + b\dot{x} + N = F \tag{19}$$

The motion in the pendulum is

$$(M+m)\ddot{x} + b\dot{x} + ml\,\ddot{\theta}\cos\theta - ml\,\dot{\theta}^2\sin\theta = F \tag{20}$$

$$\left(I + ml^2\right)\ddot{\theta} + mgl\sin\theta = -ml\,\ddot{x}\cos\theta \tag{21}$$

The system is to be linearized.
The two linearized motion of the equations are

$$\left(I + ml^2\right)\ddot{\phi} - mgl\ \phi = ml\ddot{x} \tag{22}$$

$$(M+m)\ddot{x} + b\dot{x} - ml\,\ddot{\phi} = u \tag{23}$$

The transfer function of the linearized system is

$$\frac{\Phi(s)}{U(s)} = \frac{\frac{ml}{q}s^2}{s^4 + \frac{b(I+ml^2)}{q}s^3 - \frac{(M+m)mgl}{q}s^2 - \frac{bmgl}{q}s} \tag{24}$$

$$\text{where, } q = \left[(M+m)(I+ml^2) - (ml)^2\right] \tag{24.1}$$

And the state space equation of the system is

$$\begin{bmatrix} \dot{x} \\ \ddot{x} \\ \dot{\phi} \\ \ddot{\phi} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & \frac{-(I+ml^2)}{I(M+m)+Mml^2} & \frac{m^2gl^2}{I(M+m)+Mml^2} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & \frac{-mlb}{I(M+m)+Mml^2} & \frac{mgl(M+m)}{I(M+m)+Mml^2} & 0 \end{bmatrix} \begin{bmatrix} x \\ \dot{x} \\ \phi \\ \dot{\phi} \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{(I+ml^2)}{I(M+m)+Mml^2} \\ 0 \\ \frac{ml}{I(M+m)+Mml^2} \end{bmatrix} u \tag{25}$$

$$y = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ \dot{x} \\ \phi \\ \dot{\phi} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix} u \tag{26}$$

Here the nonlinear plant is an Inverted Pendulum (IP) subject to parameter uncertainty without considering the cart movement for demonstration process. The proposed fuzzy controller is engaged for stabilizing the IP with IT2FLC. The dynamical equation of an IP is defined as follows.

$$\ddot{\theta}(t) = \frac{g\ \sin(\theta(t)) - a\, m_p L\dot{\theta}(t)^2 \sin(2\theta(t))/2 - a\cos(\theta(t))u(t)}{4L/3 - a\, m_p L\cos^2(\theta(t))} \tag{27}$$

where $\theta(t)$ is the angular displacement of the pendulum, g is the acceleration due to gravity, m_p is the mass of the pendulum, $m_p \in \left[m_{p_{\min}}\ m_{p_{\max}}\right]$, $a = \frac{1}{(m_p + M_c)}$, $M_p \in [M_{c_{\min}}\ M_{c_{\max}}]$, M_c is the mass of the cart, $2L = 1m$ is the length of the pendulum, $u(t)$ is the force applied to the cart and m_p, M_c are regarded as the parameter uncertainties.

6.1 Interval Type-2 Fuzzy Logic Controller (IT2FLC)

To fix the position of the input MFs and uniformly distributed between −1 and +1. Limit these inputs to a minimum and maximum values using two saturation blocks Saturation 1 and saturation consecutively. The fuzzy controller is tuned by scaling gains. Control the spread of the input MFs by the input gains ‘Gain 1’ and ‘Gain 2’. To rescale the axes we can change gains. The MFs are uniformly spread out and contracted for the gains, which is less than 1 and greater than 1 respectively. The spread of the output MFs controlled by the output gain ‘Gain’ and the changes in it will lead to scale the vertical axis of the controller surface. If we increase the gains ‘Gain 1’ and ‘Gain 2’ then the proportional gain and the derivative gain in a PD controller will be increased respectively. If the proportional gain is increased then the system respond will be faster (Fig. 4).

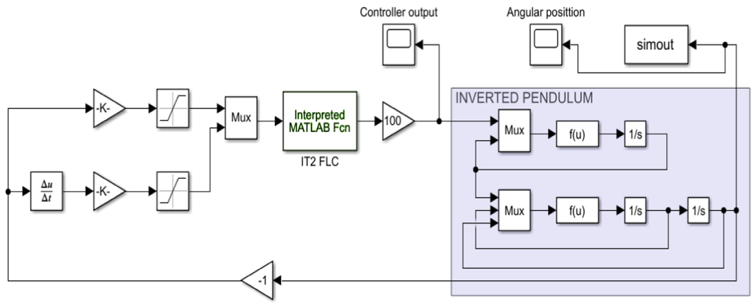

Fig. 4. IT2FLC

6.2 Controller Output

It shows the optimized control output of IT2FLC system (Fig. 5).

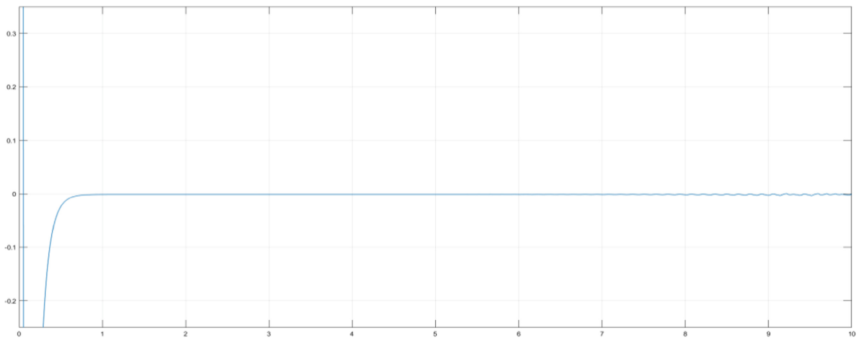

Fig. 5. Chart for controller output

6.3 Angular Position

It shows its varying between the angular positions from 0 to 1 (Fig. 6).

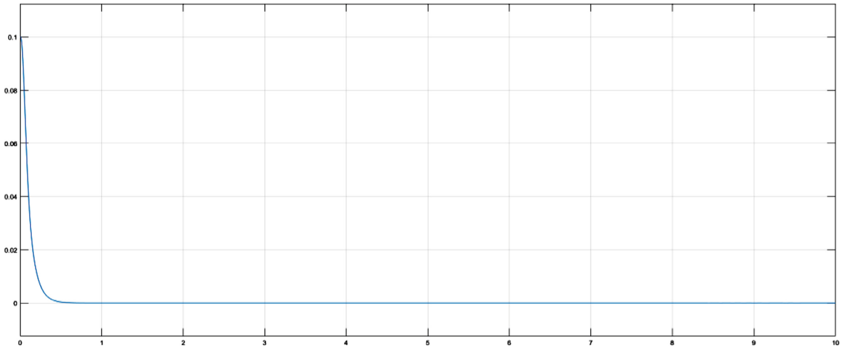

Fig. 6. Chart for angular position

6.4 PID Control System

The Transfer function is $P + \frac{I}{S} + D\left(\frac{N}{1+\frac{N}{S}}\right)$ (Fig. 7).

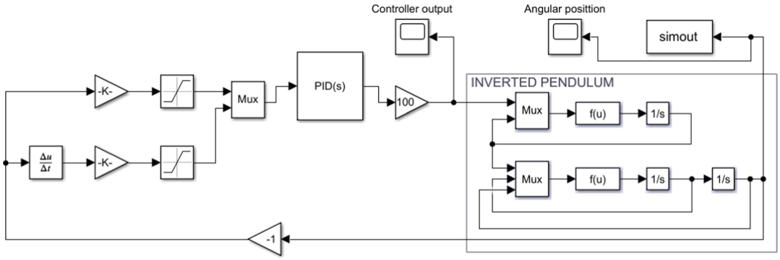

Fig. 7. PID control system

6.5 Controller Output

It shows the poor response (Fig. 8).

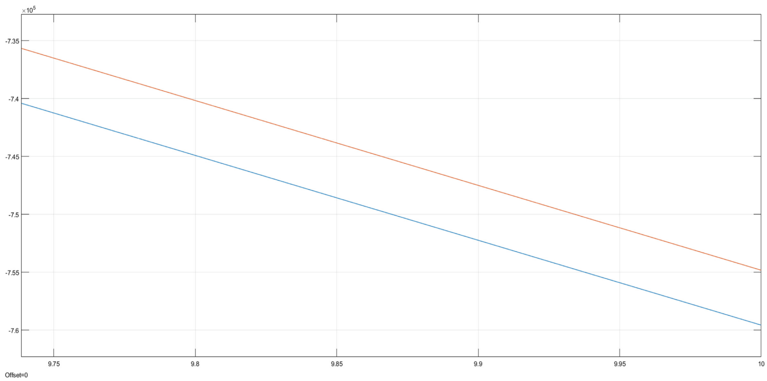

Fig. 8. Chart for controller output

P = 286.7, I = 733.234, D = 10081, Filter N = 269.93.
The control output is not stationary and it's getting decaying.

6.6 Angular Position

See Fig. 9.

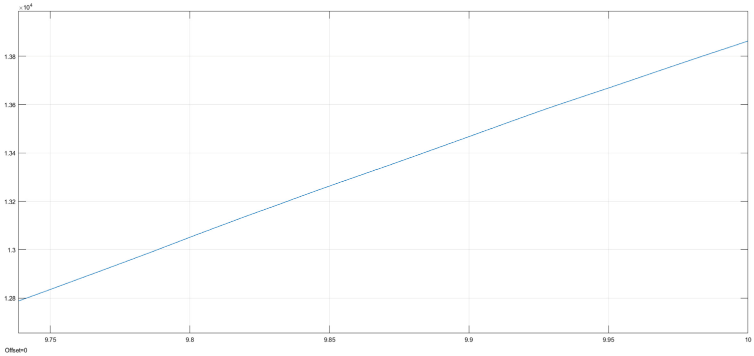

Fig. 9. Chart for angular position

6.7 Comparison of Type-2 Fuzzy Controller and PID Controller

In the field of control system, PID controller has been widely used but it has poor capability of handling the relation between input and output parameters, whereas Type-2 fuzzy logic controller has a very good capability of analyzing stability of the system with uncertain parameters. From this present work, it is proved that IT2FLC gives better stability than PID controller.

7 Conclusion

The results reveal that Gaussian membership function gives the exact result with smoothness and T2FSs with interval MF, handles more uncertainties and less mathematical complexity than Type-1. Therefore GIT2MF is used and the mathematical properties of aggregation operator using IT2GWG operator are proved. These properties play an important role in control system for the characteristics like continuity, robustness and stability. In this research, IT2FLC is used to check the stability for an inverted pendulum and compared the result with PID controller which proved IT2FLC is better than PID controller for this system. In future work, stability analysis can be done using neutrosophic controller and the comparative study can be made with the PID controller and fuzzy controller.

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ORIGINAL ARTICLE

The shortest path problem in interval valued trapezoidal and triangular neutrosophic environment

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Abstract
Real-life decision-making problem has been demonstrated to cover the indeterminacy through single valued neutrosophic set. It is the extension of interval valued neutrosophic set. Most of the problems of real life involve some sort of uncertainty in it among which, one of the famous problem is finding a shortest path of the network. In this paper, a new score function is proposed for interval valued neutrosophic numbers and SPP is solved using interval valued neutrosophic numbers. Additionally, novel algorithms are proposed to find the neutrosophic shortest path by considering interval valued neutrosophic number, trapezoidal and triangular interval valued neutrosophic numbers for the length of the path in a network with illustrative example. Further, comparative analysis has been done for the proposed algorithm with the existing method with the shortcoming and advantage of the proposed method and it shows the effectiveness of the proposed algorithm.

Keywords Interval valued triangular neutrosophic number · Interval valued trapezoidal neutrosophic number · Ranking methods · Deneutrosophication · Neutrosophic shortest path problem · Network

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Introduction and literature of review

In this part, introduction to the objective of the paper is given by presenting basic concepts and procedure of the shortest path problem (SPP) and the literature of review have been collected to know the recent work related to the presented concept which shows the novelty of the presented work

Introduction

SPP is the ultimate and popular problem in the different areas also it is the heart of the network flows. In conventional problem, the distance between the nodes is considered to be certain and for the uncertain environment fuzzy numbers can be adopted to get an optimized result. Computing the minimum cost of the path from every vertex is called single source SPP. Especially in the process of finding shortest path, finding the path which has minimum number of bends is very important and will give the most optimized result. And the cost is the mapping of length and bends. The conventional SPP is to catch the minimum cost path from initial to end node and the cost is the addition of the costs of the curves on the path [1, 2, 4].

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While applying in real time situations the vertices and the edges will be considered as follows. In transmission networks, telephone exchange, communication proficiency, satellites, work stations terminals and computers will be considered as the vertices and cables, wires and fiber optics will be treated as the arcs or paths and it is expected to meet transmission requirements at the minimum cost whereas in traffic control management the cost is due to only the paths with heavy traffic [8]. In the established network every path has a weight which will extend the flow in a recurrence fashion. The fusion of costs and weights proposes different ways of cost minimizing cycles. There may be cycles with negative cost which allow raise to perpetual instances and cost of minimum infinity and weight minimizing cycles which permits rise to a sink in such a way that it is inexpensive to consume a flow in an infinite cycle rather than transit to the station.

SPP plays an essential role in combinatorial optimization due to its elemental aspects and a broad range of applications. Investigating shortest paths is an essential thing in communication, computer networks, manufacturing systems and transportation. The weight of the path will represent the transportation timing from one end to other, i.e., the traveling time from the source to the destination. The efficiency of the transmission can be improved by speed up some of the routes to reduce the traveling time between some of the pairs of sources and terminals by minimizing the weights of the links. One needs some amount to reduce the traveling time by improving the road conditions for the faster traveling and the total cost supposed to be less to face the needs of the speedup [9].

In all the SPP, the source and terminal nodes should satisfy a set of conditions defined over a set of resources which associates to a quantity like the time, pickup of load by the vehicle or the duration of the break. The constraint of the resource will be given in the form of intervals which regulate the values that can be considered by the resources at each node on the path. SPP using complete graph can be encrypted as an assignment problem and is equivalent to an exceptional case of the assignment problem. Providing the shortest path is a necessary thing to the system of transport management, from a particular source node to the terminal node. The arc lengths are stimulated to represent time or cost of the transportation rather the geographical distances [10, 11].

The technique of using fuzzy numbers can be adopted for the environment with uncertainty. Crisp number is obtained from fuzzy number using defuzzification function and it is widely used in an optimization methods. SPP is not restricted to the geometric distance. Even though it is fixed, the traveling time within the cities may be represented by fuzzy variable. Since the weight of the arcs is uncertain in almost all the communication and transportation networks, it cannot be designed into crisp graphs. Dubois and Prade solved fuzzy shortest path problem for the first time. The most crucial combinatorial optimization problem is to find the SP to the directed graph and its primary format unable to represent the situations where the value of the detached function should be found not only by the preference of each single arc [15–19].

Shortest path of the network can be found using neutrosophic set (NS) by considering edge weight as neutrosophic numbers (NNs) and that may be single and interval valued, and bipolar as well [21, 22]. Samarandache described about neutrosophic for the first time in the year 1995 and proposed an important mathematical mechanism called neutrosophic set theory to handle imprecise, uncertain and indeterminate problems which cannot be dealt by fuzzy and its various type. NS is obtained by three autonomous mapping such as truth (T), indeterminacy (I) and falsity (F) and takes values from $]0^{-}, 1^{+}[$. It is very difficult to utilize NS directly.

While getting uncertainty in the set of vertices and edge then fuzzy graph can be adopted for SPP, but if there is indeterminacy exist between the relation of nodes and vertices then neutrosophic will be the appropriate concept to deal the real life problems [23]. Since indeterminacy is also treated seriously, NSs can be able to handle uncertainty in a better way [35]. The model of the NS is an important mechanism to deal with real scientific and engineering as it is able to deal uncertain, inconsistent and also indeterminate information [36]. Route maintenance or supply with uncertainty is playing a primary role in intelligent transport systems.

Due to inadequate data, as the stochastic shortest path needs accurate probability distributions, it is unable give the optimized result. Due to accuracy, adoptability and rapport to a system, single valued neutrosophic graph (SVNG) gets more attention and produce optimized solution than other types of fuzzy sets. Application of probabilities in machine learning is done by the score function. These functions play an essential role to find the minimum cost path in SPP and minimum spanning tree (MST) to UIVNGs (undirected interval valued neutrosophic graphs). When the data are in the form of intervals then that can dealt effectively by considering interval valued neutrosophic setting [40, 41]. Many group decision making methods including hybrid methods have been proposed to solve decision making problems such as supplier selection, project selection under triangular and trapezoidal neutrosophic environment [55–64].

The rest of the paper is arranged as follows. In Sect. 1.2, literature of review has been collected. In Sect. 2, over view of interval valued neutrosophic set is given. In Sect. 3, novel algorithms are proposed to find the neutrosophic shortest path under interval valued neutrosophic environment and interval valued triangular and trapezoidal neutrosophic environments with the help of proposed score function. In Sect. 4, shortcoming of the existing methods, advantages of the proposed method and comparative analysis are presented for the proposed method with the existing method. In Sect. 5, conclusion of the presented work is given.

Literature of review

The authors of, Ahuja et al. [1] proposed a different model redistributive heap as a rapid algorithm to find SP of the network. Yang et al. [2] presented a graph-theoretic strategy of rectilinear paths on bends and lengths. Ibarra and Zheng [3] proved that the single-origin shortest path problem for permutation graphs can be determined by order of the logarithmic of n. Arsham [4] examined the robustness of the shortest path problem. Tzoreff [5] examined the disconnected SPP with group path lengths. Batagelj et al. [6] proposed generalized SPP.

Zhang and Lin [7] introduced the calculation of the reverse SPP. Vasantha and Samaranadache [8] proposed primary neutrosophic algebraic framework. Also their utilization to fuzzy and NEUTROSOPHIC models as well. Roditty and Zwick [9] acquired some results associated with effective forms of the SPP. Irnich and Desaulniers [10] proposed SPP with support force. Buckley and Jowers [11] introduced SPP using the concept of fuzzy logic. Wastlund [12] analyzed the relationship between random assignment and SPP problem on the complete graph. Turner [13] attained strongly polynomial algorithms for a collection of SPP on acyclic and normal digraphs. Deng et al. [14] proposed fuzzy Dijkstra algorithm for SPP for imprecise environment.

Biswas et al. [15] introduced an algorithm for deriving shortest path in intuitionistic fuzzy environment. Arnautovic et al. [16] obtained the complement of the ant colony development for the SPP using open MP and CUDA. Gabrel and Murat [17] presented different models, methods and principle for the stability of the SPP. Grigoryan and Harutyunyan [18] proposed SPP in the Knodel graph. Rostami et al. [19] proposed quadratic SPP. Randour et al. [20] presented algorithms to incorporate the approaches with various securities on the length allocation of the paths instead of decreasing its normal value. Broumi et al. [21] solved SPP under neutrosophic setting using Dijkstra algorithm. Broumi et al. [22] introduced SPP based on triangular fuzzy neutrosophic environment.

Broumi et al. [23] proposed assertive types of SVNGs and examination of properties with validation and examples. Nancy and Harish [24] proposed an improved score function and applied in decision making process. Sahin and Liu [25] maximized method of deviation for neutrosophic decision making problem with a support of incomplete weight. Broumi et al. [26] proposed the measurements for SPP using SV-triangular neutrosophic numbers. Broumi et al. [27] calculated MST in interval valued bipolar neutrosophic (IVBN) setting. Hu and Sotirov [28] proposed amenity of semi definite programming for the quadratic SPP and performed some arithmetic operations to solve the QSPP using branch and bound algorithm. Dragan and Leitert [29] solved SPP on minimal peculiarity. Zhang et al. [30] proposed stable SPP with circulated uncertainty.

Broumi et al. [31] solved SPP using SVNG. Broumi et al. [32] solved SSP under bipolar neutrosophic environment. Peng and Dai [33] proposed interval-based algorithms based on neutrosophic environment for decision making process. Liu and You [34] proposed muirhead mean operators and employed them in decision making problem. Smarandache [35] solved SPP using trapezoidal neutrosophic knowledge. Wang et al. [36] applied SV-trapezoidal neutrosophic preference in decision making problem. Deli and Subas [37] proposed a ranking method of SVNNs and applied in decision making problem. Broumi et al. [38] proposed matrix algorithm for MST in undirected IVNG. Enayattabar et al. [39] applied Dijkstra algorithm to find the shortest path under IV Pythagorean fuzzy setting. Broumi et al. [40] proposed IVN soft graphs. Broumi et al. [41] proposed some notion with respect to neutrosophic set with triangular and trapezoidal concept and primary operations as well. Also done a contingent analysis with the existing concepts and proposed neutrosophic numbers.

Broumi et al. [42] proposed an innovative system and technique for the planning of telephone network using NG. Broumi et al. [43] proposed SPP under interval valued neutrosophic setting. Bolturk and Kahraman [44] presented a novel IVN AHP with cosine similarity measure. Wang et al. [45] proposed interval neutrosophic set and logic in detail. Biswas et al. [46] proposed distance measure using interval trapezoidal neutrosophic numbers. Deli [47] given detailed work on expansion and contraction on conventional neutrosophic soft set. Deli [48] solved a decision making problem using interval valued neutrosophic soft numbers.

Deli [49] proposed theory of npn-soft set and its application. Deli [50] proposed single valued trapezoidal neutrosophic operators and applied them in a decision making problem. Deli and Subas [51] proposed weighted geometric operators under single valued triangular neutrosophic numbers and applied in a decision making problem. Deli et al. [52] solved a decision making problem using neutrosophic soft sets. Basset et al. [53] proposed framework of hybrid neutrosophic group AND-TOPSIS for supplier selection. Chang et al. [54] experimented in detail about framework for the pattern of reuse necessary decision from theoretical perspective to practices.

Basset et al. [55] proposed a hybrid method of neutrosophic sets and method of DEMATEL to develop criteria for supplier selection. Basset et al. [56] proposed a structure based on VIKOR technique for e-government website evaluation. Basset et al. [57] Introduced a framework to evaluate cloud computing services. Basset et al. [58] proposed a hybrid method for project selection under neutrosophic environment. Basset et al. [59] proposed a new method for a neutrosophic linear programming problem.

Basset et al. [60] proposed an economic tool for risk quantification for supply chain. Basset et al. [61] proposed a framework for AHP-QFD to solve a supplier selection. Basset et al. [62] proposed neutrosophic AHP-Delphi group decision model under trapezoidal neutrosophic numbers. Basset et al. [63] solved a group decision making problem using neutrosophic analytic hierarchy process. Basset et al. [64] proposed a group decision making problem using triangular neutrosophic numbers. Kumar et al. [65] proposed an algorithm to solve neutrosophic shortest path problem under triangular and trapezoidal neutrosophic environment.

From this literature review, to the best of our knowledge, there is no contribution of research for SPP using interval neutrosophic numbers under triangular and trapezoidal environments. Additionally, this is the first study that SPP is solved by considering interval valued triangular and trapezoidal neutrosophic numbers for the length of the arc for a given network.

Overview on interval valued neutrosophic set

Here, a brief description of some basic concepts on NSs, SVNSs, IVNSs and some existing ranking functions for IVNNs are given.

Definition 2.1 [35] NS is constructed by $N = \{< x; T_N(x), I_N(x), F_N(x) >, x \in X\}$, where X be an universal set of elements x and $T_N(x), I_N(x), F_N(x) : X \to]^-0, 1^+[$ are the truth, indeterminacy and also falsity membership functions and satisfies the criterion,

$$^-0 \leq T_N(x) + I_N(x) + F_N(x) \leq 3^+. \tag{1}$$

Definition 2.2 [36] SVNS is defined by $\dot{N} = \{< x; T_{\dot{N}}(x), I_{\dot{N}}(x), F_{\dot{N}}(x) >, x \in X\}$ and for every

$$x \in X, \quad T_{\dot{N}}(x), I_{\dot{N}}(x), F_{\dot{N}}(x) \in [0, 1], \tag{2}$$

and the sum of these three is less than or equal to 3.

Definition 2.3 [45] An interval valued NS is defined by $\dot{N} = \left\{< x : \left[T^L_{\dot{N}}(x), T^U_{\dot{N}}(x)\right], \left[I^L_{\dot{N}}(x), I^U_{\dot{N}}(x)\right], \left[F^L_{\dot{N}}(x), F^U_{\dot{N}}(x)\right] >, x \in X\right\}$, where $T_{\dot{N}}(x) = \left[T^L_{\dot{N}}(x), T^U_{\dot{N}}(x)\right] \subseteq [0, 1]$,

$$I_{\dot{N}}(x) = \left[I^L_{\dot{N}}(x), I^U_{\dot{N}}(x)\right] \subseteq [0, 1], \quad F_{\dot{N}}(x) = \left[F^L_{\dot{N}}(x), F^U_{\dot{N}}(x)\right] \subseteq [0, 1] \text{ and} \tag{3}$$

$$0 \leq \sup T_{\dot{N}}(x) + \sup I_{\dot{N}}(x) + \sup F_{\dot{N}}(x) \leq 3. \tag{4}$$

Now we assume some mathematical operations on IVNNs (interval valued neutrosophic numbers).

Definition 2.4 [45] Let $\dot{N}_1 = \left\{< x : \left[T^L_{\dot{N}_1}, T^U_{\dot{N}_1}\right], \left[I^L_{\dot{N}_1}, I^U_{\dot{N}_1}\right], \left[F^L_{\dot{N}_1}, F^U_{\dot{N}_1}\right] >, x \in X\right\}$ and $\dot{N}_2 = \left\{< x : \left[T^L_{\dot{N}_2}, T^U_{\dot{N}_2}\right], \left[I^L_{\dot{N}_2}, I^U_{\dot{N}_2}\right], \left[F^L_{\dot{N}_2}, F^U_{\dot{N}_2}\right] >, x \in X\right\}$ be two IVNNs and $\delta > 0$ then we have the following operational laws.

$$\dot{N}_1 \oplus \dot{N}_2 = \left\langle \left[T^L_{\dot{N}_1} + T^L_{\dot{N}_2} - T^L_{\dot{N}_1} T^L_{\dot{N}_2}, T^U_{\dot{N}_1} + T^U_{\dot{N}_2} - T^U_{\dot{N}_1} T^U_{\dot{N}_2}\right], \left[I^L_{\dot{N}_1} I^L_{\dot{N}_2}, I^U_{\dot{N}_1} I^U_{\dot{N}_2}\right], \left[F^L_{\dot{N}_1} F^L_{\dot{N}_2}, F^U_{\dot{N}_1} F^U_{\dot{N}_2}\right] \right\rangle \tag{5}$$

$$\dot{N}_1 \otimes \dot{N}_2 = \left\langle \left[T^L_{\dot{N}_1} T^L_{\dot{N}_2}, T^U_{\dot{N}_1} T^U_{\dot{N}_2}\right], \left[I^L_{\dot{N}_1} + I^L_{\dot{N}_2} - I^L_{\dot{N}_1} I^L_{\dot{N}_2}, I^U_{\dot{N}_1} + I^U_{\dot{N}_2} - I^U_{\dot{N}_1} I^U_{\dot{N}_2}\right], \left[F^L_{\dot{N}_1} + F^L_{\dot{N}_2} - F^L_{\dot{N}_1} F^L_{\dot{N}_2}, F^U_{\dot{N}_1} + F^U_{\dot{N}_2} - F^U_{\dot{N}_1} F^U_{\dot{N}_2}\right] \right\rangle \tag{6}$$

$$\delta \dot{N} = \left\langle \left[1 - \left(1 - T^L_N\right)^\delta, 1 - \left(1 - T^U_N\right)^\delta\right], \left[\left(T^L_N\right)^\delta, \left(T^U_N\right)^\delta\right], \left[\left(F^L_N\right)^\delta, \left(F^U_N\right)^\delta\right] \right\rangle \tag{7}$$

$$\dot{N}^\delta = \left\langle \left[\left(T^L_N\right)^\delta, \left(T^U_N\right)^\delta\right], \left[1 - \left(1 - I^L_N\right)^\delta, 1 - \left(1 - I^U_N\right)^\delta\right], \left[1 - \left(1 - F^L_N\right)^\delta, 1 - \left(1 - F^U_N\right)^\delta\right] \right\rangle. \tag{8}$$

Deneutrosophication formulas for IVNNs: To compare two IVNNs $\dot{N}_1$ and $\dot{N}_2$. We use the score function (SF) which represents a map from $[N\,(R)]$ into the real line. In the literature there are some deneutrosophication formulas, here paper, we focus on some of types [24, 25, 33, 34, 44] defined as follows:

$$S_{\text{Bolturk}}\left(\dot{N}_1\right) = \left(\frac{\left(T^L_x + T^U_x\right)}{2} + \left(1 - \frac{\left(I^L_x + I^U_x\right)}{2}\right) * \left(I^U_x\right) - \left(\frac{\left(F^L_x + F^U_x\right)}{2}\right) * \left(1 - F^U_x\right)\right) \tag{9}$$

$$S_{\text{Ridvan}}\left(\dot{N}_1\right) = \left(\frac{1}{4}\right) \times \left(2 + T^L_x + T^U_x - 2I^L_x - 2I^U_x - F^L_x - F^U_x\right) \tag{10}$$

$$S_{\text{Peng}}\left(\dot{N}_1\right) = \left[\frac{2}{3} + \frac{\left(T_x^L + T_x^U\right)}{6} - \frac{\left(I_x^L + I_x^U\right)}{6} - \frac{\left(F_x^L + F_x^U\right)}{6}\right] \tag{11}$$

$$S_{\text{Liu}}\left(\dot{N}_1\right) = \left[2 + \frac{\left(T_x^L + T_x^U\right)}{2} - \frac{\left(I_x^L + I_x^U\right)}{2} - \frac{\left(F_x^L + F_x^U\right)}{2}\right] \tag{12}$$

$$S_{\text{Harish}}\left(\dot{N}_1\right) = \left(\frac{1}{8}\right) \times \left[4 + \left(T_x^L + T_x^U - F_x^L - F_x^U - 2I_x^L - 2I_x^U\right)\left(4 - T_x^L - T_x^U - F_x^L - F_x^U\right)\right]. \tag{13}$$

The ranking of $\dot{N}_1$ and $\dot{N}_2$ by SF is defined as follows:

(i) $\dot{N}_1 \prec \dot{N}_2$ if $\mathbb{S}\left(\dot{N}_1\right) \prec \mathbb{S}\left(\dot{N}_2\right)$

(ii) $\dot{N}_1 \succ \dot{N}_2$ if $\mathbb{S}\left(\dot{N}_1\right) \succ \mathbb{S}\left(\dot{N}_2\right)$

(iii) $\dot{N}_1 = \dot{N}_2$ if $\mathbb{S}\left(\dot{N}_1\right) = \mathbb{S}\left(\dot{N}_2\right)$

Definition 2.5 [36] Let $R_N = \left\langle \left[R_T, R_I, R_M, R_E\right], \left(T_R, I_R, F_R\right) \right\rangle$ and $S_N = \left\langle \left[S_T, S_I, S_M, S_E\right], \left(T_S, I_S, F_S\right) \right\rangle$ be two trapezoidal neutrosophic numbers (TpNNs) and $\theta \geq 0$, then

$$R_N \oplus S_N = \left\langle \left[R_T + S_T, R_I + S_I, R_M + S_M, R_E + S_E\right], \left(T_R + T_S - T_R T_S, I_R I_S, F_R F_S\right) \right\rangle \tag{14}$$

$$R_N \otimes S_N = \left\langle \left[R_T \cdot S_T, R_I \cdot S_I, R_M \cdot S_M, R_E \cdot S_E\right], \left(T_R \cdot T_S, I_R + I_S - I_R \cdot I_S, F_R + F_S - F_R \cdot F_S\right) \right\rangle \tag{15}$$

$$\theta R_N = \left\langle \left[\theta R_T, \theta R_I, \theta R_M, \theta R_E\right], \left(1 - \left(1 - T_R\right)^\theta, \left(I_R\right)^\theta, \left(F_R\right)^\theta\right) \right\rangle. \tag{16}$$

Definition 2.6 [36] Let $R = \left[R_T, R_I, R_M, R_E\right]$ and $R_T \leq R_I \leq R_M \leq R_E$ then the centre of gravity (COG) in R is

$$\text{COG}\,(R) = \begin{cases} R \text{ if } & R_T = R_I = R_M = R_E \\ \frac{1}{3}\left[R_T + R_I + R_M + R_E - \frac{R_E R_M - R_I R_T}{R_E + R_M - R_I - R_T}\right] & \text{otherwise} \end{cases}. \tag{17}$$

Definition 2.7 [36] Let $S_N = \left\langle \left[S_T, S_I, S_M, S_E\right], \left(T_S, I_S, F_S\right) \right\rangle$ be a TpNN then the score, accuracy and certainty functions are as follows

$$\mathbb{S}\left(S_N\right) = \text{COG}(R) \times \frac{\left(2 + T_S - I_S - F_S\right)}{3} \tag{18}$$

$$a\left(S_N\right) = \text{COG}(R) \times \left(T_S - F_S\right) \tag{19}$$

$$C\left(S_N\right) = \text{COG}(R) \times \left(T_S\right). \tag{20}$$

Definition 2.8 [36] Let $R_N = \left\langle \left[R_T, R_I, R_P\right], \left(T_R, I_R, F_R\right) \right\rangle$ be a triangular neutrosophic number then the score and accuracy function are,

$$\mathbb{S}\left(R_N\right) = \frac{1}{12}\left[R_T + 2 \cdot R_T + R_P\right] \times \left[2 + T_R - I_R - F_R\right] \tag{21}$$

$$a\left(R_N\right) = \frac{1}{12}\left[R_T + 2 \cdot R_T + R_P\right] \times \left[2 + T_R - I_R + F_R\right]. \tag{22}$$

Definition 2.9 [46] Let N be a trapezoidal neutrosophic number in the set of real numbers with the truth, indeterminacy and falsity membership functions are defined by

$$T_N(x) = \begin{cases} \frac{(x-a)t_N}{b-a} & , a \leq x < b \\ t_N & , b \leq x \leq c \\ \frac{(d-x)t_N}{d-c} & , c < x \leq d \\ 0 & , \text{otherwise} \end{cases} \tag{23}$$

$$I_N(x) = \begin{cases} \frac{b-x+(x-a)i_N}{b-a} & , a \leq x < b \\ i_N & , b \leq x \leq c \\ \frac{x-c+(d-x)i_N}{d-c} & , c < x \leq d \\ 0 & , \text{otherwise} \end{cases} \tag{24}$$

$$F_N(x) = \begin{cases} \frac{b-x+(x-a)f_N}{b-a} & , a \leq x < b \\ f_N & , b \leq x \leq c \\ \frac{x-c+(d-x)f_N}{d-c} & , c < x \leq d \\ 0 & , \text{otherwise} \end{cases}, \tag{25}$$

where $t_N = [t^L, t^U] \subset [0,1], i_N = [i^L, i^U] \subset [0,1]$ and $f_N = [f^L, f^U] \subset [0,1]$ are interval numbers. Then the number N can be denoted by $\left([a,b,c,d];[t^L,t^U],[i^L,i^U],[f^L,f^U]\right)$ and is called interval valued trapezoidal neutrosophic number.

- If $b = c$ in interval valued trapezoidal neutrosophic number then it becomes interval valued triangular neutrosophic number.

Proposed improved algorithm and score function

To find the length of the arc, the following algorithm and score function are proposed as follows.

Improved algorithm to solve SPP under interval valued neutrosophic number

Step 1: Determine the source node (SN) arc length $l_1 = \langle [1,1],[0,0],[0,0] \rangle$ and classify SN, node 1 by

$$\left[l_1 = \langle [1,1],[0,0],[0,0] \rangle, -\right]$$

Step 2: Find the minimum of the length of n_1 with its acquaintance node using $l_i = \min\left\{ l_i \oplus l_{ij} \right\}, \quad j = 2,3,\ldots,r.$

Step 3: If there is a minimum in the node and equating to the singular measure of i (i.e., $i = k$), then classify that node j as $[l_j, k]$.

Step 4: If the minimum value exists in the node matching to more values from i then it can be concluded that there are more IVN paths between SN (i) and DN (j) and select any value of i.

Step 5: Classify the destination node (DN) (node r) by $[l_r, 1]$. Then the interval valued neutrosophic distance (IVND) among SN l_r.

Step 6: Find the IVNSP between initial and terminal node according to $[l_r, 1]$ and check the label of n_1 and is denoted by $[l_a, d]$. Classify node a and so on. Rerun the process until get n_1.

Step 7: By connecting all the nodes acquired by repeating the process in step 4, IVNSP can be found.

Note: If $\mathbb{S}\left(N_i\right) < \mathbb{S}\left(N_p\right)$ then the interval valued neutrosophic number (IVNN) is the minimum of N_p, where $N_i, i = 1,2,\ldots,r$ is the set of IVNN and $\mathbb{S}$ is the score function.

Proposed score function

The novel SF for finding the minimum cost path under interval valued neutrosophic shortest path (IVNSP) problem is provided as follows

$$\mathbb{S}_{\text{Nagarajan}}\left(\dot{N}_1\right) = \frac{1}{2}\left[\left(T_x^L + T_x^U\right) - \left(I_x^L . I_x^U\right) + \left(I_x^U - 1\right)^2 + \left(F_x^U\right)\right]. \tag{26}$$

Numerical example:

For the edge 1–2: $S_{\text{Nagarajan}}(\ddot{A}_1) = \frac{1}{2}[(0.1+0.2) - (0.2)(0.3) + (0.3-1)^2 + (0.5)\big] = 0.125$

For the edge 1–3: $S_{\text{Nagarajan}}(\ddot{A}_1) = \frac{1}{2}[(0.2+0.4) - (0.3)(0.5) + (0.5-1)^2 + (0.2)\big] = 0.2.$

Similarly for other edges.

Note: Formulas used in the proposed algorithms.

Score function used in the proposed algorithm under IVN environment and COG for TFN are

$$\mathbb{S}(\theta) = \mathrm{COG}(R) \times \frac{1}{2}\left[T^L + T^U - \left(I^L \cdot I^U\right) + \left(I^U - 1\right)^2 + F^U\right] \tag{27}$$

$$\text{COG for TFN is } \frac{1}{3}\left[R_T + 2R_M + R_E - \frac{R_M\left(R_E - R_I\right)}{\left(R_E - R_I\right)}\right]. \tag{28}$$

Computation of shortest path using IVNNs

Illustrate to the basic process of the improved algorithm, one simple example is shown.

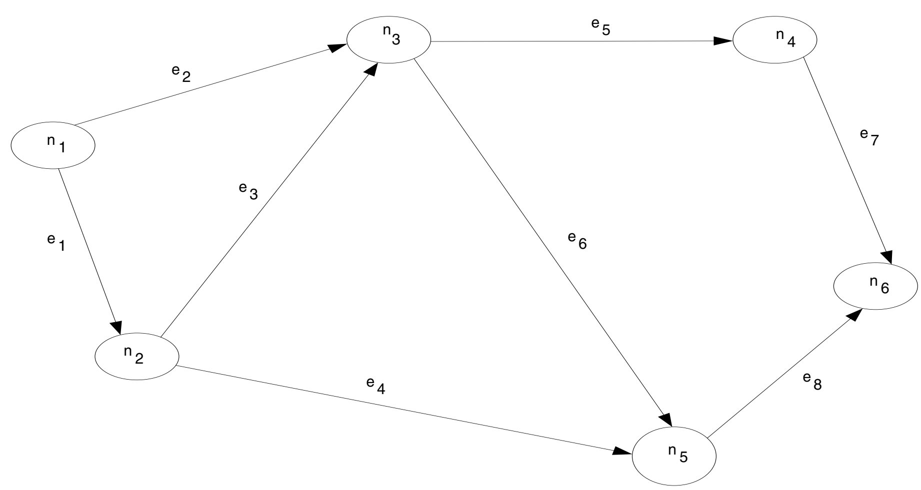

Fig. 1 Interval-valued neutrosophic network

Table 1 The details of edges information in term of IVNNs

Edges	Interval valued neutrosophic distance	Edges	Interval valued neutrosophic distance
1–2 (e_1)	$\langle[0.1, 0.2], [0.2, 0.3], [0.4, 0.5]\rangle$	3–4 (e_5)	$\langle[0.2, 0.3], [0.2, 0.5], [0.4, 0.5]\rangle$
1–3 (e_2)	$\langle[0.2, 0.4], [0.3, 0.5], [0.1, 0.2]\rangle$	3–5 (e_6)	$\langle[0.3, 0.6], [0.1, 0.2], [0.1, 0.4]\rangle$
2–3 (e_3)	$\langle[0.3, 0.4], [0.1, 0.2], [0.3, 0.5]\rangle$	4–6 (e_7)	$\langle[0.4, 0.6], [0.2, 0.4], [0.1, 0.3]\rangle$
2–5 (e_4)	$\langle[0.1, 0.3], [0.3, 0.4], [0.2, 0.3]\rangle$	5–6 (e_8)	$\langle[0.2, 0.3], [0.3, 0.4], [0.1, 0.5]\rangle$

Illustrative example

This section is based on a numerical problem adapted from Broumi et al. [40] to show the potential application of the proposed algorithm and score function.

Consider a network Fig. 1 with six nodes and eight edges with SN, node 1 and DN, node 6. The interval valued neutrosophic distance is given in Table 1.

In this situation, we need to evaluate the shortest distance from SN, i.e., node 1 to DN, i.e., node 6.

Calculating the shortest path using proposed algorithm of interval valued neutrosophic path problem is given as follows.

Here $r = 6$, since there are totally 6 nodes.

Let, $l_1 = \langle[1, 1], [0, 0], [0, 0]\rangle$ and classify the SN $n_1 = \left[\langle[1, 1], [0, 0], [0, 0]\rangle, -\right]$.

To find the value of l_j, $\quad j = 2, 3, 4, 5, 6$.

Iteration no. 1:

Since n_2 has only n_1 as the predecessor, let $i = 1, \quad j = 2$ in step 2.

To find l_2:

$$l_2 = \min\{l_1 \oplus l_{12}\}$$

$$= \min\{\langle[1, 1], [0, 0], [0, 0]\rangle \oplus \langle[0.1, 0.2], [0.2, 0.3], [0.4, 0.5]\rangle\}$$
$$= \langle[0.1, 0.2], [0.2, 0.3], [0.4, 0.5]\rangle.$$

Since, minimum occurs for $i = 1$, classify the node $n_2 = \left[\langle[0.1, 0.2], [0.2, 0.3], [0.4, 0.5]\rangle, 1\right]$.

Iteration no. 2:

Since n_3 has two predecessors n_1 and n_2, let $i = 1, 2 \,\&\, j = 3$ in step 2.

To find l_3:

$$l_3 = \min\{l_1 \oplus l_{13}, l_2 \oplus l_{23}\}$$

$$= \min\{\langle[1, 1], [0, 0], [0, 0]\rangle \oplus \langle[0.2, 0.4], [0.3, 0.5], [0.1, 0.2]\rangle, \langle[0.1, 0.2], [0.2, 0.3], [0.4, 0.5]\rangle \oplus \langle[0.3, 0.4], [0.1, 0.2], [0.3, 0.5]\rangle\}$$

$$= \min\{\langle[1 + 0.2 - 1(0.2), 1 + 0.4 - 1(0.4)], [0(0.3), 0(0.5)], [0(0.1), 0(0.2)]\rangle, \langle[0.1 + 0.3 - (0.1)(0.3), 0.2 + 0.4 - (0.2)(0.4)], [(0.2)(0.1), (0.3)(0.2)], [(0.4)(0.5), (0.5)(0.5)]\rangle\}$$

$$= \min\{\langle[1, 1], [0, 0], [0, 0]\rangle, \langle[0.37, 0.52], [0.02, 0.06], [0.12, 0.25]\rangle\} = \langle[0.37, 0.52], [0.02, 0.06], [0.12, 0.25]\rangle.$$

Since the score function values are,

$$\mathbb{S}(\langle[1, 1], [0, 0], [0, 0]\rangle = \frac{1}{2}\left[(1 + 1) - (0 \times 0) + (0 - 1)^2 + 0\right] = 1.5$$

$$\mathbb{S}(\langle[0.37, 0.52], [0.02, 0.06], [0.12, 0.25]\rangle) = \frac{1}{2}\left[(0.37 + 0.52) - (0.02 \times 0.06) + (0.06 - 1)^2 + 0.25\right] = 0.9$$

and the minimum occurs for $i = 2$, then classify the node $n_3 = \left[\langle[0.37, 0.52], [0.02, 0.06], [0.12, 0.25]\rangle, 2\right]$.

Iteration no. 3:

Since n_4 has one predecessors n_3, let $i = 3 \,\&\, j = 4$ in step 2.

To find the value of l_4:

$$l_4 = \min\{l_3 \oplus l_{34}\}$$
$$= \min\{\langle[0.37, 0.52], [0.02, 0.06], [0.12, 0.25]\rangle \oplus\langle[0.2, 0.3], [0.2, 0.5], [0.4, 0.5]\rangle\}$$
$$= \langle[0.6, 0.67], [0.004, 0.018], [0.048, 0.125]\rangle.$$

Since, minimum occurs for $i = 3$, hence classify the node $n_4 = \left[\langle[0.6, 0.67], [0.004, 0.018], [0.048, 0.125]\rangle, 3\right]$.

Iteration no. 4:

Since n_5 has two predecessors n_2 and n_3, let $i = 2, 3 \& j = 5$ in step 2.

To find the value of l_5:

$$l_5 = \min\{l_2 \oplus l_{25}, l_3 \oplus l_{35}\}$$

$$= \min\{\langle[0.1, 0.2], [0.2, 0.3], [0.4, 0.5]\rangle \oplus \langle[0.1, 0.3], [0.3, 0.4], [0.2, 0.3]\rangle, \langle[0.37, 0.52], [0.02, 0.06], [0.12, 0.25]\rangle \oplus\langle[0.3, 0.6], [0.1, 0.2], [0.1, 0.4]\rangle\}$$

$$= \min\{\langle[0.19, 0.47], [0.06, 0.12], [0.08, 0.15]\rangle, \langle[0.56, 0.81], [0.002, 0.012], [0.012, 0.1]\rangle\}$$
$$= \langle[0.19, 0.47], [0.06, 0.12], [0.08, 0.15]\rangle.$$

Since the score function values are,

$$\mathbb{S}(\langle[0.19, 0.47], [0.06, 0.12], [0.08, 0.15]\rangle) = 0.75$$

$\mathbb{S}(\langle[0.56, 0.81], [0.002, 0.012], [0.012, 0.1]\rangle) = 1$

and the minimum occurs for $i = 2$, hence classify the node $n_5 = \left[\langle[0.19, 0.47], [0.06, 0.12], [0.08, 0.15]\rangle, 2\right]$

Iteration no. 5:

Since n_6 has two predecessors n_4 and n_5, let $i = 4, 5$ & $j = 6$ in step 2.

To find the value of l_6:

$$l_6 = \min\left\{l_4 \oplus l_{46}, l_5 \oplus l_{56}\right\}$$

$$\begin{aligned}
&= \min\{\langle[0.6, 0.67], [0.004, 0.018], [0.048, 0.125]\rangle \\
&\quad \oplus \langle[0.4, 0.6], [0.2, 0.4], [0.1, 0.3]\rangle, \\
&\quad \langle[0.19, 0.47], [0.06, 0.12], [0.08, 0.15]\rangle \\
&\quad \oplus\langle[0.2, 0.3], [0.3, 0.4], [0.1, 0.5]\rangle\}
\end{aligned}$$

$$\begin{aligned}
&= \min\{\langle[0.76, 0.87], [0.008, 0.0018], [0.0048, 0.0375]\rangle, \\
&\quad \langle[0.352, 0.63], [0.018, 0.048], [0.008, 0.075]\rangle\} \\
&= \langle[0.35, 0.63], [0.018, 0.048], [0.008, 0.075]\rangle.
\end{aligned}$$

Since the score function values are,

$$\mathbb{S}(\langle[0.76, 0.87], [0.008, 0.0018], [0.0048, 0.0375]\rangle) = 1$$

$$\mathbb{S}(\langle\langle[0.352, 0.63], [0.018, 0.048], [0.008, 0.075]\rangle\rangle) = 0.82$$

and the minimum occurs for $i = 5$ hence classify $n_6 = \left[\langle[0.35, 0.63], [0.018, 0.048], [0.008, 0.075]\rangle, 5\right]$.

Since n_6 is the DN of the given network, IVNSP between n_1 and n_6 is $\langle[0.35, 0.63], [0.018, 0.048], [0.008, 0.075]\rangle$.

Now, IVNSP from n_1 and n_6 is obtained as follows.

Since, $n_6 = \left[\langle[0.35, 0.63], [0.018, 0.048], [0.008, 0.075]\rangle, 5\right] \Rightarrow$ a person is coming from $5 \to 6 n_5 = \left[\langle[0.19, 0.47], [0.06, 0.12], [0.08, 0.15]\rangle, 2\right] \Rightarrow$ a person is coming from $2 \to 5$ $n_2 = \left[\langle[0.1, 0.2], [0.2, 0.3], [0.4, 0.5]\rangle, 1\right] \Rightarrow$ a person is coming from $1 \to 2$.

By joining all the acquired nodes, interval valued neutrosophic shortest path from n_1 and n_6 is obtained.

Hence IVNSP of the given network is $1 \to 2 \to 5 \to 6$.

The IVNS distance and IVNSP of all the nodes from SN node 1 in the below Table 2 and the classification of all the nodes are shown in Fig. 2.

The following table is formed using different deneutrosophic functions called score functions for all the possible edges and using proposed improved score function in the last column (Table 3).

According to the improved score function proposed in Sect. 3, the shortest path from node one to node six can be computed as follows (Table 4).

Therefore, the path $P : 1 \to 2 \to 5 \to 6.$ is identified as the neutrosophic shortest path.

Algorithm: a new approach to find SPP using TpIVNN and TIVNN

Consider a directed and noncyclic graph, where the length of the arcs is represented by IVNN. The introduced algorithm

Table 2 Interval valued neutrosophic shortest path

Node number (j)	l_i	IVNSP between jth and node 1
2	$\langle[0.1, 0.2], [0.2, 0.3], [0.4, 0.5]\rangle$	$1 \to 2$
3	$\langle[0.37, 0.52], [0.02, 0.06], [0.12, 0.25]\rangle$	$1 \to 2 \to 3$
4	$\langle[0.6, 0.67], [0.004, 0.018], [0.048, 0.125]\rangle$	$1 \to 2 \to 3 \to 4$
5	$\langle[0.19, 0.47], [0.06, 0.12], [0.08, 0.15]\rangle$	$1 \to 2 \to 5$
6	$\langle[0.35, 0.63], [0.018, 0.048], [0.008, 0.075]\rangle$	$1 \to 2 \to 5 \to 6$

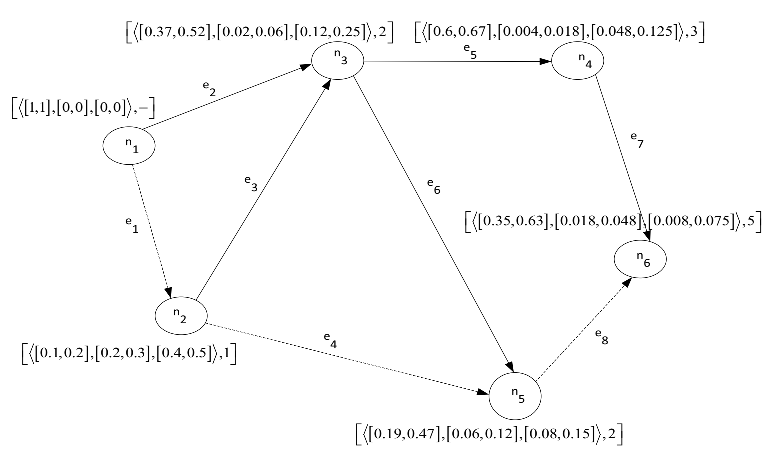

Fig. 2 Interval-valued neutrosophic shortest path

Table 3 Different deneutrosophication value of edge (i, j)

Edges	S_{Ridvan} [43]	$\mathbb{S}_{\text{Nagarajan}}$
1–2	0.1	**0.125**
1–3	0.175	**0.2**
2–3	0.325	**0.17**
2–5	0.125	**0.11**
3–4	0.05	**0.325**
3–5	0.45	**0.32**
4–6	0.35	**0.43**
5–6	0.125	**0.26**

Table 4 Crisp path length for proposed algorithm

The proposed algorithm based $\mathbb{S}_{\text{Nagarajan}}$	Crisp path length	Ranking
$1 \to 2 \to 5 \to 6$	**0.485**	**1**
$\mathbf{1 \to 3 \to 5 \to 6}$	0.78	2
$\mathbf{1 \to 2 \to 3 \to 5 \to 6}$	0.875	3
$\mathbf{1 \to 3 \to 4 \to 6}$	0.955	4
$\mathbf{1 \to 2 \to 3 \to 4 \to 6}$	1.05	5

determines the shortest path from the initial node to the terminal node. The algorithm is described as follows.

Step 1: Let n be the total number of paths from the initial node to terminal one. Find the score function of every arc length for the given network using Eqs. (18), (19) and (24), (25).

Step 2: Find all the available paths P_i, $\quad i = 1, 2, \dots, n$ and the corresponding path length. Also every n paths can be dealt as an arc which are represented by IVNN.

Step 3: Find the sum of all score functions $\mathbb{S}(\theta_i)$ of each available path.

Step 4: The path which have minimum score value will represent an optimized interval valued shortest path by ranking in ascending order.

End

Note: TpIVNN-Trapezoidal interval valued neutrosophic number.

TIVNN-Triangular interval valued neutrosophic number.

Table 6 Available paths and its score value

Available path	$\mathbb{S}(\theta_i)$	Ranking
$P_1 : 1 \to 2 \to 5 \to 6$	4.18	1
$P_2 : 1 \to 3 \to 5 \to 6$	8.25	2
$P_4 : 1 \to 3 \to 4 \to 6$	12.43	3
$P_3 : 1 \to 2 \to 3 \to 5 \to 6$	13.31	4
$P_5 : 1 \to 2 \to 3 \to 4 \to 6$	17.5	5

Illustrative example to find the shortest path using TpIVNN

For the validation of the proposed algorithm, a network is adopted from Broumi et al. [43] and Kumar et al. [65].

Consider a network with six nodes and eight edges. The TpIVN cost is given below (Tables 5, 6).

Applying steps 1–4 of the proposed algorithm, it if found that $1 \to 2 \to 5 \to 6$ is IVNP with lowest cost 4.18 and the IVNP is $\langle(4, 11, 15, 20); [0.35, 0.608], [0.018, 0.048], [0.008, 0.075]\rangle$.

Illustrative example to find the shortest path using TIVNN

For the validation of the proposed algorithm, an example network is adopted from Broumi et al. [26, 35].

Consider a network with six nodes and eight edges. The TIVN cost is given below (Tables 7, 8).

Applying steps 1–4 of the proposed algorithm, it if found that $1 \to 2 \to 5 \to 6$ is IVNP with lowest cost 4.18 and the IVNP is $\langle(4, 11, 15); [0.35, 0.61], [0.02, 0.05], [0.01, 0.08]\rangle$.

Comparative study of the proposed algorithm

In this section, a comparative study is carried out with the shortcomings and advantage of the proposed algorithm and it shows the effectiveness of the proposed algorithm

Shortcoming of the existing method

The compared existing method is unable to handle the interval-based information about the length of the arc and

Table 5 Trapezoidal interval valued neutrosophic distance

Edges	Trapezoidal interval valued neutrosophic distance	Edges	Trapezoidal interval valued neutrosophic distance
1–2 (e_1)	$\langle(1, 2, 3, 4); [0.1, 0.2], [0.2, 0.3], [0.4, 0.5]\rangle$	3–4 (e_5)	$\langle(2, 4, 8, 9); [0.2, 0.3], [0.2, 0.5], [0.4, 0.5]\rangle$
1–3 (e_2)	$\langle(2, 5, 7, 8); [0.2, 0.4], [0.3, 0.5], [0.1, 0.2]\rangle$	3–5 (e_6)	$\langle(3, 4, 5, 10); [0.3, 0.6], [0.1, 0.2], [0.1, 0.4]\rangle$
2–3 (e_3)	$\langle(3, 7, 8, 9); [0.3, 0.4], [0.1, 0.2], [0.3, 0.5]\rangle$	4–6 (e_7)	$\langle(7, 8, 9, 10); [0.4, 0.6], [0.2, 0.4], [0.1, 0.3]\rangle$
2–5 (e_4)	$\langle(1, 5, 7, 9); [0.1, 0.3], [0.3, 0.4], [0.2, 0.3]\rangle$	5–6 (e_8)	$\langle(2, 4, 5, 7); [0.2, 0.3], [0.3, 0.4], [0.1, 0.5]\rangle$

Table 7 Triangular interval valued neutrosophic distance

Edges	Triangular interval valued neutrosophic distance	Edges	Triangular interval valued neutrosophic distance
1–2 (e_1)	$\langle(1, 2, 3); [0.1, 0.2], [0.2, 0.3], [0.4, 0.5]\rangle$	3–4 (e_5)	$\langle(2, 4, 8); [0.2, 0.3], [0.2, 0.5], [0.4, 0.5]\rangle$
1–3 (e_2)	$\langle(2, 5, 7); [0.2, 0.4], [0.3, 0.5], [0.1, 0.2]\rangle$	3–5 (e_6)	$\langle(3, 4, 5); [0.3, 0.6], [0.1, 0.2], [0.1, 0.4]\rangle$
2–3 (e_3)	$\langle(3, 7, 8); [0.3, 0.4], [0.1, 0.2], [0.3, 0.5]\rangle$	4–6 (e_7)	$\langle(7, 8, 9); [0.4, 0.6], [0.2, 0.4], [0.1, 0.3]\rangle$
2–5 (e_4)	$\langle(1, 5, 7); [0.1, 0.3], [0.3, 0.4], [0.2, 0.3]\rangle$	5–6 (e_8)	$\langle(2, 4, 5); [0.2, 0.3], [0.3, 0.4], [0.1, 0.5]\rangle$

Table 8 Available paths and its score value

Available path	$\mathbb{S}(\theta_i)$	Ranking
$P_1 : 1 \to 2 \to 5 \to 6$	4.9	1
$P_2 : 1 \to 3 \to 5 \to 6$	8.27	2
$P_4 : 1 \to 3 \to 4 \to 6$	11.1	3
$P_3 : 1 \to 2 \to 3 \to 5 \to 6$	12.86	4
$P_5 : 1 \to 2 \to 3 \to 4 \to 6$	15.69	5

shortest path cannot be obtained for interval-based neutrosophic network.

Advantage of the proposed algorithm

If the length of the path is interval-based one then the shortest path of the given network can be obtained by interval valued neutrosophic numbers for an optimized path. Since triangular and trapezoidal numbers are widely used in many of the real world applications for their simplicity of computation, interval valued triangular and trapezoidal neutrosophic numbers have been used to find the neutrosophic shortest path. This is the advantage of the proposed algorithm.

Comparative study of algorithm

This section provides a comparative study of the proposed algorithm with the existing method of for neutrosophic shortest path problems.

A comparison of the results between existing and new techniques is shown in Table 9.

The result shows that the proposed algorithm provides sequence of visited nodes which shown to be similar with neutrosophic shortest path.

Table 9 Comparison of sequence of nodes using neutrosophic shortest path and our proposed algorithm

Algorithm of Broumi	Path	Crisp path length
S_{Ridvan} [43]	$1 \to 2 \to 5 \to 6$	0.35
$S_{\text{Nagarajan}}$	$1 \to 2 \to 5 \to 6$	0.485

The neutrosophic shortest path (abbr.NSP) remains the same namely $1 \to 2 \to 5 \to 6$, but the crisp shortest path length (CSPL) differs namely $\langle[0.35, 0.60], [0.01, 0.04], [0.008, 0.075]\rangle$, respectively. From here we come to the conclusion that there exists no unique method for comparing neutrosophic numbers and different methods may satisfy different desirable criteria (Table 10).

Conclusion and future implication

The heart of the network community is nothing but the SPP. The objective of this problem is finding the minimum cost path among all other paths. This issue has been solved using many methods starts from conventional SPP with crisp weights. As many of the real world applications have uncertain vertices and edges fuzzy environment was useful to handle this problem. But still fuzzy setting cannot handle indeterminacy of the information, neutrosophic sets are found to be the best choice to handle this issue and has applied successfully. In this paper, neutrosophic SPP has been solved under interval valued neutrosophic, trapezoidal and triangular interval valued neutrosophic environments as it handles interval values. Also an improved score function and center of gravity has been proposed and applied to find the minimum cost of the path. Our proposed score function is without having the lower membership of falsity and which saves the time naturally. Further comparative analysis is done for Broumi's algorithm with different

Table 10 Sequence of nodes with shortest path length

Possible path	Sequence of nodes	Neutrosophic shortest path length
Neutrosophic shortest path with interval valued neutrosophic numbers [43]	$1 \to 2 \to 5 \to 6$	$\langle[0.35, 0.60], [0.01, 0.04], [0.008, 0.075]\rangle$
Proposed algorithm on $S_{\text{Nagarajan}}$	$1 \to 2 \to 5 \to 6$	$\langle[0.35, 0.60], [0.01, 0.04], [0.008, 0.075]\rangle$

deneutrosophication function and proposed one. It is found that minimum cost is less compare than other existing method using proposed algorithms and score function. Also the proposed algorithm and improved score function have less computational complexity and saves the time. In future, the SPP would be extended to neutrosophic soft and rough set environments for interval-based path lengths. Also the proposed concept will be extended to complex neutrosophic environment.

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ORIGINAL ARTICLE

Shortest path problem using Bellman algorithm under neutrosophic environment

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Abstract

An elongation of the single-valued neutrosophic set is an interval-valued neutrosophic set. It has been demonstrated to deal indeterminacy in a decision-making problem. Real-world problems have some kind of uncertainty in nature and among them; one of the influential problems is solving the shortest path problem (SPP) in interconnections. In this contribution, we consider SPP through Bellman's algorithm for a network using interval-valued neutrosophic numbers (IVNNs). We proposed a novel algorithm to obtain the neutrosophic shortest path between each pair of nodes. Length of all the edges is accredited an IVNN. Moreover, for the validation of the proposed algorithm, a numerical example has been offered. Also, a comparative analysis has been done with the existing methods which exhibit the advantages of the new algorithm.

Keywords Interval-valued neutrosophic numbers · Ranking methods · Shortest path problem · Bellman's algorithm · Directed graph network

Introduction and review of the literature

A tool which represents the partnership or relationship function is called a Fuzzy Set (FS) and handles the real-world problems in which generally some type of uncertainty exists [1]. This concept was generalized by Atanassov [2] to intuitionistic fuzzy set (IFS) which is determined in terms of membership (MS) and non-membership (NMS) functions, the characteristic functions of the set. Beside this, several theories have been developed for uncertainties, including generalized orthopair FSs [3], Pythagorean FSs [4], picture FSs [5], hesitant interval-based neutrosophic linguistic sets [6], N-valued interval neutrosophic sets (NVINSs) [7], generalized interval-valued triangular intuitionistic FSs [8], interval-valued trapezoidal intuitionistic FSs [9],

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interval-valued Pythagorean FSs [10], interval-valued IFSs [11], and interval type 2 FSs [12].

In 1995, Smarandache [13] premises the theme of neutrosophic sets (NS). The NS is to be a set of elements having a membership degree, indeterminate membership and also non-membership with the criterion less than or equal to 3. The neutrosophic number is an exceptional type of neutrosophic sets that extend the domain of numbers from those of real numbers to neutrosophic numbers. By generalizing SVNSs [14], Wang et al. premised the idea of IVNS. The IVNS [15] is a more general database to generalize the concept of different types of sets to express membership degrees' truth, indeterminacy, and a false degree in terms of intervals. Thus, several papers are published in the field of fuzzy and neutrosophic sets [46–62].

Harish [16] proposed and analyzed an extension of the score function by incorporating hesitance. The authors presented an algorithm for the function including qualitative examples. Jun et al. [17] discuss INSs in algebra of BCK/BCI. Mehmet [18] put forward for analyzing the concept of the interval cut set (ICS) and strong ICS (α, β, γ) of IVNSs with proof and examples. Also, there are other several extensions of NSs described in the literature including interval-valued bipolar neutrosophic sets [19], hesitant interval neutrosophic linguistic set [20], and interval neutrosophic hesitant fuzzy sets [21]; for more details of neutrosophic set and their extensions, we refer the reader to [22–28].

Among humanistic problems of computer science, finding the shortest path is one of the significant problems. Many of the algorithms existing for optimization assumed the edge weights as the absolute real numbers. Despite this, we need to deal inexplicit parameters such as scope, costs, time and requirements in real-world problems. For example, a substantial length of any road is permanent; still, traveling time along the road varies according to weather and traffic conditions. An uncertain fact of those cases directs us to adopt fuzzy logic, fuzzy numbers, intuitionistic fuzzy and so on. The SPP using fuzzy numbers is called fuzzy shortest path problem (FSPP). Several researchers are paying attention in fuzzy shortest path (FSP) and intuitionistic FSP algorithms.

Das and De [29] employed Bellman dynamic programming problem for solving FSP based on value and ambiguity of trapezoidal intuitionistic fuzzy numbers. De and Bhincher [30] have studied the FSP in a network under triangular fuzzy number (TFN) and trapezoidal fuzzy number (TpFN) using two approaches such as influential programming of Bellman and linear programming with multi-objective. Kumar et al. [31] proposed a model to find the SP of the network under intuitionistic trapezoidal fuzzy number based on interval value. Meenakshi and Kaliraja [32] formulated interval-valued FSPP for interval-valued type and developed a technique to solve SPP.

Elizabeth and Sujatha [33] solved FSPP using interval-valued fuzzy matrices. Based on traditional Dijkstra algorithm, Enayattabar et al. [34] solved SPP in the interval-valued pythagorean fuzzy setting. Dey et al. [35] formulated fuzzy shortest path problem with interval type 2 fuzzy numbers. But, if the indeterminate information has appeared, all these kinds of shortest path problems failed. For this reason, some new approaches have been developed using neutrosophic numbers. Then neutrosophic shortest path was first developed by Broumi et al. [36]. The authors in [36] constructed an extension of Dijkstra algorithm to solve neutrosophic SPP. Then they used the extended version to treat the NSPP where the edge weight is characterized by IVNNs [37].

Broumi et al. [38–40] first introduced a technique of finding SP under SV-trapezoidal and triangular fuzzy neutrosophic environment. In [41], the authors proposed another approach to solve SPP on a network using trapezoidal neutrosophic numbers. Broumi et al. [42] developed a new algorithm to solve SPP using bipolar neutrosophic setting. In another paper, Broumi et al. [43] discussed an algorithmic approach based on a score function defined in [44] for

Table 1 Authors' contributions towards neutrosophic shortest path problem

Author and references	Year	Contribution
Broumi et al. [36]	2016	Solved NSPP using Dijkstra algorithm
Broumi et al. [37]	2016	Solved NSPP for interval-based data using Dijkstra algorithm
Broumi et al. [38]	2016	Discovered the SP using SV-TpNNs
Broumi et al. [40]	2016	Worked out SPP using single-valued neutrosophic graphs
Broumi et al. [41]	2017	Solved SPP under neutrosophic setting as well as trapezoidal fuzzy
Broumi et al. [42]	2017	Solved SPP under bipolar neutrosophic environment.
Broumi et al. [43]	2017	Dealt SPP under interval-valued neutrosophic setting
Broumi et al. [44]	2018	Proposed maximizing deviation method with partial weight in a decision-making problem under the neutrosophic environment
This paper	–	Introduction of the neutrosophic version of a Bellman's algorithm

IVN interval-valued neutrosophic, *PA* proposed algorithm solving NSPP on a network with IVNN as the edges. Liu and You proposed interval neutrosophic Muirhead mean operators and their applications in multiple-attribute group decision-making [45]. Thus, several papers are published in the field of neutrosophic sets [46–55]. Table 1 summarizes some contributions towards NSPP. Based on the idea of Bellman's algorithm, SPP is solved for fuzzy network [29–32]. This algorithm is not applied yet on neutrosophic network. Therefore, there is a need to establish a neutrosophic version of Bellman's algorithm for neutrosophic shortest path problems.

The main motivation of this study is to introduce an algorithmic approach for SPP in an uncertain environment which will be simple enough and effective in real-life problem. The main contributions of this paper are as follows.

- We concentrate on a NSP on a neutrosophic graph in which an IVNN, instead of a real number/fuzzy number, is assigned to each arc length.
- A modified Bellman's algorithm is introduced to deal the shortest path problem in an uncertain environment.
- Based on the idea discussed in [15], we use an addition operation for adding the IVNNs corresponding to the edge weights present in the path. It is used to find the path length between source and destination nodes. We also use a ranking method to choose the shortest path associated with the lowest value of rank.

In this work, we are motivated to solve SPP by introducing a new version of BA where the edge weight is represented by IVNNs. The remaining part of the paper is presented as follows. The next section contains a few of the ideas and theories as overview of interval neutrosophic set followed by which the Bellman algorithm is discussed. In the subsequent section, an analytical illustration is presented, where our algorithm is applied. Then contingent study has been done with existing methods. Before the concluding section, advantages of the proposed algorithm are presented. Finally, conclusive observations are given.

Overview on interval-valued neutrosophic set

In this part, we recall few primary notions pertaining to NSs, SVNSs, IVNSs and some existing ranking functions for IVNNs which are the background of this study and will help us to further research.

Definition 1 [13] Let X be a set of elements and its universal elements denoted by x; we define the neutrosophic set A (NS A) by $A = \{< x: T_A(x), I_A(x), F_A(x) >, x \in X\}$, where the functions $T, I, F: X \to]^{-}0,1^{+}[$ are called the truth, indeterminate and false MS functions, respectively, and they satisfy the following condition:

$$^{-}0 \leq T_A(x) + I_A(x) + F_A(x) \leq 3^{+}. \tag{1}$$

The values of the three MS functions are taken from $]^{-}0,1^{+}[$. As we have difficulty of applying NSs to real-time issues, Wang et al. [14] put forward the approach of a SVNS, which is the simplification of a NS and can be applied to any real-world topic.

Definition 2 [14] $\ddot{A}$ is the SVNS in X and is described by the set:

$$\ddot{A} = \left\{ \langle x : T_{\ddot{A}}(x), I_{\ddot{A}}(x), F_{\ddot{A}}(x) \rangle,\ x \in X \right\}, \tag{2}$$

where $T_{\ddot{A}}(x), I_{\ddot{A}}(x), F_{\ddot{A}}(x) \in [0, 1]$ satisfying the condition:

$$0 \leq T_A(x) + I_A(x) + F_A(x) \leq 3. \tag{3}$$

Definition 3 [15] An IVNS in X, which represented by:

$$\ddot{A} = \left\{ \langle x : \tilde{T}_{\ddot{A}}(x), \tilde{I}_{\ddot{A}}(x), \tilde{F}_{\ddot{A}}(x) \rangle, x \in X \right\}, \tag{4}$$

$$\ddot{A} = \left\{ \langle x : [T^L_{\ddot{A}}(x), T^U_{\ddot{A}}(x)\Big], [I^L_{\ddot{A}}(x), I^U_{\ddot{A}}(x)\Big], [F^L_{\ddot{A}}(x), F^U_{\ddot{A}}(x)] \rangle, x \in X \right\}, \tag{5}$$

where $[T^L_{\ddot{A}}(x), T^U_{\ddot{A}}(x)\Big], [I^L_{\ddot{A}}(x), I^U_{\ddot{A}}(x)\Big], [F^L_{\ddot{A}}(x), F^U_{\ddot{A}}(x)] \subseteq [0,\ 1]$ are the interval numbers satisfying the condition:

$$0 \leq \sup T_A(x) + \sup I_A(x) + \sup F_A(x) \leq 3. \tag{6}$$

Now we consider a few mathematical operations on interval-valued neutrosophic numbers (IVNNs)s.

Definition 4 [15] Let

$$\ddot{A} = \left\langle \left[T^L_a, T^U_a\right], \left[I^L_a, I^U_a\right], \left[F^L_a, F^U_a\right] \right\rangle \text{ and}$$
$$\ddot{B} = \left\langle \left[T^L_b, T^U_b\right], \left[I^L_b, I^U_b\right], \left[F^L_b, F^U_b\right] \right\rangle,$$

be two IVNNs and $\eta > 0$.

Then

$$\ddot{A} \oplus \ddot{B} = \left\langle \left[T^L_a + T^L_b - T^L_a T^L_b, T^U_a + T^U_b - T^U_a T^U_b\right], \left[I^L_a I^L_b, I^U_a I^U_b\right], \left[F^L_a F^L_b, F^U_a F^U_b\right] \right\rangle, \tag{7}$$

$$\ddot{A} \otimes \ddot{B} = \left\langle \left[T^L_a T^L_b, T^U_a T^U_b\right], \left[I^L_a + I^L_b - I^L_a I^L_b, I^U_a + I^U_b - I^U_a I^U_b\right] \left[F^L_a + F^L_b - F^L_a F^L_b, F^U_a + F^U_b - F^U_a F^U_b\right] \right\rangle, \tag{8}$$

$$\eta\ddot{A} = \left\langle \left[1 - \left(1 - T^L_a\right)^\eta, 1 - \left(1 - T^U_a\right)^\eta\right], \left[\left(I^L_a\right)^\eta, \left(I^U_a\right)^\eta\right], \left[\left(F^L_a\right)^\eta, \left(F^U_a\right)^\eta\right] \right\rangle, \tag{9}$$

$$\ddot{A}^{\eta} = \left\langle \left[\left(T_a^L\right)^{\eta}, \left(T_a^U\right)^{\eta}\right], \left[1-\left(1-I_a^L\right)^{\eta}, 1-\left(1-I_a^U\right)^{\eta}\right], \left[1-\left(1-F_a^L\right)^{\eta}, 1-\left(1-F_a^U\right)^{\eta}\right] \right\rangle, \tag{10}$$

where $\eta > 0$.

Deneutrosophication formulas for interval-valued neutrosophic numbers

To compare two IVNNs $\ddot{A}_1$ and $\ddot{A}_2$, a map from $[N(R)]$ to real line called score function has been used here. In the review of the literature, there are some formulas for deneutrosophication; in this paper, the following formulas have been focused [44, 45] and defined as follows:

$$S_{\text{Ridvan}}\left(\ddot{A}_1\right) = \left(\frac{1}{4}\right) \times \left[2 + T_x^L + T_x^U - 2I_x^L - 2I_x^U - F_x^L - F_x^U\right], \tag{11}$$

$$S_{\text{Liu}}\left(\ddot{A}_1\right) = \left[2 + \frac{T_x^L + T_x^U}{2} - \frac{I_x^L + I_x^U}{2} - \frac{F_x^L + F_x^U}{2}\right]. \tag{12}$$

Using score function (SF), the ranking technique is defined as below:

(i) $\ddot{A}_1 < \ddot{A}_2$ if $\mathrm{SF}(\ddot{A}_1) < \mathrm{SF}(\ddot{A}_2)$.
(ii) $\ddot{A}_1 > \ddot{A}_2$ if $\mathrm{SF}(\ddot{A}_1) > \mathrm{SF}(\ddot{A}_2)$.
(iii) $\ddot{A}_1 = \ddot{A}_2$ if $\mathrm{SF}(\ddot{A}_1) = \mathrm{SF}(\ddot{A}_2)$.

Computation of the shortest path based on neutrosophic numbers

In this section, the new algorithmic approach to solve IVNSP is provided. It is pretended that there are n nodes with the source node (SN), node 1 and destination node (DN), node n. The neutrosophic length between nodes i and j is denoted by d_{ij} and the set of all nodes having a connection with the node i is denoted by $M_{N(i)}$.

Mathematical formulation of BELLMAN dynamic programming

Consider a directed connected graph $G = (V, E)$ from SN '1' and the DN 'n' which is acyclic and they are organized by topological ordering $\left(E_{ij}; i < j\right)$. Using the Bellman powerful programming system, the shortest path can be determined by forward pass computation method. The Bellman powerful programming system is defined as follows:

$$f(i) = \begin{cases} 0, & i = 1 \\ \min\limits_{i<j} \left[f(i) + d_{ij}\right], & \text{otherwise} \end{cases}, \tag{13}$$

where d_{ij} is the weight the directed edge E_{ij}, $f(i)$ is the length of SP node i from the SN 1.

Neutrosophic Bellman–Ford algorithm:

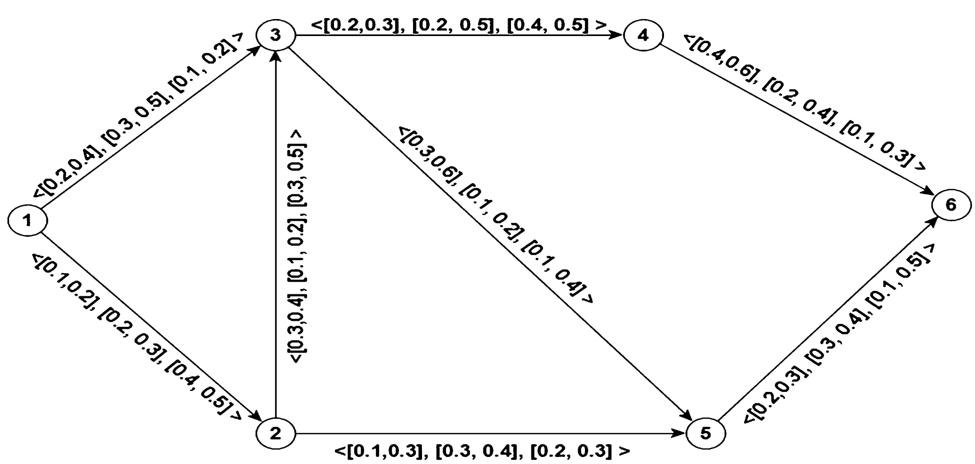

Fig. 1 Interval-valued neutrosophic network

```
nrank[s] ← 0
ndist[s] ← Emptyneutrosophic number.
Add s into Q
For every node i (excluding the s) in the neutrosophic graph G
        rank[i] ← ∞
        Add i into Q
End For
u←s
While(Q is not empty)
        eliminate the vertex u from Q
For each adjacent vertex v of  vertex u
                relaxed←False
                temp_ndist[v] ←   ndist[u] ⊕ edge_weight(u,v)        ⊕ represents the
addition of neutrosophic
                temp_nrank[v] ←rank_of_neutrosophic(temp_ndist[v])
        Iftemp_nrank[v]<nrank[v]   then
                        ndist[v] ←temp_ndist[v]
                        nrank[v] ← temp_nrank[v]
                prev[v] ←u
                End If
        End For
        If  relaxed equals False then
exit the loop
        End If
u ←Node in Q with a minimum rank value
  End While
  For each arc(u,v) in neutrosophic graph G do
      If nrank[v]>rank_of_neutrosophic(ndist[u]  ⊕ edge_weight(u,v))
return false
End If
   End For
The neutrosophic number ndist[u] is a neutrosophic number and it represents the SP from
SN  s  and DN u.
```

In the posterior section, we present a simple illustration to show the brevity of our method.

Illustrative example

This part is based on a numerical problem adapted from [43] to show the potential application of the proposed algorithm.

Example 1 Consider an interval-valued neutrosophic network whose edge weights are represented by IVNNs with SN, node 1 and DN, node 6 (Fig. 1). Table 2 represents interval-valued neutrosophic distance.

Here we need to find the shortest distance from node 1 to node 6 (Table 3).

Using the proposed algorithm in previous section, the SP from SN and DN is calculated as follows:

Table 2 The details of edge information in terms of interval-valued neutrosophic numbers

Edges	IVN distance	Edges	IVN distance
1–2	([0.1, 0.2], [0.2, 0.3], [0.4, 0.5])	3–4	([0.2, 0.3], [0.2, 0.5], [0.4, 0.5])
1–3	([0.2, 0.4], [0.3, 0.5], [0.1, 0.2])	3–5	([0.3, 0.6], [0.1, 0.2], [0.1, 0.4])
2–3	([0.3, 0.4], [0.1, 0.2], [0.3, 0.5])	4–6	([0.4, 0.6], [0.2, 0.4], [0.1, 0.3])
2–5	([0.1, 0.3], [0.3, 0.4], [0.2, 0.3])	5–6	([0.2, 0.3], [0.3, 0.4], [0.1, 0.5])

Table 3 The details of deneutrosophication value of edge (i, j)

Edges	S_{Ridvan}	S_{Liu}
1–2	0.1	1.45
1–3	0.175	1.75
2–3	0.325	1.8
2–5	0.125	1.6
3–4	0.05	1.45
3–5	0.45	2.05
4–6	0.35	2
5–6	0.125	1.6

$$f(1) = 0,$$

$$f(2) = \min_{i<2}\left\{f(1) + c_{12}\right\} = c_{12}^{*} = 0,1,$$

$$f(3) = \min_{i<3}\left\{f(i) + c_{i3}\right\} = \min\left\{f(1) + c_{13}, f(2) + c_{23}\right\}$$
$$= \{0 + 0,175, 0,1 + 0,235\} = \{0,175, 0,335\} = 0,175,$$

$$f(4) = \min_{i<4}\left\{f(i) + c_{i4}\right\} = \min\left\{f(3) + c_{34}\right\}$$
$$= \{0,175 + 0,05\} = 0,225,$$

$$f(5) = \min_{i<5}\left\{f(i) + c_{i5}\right\} = \min\left\{f(2) + c_{25}, f(3) + c_{35}\right\}$$
$$= \{0.1 + 0,125, 0,175 + 0,455\} = \{0.225, 0,625\} = 0.225,$$

$$f(6) = \min_{i<6}\left\{f(i) + c_{i6}\right\} = \min\left\{f(4) + c_{46}, f(5) + c_{56}\right\}$$
$$= \{0.225 + 0,35, 0,225 + 0,125\} = \{0.575, 0,350\} = 0.350,$$

thus

$$f(6) = f(5) + c_{56} = f(2) + c_{25} + c_{56}$$
$$= f(1) + c_{12} + c_{25} + c_{56} = c_{12} + c_{25} + c_{56}.$$

Therefore, the path P: $1 \to 2 \to 5 \to 6$ is recognized as the neutrosophic shortest path, and the crisp shortest path is 0.35.

Contingent study

In this section, the analysis of contingency for the proposed algorithm with existing approaches has been analyzed. A comparison of the results between the existing and new technique is shown in Table 4.

From the result, it is shown that the introduced algorithm contributes sequence of visited nodes which shown to be similar to neutrosophic shortest path presented in [43].

The neutrosophic shortest path (NSP) remains the same, namely $1 \to 2 \to 5 \to 6$, but the neutrosophic shortest path length (NSPL) differs, namely $([0.424, 0.608], [0.012, 0.06], [0.016, 0.125])$, respectively. From here we come to the conclusion that there exists no unique method for comparing neutrosophic numbers and different methods may satisfy different desirable criteria.

Advantages and limitations of the proposed algorithm

Advantages

By correlating our PA with Broumi et al. [43] to solve the same problem, we conclude that the proposed approach leads to the same path $1 \to 2 \to 5 \to 6$. The extended Bellman's algorithm operates on neutrosophic directed graphs with negative weight edges whereas the extended Dijkstra algorithm proposed in [37] cannot deal with. This approach can be easily extended and applied to other neutrosophic networks with the edge weight as

1. Single-value neutrosophic numbers.
2. Bipolar neutrosophic numbers.
3. Trapezoidal neutrosophic numbers.
4. Cubic neutrosophic numbers.
5. Interval bipolar neutrosophic numbers.
6. Triangular neutrosophic numbers and so on.

Limitations

1. Slow response will be observed when there is a change in the network as this change will spread node-by-node.
2. If node failure occurs then routing loops may exist.

Table 4 Comparison of the sequence of nodes using neutrosophic shortest path and our proposed algorithm

Possible path	Sequence of nodes	Crisp shortest path length
Neutrosophic shortest path with interval-valued neutrosophic numbers [43]	$1 \to 2 \to 5 \to 6$	$([0.35, 0.60], [0.01, 0.04], [0.008, 0.075])$
PA based on S_{Ridvan}	$1 \to 2 \to 5 \to 6$	0.35
PA based on S_{Liu}	$1 \to 2 \to 5 \to 6$	4.65

Conclusion

In this study, we describe the NSP, where edge weights are represented by IVNS. The advantage of using IVNSs in NSP is discussed in this paper. The classical Bellman's algorithm is modified by incorporating the uncertainty using IVNSs for NPP between source and destination nodes. We use a numerical example to illustrate the efficiency of our proposed algorithm. The main goal of this work is to describe an algorithm for NSP in the neutrosophic environment using IVNS as edge weight. The proposed algorithm is very effective for real-life problem. In this paper, we have used a simple numerical example to illustrate our proposed algorithm. Therefore, as future work, we need to consider a large-scale practical shortest path problem using our proposed algorithm and to compare our proposed algorithm with the existing algorithm in terms of strictness of optimality, efficiency, computational time, and other aspects.

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ORIGINAL ARTICLE

Shortest path problem in fuzzy, intuitionistic fuzzy and neutrosophic environment: an overview

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Abstract
In the last decade, concealed by uncertain atmosphere, many algorithms have been studied deeply to workout the shortest path problem. In this paper, we compared the shortest path problem with various existing algorithms. Finally, we concluded the best algorithm for certain environment.

Keywords Fuzzy sets · Intuitionistic fuzzy sets · Vague sets · Neutrosophic sets · Shortest path problem

Introduction

SPP is a cardinal issue among familiar connectional problems which occur in different areas of engineering and science, such as application in highway networks, portage and conquer in intelligence channels and problem of scheduling. The SPP focuses on recommending the path which has minimum length enclosed by two vertices. The length of the arc/edge produces the quantities of the real life, namely cost, time, etc. In the case of conventional method of measuring SP, the length of each bend is assumed as a crisp numbers. If there is uncertainty on the parameters in the network, then the length can be represented by fuzzy number.

In the current preceding, many of the SPPs with various types of input data have been examined in junction with fuzzy, intuitionistic, vague, interval fuzzy, interval-valued intuitionistic fuzzy and neutrosophic sets [2, 3, 8, 9, 11, 13, 14, 17–20, 23, 30, 39, 46–52, 83–92]. Up until now plenty of new algorithms have been designed.

The paper is arranged as: section "Preliminaries" comprehends the primary definitions and overviewed SPP under different sets in sections, "SPP in vague environment", "SPP in fuzzy environment", "SPP in intuitionistic fuzzy environment" and "SPP in neutrosophic environment", respectively. Lastly, conclusion has been presented for the objective of the paper.

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Preliminaries

Here, we principally recollected some of the concepts connected to neutrosophic sets (NSs), single-valued neutrosophic sets (SVNSs) related to the present work. See especially [10, 12] for further details and background.

Definition 2.1. Let X be a nonempty set. A fuzzy set A drawn from X is defined as,

$$A = \{x, \mu_A(x) | x \in X\}, \tag{1}$$

where $\mu_A : X \to [0, 1]$, is called the membership function of A and defined over a universe of discourse X.

Definition 2.2. A type-2 fuzzy set, denoted by $\overline{A}$ is characterized by a type-2 membership function $\mu_{\overline{A}}(x, u)$, where $x \in X, u \in J_x \subseteq [0, 1]$, i.e.,

$$\overline{A} = \{((x, u), \mu_{\overline{A}}(x, u)) | x \in X, \quad \forall u \in J_x \subseteq [0, 1]\}. \tag{2}$$

Definition 2.3. An interval-valued fuzzy set is a special case of type-2 fuzzy sets by representing the membership function $\mu_{\overline{A}} = \left[\underline{\mu_{\overline{A}}}, \overline{\mu_{\overline{A}}}\right]$, where $\underline{\mu_{\overline{A}}}$ is a lower membership function and $\overline{\mu_{\overline{A}}}$ is an upper membership function. The area between these lower and upper membership functions is called a footprint of uncertainty (FOU), which represents the level of uncertainty of the set.

Definition 2.4. Let X be a nonempty set. An intuitionistic fuzzy set (IFS)A in X is an object having the form

$$A = \{\langle x, \mu_A(x), \nu_A(x)\rangle | x \in X\}, \tag{3}$$

where the functions $\mu_A(x), \nu_A(x) : X \to [0, 1]$ define the degree of membership and nonmembership, respectively, of the element $x \in X$ to A, for the entire element $x \in X$ $0 \le \mu_A(x) + \nu_A(x) \le 1$.Also,$\pi_A(x) = 1 - \mu_A(x) - \nu_A(x)$ is called the index of IFS, and is the degree of indeterminacy of $x \in X$ to the IFS A, which expresses the lack of knowledge of whether x belongs to IFS or not. Also $\pi_A(x) \in [0, 1]$, i.e., $\pi_A(x) : X \to [0, 1]$ and $0 \le \pi_A(x) \le 1, \quad \forall x \in X$.

Definition 2.5. An interval-valued intuitionistic fuzzy set (IVIFS) A in X is defined as an object of the form

$$A = \{\langle x, P_A(x), Q_A(x)\rangle | x \in X\}, \tag{4}$$

where the functions $P_A(x) : X \to [0, 1], Q_A(x) : X \to [0, 1]$ denote the degree of membership and non-membership of A, respectively. Also, $P_A(x) = [P_A^L(x), P_A^U(x)]$ and $Q_A(x) = [Q_A^L(x), Q_A^U(x)]$,$0 \le P_A^U(x) + Q_A^U(x) \le 1, \forall x \in X$

Definition 2.6. Let U be the universe, $U = \{x_1, x_2, \dots, x_n\}$, with a generic element of U denoted by $x_i, i = 1, 2, \dots, n$. A vague set is defined as an object of the form $A = \{\langle x_i, T_A(x_i), F_A(x_i)\rangle | x_i \in X\}$ in U is characterized by a truth membership function T_A and a false membership function F_A, i.e., $T_A : U \to [0, 1]$, $F_A : U \to [0, 1]$, where $T_A(x_i)$ is the lower bound on the grade of membership of x_i, $F_A(x_i)$ is the lower bound on the negation of x_i, derived from the evidence against x_i and $T_A(x_i) + F_A(x_i) \le 1$. The grade of membership of x_i in the vague set A is bounded to the subinterval $[T_A(x_i), 1 - F_A(x_i)]$ of the interval $[0, 1]$. The vague value $[T_A(x_i), 1 - F_A(x_i)]$ indicates that the exact grade of membership $\mu_A(x_i)$ of x_i may be unknown. But it is bounded by $T_A(x_i) \le \mu_A(x_i) \le 1 - F_A(x_i)$.

Definition 2.7. An interval-valued vague set A over a universe of discourse X is defined as an object of the form $A = \{\langle x_i, [T_A^L, T_A^U], [F_A^L, F_A^U]\rangle | x_i \in X\}$, where $0 \le T_A^L \le T_A^U \le 1$ and $0 \le T_A^U \le T_A^L \le 1$. For each interval-valued vague set A, $\pi_A(x_i) = 1 - T_A^L(x_i) - F_A^L(x_i)$ and are called degree of hesitancy of x_i.

Definition 2.8 Consider the space X consists of universal elements characterized by x. The NS A is a phenomenon which has the structure $A = \{(T_A(x), I_A(x), F_A(x))/x \in X\}$, where the three grades of memberships are from X to $]^-0, 1^+[$ of the element x ∈ X to the set A, with the criterion:

$$^-0 \le T_A(x) + I_A(x) + F_A(x) \le 3^+. \tag{5}$$

The functions, and are the truth, indeterminate and falsity grades which lie in real standard/non-standard subsets of $]^-0, 1^+[$. Since there is a complication of applying NSs to realistic issues, Samarandache and Wang wt al. [11, 12] proposed the notion of SVNS, which is a specimen of NS and it is useful for realistic applications of all the fields.

Definition 2.9. Let X be the space of objects which contains global elements. A SVNS is represented by degrees of membership grades mentioned in Definition 2.1. For all x in X, $T_A(x), I_A(x)$ $F_A(x) \in [0, 1]$. A SVNS can be written as

$$A = \{< x : T_A(x), I_A(x), F_A(x) > /x \in X\} \tag{6}$$

Definition 3. Let X be a space of objects with generic elements in X denoted by x. An interval-valued neutrosophic set (IVNS) A in X is characterized by truth membership function, $T_A(x)$, indeterminacy membership function $I_A(x)$ and falsity membership function $F_A(x)$. For each point x in $X, T_A(x), I_A(x), F_A(x) \in [0, 1]$, and an IVNS A is defined by

$$A = \{\langle [T_A^L(x), T_A^U(x)], [I_A^L(x), I_A^U(x)], [F_A^L(x), F_A^U(x)]\rangle | x \in X\}, \tag{7}$$

where $T_A(x) = [T_A^L(x), T_A^U(x)]$,$I_A(x) = [I_A^L(x), I_A^U(x)]$ and $F_A(x) = [F_A^L(x), F_A^U(x)]$.

SPP in vague environment

A peculiar way for getting a shortest path (SP) of a given network was found by Dou et al. [26]. in 2008, where the sets are vague. Firstly, the authors recommended that the length of the SP was determined using vague sets from the source node (SN) to the destination node (DN) for conventional network with direction. Secondly, they calculated the degree of resemblance among the lengths of the vague paths under vague similarity measure. Finally, it was concluded that the path which has the greater degree of similarity is a SP. A novel algorithm was constructed to identify the SP in a directed graph (DG), where the distance between the arcs is considered as vague number in triangular measure rather than real number. In 2018, Rashmanlou et al. solved SPP using Euclidean distance for vague network [57].

SPP in fuzzy environment

This part describes about various methods to solve SPP using fuzzy arc length by many authors. SPP can be solved in an optimized way for a given network using fuzzy logic as the real-world problems are uncertain in nature. Dubois and Prade solved fuzzy SPP (FSPP) using Floyd's and Ford's algorithms firstly [5]. In the year 2000, Okada and soper introduced an algorithm to solve FSPP in terms of multiple labeling procedure [32]. Klein [27] projected a vital programming fuzzy algorithm based on recursive concept. Lin [41] constructed a technique of fuzzy linear programming to find the fuzzy SP (FSPP) length of a network. Yao [42] contemplated two different FSPP such as SPP using triangular fuzzy numbers (TFNs) and SPP using level (1–β 1–α) interval-valued fuzzy numbers (IVFNs).

In 2003, the same author solved FSSP using two different types of methods, namely TFNs and level $(1-\beta, 1-\alpha)$ interval-valued FNs (IVFNs). Nayeem and Pal introduced an algorithm to solve SPP using notoriety index, where the lengths of the arc were taken as interval numbers or TFNs [44]. In the year 2005, Chuang recommended a novel idea to identify FSP by finding the length of the FSP encompassed by all possible paths of a given network [38]. Kung et al. established new technique to handle FSPP by representing the arc length as TFN [40]. In 2009, Yadav and Biswas conferred a new method to solve SPP by considering the edge length as FN in a directed graph instead of real number. The authors constructed an algorithm to discover an optimal path by considering that both input and output are FNs [22]. Also in the same period, Lin solved SPP using interval-valued FNs and endorsed distance method of defuzzification [62]. In 2010, Pandian and Rajendran introduced path classification algorithm to find the minimal path by considering crisp or uncertain weights (TFNs) from one node to another. In this method, indeterminate nodes in the minimum path can be found without going backward and this is the major advantage. This would be very helpful for the decision-makers to omit indeterminate nodes [28]. Seda presented all-pairs SPP by applying fuzzy ranking method [25]. In 2012, Meenakshi and Kaliraja determined SP for IVFN (Interval Valued Fuzzy Network) [61].

In 2013, Shukla projected Floyd's algorithm to solve SPP using a concept of fuzzy sets which is based on graded mean unification of FNs [21]. In 2014, Elizabeth and Sujatha introduced a novel approach to solve FSPP by finding minimum arithmetic mean among IVFN matrices [31]. The same authors Huyen et al. gave a direction on establishing a design for SPP with TFNs as the edge weights. In this work, mathematical concept of the algorithm is developed on Defined Strict Comparative Relation Function for the set of TFNs [56]. Nayeem proposed a novel expected value algorithm for the FSSP [60].

In 2015, Mukerje [34] explored the fuzzy approach programming to solve FSPP. Here, the authors converted a single-objective fuzzy linear programming (SOFLP) by considering TFNs and TpFNs as the edge weight into crisp multi-objective Linear Programming (CMOLP). Anusuya and Sathya proposed a design for SPP where the arc lengths are type-2 fuzzy numbers (T2FNs) from SN to DN in a network [54]. Also, the authors established an algorithm for SPP using type reduction on the edges using centroid and center of gravity of FSwhich gives the FSP where the arc lengths are represented by discrete T2FN [55]. Mahdavi et al. [16] applied dynamic programming method for finding the shortest chain in a fuzzy network. In [33] Okada solved FSPPs by incorporating interactivity among path. Deng et al. [53] established fuzzy Dijikstra algorithm for solving SPP under uncertain environment. Dey et al. have contributed the following ideas: solved FSPP using IT2FSs (interval type-2 fuzzy set) as the edge weights, they have altered conventional Dijikstra's design by including impreciseness using IT2FSs to solve SPP from SN to DN, afford a new way for SPP in imprecise setting using IT2FSs for the edge weights and examined the path algebra and its generalized algorithm for FSPP [6]. Meenakshi and Kaliraja described the SP for a network under the notion of interval valued fuzzy (IVF) where the SP in lower limit fuzzy networks coexists with the case for upper limit [7]. In 2016, Dey et al. introduced a model to solve FSPP for using Interval Type-2 Fuzzy (IT2F) [59].In 2017, Biswas proposed IVFSP in a multi-graph [63]. In 2018, Eshaghnezhad et al. presented a first scientific paper for resolving of FSP by artificial network model which has the property of the global exponential stability [82].

SPP in intuitionistic fuzzy environment

In this part, various methods have been disclosed in literature to handle the SPP by taking intuitionistic fuzzy (IF) as the arc lengths by different authors.

In 2007, Karunambigai et al. refined an approach found on dynamic programming to solve SPP using intuitionistic fuzzy graphs (IFGs) [24]. In 2010, Gani also established a technique to identify intuitionistic fuzzy shortest path (IFSP) for a given network [3]. Mukherje pre-owned an interesting methodology to solve IFSPP using the idea of Dijikstra's algorithm and intuitionistic fuzzy hybrid Geometric (IFHG) operator [45]. Majudmder and Pal [30] solved SPP for intuitionistic fuzzy network. In 2013, Biswas modified an IF method for SPP in a realistic multigraph [35]. Rangasamy et al. proposed score-based methodology to find the shortest hyper paths for a given network where hyper edges are characterized by IF weights without describing similarity measure and Euclidean distance [43]. Babacioru conferred an algorithm to find the minimum arc length of an IF hyper path using MAPLE [15]. In [29], Jayagowri and GeethaRamani solved SPP on a network with the use of Trapezoidal Intuitionistic Fuzzy Numbers (TpFNs).In 2014, Porchelvi and Sudha recommended a minimum path labeling algorithm to solve SPP using triangular IF number (TIFN) [36]. Also, they proposed a new and different methodology to solve SPP with TIFNs, where the authors found the minimal edge using IF distance by applying graded mean integration and examined SPP from a particular vertex to all other ones in a network [37]. In 2015, Kumar et al. suggested a design to identify the SP and shortest distance in an IVIF graph where the nodes are taken as crisp numbers and edge weights are assigned by IVITpFNs (Interval Valued Intuitionistic Trapezoidal Fuzzy Numbers) [1]. Kumar et al. proposed an algorithm for SPP using IVITpFN as the weights in a network [58].

SPP in neutrosophic environment

The authors modeled a design to find the ideal path where the inputs and outputs are neutrosophic numbers (NNs) [4]. In 2016, Broumi et al.solved SPP, by considering the edge weights as, SVTpNNs as the edge weights [68], Triangular fuzzy neutrosophic numbers (TFNNs) [69], bipolar neutrosophic set [70] and applied Dijikstra's method to solve NSPP and IVNSPP [67, 72]. In 2017, Broumi et al. solved SPP using SVNGs [64], by adopting SVTNNs and SVTpNNs [65, 66]; found an optimal solution for the NSPP using trapezoidal data under neutrosophic environment [68], by SVNN; solved the MST problem [73], by Trapezoidal fuzzy neutrosophic [74]; introduced a new notion of matrix design for MST in IVNG [79]; and introduced computational method to MST in IV bipolar neutrosophic setting [80]. Also in [75], Broumi et al. proposed another algorithm to solve MST problem on a network with the use of SVTpNNs. Broumi et al. [76] solved MST problem in a bipolar neutrosophic environment. Mullai et al. solved SPP by minimal spanning tree (MST) using BNS [77]. In 2018, Broumi et al. applied IVNNs and BNS SPP for a given network [70, 71]. Dey et al. proposed a novel design for MST for NGs which are undirected [78]. Jeyanthi and Radhika [81] solved NSPP using Floyd's algorithm firstly. Basset et al. proposed a hybrid approach of neutrosophic sets and DEMATEL method for developing the criteria for supplier selection [83]. Basset et al. introduced a novel method, to solve the fully neutrosophic linear programming problems [84]. Basset et al. proposed three-way decisions based on neutrosophic sets and AHP-QFD framework for the problem supplier selection [85]. Basset et al. proposed a novel framework to evaluate cloud computing services [86]. Basset et al. introduced an extension of neutrosophic AHP-SWOT analysis for strategic planning and decision-making [87]. Basset et al. proposed an approach of hybrid neutrosophic multiple criteria group decision-making for project selection. [88]. Basset et al. proposed a framework for a group decision-making problem, based on neutrosophic VIKOR approach for e-government website evaluation [89]. Basset et al. proposed an economic tool for quantifying risks in supply chain as a framework for risk assessment, management and evaluation [90].

The following table confers four types of SPP containing FSPP, IFSPP and neutrosophic SPP (NSPP) and for the case of interval numbers to all the types of parameters.

Shortest path problem on network with	Edges/vertices	Indeterminacy	Ambiguity	Uncertainty
Crisp parameters	Crisp Number (CN)	Inadequate to handle	Inadequate to handle	Inadequate to handle
Crisp Interval Parameters	Crisp Interval Number (IN)	Inadequate to handle	Inadequate to handle	Inadequate to handle
Fuzzy parameters (FPs)	Fuzzy Number (FN)	Unable to deal	Unable to deal	Able to deal with uncertainty

Shortest path problem on network with	Edges/vertices	Indeterminacy	Ambiguity	Uncertainty
Interval FPs	Interval Fuzzy Number (IFN)	Unable to deal	Unable to deal	Able to deal with more uncertainty, as it has lower and upper membership values
Intuitionistic fuzzy parameters (IFPs)	Intuitionistic Fuzzy Number (IFN)	Inadequate to deal	Adequate to deal	Adequate to deal
Interval IFPs	Interval Intuitionistic Fuzzy Number (IFN)	Inadequate to deal	Adequate to deal clearly as it has loer and upper membership values	Adequate to deal more uncertainty as it has lower and upper membership functions
Neutrosophic parameters (NPs)	Neutrosophic Number (NN)	Able to handle	Able to handle	Able to handle
Interval NPs	Interval Neutrosophic Number (INN)	Able to handle more indeterminacy	Able to handle more ambiguity	Able to handle more uncertainty as it has lower and upper membership functions.

From the overhead table, it is seen that the available methods could not employed to solve NSPP from SN to DN for a given network with IVNN as the edge weights.

But neutrosophic environment can able to solve SPP effectively as it handles indeterminacy together with impreciseness and ambiguity to take the best decision in identifying the SP with the use of IVNN rather than single-valued NN. Effortlessly, the proposed algorithm can be adapted to any kind of NNs.

As the neutrosophic logic deals indeterminacy with the collected/given information, the algorithms proposed to find SPP may be the best one than other algorithms under fuzzy and intuitionistic fuzzy environments.

Advantages and limitations of different types of sets

The below table expresses the capacity of various types of sets as an advantage and their incapability to handle some conditions or important situations towards to realistic problems.

Various types of sets	Advantages	Limitations
Crisp sets	Can accurately determine with no hesitation	Cannot describe the uncertain Information
Fuzzy sets	Can describe the uncertain Information	Cannot describe the uncertain Information with non-membership degree
Interval valued fuzzy sets	Can able to deal interval data instead of exact data	Cannot handle the uncertain Information with non-membership degree
Intuitionistic fuzzy sets	Can describe the uncertain Information with membership (MS) and non-membership (NMS) degrees simultaneously	Cannot describe the sum of MS and NMS degrees bigger than 1
Interval valued Intuitionistic fuzzy sets	Able to handle interval data	Cannot portray the addition of MS and NMS degrees bigger than 1
Vague sets	Can describe uncertain Information with grades of MS and NMS at the same time.	Cannot describe the sum of MS and NMS degrees greater than 1.
Pythagorean fuzzy sets	It has full space to describe the sum of MS and NMS degrees greater than 1	Cannot describe the square sum of MS and NMS degrees greater than 1
Interval valued Pythagorean fuzzy sets	Capable of dealing interval data	Unable to define the square sum of MS and NMS degrees greater than 1
Neutrosophic Sets	Able to deal indeterminacy of the data and the optimized solution can be obtained completely.	Unable to handle interval data
Interval valued Neutrosophic sets	Able to deal indeterminacy of the interval data and the optimized solution can be obtained .	Unable to handle incomplete weight information

Conclusion

Crisp SPP (CSPP) can be adopted only if there exists certainty on the parameters of nodes and edges. If uncertainty exists in the arc, then the authors have been recommended to use FSPP. Later, FSPPs cannot be enforced for the certain message which is not endured and indecisive, and so the investigation invented the concept of IFSPP. Further when the information about the path is indetermined, uncertain and unreliable, neutrosophic concept has been implemented and obtained the solution for neutrosophic shortest path problem in the literature. All the existing algorithms developed by the reserachers. The algorithms have been used for various real world problems but occasionally not suitable for persuade situations. Hence, the recognized algorithms in various sets such as vague set (VS), FS, IFS and NS are forced. In real world, the researcher who has clear knowledge about the data can accept and implement the algorithms for solving SPP. This paper will be very helpful to the new researchers to propose novel concepts to solve the shortest path problem. In the future, based on this present study, new algorithms and frameworks will be designed to find the shortest path for a given network under various types of sets environments.

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Traffic control management using Gauss Jordan method under neutrosophic environment

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Traffic Control Management using Gauss Jordan Method Under Neutrosophic Environment

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Abstract. Neutrosophic is the logic which deals membership and non membership degree of the data and also indeterminacy of the data as well. Dealing indeterminacy is the advantage of the neutrosophic environment and hence helpful to find the optimized result of the system/ choosing the best alternative from the collected data in a complete manner. In this paper, traffic flow control is analyzed under neutrosophic environment using Gauss Jordan method using MATLAB.

INTRODUCTION

Predicting traffic flow, speed, length of the queue and travel time are necessary for the transportation management applications [1,10,12]. Predicting and modeling traffic flow has drawn attention from literature as it is very important for formatting intelligent transportation system theoretically and practically. The area of transportation studies has attracted interest among the researchers [2- 8,11].

Fuzzy logic is a powerful tool to handle uncertainties in measurements and information used to calculate the parameters; here the membership value for a particular traffic state is not a crisp value [9]. if the system is nonlinear designing a model for traffic flow prediction is very difficult [13]. Any number of intersections and lanes can be handled using fuzzy and interval fuzzy logic in traffic control management. Generally there are two types of signal control available, namely, fixed time control (the traffic conditions are fixed) and adaptive time control (the traffic conditions may be refined over a period of time control [14-16].

If the number of vehicles are increased and having lower phase of highways then there will be a traffic congestion problem. The general factors for traffic problems are density of the vehicles, human behavior, traffic light system and social behavior. The complex and changing traffic situations cannot be dealt by conventional traffic control methodologies or control systems.

As the flow of traffic varies from hour to hour in morning and evening. Especially during office timing the traffic flow will be high in general. The one of the advantage of fuzzy logic is, there is a possibility of computing with words [17-18]

.

Neutrosophic logic was proposed by Smarandache can express determinate as well as indeterminate of the information by neutrosophic numbers. Solving traffic flow problem is a difficult one for certain parameters as the real-time situations are uncertain in nature and can be solved easily by considering neutrosophic logic [19-25]. Fuzzy and neutrosophic logic are playing a vital role in dealing with uncertainties. But neutrosophic deals

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uncertain problems completely as it handles indeterminacy also. Traffic control management can be dealt using triangular interval type-2 fuzzy and interval neutrosophic environments [26-34]. In this paper, traffic flow of the junction has been analyzed using Gauss Jordan method with the support of MATLAB program under neutrosophic environment.

BASIC CONCEPTS

Single Valued Neutrosophic Set

Let X be the universal set of objects denoted by x. Neutrosophic set K (NS S) is denoted by K = {< y: $T_K(y)$, $I_K(y)$, $F_K(y)$>, y $\in$ X}, where the truth, indeterminacy and falsity membership are defined by T, I, F: →]$^-$0,1$^+$[respectively and satisfies the condition $^-0 \leq T_K(y) + I_K(y) + F_K(y) \leq 3^+$. The functions $T_K(y)$, $I_K(y)$ and $F_K(y)$ are real standard of]−0,1$^+$ [. If $T_K(y)$, $I_K(y)$ and $F_K(y)$ takes values from [0,1] then, neutrosophic set is called single valued neutrosophic set. [29]

Gauss Jordan Method

Using Gauss Jordan method, a system consists of linear equations (LEs) can be solved using Gauss. By finding an augmented matrix form the given system, one can find the inverse of the matrix.

PROPOSED METHODOLOGY

Gauss Jordan Method to analyze traffic flow

In this section, four roads have been considered, namely road 1 (R1), road 2 (R2), road 3(R3) and road 4 (R4). R1 and R2 can handle a maximum of 800 vehicles whereas road 3 and road 4 can handle a maximum of 2100 vehicles. FIGURE 1 shows the traffic flow on the four roads. Here z is a neutrosophic variable, and y_1, y_2 and y_3 are the unknown variables. Form neutrosophic linear equations (NLEs) for all the junctions and using MATLAB program apply Gauss Jordan method to find the inverse of the matrix formed from NLEs. Where I is the indeterminacy of the traffic flow. Here falsity considered as zero as we consider traffic flow at the four junctions.

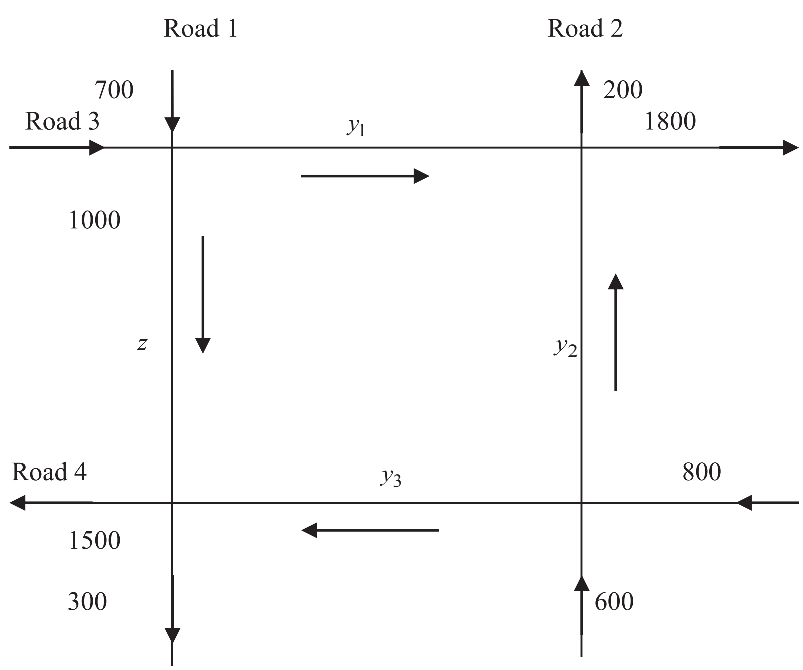

FIGURE 1. The traffic flows of four roads

The intersections between the roads (junctions), the four LEs can be formed as follows:

$$1700 = y_1 + z$$
$$2000 = y_1 + y_2$$
$$1400 = y_2 + y_3$$
$$1800 = y_3 + z$$

The equations can be rewritten as,

$$y_1 = 1700\text{-z}$$
$$y_1 + y_2 = 2000$$
$$y_2 + 2y_3 = 3200\text{-z}$$

Based on, z = 200+I, the system can also be described by the following three NLEs:

$$y_1 = 1700\text{-}(200\text{+I})$$
$$y_1 + y_2 = 2000$$
$$y_2 + 2y_3 = 3200\text{-}(200\text{+I})$$

Then, the Neutrosophic equations are:

$$y_1 = 1500\text{-I}$$
$$y_1 + y_2 = 2000$$
$$y_2 + 2y_3 = 3000\text{-I}$$

Thus the Neutrosophic matrices are:

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 2 \end{bmatrix},\ Y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix},\ B = \begin{bmatrix} 1500 - I \\ 2000 \\ 3000 - I \end{bmatrix}$$

For the system consists of NLEs, apply the Gauss Jordan method using MATLAB software, shown in the following program

```
clc
syms I;
a=[1 0 0;1 1 0;0 1 2]
b=[1500-I;2000;3000-I]
[v,j]=jordan(a)
j=inv (v)*a*v
y=a\b
```

Output of the program are as follows:

```
a=[1 0 0;1 1 0;0 1 2]

a =

    1    0    0
    1    1    0
    0    1    2

 b=[1500-I;2000;3000-I]

b =
```

```
 1500 - I
 2000
 3000 - I

 [v,j]=jordan(a)

v =

     0     0     1
     0     1     0
     1    -1    -1

j =

     2     0     0
     0     1     1
     0     0     1

 j=inv (v)*a*v

j =

     2     0     0
     0     1     1
     0     0     1

 y=a\b

y =
 1500 - I
  I + 500
 1250 – I
```

and the solution of the system is :

$$Y = \begin{bmatrix} 1500 - I \\ I + 500 \\ 1250 - I \end{bmatrix}$$

The values of Y are NNs.

In some of the real-time situations, when I ∈[0, 50] is the possible range, the solution of the system is:

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} [1450,1500] \\ [500,550] \\ [1200,1250] \end{bmatrix}$$

Corresponding to the possible traffic flow z = [200,250]

Thus the ranges of the three traffic flows are

$$y_1 = [1450,1500]$$
$$y_2 = [500,550]$$
$$y_3 = [1200,1250]$$

TABLE 1. Traffic flows according to various ranges of Indeterminacy I

I	z	y_1	y_2	y_3
$I=0$	200	1500	500	1250
$I \in [100,200]$	$[300,400]$	$[1300,1400]$	$[600,700]$	$[1050,1150]$
$I \in [200,300]$	$[400,500]$	$[1200,1300]$	$[700,800]$	$[950,1050]$
$I \in [300,400]$	$[500,600]$	$[1100,1200]$	$[800,900]$	$[850,950]$
$I \in [400,500]$	$[600,700]$	$[1000,1100]$	$[900,1000]$	$[750,850]$

CONCLUSION

Traffic control is the key role for traffic control management in day to day life. Neutrosophic environment is an optimized logic to deal real world problems as it handles indeterminacy of the data. Gauss Jordan method is a cardinal tool for finding the inverse of the matrix. In this paper, traffic flow management has been analyzed with respect to various ranges of indeterminacy under neutrosophic environment using Gauss Jordan method with the support of MATLAB program.

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Blockchain Single and Interval Valued Neutrosophic Graphs

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Abstract. Blockchain Technology (BCT) is a growing and reliable technology in various fields such as developing business deals, economic environments, social and politics as well. Without having a trusted central party this technology, gives the guarantee for safe and reliable transactions using Bitcoin or Ethereum. In this paper BCT has been considered using Bitcoins. Also Blockchain Single and Interval Valued Neutrosophic Graphs have been proposed and applied in transaction of Bitcoins. Also degree, total degree, minimum and maximum degree have been found for the proposed graphs. Further, comparative analysis is done with advantages and limitations of different types of Blockchain graphs.

Keywords: Blockchian Technology, Bitcoins, Fuzzy Graph, Neutrosophic Graphs, Properties

1. Introduction

A completely peer-to-peer form of electronic cash will permit payments through online and direct transaction can be done from one participant to another without facing any financial organization. If a central party wants to avoid double-spending then the main gain will be lost even though digital signatures contribute part of the solution. This issue was the reason of bargain a solution to this problem based on peer-to-peer network. For direct transaction of two willing parties without having a trusted third party, an electronic system using cryptographic proof (signaling code) can be used. Fuzzy logic is introduced by Zadeh to deal uncertainty of the problem. Fuzzy graphs are playing an important role in network where impreciseness exists on the vertices and egdes. Yeh and Banh also proposed the fuzzy graph independently and examined various connectedness theories [1-4].

The universal problems namely sustainable development or transformation of assets can be dealt effectively by Block chain technology than the existing financial systems. The financial sector acquires in various operative costs for the smooth and effective functioning of the entire system. These costs consist of time and money needed for investment in framework, electricity cost spent for operation and from Automated Teller Machines, consumption of water and gas by the employees and wastage production.

Also there is no possibility of creating fiat currency without costs. In order to give assurance in a regular basis in the quality standards for the bank notes in circulation, the used ones are shredded. To find an overview of the overall cost of an existing financial system, the cost for the production of coins and noted will be included. Whereas in BCT, one needs only to connect to the network and do not obtain the electricity cost for any source. Also the production of the crypto currency (a digitalized currency, where encoding method is applied to control the production of currency and funds transference verification) [5-7].

Platforms of Central banking, improvement of business processing, automotive ownership, sharing of health information, deals and voting can be potentially replaced by Block chain Technology. BCT plays an important role, in political components namely governmental interference, control leadership and taxation. Also BCT is very useful in Exchange rates of currency market growth and monetary as an economic component. BCT is very helpful in social components namely environmental situation, culture, behavior of the customer and demand. In the same way, BCT has a potential action in modern technologies and tendency [8-9].

BCT permits an emerging set of participants to continue with a secure and alter–proof ledger for all the activities without having a third trusted party. Here, transactions are not actually documented but instead, every participant keeps a provincial copy of the ledger which is a related listing of blocks and they comprise agreed transactions [19]. Nagoorgani and Radha introduced the concept of degree of fuzzy vertex. A crypto currency is nothing a Bitcoin which is a universal payment system and also the initial decentralized digital currency since the system works without a single administrator or central bank. Bitcoins made as a payment for a process called mining and can be exchanged for different currencies. Nakamoto and Satoshi were introduced the concept of Blockchain and applied as an important component of Bitcoin where it act as a public ledger for all the transactions. To solve double-spending problem, Blockchain for the Bitcoin has been an appropriate choice without the help of trusted third part as a central server. Block chain transactions will be done on the interchangeable ledger data saved at every node [42].

A Blockchain network can be seen as a reliable computer whose private states are auditable by anyone. A ledger of transactions may call as a Blockchain. Generally a physical ledger will be maintained by a centralized party whereas in Blockchain is a distributed ledger which locates on the device of every participants. Bitcoins are believable and best used [40, 45]. A Fuzzy Set (FS) can be described mathematically by assigning a value, a grade of membership to every desirable person in the universe of discourse. This grade of membership associates a degree for the participant is either identical or appropriate to the approach performed by FS. A fuzzy subset of FS, X is a function from membership to non-membership and is defined by $\eta : X \to [0,1]$ continuous rather than unexpected. Fuzzy relationships are popular and essential in the fields of computer chains, decision making, neural network, expert systems etc. Direct relationship and also indirect relationship also will be considered in graph theory.

Model of relation is nothing but a graph which is a comfortable way of describing information about the connection between two objects. In graph, points and relations are defined by vertices and edges respectively. While impreciseness exists in the statement of the phenomenon or in the communication or both, fuzzy graph model can be designed for getting an optimized output. Maximizing the Utility of the application is always done by the researchers during the constructing of a model with a key characteristics reliability, complexity and impreciseness. Among these, impreciseness is a considerable one in maximizing the utility of the technique. This situation can be described by fuzzy sets, introduced by Lotfi. A. Zadeh [24, 25].

Zadeh formulated, grade of membership in order to handle with impreciseness. Atannasov introduced intuitionistic fuzzy set by including the grade of non-membership in FS as a separate element. Samarandache introduced Neutrosophic set (NS) by finding the membership degree of indeterminacy, it can be viewed from the logical point of view as a self-ruling component to handle with uncertain, undetermined and unpredictable data which are exist in the real world problem. The NSs are defined by the membership functions of truth, indeterminacy and falsity whose values take from the real standard interval. Wang et al. proposed the theoretical concept of single-valued Neutrosophic sets (SVNS) and Interval valued Neutrosophic Sets (IVNSs) as well [26-34].

If uncertainty exists in vertices or edges set or both then the structure turns into a fuzzy graph. It can be established by taking the set of vertices and edges as FS, in the same way one can model any other types of fuzzy graphs [21-15, 32].Graph theory defines the relationship between various individuals and has got many number of applications in different fields namely database theory, modern discipline and technology, neural networks, data scooping cluster analysis, knowledge systems image capturing and control theory. Handling Indeterminacy on the object or edge or both cannot be handled fuzzy graphs and hence Neutrosophic graphs have been introduced. [44, 47-48].

A new perspective for neutrosophic theory and its applications also proposed [49]. There are many methods have been proposed under single valued neutrosophic, interval valued neutrosophic and neutrosophic environments by colloborating with other methods such as TOPSIS, DEMATEL, VIKOR. Also all these hybrid and extension methods applied in the process of decision making. Further, NS-cross entropy, hyperbolic sine similarity measure, hybrid binary algorithm similarity measure method and single-valued co-neutrosophic graphs play an important role in decision making. In fuzzy graph all the edges are represented by fuzzy numbers and that may be interval valued fuzzy number also. Whereas in neutrososphic graph the edges are represented by single valued neutrosophic numbers [50-62].

The remaining part of the paper is organized as follows. In section 2, review of literature is presented. In section 3, basic concepts related to the presented work is given. In section 4, Blockchain single valued and interval valued neutroosphic graphs are proposed and applied for Bitcoin transaction. Also degree, total degree, minimum and maximum degree have been found. In section 5, qualitative analysis has been done with the limitations and advantages of various types of graphs. In section 6, conclusion of the paper is given with the future work.

2. Review of Literature

[Yeh and Bang 1] proposed fuzzy relations, fuzzy graphs and applied them in cluster analysis. [Satoshi 2] presented a solution for the problem of double-sending using a peer-to-peer network. [Leroy 3] portrayed the evolution and proof of linguistic care of an accumulator back-end. [Dey et al.4] have done a vertex colouring of a fuzzy graph. [Dey et al. 5] applied the concept of fuzzy graph in light control in traffic control management. [Ober et al. 6] proposed a model and obscurity of the Bitcoin transaction graph. [Decker and Wattenhofer 7] examined about knowledge reproduction in the network of Bitcoin. [Fleder et al. 8] linked bit coin public keys to real people and commented about the public transaction graph and hence done a graph analysis scheme to find and compiled activity of known as well as unknown users.

[Stanfill and Wholey 9] proposed a transactional graph on the basis of computation with error management. [Ye 10] proposed aggregation operators under simplified neutrosophic environment and applied them in a decision making problem. [Biswas et al. 11] introduced a new methodology for dealing unknown weight information and applied in a decision making problem. [Biswas et al. 12] proposed grey relational analysis based on entropy under single valued neutrosophic setting and applied in a decision making process with multi attribute.

[Mondal and Pramanik 13] introduced a model for clay-brick selection based on grey relational analysis for neutrosophic decision making. [Mondal and Pramanik 14] proposed neutrosophic tangent similarity measure and applied in multiple attribute decision making. [Biswas et al. 15] introduced cosine similarity measure with trapezoidal fuzzy neutrosophic numbers and applied in a decision making problem. [Broumi et al. 16] introduced an extended TOPSIS methodology using interval neutrosophic uncertain linguistic variables. [Greaves and Au 17] investigated the prognostic power of Blockchain network using lineaments on the future price of Bitcoin. [Pilkington 18] clarified the main ethics behind block chain technique and few of its application of cutting edge.

[Bonneau et al. 19] Analyzed invisibility problems in Bitcoin and contribute an evaluation plan for private- enlarging proposals and contributed a new intuition on language disintermediation protocols. [Smarandache and Pramanik 20] introduced a new direction for neutrosophic theory and applications. [Biswas et al. 21] proposed TOPSIS methodology under single-valued neutrosophic setting for multi-attribute group decision making. [Biswas et al. 22] proposed aggregation operators for triangular fuzzy neutrosophic set information and used for a decision making problem. [Biswas et al. 23] introduced a ranking method based on value and ambiguity index using single-valued trapezoidal neutrosophic numbers and its application to decision making problem. [Eyal et al. 24] designed a block chain protocol called Bitcoin –next generation. [Broumi et al. 25] introduced operational laws on interval valued neutrosophic graphs.

[Broumi et al. 26] proposed the formulas to find degree, size and order of a single valued neutrosophic graphs. [Pramanik et al. 27] proposed hybrid similarity measures under neutrosophic environment and applied them in decision making problem. [Dalapati et al. 28] introduced IN-cross entropy for interval neutrosophic set environment and applied in multi attribute group decision making process. [Broumi et al. 29] proposed uniform single valued neutrosophic graphs. [Cocco et al. 30] paid attention at the threats and opportunities of carrying out Blockchain mechanism across banking. [Jeoseph et al. 31] reviewed the approval and future use of block chain technology.

[Chan and Olmsted 32] proposed a design for prevailing transactions from Ethereum into a graph database namely leveraging graph computer. [Illgner 33] proposed a blockchian to fix all Blockchains. [Swan and Filippi 34] explained about the philosophy of Bockchain technology. [Banuelos et al. 35] proposed an advanced method to implement business developments on top of commodity Blockchain technology. [Dinh et al. 36] surveyed the case of the art targeting on private Blockchain where the parties are authenticated. [Desai 37] analysed industry application and have legal perspectives for Blockchain technology. [Jain et al. 38] analyzed asymmetrical associations using fuzzy graph and finding hidden connections in Facebook. [Raikwar et al. 39] proposed a framework of Blockchain for insurance processes.

[Ramkumar 40] proposed Blockchain integrity framework. [Hill 41] presented a review on Blockchain [Arockiaraj and Charumathi 42] introduced the Blockchain fuzzy graph and its concepts and properties. [Halaburda 43] answered for the question, Blockchain transformation without the Blockchain. [Gupta and Sadoghi 44] explained about Blockchain process in detailed manner. [Ramkumar 45] accomplished large scale measure in Blockchian. [Asraf et al. 46] proposed Dombi fuzzy graphs. [Marapureddy 47] introduced fuzzy graph for the semi group. [Quek et al. 48] introduced a few of the results for complex Neutrosophic sets on graph theory. [Smarandache and Pramanik 49] introduced a new perspective to neutrosophic theory and its applications.

[Basset et al. 50] proposed an extended neutrosophic AHP-SWOT analysis for critical planning and decision making. [Basset et al. 51] proposed association rule mining algorithm to analyze big data. [Basset et al. 52] introduced Group ANP-TOPSIS framework under hybrid neutrosophic setting for supplier selection problem. [Basset et al. 53] presented a hybrid approach of neutrosophic sets and DEMATEL method to enhance the criteria for supplier selection. The same authors presented a series of article[63-69]. ([Pramanik et al. 54] proposed NS-cross entropy under single valued neutrosophic environment and applied in a MAGDM problem. [Biswas et al. 55] proposed neutrosophic TOPSIS method and solved group decision making problem.

[Pramanik and Mallick 56] proposed VIKOR method using trapezoidal neutrosophic numbers and solved MAGDM problem using proposed method. [Biswas et al. 57] solved MADM problem by introducing distance measure using interval trapezoidal neutrosophic numbers. [Biswas et al. 58] introduced TOPSIS strategy for solving MADM problem with trapezoidal numbers. [Biswas et al. 59] solved MAGDM problem using expected value of neutrosophic trapezoidal numbers. [Mondal et al. 60] introduced hyperbolic sine similarity measure based MADM strategy under single valued neutrosophic environment. [Mondal et al. 61] proposed hybrid binary algorithm similarity measure under single valued neutrosophic set assessments for MAGDM problem. [Dhavaseelan et al. 62] proposed single-valued co-neutrosophic graphs.

The above literature survey motivated to propose Blockchain single and interval valued Neutrosophic Graphs and applied them in Blockchain technology using Bitcoins.

3. Basic Concepts

Some basic concepts needed for the proposed concepts and their application, are listed below.

3.1 Bitcoins [40]

Bitcoin is the digital currency and worldwide payment system and are believable and best used when,

- There are a series of transaction
- Need to be recorded
- Need to be verified with respect to purity of the information and the order of the events.

3.2 Blockchain [42]

A Blockchain is a network and can be seen as a reliable computer whose private states are auditable by anyone. It can also be defined as follows.

- Cryptographic approach for modeling an unalterable append-only public ledger
- It includes a methodology for obtaining an open general agreement on each entry
- Ledger entries are mappings of the states of processes by the Blockchain network.

Uses of Blockchain

- A uniform approach to execute a variety of application processes
- Reliable and efficient Low upward approaches for stakeholders/users namely states with query application and audit correctness of changes of states.

3.3 Graph [46]

A mathematical system $G=(V,E)$ is called a graph, where a vertex set is $V=V(G)$ and an edge set is $E=E(G)$. In this paper, undirected graph has been considered and hence every edge is considered as an unordered pair of different vertices.

3.4 Fuzzy Graph [47]

Consider a non-empty finite set V , λ be a fuzzy subsets on V and δ be a fuzzy subsets on $\mathrm{V}\times\mathrm{V}$ .A fuzzy graph is a pair $\mathrm{G}=(\lambda,\delta)$ over the set V if $\delta(a,b)\le\min\{\lambda(a),\lambda(b)\}$ for all $(a,b)\in\mathrm{V}\times\mathrm{V}$ where λ is a fuzzy vertex and δ is a fuzzy edge. Where:

1. A fuzzy subset is a mapping $\lambda:\mathrm{V}\to[0,1]$ of V .
2. A fuzzy relation is a mapping $\delta:\mathrm{V}\times\mathrm{V}\to[0,1]$ on λ of V if $\delta(a,b)\le\min\{\lambda(a),\lambda(b)\}$
3. If $\delta(a,b)=\min\{\lambda(a),\lambda(b)\}$ then G is a strong fuzzy graph.

3.5 Blockchain Fuzzy Graph (BCFG) [42]

The pair $\mathrm{G}=(\lambda,\delta)$ is a BCFG, where λ is a fuzzy vertex set and δ is symmetric on λ such that $\delta(a,b)\le\min\{\lambda(a),\lambda(b)\}$, $\forall\ a,b\in\mathrm{V}$ with the following criterion.

1. If $i\neq j$ then $\sum\left[\delta(a_i,b_j)\le\min\left(\lambda(a_i),\lambda(b_j)\right)\right]=1$
2. If $i\neq j$ then $\sum\left[\delta(a_i,b_j)\le\max\left(\lambda(a_i),\lambda(b_j)\right)\right]=1$
3. If $i=j$ then $\sum\left[\delta(a_i,b_j)\le\min\left(\lambda(a_i),\lambda(b_j)\right)\right]=0$

3.6 Single Valued Neutrosophic Graph (SVNG) [26]

A pair $G=(R,S)$ is SVNG with elemental set V . Where:

1. Grade of truth, indeterminacy and falsity memberships of $a_i \in \mathbf{V}$ are defined by $T_R : \mathrm{V} \to [0,1]$, $I_R : \mathrm{V} \to [0,1]$ and $F_R : \mathrm{V} \to [0,1]$ respectively and $0 \le T_R(a_i) + I_R(a_i) + F_R(a_i) \le 3, \forall a_i \in \mathbf{V}, i = 1,2,3,\ldots,n$
2. The above three memberships of the edge $(a_i, b_j) \in \mathbf{E}$ are denoted by $T_S : \mathrm{E} \subseteq \mathrm{V} \times \mathrm{V} \to [0,1]$, $I_S : \mathrm{E} \subseteq \mathrm{V} \times \mathrm{V} \to [0,1]$ and $F_S : \mathrm{E} \subseteq \mathrm{V} \times \mathrm{V} \to [0,1]$ respectively and are defined by
   - $T_S(\{a_i, b_j\}) \le \min[T_R(a_i), T_R(b_j)]$
   - $I_S(\{a_i, b_j\}) \ge \max[I_R(a_i), I_R(b_j)]$
   - $F_S(\{a_i, b_j\}) \ge \max[F_R(a_i), F_R(b_j)]$

where $0 \le T_S(\{a_i, b_j\}) + I_S(\{a_i, b_j\}) + F_S(\{a_i, b_j\}) \le 3, \forall \{a_i, b_j\} \in \mathbf{E}(i, j = 1,2,\ldots,n)$.

Also Rand S are the single valued Neutrosophic vertex and edge set of V and $\mathbf{E}$ respectively. S is symmetric on R.

3.7 Interval Valued Neutrosophic Graph (IVNG) [25]

A pair $G=(R,S)$ is IVNG, where $R = \langle [T_R^L, T_R^U], [I_R^L, I_R^U], [F_R^L, F_R^U] \rangle$, is an IVN set on V and $S = \langle [T_S^L, T_S^U], [I_S^L, I_S^U], [F_S^L, F_S^U] \rangle$ is an IVN edge set on $\mathbf{E}$ satisfying the following conditions:

1. Here the lower and upper memberships functions of $a_i \in \mathbf{V}$ are defined by $T_R^L : \mathrm{V} \to [0,1]$, $T_R^U : \mathrm{V} \to [0,1]$, $I_R^L : \mathrm{V} \to [0,1]$, $I_R^U : \mathrm{V} \to [0,1]$ and $F_R^L : \mathrm{V} \to [0,1]$, $F_R^U : \mathrm{V} \to [0,1]$ respectively and $0 \le T_R(a_i) + I_P(a_i) + F_P(a_i) \le 3, \forall a_i \in \mathbf{V}, i = 1,2,3,\ldots,n$
2. And the same for edge $(a_i, b_j) \in \mathbf{E}$ are denoted by $T_S^L : \mathrm{V} \times \mathrm{V} \to [0,1]$, $T_S^U : \mathrm{V} \times \mathrm{V} \to [0,1]$ $I_S^L : \mathrm{V} \times \mathrm{V} \to [0,1]$, $I_S^U : \mathrm{V} \times \mathrm{V} \to [0,1]$ and $F_S^L : \mathrm{V} \times \mathrm{V} \to [0,1]$, $F_S^U : \mathrm{V} \times \mathrm{V} \to [0,1]$ respectively and are defined by
   - $T_S^L(\{a_i, b_j\}) \le \min[T_R^L(a_i), T_R^L(b_j)]$
   - $T_S^U(\{a_i, b_j\}) \le \min[T_R^U(a_i), T_R^U(b_j)]$
   - $I_S^L(\{a_i, b_j\}) \ge \max[I_R^L(a_i), I_R^L(b_j)]$
   - $I_S^U(\{a_i, b_j\}) \ge \max[I_R^U(a_i), I_R^U(b_j)]$
   - $F_S^L(\{a_i, b_j\}) \ge \max[F_R^L(a_i), F_R^L(b_j)]$
   - $F_S^U(\{a_i, b_j\}) \ge \max[F_R^U(a_i), F_R^U(b_j)]$

where $0 \le T_S(\{a_i, b_j\}) + I_S(\{a_i, b_j\}) + F_S(\{a_i, b_j\}) \le 3, \forall \{a_i, b_j\} \in \mathbf{E}(i, j = 1,2,\ldots,n)$.

Also R and S are the interval valued Neutrosophic vertex and edge set of V and $\mathbf{E}$ respectively. S is symmetric on R.

4. Proposed Concepts

In this section, Blockchian single valued neutrosophic graph is proposed and applied in Blockchain technology with Bitcoin transaction.

4.1 Blockchain Single Valued Neutrosophic Graph (BCSVNG)

A pair $G=(R,S)$ is BCSVNG with elemental set V . Where:

1. $T_R : \mathrm{V} \to [0,1]$, $I_R : \mathrm{V} \to [0,1]$ and $F_R : \mathrm{V} \to [0,1]$ and $0 \le T_R(x_i) + I_R(x_i) + F_R(x_i) \le 3, \forall x_i \in \mathbf{V}, i = 1,2,3,\ldots,n$
2. $T_S : \mathrm{E} \subseteq \mathrm{V} \times \mathrm{V} \to [0,1]$, $I_S : \mathrm{E} \subseteq \mathrm{V} \times \mathrm{V} \to [0,1]$ and $F_S : \mathrm{E} \subseteq \mathrm{V} \times \mathrm{V} \to [0,1]$ are defined by

Case (i): If $i \neq j$ then

$$\sum\left[T_S\left(x_i, y_j\right) \leq \min\left[T_R\left(x_i\right), T_R\left(y_j\right)\right]\right] = 1$$
$$\sum\left[I_S\left(x_i, y_j\right) \geq \max\left[I_R\left(x_i\right), I_R\left(y_j\right)\right]\right] = 1$$
$$\sum\left[F_S\left(x_i, y_j\right) \geq \max\left[F_R\left(x_i\right), F_R\left(y_j\right)\right]\right] = 1$$

Case (ii): If $i = j$ then the above values are 0.

Where, $0 \leq T_S\left(\left\{x_i, y_j\right\}\right) + I_S\left(\left\{x_i, y_j\right\}\right) + F_S\left(\left\{x_i, y_j\right\}\right) \leq 3, \forall \left\{x_i, y_j\right\} \in \mathbf{E}\left(i, j = 1, 2, \ldots, n\right)$

Also R is a single valued Neutrosophic vertex of V and S is a single valued Neutrosophic edge set of $\mathbf{E}$. S is a symmetric single valued Neutrosophic relation on R.

4.1.1 Blockchain Single Valued Neutrosophic Graph in Bitcoin Transaction

Let us consider there are 4 persons in the Blockchain and everyone is doing a transaction using Bitcoin and they are saving 40% and investing the remaining 60% in Bitcoin.

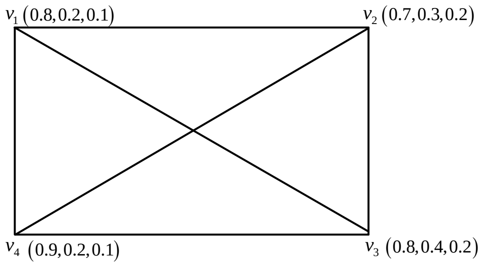

Figure 1: BCSVNG

Party 1: investing 20 lakhs and doing 3 transactions
Party 2: investing 15 lakhs and doing 3 transactions
Party 3: investing 10 lakhs and doing 3 transactions
Party 4: investing 5.5 lakhs and doing 3 transactions
For example, assume that the party-1 (v_1) has the total amount of 20 lakhs, from this he is saving 40% and invest the remaining 60% as Bitcoins for his crypto currencies.

The following are the transactions of Party-1:

Transaction 1: Party-1 to Party-2 : (v_1 to v_2)

$\left(0.7, 0.3, 0.2\right) \times 12,00,000$

$= \left\langle\left(1-\left(1-T_R\right)^k\right), \left(1-\left(1-T_R\right)^k\right), \left(1-\left(1-T_R\right)^k\right)\right\rangle$ **,** $k > 0$ (any arbitrary number) **[10]**

$= \left\langle\left(1-\left(1-0.7\right)^{12,00,000}\right), \left(1-\left(1-0.3\right)^{12,00,000}\right), \left(1-\left(1-0.2\right)^{12,00,000}\right)\right\rangle$

$= \left\langle\left(1-\left(1-0.7\right)^{12,00,000}\right), \left(1-\left(1-0.3\right)^{12,00,000}\right), \left(1-\left(1-0.2\right)^{12,00,000}\right)\right\rangle$

$= \left\langle 1, 1, 1\right\rangle$

Similarly for other transactions namely

Transaction 2: Party-1 to Party-3 : (v_1 to v_3)
Transaction 3: Party-1 to Party-4 : (v_1 to v_4)

All the possible transaction are listed out in Table 1 with the sum value of each row.

			$(0.8,0.2,0.1)$	$(0.7,0.3,0.2)$	$(0.8,0.4,0.2)$	$(0.9,0.2,0.1)$	sum
			v_1	v_2	v_3	v_4	
	$(0.8,0.2,0.1)$	v_1	0	$(0.4,0.38,0.3)$	$(0.3,0.41,0.4)$	$(0.3,0.21,0.3)$	$(1,1,1)$
	$(0.7,0.3,0.2)$	v_2	$(0.4,0.38,0.3)$	0	$(0.4,0.37,0.3)$	$(0.2,0.25,0.4)$	$(1,1,1)$
	$(0.8,0.4,0.2)$	v_3	$(0.3,0.41,0.4)$	$(0.4,0.37,0.3)$	0	$(0.3,0.54,0.3)$	$(1,1,1)$
	$(0.9,0.2,0.1)$	v_4	$(0.3,0.21,0.3)$	$(0.2,0.25,0.4)$	$(0.3,0.54,0.3)$	0	$(1,1,1)$
sum			$(1,1,1)$	$(1,1,1)$	$(1,1,1)$	$(1,1,1)$	

Table 1: Transaction Table for BCSVNG

Where sum= $\sum(v_i, v_j)$

4.1.2 Properties of Blockchain Single Valued Neutrosophic Graph

In this section, degree, total degree, minimum and maximum degrees are found for Blockchain Single Valued Neutrosophic Graph.

(i). Degree of Single Valued Neutrosophic Graph (SVNG)

$d(v_1) = (d_T(v_1), d_I(v_1), d_F(v_1))$ **[26]**

$= (1,1,1)$

Where, $d_T(v_1) = T_S(v_1,v_2) + T_S(v_1,v_3) + T_S(v_1,v_4) = 0.4 + 0.3 + 0.3 = 1$

$d_I(v_1) = I_S(v_1,v_2) + I_S(v_1,v_3) + I_S(v_1,v_4) = 0.38 + 0.41 + 0.21 = 1$

$d_F(v_1) = F_S(v_1,v_2) + F_S(v_1,v_3) + F_S(v_1,v_4) = 0.3 + 0.4 + 0.3 = 1$

Similarly $d(v_2) = (d_T(v_2), d_I(v_2), d_F(v_2)) = (1,1,1)$

$d(v_3) = (d_T(v_3), d_I(v_3), d_F(v_3)) = (1,1,1)$

$d(v_4) = (d_T(v_4), d_4(v_4), d_F(v_4)) = (1,1,1)$

And $\sum d(v_1) = \left(2\sum_{v_1 \neq v_j} T_S(v_1,v_j), 2\sum_{v_1 \neq v_j} I_S(v_1,v_j), 2\sum_{v_1 \neq v_j} F_S(v_1,v_j) \right)$

$= (2(1), 2(1), 2(1)) = (2,2,2)$

(ii). Total Degree of SVNG

$td(v_i) = (td_T(v_i), td_I(v_i), td_F(v_i))$ **[26]**

Where $td_T(v_i) = \sum T_S(v_i,v_j) + T_R(v_i)$

$td_T(v_1) = \sum T_S(v_1,v_j) + T_R(v_1) = 1 + 0.8 = 1.8$

$td_I(v_1) = \sum I_S(v_1,v_j) + I_R(v_1) = 1 + 0.2 = 1.2$

$td_F(v_1) = \sum F_S(v_1,v_j) + F_R(v_1) = 1 + 0.1 = 1.1$

Therefore, $td(v_1) = (td_T(v_1), td_I(v_1), td_F(v_1)) = (1.8,1.2,1.1)$

Similarly, $td(v_2) = (td_T(v_2), td_I(v_2), td_F(v_2)) = (1.7,1.3,1.2)$

$$td(v_3)=\left(td_T(v_3),td_I(v_3),td_F(v_3)\right)=(1.8,1.4,1.2)$$
$$td(v_4)=\left(td_T(v_4),td_I(v_4),td_F(v_4)\right)=(1.9,1.2,1.1)$$

(iii). Minimum degree of SVNG

It is $\xi(\mathrm{G})=\left(\xi_T(\mathrm{G}),\xi_I(\mathrm{G}),\xi_F(\mathrm{G})\right)$, where
$\xi_T(\mathrm{G})=\min\left\{d_T(v)/v\in\mathrm{V}\right\}$, $\xi_I(\mathrm{G})=\min\left\{d_I(v)/v\in\mathrm{V}\right\}$ and $\xi_F(\mathrm{G})=\min\left\{d_F(v)/v\in\mathrm{V}\right\}$ **[15]**

For the Fig. 1,
$\xi_T(\mathrm{G})=\min\left\{d_T(v)/v\in\mathrm{V}\right\}=1$
$\xi_I(\mathrm{G})=\min\left\{d_I(v)/v\in\mathrm{V}\right\}=1$
$\xi_F(\mathrm{G})=\min\left\{d_F(v)/v\in\mathrm{V}\right\}=1$

(iv). Maximum degree of SVNG

It is defined by $\eta(\mathrm{G})=\left(\eta_T(\mathrm{G}),\eta_I(\mathrm{G}),\eta_F(\mathrm{G})\right)$, where
$\eta_T(\mathrm{G})=\max\left\{d_T(v)/v\in\mathrm{V}\right\},\eta_T(\mathrm{G})=\max\left\{d_T(v)/v\in\mathrm{V}\right\},\eta_F(\mathrm{G})=\max\left\{d_F(v)/v\in\mathrm{V}\right\}$ **[26]**
For the Fig. 1,
$\eta_T(\mathrm{G})=\max\left\{d_T(v)/v\in\mathrm{V}\right\}=1$
$\eta_I(\mathrm{G})=\max\left\{d_I(v)/v\in\mathrm{V}\right\}=1$
$\eta_F(\mathrm{G})=\max\left\{d_F(v)/v\in\mathrm{V}\right\}=1$
For the Fig. 1,
$\eta_T(\mathrm{G})=\max\left\{d_T(v)/v\in\mathrm{V}\right\}=\eta_T(\mathrm{G})=\max\left\{d_T(v)/v\in\mathrm{V}\right\}=\eta_F(\mathrm{G})=\max\left\{d_F(v)/v\in\mathrm{V}\right\}=1$

4.2 Blockchain Interval Valued Neutrosophic Graph (BCIVNG)

A pair $\mathrm{G}=(R,S)$ is BCIVNG, where $R=\left\langle\left[T_R^L,T_R^U\right],\left[I_R^L,I_R^U\right],\left[F_R^L,F_R^U\right]\right\rangle$, is an IVN set on V and $S=\left\langle\left[T_S^L,T_S^U\right],\left[I_S^L,I_S^U\right],\left[F_S^L,F_S^U\right]\right\rangle$ is an IVN edge set on $\mathbf{E}$ satisfying conditions 1 and 2 as in the definition of IVNG and with the following criterions.

Case (i): If $i\neq j$ then

$$\sum\left[T_S^L\left(x_i,y_j\right)\leq\min\left[T_R^L(x_i),T_R^L\left(y_j\right)\right]\right]=0.5$$

$$\sum\left[T_S^U\left(x_i,y_j\right)\leq\min\left[T_R^U(x_i),T_R^U\left(y_j\right)\right]\right]=0.5$$

$$\sum\left[I_S^L\left(x_i,y_j\right)\geq\max\left[I_R^L(x_i),I_R^L\left(y_j\right)\right]\right]=0.5$$

$$\sum\left[I_S^U\left(x_i,y_j\right)\geq\max\left[I_R^U(x_i),I_R^U\left(y_j\right)\right]\right]=0.5$$

$$\sum\left[F_S^L\left(x_i,y_j\right)\geq\max\left[F_R^L(x_i),F_R^L\left(y_j\right)\right]\right]=0.5$$

$$\sum\left[F_S^U\left(x_i,y_j\right)\geq\max\left[F_R^U(x_i),F_R^U\left(y_j\right)\right]\right]=0.5$$

Case (ii): If $i=j$ then the above six values are 0.
Where $0\leq T_S\left(\left\{x_i,y_j\right\}\right)+I_S\left(\left\{x_i,y_j\right\}\right)+F_S\left(\left\{x_i,y_j\right\}\right)\leq 3,\forall\left\{x_i,y_j\right\}\in\mathbf{E}(i,j=1,2,...,n)$

Also R is an interval valued Neutrosophic vertex of V and S is an interval valued Neutrosophic edge set of $\mathbf{E}$. S is a symmetric interval valued Neutrosophic relation on R.

4.2.1 Blockchain Interval Valued Neutrosophic Graph in Bitcoin Transaction

Let us consider there are 3 persons in the Blockchain and everyone is doing a transaction using Bitcoin and they are saving 40% and investing the remaining 60% in Bitcoin.

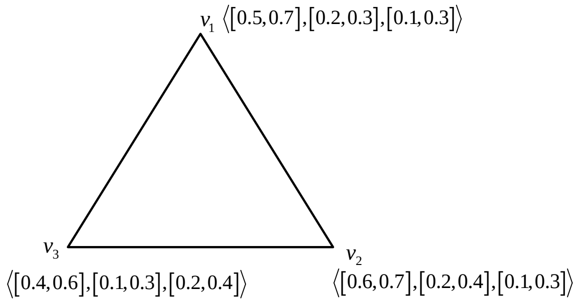

Figure 2: BCIVNG

Party 1: investing 20 lakhs and doing 2 transactions
Party 2: investing 15 lakhs and doing 2 transactions
Party 3: investing 10 lakhs and doing 2 transactions

For example, assume that the party-1 (v_1) has the total amount of 20 lakhs, from this he is saving 40% and invest the remaining 60% as Bitcoins for his crypto currencies.

The following are the transactions of Party-1:

Transaction 1: Party-1 to Party-2: (v_1 to v_2)

$$\langle[0.6,0.7],[0.2,0.4],[0.1,0.3]\rangle\times 12,00,000$$

$$=\left\{\left[1-\left(1-T_R^L\right)^k,1-\left(1-T_R^U\right)^k\right],\left[\left(I_R^L\right)^k,\left(I_R^U\right)^k\right],\left[\left(F_R^L\right)^k,\left(F_R^U\right)^k\right]\right\}$$ **[25]**

$$=\left\{\left[1-(1-0.6)^{12,00,000},1-(1-0.7)^{12,00,000}\right],\left[(0.2)^{12,00,000},(0.4)^{12,00,000}\right],\left[(0.1)^{12,00,000},(0.3)^{12,00,000}\right]\right\}$$

$$=\left\{\left[1-(0.4)^{12,00,000},1-(0.3)^{12,00,000}\right],\left[(0.2)^{12,00,000},(0.4)^{12,00,000}\right],\left[(0.1)^{12,00,000},(0.3)^{12,00,000}\right]\right\}$$

$$=\{[1-0,1-0],[0,0],[0,0]\}$$

$$=\{[1,1],[0,0],[0,0]\}$$

Transaction 2: Party-1 to Party-3: (v_1 to v_3) $=\{[1,1],[0,0],[0,0]\}$

Table 1represent all possible transactions from one vertex to all other verteices. Here edge represents the transaction and the vertex represents the parties.

		$\langle[0.5,0.7],[0.2,0.3],[0.1,0.3]\rangle$	$\langle[0.6,0.7],[0.2,0.4],[0.1,0.3]\rangle$	$\langle[0.4,0.6],[0.1,0.3],[0.2,0.4]\rangle$	$\sum(v_i,v_j)$
		v_1	v_2	v_3	
$\langle[0.5,0.7],[0.2,0.3],[0.1,0.3]\rangle$	v_1	0	$\langle[0.217,0.283],[0.211,0.289],[0.302,0.313]\rangle$	$\langle[0.281,0.282],[0.198,0.199],[0.208,0.209]\rangle$	$(1,1,1)$
$\langle[0.6,0.7],[0.2,0.4],[0.1,0.3]\rangle$	v_2	$\langle[0.217,0.283],[0.211,0.289],[0.302,0.313]\rangle$	0	$\langle[0.28,0.283],[0.197,0.198],[0.208,0.209]\rangle$	$(1,1,1)$
$\langle[0.4,0.6],[0.1,0.3],[0.2,0.4]\rangle$	v_3	$\langle[0.281,0.282],[0.198,0.199],[0.208,0.209]\rangle$	$\langle[0.217,0.283],[0.302,0.313],[0.292,0.302]\rangle$	0	$(1,1,1)$
$\sum(v_i,v_j)$		$(1,1,1)$	$(1,1,1)$	$(1,1,1)$	

Table 2: Transaction Table for BCIVNG

From table 1 and table 2 it is observed that sum of all single /interval valued Neutrosophic edges of a particular Neutrosophic vertex is equal to (1, 1, 1). Hence the proposed method is an optimized one to deal indeterminacy of the data in Bitcoin transaction.

5. Comparative Analysis (Qualitative)

Blockchain approach has been applied in various fields as a growing technique. Here the advantages and limitations are listed out for Blockchian crisp, fuzzy and Neutrosophic graphs. This analysis will be very useful to understand the concept of Blockchian under different environments.

Type of Blockchain Graph	Advantages	Limitations
Blockchian Crisp Graph	• Faster Process with purity and detectable • Process clarity • Data will be permanent	• Unable to handle uncertainties • Size of the network will decide the security level
Blockchain Fuzzy Graph	• Can handle uncertainty exists in the vertex and edge sets • Invariable data	• Incapable to handle indeterminacy of the data and interval data • Large network will give more level of security
Blockchain Interval Valued Fuzzy Graph	• Can able to deal with data in terms of range	• Inadequate to handle undetermined data
Blockchain Single Valued Neutrosophic Graph	• Able to handle indeterminacy of the data	• Unfit to handle interval data
Blockchain Interval Valued	• Capable to handle interval	• Unsuited to handle criterion

Neutrosophic Graph	data as the participant's decision is always lie in a range.	insufficient information of the weights.

6. Conclusion

Reliability and assurance of the dealing is very important for any business transaction. Blockchain technology is such a technology and recently it is widely applied in many fields. In any field uncertainty is unavoidable one as the human behavior always uncertainty in nature. Also indeterminacy does not deal in any area field of mathematics whereas Neutrosophic set deals indeterminacy and hence an optimized solution can be obtain for any problem. In this paper Blockchain network has been used in terms of Bitcoin transaction where the vertex and edges have been considered as single and interval valued Neutrosophic sets. Also the degree, total degree, minimum and maximum degree have been found for the proposed Blockchain single valued Neutrosophic graph. In addition to this, contingent study has been done for various types of Blockchain graphs.

Notes

Compliance with Ethical Standards

Conflict of Interest
The authors declare that they have no conflict of interest

Ethical Approval
The article does not contain any studies with human participants or animal performed by any of the authors

Informed Consent
Informed consent was obtained from all individual participants included in the study

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Neutrosophic General Finite Automata

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Abstract: The constructions of finite switchboard state automata is known to be an extension of finite automata in the view of commutative and switching automata. In this research, the idea of a neutrosophic is incorporated in the general fuzzy finite automata and general fuzzy finite switchboard automata to introduce neutrosophic general finite automata and neutrosophic general finite switchboard automata. Moreover, we define the notion of the neutrosophic subsystem and strong neutrosophic subsystem for both structures. We also establish the relationship between the neutrosophic subsystem and neutrosophic strong subsystem.

Keywords: Neutrosophic set, General fuzzy automata; switchboard; subsystems.

1 Introduction

It is well-known that the simplest and most important type of automata is finite automata. After the introduction of fuzzy set theory by [47] Zadeh in 1965, the first mathematical formulation of fuzzy automata was proposed by[46] Wee in 1967, considered as a generalization of fuzzy automata theory. Consequently, numerous works have been contributed towards the generalization of finite automata by many authors such as Cao and Ezawac [9], Jin et al [18], Jun [20], Li and Qiu [27], Qiu [34], Sato and Kuroki [36], Srivastava and Tiwari [41], Santos [35], Jun and Kavikumar [21], Kavikumar et al, [22, 23, 24] especially the simplest one by Mordeson and Malik [29]. In 2005, the theory of general fuzzy automata was firstly proposed by Doostfatemeh and Kermer [11] which is used to resolve the problem of assigning membership values to active states of the fuzzy automaton and its multi-membership. Subsequently, as a generalization, the concept of intuitionistic general fuzzy automata has been introduced and studied by Shamsizadeh and Zahedi [37], while Abolpour and Zahedi [6] proposed general fuzzy automata theory based on the complete residuated lattice-valued. As a further extension, Kavikumar et al [25] studied the notions of general fuzzy switchboard automata. For more details see the recent literature as [5, 12, 13, 14, 15, 16, 17].

The notions of neutrosophic sets was proposed by Smarandache [38, 39], generalizing the existing ordinary fuzzy sets, intuitionistic fuzzy sets and interval-valued fuzzy set in which each element of the universe has the degrees of truth, indeterminacy and falsity and the membership values are lies in $]0^-, 1^+[$, the nonstandard unit interval [40] it is an extension from standard interval [0,1]. It has been shown that fuzzy sets provides limited platform for computational complexity but neutrosophic sets is suitable for it. The neutrosophic sets is an appropriate mechanism for interpreting real-life philosophical problems but not for scientific problems since it is difficult to consolidate. In neutrosophic sets, the degree of indeterminacy can be defined independently since it is quantified explicitly which led to different from intuitionistic fuzzy sets. Single-valued neutrosophic set and interval neutrosophic set are the subclasses of the neutrosophic sets which was introduced by Wang et al. [44, 45] in order to examine kind of real-life and scientific problems. The applications of fuzzy sets have been found very useful in the domain of mathematics and elsewhere. A number of authors have been applied the concept of the neutrosophic set to many other structures especially in algebra [19, 28], decision-making [1, 2, 10, 30], medical [3, 4, 8], water quality management [33] and traffic control management [31, 32].

1.1 Motivation

In view of exploiting neutrosophic sets, Tahir et al. [43] introduced and studied the concept of single valued Neutrosophic finite state machine and switchboard state machine. Moreover, the fuzzy finite switchboard state machine is introduced into the context of the interval neutrosophic set in [42]. However, the realm of general structure of fuzzy automata in the neutrosophic environment has not been studied yet in the literature so far. Hence, it is still open to many possibilities for innovative research work especially in the context of neutrosophic general automata and its switchboard automata. The fundamental advantage of incorporating neutrosophic sets into general fuzzy automata is the ability to bring indeterminacy membership and nonmembership in each transitions and active states which help us to overcome the uncertain situation at the time of predicting next active state. Motivated by the work of [11], [36] and [38] the concept of neutrosophic general automata and neutrosophic general switchboard automata are introduced in this paper.

1.2 Main Contribution

The purpose of this paper is to introduce the primary algebraic structure of neutrosophic general finite automata and neutrosophic switchboard finite automata. The subsystem and strong subsystem of neutrosophic general finite automata and neutrosophic general finite switchboard f automata are exhibited. The relationship between these subsystems have been discussed and the characterizations of switching and commutative are discussed in the neutrosophic backdrop. We prove that the implication of a strong subsystem is a subsystem of neutrosophic general finite automata. The remainder of this paper is organised as follows. Section 2 provides the results and definitions concerning the general fuzzy automata. Section 3 describes the algebraic properties of the neutrosophic general finite automata. Finally, in section 4, the notion of the neutrosophic general finite switchboard automata is introduced. The paper concludes with Section 5.

2 Preliminaries

"For a nonempty set X, $\tilde{P}(X)$ denotes the set of all fuzzy sets on X.

Definition 2.1. [11] A general fuzzy automaton (GFA) is an eight-tuple machine $\tilde{F} = (Q, \Sigma, \tilde{R}, Z, \tilde{\delta}, \omega, F_1, F_2)$ where

(a) Q is a finite set of states, $Q = \{q_1, q_2, \cdots, q_n\}$,

(b) Σ is a finite set of input symbols, $\Sigma = \{a_1, a_2, \cdots, a_m\}$,

(c) $\tilde{R}$ is the set of fuzzy start states, $\tilde{R} \subseteq \tilde{P}(Q)$,

(d) Z is a finite set of output symbols, $Z = \{b_1, b_2, \cdots, b_k\}$,

(e) $\omega : Q \to Z$ is the non-fuzzy output function,

(f) $F_1 : [0,1] \times [0,1] \to [0,1]$ is the membership assignment function,

(g) $\tilde{\delta} : (Q \times [0,1]) \times \Sigma \times Q \xrightarrow{F_1(\mu,\delta)} [0,1]$ is the augmented transition function,

(h) $F_2 : [0,1]^* \to [0,1]$ is a multi-membership resolution function.

Noted that the function $F_1(\mu, \delta)$ has two parameters μ and δ, where μ is the membership value of a predecessor and δ is the weight of a transition. In this definition, the process that takes place upon the transition from state q_i to q_j on input a_k is represented as:

$$\mu^{t+1}(q_j) = \tilde{\delta}((q_i, \mu^t(q_i)), a_k, q_j) = F_1(\mu^t(q_i), \delta(q_i, a_k, q_j)).$$

This means that the membership value of the state q_j at time $t+1$ is computed by function F_1 using both the membership value of q_i at time t and the weight of the transition. The usual options for the function $F(\mu, \delta)$ are $\max\{\mu, \delta\}, \min\{\mu, \delta\}$ and $(\mu + \delta)/2$. The multi-membership resolution function resolves the multi-membership active states and assigns a single membership value to them.

Let $Q_{act}(t_i)$ be the set of all active states at time $t_i, \forall i \geq 0$. We have $Q_{act}(t_0) = \tilde{R}$,

$$Q_{act}(t_i) = \{(q, \mu^{t_i}(q)) : \exists q' \in Q_{act}(t_{i-1}), \exists\, a \in \Sigma, \delta(q', a, q) \in \Delta\}, \forall i \geq 1.$$

Since $Q_{act}(t_i)$ is a fuzzy set, in order to show that a state q belongs to $Q_{act}(t_i)$ and T is a subset of $Q_{act}(t_i)$, we should write: $q \in Domain(Q_{act}(t_i))$ and $T \subset Domain(Q_{act}(t_i))$. Hereafter, we simply denote them as: $q \in Q_{act}(t_i)$ and $T \subset Q_{act}(t_i)$. The combination of the operations of functions F_1 and F_2 on a multi-membership state q_j leads to the multi-membership resolution algorithm.

Algorithm 2.2. [11] (Multi-membership resolution) If there are several simultaneous transitions to the active state q_j at time $t+1$, the following algorithm will assign a unified membership value to it:

1. Each transition weight $\tilde{\delta}(q_i, a_k, q_j)$ together with $\mu^t(q_i)$, will be processed by the membership assignment function F_1, and will produce a membership value. Call this v_i,

$$v_i = \tilde{\delta}((q_i, \mu^t(q_i)), a_k, q_j) = F_1(\mu^t(q_i), \delta(q_i, a_k, q_j)).$$

2. These membership values are not necessarily equal. Hence, they need to be processed by the multi-membership resolution function F_2.

3. The result produced by F_2 will be assigned as the instantaneous membership value of the active state q_j,

$$\mu^{t+1}(q_j) = F_2{}_{i=1}^{n}[v_i] = F_2{}_{i=1}^{n}[F_1(\mu^t(q_i), \delta(q_i, a_k, q_j))],$$

where

- n is the number of simultaneous transitions to the active state q_j at time $t+1$.
- $\delta(q_i, a_k, q_j)$ is the weight of a transition from q_i to q_j upon input a_k.
- $\mu^t(q_i)$ is the membership value of q_i at time t.
- $\mu^{t+1}(q_j)$ is the final membership value of q_j at time $t+1$.

Definition 2.3. Let $\tilde{F} = (Q, \Sigma, \tilde{R}, Z, \tilde{\delta}, \omega, F_1, F_2)$ be a general fuzzy automaton, which is defined in Definition 2.1. The max-min general fuzzy automata is defined of the form:

$$\tilde{F}^* = (Q, \Sigma, \tilde{R}, Z, \tilde{\delta}^*, \omega, F_1, F_2),$$

where $Q_{act} = \{Q_{act}(t_0), Q_{act}(t_1), \cdots\}$ and for every i, $i \geq 0$:

$$\tilde{\delta}^*((q, \mu^{t_i}(q)), \Lambda, p) = \begin{cases} 1, & q = p \\ 0, & \text{otherwise} \end{cases}$$

and for every i, $i \geq 1$: $\tilde{\delta}^*((q, \mu^{t_{i-1}}(q)), u_i, p) = \tilde{\delta}((q, \mu^{t_{i-1}}(q)), u_i, p)$,

$$\tilde{\delta}^*((q, \mu^{t_{i-1}}(q)), u_i u_{i+1}, p) = \bigvee_{q' \in Q_{act}(t_i)} (\tilde{\delta}((q, \mu^{t_{i-1}}(q)), u_i, q') \wedge \tilde{\delta}((q', \mu^{t_i}(q')), u_{i+1}, p))$$

and recursively

$$\tilde{\delta}^*((q, \mu^{t_0}(q)), u_1 u_2 \cdots u_n, p) = \bigvee \{\tilde{\delta}((q, \mu^{t_0}(q)), u_1, p_1) \wedge \tilde{\delta}((p_1, \mu^{t_1}(p_1)), u_2, p_2) \wedge \cdots \wedge \tilde{\delta}((p_{n-1}, \mu^{t_{n-1}}(p_{n-1})), u_n, p) | p_1 \in Q_{act}(t_1), p_2 \in Q_{act}(t_2), \cdots, p_{n-1} \in Q_{act}(t_{n-1})\},$$

in which $u_i \in \Sigma, \forall 1 \leq i \leq n$ and assuming that the entered input at time t_i be $u_i, \forall 1 \leq i \leq n-1$.

Definition 2.4. [13] Let $\tilde{F}^*$ be a max-min GFA, $p \in Q, q \in Q_{act}(t_i), i \geq 0$ and $0 \leq \alpha < 1$. Then p is called a successor of q with threshold α if there exists $x \in \Sigma^*$ such that $\tilde{\delta}^*((q, \mu^{t_j}(q)), x, p) > \alpha$.

Definition 2.5. [13] Let $\tilde{F}^*$ be a max-min GFA, $q \in Q_{act}(t_i), i \geq 0$ and $0 \leq \alpha < 1$. Also let $S_\alpha(q)$ denote the set of all successors of q with threshold α. If $T \subseteq Q$, then $S_\alpha(T)$ the set of all successors of T with threshold α is defined by $S_\alpha(T) = \bigcup \{S_\alpha(q) : q \in T\}$.

Definition 2.6. [38] Let X be an universe of discourse. The neutrosophic set is an object having the form $A = \{\langle x, \mu_1(x), \mu_2(x), \mu_3(x)\rangle | \forall x \in X\}$ where the functions can be defined by $\mu_1, \mu_2, \mu_3 : X \to]0, 1[$ and μ_1 is the degree of membership or truth, μ_2 is the degree of indeterminacy and μ_3 is the degree of non-membership or false of the element $x \in X$ to the set A with the condition $0 \leq \mu_1(x) + \mu_2(x) + \mu_3(x) \leq 3$."

3 Neutrosophic General Finite Automata

Definition 3.1. An eight-tuple machine $\tilde{F} = (Q, \Sigma, \tilde{R}, Z, \tilde{\delta}, \omega, F_1, F_2)$ is called neutrosophic general finite automata (*NGFA* for short), where

1. Q is a finite set of states, $Q = \{q_1, q_2, \cdots, q_n\}$,
2. Σ is a finite set of input symbols, $\Sigma = \{u_1, u_2, \cdots, u_m\}$,
3. $\tilde{R} = \{(q, \mu_1^{t_0}(q), \mu_2^{t_0}(q), \mu_3^{t_0}(q))|q \in R\}$ is the set of fuzzy start states, $R \subseteq \tilde{P}(Q)$,
4. Z is a finite set of output symbols, $Z = \{b_1, b_2, \cdots, b_k\}$,
5. $\tilde{\delta} : (Q \times [0,1] \times [0,1] \times [0,1])) \times \Sigma \times Q \stackrel{F_1(\mu,\delta)}{\longrightarrow} [0,1] \times [0,1] \times [0,1]$ is the neutrosophic augmented transition function,
6. $\omega : (Q \times [0,1] \times [0,1] \times [0,1]) \to Z$ is the non-fuzzy output function,
7. $F_1 = (F_1^{\wedge}, F_1^{\wedge\vee}, F_1^{\vee})$, where $F_1^{\wedge} : [0,1] \times [0,1] \to [0,1]$, $F_2^{\wedge\vee} : [0,1] \times [0,1] \to [0,1]$ and $F_3^{\vee} : [0,1] \times [0,1] \to [0,1]$ are the truth, indeterminacy and false membership assignment functions, respectively. $F_1^{\wedge}(\mu_1, \tilde{\delta}_1)$, $F_2^{\wedge\vee}(\mu_2, \tilde{\delta}_2)$ and $F_3^{\vee}(\mu_3, \tilde{\delta}_3)$ are motivated by two parameters μ_1, μ_2, μ_3 and $\tilde{\delta}_1$, $\tilde{\delta}_2$, $\tilde{\delta}_3$ where μ_1, μ_2 and μ_3 are the truth, indeterminacy and false membership value of a predecessor and $\tilde{\delta}_1$, $\tilde{\delta}_2$ and $\tilde{\delta}_3$ are the truth, indeterminacy and false membership value of a transition,
8. $F_2 = (F_2^{\wedge}, F_2^{\wedge\vee}, F_2^{\vee})$, where $F_2^{\wedge} : [0,1]^* \to [0,1]$, $F_2^{\wedge\vee} : [0,1]^* \to [0,1]$ and $F_2^{\vee} : [0,1]^* \to [0,1]$ are the truth, indeterminacy and false multi-membership resolution function.

Remark 3.2. In Definition 3.1, the process that takes place upon the transition from the state q_i to q_j on an input u_k is represented by

$$\mu_1^{t_{k+1}}(q_j) = \tilde{\delta}_1((q_i, \mu_1^{t_k}(q_i)), u_k, q_j) = F_1^{\wedge}(\mu_1^{t_k}(q_i), \delta_1(q_i, u_k, q_j)) = \bigwedge(\mu_1^{t_k}(q_i), \delta_1(q_i, u_k, q_j)),$$

$$\mu_2^{t_{k+1}}(q_j) = \tilde{\delta}_2((q_i, \mu_2^{t_k}(q_i)), u_k, q_j) = F_1^{\wedge\vee}(\mu_2^{t_k}(q_i), \delta_2(q_i, u_k, q_j)) = \begin{cases} \bigvee(\mu_2^{t_k}(q_i), \delta_2(q_i, u_k, q_j)) & \text{if } t_k < t_{k+1} \\ \bigwedge(\mu_2^{t_k}(q_i), \delta_2(q_i, u_k, q_j)) & \text{if } t_k \geq t_{k+1} \end{cases},$$

$$\mu_3^{t_{k+1}}(q_j) = \tilde{\delta}_3((q_i, \mu_3^{t_k}(q_i)), u_k, q_j) = F_1^{\vee}(\mu_3^{t_k}(q_i), \delta_3(q_i, u_k, q_j)) = \bigvee(\mu_3^{t_k}(q_i), \delta_3(q_i, u_k, q_j)),$$

where

$$\tilde{\delta}((q_i.\mu^t(q_i)), u_k, q_j) = (\tilde{\delta}_1((q_i, \mu_1^t(q_i)), u_k, q_j), \tilde{\delta}_2((q_i, \mu_2^t(q_i)), u_k, q_j), \tilde{\delta}_3((q_i, \mu_3^t(q_i)), u_k, q_j)) \text{ and}$$

$$\delta(q_i, u_k, q_j) = (\delta_1(q_i, u_k, q_j), \delta_2(q_i, u_k, q_j), \delta_3(q_i, u_k, q_j)).$$

Remark 3.3. The algorithm for truth, indeterminacy and false multi-membership resolution for transition function is same as Algorithm 2.2 but the computation depends (see Remark 3.2) on the truth, indeterminacy and false membership assignment function.

Definition 3.4. Let $\tilde{F} = (Q, \Sigma, \tilde{R}, Z, \tilde{\delta}, \omega, F_1, F_2)$ be a NGFA. We define the max-min neutrosophic general fuzzy automaton $\tilde{F}^* = (Q, \Sigma, \tilde{R}, Z, \tilde{\delta}^*, \omega, F_1, F_2)$, where $\tilde{\delta}^* : (Q \times [0,1] \times [0,1] \times [0,1]) \times \Sigma^* \times Q \to [0,1] \times [0,1] \times [0,1]$ and define a neutrosophic set $\tilde{\delta}^* = \langle \tilde{\delta}_1^*, \tilde{\delta}_2^*, \tilde{\delta}_3^* \rangle$ in $(Q \times [0,1] \times [0,1] \times [01]) \times \Sigma^* \times Q$ and for every i, $i \geq 0$:

$$\tilde{\delta}_1^*((q, \mu^{t_i}(q)), \Lambda, p) = \begin{cases} 1, & q = p \\ 0, & q \neq p \end{cases},$$

$$\tilde{\delta}_2^*((q, \mu^{t_i}(q)), \Lambda, p) = \begin{cases} 0, & q = p \\ 1, & q \neq p \end{cases},$$

$$\tilde{\delta}_3^*((q, \mu^{t_i}(q)), \Lambda, p) = \begin{cases} 0, & q = p \\ 1, & q \neq p \end{cases},$$

and for every i, $i \geq 1$:

$$\tilde{\delta}_1^*((q, \mu^{t_{i-1}}(q)), u_i, p) = \tilde{\delta}_1((q, \mu^{t_{i-1}}(q)), u_i, p), \tilde{\delta}_2^*((q, \mu^{t_{i-1}}(q)), u_i, p) = \tilde{\delta}_2((q, \mu^{t_{i-1}}(q)), u_i, p)$$
$$\tilde{\delta}_3^*((q, \mu^{t_{i-1}}(q)), u_i, p) = \tilde{\delta}_3((q, \mu^{t_{i-1}}(q)), u_i, p)$$

and recursively,

$$\begin{aligned}
\tilde{\delta}_1^*((q, \mu^{t_0}(q)), u_1 u_2 \cdots u_n, p) = \bigvee \{ & \tilde{\delta}_1((q, \mu^{t_0}(q)), u_1, p_1) \wedge \tilde{\delta}_1((p_1, \mu^{t_1}(p_1)), u_2, p_2) \wedge \cdots \wedge \\
& \tilde{\delta}_1((p_{n-1}, \mu^{t_{n-1}}(p_{n-1})), u_n, p) | p_1 \in Q_{act}(t_1), p_2 \in Q_{act}(t_2), \cdots, p_{n-1} \in Q_{act}(t_{n-1}) \}, \\
\tilde{\delta}_2^*((q, \mu^{t_0}(q)), u_1 u_2 \cdots u_n, p) = \bigwedge \{ & \tilde{\delta}_2((q, \mu^{t_0}(q)), u_1, p_1) \vee \tilde{\delta}_2((p_1, \mu^{t_1}(p_1)), u_2, p_2) \vee \cdots \vee \\
& \tilde{\delta}_2((p_{n-1}, \mu^{t_{n-1}}(p_{n-1})), u_n, p) | p_1 \in Q_{act}(t_1), p_2 \in Q_{act}(t_2), \cdots, p_{n-1} \in Q_{act}(t_{n-1}) \}, \\
\tilde{\delta}_3^*((q, \mu^{t_0}(q)), u_1 u_2 \cdots u_n, p) = \bigwedge \{ & \tilde{\delta}_3((q, \mu^{t_0}(q)), u_1, p_1) \vee \tilde{\delta}_3((p_1, \mu^{t_1}(p_1)), u_2, p_2) \vee \cdots \vee \\
& \tilde{\delta}_3((p_{n-1}, \mu^{t_{n-1}}(p_{n-1})), u_n, p) | p_1 \in Q_{act}(t_1), p_2 \in Q_{act}(t_2), \cdots, p_{n-1} \in Q_{act}(t_{n-1}) \},
\end{aligned}$$

in which $u_i \in \Sigma, \forall 1 \leq i \leq n$ and assuming that the entered input at time t_i be $u_i, \forall 1 \leq i \leq n-1$.

Example 3.5. Consider the NGFA in Figure 1 with several transition overlaps. Let $\tilde{F} = (Q, \Sigma, \tilde{R}, Z, \tilde{\delta}, \omega, F_1, F_2)$, where

- $Q = \{q_0, q_1, q_2, q_3, q_4, q_5, q_6, q_7, q_8, q_9\}$ be a set of states,
- $\Sigma = \{a, b\}$ be a set of input symbols,
- $\tilde{R} = \{(q_0, 0.7, 0.5, 0.2), (q_4, 0.6, 0.2, 0.45)\}$, set of initial states,
- the operation of $F_1^{\wedge}, F_1^{\wedge\vee}$ and $F_1^{\vee}$ are according to Remark 3.2,
- $Z = \emptyset$ and ω are not applicable (output mapping is not of our interest in this paper),
- $\tilde{\delta} : (Q \times [0,1] \times [0,1] \times [0,1])) \times \Sigma \times Q \xrightarrow{F_1(\mu, \delta)} [0,1] \times [0,1] \times [0,1]$, the neutrosophic augmented transition function.

Assuming that $\tilde{F}$ starts operating at time t_0 and the next three inputs are a, b, b respectively (one at a time), active states and their membership values at each time step are as follows:

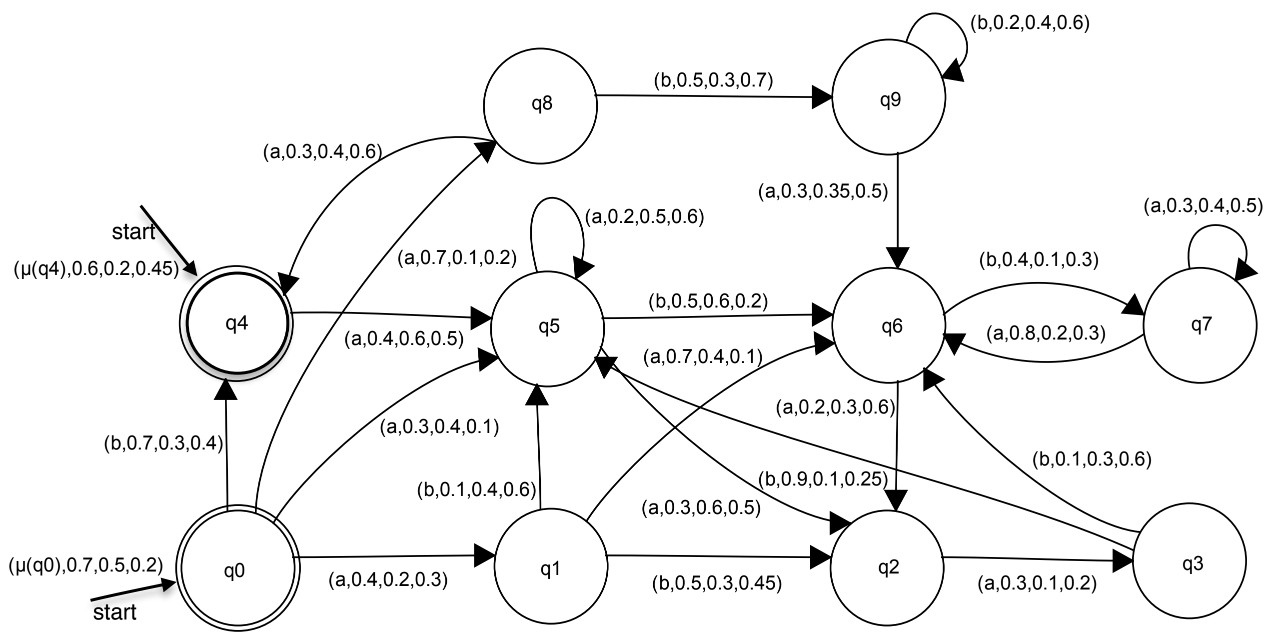

Figure 1: The NGFA of Example 3.5

- At time t_0: $Q_{act}(t_0) = \tilde{R} = \{(q_0, 0.7, 0.5, 0.2), (q_4, 0.6, 0.2, 0.45)\}$

- At time t_1, input is a. Thus q_1, q_5 and q_8 get activated. Then:

$$\begin{aligned}
\mu^{t_1}(q_1) &= \tilde{\delta}((q_0, \mu_1^{t_0}(q_0), \mu_2^{t_0}(q_0), \mu_3^{t_0}(q_0)), a, q_1) \\
&= \left[F_1^{\wedge}(\mu_1^{t_0}(q_0), \delta_1(q_0, a, q_1)), F_1^{\wedge\vee}(\mu_2^{t_0}(q_0), \delta_2(q_0, a, q_1)), F_1^{\vee}(\mu_3^{t_0}(q_0), \delta_3(q_0, a, q_1))\right] \\
&= [F_1^{\wedge}(0.7, 0.4), F_1^{\wedge\vee}(0.5, 0.2), F_1^{\vee}(0.2, 0.3)] = (0.4, 0.2, 0.3),
\end{aligned}$$

$$\begin{aligned}
\mu^{t_1}(q_8) &= \tilde{\delta}((q_0, \mu_1^{t_0}(q_0), \mu_2^{t_0}(q_0), \mu_3^{t_0}(q_0)), a, q_8) \\
&= \left[F_1^{\wedge}(\mu_1^{t_0}(q_0), \delta_1(q_0, a, q_8)), F_1^{\wedge\vee}(\mu_2^{t_0}(q_0), \delta_2(q_0, a, q_8)), F_1^{\vee}(\mu_3^{t_0}(q_0), \delta_3(q_0, a, q_8))\right] \\
&= [F_1^{\wedge}(0.7, 0.7), F_1^{\wedge\vee}(0.5, 0.1), F_1^{\vee}(0.2, 0.2)] = (0.7, 0.1, 0.2),
\end{aligned}$$

but q_5 is multi-membership at t_1. Then

$$\begin{aligned}
\mu^{t_1}(q_5) &= \underset{i=0\&4}{F_2}\left[F_1[\mu^{t_0}(q_i), \delta(q_i, a, q_5)]\right] \\
&= F_2\left[F_1[\mu^{t_0}(q_0), \delta(q_0, a, q_5)], F_1[\mu^{t_0}(q_0), \delta(q_4, a, q_5)]\right] \\
&= F_2\left[F_1[(0.7, 0.5, 0.2), (0.3, 0.4, 0.1)], F_1[(0.6, 0.2, 0.45), (0.4, 0.6, 0.5)]\right] \\
&= (F_2^{\wedge}[F_1^{\wedge}(0.7, 0.3), F_1^{\wedge}(0.6, 0.4)], F_2^{\wedge\vee}[F_1^{\wedge\vee}(0.5, 0.4), F_1^{\wedge\vee}(0.2, 0.6)], \\
&\qquad F_2^{\vee}[F_1^{\vee}(0.2, 0.1), F_1^{\vee}(0.45, 0.5)]) \\
&= (F_2^{\wedge}(0.3, 0.4), F_2^{\wedge\vee}(0.4, 0.2), F_2^{\vee}(0.2, 0.5)) = (0.3, 0.2, 0.5).
\end{aligned}$$

Then we have:

$$\begin{aligned} Q_{act}(t_1) &= \{(q_1, \mu^{t_1}(q_1)), (q_5, \mu^{t_1}(q_5)), (q_8, \mu^{t_1}(q_8))\} \\ &= \{(q_1, 0.4, 0.2, 0.3), (q_5, 0.3, 0.2, 0.5), (q_8, 0.7, 0.1, 0.2)\}. \end{aligned}$$

- At t_2 input is b. q_2, q_5, q_6 and q_9 get activated. Then

$$\begin{aligned} \mu^{t_2}(q_5) &= \tilde{\delta}((q_1, \mu_1^{t_1}(q_1), \mu_2^{t_1}(q_1), \mu_3^{t_1}(q_1)), b, q_5) \\ &= \left[F_1^{\wedge}(\mu_1^{t_1}(q_1), \delta_1(q_1, b, q_5)), F_1^{\wedge\vee}(\mu_2^{t_1}(q_1), \delta_2(q_1, b, q_5)), F_1^{\vee}(\mu_3^{t_1}(q_1), \delta_3(q_1, b, q_5))\right] \\ &= [F_1^{\wedge}(0.4, 0.1), F_1^{\wedge\vee}(0.2, 0.4), F_1^{\vee}(0.3, 0.6)] = (0.1, 0.2, 0.6), \end{aligned}$$

$$\begin{aligned} \mu^{t_2}(q_6) &= \tilde{\delta}((q_5, \mu_1^{t_1}(q_5), \mu_2^{t_1}(q_5), \mu_3^{t_1}(q_5)), b, q_6) \\ &= \left[F_1^{\wedge}(\mu_1^{t_1}(q_5), \delta_1(q_5, b, q_6)), F_1^{\wedge\vee}(\mu_2^{t_1}(q_5), \delta_2(q_5, b, q_6)), F_1^{\vee}(\mu_3^{t_1}(q_5), \delta_3(q_5, b, q_6))\right] \\ &= [F_1^{\wedge}(0.3, 0.5), F_1^{\wedge\vee}(0.2, 0.6), F_1^{\vee}(0.5, 0.2)] = (0.3, 0.2, 0.5), \end{aligned}$$

$$\begin{aligned} \mu^{t_2}(q_9) &= \tilde{\delta}((q_8, \mu_1^{t_1}(q_8), \mu_2^{t_1}(q_8), \mu_3^{t_1}(q_8)), b, q_9) \\ &= \left[F_1^{\wedge}(\mu_1^{t_1}(q_8), \delta_1(q_8, b, q_9)), F_1^{\wedge\vee}(\mu_2^{t_1}(q_8), \delta_2(q_8, b, q_9)), F_1^{\vee}(\mu_3^{t_1}(q_8), \delta_3(q_8, b, q_9))\right] \\ &= [F_1^{\wedge}(0.7, 0.5), F_1^{\wedge\vee}(0.1, 0.3), F_1^{\vee}(0.2, 0.7)] = (0.5, 0.1, 0.7), \end{aligned}$$

but q_2 is multi-membership at t_2. Then:

$$\begin{aligned} \mu^{t_2}(q_2) &= \underset{i=1\&5}{F_2}\left[F_1[\mu^{t_1}(q_i), \delta(q_i, b, q_2)]\right] \\ &= F_2\left[F_1[\mu^{t_1}(q_1), \delta(q_1, b, q_2)], F_1[\mu^{t_1}(q_5), \delta(q_5, b, q_2)]\right] \\ &= F_2\left[F_1[(0.4, 0.2, 0.3), (0.5, 0.3, 0.45)], F_1[(0.3, 0.2, 0.5), (0.1, 0.4, 0.6)]\right] \\ &= (F_2^{\wedge}[F_1^{\wedge}(0.4, 0.5), F_1^{\wedge}(0.3, 0.1)], F_2^{\wedge\vee}[F_1^{\wedge\vee}(0.2, 0.3), F_1^{\wedge\vee}(0.2, 0.4)], \\ &\qquad F_2^{\vee}[F_1^{\vee}(0.3, 0.45), F_1^{\vee}(0.5, 0.6)]) \\ &= (F_2^{\wedge}(0.4, 0.1), F_2^{\wedge\vee}(0.2, 0.2), F_2^{\vee}(0.3, 0.5)) = (0.1, 0.2, 0.5). \end{aligned}$$

Then we have:

$$\begin{aligned} Q_{act}(t_2) &= \{(q_2, \mu^{t_2}(q_2)), (q_5, \mu^{t_2}(q_5)), (q_6, \mu^{t_2}(q_6)), (q_9, \mu^{t_2}(q_9))\} \\ &= \{(q_2, 0.1, 0.2, 0.5), (q_5, 0.1, 0.2, 0.6), (q_6, 0.3, 0.2, 0.5), (q_9, 0.5, 0.1, 0.7)\}. \end{aligned}$$

- At t_3 input is b. q_2, q_6, q_7 and q_9 get activated and none of them is multi-membership. It is easy to verify that:

$$\begin{aligned} Q_{act}(t_3) &= \{(q_2, \mu^{t_3}(q_2)), (q_6, \mu^{t_3}(q_6)), (q_7, \mu^{t_3}(q_7)), (q_9, \mu^{t_3}(q_9))\} \\ &= \{(q_2, 0.1, 0.1, 0.6), (q_6, 0.1, 0.2, 0.6), (q_7, 0.3, 0.1, 0.5), (q_9, 0.3, 0.1, 0.5)\}. \end{aligned}$$

Proposition 3.6. *Let $\tilde{F}$ be a NGFA, if $\tilde{F}^*$ is a max-min NGFA, then for every $i \geq 1$,*

$$\tilde{\delta}_1^*((q,\mu^{t_{i-1}}(q)),xy,p) = \bigvee_{r\in Q_{act}(t_i)} \left[\tilde{\delta}_1^*((p,\mu^{t_{i-1}}(p)),x,r) \wedge \tilde{\delta}_1^*((r,\mu^{t_{i-1}}(r)),y,q)\right],$$
$$\tilde{\delta}_2^*((q,\mu^{t_{i-1}}(q)),xy,p) = \bigwedge_{r\in Q_{act}(t_i)} \left[\tilde{\delta}_2^*((p,\mu^{t_{i-1}}(p)),x,r) \vee \tilde{\delta}_2^*((r,\mu^{t_{i-1}}(r)),y,q)\right],$$
$$\tilde{\delta}_3^*((q,\mu^{t_{i-1}}(q)),xy,p) = \bigwedge_{r\in Q_{act}(t_i)} \left[\tilde{\delta}_3^*((p,\mu^{t_{i-1}}(p)),x,r) \vee \tilde{\delta}_3^*((r,\mu^{t_{i-1}}(r)),y,q)\right],$$

for all $p,q \in Q$ and $x,y \in \Sigma^*$.

Proof. Since $p,q \in Q$ and $x,y \in \Sigma^*$, we prove the result by induction on $|y| = n$. First, we assume that $n = 0$, then $y = \Lambda$ and so $xy = x\Lambda = x$. Thus, for all $r \in Q_{act}(t_i)$

$$\begin{aligned}
\bigvee\left[\tilde{\delta}_1^*((p,\mu^{t_{i-1}}(p)),x,r) \wedge \tilde{\delta}_1^*((r,\mu^{t_{i-1}}(r)),y,q)\right] &= \bigvee\left[\tilde{\delta}_1^*((p,\mu^{t_{i-1}}(p)),x,r) \wedge \tilde{\delta}_1^*((r,\mu^{t_{i-1}}(r)),\Lambda,q)\right]\\
&= \tilde{\delta}_1^*((p,\mu^{t_{i-1}}(p)),x,r) = \tilde{\delta}_1^*((q,\mu^{t_{i-1}}(q)),xy,p),\\
\bigwedge\left[\tilde{\delta}_2^*((p,\mu^{t_{i-1}}(p)),x,r) \vee \tilde{\delta}_2^*((r,\mu^{t_{i-1}}(r)),y,q)\right] &= \bigwedge\left[\tilde{\delta}_2^*((p,\mu^{t_{i-1}}(p)),x,r) \vee \tilde{\delta}_2^*((r,\mu^{t_{i-1}}(r)),\Lambda,q)\right]\\
&= \tilde{\delta}_2^*((p,\mu^{t_{i-1}}(p)),x,r) = \tilde{\delta}_2^*((q,\mu^{t_{i-1}}(q)),xy,p),\\
\bigwedge\left[\tilde{\delta}_3^*((p,\mu^{t_{i-1}}(p)),x,r) \vee \tilde{\delta}_3^*((r,\mu^{t_{i-1}}(r)),y,q)\right] &= \bigwedge\left[\tilde{\delta}_3^*((p,\mu^{t_{i-1}}(p)),x,r) \vee \tilde{\delta}_3^*((r,\mu^{t_{i-1}}(r)),\Lambda,q)\right]\\
&= \tilde{\delta}_3^*((p,\mu^{t_{i-1}}(p)),x,r) = \tilde{\delta}_3^*((q,\mu^{t_{i-1}}(q)),xy,p).
\end{aligned}$$

The result holds for $n = 0$. Now, continue the result is true for all $u \in \Sigma^*$ with $|u| = n-1$, where $n > 0$. Let $y = ua$, where $a \in \Sigma$ and $u \in \Sigma^*$. Then

$$\begin{aligned}
&\tilde{\delta}_1^*((q,\mu^{t_{i-1}}(q)),xy,p) = \tilde{\delta}_1^*((q,\mu^{t_{i-1}}(q)),xua,p) = \bigvee_{r\in Q_{act}(t_i)} \Big(\tilde{\delta}_1((q,\mu^{t_{i-1}}(q)),xu,r) \wedge \tilde{\delta}_1((r,\mu^{t_i}(r)),a,p)\Big)\\
&= \bigvee_{r\in Q_{act}(t_i)} (\bigvee_{s\in Q_{act}(t_i)} (\tilde{\delta}_1((q,\mu^{t_{i-1}}(q)),x,s) \wedge \tilde{\delta}_1((s,\mu^{t_{i-1}}(s)),u,r)) \wedge \tilde{\delta}_1((r,\mu^{t_i}(r)),a,p))\\
&= \bigvee_{r,s\in Q_{act}(t_i)} (\tilde{\delta}_1((q,\mu^{t_{i-1}}(q)),x,s) \wedge \tilde{\delta}_1((s,\mu^{t_{i-1}}(s)),u,r) \wedge \tilde{\delta}_1((r,\mu^{t_i}(r)),a,p))\\
&= \bigvee_{s\in Q_{act}(t_i)} (\tilde{\delta}_1((q,\mu^{t_{i-1}}(q)),x,s) \wedge (\bigvee_{r\in Q_{act}(t_i)} (\tilde{\delta}_1((s,\mu^{t_{i-1}}(s)),u,r) \wedge \tilde{\delta}_1((r,\mu^{t_i}(r)),a,p))))\\
&= \bigvee_{s\in Q_{act}(t_i)} (\tilde{\delta}_1((q,\mu^{t_{i-1}}(q)),x,s) \wedge \tilde{\delta}_1((s,\mu^{t_i}(r)),ua,p))) = \bigvee_{s\in Q_{act}(t_i)} (\tilde{\delta}_1((q,\mu^{t_{i-1}}(q)),x,s) \wedge \tilde{\delta}_1((s,\mu^{t_i}(r)),y,p))),
\end{aligned}$$

$$\begin{aligned}
&\tilde{\delta}_2^*((q,\mu^{t_{i-1}}(q)),xy,p) = \tilde{\delta}_2^*((q,\mu^{t_{i-1}}(q)),xua,p) = \bigwedge_{r\in Q_{act}(t_i)} \Big(\tilde{\delta}_2((q,\mu^{t_{i-1}}(q)),xu,r) \vee \tilde{\delta}_2((r,\mu^{t_i}(r)),a,p)\Big)\\
&= \bigwedge_{r\in Q_{act}(t_i)} (\bigwedge_{s\in Q_{act}(t_i)} (\tilde{\delta}_2((q,\mu^{t_{i-1}}(q)),x,s) \vee \tilde{\delta}_2((s,\mu^{t_{i-1}}(s)),u,r)) \vee \tilde{\delta}_2((r,\mu^{t_i}(r)),a,p))\\
&= \bigwedge_{r,s\in Q_{act}(t_i)} (\tilde{\delta}_2((q,\mu^{t_{i-1}}(q)),x,s) \vee \tilde{\delta}_2((s,\mu^{t_{i-1}}(s)),u,r) \vee \tilde{\delta}_2((r,\mu^{t_i}(r)),a,p))\\
&= \bigwedge_{s\in Q_{act}(t_i)} (\tilde{\delta}_2((q,\mu^{t_{i-1}}(q)),x,s) \vee (\bigwedge_{r\in Q_{act}(t_i)} (\tilde{\delta}_2((s,\mu^{t_{i-1}}(s)),u,r) \vee \tilde{\delta}_2((r,\mu^{t_i}(r)),a,p))))\\
&= \bigwedge_{s\in Q_{act}(t_i)} (\tilde{\delta}_2((q,\mu^{t_{i-1}}(q)),x,s) \vee \tilde{\delta}_2((s,\mu^{t_i}(r)),ua,p))) = \bigwedge_{s\in Q_{act}(t_i)} (\tilde{\delta}_2((q,\mu^{t_{i-1}}(q)),x,s) \vee \tilde{\delta}_2((s,\mu^{t_i}(r)),y,p))),
\end{aligned}$$

$$\begin{aligned}
&\tilde{\delta}_3^*((q,\mu^{t_{i-1}}(q)),xy,p)=\tilde{\delta}_3^*((q,\mu^{t_{i-1}}(q)),xua,p)=\bigwedge_{r\in Q_{act}(t_i)}\Big(\tilde{\delta}_3((q,\mu^{t_{i-1}}(q)),xu,r)\vee\tilde{\delta}_3((r,\mu^{t_i}(r)),a,p)\Big)\\
&=\bigwedge_{r\in Q_{act}(t_i)}(\bigwedge_{s\in Q_{act}(t_i)}(\tilde{\delta}_3((q,\mu^{t_{i-1}}(q)),x,s)\vee\tilde{\delta}_3((s,\mu^{t_{i-1}}(s)),u,r))\vee\tilde{\delta}_3((r,\mu^{t_i}(r)),a,p))\\
&=\bigwedge_{r,s\in Q_{act}(t_i)}(\tilde{\delta}_3((q,\mu^{t_{i-1}}(q)),x,s)\vee\tilde{\delta}_3((s,\mu^{t_{i-1}}(s)),u,r)\vee\tilde{\delta}_3((r,\mu^{t_i}(r)),a,p))\\
&=\bigwedge_{s\in Q_{act}(t_i)}(\tilde{\delta}_3((q,\mu^{t_{i-1}}(q)),x,s)\vee(\bigwedge_{r\in Q_{act}(t_i)}(\tilde{\delta}_3((s,\mu^{t_{i-1}}(s)),u,r)\vee\tilde{\delta}_3((r,\mu^{t_i}(r)),a,p))))\\
&=\bigwedge_{s\in Q_{act}(t_i)}(\tilde{\delta}_3((q,\mu^{t_{i-1}}(q)),x,s)\vee\tilde{\delta}_3((s,\mu^{t_i}(r)),ua,p)))=\bigwedge_{s\in Q_{act}(t_i)}(\tilde{\delta}_3((q,\mu^{t_{i-1}}(q)),x,s)\vee\tilde{\delta}_3((s,\mu^{t_i}(r)),y,p))).
\end{aligned}$$

Hence the result is valid for $|y|=n$. This completes the proof. □

Definition 3.7. Let $\tilde{F}^*$ be a max-min NGFA, $p\in Q, q\in Q_{act}(t_i), i\geq 0$ and $0\leq\alpha<1$. Then p is called a successor of q with threshold α if there exists $x\in\Sigma^*$ such that $\tilde{\delta}_1^*((q,\mu_1^{t_j}(q)),x,p)>\alpha$, $\tilde{\delta}_2^*((q,\mu_2^{t_j}(q)),x,p)<\alpha$ and $\tilde{\delta}_3^*((q,\mu_3^{t_j}(q)),x,p)<\alpha$.

Definition 3.8. Let $\tilde{F}^*$ be a max-min NGFA, $q\in Q_{act}(t_i), i\geq 0$ and $0\leq\alpha<1$. Also let $S_\alpha(q)$ denote the set of all successors of q with threshold α. If $T\subseteq Q$, then $S_\alpha(T)$ the set of all successors of T with threshold α is defined by $S_\alpha(T)=\bigcup\{S_\alpha(q):q\in T\}$.

Definition 3.9. Let $\tilde{F}^*$ be a max-min NGFA. Let $\mu=\langle\mu_1,\mu_2,\mu_3\rangle$ and $\tilde{\delta}^*=\langle\tilde{\delta}_1^*,\tilde{\delta}_2^*,\tilde{\delta}_3^*\rangle$ in $(Q\times[0,1]\times[0,1]\times[0,1])\times\Sigma^*\times Q$ be a neutrosophic set in Q. Then μ is a neutrosophic subsystem of $\tilde{F}^*$, say $\mu\subseteq\tilde{F}^*$ if for every j, $1\leq j\leq k$ such that $\mu_1^{t_j}(p)\geq\tilde{\delta}_1^*((q,\mu_1^{t_j}(q)),x,p)$, $\mu_2^{t_j}(p)\leq\tilde{\delta}_2^*((q,\mu_2^{t_j}(q)),x,p)$, $\mu_3^{t_j}(p)\leq\tilde{\delta}_3^*((q,\mu_3^{t_j}(q)),x,p)$. $\forall q,p\in Q$ and $x\in\Sigma^*$.

Example 3.10. Let $Q=\{p,q\}$, $\Sigma=\{a\}$. Let $\mu=\langle\mu_1,\mu_2,\mu_3\rangle$ and $\tilde{\delta}^*=\langle\tilde{\delta}_1^*,\tilde{\delta}_2^*,\tilde{\delta}_3^*\rangle$ in $(Q\times[0,1]\times[0,1]\times[0,1])\times\Sigma^*\times Q$ be a neutrosophic set in Q such that $\mu_1^{t_j}(p)=0.8$, $\mu_2^{t_j}(p)=0.7$, $\mu_3^{t_j}(p)=0.5$, $\mu_1^{t_j}(q)=0.5$, $\mu_2^{t_j}(q)=0.6$, $\mu_3^{t_j}(q)=0.8$, $\delta_1(q,x,p)=0.7$, $\delta_2(q,x,p)=0.9$ and $\delta_3(q,x,p)=0.7$. Then

$$\begin{aligned}
\tilde{\delta}_1^*((q,\mu_1^{t_j}(q)),x,p)&=F_1^{\wedge}(\mu_1^{t_j}(q),\delta_1(q,x,p))=\min\{0.5,0.7\}=0.5\leq\mu_1^{t_j}(p),\\
\tilde{\delta}_2^*((q,\mu_2^{t_j}(q)),x,p)&=F_2^{\wedge\vee}(\mu_2^{t_j}(q),\delta_2(q,x,p))=\max\{0.6,0.9\}=0.9\geq\mu_2^{t_j}(p),\ \ (\text{since } t<t_j)\\
\tilde{\delta}_3^*((q,\mu_3^{t_j}(q)),x,p)&=F_3^{\vee}(\mu_3^{t_j}(q),\delta_3(q,x,p))=\max\{0.8,0.7\}=0.8\geq\mu_3^{t_j}(p).
\end{aligned}$$

Hence μ is a neutrosophic subsystem of $\tilde{F}^*$.

Theorem 3.11. *Let $\tilde{F}^*$ be a NGFA and let $\mu=\langle\mu_1,\mu_2,\mu_3\rangle$ and $\tilde{\delta}^*=\langle\tilde{\delta}_1^*,\tilde{\delta}_2^*,\tilde{\delta}_3^*\rangle$ in $(Q\times[0,1]\times[0,1]\times[0,1])\times\Sigma^*\times Q$ be a neutrosophic set in Q. Then μ is a neutrosophic subsystem of $\tilde{F}^*$ if and only if $\mu_1^{t_j}(p)\geq\tilde{\delta}_1^*((q,\mu_1^{t_j}(q)),x,p)$, $\mu_2^{t_j}(p)\leq\tilde{\delta}_2^*((q,\mu_2^{t_j}(q)),x,p)$, $\mu_3^{t_j}(p)\leq\tilde{\delta}_3^*((q,\mu_3^{t_j}(q)),x,p)$, for all $q\in Q_{(act)}(t_j)$, $p\in Q$ and $x\in\Sigma^*$.*

Proof. Suppose that μ is a neutrosophic subsystem of $\tilde{F}^*$. Let $q\in Q_{(act)}(t_j)$, $p\in Q$ and $x\in\Sigma^*$. The proof is by induction on $|x|=n$. If $n=0$, then $x=\Lambda$. Now if $q=p$, then $\tilde{\delta}_1^*((p,\mu_1^{t_i}(p)),\Lambda,p)=F_1^{\wedge}(\mu_1^{t_i}(p),\tilde{\delta}_1(p,\Lambda,p))=\mu_1^{t_i}(p)$, $\tilde{\delta}_2^*((p,\mu_2^{t_i}(p)),\Lambda,p)=F_1^{\wedge\vee}(\mu_2^{t_i}(p),\tilde{\delta}_2(p,\Lambda,p))=\mu_2^{t_i}(p)$, $\tilde{\delta}_3^*((p,\mu_3^{t_i}(p)),\Lambda,p)=F_1^{\vee}(\mu_3^{t_i}(p),\tilde{\delta}_3(p,\Lambda,p))=\mu_3^{t_i}(p)$.

If $q\neq p$, then $\tilde{\delta}_1^*((q,\mu_1^{t_i}(q)),\Lambda,p)=F_1^{\wedge}(\mu_1^{t_i}(q),\tilde{\delta}_1(q,\Lambda,p))=0\leq\mu_1^{t_j}(p)$, $\tilde{\delta}_2^*((q,\mu_2^{t_i}(q)),\Lambda,p)=F_1^{\wedge\vee}(\mu_2^{t_i}(q),\tilde{\delta}_2(q,\Lambda,p))=1\geq\mu_2^{t_j}(p)$, $\tilde{\delta}_3^*((q,\mu_3^{t_i}(q)),\Lambda,p)=F_1^{\vee}(\mu_3^{t_i}(q),\tilde{\delta}_3(q,\Lambda,p))=1\geq\mu_3^{t_j}(p)$.

Hence the result is true for $n = 0$. For now, we assume that the result is valid for all $y \in \Sigma^*$ with $|y| = n-1$, $n > 0$. For the y above, let $x = u_1 \cdots u_n$ where $u_i \in \Sigma, i = 1, 2, \cdots n$. Then

$$\tilde{\delta}_1^*((q, \mu_1^{t_i}(q)), x, p) = \tilde{\delta}_1^*((q, \mu_1^{t_i}(q)), u_1 \cdots u_n, p) = \bigvee \left(\tilde{\delta}_1^*((q, \mu_1^{t_i}(q)), u_1, r_1) \wedge \cdots \wedge \tilde{\delta}_1^*((r_{n-1}, \mu_1^{t_{i+n}}(r_{n-1})), u_n, p)\right)$$
$$\leq \bigvee \left(\tilde{\delta}_1^*((r_{n-1}, \mu_1^{t_{i+n}}(r_{n-1})), u_n, p) | r_{n-1} \in Q_{(act)}(t_{i+n})\right) \leq \bigvee \mu_1^{t_j}(p) = \mu_1^{t_j}(p),$$

$$\tilde{\delta}_2^*((q, \mu_2^{t_i}(q)), x, p) = \tilde{\delta}_2^*((q, \mu_2^{t_i}(q)), u_1 \cdots u_n, p) = \bigwedge \left(\tilde{\delta}_2^*((q, \mu_2^{t_i}(q)), u_1, r_1) \vee \cdots \vee \tilde{\delta}_2^*((r_{n-1}, \mu_2^{t_{i+n}}(r_{n-1})), u_n, p)\right)$$
$$\leq \bigwedge \left(\tilde{\delta}_2^*((r_{n-1}, \mu_2^{t_{i+n}}(r_{n-1})), u_n, p) | r_{n-1} \in Q_{(act)}(t_{i+n})\right) \leq \bigwedge \mu_2^{t_j}(p) = \mu_2^{t_j}(p),$$

$$\tilde{\delta}_3^*((q, \mu_3^{t_i}(q)), x, p) = \tilde{\delta}_3^*((q, \mu_3^{t_i}(q)), u_1 \cdots u_n, p) = \bigwedge \left(\tilde{\delta}_3^*((q, \mu_3^{t_i}(q)), u_1, r_1) \vee \cdots \vee \tilde{\delta}_3^*((r_{n-1}, \mu_3^{t_{i+n}}(r_{n-1})), u_n, p)\right)$$
$$\leq \bigwedge \left(\tilde{\delta}_3^*((r_{n-1}, \mu_3^{t_{i+n}}(r_{n-1})), u_n, p) | r_{n-1} \in Q_{(act)}(t_{i+n})\right) \leq \bigwedge \mu_3^{t_j}(p) = \mu_3^{t_j}(p),$$

where $r_1 \in Q_{(act)}(t_{i+1}) \cdots r_{n-1} \in Q_{(act)}(t_{i+n})$. Hence $\mu_1^{t_j}(p) \geq \tilde{\delta}_1^*((q, \mu_1^{t_j}(q)), x, p)$, $\mu_2^{t_j}(p) \leq \tilde{\delta}_2^*((q, \mu_2^{t_j}(q)), x, p)$, $\mu_3^{t_j}(p) \leq \tilde{\delta}_3^*((q, \mu_3^{t_j}(q)), x, p)$. The converse is trivial. This proof is completed. $\square$

Definition 3.12. Let $\tilde{F}^*$ be a NGFA. Let $\mu = \langle \mu_1, \mu_2, \mu_3 \rangle$ and $\tilde{\delta}^* = \langle \tilde{\delta}_1^*, \tilde{\delta}_2^*, \tilde{\delta}_3^* \rangle$ in $(Q \times [0,1] \times [0,1] \times [0,1]) \times \Sigma^* \times Q$ be a neutrosophic set in Q. Then μ is a neutrosophic strong subsystem of $\tilde{F}^*$, say $\mu \subseteq \tilde{F}^*$, if for every i, $1 \leq i \leq k$ such that $p \in S_\alpha(q)$, then for $q, p \in Q$ and $x \in \Sigma$, $\mu_1^{t_j}(p) \geq \mu_1^{t_j}(q)$, $\mu_2^{t_j}(p) \leq \mu_2^{t_j}(q)$, $\mu_3^{t_j}(p) \leq \mu_3^{t_j}(q)$, for every $1 \leq j \leq k$.

Theorem 3.13. *Let $\tilde{F}^*$ be a NGFA and let $\mu = \langle \mu_1, \mu_2, \mu_3 \rangle$ and $\tilde{\delta}^* = \langle \tilde{\delta}_1^*, \tilde{\delta}_2^*, \tilde{\delta}_3^* \rangle$ in $(Q \times [0,1] \times [0,1] \times [0,1]) \times \Sigma^* \times Q$ be a neutrosophic set in Q. Then μ is a strong neutrosophic subsystem of $\tilde{F}^*$ if and only if there exists $x \in \Sigma^*$ such that $p \in S_\alpha(q)$, then $\mu_1^{t_j}(p) \geq \mu_1^{t_j}(q)$, $\mu_2^{t_j}(p) \leq \mu_2^{t_j}(q)$, $\mu_3^{t_j}(p) \leq \mu_3^{t_j}(q)$, for all $q \in Q_{(act)}(t_j)$, $p \in Q$.*

Proof. Suppose that μ is a strong neutrosophic subsystem of $\tilde{F}^*$. Let $q \in Q_{(act)}(t_j)$, $p \in Q$ and $x \in \Sigma^*$. The proof is by induction on $|x| = n$. If $n = 0$, then $x = \Lambda$. Now if $q = p$, then $\delta_1^*((p, \mu_1^{t_i}(p)), \Lambda, p) = 1$, $\delta_2^*((p, \mu_2^{t_i}(p)), \Lambda, p) = 0$, $\delta_3^*((p, \mu_3^{t_i}(p)), \Lambda, p) = 0$ and $\mu_1^{t_j}(p) = \mu_1^{t_j}(p)$, $\mu_2^{t_j}(p) = \mu_2^{t_j}(p)$, $\mu_3^{t_j}(p) = \mu_3^{t_j}(p)$. If $q \neq p$, then $\tilde{\delta}_1^*((q, \mu_1^{t_i}(q)), \Lambda, p) = F_1^{\wedge}(\mu_1^{t_i}(q), \tilde{\delta}_1(q, \Lambda, p)) = c \leq \mu_1^{t_j}(p)$, $\tilde{\delta}_2^*((q, \mu_2^{t_i}(q)), \Lambda, p) = F_1^{\wedge\vee}(\mu_2^{t_i}(q), \tilde{\delta}_2(q, \Lambda, p)) = d \geq \mu_2^{t_j}(p)$, $\tilde{\delta}_3^*((q, \mu_3^{t_i}(q)), \Lambda, p) = F_1^{\vee}(\mu_3^{t_i}(q), \tilde{\delta}_3(q, \Lambda, p)) = e \geq \mu_3^{t_j}(p)$. Hence the result is true for $n = 0$. For now, we assume that the result is valid for all $u \in \Sigma^*$ with $|u| = n-1$, $n > 0$. For the u above, let $x = u_1 \cdots u_n$ where $u_i \in \Sigma^*, i = 1, 2, \cdots n$. Suppose that $\tilde{\delta}_1^*((q, \mu_1^{t_i}(q)), x, p) > c$, $\tilde{\delta}_2^*((q, \mu_2^{t_i}(q)), x, p) < d$, $\tilde{\delta}_3^*((q, \mu_3^{t_i}(q)), x, p) < e$. Then

$$\tilde{\delta}_1^*((q, \mu_1^{t_i}(q)), u_1 \cdots u_n, p) = \bigvee \left\{\tilde{\delta}_1^*((q, \mu_1^{t_i}(q)), u_1, p_1) \wedge \cdots \wedge \tilde{\delta}_1^*((p_{n-1}, \mu_1^{t_{i+n}}(p_{n-1})), u_n, p)\right\} > c,$$
$$\tilde{\delta}_2^*((q, \mu_2^{t_i}(q)), u_1 \cdots u_n, p) = \bigwedge \left\{\tilde{\delta}_2^*((q, \mu_2^{t_i}(q)), u_1, p_1) \vee \cdots \vee \tilde{\delta}_2^*((p_{n-1}, \mu_2^{t_{i+n}}(p_{n-1})), u_n, p)\right\} < d,$$
$$\tilde{\delta}_3^*((q, \mu_3^{t_i}(q)), u_1 \cdots u_n, p) = \bigwedge \left\{\tilde{\delta}_3^*((q, \mu_3^{t_i}(q)), u_1, p_1) \vee \cdots \vee \tilde{\delta}_3^*((p_{n-1}, \mu_3^{t_{i+n}}(p_{n-1})), u_n, p)\right\} < e,$$

where $p_1 \in Q_{(act)}(t_i), \cdots, p_{n-1} \in Q_{(act)}(t_{i+n})$.

This implies that $\tilde{\delta}_1^*((q,\mu_1^{t_i}(q)),u_1,p_1) > c, \cdots, \tilde{\delta}_1^*((p_{n-1},\mu_1^{t_{i+n}}(p_{n-1})),u_n,p) > c$, $\tilde{\delta}_2^*((q,\mu_2^{t_i}(q)),u_1,p_1) < d, \cdots, \tilde{\delta}_2^*((p_{n-1},\mu_2^{t_{i+n}}(p_{n-1})),u_n,p) < d$, $\tilde{\delta}_3^*((q,\mu_3^{t_i}(q)),u_1,p_1) < e, \cdots, \tilde{\delta}_3^*((p_{n-1},\mu_3^{t_{i+n}}(p_{n-1})),u_n,p) < e$. Hence $\mu_1^{t_j}(p) \geq \mu_1^{t_{i+n}}(p_{n-1}), \mu_1^{t_{i+n}}(p) \geq \mu_1^{t_{i+n-1}}(p_{n-2}), \cdots, \mu_1^{t_i}(p_1) \geq \mu_1^{t_j}(q)$, $\mu_2^{t_j}(p) \leq \mu_2^{t_{i+n}}(p_{n-1}), \mu_2^{t_{i+n}}(p) \leq \mu_2^{t_{i+n-1}}(p_{n-2}), \cdots, \mu_2^{t_i}(p_1) \leq \mu_2^{t_j}(q)$, $\mu_3^{t_j}(p) \leq \mu_3^{t_{i+n}}(p_{n-1}), \mu_3^{t_{i+n}}(p) \leq \mu_3^{t_{i+n-1}}(p_{n-2}), \cdots, \mu_3^{t_i}(p_1) \leq \mu_3^{t_j}(q)$. Thus $\mu_1^{t_j}(p) \geq \mu_1^{t_j}(q)$, $\mu_2^{t_j}(p) \leq \mu_2^{t_j}(q)$, $\mu_3^{t_j}(p) \leq \mu_3^{t_j}(q)$. The converse is trivial. The proof is completed. □

4 Neutrosophic General Finite Switchboard Automata

Definition 4.1. Let $\tilde{F}^*$ be a max-min NGFA. Let $\mu = \langle \mu_1, \mu_2, \mu_3 \rangle$ and $\tilde{\delta}^* = \langle \tilde{\delta}_1^*, \tilde{\delta}_2^*, \tilde{\delta}_3^* \rangle$ be a neutrosophic set in $(Q \times [0,1] \times [0,1] \times [0,1]) \times \Sigma \times Q$ in Q. Then

1. $\tilde{F}^*$ is switching, if it satisfies $\forall p, q \in Q$, $a \in \Sigma$ and for every i, $i \geq 0$,
$\tilde{\delta}_1^*((q,\mu_1^{t_i}(q)),a,p) = \tilde{\delta}_1^*((p,\mu_1^{t_i}(p)),a,q)$, $\tilde{\delta}_2^*((q,\mu_2^{t_i}(q)),a,p) = \tilde{\delta}_2^*((p,\mu_2^{t_i}(p)),a,q)$,
$\tilde{\delta}_3^*((q,\mu_3^{t_i}(q)),a,p) = \tilde{\delta}_3^*((p,\mu_3^{t_i}(p)),a,q)$.

2. $\tilde{F}^*$ is commutative, if it satisfies $\forall p, q \in Q$, $a, b \in \Sigma$ and for every i, $i \geq 1$, $\tilde{\delta}_1^*((q,\mu_1^{t_{i-1}}(q)),ab,p) = \tilde{\delta}_1^*((q,\mu_1^{t_{i-1}}(q)),ba,p)$, $\tilde{\delta}_2^*((q,\mu_2^{t_{i-1}}(q)),ab,p) = \tilde{\delta}_2^*((q,\mu_2^{t_{i-1}}(q)),ba,p)$,
$\tilde{\delta}_3^*((q,\mu_3^{t_{i-1}}(q)),ab,p) = \tilde{\delta}_3^*((q,\mu_3^{t_{i-1}}(q)),ba,p)$.

3. $\tilde{F}^*$ is Neutrosophic General Finite Switchboard Automata (NGFSA, for short), if $\tilde{F}^*$ satisfies both switching and commutative.

Proposition 4.2. *Let $\tilde{F}$ be a NGFA, if $\tilde{F}^*$ is a commutative NGFSA, then for every $i \geq 1$,*

$$\begin{aligned}
\tilde{\delta}_1^*((q,\mu_1^{t_{i-1}}(q)),xa,p) &= \tilde{\delta}_1^*((q,\mu_1^{t_{i-1}}(q)),ax,p),\\
\tilde{\delta}_2^*((q,\mu_2^{t_{i-1}}(q)),xa,p) &= \tilde{\delta}_2^*((q,\mu_2^{t_{i-1}}(q)),ax,p),\\
\tilde{\delta}_3^*((q,\mu_3^{t_{i-1}}(q)),xa,p) &= \tilde{\delta}_3^*((q,\mu_3^{t_{i-1}}(q)),ax,p),
\end{aligned}$$

for all $q \in Q_{act}(t_{i-1}), p \in S_c(q)$, $a \in \Sigma$ and $x \in \Sigma^*$.

Proof. Since $p \in S_c(q)$ then $q \in Q_{act}(t_{i-1})$ and $|x| = n$. If $n = 0$, then $x = \Lambda$. Thus

$$\begin{aligned}
\tilde{\delta}_1^*((q,\mu_1^{t_{i-1}}(q)),xa,p) = \tilde{\delta}_1^*((q,\mu_1^{t_{i-1}}(q)),\Lambda a,p) = \tilde{\delta}_1^*((q,\mu_1^{t_{i-1}}(q)),a,p) &= \tilde{\delta}_1^*((q,\mu_1^{t_{i-1}}(q)),a\Lambda,p)\\
&= \tilde{\delta}_1^*((q,\mu_1^{t_{i-1}}(q)),ax,p),\\
\tilde{\delta}_2^*((q,\mu_2^{t_{i-1}}(q)),xa,p) = \tilde{\delta}_2^*((q,\mu_2^{t_{i-1}}(q)),\Lambda a,p) = \tilde{\delta}_2^*((q,\mu_2^{t_{i-1}}(q)),a,p) &= \tilde{\delta}_2^*((q,\mu_2^{t_{i-1}}(q)),a\Lambda,p)\\
&= \tilde{\delta}_2^*((q,\mu_2^{t_{i-1}}(q)),ax,p),\\
\tilde{\delta}_3^*((q,\mu_3^{t_{i-1}}(q)),xa,p) = \tilde{\delta}_3^*((q,\mu_3^{t_{i-1}}(q)),\Lambda a,p) = \tilde{\delta}_3^*((q,\mu_3^{t_{i-1}}(q)),a,p) &= \tilde{\delta}_3^*((q,\mu_3^{t_{i-1}}(q)),a\Lambda,p)\\
&= \tilde{\delta}_3^*((q,\mu_3^{t_{i-1}}(q)),ax,p).
\end{aligned}$$

Suppose the result is true for all $u \in \Sigma^*$ with $|u| = n-1$, where $n > 0$. Let $x = ub$, where $b \in \Sigma$. Then

$$\tilde{\delta}_1^*((q, \mu_1^{t_{i-1}}(q)), xa, p) = \tilde{\delta}_1^*((q, \mu_1^{t_{i-1}}(q)), uba, p) = \bigvee_{r \in Q_{act}(t_i)} \Big(\tilde{\delta}_1((q, \mu_1^{t_{i-1}}(q)), u, r) \wedge \tilde{\delta}_1((r, \mu_1^{t_i}(r)), ba, p) \Big)$$
$$= \bigvee_{r \in Q_{act}(t_i)} \Big(\tilde{\delta}_1((q, \mu_1^{t_{i-1}}(q)), u, r) \wedge \tilde{\delta}_1((r, \mu_1^{t_i}(r)), ab, p) \Big) = \tilde{\delta}_1^*((q, \mu_1^{t_{i-1}}(q)), uab, p)$$
$$= \bigvee_{r \in Q_{act}(t_i)} \Big(\tilde{\delta}_1((q, \mu_1^{t_{i-1}}(q)), ua, r) \wedge \tilde{\delta}_1((r, \mu_1^{t_i}(r)), b, p) \Big)$$
$$= \bigvee_{r \in Q_{act}(t_i)} \Big(\tilde{\delta}_1((q, \mu_1^{t_{i-1}}(q)), au, r) \wedge \tilde{\delta}_1((r, \mu_1^{t_i}(r)), b, p) \Big) = \tilde{\delta}_1^*((q, \mu_1^{t_{i-1}}(q)), aub, p) = \tilde{\delta}_1^*((q, \mu_1^{t_{i-1}}(q)), ax, p),$$

$$\tilde{\delta}_2^*((q, \mu_2^{t_{i-1}}(q)), xa, p) = \tilde{\delta}_2^*((q, \mu_2^{t_{i-1}}(q)), uba, p) = \bigwedge_{r \in Q_{act}(t_i)} \Big(\tilde{\delta}_2((q, \mu_2^{t_{i-1}}(q)), u, r) \vee \tilde{\delta}_2((r, \mu_2^{t_i}(r)), ba, p) \Big)$$
$$= \bigwedge_{r \in Q_{act}(t_i)} \Big(\tilde{\delta}_2((q, \mu_2^{t_{i-1}}(q)), u, r) \vee \tilde{\delta}_2((r, \mu_2^{t_i}(r)), ab, p) \Big) = \tilde{\delta}_2^*((q, \mu_2^{t_{i-1}}(q)), uab, p)$$
$$= \bigwedge_{r \in Q_{act}(t_i)} \Big(\tilde{\delta}_2((q, \mu_2^{t_{i-1}}(q)), ua, r) \vee \tilde{\delta}_2((r, \mu_2^{t_i}(r)), b, p) \Big)$$
$$= \bigwedge_{r \in Q_{act}(t_i)} \Big(\tilde{\delta}_2((q, \mu_2^{t_{i-1}}(q)), au, r) \vee \tilde{\delta}_2((r, \mu_2^{t_i}(r)), b, p) \Big) = \tilde{\delta}_2^*((q, \mu_2^{t_{i-1}}(q)), aub, p) = \tilde{\delta}_2^*((q, \mu_2^{t_{i-1}}(q)), ax, p),$$

$$\tilde{\delta}_3^*((q, \mu_3^{t_{i-1}}(q)), xa, p) = \tilde{\delta}_3^*((q, \mu_3^{t_{i-1}}(q)), uba, p) = \bigwedge_{r \in Q_{act}(t_i)} \Big(\tilde{\delta}_3((q, \mu_3^{t_{i-1}}(q)), u, r) \vee \tilde{\delta}_3((r, \mu_3^{t_i}(r)), ba, p) \Big)$$
$$= \bigwedge_{r \in Q_{act}(t_i)} \Big(\tilde{\delta}_3((q, \mu_3^{t_{i-1}}(q)), u, r) \vee \tilde{\delta}_3((r, \mu_3^{t_i}(r)), ab, p) \Big) = \tilde{\delta}_3^*((q, \mu_3^{t_{i-1}}(q)), uab, p)$$
$$= \bigwedge_{r \in Q_{act}(t_i)} \Big(\tilde{\delta}_3((q, \mu_3^{t_{i-1}}(q)), ua, r) \vee \tilde{\delta}_3((r, \mu_3^{t_i}(r)), b, p) \Big)$$
$$= \bigwedge_{r \in Q_{act}(t_i)} \Big(\tilde{\delta}_3((q, \mu_3^{t_{i-1}}(q)), au, r) \vee \tilde{\delta}_3((r, \mu_3^{t_i}(r)), b, p) \Big) = \tilde{\delta}_3^*((q, \mu_3^{t_{i-1}}(q)), aub, p) = \tilde{\delta}_3^*((q, \mu_3^{t_{i-1}}(q)), ax, p).$$

This completes the proof. □

Proposition 4.3. *Let $\tilde{F}$ be a NGFA, if $\tilde{F}^*$ is a switching NGFSA, then for every $i \geq 0$, $\tilde{\delta}_1^*((q, \mu_1^{t_i}(q)), x, p) = \tilde{\delta}_1^*((p, \mu_1^{t_i}(p)), x, q)$, $\tilde{\delta}_2^*((q, \mu_2^{t_i}(q)), x, p) = \tilde{\delta}_2^*((p, \mu_2^{t_i}(p)), x, q)$, $\tilde{\delta}_3^*((q, \mu_3^{t_i}(q)), x, p) = \tilde{\delta}_3^*((p, \mu_3^{t_i}(p)), x, q)$, for all $p, q \in Q_{act}(t_i)$ and $x \in \Sigma^*$.*

Proof. Since $p, q \in Q_{act}(t_i)$ and $x \in \Sigma^*$, we prove the result by induction on $|x| = n$. First, we assume that $x = \Lambda$, whenever $n = 0$. Then we have $\tilde{\delta}_1^*((q, \mu_1^{t_i}(q)), x, p) = \tilde{\delta}_1^*((q, \mu_1^{t_i}(q)), \Lambda, p) = \tilde{\delta}_1^*((p, \mu_1^{t_i}(p)), \Lambda, q) = \tilde{\delta}_1^*((p, \mu_1^{t_i}(p)), x, q)$, $\tilde{\delta}_2^*((q, \mu_2^{t_i}(q)), x, p) = \tilde{\delta}_2^*((q, \mu_2^{t_i}(q)), \Lambda, p) = \tilde{\delta}_2^*((p, \mu_2^{t_i}(p)), \Lambda, q) = \tilde{\delta}_2^*((p, \mu_2^{t_i}(p)), x, q)$ $\tilde{\delta}_3^*((q, \mu_3^{t_i}(q)), x, p) = \tilde{\delta}_3^*((q, \mu_3^{t_i}(q)), \Lambda, p) = \tilde{\delta}_3^*((p, \mu_3^{t_i}(p)), \Lambda, q) = \tilde{\delta}_3^*((p, \mu_3^{t_i}(p)), x, q)$. Thus, the theorem holds for $x = \Lambda$. Now, we assume that the results holds for all $u \in \Sigma^*$ such that $|u| = n-1$ and $n > 0$. Let

$a \in \Sigma$ and $x \in \Sigma^*$ be such that $x = ua$. Then

$$\tilde{\delta}_1^*((q, \mu_1^{t_i}(q)), x, p) = \tilde{\delta}_1^*((q, \mu_1^{t_i}(q)), ua, p) = \bigvee_{r \in Q_{act}(t_{i+1})} \Big(\tilde{\delta}_1((q, \mu_1^{t_i}(q)), u, r) \wedge \tilde{\delta}_1((r, \mu_1^{t_{i+1}}(r)), a, p) \Big)$$
$$= \bigvee_{r \in Q_{act}(t_i)} \Big(\tilde{\delta}_1((r, \mu_1^{t_i}(r)), u, q) \wedge \tilde{\delta}_1(p, \mu_1^{t_{i+1}}(p)), a, r) \Big) = \bigvee_{r \in Q_{act}(t_{i+1})} \Big(\tilde{\delta}_1((p, \mu_1^{t_i}(p)), a, r) \wedge \tilde{\delta}_1((r, \mu_1^{t_{i+1}}(r)), u, q) \Big)$$
$$= \tilde{\delta}_1^*((p, \mu_1^{t_i}(p)), au, q) = \tilde{\delta}_1^*((p, \mu_1^{t_i}(p)), ua, q) = \tilde{\delta}_1^*((p, \mu_1^{t_i}(p)), x, q),$$

$$\tilde{\delta}_2^*((q, \mu_2^{t_i}(q)), x, p) = \tilde{\delta}_2^*((q, \mu_2^{t_i}(q)), ua, p) = \bigwedge_{r \in Q_{act}(t_{i+1})} \Big(\tilde{\delta}_2((q, \mu_2^{t_i}(q)), u, r) \vee \tilde{\delta}_2((r, \mu_2^{t_{i+1}}(r)), a, p) \Big)$$
$$= \bigwedge_{r \in Q_{act}(t_i)} \Big(\tilde{\delta}_2((r, \mu_2^{t_i}(r)), u, q) \vee \tilde{\delta}_2(p, \mu_2^{t_{i+1}}(p)), a, r) \Big) = \bigwedge_{r \in Q_{act}(t_{i+1})} \Big(\tilde{\delta}_2((p, \mu_2^{t_i}(p)), a, r) \vee \tilde{\delta}_2((r, \mu_2^{t_{i+1}}(r)), u, q) \Big)$$
$$= \tilde{\delta}_2^*((p, \mu_2^{t_i}(p)), au, q) = \tilde{\delta}_2^*((p, \mu_2^{t_i}(p)), ua, q) = \tilde{\delta}_2^*((p, \mu_2^{t_i}(p)), x, q),$$

$$\tilde{\delta}_3^*((q, \mu_3^{t_i}(q)), x, p) = \tilde{\delta}_3^*((q, \mu_3^{t_i}(q)), ua, p) = \bigwedge_{r \in Q_{act}(t_{i+1})} \Big(\tilde{\delta}_3((q, \mu_3^{t_i}(q)), u, r) \vee \tilde{\delta}_3((r, \mu_3^{t_{i+1}}(r)), a, p) \Big)$$
$$= \bigwedge_{r \in Q_{act}(t_i)} \Big(\tilde{\delta}_3((r, \mu_3^{t_i}(r)), u, q) \vee \tilde{\delta}_3(p, \mu_3^{t_{i+1}}(p)), a, r) \Big) = \bigwedge_{r \in Q_{act}(t_{i+1})} \Big(\tilde{\delta}_3((p, \mu_3^{t_i}(p)), a, r) \vee \tilde{\delta}_3((r, \mu_3^{t_{i+1}}(r)), u, q) \Big)$$
$$= \tilde{\delta}_3^*((p, \mu_3^{t_i}(p)), au, q) = \tilde{\delta}_3^*((p, \mu_3^{t_i}(p)), ua, q) = \tilde{\delta}_3^*((p, \mu_3^{t_i}(p)), x, q).$$

Hence, the result is true for $|u| = n$. This completes the proof. □

Proposition 4.4. *Let $\tilde{F}$ be a NGFA, if $\tilde{F}^*$ is a NGFSA, then for every $i \geq 1$, $\tilde{\delta}_1^*((q, \mu_1^{t_{i-1}}(q)), xy, p) = \tilde{\delta}_1^*((p, \mu_1^{t_{i-1}}(p)), yx, q)$, $\tilde{\delta}_2^*((q, \mu_2^{t_{i-1}}(q)), xy, p) = \tilde{\delta}_2^*((p, \mu_2^{t_{i-1}}(p)), yx, q)$, $\tilde{\delta}_3^*((q, \mu_3^{t_{i-1}}(q)), xy, p) = \tilde{\delta}_3^*((p, \mu_3^{t_{i-1}}(p)), yx, q)$ for all $p, q \in Q$ and $x, y \in \Sigma^*$.*

Proof. Since $p, q \in Q$ and $x, y \in \Sigma^*$, we prove the result by induction on $|x| = n$. First, we assume that $n = 0$, then $x = \Lambda$. Thus
$\tilde{\delta}_1^*((q, \mu_1^{t_{i-1}}(q)), xy, p) = \tilde{\delta}_1^*((q, \mu_1^{t_{i-1}}(q)), x\Lambda, p) = \tilde{\delta}_1^*((q, \mu_1^{t_{i-1}}(q)), \Lambda x, p) = \tilde{\delta}_1^*((q, \mu_1^{t_{i-1}}(q)), yx, p)$,
$\tilde{\delta}_2^*((q, \mu_2^{t_{i-1}}(q)), xy, p) = \tilde{\delta}_2^*((q, \mu_2^{t_{i-1}}(q)), x\Lambda, p) = \tilde{\delta}_2^*((q, \mu_2^{t_{i-1}}(q)), \Lambda x, p) = \tilde{\delta}_2^*((q, \mu_2^{t_{i-1}}(q)), yx, p)$,
$\tilde{\delta}_3^*((q, \mu_3^{t_{i-1}}(q)), xy, p) = \tilde{\delta}_3^*((q, \mu_3^{t_{i-1}}(q)), x\Lambda, p) = \tilde{\delta}_3^*((q, \mu_3^{t_{i-1}}(q)), \Lambda x, p) = \tilde{\delta}_3^*((q, \mu_3^{t_{i-1}}(q)), yx, p)$.

Suppose that
$\tilde{\delta}_1^*((q, \mu_1^{t_{i-1}}(q)), xu, p) = \tilde{\delta}_1^*((p, \mu_1^{t_{i-1}}(p)), ux, q)$, $\tilde{\delta}_2^*((q, \mu_2^{t_{i-1}}(q)), xu, p) = \tilde{\delta}_2^*((p, \mu_2^{t_{i-1}}(p)), ux, q)$,
$\tilde{\delta}_3^*((q, \mu_3^{t_{i-1}}(q)), xu, p) = \tilde{\delta}_3^*((p, \mu_3^{t_{i-1}}(p)), ux, q)$, for every $u \in \Sigma^*$.

Now, continue the result is true for all $u \in \Sigma^*$ with $|u| = n - 1$, where $n > 0$. Let $y = ua$, where $a \in \Sigma$

and $u \in \Sigma^*$. Then

$$\tilde{\delta}_1^*((q, \mu_1^{t_{i-1}}(q)), xy, p) = \tilde{\delta}_1^*((q, \mu_1^{t_{i-1}}(q)), xua, p) = \bigvee_{r \in Q_{act}(t_i)} \Big(\tilde{\delta}_1((q, \mu_1^{t_{i-1}}(q)), xu, r) \wedge \tilde{\delta}_1((r, \mu_1^{t_i}(r)), a, p)\Big)$$
$$= \bigvee_{r \in Q_{act}(t_i)} \Big(\tilde{\delta}_1((q, \mu_1^{t_{i-1}}(q)), ux, r) \wedge \tilde{\delta}_1((r, \mu_1^{t_i}(r)), a, p)\Big)$$
$$= \bigvee_{r \in Q_{act}(t_{i-1})} \Big(\tilde{\delta}_1((r, \mu_1^{t_{i-1}}(r)), ux, q) \wedge \tilde{\delta}_1((p, \mu_1^{t_i}(p)), a, r)\Big)$$
$$= \bigvee_{r \in Q_{act}(t_i)} \Big(\tilde{\delta}_1((p, \mu_1^{t_{i-1}}(p)), a, r) \wedge \tilde{\delta}_1((r, \mu_1^{t_i}(r)), ux, q)\Big)$$
$$= \tilde{\delta}_1^*((p, \mu_1^{t_{i-1}}(p)), aux, q) = \bigvee_{r \in Q_{act}(t_i)} \Big(\tilde{\delta}_1((p, \mu_1^{t_{i-1}}(p)), au, r) \wedge \tilde{\delta}_1((r, \mu_1^{t_i}(r)), x, q)\Big)$$
$$= \bigvee_{r \in Q_{act}(t_i)} \Big(\tilde{\delta}_1((p, \mu_1^{t_{i-1}}(p)^), ua, r) \wedge \tilde{\delta}_1((r, \mu_1^{t_i}(r)), x, q)\Big)$$
$$= \tilde{\delta}_1^*((p, \mu_1^{t_{i-1}}(p)), uax, q) = \tilde{\delta}_1^*((q, \mu_1^{t_{i-1}}(q)), uax, p) = \tilde{\delta}_1^*((q, \mu_1^{t_{i-1}}(q)), yx, p),$$

$$\tilde{\delta}_2^*((q, \mu_2^{t_{i-1}}(q)), xy, p) = \tilde{\delta}_2^*((q, \mu_2^{t_{i-1}}(q)), xua, p) = \bigwedge_{r \in Q_{act}(t_i)} \Big(\tilde{\delta}_2((q, \mu_2^{t_{i-1}}(q)), xu, r) \vee \tilde{\delta}_2((r, \mu_2^{t_i}(r)), a, p)\Big)$$
$$= \bigwedge_{r \in Q_{act}(t_i)} \Big(\tilde{\delta}_2((q, \mu_2^{t_{i-1}}(q)), ux, r) \vee \tilde{\delta}_2((r, \mu_2^{t_i}(r)), a, p)\Big)$$
$$= \bigwedge_{r \in Q_{act}(t_{i-1})} \Big(\tilde{\delta}_2((r, \mu_2^{t_{i-1}}(r)), ux, q) \vee \tilde{\delta}_2((p, \mu_2^{t_i}(p)), a, r)\Big)$$
$$= \bigwedge_{r \in Q_{act}(t_i)} \Big(\tilde{\delta}_2((p, \mu_2^{t_{i-1}}(p)), a, r) \vee \tilde{\delta}_2((r, \mu_2^{t_i}(r)), ux, q)\Big)$$
$$= \tilde{\delta}_2^*((p, \mu_2^{t_{i-1}}(p)), aux, q) = \bigwedge_{r \in Q_{act}(t_i)} \Big(\tilde{\delta}_2((p, \mu_2^{t_{i-1}}(p)), au, r) \wedge \tilde{\delta}_2((r, \mu_2^{t_i}(r)), x, q)\Big)$$
$$= \bigwedge_{r \in Q_{act}(t_i)} \Big(\tilde{\delta}_2((p, \mu_2^{t_{i-1}}(p)^), ua, r) \vee \tilde{\delta}_2((r, \mu_2^{t_i}(r)), x, q)\Big)$$
$$= \tilde{\delta}_2^*((p, \mu_2^{t_{i-1}}(p)), uax, q) = \tilde{\delta}_2^*((q, \mu_2^{t_{i-1}}(q)), uax, p) = \tilde{\delta}_2^*((q, \mu_2^{t_{i-1}}(q)), yx, p),$$

$$\tilde{\delta}_3^*((q, \mu_3^{t_{i-1}}(q)), xy, p) = \tilde{\delta}_3^*((q, \mu_3^{t_{i-1}}(q)), xua, p) = \bigwedge_{r \in Q_{act}(t_i)} \Big(\tilde{\delta}_3((q, \mu_3^{t_{i-1}}(q)), xu, r) \vee \tilde{\delta}_3((r, \mu_3^{t_i}(r)), a, p)\Big)$$
$$= \bigwedge_{r \in Q_{act}(t_i)} \Big(\tilde{\delta}_3((q, \mu_3^{t_{i-1}}(q)), ux, r) \vee \tilde{\delta}_3((r, \mu_3^{t_i}(r)), a, p)\Big)$$
$$= \bigwedge_{r \in Q_{act}(t_{i-1})} \Big(\tilde{\delta}_3((r, \mu_3^{t_{i-1}}(r)), ux, q) \vee \tilde{\delta}_3((p, \mu_3^{t_i}(p)), a, r)\Big)$$
$$= \bigwedge_{r \in Q_{act}(t_i)} \Big(\tilde{\delta}_3((p, \mu_3^{t_{i-1}}(p)), a, r) \vee \tilde{\delta}_3((r, \mu_3^{t_i}(r)), ux, q)\Big)$$
$$= \tilde{\delta}_3^*((p, \mu_3^{t_{i-1}}(p)), aux, q) = \bigwedge_{r \in Q_{act}(t_i)} \Big(\tilde{\delta}_3((p, \mu_3^{t_{i-1}}(p)), au, r) \wedge \tilde{\delta}_3((r, \mu_3^{t_i}(r)), x, q)\Big)$$
$$= \bigwedge_{r \in Q_{act}(t_i)} \Big(\tilde{\delta}_3((p, \mu_3^{t_{i-1}}(p)^), ua, r) \vee \tilde{\delta}_3((r, \mu_3^{t_i}(r)), x, q)\Big)$$
$$= \tilde{\delta}_3^*((p, \mu_3^{t_{i-1}}(p)), uax, q) = \tilde{\delta}_3^*((q, \mu_3^{t_{i-1}}(q)), uax, p) = \tilde{\delta}_3^*((q, \mu_3^{t_{i-1}}(q)), yx, p).$$

This completes the proof. □

Definition 4.5. Let $\tilde{F}^*$ be a GNFSA. Let $\mu = \langle \mu_1, \mu_2, \mu_3 \rangle$ and $\tilde{\delta}^* = \langle \tilde{\delta}_1^*, \tilde{\delta}_2^*, \tilde{\delta}_3^* \rangle$ in $(Q \times [0, 1]) \times \Sigma^* \times Q$ be a neutrosophic set in Q. Then μ is a neutrosophic switchboard subsystem of $\tilde{F}^*$, say $\mu \subseteq \tilde{F}^*$, if for every j, $1 \leq j \leq k$ such that $\mu_1^{t_j}(p) \geq \tilde{\delta}_1^*((q, \mu_1^{t_j}(q)), x, p)$, $\mu_2^{t_j}(p) \leq \tilde{\delta}_2^*((q, \mu_2^{t_j}(q)), x, p)$, $\mu_3^{t_j}(p) \leq \tilde{\delta}_3^*((q, \mu_3^{t_j}(q)), x, p)$. $\forall q, p \in Q$ and $x \in \Sigma$.

Theorem 4.6. *Let $\tilde{F}^*$ be a NGFSA and let $\mu = \langle \mu_1, \mu_2, \mu_3 \rangle$ and $\tilde{\delta}^* = \langle \tilde{\delta}_1^*, \tilde{\delta}_2^*, \tilde{\delta}_3^* \rangle$ in $(Q \times [0,1] \times [0,1] \times [0,1]) \times \Sigma^* \times Q$ be a neutrosophic set in Q. Then μ is a neutrosophic switchboard subsystem of $\tilde{F}^*$ if and only if $\mu_1^{t_j}(p) \geq \tilde{\delta}_1^*((q, \mu_1^{t_j}(q)), x, p)$, $\mu_2^{t_j}(p) \leq \tilde{\delta}_2^*((q, \mu_2^{t_j}(q)), x, p)$, $\mu_3^{t_j}(p) \leq \tilde{\delta}_3^*((q, \mu_3^{t_j}(q)), x, p)$, for all $q \in Q_{(act)}(t_j)$, $p \in Q$ and $x \in \Sigma^*$.*

Proof. The proof of the theorem is similar to Theorem 3.11 and it is clear that μ satisfies switching and commutative, since $\tilde{F}^*$ is NGFSA. This proof is completed. $\square$

Definition 4.7. Let $\tilde{F}^*$ be a NGFSA. Let $\mu = \langle \mu_1, \mu_2, \mu_3 \rangle$ and $\tilde{\delta}^* = \langle \tilde{\delta}_1^*, \tilde{\delta}_2^*, \tilde{\delta}_3^* \rangle$ in $(Q \times [0,1] \times [0,1] \times [0,1]) \times \Sigma^* \times Q$ be a neutrosophic set in Q. Then μ is a neutrosophic strong switchboard subsystem of $\tilde{F}^*$, say $\mu \subseteq \tilde{F}^*$, if for every i, $1 \leq i \leq k$ such that $p \in S_\alpha(q)$, then for $q, p \in Q$ and $x \in \Sigma$, $\mu_1^{t_j}(p) \geq \mu_1^{t_j}(q)$, $\mu_2^{t_j}(p) \leq \mu_2^{t_j}(q)$, $\mu_3^{t_j}(p) \leq \mu_3^{t_j}(q)$, for every $1 \leq j \leq k$.

Theorem 4.8. *Let $\tilde{F}^*$ be a NGFA and let $\mu = \langle \mu_1, \mu_2, \mu_3 \rangle$ and $\tilde{\delta}^* = \langle \tilde{\delta}_1^*, \tilde{\delta}_2^*, \tilde{\delta}_3^* \rangle$ in $(Q \times [0,1] \times [0,1] \times [0,1]) \times \Sigma^* \times Q$ be a neutrosophic set in Q. Then μ is a strong neutrosophic switchboard subsystem of $\tilde{F}^*$ if and only if there exists $x \in \Sigma^*$ such that $p \in S_\alpha(q)$, then $\mu_1^{t_j}(p) \geq \mu_1^{t_j}(q)$, $\mu_2^{t_j}(p) \leq \mu_2^{t_j}(q)$, $\mu_3^{t_j}(p) \leq \mu_3^{t_j}(q)$, for all $q \in Q_{(act)}(t_j)$, $p \in Q$.*

Proof. The proof of the theorem is similar to Theorem 3.13 and it is clear that μ satisfies switching and commutative, since $\tilde{F}^*$ is NGFSA. The proof is completed. $\square$

Theorem 4.9. *Let $\tilde{F}^*$ be a NGFSA and let $\mu = \langle \mu_1, \mu_2, \mu_3 \rangle$ be a neutrosophic subset of Q. If μ is a neutrosohic switchboard subsystem of $\tilde{F}^*$, then μ is a strong neutrosophic switchboard subsystem of $\tilde{F}^*$.*

Proof. Assume that $\tilde{\delta}_1^*((q, \mu_1^{t_i}(q)), x, p) > 0$, $\tilde{\delta}_2^*((q, \mu_2^{t_i}(q)), x, p) < 1$ and $\tilde{\delta}_3^*((q, \mu_3^{t_i}(q)), x, p) < 1$, for all $x \in \Sigma$. Since μ is a neutrosophic switchboard subsystem of $\tilde{F}^*$, we have

$$\mu_1^{t_j}(p) \geq \tilde{\delta}_1^*((q, \mu_1^{t_i}(q)), x, p), \quad \mu_2^{t_j}(p) \leq \tilde{\delta}_2^*((q, \mu_2^{t_i}(q)), x, p), \quad \mu_3^{t_j}(p) \leq \tilde{\delta}_3^*((q, \mu_3^{t_i}(q)), x, p).$$

for all $q \in Q_{(act)}(t_j)$, $p \in Q$ and $x \in \Sigma$. As μ is switching, then we have

$$\begin{aligned}
\mu_1^{t_j}(p) &\geq \tilde{\delta}_1^*((q, \mu_1^{t_i}(q)), x, p) = \tilde{\delta}_1^*((p, \mu_1^{t_i}(p)), x, q) = \mu_1^{t_j}(q), \\
\mu_2^{t_j}(p) &\leq \tilde{\delta}_2^*((q, \mu_2^{t_i}(q)), x, p) = \tilde{\delta}_2^*((p, \mu_2^{t_i}(p)), x, q) = \mu_2^{t_j}(q), \\
\mu_3^{t_j}(p) &\leq \tilde{\delta}_3^*((q, \mu_3^{t_i}(q)), x, p) = \tilde{\delta}_3^*((p, \mu_3^{t_i}(p)), x, q) = \mu_3^{t_j}(q).
\end{aligned}$$

As μ is commutative, then $x = uv$, we have

$$\begin{aligned}
\mu_1^{t_j}(p) &\geq \tilde{\delta}_1^*((q, \mu_1^{t_i}(q)), x, p) = \tilde{\delta}_1^*((q, \mu_1^{t_i}(q)), uv, p) \\
&= \bigvee \left\{ \tilde{\delta}_1^*((q, \mu_1^{t_i}(q)), u, r) \wedge \tilde{\delta}_1^*((r, \mu_1^{t_{i+1}}(r)), v, p) | r \in Q_{1(act)}(t_{i+1}) \right\} \\
&= \bigvee \left\{ \tilde{\delta}_1^*((r, \mu_1^{t_i}(r)), u, q) \wedge \tilde{\delta}_1^*((p, \mu_1^{t_{i+1}}(p)), v, r) | r \in Q_{1(act)}(t_{i+1}) \right\} \\
&= \bigvee \left\{ \tilde{\delta}_1^*((p, \mu_1^{t_{i+1}}(p)), v, r) \wedge \tilde{\delta}_1^*((r, \mu_1^{t_i}(r)), u, q) | r \in Q_{1(act)}(t_{i+1}) \right\} \\
&= \tilde{\delta}_1^*((p, \mu_1^{t_{i+1}}(p)), vu, q) = \tilde{\delta}_1^*((p, \mu_1^{t_{i+1}}(p)), uv, q) = \tilde{\delta}_1^*((p, \mu_1^{t_{i+1}}(p)), x, q) \geq \mu_1^{t_j}(q), \\
\mu_2^{t_j}(p) &\leq \tilde{\delta}_2^*((q, \mu_2^{t_i}(q)), x, p) = \tilde{\delta}_2^*((q, \mu_2^{t_i}(q)), uv, p) \\
&= \bigwedge \left\{ \tilde{\delta}_2^*((q, \mu_2^{t_i}(q)), u, r) \vee \tilde{\delta}_2^*((r, \mu_2^{t_{i+1}}(r)), v, p) | r \in Q_{1(act)}(t_{i+1}) \right\} \\
&= \bigwedge \left\{ \tilde{\delta}_2^*((r, \mu_2^{t_i}(r)), u, q) \vee \tilde{\delta}_2^*((p, \mu_2^{t_{i+1}}(p)), v, r) | r \in Q_{1(act)}(t_{i+1}) \right\} \\
&= \bigwedge \left\{ \tilde{\delta}_2^*((p, \mu_2^{t_{i+1}}(p)), v, r) \vee \tilde{\delta}_2^*((r, \mu_2^{t_i}(r)), u, q) | r \in Q_{1(act)}(t_{i+1}) \right\} \\
&= \tilde{\delta}_2^*((p, \mu_2^{t_{i+1}}(p)), vu, q) = \tilde{\delta}_2^*((p, \mu_2^{t_{i+1}}(p)), uv, q) = \tilde{\delta}_2^*((p, \mu_2^{t_{i+1}}(p)), x, q) \leq \mu_2^{t_j}(q), \\
\mu_3^{t_j}(p) &\leq \tilde{\delta}_3^*((q, \mu_3^{t_i}(q)), x, p) = \tilde{\delta}_3^*((q, \mu_3^{t_i}(q)), uv, p) \\
&= \bigwedge \left\{ \tilde{\delta}_3^*((q, \mu_3^{t_i}(q)), u, r) \vee \tilde{\delta}_3^*((r, \mu_3^{t_{i+1}}(r)), v, p) | r \in Q_{1(act)}(t_{i+1}) \right\} \\
&= \bigwedge \left\{ \tilde{\delta}_3^*((r, \mu_3^{t_i}(r)), u, q) \vee \tilde{\delta}_3^*((p, \mu_3^{t_{i+1}}(p)), v, r) | r \in Q_{1(act)}(t_{i+1}) \right\} \\
&= \bigwedge \left\{ \tilde{\delta}_3^*((p, \mu_3^{t_{i+1}}(p)), v, r) \vee \tilde{\delta}_3^*((r, \mu_3^{t_i}(r)), u, q) | r \in Q_{1(act)}(t_{i+1}) \right\} \\
&= \tilde{\delta}_3^*((p, \mu_3^{t_{i+1}}(p)), vu, q) = \tilde{\delta}_3^*((p, \mu_3^{t_{i+1}}(p)), uv, q) = \tilde{\delta}_3^*((p, \mu_3^{t_{i+1}}(p)), x, q) \leq \mu_3^{t_j}(q).
\end{aligned}$$

Hence μ is a strong neutrosophic switchboard subsystem of $\tilde{F}^*$. □

Theorem 4.10. *Let $\tilde{F}^*$ be a NGFSA and let $\mu = \langle \mu_1, \mu_2, \mu_3 \rangle$ be a neutrosophic subset of Q. If μ is a strong neutrosophic switchboard subsystem of $\tilde{F}^*$, then μ is a neutrosophic switchboard subsystem of $\tilde{F}^*$.*

Proof. Let $q, p \in Q$. Since μ is a strong neutrosophic switchboard subsystem of $\tilde{F}^*$ and μ is switching, we have for all $x \in \Sigma$, since $\tilde{\delta}_1^*((q, \mu_1^{t_i}(q)), x, p) > 0$, $\tilde{\delta}_2^*((q, \mu_2^{t_i}(q)), x, p) < 1$ and $\tilde{\delta}_3^*((q, \mu_3^{t_i}(q)), x, p) < 1$, $\forall x \in \Sigma$,
$\mu_1^{t_j}(p) \geq \mu_1^{t_j}(q) \geq \tilde{\delta}_1^*((p, \mu_1^{t_i}(p)), x, q) \geq \tilde{\delta}_1^*((q, \mu_1^{t_i}(q)), x, p)$,
$\mu_2^{t_j}(p) \leq \mu_2^{t_j}(q) \leq \tilde{\delta}_2^*((p, \mu_2^{t_i}(p)), x, q) \leq \tilde{\delta}_2^*((q, \mu_2^{t_i}(q)), x, p)$,
$\mu_3^{t_j}(p) \leq \mu_3^{t_j}(q) \leq \tilde{\delta}_3^*((p, \mu_3^{t_i}(p)), x, q) \leq \tilde{\delta}_3^*((q, \mu_3^{t_i}(q)), x, p)$.

It is clear that μ is commutative. Thus μ is a neutrosophic switchboard subsystem of $\tilde{F}^*$. □

5 Conclusions

This paper attempt to develop and present a new general definition for neutrosophic finite automata. The general definition for (strong) subsystem also examined and discussed their properties. A comprehensive analysis and an appropriate methodology to manage the essential issues of output mapping in general fuzzy automata were studied by Doostfatemen and Kremer [11]. Their approach is consistent with the output which is either associated with the states (Moore model) or with the transitions (Mealy model). Interval-valued fuzzy subsets have many applications in several areas. The concept of interval-valued fuzzy sets have been studied in various algebraic structures, see [7, 26]. On the basis [11] and [7], the future work will focus on general interval-valued neutrosophic finite automata with output respond to input strings.

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Implementation of Neutrosophic Function Memberships Using MATLAB Program

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Abstract. Membership function (MF) plays a key role for getting an output of a system and hence it influences system's performance directly. Therefore choosing a MF is an essential task in fuzzy logic and neutrosophic logic as well. Uncertainty is usually represented by MFs. In this paper, a novel Matlab code is derived for trapezoidal neutrosophic function and the validity of the proposed code is proved with illustrative graphical representation

Keywords: Membership function, Matlab code, Trapezoidal neutrosophic function, Graphical representation

1 Introduction

The membership function (MF) designs a structure of practical relationship to relational structure numerically where the elements lies between 0 and 1. By determining the MFs one can model the relationship between the cognitive and stimuli portrayal in fuzzy set theory [1]. The computed MF will provide a solution to the problem and the complete process can be observed as a training and acceptable approximation to the function from the behavior of the objects [2]. This kind of MFs can be utilized for the fuzzy implication appeared in the given rules to examine more examples [3].

The MFs of fuzzy logic is nothing but a stochastic representation and are used to determine a probability space and its value may be explained as probabilities. The stochastic representation will to know the reasoning and capability of fuzzy control [4]. MFs which are characterized in a single domain where the functions are in terms of single variable are playing a vital role in fuzzy logic system. FMFs determine the degree of membership (M/S) which is a crisp value. Generally MFs are considered as either triangular or trapezoidal as they are adequate, can be design easily and flexible [5].

MFs can be carried out using hardware [6]. MFs are taking part in most of the works done under fuzzy environment without checking their existence for sure and also in the connection between a studied characteristic for sure and its reference set won't be problematic as it is a direct measurement [7]. It is adorable to have continuously differentiable MFs with less parameters [8]. MFs plays an important role in fuzzy classifier (FC). In traditional FC, the domain of every input variable is separated into various intervals. All these intervals is assumed to be a FS and a correlated MF is determined. Hence the input space is separated again into various sub regions which are all parallel in to input axes and a fuzzy rule is defined for all these sub regions if the input belongs to the sub region then it is also belongs to the associated class with the sub region.

Further the degrees of M/S of an unidentified input for all the FSs are evaluated and the input is restricted into the class with maximum degree of M/S. Thus the MFs are directly control the performance of the fuzzy classifier [10]. If the position of the MF is changed then the direct methods maximize the understanding rate of the training data by calculating the total increase directly [11]. Estimation of the MF is usually based on the level of information gained with the experiment transferred by the numerical data [12]. Due to the important role of MFs, concepts of fuzzy logic have been applied in many of the control systems for controlling the robot, nuclear reactor, climate, speed of the car, power systems, memory device under fuzzy logic, aircraft flight, mobile robots and focus of a camcorder.

There has been a habit of restrain the MFs into a well-known formats like triangular, trapezoidal and standard Gaussian or sigmoid types [13]. In information systems the incomplete information can be designed by rough sets [20]. Neutrosophy has established the base for the entire family of novel mathematical theories which generalizes the counterparts of the conventional and fuzzy sets [21]. The success of an approach depends on the MFs and hence designing MFs is an important task for the process and the system. Theory of FSs contributes the way of handling impreciseness, uncertainty and vagueness in the software metrics. The uncertainty of the problem can be solved b considering MFs in an expert system under fuzzy setting. Triangular and trapezoidal MFs are flexible representation of domain expert knowledge and where the computational complexity is less. Hence the derivation of the MF is need to be clarified.

The MFs are continuous and maps from any closed interval to [0,1]. Also which are all either monotonically decreasing or increasing or both [22]. A connectively flexible aggregation of crisp and imprecise knowledge is possible with the horizontal MFs which are capable of introducing uncertainty directly [23]. There are effective methods for calculating MFs of FSs connected with few multi criteria decision making problem [25]. Due to the possibility of having some degree of hesitation, one could not define the non-membership degree by subtracting membership degree from 1 [26]. The degree of the fuzzy sets will be determined by FMFs. [30] Crisp value is converted into fuzzy during fuzzification process. If uncertainty exists on the variable then becomes fuzzy and could be characterized by MFs. The degree of MF is determined by fuzzification.

In the real world problems satisfaction of the decision maker is not possible at most of the time due to impreciseness and incompleteness of the information of the data. Fuzziness exist in the FS is identified by the MF [27]. he uncertainty measure is the possible MF of the FS and is interpreted individually. This is the advantage of MFs especially one needs to aggregate the data and human expert knowledge. Designing MFs vary according to the ambition of their use. Membership functions influence a quality of inference [31].

Neutrosophy is the connecting idea with its opposite idea also with non-committal idea to get the common parts with unknown things [36]. Artificial network, fuzzy clustering, genetic algorithm are some methods to determine the MFs and all these consume time with complexities. The MFs plays a vital role in getting the output. The methods are uncertain due to noisy data and difference of opinion of the people. The most suitable shape and widely used MFs in fuzzy systems are triangular and trapezoidal [37]. Properties and relations of multi FSs and its extension are depending on the order relations of the MFs [38]. FS is the class of elements with a continuum of grades of M/S [39].

The logic of neutrosophic concept is an explicit frame trying to calculate the truth, IIndetrminacy and falsity. Smarandache observes the dissimilarity of intuitionistic fuzzy logic (IFL) and neutrosophic logic (NL). NL could differentiate absolute truth (AT) and relative truth (RT) by assigning 1^+ for AT and 1 for RT and is also applied in the field of philosophy. Hence the standard interval [0,1] used in IFS is extended to non-standard $]^-0,1^+[$ in NL. There is not condition on truth, indeterminacy and falsity which are all the subsets of non-standard unitary interval. This is the reason of considering $^-0\,^-0 \le \inf T \le \inf I \le \inf F \le \sup T \le \sup I \le \sup F \le 3^+$ and which is useful to characterize para consistent and incomplete information [40]. The generalized form of trapezoidal FNs, trapezoidal IFNs, triangular FN and TIFNs are the trapezoidal and triangular neutrosophic fuzzy number [48].

2 Review of Literature

The authors of, [Zysno 1] presented a methodology to determine the MFs analytically. [Sebag and Schoenauer 2] Established algorithms to determine functions from examples. [Bergadano and Cutello 3] proposed an effective technique to learn MFs for fuzzy predicates. [Hansson 4] introduce a stochastic perception of the MFs based on fuzzy logic. [Kelly and Painter 5] proposed a methodology to define N-dimensional fuzzy MFs (FMFs) which is a generalized form of one dimensional MF generally used in fuzzy systems. [Peterson et al. 6] presented a hardware implementation of MF. [Royo and Verdegay 7] examined about the characterization of the different cases where the endurance of the MF is assured.

[Grauel and L. A. Ludwig 8] proposed a class of MFs for symmetrically and asymmetrically in exponential order and constructed a more adaptive MFs. [Straszecka 9] presented preliminaries and methodology to define the MFs of FSs and discussed about application of FS with its universe, certainty of MFs and format. [Abe 10] examined the influence of the MFs in fuzzy classifier. [Abe 11] proved that by adjusting the slopes and positions the performance of the fuzzy rule classification can be improved [Pedrycz and G. Vukovich 12] imposed on an influential issue of determining MF. [J. M. Garibaldi and R. I. John 13] focused more MFs which considered as the alternatives in fuzzy systems [T. J. Ross 14] established the methodology of MFs.

[Brennan, E. Martin 15] proposed MFs for dimensional proximity. [Hachani et al. 16] Proposed a new incremental method to represent the MFs for linguistic terms. [Gasparovica et al. 17] examined about the suitable MF for data analysis in bioinformatics. [Zade and Ismayilova 18] investigated a class of MFs which conclude the familiar types of MFs for FSs. [Bilgic 19] proposed a method of measuring MFs. [Broumi et al. 20] established rough neutrosophic sets and their properties. [Salama et al. 21] proposed a technique for constructing. [Yadava and Yadav 22] proposed an approach for constructing the MFs of software metrics. [Piegat and M. Landowski 23] proposed horizontal MFs to determine the FS instead of usual vertical MFs. [Mani 24] reviewed the relation between different meta theoretical concepts of probability and rough MFs critically.

[Sularia 25] showed their interest of multi-criteria decision analysis under fuzzy environment. [Ali and F. Smarandache 26] Introduced complex NS. [Goyal et al. 27] proposed a circuit model for Gaussian MF. [Can and Ozguven 28] proposed fuzzy logic controller with neutrosophic MFs. [Ali et al. 29] introduced δ -equalities and their properties of NSs. [Radhika and Parvathi 30] introduced different fuzzification methods for intuitionistic fuzzy environment. [Porebski and Straszecka 31] examined diagnosing rules for driving data which can be described by human experts. [Hong et al. 32] accumulated the concepts of fuzzy MFs using fuzzy c-means clustering method.

[Kundu 33] proposed an improved method of approximation of piecewise linear MFs with the support of approximation of cut function obtained by sigmoid function. [Wang 34] proposed the operational laws of fuzzy ellipsoid numbers and straight connection between the MFs which are located on the junctions and edges. [Mani 35] studied the contemplation of theory of probability over rough MFs. [Christianto and Smarandache 36] offered a new perception at Liquid church and neutrosophic MF. [Asanka and A. S. Perera 37] introduced a new approach of using box plot to determine fuzzy Function with some conditions. [Sebastian and F. Smarandache 38] generalized the concepts of NSs and its extension method. [Reddy 39] proposed a FS with two MFs such as Belief and Disbelief. [Lupianeza 40] determined NSs and Topology.

[Zhang et al. 41] derived FMFs analytically. [Wang 42] framed a framework theoretically to construct MFs in a hierarchical order. [Germashev et al. 43] proposed convergence of series of FNs along with Unimodal membership. [Marlen and Dorzhigulov 44] implemented FMF with Memristor. [Ahmad et al. 45] introduced MFs and fuzzy rules for Harumanis examinations [Buhentala et al. 46] explained about the procedure and process of the Takagi-Sugeno fuzzy model. [Broumi et al. 47-55] proposed few concepts of NSs, triangular and trapezoidal NNs.

From this literature study, to the best our knowledge there is no contribution of work on deriving membership function using Matlab under neutrosophic environment and hence it's a motivation of the present work.

3 Preliminaries

Definition:A trapezoidal neutrosophic number $a=\langle (a,b,c,d);w_a,u_a,y_a\rangle$ is a special neutrosophic set on the real number set R, whose truth-membership, indeterminacy– membership and falsity-membership functions are defined as follows:

$$\mu_a(x)=\begin{cases}\dfrac{(x-a)}{(b-a)}w_a & , \quad a\le x\le b\\ w_a & , \quad b\le x\le c\\ \dfrac{(d-x)}{(d-c)}w_a & , \quad c\le x\le d\\ 0 & , \quad otherwise\end{cases}\qquad v_a(x)=\begin{cases}\dfrac{(b-x)+u_a(x-a)}{(b-a)} & , \quad a\le x\le b\\ u_a & , \quad b\le x\le c\\ \dfrac{(x-c)+u_a(d-x)}{(d-c)} & , \quad c\le x\le d\\ 1 & , \quad otherwise\end{cases}$$

$$\lambda_a(x)=\begin{cases}\dfrac{(b-x)+y_a(x-a)}{(b-a)} & , \quad a\le x\le b\\ y_a & , \quad b\le x\le c\\ \dfrac{(x-c)+y_a(d-x)}{(d-c)} & , \quad c\le x\le d\\ 1 & , \quad otherwise\end{cases}$$

4. Proposed Matlab code to find Trapezoidal Neutrosophic Function

In this section, trapezoidal neutrosophic function has been proposed using Matlab program and for the differennt membership values, pictorical representation is given and the Matlab code is designed as follows.

```
Trapezoidal neutrosophic Function (trin)
%x=45:70;
%[y,z]=trin(x,50,55,60,65, 0.6, 0.4,0.6)%

U truth membership
V indterminacy membership
W :falsemembership

function [y,z,t]=trin(x,a,b,c,d,u,v,w)
y=zeros(1,length(x));
z=zeros(1,length(x));
t=zeros(1,length(x));
for j=1:length(x)
if(x(j)<=a)
   y(j)=0;
   z(j)=1;
   t(j)=1;
elseif(x(j)>=a)&&(x(j)<=b)
y(j)=u*(((x(j)-a)/(b-a)));
z(j)=(((b-x(j))+v*(x(j)-a))/(b-a));
t(j)=(((b-x(j))+w*(x(j)-a))/(b-a));
elseif(x(j)>=b)&&(x(j)<=c)
y(j)=u;
   z(j)=v;
   t(j)=w;
elseif(x(j)>=c)&&(x(j)<=d)
    y(j)=u*(((d-x(j))/(d-c)));
    z(j)=(((x(j)-c)+v*(d-x(j)))/(d-c));
    t(j)=(((x(j)-c)+w*(d-x(j)))/(d-c));
elseif(x(j)>=d)
   y(j)=0;
z(j)=1;
t(j)=1;
end
end
plot(x,y,x,z,x,t)
legend('Membership function','indeterminate function','Non-membership function')
end
```

4.1 Example

The figure 1 portrayed the pictorical representation of the trapezoidal neutrosophic function $a=\langle(0.3,0.5,0.6,0.7);0.4,0.2,0.3\rangle$

The line command to show this function in Matlab is written below:

```
x=0:0.01:1;
[y,z,t]=trin(x,0.3,0.5,0.6,0.7, 0.4, 0.2,0.3)
```

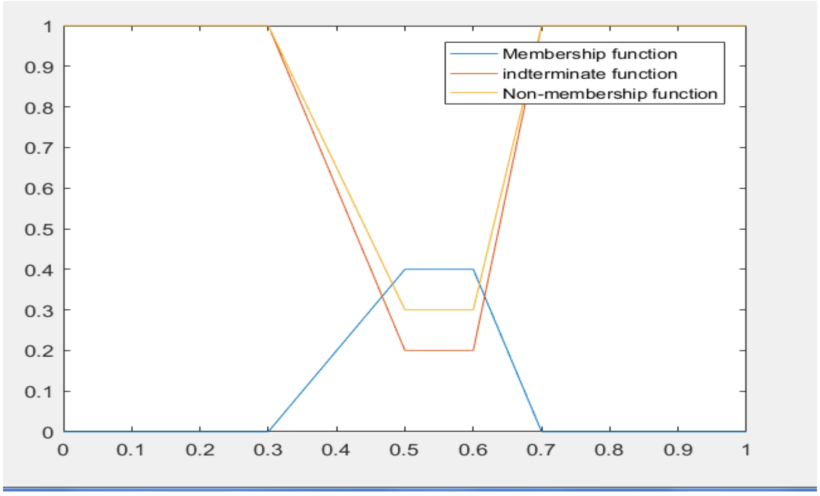

Figure 1: Trapezoidal neutrosophic function for example 4.1

4.2 Example

The figure 2 portrayed the trapezoidal neutrosophic function of $a = \langle (50, 55, 60, 65); 0.6, 0.4, 0.3 \rangle$

The line command to show this function in Matlab is written below:

```
>> x=45:70;
[y,z]=trin(x,50,55,60,65, 0.6, 0.4,0.3)
```

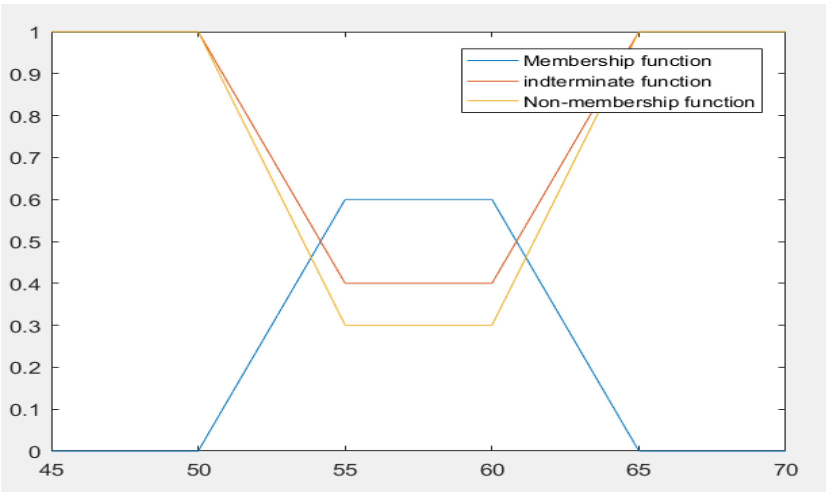

Figure 2: Trapezoidal neutrosophic function for example 4.2

4.3 Example

The figure 3 portrayed the triangular neutrosophic function of $a = \langle (0.3, 0.5, 0.5, 0.7); 0.4, 0.2, 0.3 \rangle$

The line command to show this function in Matlab is written below:

```
x= 0:0.01:1;
[y,z,t]=trin(x,0.3, 0.5,0.5,0.7, 0.4, 0.2,0.3)
```

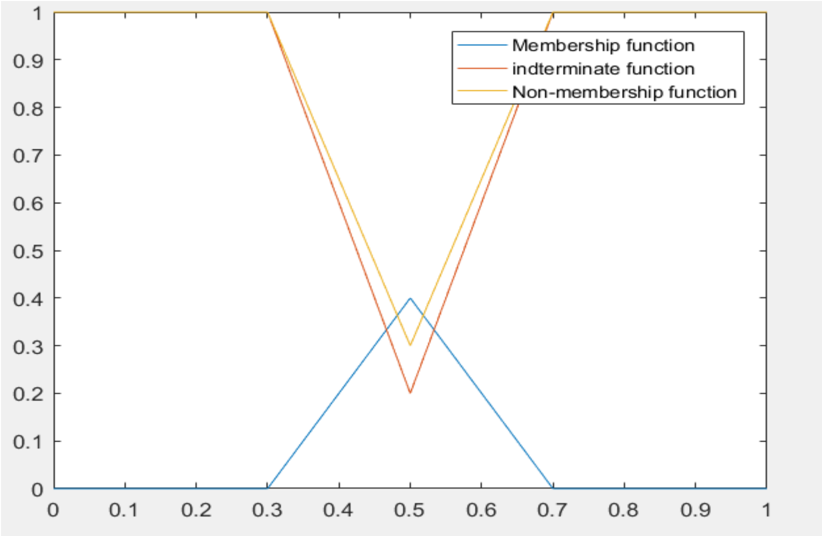

Figure 3: Triangular neutrosophic function for example 3

Remark: if b= c, the trapezoidal neutrosophic function degenerate to triangular neutrosophic function as protrayed in figure 3.

5. Qualitative analysis of different types of graphs

The following analysis helps to know the importance of the neutrosophic graph where the limitations are possible as mentioned in the table for fuzzy and intuitionistic fuzzy graphs.

Types of graphs	Advantages	Limitations
Graphs	• Models of relations • describing information involving relationship between objects • Objects are represented by vertices and relations by edges • Vertex and edge sets are crisp	• Unable to handle fuzzy relation (FR)
Fuzzy graphs (FGs)	• Symmetric binary fuzzy relation on a fuzzy subset • Uncertainty exist in the description of the objects or in the relationships or in both • Able to handle FR with membership value • FGs models are more useful and practical in nature	• Not able to deal interval data
Interval valued FGs	• Edge set of a graphs is a collection of intervals	• Unable to deal the case of non membership
Intuitionistic fuzzy graphs (IntFGs)	• Gives more certainty into the problems • Minimize the cost of operation and enhance efficiency • Contributes a adjustable model to define uncertainty and vagueness exists in decision making • Able to deal non membership of a relation	• Unable to handle interval data

Interval valued IntFGs	• Capable of dealing interval data	• Unable to deal indeterminacy

6. Conclusion

Choosing a MF is an essential task of all the fuzzy and neutrosophic system (Control system or decision making process). Due to the simplicity (less computational complexity) and flexibility triangular and trapezoidal membership functions are widely used in many real world applications. In this paper, trapezoidal neutrosophic membership function is derived using Matlab with illustrative example. In future, this work may be extended to interval valued trapezoidal and triangular neutrosophic membership functions.

Notes

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Dombi Interval Valued Neutrosophic Graph and its Role in Traffic Control Management

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Abstract. An advantage of dealing indeterminacy is possible only with Neutrosophic Sets. Graph theory plays a vital role in the field of networking. If uncertainty exist in the set of vertices and edge then that can be dealt by fuzzy graphs in any application and using Neutrosophic Graph uncertainty of the problems can be completely dealt with the concept of indeterminacy. In this paper, Dombi Interval Valued Neutrosophic Graph has been proposed and Cartesian product and composition of the proposed graphs have been derived. The validity of the derived results have been proved with the numerical example. This paper expose the use of Dombi triangular norms in the area of Neutrosophic graph theory. Advantages and limitations has been discussed for Crisp, Fuzzy, Type-2 Fuzzy, Neutrosophic Set, Interval Neutrosophic Set, Neutrosophic Graph and Interval Neutrosophic Graph.

Keywords: Dombi Triangular Norms, Fuzzy Graphs, Interval valued Neutrosophic Graph, Dombi Interval Valued Neutrosophic Graph, Cartesian product, Composition, Traffic Control Management.

1. Introduction

The generalized Dombi operator family was introduced by Dombi and applied to speech Recognition Task [1, 2]. Graph theory plays a vital role in different fields namely computer science, engineering, physics and biology to deal with complex networks [3, 4]. It is also used to solve various optimization problems in transportation where network is nothing but the logical sequence of the method and visualization possibility, permits surveys to be afforded. Also it responses to two equations simultaneously for the purpose and the procedure [5]. Permanent growth of the population is one of the main face of modern cities and it is the reason for building new roads and highways to avoid traffic problems and for stress free life of the people [6, 13].

A fuzzy set can be described mathematically by assigning a value, a grade of membership to each possible individual in the universe of discourse. This grade of membership associates a degree to which that individual either is similar or appropriate with the concept performed by the fuzzy set. A fuzzy subset of a set X is a mapping from membership to non-membership and is defined by $\eta : X \to [0,1]$ continuous rather than unexpected. Fuzzy relations are popular and important in the fields of computer networks, decision making, neural network, expert systems etc. [7, 16].Direct relationship and also indirect relationship also will be considered in graph theory [8].

Model of relation is nothing but a graph and it is a comfortable way of describing information involving connection between objects [9]. In graph, vertices are represented by vertices and relations by edges. While there is an impreciseness in the statement of the objects or it its communication or in both, fuzzy graph model can be designed for getting an optimized output. Maximizing the Utility of the application is always done by the researchers during the constructing of a model with a key characteristics reliability, complexity and impreciseness. Among these impreciseness plays an important role in maximizing the utility of the model. This situation can be described by fuzzy sets, introduced by Lotfi. A. Zadeh. Fuzzy graphs will be very useful, since for the real world problems, one gets the partial information [10, 11].

Performance evaluation can be defined as modeling and application and should be done before using them using graph theory[12]. When the system is huge and complex, it is challenging to extract the information about the system using classical graph theory and at this junction fuzzy graph can be used to examine the system [14]. A situation in which goods is shifted from one location to another can be dealt by graphs. For example, one can consider water supply, where water users and pipe join etc. are vertices and pipelines are edges [15]. The concept of graph theory was introduced by Euler in 1736 and it is a branch of combinatorics [16]. Fuzzy graph was introduced by Rosenfeld who has defined the fuzzy correlation of various graph theoretic notions such as cycles, paths, trees and connectedness and set some of their properties [18].

Zadeh formulated the term degree of membership and describe the notion of fuzzy set in order to deal with an impreciseness. Atanassov introduced intuitionistic fuzzy set by including the degree of non-membership in the concept of fuzzy set as an independent component. Samarandache introduced Neutrosophic set(NS) by finding the term degree of indeterminacy from the logical point of view as an independent component to handle with imprecise, indeterminate and unpredictable information which are exist in the real world problems The NSs are defined by truth, indeterminacy and false membership functions which are taking the values in the real standard interval. Wang et al. proposed the concept of single-valued Neutrosophic sets (SVNS) and Interval valued Neutrosophic Sets (IVNSs) as well, where the three membership functions are independent and takes value in the unit interval [0,1] [19, 20, 30, 32].

If uncertainty exists in the set of vertices or edges or both then the model becomes a fuzzy graph. Fuzzy graphs can be established by considering the vertex and edge sets as fuzzy, in the same way one can model interval valued fuzzy graphs, intuitionistic fuzzy graphs, interval valued fuzzy graphs, Neutrosophic graphs, single valued Neutrosophic graphs and interval valued Neutrosophic graphs [23]. Network of the brain is a Neutrosophic graph especially strong Neutrosophic graph[25]. Intelligent transport systems is a universal aspect gets the attention of worldwide interest from professionals in transportation, political decision makers and computerized industry. It is developed by understanding the progress of the road traffic in the interval of time and communication between the participants and structural elements available in the situation [26, 31].

Graph theory defines the relationship between various individuals and has got many number of applications in different fields namely database theory, modern sciences and technology, neural networks, data mining cluster analysis, expert systems image capturing and control theory [27]. The strength of the relationship in social networks can be analyzed by fuzzy graph theory and has got important potential [28]. While the network is large, analysis and evaluation of traffic will be very challenging one for the network managers and it can be done using dynamic Bandwidth [29]. Indeterminacy of the object or edge or both cannot be handled by fuzzy, intuitionistic fuzzy, bipolar fuzzy or interval valued fuzzy graphs and hence Neutrosophic graphs have been introduced [33].

Menger proposed triangular norms in the structure of probabilistic metric spaces and discussed by Schweizer Sklar. Also intersection and union of fuzzy sets have been proved by Alsina et al. Triangular norms play an important role application of fuzzy logic namely fuzzy graph and decision making process [34]. Some of the real world applications can be modelled in a better way with triangular norms especially t-norm than using minimum operations. Using this concept awareness of tracking in person for networks is possible [36]

Since intervals plays an essential role in graph theory and useful in the study of properties of fuzzy graphs which applied to surveying the land based on the concept of the distance between the vertices, using the concepts of Interval Valued Neutrosophic Graphs and Dombi fuzzy graphs, Dombi Single Valued and Dombi Interval Valued Neutrosophic Graphs have been proposed. Also Cartesian product and composition of Dombi Interval Valued Neutrosophic Graph have been derived. Numerical example also has been given for the validity of the results. The main goal of this paper is to emphasis that the minimum and maximum operators are not the only applicant for the logical reasoning of the classical graphs to Neutrosophic graphs.

2. Review of Literature

The authors of [1] introduced the generalized Dombi operator family and the multiplicative utility function. [2] applied the Generalized Dombi Operator Family to the Speech Recognition Task. [3] explained about Neutrosophic model and Control. [4] examined biological networks using graph theory. [5] established a computerized technique to solve network problems using graph theory. [6] analyzed a traffic control problem using cut-set of a graph and applied in arbitrary intersection to reduce the waiting time of the people.[7] proposed operation on a complement of a fuzzy graph. [8] proposed functional consistency of an input Gene Network. [9] proposed a program for coloring the vertex of a fuzzy graph. [10] applied vertex coloring function of a fuzzy graph to the traffic light problem.

[11] reviewed about Fuzzy Graph Theory.[12] evaluated an impreciseness produced in performance measures during the procedure of performance evaluation using graph theory. [13] introduced graph for the problem and circular arcs and applied in traffic management. [15] applied the concept of graph in traffic control management in city and airport.[16] described certain types of Neutrosophic graphs.[17] proposed a new dimension to graph theory. [18] proposed strong domination number using membership values of strong arcs in fuzzy graphs. [19] gave introduction to bipolar single valued Neutrosophic Graph theory. (Broumi et al. 2016) proposed an isolated Interval valued Neutrosophic graph.

[20] applied interval valued Neutrosophic in decision making problem to invest the money in the best company. [21] proved the necessary and sufficient condition for a Neutrosophic graph to be an isolated single valued Neutrosophic graph. [22] examined the properties of different types of degrees size and order of SVNGs and proposed the definition of regular SVNG. [25] proposed strong NGs and Sub graph Topological Subspaces. [26] presented a complete study of all existing Intelligent Transport systems namely research models and open systems.[27] applied graph theory concepts to SVNGs and examine a new type of graph model and concluded the result to crisp graphs, fuzzy graphs and intuitionistic fuzzy graphs and characterized their properties.

[28] examined asymmetrical partnership using fuzzy graph and detect hidden connections in Facebook. [29] proposed an optimized algorithm using the approach of rating of web pages and it assigns a minimum approved bandwidth to every connected user.[30] represented a graph model based on IVN sets.[31] proposed dimensional modeling of traffic in urban road using graph theory. [32] proposed uniform SVNGs. [33] proposed some of the results on the graph theory for complex NSs. [34] proposed Dombi fuzzy graphs and proved the standard operations on Dombi fuzzy graphs. [35] proposed fuzzy graph of semigroup.[36] proposed t-norm fuzzy graphs and discussed the importance of t-norm in network system.

3. Basic Concepts

Some basic concepts needed for proposing and deriving the results are listed below.

3.1 Graph (Ashraf et al. 2018)

A mathematical system $G=(V,E)$ is called a graph, where $V = V(G)$, a vertex set and $E=E(G)$ is anedge set. In this paper, undirected graph has been considered and hence every edge is considered as an unordered pair of different vertices.

3.2 Fuzzy Graph (Marapureddy 2018)

Let V be a non-empty finite set, λ be a fuzzy subsets on V and δbe a fuzzy subsets on $\mathrm{V}\times\mathrm{V}$. The pair $\mathrm{G}=(\lambda,\delta)$ is a fuzzy graph over the set V if $\delta(x,y)\leq\min\{\lambda(x),\lambda(y)\}$ for all $(x,y)\in\mathrm{V}\times\mathrm{V}$ where λ is a fuzzy vertex and $\delta$is a fuzzy edge. Where:

1. A mapping $\lambda:\mathrm{V}\rightarrow[0,1]$ is called a **fuzzy subset** of V, where V is the non-empty set.
2. A mapping $\delta:\mathrm{V}\times\mathrm{V}\rightarrow[0,1]$ is a **fuzzy relation** on λ of V if $\delta(x,y)\leq\min\{\lambda(x),\lambda(y)\}$
3. If $\delta(x,y)=\min\{\lambda(x),\lambda(y)\}$ then G is a **strong fuzzy graph**.

3.3 Dombi Fuzzy Graph (Ashraf et al. 2018)

A pair $G=(\lambda,\delta)$ is a Dombi fuzzy graph if $\delta(xy)\le\frac{\lambda(x)\lambda(y)}{\lambda(x)+\lambda(x)-\lambda(x)\lambda(y)}$, for all $x,y\in V$, where the Dombi fuzzy vertex set, $\lambda:V\to[0,1]$ is a fuzzy subset in V and the Dombi fuzzy edge set, $\delta:V\times V\to[0,1]$ is a symmetric fuzzy relation on λ.

3.4 Single Valued Neutrosophic Graph (SVNG) (Broumi et al. 2016)

A pair $G_N=(P,Q)$ is SVNG with elemental set v . Where:

1. Degree of truth membership, indeterminacy membership and falsity membership of the element $x_i\in\mathbf{V}$ are defined by $T_P:V\to[0,1]$, $I_P:V\to[0,1]$ and $F_P:V\to[0,1]$ respectively and $0\le T_P(x_i)+I_P(x_i)+F_P(x_i)\le 3,\forall x_i\in\mathbf{V}, i=1,2,3,...,n$
2. Degree of truth membership, indeterminacy membership and falsity membership of the edge $(x_i,y_j)\in\mathbf{E}$ are denoted by $T_Q:E\subseteq V\times V\to[0,1]$, $I_Q:E\subseteq V\times V\to[0,1]$ and $F_Q:E\subseteq V\times V\to[0,1]$ respectively and are defined by
   - $T_Q\left(\{x_i,y_j\}\right)\le\min\left[T_P(x_i),T_P(y_j)\right]$
   - $I_Q\left(\{x_i,y_j\}\right)\ge\max\left[I_P(x_i),I_P(y_j)\right]$
   - $F_Q\left(\{x_i,y_j\}\right)\ge\max\left[F_P(x_i)F_P(y_j)\right]$

where $0\le T_Q\left(\{x_i,y_j\}\right)+I_Q\left(\{x_i,y_j\}\right)+F_Q\left(\{x_i,y_j\}\right)\le 3,\forall\{x_i,y_j\}\in\mathbf{E}\,(i,j=1,2,...,n)$.

Also P is a single valued Neutrosophic vertex of V and Q is a single valued Neutrosophic edge set of $\mathbf{E}$. Q is a symmetric single valued Neutrosophic relation on P.

3.5Interval Valued Neutrosophic Graph (IVNG) (Broumi et al. 2016)

A pair $G_N=(P,Q)$ is IVNG, where $P=\left\langle\left[T_P^L,T_P^U\right],\left[I_P^L,I_P^U\right],\left[F_P^L,F_P^U\right]\right\rangle$, an IVN is set on V and $Q=\left\langle\left[T_Q^L,T_Q^U\right],\left[I_Q^L,I_Q^U\right],\left[F_Q^L,F_Q^U\right]\right\rangle$ is an IVN edge set on $\mathbf{E}$ satisfying the following conditions:

1. Degree of truth membership, indeterminacy membership and falsity membership of the element $x_i\in\mathbf{V}$ are defined by $T_P^L:V\to[0,1]$, $T_P^U:V\to[0,1]$, $I_P^L:V\to[0,1]$, $I_P^L:V\to[0,1]$, $I_P^U:V\to[0,1]$ and $F_P^L:V\to[0,1], F_P^U:V\to[0,1]$ respectively and $0\le T_P(x_i)+I_P(x_i)+F_P(x_i)\le 3,\forall x_i\in\mathbf{V}, i=1,2,3,...,n$
2. Degree of truth membership, indeterminacy membership and falsity membership of the edge $(x_i,y_j)\in\mathbf{E}$ are denoted by $T_Q^L:V\times V\to[0,1]$, $T_Q^U:V\times V\to[0,1]$ $I_Q^L:V\times V\to[0,1]$, $I_Q^U:V\times V\to[0,1]$ and $F_Q^L:V\times V\to[0,1]$, $F_Q^U:V\times V\to[0,1]$ respectively and are defined by
   - $T_Q^L\left(\{x_i,y_j\}\right)\le\min\left[T_P^L(x_i),T_P^L(y_j)\right]$
   - $T_Q^U\left(\{x_i,y_j\}\right)\le\min\left[T_P^U(x_i),T_P^U(y_j)\right]$
   - $I_Q^L\left(\{x_i,y_j\}\right)\ge\max\left[T_P^L(x_i),T_P^L(y_j)\right]$
   - $I_Q^U\left(\{x_i,y_j\}\right)\ge\max\left[I_P^U(x_i),I_P^U(y_j)\right]$
   - $F_Q^L\left(\{x_i,y_j\}\right)\ge\max\left[F_P^L(x_i),F_P^L(y_j)\right]$
   - $F_Q^U\left(\{x_i,y_j\}\right)\ge\max\left[F_P^U(x_i),F_P^U(y_j)\right]$

where $0\le T_Q\left(\{x_i,y_j\}\right)+I_Q\left(\{x_i,y_j\}\right)+F_Q\left(\{x_i,y_j\}\right)\le 3,\forall\{x_i,y_j\}\in\mathbf{E}\,(i,j=1,2,...,n)$.

3.6 Triangular Norms (Ashraf et al. 2018)

A mapping $T:[0,1]^2 \to [0,1]$ is a binary operation and is called a triangular norm or T-Norm if $\forall x, y, z \in [0,1]$ it satisfies the following:

(1). $T(1,x) = x$ (Boundary condition)
(2). $T(x,y) = T(y,x)$ (Commutativity)
(3). $T(x,T(y,z)) = T(T(x,y),z)$ (Associativity)
(4). $T(x,y) \le T(x,z)$, if $y \le z$ (Monotonicity)

- Triangular conorm or T-Conorm is also a binary operation $TC:[0,1]^2 \to [0,1]$ and is defined by $TC(x,y) = 1 - T(1-x, 1-y)$

3.7 Dombi Triangular Norms (Dombi 2009, Dombi and Kocsor 2009, Ashraf et al. 2018)

Dombi product or T-Norm and T-Conorm are denoted by $\otimes$ and $\oplus$ respectively and defined by

$$TN(x,y) = x \underset{D}{\otimes} y = \frac{1}{1+\left[\left(\frac{1-x}{y}\right)^{\xi} + \left(\frac{1-y}{y}\right)^{\xi}\right]^{1/\xi}}, \xi > 0$$

$$TCN(x,y) = x \underset{D}{\oplus} y = \frac{1}{1+\left[\left(\frac{1-x}{y}\right)^{-\xi} + \left(\frac{1-y}{y}\right)^{-\xi}\right]^{1/-\xi}}, \xi > 0$$

This triangular norm contains the product, Hamacher operators, and Einstein operators and as the limiting case, minimum and maximum operators can be obtained. Multivariable case can be dealt easily by the new form of Hamacher family. Dombi operators have flexible parameters and hence the success rate will be greater one.

3.8 Hamacher Triangular Norms (Ashraf et al. 2018)

Hamacher product or T-Norm and T-Conorm are denoted by $\otimes$ and $\oplus$ respectively and defined by

$$T(x,y) = x \underset{H}{\otimes} y = \frac{xy}{\xi + (1-\xi)(x+y-xy)}, \xi > 0$$

$$TC(x,y) = x \underset{H}{\otimes} y = \frac{x+y+(\xi-2)xy}{1+(\xi-1)xy}, \xi > 0$$

3.9 Special Cases of Dombi and Hamacher Triangular Norms (Ashraf et al. 2018)

If we replace $\xi = 0$ in Hamacher family of triangular norms and $\xi = 1$ in Dombi family of triangular norms then

$T(x,y) = x \otimes y = \frac{xy}{(x+y-xy)}$ and $TC(x,y) = x \oplus y = \frac{x+y-2xy}{(1-xy)}$.

3.10 Standard Products of graphs (Ashraf et al. 2018)

Consider two $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ then the following products are defined by

1. Direct product : $E(G_1 \square G_2) = \{(x_1,x_2)(y_1,y_2) \mid x_1y_1 \in E_1 \ \& \ x_2y_2 \in E_1\}$
2. Cartesian Product: $E(G_1 \times G_2) = \{(x_1,x_2)(y_1,y_2) \mid x_1 = y_1 \ \& \ x_2y_2 \in E_2, or\, x_1y_1 \in E_1 \ \& \ x_2 = y_2\}$
3. Strong Product: $E(G_1 \times G_2) = \{(x_1,x_2)(y_1,y_2) \mid x_1 = y_1 \ \& \ x_2y_2 \in E_2, or\, x_1y_1 \in E_1 \ \& \ x_2y_2 \in E_2\}$
4. Dombi fuzzy Cartesian product:
$$(\lambda_1 \times \lambda_2)(x_1,x_2) = \frac{\lambda_1(x_1)\lambda_2(x_2)}{\lambda_1(x_1)+\lambda_2(x_2)-\lambda_1(x_1)\lambda_2(x_2)}, \forall (x_1,x_2) \in \mathbf{V}_1 \times \mathbf{V}_2$$

$$(\delta_1\times\delta_2)((x,x_2)(x,y_2))=\frac{\lambda_1(x)\delta_2(x_2y_2)}{\lambda_1(x)+\delta_2(x_2y_2)-\lambda_1(x_1)\delta_2(x_2y_2)},\forall x\in \mathbf{V}_1, x_2y_2\in \mathbf{E}_2$$

$$(\delta_1\times\delta_2)((x_1,z)(y_1,z))=\frac{\lambda_2(z)\delta_1(x_1y_1)}{\lambda_2(z)+\delta_1(x_1y_1)-\lambda_2(z)\delta_1(x_1y_1)},\forall z\in \mathbf{V}_2, x_1y_1\in \mathbf{E}_1$$

3.11 Neutrosophic Controllers (Aggarwal et al. 2010)

In the field of logic, fuzzy logic is a powerful one due its capacity of extracting the information from imprecise data. Human interpretations and computer simulation are the active and interested area among the researchers where the computers are managing only precise assessments. But the Neutrosophic logic has the capacity of recovering all sorts of logics. It is an appropriate choice to imitate the action of human brain which is assembled with handling with uncertainties and impreciseness.
Neutrosophic Controllers are giving an optimized results than the fuzzy supplement since they are more hypothesized and indeterminacy tolerant. These controllers would modify considerably according to the nature of the problem.

Modeling a proper mathematical structure of a control problem that would simulate the behavior of the system is very difficult. Due to impreciseness and indeterminacy in the data, unpredictable environmental disturbances, corrupt sensor getting a complete and determinate date is not possible in the real world problems. These situations can be handled by considering Neutrosophic linguistic terms and can be dealt effectively by considering interval valued Neutrosophic environment as it handles more impreciseness using lower and upper membership functions.

4. Proposed Dombi Interval Valued Neutrosophic Graph

Using the above special triangular norms Dombi Interval Valued Neutrosophic Graph (DIVNG) has been proposed and derived the Cartesian and composite products of DIVNG had been derived along with the numerical example.

4.1 Dombi Single Valued Neutrosophic Graph (DSVNG)

A single Valued Neutrosophic Graph is a DSVNG on V is defined to be a pair $\mathrm{G_N}=(P,Q)$, where:

1. The functions $T_P:\mathrm{V}\to[0,1]$, $I_P:\mathrm{V}\to[0,1]$ and $F_P:\mathrm{V}\to[0,1]$ are the degree of truth membership. Indeterminacy membership and falsity membership of the element $x_i\in\mathrm{V}$ respectively and $0\le T_P(x_i)+I_P(x_i)+F_P(x_i)\le 3, \forall x_i\in \mathbf{V}, i=1,2,3,...,n$.
2. The functions $T_Q:\mathrm{E}\subseteq\mathrm{V}\times\mathrm{V}\to[0,1]$, $I_Q:\mathrm{E}\subseteq\mathrm{V}\times\mathrm{V}\to[0,1]$ and $F_Q:\mathrm{E}\subseteq\mathrm{V}\times\mathrm{V}\to[0,1]$ are defined by

$$T_Q(\{x_i,y_j\})\le\frac{T_P(x_i)T_P(y_j)}{T_P(x_i)+T_P(y_j)-T_P(x_i)T_P(y_j)}$$

$$I_Q(\{x_i,y_j\})\ge\frac{I_P(x_i)+I_P(y_j)-2I_P(x_i)I_P(y_j)}{1-I_P(x_i)I_P(y_j)}$$

$$F_Q(\{x_i,y_j\})\ge\frac{F_P(x_i)+F_P(y_j)-2F_P(x_i)F_P(y_j)}{1-F_P(x_i)F_P(y_j)}$$

Numerical Example:

Figure 1 is an example of DSVNG $\mathrm{G_N}=(P,Q)$ of the graph $G=(V,E)$ such that the vertex set is $P=\{\langle i,(0.5,0.1,0.4)\rangle,\langle j,(0.6,0.3,0.2)\rangle,\langle k,(0.2,0.3,0.4)\rangle,\langle l,(0.4,0.2,0.5)\rangle\}$
and the edge set is given by
$Q=\{\langle ij,(0.4,0.3,0.5)\rangle,\langle jk,(0.2,0.4,0.5)\rangle\},\langle kl,(0.2,0.4,0.6)\rangle,\langle il,(0.3,0.4,0.6)\rangle$

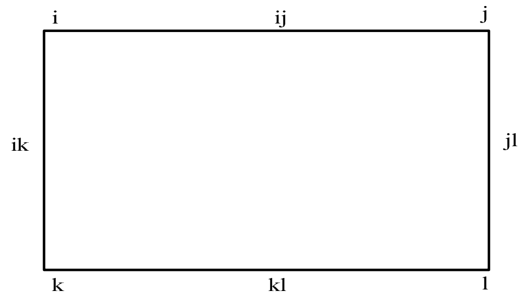

Fig.1.Dombi Single Valued Neutrosophic Graph

Numerical Computation:

To find $T_Q(ij)$:

$$T_Q(ij) \le \frac{(0.5)(0.6)}{(0.5)+(0.6)-(0.5)(0.6)} \le 0.4 \text{, hence } T_Q(ij)=0.4$$

To find $I_Q(ij)$:

$$I_Q(ij) \ge \frac{(0.1)+(0.3)-2(0.1)(0.3)}{1-(0.1)(0.3)} \ge 0.3 \text{, hence } I_Q(ij)=0.3$$

To find $F_Q(ij)$:

$$F_Q(ij) \ge \frac{(0.4)+(0.2)-2(0.4)(0.2)}{1-(0.4)(0.2)} \ge 0.5 \text{, hence } I_Q(ij)=0.5$$

Similarly other values can be found.

- While the membership values of truth, indeterminacy and falsity are not in the interval form then the DSVNG can be used to get the solution or output of the system for the linguistic terms. Hence DSVNG is a special case of interval based Neutrosophic graph which is proposed below.

4.2 Dombi Interval Valued Neutrosophic Graph (DIVNG)

A pair $G_N=(P,Q)$ is DIVNG, where $P=\left\langle\left[T_P^L,T_P^U\right],\left[I_P^L,I_P^U\right],\left[F_P^L,F_P^U\right]\right\rangle$, is an IVN set on $\mathbf{V}$ and $Q=\left\langle\left[T_Q^L,T_Q^U\right],\left[I_Q^L,I_Q^U\right],\left[F_Q^L,F_Q^U\right]\right\rangle$ is an IVN edge set on $\mathbf{E}$ satisfying the following conditions:

1. Degree of truth membership, indeterminacy membership and falsity membership of the element $x_i\in\mathbf{V}$ are defined by $T_P^L:\mathrm{V}\to[0,1]$, $T_P^U:\mathrm{V}\to[0,1]$, $I_P^L:\mathrm{V}\to[0,1]$, $I_P^L:\mathrm{V}\to[0,1]$, $I_P^U:\mathrm{V}\to[0,1]$ and $F_P^L:\mathrm{V}\to[0,1]$, $F_P^U:\mathrm{V}\to[0,1]$ respectively and $0\le T_P(x_i)+I_P(x_i)+F_P(x_i)\le 3, \forall x_i\in\mathbf{V}, i=1,2,3,...,n$
2. Degree of truth membership, indeterminacy membership and falsity membership of the edge $\left(x_i,y_j\right)\in\mathbf{E}$ are denoted by the functions, $T_Q^L:\mathrm{V}\times\mathrm{V}\to[0,1]$, $T_Q^U:\mathrm{V}\times\mathrm{V}\to[0,1]$ $I_Q^L:\mathrm{V}\times\mathrm{V}\to[0,1]$, $I_Q^U:\mathrm{V}\times\mathrm{V}\to[0,1]$ and $F_Q^L:\mathrm{V}\times\mathrm{V}\to[0,1]$, $F_Q^U:\mathrm{V}\times\mathrm{V}\to[0,1]$ respectively and are defined by

- $T_Q^L\left(\{x_i,y_j\}\right)\le\dfrac{T_P^L(x_i)T_P^L(y_j)}{T_P^L(x_i)+T_P^L(y_j)-T_P^L(x_i)T_P^L(y_j)}$
- $T_Q^U\left(\{x_i,y_j\}\right)\le\dfrac{T_P^U(x_i)T_P^U(y_j)}{T_P^U(x_i)+T_P^U(y_j)-T_P^U(x_i)T_P^U(y_j)}$
- $I_Q^L\left(\{x_i,y_j\}\right)\ge\dfrac{I_P^L(x_i)+I_P^L(y_j)-2I_P^L(x_i)I_P^L(y_j)}{1-I_P^L(x_i)I_P^L(y_j)}$
- $I_Q^U\left(\{x_i,y_j\}\right)\ge\dfrac{I_P^U(x_i)+I_P^U(y_j)-2I_P^U(x_i)I_P^U(y_j)}{1-I_P^U(x_i)I_P^U(y_j)}$
- $F_Q^L\left(\{x_i,y_j\}\right)\ge\dfrac{F_P^L(x_i)+F_P^L(y_j)-2F_P^L(x_i)F_P^L(y_j)}{1-F_P^L(x_i)F_P^L(y_j)}$

- $F_Q^U\left(\left\{x_i, y_j\right\}\right) \geq \dfrac{F_P^U\left(x_i\right)+F_P^U\left(y_j\right)-2F_P^U\left(x_i\right)F_P^U\left(y_j\right)}{1-F_P^U\left(x_i\right)F_P^U\left(y_j\right)}$

Where $0 \leq T_Q\left(\left\{x_i, y_j\right\}\right)+I_Q\left(\left\{x_i, y_j\right\}\right)+F_Q\left(\left\{x_i, y_j\right\}\right) \leq 3, \forall\left\{x_i y_j\right\} \in \mathbf{E}\left(i, j=1,2,\ldots,n\right)$.

Numerical Example: Fig. 1. is an example of Dombi Interval Valued Neutrosophic Graph

$$P=\left\{\langle i,[0.5,0.7],[0.2,0.3],[0.1,0.3]\rangle,\langle j,[0.6,0.7],[0.2,0.4],[0.1,0.3]\rangle\right.$$
$$\left.\langle k,[0.4,0.6],[0.1,0.3],[0.2,0.4]\rangle\right\}$$
$$Q=\left\{\langle ij,[0.4,0.5],[0.4,0.5],[0.3,0.5]\rangle,\langle jk,[0.3,0.5],[0.3,0.5],[0.3,0.5]\rangle\right.$$
$$\left.\langle ik,[0.3,0.5],[0.3,0.5],[0.3,0.5]\rangle\right\}$$

Numerical Computation:

To find $T_Q^L(ij)$:

$T_Q^L(ij) \leq \dfrac{(0.5)(0.4)}{(0.5)+(0.4)-(0.5)(0.4)} \leq 0.4$, hence $T_Q^L(ij)=0.4$

To find $I_Q^L(ij)$:

$I_Q^L(ij) \geq \dfrac{(0.2)+(0.2)-2(0.2)(0.2)}{1-(0.2)(0.2)} \geq 0.3$, hence $I_Q^L(ij)=0.4$

To find $F_Q^L(ij)$:

$F_Q^L(ij) \geq \dfrac{(0.1)+(0.1)-2(0.1)(0.1)}{1-(0.1)(0.1)} \geq 0.2$, hence $I_Q^L(ij)=0.3$

Similarly other values can be found.

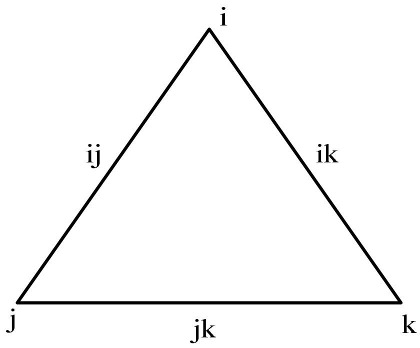

Fig.2. Dombi Interval Valued Neutrosophic Graph

- From the definition and numerical examples of DSVNG and DIVNG, it is found that Dombi Fuzzy Graph is a special case of DSVNG and DIVNG.

4.3Definition: Cartesian product of Dombi Interval Valued Neutrosophic Graphs

Consider λ_i, a Neutrosophic fuzzy subset of $\mathbf{V}_i$ and δ_i, a fuzzy subset of $\mathbf{E}_i, i=1,2$. Let $\mathbf{G}_{\mathbf{N}1}(\lambda_1,\delta_1)$ and $\mathbf{G}_{\mathbf{N}2}(\lambda_2,\delta_2)$ be two Dombi Neutrosophic Fuzzy Graphs of the crisp graphs $G_1^*(V_1,E_1)$ and $G_2^*(V_2,E_2)$ respectively and are defined by

$\left(\lambda_1^L \times \lambda_2^L\right)(x_1, x_2) = \dfrac{\lambda_1^L(x_1)\lambda_2^L(x_2)}{\lambda_1^L(x_1)+\lambda_2^L(x_2)-\lambda_1^L(x_1)\lambda_2^L(x_2)}$, for all $(x_1, x_2) \in V_1 \times V_2$ and

$\left(\delta_1^L \times \delta_2^L\right)((x, x_2)(x, y_2)) = \dfrac{\lambda_1^L(x)\lambda_2^L(x_2y_2)}{\lambda_1^L(x)+\lambda_2^L(x_2y_2)-\lambda_1^L(x)\lambda_2^L(x_2y_2)}$, for all $x \in V_1$, $x_2y_2 \in \mathbf{E}_2$

$\left(\delta_1^L \times \delta_2^L\right)((x_1, z)(y_1, z)) = \dfrac{\lambda_2^L(z)\delta_1^L(x_1y_1)}{\lambda_2^L(z)+\delta_1^L(x_1y_1)-\lambda_2^L(z)\delta_1^L(x_1y_1)}$, for all $z \in \mathbf{V}_2$, $x_1y_1 \in E_1$

Similarly for Indeterminacy and Falsity memberships with upper and lower membership values.

4.3.1Proposition

Let $\mathbf{G_{N1}}$ and $\mathbf{G_{N2}}$ be the Dombi IVN edge graphs of G_1 and G_2 respectively. Then Cartesian product of two Dombi IVN edge graphs is the Dombi Interval Valued Neutrosophic edge graph.

Proof:

Let $\mathbf{E} = \{(x, x_2)(x, y_2) / x \in V_1, x_2y_2 \in \mathbf{E}_2\} \cup \{(x_1, z)(y_1, z) / z \in V_2, x_1y_1 \in \mathbf{E}_1\}$

Consider $x \in V_1$, $x_2y_2 \in E_2$

By the definition of Cartesian product of IVNG **(Broumi et al. 2016)**

$$\left(T_{Q_1}^L \times T_{Q_2}^L\right)((x, x_2)(x, y_2))$$

$$= \min\left(T_{P_1}^L(x), T_{Q_2}^L(x_2y_2)\right) = TN\left(1, T_{Q_2}^L(x_2y_2)\right) \le \frac{T_{P_2}^L(x_2)T_{P_2}^L(y_2)}{T_{P_2}^L(x_2)+T_{P_2}^L(y_2)-T_{P_2}^L(x_2)T_{P_2}^L(y_2)}$$

$$\le \min\left(T_{P_1}^L(x), \min\left(T_{P_2}^L(x_2), T_{P_2}^L(y_2)\right)\right) = \min\left(\min\left(T_{P_1}^L(x), T_{P_2}^L(x_2)\right), \min\left(T_{P_1}^L(x), T_{A_2}^L(y_2)\right)\right)$$

$$= \min\left(\left(T_{P_1}^L \times T_{P_2}^L\right)(x, x_2), \left(T_{P_1}^L \times T_{P_2}^L\right)(x, y_2)\right)$$

$$\le \frac{\left(T_{P_1}^L \times T_{P_2}^L\right)((x, x_2))\left(T_{P_1}^L \times T_{P_2}^L\right)((x, y_2))}{\left(T_{P_1}^L \times T_{P_2}^L\right)((x, x_2))+\left(T_{P_1}^L \times T_{P_2}^L\right)((x, y_2))-\left(T_{P_1}^L \times T_{P_2}^L\right)((x, x_2))\left(T_{P_1}^L \times T_{P_2}^L\right)((x, y_2))}$$

$$\left(T_{Q_1}^U \times T_{Q_2}^U\right)((x, x_2)(x, y_2))$$

$$= \min\left(T_{P_1}^U(x), T_{Q_2}^U(x_2y_2)\right) = TN\left(1, T_{Q_2}^U(x_2y_2)\right) \le \frac{T_{P_2}^U(x_2)T_{P_2}^U(y_2)}{T_{P_2}^U(x_2)+T_{P_2}^U(y_2)-T_{P_2}^U(x_2)T_{P_2}^U(y_2)}$$

$$\le \min\left(T_{P_1}^U(x), \min\left(T_{P_2}^U(x_2), T_{P_2}^U(y_2)\right)\right) = \min\left(\min\left(T_{P_1}^U(x), T_{P_2}^U(x_2)\right), \min\left(T_{P_1}^U(x), T_{A_2}^U(y_2)\right)\right)$$

$$= \min\left(\left(T_{P_1}^U \times T_{P_2}^U\right)(x, x_2), \left(T_{P_1}^U \times T_{P_2}^U\right)(x, y_2)\right)$$

$$= \frac{\left(T_{P_1}^U \times T_{P_2}^U\right)((x, x_2))\left(T_{P_1}^U \times T_{P_2}^U\right)((x, y_2))}{\left(T_{P_1}^U \times T_{P_2}^U\right)((x, x_2))+\left(T_{P_1}^U \times T_{P_2}^U\right)((x, y_2))-\left(T_{P_1}^U \times T_{P_2}^U\right)((x, x_2))\left(T_{P_1}^U \times T_{P_2}^U\right)((x, y_2))}$$

$$\left(I_{Q_1}^L \times I_{Q_2}^L\right)((x, x_2)(x, y_2)) = \max\left(I_{P_1}^L(x), I_{Q_2}^L(x_2y_2)\right)$$

$$\ge TCN\left(I_{P_1}^L(x), I_{Q_2}^L(x_2y_2)\right) = \max\left(\max\left(I_{P_1}^L(x), I_{P_2}^L(x_2)\right), \max\left(I_{P_1}^L(x), T_{P_2}^L(y_2)\right)\right) = \max\left(\left(I_{P_1}^L \times I_{P_2}^L\right)((x, x_2)), \left(I_{P_1}^L \times I_{P_2}^L\right)((x, y_2))\right)$$

$$= \frac{\left(I_{P_1}^L \times I_{P_2}^L\right)((x, x_2))+\left(I_{P_1}^L \times I_{P_2}^L\right)((x, y_2))-2\left(I_{P_1}^L \times I_{P_2}^L\right)((x, x_2))\left(I_{P_1}^L \times I_{P_2}^L\right)((x, y_2))}{1-\left(I_{P_1}^L \times I_{P_2}^L\right)((x, x_2)).\left(I_{P_1}^L \times I_{P_2}^L\right)((x, y_2))}$$

$$\left(I_{Q_1}^U \times I_{Q_2}^U\right)((x, x_2)(x, y_2)) = \max\left(I_{P_1}^U(x), I_{Q_2}^U(x_2y_2)\right)$$

$$\ge TCN\left(I_{P_1}^U(x), I_{Q_2}^U(x_2y_2)\right) = \max\left(\max\left(I_{P_1}^U(x), I_{P_2}^U(x_2)\right), \max\left(I_{P_1}^U(x), T_{P_2}^U(y_2)\right)\right) = \max\left(\left(I_{P_1}^U \times I_{P_2}^U\right)((x, x_2)), \left(I_{P_1}^U \times I_{P_2}^U\right)((x, y_2))\right)$$

$$=\frac{\left(I_{P_1}^U\times I_{P_2}^U\right)((x,x_2))+\left(I_{P_1}^U\times I_{P_2}^U\right)((x,y_2))-2\left(I_{P_1}^U\times I_{P_2}^U\right)((x,x_2))\left(I_{P_1}^U\times I_{P_2}^U\right)((x,y_2))}{1-\left(I_{P_1}^U\times I_{P_2}^U\right)((x,x_2)).\left(I_{P_1}^U\times I_{P_2}^U\right)((x,y_2))}$$

$$\left(F_{Q_1}^L\times F_{Q_2}^L\right)((x,x_2)(x,y_2)) = \max\left(F_{P_1}^L(x),F_{Q_2}^L(x_2y_2)\right)$$
$$\geq TCN\left(F_{P_1}^L(x),F_{Q_2}^L(x_2y_2)\right) = \max\left(\max\left(F_{P_1}^L(x),F_{P_2}^L(x_2)\right),\max\left(F_{P_1}^L(x),F_{P_2}^L(y_2)\right)\right) = \max\left(\left(F_{P_1}^L\times F_{P_2}^L\right)((x,x_2)),\left(F_{P_1}^L\times F_{P_2}^L\right)((x,y_2))\right)$$
$$\geq\frac{\left(F_{P_1}^L\times F_{P_2}^L\right)((x,x_2))+\left(F_{P_1}^L\times F_{P_2}^L\right)((x,y_2))-2\left(F_{P_1}^L\times F_{P_2}^L\right)((x,x_2))\left(F_{P_1}^L\times F_{P_2}^L\right)((x,y_2))}{1-\left(F_{P_1}^L\times F_{P_2}^L\right)((x,x_2)).\left(F_{P_1}^L\times F_{P_2}^L\right)((x,y_2))}$$

$$\left(F_{Q_1}^U\times F_{Q_2}^U\right)((x,x_2)(x,y_2)) = \max\left(F_{P_1}^U(x),F_{Q_2}^U(x_2y_2)\right)$$
$$\geq TCN\left(F_{P_1}^U(x),F_{Q_2}^U(x_2y_2)\right) = \max\left(\max\left(F_{P_1}^U(x),F_{P_2}^U(x_2)\right),\max\left(F_{P_1}^U(x),F_{P_2}^U(y_2)\right)\right) = \max\left(\left(F_{P_1}^U\times F_{P_2}^U\right)((x,x_2)),\left(F_{P_1}^U\times F_{P_2}^U\right)((x,y_2))\right)$$
$$\geq\frac{\left(F_{P_1}^U\times F_{P_2}^U\right)((x,x_2))+\left(F_{P_1}^U\times F_{P_2}^U\right)((x,y_2))-2\left(F_{P_1}^U\times F_{P_2}^U\right)((x,x_2))\left(F_{P_1}^U\times F_{P_2}^U\right)((x,y_2))}{1-\left(F_{P_1}^U\times F_{P_2}^U\right)((x,x_2)).\left(F_{P_1}^U\times F_{P_2}^U\right)((x,y_2))}$$

Consider $z\in V_2$, $x_1y_1\in E_1$

$$\left(T_{Q_1}^L\times T_{Q_2}^L\right)((x_1,z)(y_1,z))$$
$$=\min\left(T_{Q_1}^L(x_1y_1),T_{P_2}^L(z)\right) = TN\left(T_{Q_1}^L(x_1y_1),1\right)=T_{Q_1}^L(x_1y_1) \leq\frac{T_{P_1}^L(x_1)T_{P_1}^L(y_1)}{T_{P_1}^L(x_1)+T_{P_1}^L(y_1)-T_{P_1}^L(x_1).T_{P_1}^L(y_1)}$$
$$\leq\min\left(\min\left(T_{P_1}^L(x_1),T_{P_1}^L(y_1)\right),T_{P_2}^L(z)\right) = \min\left(\min\left(T_{P_1}^L(x_1),T_{P_2}^L(z)\right),\min\left(T_{P_1}^L(y_1),T_{P_2}^L(z)\right)\right)$$
$$=\min\left(\left(T_{P_1}^L\times T_{P_2}^L\right)(x_1,z),\left(T_{P_1}^L\times T_{P_2}^L\right)(y_1,z)\right)$$
$$\leq\frac{\left(T_{P_1}^L\times T_{P_2}^L\right)(x_1,z)\left(T_{P_1}^L\times T_{P_2}^L\right)(y_1,z)}{\left(T_{P_1}^L\times T_{P_2}^L\right)(x_1,z)+\left(T_{P_1}^L\times T_{P_2}^L\right)(y_1,z)-\left(T_{P_1}^L\times T_{P_2}^L\right)(x_1,z)\left(T_{P_1}^L\times T_{P_2}^L\right)(y_1,z)}$$

$$\left(T_{Q_1}^U\times T_{Q_2}^U\right)((x_1,z)(y_1,z))$$
$$=\min\left(T_{Q_1}^U(x_1y_1),T_{P_2}^U(z)\right) = TN\left(T_{Q_1}^U(x_1y_1),1\right)=T_{Q_1}^U(x_1y_1) \leq\frac{T_{P_1}^U(x_1)T_{P_1}^U(y_1)}{T_{P_1}^U(x_1)+T_{P_1}^U(y_1)-T_{P_1}^U(x_1).T_{P_1}^U(y_1)}$$
$$\leq\min\left(\min\left(T_{P_1}^U(x_1),T_{P_1}^U(y_1)\right),T_{P_2}^U(z)\right) = \min\left(\min\left(T_{P_1}^U(x_1),T_{P_2}^U(z)\right),\min\left(T_{P_1}^U(y_1),T_{P_2}^U(z)\right)\right)$$
$$=\min\left(\left(T_{P_1}^U\times T_{P_2}^U\right)(x_1,z),\left(T_{P_1}^U\times T_{P_2}^U\right)(y_1,z)\right)$$
$$\leq\frac{\left(T_{P_1}^U\times T_{P_2}^U\right)(x_1,z)\left(T_{P_1}^U\times T_{P_2}^U\right)(y_1,z)}{\left(T_{P_1}^U\times T_{P_2}^U\right)(x_1,z)+\left(T_{P_1}^U\times T_{P_2}^U\right)(y_1,z)-\left(T_{P_1}^U\times T_{P_2}^U\right)(x_1,z)\left(T_{P_1}^U\times T_{P_2}^U\right)(y_1,z)}$$

$$\left(I_{Q_1}^L\times I_{Q_2}^L\right)((x_1,z)(y_1,z))$$
$$=\max\left(T_{Q_1}^L(x_1y_1),T_{P_2}^L(z)\right) \geq \max\left(\max\left(I_{P_1}^L(x_1),I_{P_1}^L(y_1)\right),I_{P_2}^L(z)\right) = \max\left(\max\left(I_{P_1}^L(x_1),I_{P_2}^L(z)\right),\max\left(T_{P_1}^L(y_1),T_{P_2}^L(z)\right)\right)$$
$$=\max\left(\left(I_{P_1}^L\times I_{P_2}^L\right)(x_1,z),\left(I_{P_1}^L\times I_{P_2}^L\right)(y_1,z)\right)$$
$$\geq\frac{\left(I_{P_1}^L\times I_{P_2}^L\right)(x_1,z)+\left(I_{P_1}^L\times I_{P_2}^L\right)(y_1,z)-2\left(I_{P_1}^L\times I_{P_2}^L\right)(x_1,z)\left(I_{P_1}^L\times I_{P_2}^L\right)(y_1,z)}{1-\left(I_{P_1}^L\times I_{P_2}^L\right)(x_1,z)\left(I_{P_1}^L\times I_{P_2}^L\right)(y_1,z)}$$

$$\left(I_{Q_1}^{U}\times I_{Q_2}^{U}\right)\left((x_1,z)(y_1,z)\right)$$
$$=\max\left(T_{Q_1}^{U}(x_1y_1),T_{P_2}^{U}(z)\right)\geq\max\left(\max\left(I_{P_1}^{U}(x_1),I_{P_1}^{U}(y_1)\right),I_{P_2}^{U}(z)\right)$$
$$=\max\left(\max\left(I_{P_1}^{U}(x_1),I_{P_2}^{U}(z)\right),\max\left(T_{P_1}^{U}(y_1),T_{P_2}^{U}(z)\right)\right)=\max\left(\left(I_{P_1}^{U}\times I_{P_2}^{U}\right)(x_1,z),\left(I_{P_1}^{U}\times I_{P_2}^{U}\right)(y_1,z)\right)$$
$$\geq\frac{\left(I_{P_1}^{U}\times I_{P_2}^{U}\right)(x_1,z)+\left(I_{P_1}^{U}\times I_{P_2}^{U}\right)(y_1,z)-2\left(I_{P_1}^{U}\times I_{P_2}^{U}\right)(x_1,z)\left(I_{P_1}^{U}\times I_{P_2}^{U}\right)(y_1,z)}{1-\left(I_{P_1}^{U}\times I_{P_2}^{U}\right)(x_1,z)\left(I_{P_1}^{U}\times I_{P_2}^{U}\right)(y_1,z)}$$

$$\left(F_{Q_1}^{L}\times F_{Q_2}^{L}\right)\left((x_1,z)(y_1,z)\right)$$
$$=\max\left(F_{Q_1}^{L}(x_1y_1),F_{P_2}^{L}(z)\right)\geq\max\left(\max\left(F_{P_1}^{L}(x_1),F_{P_1}^{L}(y_1)\right),F_{P_2}^{L}(z)\right)$$
$$=\max\left(\max\left(F_{P_1}^{L}(x_1),F_{P_2}^{L}(z)\right),\max\left(F_{P_1}^{L}(y_1),F_{P_2}^{L}(z)\right)\right)=\max\left(\left(F_{P_1}^{L}\times F_{P_2}^{L}\right)(x_1,z),\left(F_{P_1}^{L}\times F_{P_2}^{L}\right)(y_1,z)\right)$$
$$\geq\frac{\left(F_{P_1}^{L}\times F_{P_2}^{L}\right)(x_1,z)+\left(F_{P_1}^{L}\times F_{P_2}^{L}\right)(y_1,z)-2\left(F_{P_1}^{L}\times F_{P_2}^{L}\right)(x_1,z)\left(F_{P_1}^{L}\times F_{P_2}^{L}\right)(y_1,z)}{1-\left(F_{P_1}^{L}\times F_{P_2}^{L}\right)(x_1,z)\left(F_{P_1}^{L}\times F_{P_2}^{L}\right)(y_1,z)}$$

$$\left(F_{Q_1}^{U}\times F_{Q_2}^{U}\right)\left((x_1,z)(y_1,z)\right)=\max\left(F_{Q_1}^{U}(x_1y_1),F_{P_2}^{U}(z)\right)\geq\max\left(\max\left(F_{P_1}^{U}(x_1),F_{P_1}^{U}(y_1)\right),F_{P_2}^{U}(z)\right)$$
$$=\max\left(\max\left(F_{P_1}^{U}(x_1),F_{P_2}^{U}(z)\right),\max\left(F_{P_1}^{U}(y_1),F_{P_2}^{U}(z)\right)\right)=\max\left(\left(F_{P_1}^{U}\times F_{P_2}^{U}\right)(x_1,z),\left(F_{P_1}^{U}\times F_{P_2}^{U}\right)(y_1,z)\right)$$
$$\geq\frac{\left(F_{P_1}^{U}\times F_{P_2}^{U}\right)(x_1,z)+\left(F_{P_1}^{U}\times F_{P_2}^{U}\right)(y_1,z)-2\left(F_{P_1}^{U}\times F_{P_2}^{U}\right)(x_1,z)\left(F_{P_1}^{U}\times F_{P_2}^{U}\right)(y_1,z)}{1-\left(F_{P_1}^{U}\times F_{P_2}^{U}\right)(x_1,z)\left(F_{P_1}^{U}\times F_{P_2}^{U}\right)(y_1,z)}$$

Note: Cartesian product of two IVNEGs need not be IVNEG.

Numerical Example:

Consider two crisp graphs $G_1=(A_1,B_1)$ and $G_2=(A_2,B_2)$, where $V_1=\{a,b\}$, $V_2=\{c,d\}$, $E_1=\{a,b\}$ and $E_2=\{c,d\}$. Consider two Interval Valued Neutrosophic Graphs $\mathbf{G_{N1}}=(P_1,Q_1)$ and $\mathbf{G_{N2}}=(P_2,Q_2)$

$$P_1=\left\{\left\langle i,[0.5,0.7],[0.2,0.3],[0.1,0.3]\right\rangle,\left\langle j,[0.6,0.7],[0.2,0.4],[0.1,0.3]\right\rangle\right\}$$

$Q_1=\left\{\left\langle ij,[0.3,0.6],[0.2,0.4],[0.2,0.4]\right\rangle\right\}$ **(Broumi et al.2016)**

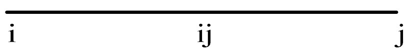

Fig. 3. 1 IVNG $\mathbf{G_{N1}}$

$$P_2=\left\{\left\langle k,[0.4,0.6],[0.2,0.3],[0.1,0.3]\right\rangle,\left\langle l,[0.4,0.7],[0.2,0.4],[0.1,0.3]\right\rangle\right\}$$
$$Q_2=\left\{\left\langle kl,[0.3,0.5],[0.4,0.5],[0.3,0.5]\right\rangle\right\}$$

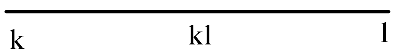

Fig. 4. IVNG $\mathbf{G_{N2}}$

Cartesian product of $\mathbf{G_{N1}}$ and $\mathbf{G_{N2}}$ $(\mathbf{G_{N1}}\times\mathbf{G_{N2}})$

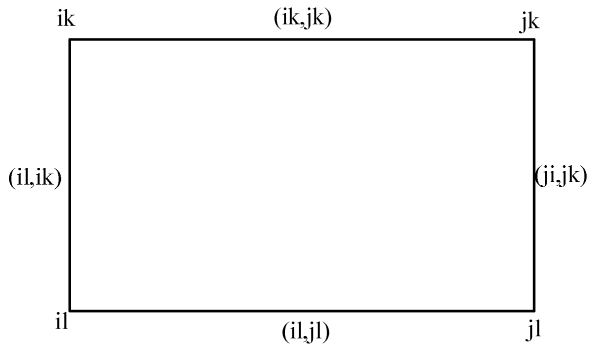

Fig. 5. Cartesian product of DIVNGs $\left(\mathbf{G_{N1}} \times \mathbf{G_{N2}}\right)$

Interval Neutrosophic Vertices and Edges

$ik = \langle [0.4, 0.6], [0.2, 0.3], [0.1, 0.3] \rangle$, $jk = \langle [0.4, 0.6], [0.2, 0.4], [0.1, 0.3] \rangle$

$il = \langle [0.4, 0.7], [0.2, 0.4], [0.1, 0.3] \rangle$, $jl = \langle [0.4, 0.7], [0.2, 0.4], [0.1, 0.3] \rangle$

$(ik, jk) = \langle [0.3, 0.6], [0.2, 0.4], [0.2, 0.4] \rangle$, $(jl, jk) = \langle [0.3, 0.5], [0.4, 0.5], [0.3, 0.5] \rangle$

$(il, jl) = \langle [0.3, 0.6], [0.2, 0.4], [0.2, 0.4] \rangle$, $(il, ik) = \langle [0.3, 0.5], [0.4, 0.5], [0.3, 0.5] \rangle$

To find ik:

$\left(T^L_{P_1} \times T^L_{P_2}\right)(i,k) = \min\left(T^L_{P_1}(i), T^L_{P_2}(k)\right) = \min(0.5, 0.4) = 0.4$

$\left(I^L_{P_1} \times I^L_{P_2}\right)(i,k) = \max\left(I^L_{P_1}(i), I^L_{P_2}(k)\right) = \max(0.2, 0.2) = 0.2$

$\left(F^L_{P_1} \times F^L_{P_2}\right)(i,k) = \min\left(F^L_{P_1}(i), F^L_{P_2}(k)\right) = \max(0.1, 0.1) = 0.1$

Similarly for other values.

Interval Neutrosophic Edges:

To find (ik, jk):

$\left(T^L_{Q_1} \times T^L_{Q_2}\right)\left((i,k)(j,k)\right) = \min\left(T^L_{Q_1}(ij), T^L_{P_2}(k)\right) = \min(0.3, 0.4) = 0.3$

$\left(I^L_{Q_1} \times I^L_{Q_2}\right)\left((i,k)(j,k)\right) = \max\left(I^L_{Q_1}(ij), I^L_{P_2}(k)\right) = \max(0.2, 0.2) = 0.2$

$\left(F^L_{Q_1} \times F^L_{Q_2}\right)\left((i,k)(j,k)\right) = \max\left(F^L_{Q_1}(ij), F^L_{P_2}(k)\right) = \max(0.2, 0.1) = 0.2$

To check the condition of Cartesian product of Dombi Interval Neutrosophic Edge Graph

$$\left(T^L_{Q_1} \times T^L_{Q_2}\right)\left((i,k)(j,k)\right) \le \frac{\left(T^L_{P_1} \times T^L_{P_2}\right)\left((i,k)\right)\left(T^L_{P_1} \times T^L_{P_2}\right)\left((j,k)\right)}{\left(T^L_{P_1} \times T^L_{P_2}\right)\left((i,k)\right) + \left(T^L_{P_1} \times T^L_{P_2}\right)\left((j,k)\right) - \left(T^L_{P_1} \times T^L_{P_2}\right)\left((i,k)\right)\left(T^L_{P_1} \times T^L_{P_2}\right)\left((j,k)\right)}$$

$0.3 \le \dfrac{(0.4)(0.4)}{(0.4)+(0.4)-(0.4)(0.4)} = 0.3$, hence satisfied.

$$\left(I_{Q_1}^L \times I_{Q_2}^L\right)((i,k)(j,k)) \geq \frac{\left(I_{P_1}^L \times I_{P_2}^L\right)((i,k)) + \left(I_{P_1}^L \times I_{P_2}^L\right)((j,k)) - 2\left(I_{P_1}^L \times I_{P_2}^L\right)((i,k))\left(I_{P_1}^L \times I_{P_2}^L\right)((j,k))}{1-\left(I_{P_1}^L \times I_{P_2}^L\right)((i,k))\left(I_{P_1}^L \times I_{P_2}^L\right)((j,k))}$$

$0.2 \geq \dfrac{(0.2)+(0.2)-2(0.2)(0.2)}{1-(0.2)(0.2)} = 0.33$, hence not satisfied

$$\left(F_{Q_1}^L \times F_{Q_2}^L\right)((i,k)(j,k)) \geq \frac{\left(F_{P_1}^L \times F_{P_2}^L\right)((i,k)) + \left(F_{P_1}^L \times F_{P_2}^L\right)((j,k)) - 2\left(F_{P_1}^L \times F_{P_2}^L\right)((i,k))\left(F_{P_1}^L \times F_{P_2}^L\right)((j,k))}{1-\left(F_{P_1}^L \times F_{P_2}^L\right)((i,k))\left(F_{P_1}^L \times F_{P_2}^L\right)((j,k))}$$

$0.2 \geq \dfrac{(0.1)+(0.1)-2(0.1)(0.1)}{1-(0.1)(0.1)} = 0.2$, hence satisfied.

Similarly for other edges.
Hence Cartesian product of two Dombi Interval Valued Neutrosophic Edge graphs need not be a DIVNEG.

4.4Definition: Composite product of Dombi Interval Valued Neutrosophic Graphs

Consider λ_i , a Neutrosophic fuzzy subset of $\mathbf{V}_i$ and δ_i , a fuzzy subset of $\mathbf{E}_i, i = 1,2$. Let $\mathbf{G_{N1}}(\lambda_1,\delta_1)$ and $\mathbf{G_{N2}}(\lambda_2,\delta_2)$ be two Dombi Interval Valued Neutrosophic Graphs of the crisp graphs $G_1^*(V_1,E_1)$ and $G_2^*(V_2,E_2)$ respectively and are defined by

$\left(\lambda_1^L \circ \lambda_2^L\right)(x_1,x_2) = \dfrac{\lambda_1^L(x_1)\lambda_2^L(x_2)}{\lambda_1^L(x_1)+\lambda_2^L(x_2)-\lambda_1^L(x_1)\lambda_2^L(x_2)}$, for all $(x_1,x_2) \in \mathrm{V}_1 \times \mathrm{V}_2$ and

$\left(\delta_1^L \circ \delta_2^L\right)((x,x_2)(x,y_2)) = \dfrac{\lambda_1^L(x)\lambda_2^L(x_2y_2)}{\lambda_1^L(x)+\lambda_2^L(x_2y_2)-\lambda_1^L(x)\lambda_2^L(x_2y_2)}$, for all $x \in \mathrm{V}_1,\ x_2y_2 \in \mathbf{E}_2$

$\left(\delta_1^L \circ \delta_2^L\right)((x_1,z)(y_1,z)) = \dfrac{\lambda_2^L(z)\delta_1^L(x_1y_1)}{\lambda_2^L(z)+\delta_1^L(x_1y_1)-\lambda_2^L(z)\delta_1^L(x_1y_1)}$, for all $z \in \mathbf{V}_2,\ x_1y_1 \in \mathrm{E}_1$

$$\left(\delta_1^L \circ \delta_2^L\right)((x_1,x_2)(y_1,y_2))$$
$$= \frac{\lambda_2^L(x_2)\lambda_2^L(y_2)\delta_1^L(x_1y_1)}{\lambda_2^L(x_2)\lambda_2^L(y_2)+\lambda_2^L(y_2)\delta_1^L(x_1y_1)+\lambda_2^L(x_2)\delta_1^L(x_1y_1)-2\lambda_2^L(x_2)\lambda_2^L(y_2)\delta_1^L(x_1y_1)},$$

for all $x_1y_1 \in \mathrm{E}_1$ and $x_2 \neq y_2$

Similarly for Indeterminacy and Falsity memberships with upper and lower membership values.

4.4.1 Proposition

The composite product of two Dombi Interval Valued Neutrosophic Edge graphs (DIVNEGs) of G_1 and G_2 is the DIVNEG.

Proof:

From the proof of 4.1.1

$$\left(T_{Q_1}^L \times T_{Q_2}^L\right)((x,x_2)(x,y_2)) \leq \frac{\left(T_{P_1}^L \times T_{P_2}^L\right)((x,x_2))\left(T_{P_1}^L \times T_{P_2}^L\right)((x,y_2))}{\left(T_{P_1}^L \times T_{P_2}^L\right)((x,x_2))+\left(T_{P_1}^L \times T_{P_2}^L\right)((x,y_2))-\left(T_{P_1}^L \times T_{P_2}^L\right)((x,x_2))\left(T_{P_1}^L \times T_{P_2}^L\right)((x,y_2))}$$

$$\left(T_{Q_1}^U \times T_{Q_2}^U\right)((x,x_2)(x,y_2)) \leq \frac{\left(T_{P_1}^U \times T_{P_2}^U\right)((x,x_2))\left(T_{P_1}^U \times T_{P_2}^U\right)((x,y_2))}{\left(T_{P_1}^U \times T_{P_2}^U\right)((x,x_2))+\left(T_{P_1}^U \times T_{P_2}^U\right)((x,y_2))-\left(T_{P_1}^U \times T_{P_2}^U\right)((x,x_2))\left(T_{P_1}^U \times T_{P_2}^U\right)((x,y_2))}$$

$$\left(I_{Q_1}^L \times I_{Q_2}^L\right)((x,x_2)(x,y_2))$$
$$\geq \frac{\left(I_{P_1}^L \times I_{P_2}^L\right)((x,x_2))+\left(I_{P_1}^L \times I_{P_2}^L\right)((x,y_2))-2\left(I_{P_1}^L \times I_{P_2}^L\right)((x,x_2))\left(I_{P_1}^L \times I_{P_2}^L\right)((x,y_2))}{1-\left(I_{P_1}^L \times I_{P_2}^L\right)((x,x_2)).\left(I_{P_1}^L \times I_{P_2}^L\right)((x,y_2))}$$

$$\left(I_{Q_1}^U \times I_{Q_2}^U\right)((x,x_2)(x,y_2))$$

$$\geq \frac{\left(I_{P_1}^U \times I_{P_2}^U\right)((x,x_2))+\left(I_{P_1}^U \times I_{P_2}^U\right)((x,y_2))-2\left(I_{P_1}^U \times I_{P_2}^U\right)((x,x_2))\left(I_{P_1}^U \times I_{P_2}^U\right)((x,y_2))}{1-\left(I_{P_1}^U \times I_{P_2}^U\right)((x,x_2)).\left(I_{P_1}^U \times I_{P_2}^U\right)((x,y_2))}$$

$$\left(F_{Q_1}^L \times F_{Q_2}^L\right)((x,x_2)(x,y_2))$$

$$\geq \frac{\left(F_{P_1}^L \times F_{P_2}^L\right)((x,x_2))+\left(F_{P_1}^L \times F_{P_2}^L\right)((x,y_2))-2\left(F_{P_1}^L \times F_{P_2}^L\right)((x,x_2))\left(F_{P_1}^L \times F_{P_2}^L\right)((x,y_2))}{1-\left(F_{P_1}^L \times F_{P_2}^L\right)((x,x_2)).\left(F_{P_1}^L \times F_{P_2}^L\right)((x,y_2))}$$

$$\left(F_{Q_1}^U \times F_{Q_2}^U\right)((x,x_2)(x,y_2))$$

$$\geq \frac{\left(F_{P_1}^U \times F_{P_2}^U\right)((x,x_2))+\left(F_{P_1}^U \times F_{P_2}^U\right)((x,y_2))-2\left(F_{P_1}^U \times F_{P_2}^U\right)((x,x_2))\left(F_{P_1}^U \times F_{P_2}^U\right)((x,y_2))}{1-\left(F_{P_1}^U \times F_{P_2}^U\right)((x,x_2)).\left(F_{P_1}^U \times F_{P_2}^U\right)((x,y_2))}$$

Similarly for $z\in V_2$, $x_1y_1\in E_1$.

Now consider $x_1y_1\in E_1$, $x_2\neq y_2$

$\left(T_{Q_1}^L \circ T_{Q_2}^L\right)((x_1,x_2)(y_1,y_2))$ **(Broumi et al. 2016)**

$$\leq \min\left(T_{P_2}^L(x_2),T_{P_2}^L(y_2),T_{Q_1}^L(x_1y_1)\right)$$

$$\leq \min\left(T_{P_2}^L(x_2),T_{P_2}^L(y_2),\min\left(T_{P_1}^L(x_1),T_{P_1}^L(y_1)\right)\right)=\min\left(\min\left(T_{P_1}^L(x_1),T_{P_2}^L(x_2)\right),\min\left(T_{P_1}^L(y_1),T_{P_1}^L(y_2)\right)\right)$$

$$=\min\left(\left(T_{P_1}^L \circ T_{P_2}^L\right)(x_1,x_2),\left(T_{P_1}^L \circ T_{P_2}^L\right)(y_1,y_2)\right)$$

$$=TN\left(\left(T_{P_1}^L \circ T_{P_2}^L\right)((x_1,x_2)),\left(T_{P_1}^L \circ T_{P_2}^L\right)((y_1,y_2))\right)$$

$$\leq \frac{\left(T_{P_1}^L \circ T_{P_2}^L\right)((x_1,x_2))\left(T_{P_1}^L \circ T_{P_2}^L\right)((y_1,y_2))}{\left(T_{P_1}^L \circ T_{P_2}^L\right)((x_1,x_2))+\left(T_{P_1}^L \circ T_{P_2}^L\right)((y_1,y_2))-\left(T_{P_1}^L \circ T_{P_2}^L\right)((x_1,x_2))\left(T_{P_1}^L \circ T_{P_2}^L\right)((y_1,y_2))}$$

$$\left(T_{Q_1}^U \circ T_{Q_2}^U\right)((x_1,x_2)(y_1,y_2))$$

$$=\min\left(T_{P_2}^U(x_2),T_{P_2}^U(y_2),T_{Q_1}^U(x_1y_1)\right)$$

$$\leq \min\left(T_{P_2}^U(x_2),T_{P_2}^U(y_2),\min\left(T_{P_1}^U(x_1),T_{P_1}^U(y_1)\right)\right)=\min\left(\min\left(T_{P_1}^U(x_1),T_{P_2}^U(x_2)\right),\min\left(T_{P_1}^U(y_1),T_{P_1}^U(y_2)\right)\right)$$

$$=\min\left(\left(T_{P_1}^U \circ T_{P_2}^U\right)(x_1,x_2),\left(T_{P_1}^U \circ T_{P_2}^U\right)(y_1,y_2)\right)$$

$$=TN\left(\left(T_{P_1}^U \circ T_{P_2}^U\right)((x_1,x_2)),\left(T_{P_1}^U \circ T_{P_2}^U\right)((y_1,y_2))\right)$$

$$\leq \frac{\left(T_{P_1}^U \circ T_{P_2}^U\right)((x_1,x_2))\left(T_{P_1}^U \circ T_{P_2}^U\right)((y_1,y_2))}{\left(T_{P_1}^U \circ T_{P_2}^U\right)((x_1,x_2))+\left(T_{P_1}^U \circ T_{P_2}^U\right)((y_1,y_2))-\left(T_{P_1}^U \circ T_{P_2}^U\right)((x_1,x_2))\left(T_{P_1}^U \circ T_{P_2}^U\right)((y_1,y_2))}$$

$$\left(I_{Q_1}^L \circ I_{Q_2}^L\right)((x_1,x_2)(y_1,y_2))$$

$$=\max\left(I_{P_2}^L(x_2),I_{P_2}^L(y_2),I_{Q_1}^L(x_1y_1)\right)$$

$$\geq \max\left(I_{P_2}^L(x_2),I_{P_2}^L(y_2),\max\left(I_{P_1}^L(x_1),I_{P_1}^L(y_1)\right)\right)=\max\left(\max\left(I_{P_1}^L(x_1),I_{P_2}^L(x_2)\right),\max\left(I_{P_1}^L(y_1),I_{P_1}^L(y_2)\right)\right)$$

$$=\max\left(\left(I_{P_1}^L \circ I_{P_2}^L\right)(x_1,x_2),\left(I_{P_1}^L \circ I_{P_2}^L\right)(y_1,y_2)\right)$$

$$=TCN\left(\left(I_{P_1}^L \circ I_{P_2}^L\right)((x_1,x_2)),\left(I_{P_1}^L \circ I_{P_2}^L\right)((y_1,y_2))\right)$$

$$\geq \frac{\left(I_{P_1}^L \circ I_{P_2}^L\right)((x_1,x_2))+\left(I_{P_1}^L \circ I_{P_2}^L\right)((y_1,y_2))-2\left(I_{P_1}^L \circ I_{P_2}^L\right)((x_1,x_2)).\left(I_{P_1}^L \circ I_{P_2}^L\right)((y_1,y_2))}{1-\left(I_{P_1}^L \circ I_{P_2}^L\right)((x_1,x_2)).\left(I_{P_1}^L \circ I_{P_2}^L\right)((y_1,y_2))}$$

$$\left(I_{Q_1}^U \circ I_{Q_2}^U\right)((x_1,x_2)(y_1,y_2))$$

$$=\frac{I_{P_2}^U(x_2)I_{P_2}^U(y_2)I_{Q_1}^U(x_1y_1)}{I_{P_2}^U(x_2)I_{P_2}^U(y_2)+I_{P_2}^U(y_2)I_{Q_1}^U(x_1y_1)+I_{P_2}^U(x_2)I_{Q_1}^U(x_1y_1)-2I_{P_2}^U(x_2)I_{P_2}^U(y_2)I_{Q_1}^U(x_1y_1)},$$

$$=\max\left(I_{P_2}^U(x_2),I_{P_2}^U(y_2),I_{Q_1}^U(x_1y_1)\right)$$

$$=TCN\left(\left(I_{P_1}^U\circ I_{P_2}^U\right)((x_1,x_2)),\left(I_{P_1}^U\circ I_{P_2}^U\right)((y_1,y_2))\right)$$

$$\geq\frac{\left(I_{P_1}^U\circ I_{P_2}^U\right)((x_1,x_2))+\left(I_{P_1}^U\circ I_{P_2}^U\right)((y_1,y_2))-2\left(I_{P_1}^U\circ I_{P_2}^U\right)((x_1,x_2)).\left(I_{P_1}^U\circ I_{P_2}^U\right)((y_1,y_2))}{1-\left(I_{P_1}^U\circ I_{P_2}^U\right)((x_1,x_2)).\left(I_{P_1}^U\circ I_{P_2}^U\right)((y_1,y_2))}$$

$$\left(F_{Q_1}^L\circ F_{Q_2}^L\right)((x_1,x_2)(y_1,y_2))$$

$$=\max\left(F_{P_2}^L(x_2),F_{P_2}^L(y_2),F_{Q_1}^L(x_1y_1)\right)$$

$$\geq\max\left(F_{P_2}^L(x_2),F_{P_2}^L(y_2),\max\left(F_{P_1}^L(x_1),F_{P_1}^L(y_1)\right)\right)=\max\left(\max\left(F_{P_1}^L(x_1),F_{P_2}^L(x_2)\right),\max\left(F_{P_1}^L(y_1),F_{P_1}^L(y_2)\right)\right)$$

$$=\max\left(\left(F_{P_1}^L\circ F_{P_2}^L\right)(x_1,x_2),\left(F_{P_1}^L\circ F_{P_2}^L\right)(y_1,y_2)\right)$$

$$=TCN\left(\left(F_{P_1}^L\circ F_{P_2}^L\right)((x_1,x_2)),\left(F_{P_1}^L\circ F_{P_2}^L\right)((y_1,y_2))\right)$$

$$\geq\frac{\left(F_{P_1}^L\circ F_{P_2}^L\right)((x_1,x_2))+\left(F_{P_1}^L\circ F_{P_2}^L\right)((y_1,y_2))-2\left(F_{P_1}^L\circ F_{P_2}^L\right)((x_1,x_2)).\left(F_{P_1}^L\circ F_{P_2}^L\right)((y_1,y_2))}{1-\left(F_{P_1}^L\circ F_{P_2}^L\right)((x_1,x_2)).\left(F_{P_1}^L\circ F_{P_2}^L\right)((y_1,y_2))}$$

$$\left(F_{Q_1}^U\circ F_{Q_2}^U\right)((x_1,x_2)(y_1,y_2))$$

$$=\max\left(F_{P_2}^U(x_2),F_{P_2}^U(y_2),F_{Q_1}^U(x_1y_1)\right)$$

$$=TCN\left(\left(F_{P_1}^U\circ F_{P_2}^U\right)((x_1,x_2)),\left(F_{P_1}^U\circ F_{P_2}^U\right)((y_1,y_2))\right)$$

$$\geq\frac{\left(F_{P_1}^U\circ F_{P_2}^U\right)((x_1,x_2))+\left(F_{P_1}^U\circ F_{P_2}^U\right)((y_1,y_2))-2\left(F_{P_1}^U\circ F_{P_2}^U\right)((x_1,x_2)).\left(F_{P_1}^U\circ F_{P_2}^U\right)((y_1,y_2))}{1-\left(F_{P_1}^U\circ F_{P_2}^U\right)((x_1,x_2)).\left(F_{P_1}^U\circ F_{P_2}^U\right)((y_1,y_2))}$$

Hence the proposition.

Numerical Example:

Consider the same example as in numerical example for Cartesian product.

Composition of two IVNGs (Broumi et al. 2016)

$$P_1=\{\langle i,[0.5,0.7],[0.2,0.5],[0.1,0.3]\rangle,\langle j,[0.6,0.7],[0.2,0.4],[0.1,0.3]\rangle\}$$
$$Q_1=\{\langle ij,[0.3,0.6],[0.2,0.4],[0.2,0.4]\rangle\}$$
$$P_2=\{\langle k,[0.4,0.6],[0.3,0.4],[0.1,0.3]\rangle,\langle l,[0.4,0.7],[0.2,0.4],[0.1,0.3]\rangle\}$$
$$Q_2=\{\langle kl,[0.3,0.5],[0.4,0.5],[0.3,0.5]\rangle\}$$

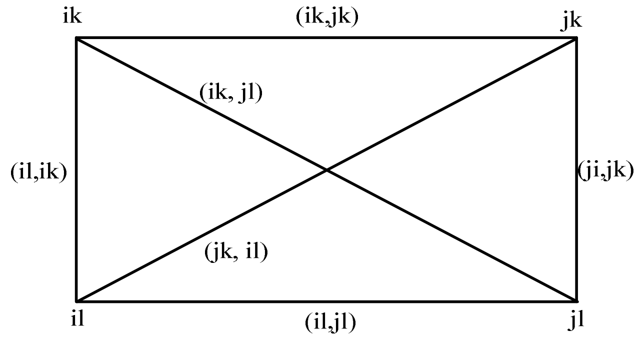

Fig. 6. Composition of G_{N1} and G_{N2} $(G_{N1} \circ G_{N2})$

$(ik, jk) = \langle [0.3,0.6],[0.3,0.4],[0.2,0.4] \rangle$, $(jl, jk) = \langle [0.3,0.5],[0.4,0.5],[0.3,0.5] \rangle$

$(il, jl) = \langle [0.3,0.6],[0.2,0.4],[0.2,0.4] \rangle$, $(il, ik) = \langle [0.3,0.5],[0.4,0.5],[0.3,0.5] \rangle$

$(ik, jl) = \langle [0.3,0.6],[0.3,0.4],[0.2,0.4] \rangle$, $(jk, il) = \langle [0.3,0.6],[0.3,0.4],[0.2,0.4] \rangle$

Other four edges has been given in Numerical Example 1.

To check for Dombi Composition of two IVNEGs is an IVNEG.
Consider $ij \in E_1$,

$$\left(T^L_{Q_1} \circ T^L_{Q_2}\right)((i,k)(j,l)) = \min\left[T^L_{P_2}(k), T^L_{P_2}(l), T^L_{Q_1}(ij)\right] = \min[0.4,0.4,0.3] = 0.3$$

$$\leq \frac{\left(T^L_{P_1} \circ T^L_{P_1}\right)((i,k))\left(T^L_{P_1} \circ T^L_{P_1}\right)((j,l))}{\left(T^L_{P_1} \circ T^L_{P_1}\right)((i,k)) + \left(T^L_{P_1} \circ T^L_{P_1}\right)((j,l)) - \left(T^L_{P_1} \circ T^L_{P_1}\right)((i,k))\left(T^L_{P_1} \circ T^L_{P_1}\right)((j,l))}$$

$= \frac{(0.4)(0.4)}{(0.4)+(0.4)-(0.4)(0.4)} = 0.3$, hence satisfied

$$\left(I^L_{Q_1} \circ I^L_{Q_2}\right)((i,k)(j,l)) = \max\left[I^L_{P_2}(k), I^L_{P_2}(l), I^L_{Q_1}(ij)\right] = \max[0.3,0.2,0.2] = 0.3$$

$$\geq \frac{\left(I^L_{P_1} \circ I^L_{P_1}\right)((i,k)) + \left(I^L_{P_1} \circ I^L_{P_1}\right)((j,l)) - 2\left(I^L_{P_1} \circ I^L_{P_1}\right)((i,k))\left(I^L_{P_1} \circ I^L_{P_1}\right)((j,l))}{1 - \left(I^L_{P_1} \circ I^L_{P_1}\right)((i,k))\left(I^L_{P_1} \circ I^L_{P_1}\right)((j,l))}$$

$= \frac{(0.2)+(0.2)-2(0.2)(0.2)}{1-(0.2)(0.2)} = 0.3$, hence satisfied

$$\left(F^L_{Q_1} \circ F^L_{Q_2}\right)((i,k)(j,l)) = \max\left[F^L_{P_2}(k), F^L_{P_2}(l), F^L_{Q_1}(ij)\right] = \max[0.1,0.1,0.2] = 0.2$$

$$\geq \frac{\left(F^L_{P_1} \circ F^L_{P_1}\right)((i,k)) + \left(F^L_{P_1} \circ F^L_{P_1}\right)((j,l)) - 2\left(F^L_{P_1} \circ F^L_{P_1}\right)((i,k))\left(F^L_{P_1} \circ F^L_{P_1}\right)((j,l))}{1 - \left(F^L_{P_1} \circ F^L_{P_1}\right)((i,k))\left(F^L_{P_1} \circ F^L_{P_1}\right)((j,l))}$$

$= \frac{(0.1)+(0.1)-2(0.1)(0.1)}{1-(0.1)(0.1)} = 0.2$, hence satisfied

Similarly for other edges.
Hence composition two Dombi Interval Valued Neutrosophic Edge graphs is a DIVNEG.

5. Comparison of Traffic Control Management using different types of set and Graph theory

The below table expresses the advantage and limitations of crisp sets, Fuzzy sets (Type-1 Fuzzy Sets), Type-2 Fuzzy sets, Neutrosophic Sets (Single Valued Neutrosophic Graphs), Interval Valued Neutrosophic Sets, Dombi Fuzzy Graphs, Dombi Neutrosophic Graphs and Dombi Interval Valued Neutrosophic Graphs in traffic control management. This table also describe the role of triangular norms in control theory. In traffic control system, detection of latency of the vehicles in a roadway, estimation of the density will be done by sensors and it will send an interrupt signal to the control unit. After that using the controllers with logical operations, the roadway will be decided for giving the service first. At this junction, the logical operations of Dombi Interval Valued Neutrosophic Graphs can be used.

Traffic Control Management	Advantages	Limitations
Using Crisp Sets	• Fixed time period for all the traffic density • Achieved to characterize the real situation	• Cannot act while there is a fluctuation in traffic density • Unable to react immediately to unpredictable changes like driver's behavior. • Unable to handle with rapid momentous changes which disturb the continuity of the traffic.
Using Fuzzy Sets	• Different time duration can be considered according to the traffic density • Follow rule based approach which accepts uncertainties • Able to model the reasoning of an experienced human being • Adaptive and intelligent • Able to apply and handle real life rules identical to human thinking • Admits fuzzy terms and conditions • Performs the best security • It makes simpler to convert knowledge beyond the domain	• Adaptiveness is missing to compute the connectedness of the interval based input • Cannot be used to show uncertainty as it apply crisp and accurate functions • Cannot handle the uncertainties such as stability, flexibility and on-line planning completely since consequents can be uncertain
Using Type-2 Fuzzy Sets	• Rule based approach which accepts uncertainties completely • Adaptiveness (Fixed Type-1 fuzzy sets are used to calculate the bounds of the type reduced interval change as input changes) • Novelty (the upper and lower membership functions may be used concurrently in calculating every bound of the type reduced interval)	• Computational complexity is high as the membership functions themselves fuzzy.
Neutrosophic Sets	• Deals not only uncertainty but also indeterminacy due to unpredictable environmental disturbances	• Unable to rounding up and down errors of calculations
Interval Valued Neutrosophic Sets	• Deals with more uncertainties and indeterminacy • Flexible and adaptability • Able to address issues with a set of numbers in the real unit interval, not just a particular number. • Able to rounding up and down errors of calculations	• Unable to deal criterion incomplete weight information.
Neutrosophic Graphs	• When the terminal points and the paths are uncertain, optimized output is possible	• Unable to handle more uncertainties.
Interval Valued Neutrosophic Graphs	• Able to handle more uncertainties exist in the terminal points (vertices) and paths (edges)	• Unable to deal criterion incomplete weight information.
Dombi Fuzzy Graphs	• The Dombi Fuzzy Graph can portray the impreciseness strongly for all types of networks like traffic control.	• Indeterminacy cannot be dealt by Dombi Fuzzy Graph.
Dombi Neutrosophic Graphs	• Indeterminacy can be dealt by Dombi Neutrosophic Graph.	• Unable to handle uncertainty for interval values provided by the expert's

Dombi Interval Valued Neutrosophic Graphs	• Able to handle uncertainty properly for interval values provided by the expert's	• Unable to deal criterion incomplete weight information.

6. Conclusion

Dealing indeterminacy is an essential work to get an optimized output in any problem and control system as well. It is possible only with Neutrosophic logic and that too effectively by interval valued Neutrosophic setting as it has lower and upper membership function for three independent membership functions namely truth, indeterminacy and falsity. In this paper, Dombi Single valued Neutrosophic Graph and Dombi Interval valued Neutrosophic Graph have been proposed. Also it has been proved that Cartesian product and composition of two DIVNGs are DIVNG with numerical example. Also an importance of the Neutrosophic Controllers has been given theoretically and its use in traffic control management. It has been pointed out that instead of using minimum and maximum operations, triangular norms namely T Norm and T-Conorm can be used in control system such as traffic control management. Advantage and limitations has been discussed for crisp sets, fuzzy sets, and type-2 fuzzy sets, Neutrosophic Sets, Interval Valued Neutrosophic Sets, Neutrosophic Graphs, Interval Valued Neutrosophic Graphs, Dombi Fuzzy Graphs, Dombi Neutrosophic Graphs and Dombi Interval Valued Neutrosophic Graphs.

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Edge Detection on DICOM Image using Triangular Norms in Type-2 Fuzzy

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***Abstract*—In image processing, edge detection is an important venture. Fuzzy logic plays a vital role in image processing to deal with lacking in quality of an image or imprecise in nature. This present study contributes an authentic method of fuzzy edge detection through image segmentation. Gradient of the image is done by triangular norms to extract the information. Triangular norms (T norms) and triangular conorms (T conorms) are specialized in dealing uncertainty. Therefore triangular norms are chosen with minimum and maximum operators for the purpose of morphological operations. Also, mathematical properties of aggregation operator to represent the role of morphological operations using Triangular Interval Type-2 Fuzzy Yager Weighted Geometric (TIT2FYWG) and Triangular Interval Type-2 Fuzzy Yager Weighted Arithmetic (TIT2FYWA) operators are derived. These properties represent the components of image processing. Here Edge detection is done for DICOM image by converting into 2D gray scale image, using Type-2 fuzzy MATLAB and which is the novelty of this work.**

Keywords—Aggregation operators; T norm; T conorm; triangular interval type-2 fuzzy number (TIT2FN); fuzzy morphology; gray scale Image; medical image processing

I. Introduction

In the field of optimization problems in Mathematics, Statistics, Economics and Information Science, the max and min operators are very useful for any dimension. Uncertainty convoluted in most of the real world problems. Fuzzy theory has been developed as an efficient and powerful mechanism in mathematical design of many engineering and objective phenomena [1-5].To deal uncertainty in any field one needs an effective and predictable incentive. Usually incomplete data and errors in the analyzing stage will be the reason for getting vague situation and this can be dealt with fuzzy theory. Mathematical devices may figure out an impreciseness. The largest and the smallest elements of a precise set of real numbers is the maximum and the minimum and so Yager triangular norm is chosen for this work [6-10]. We are facing many problems to add, melt and synthesize the datum from different sources to get a conclusion. The operators may be chosen according to the characteristic properties and then the operations for minimum and maximum can be applied [11-14].

The triangular norms with maximum and minimum operators could be used for an image processing since these norms play as the synthesize operators for which these maximum and minimum operators are just an exclusive choice [15-17]. A Fuzzy Set (FS) is defined from a universal set to [0, 1] and the membership values (MVs) of every element is a crisp value between 0 and 1. This kind of system is called Type-1 Fuzzy Set (T1FS) system. In many of the real world problems it is necessary to have a MV itself fuzzy instead of crisp value which is called T2FS [18-19]. The generalization of union and intersection operators are triangular norms. Though the general case is important there is an equal important for the particular cases which provide efficient algorithm and more understanding missing in the general case [20].

T2FS is used when T1FS is blurred. In T2FS, the MVs lies in an interval so it is useful in image processing as many of the images are not properly visible. The parameter η in Yager triangular norms, accepts for tuning the norm between the other norms [21]. Yager norms covers all the continuous norms by changing the parameter where as other major norms can't do the same and have more time complexity [22]. In automation, visual sense, remotely second scene analysis and bio medical image processing, Fuzzy image analysis has been applied. When the images with low brightness, the structure will not be evidently visible. In this situation, the sets which have better and naturally include different types of uncertainties might be useful for image analysis in any field.

To deal this complication Fuzzy Sets and their advanced extensions like T2FS Sets are suitable since it handles the uncertainty in a better way. Using Type-2 Fuzzy thresholding techniques, different regions and abnormal lesions can be separated. Image processing can be done by FMM using triangular norms. Using T2FS, collection of undesirable scraps can be made while noise exist. In image processing, image enrichment, clustering, thresholding, edge detection and morphological image processing are easy to be done using T2FS. Application of single image analysis is always not reliable and therefore image processing based on T2 Fuzzy system has been considered [23, 24]. Borderline between two compatible regions is called an edge.

Using unit of the regional array, sense of the trial edge will be done at different points. Real world issues are levelheaded of various structures at various scales and an ideal image cannot be expected. The technique of selecting and detecting acute disruption in an image is called edge detection. DICOM is worn to store, transfer and pass on the medical images (MIs). Most of the MIs are saved in DICOM pattern where one can store data of an image and header as well and per file there is one slice in general. Singe color images are called gray scale which accommodate the knowledge of only gray level but not about color. Every pixel has some number of bits that determines available number of various gray levels [25-30].

The paper is organized in the following manner. In section II, literature review has been done related to the present work. In section III, basic definitions required for developing the concept have been described. In section IV, operational laws have been proposed for TIT2FN. In section V, aggregation properties have been proved using weighted arithmetic and geometric operators. In section VI, the theory of image processing and the role of T2FS and Yager norms is presented. In section VII, applied Type-2 fuzzy logic in edge detection for DICOM image in two dimensional through MATLAB. In section VIII, conclusion and future work is given.

II. Review of Literature

The authors of, [1] described Aggregation operators elaborately with their advanced direction and applications. [2] explained about gathering of the information and its related aggregation operators. [3] proposed Frank Aggregation Operators (AOs) and its mathematical properties for TIT2FSs and applied in a decision making problem. [4] studied t norms of Yager and Hamacher and also metric space on fuzzy logic. [5] utilized AOs in the process of decision making under the environment of probabilistic fuzzy.

[17] proposed fuzzy image processing (FIP) using Dubois and Prade triangular norm. [22] proposed a methodology for an image condensation and rehabilitation on a Lossy image using fuzzy relational equations. [23] proposed a technique for image analysis with the application of morphological operators with the support of uninorms.

[24] described and explained very clearly about the role of theoretical fuzzy logic strategies in medical image processing. [25] reviewed the applications of type-2 fuzzy systems in the field of image processing. [26] presented a comprehensive depiction of imitation of an image with the help of fuzzy logic. [27] established an algorithm for edge detection under fuzzy environment where instability of a digital image for every pixel has been calculated.

[28] proposed a methodology for fusion of image under intuitionistic fuzzy setting. [29] introduced a new technique for edge detection with the support of representation of fuzzy image and pixels. [30] examined and done a comparative analysis of various techniques of edge detection. From this review it is found that there is no work has been done for edge detection on DICOM image using Type-2 fuzzy logic. This is the motivation of the present work.

III. Basic Definitions

The following basic concepts are given for the better understanding of the paper.

A. Aggregation Operator [3]

Let $(M_\alpha)_{\alpha\in[0,1]}$ be a group of aggregation operators (AOs) which is non-decreasing. If A is an AO then

$$M_A : \bigcup_{n\in N}[0,1]^n \to [0,1].$$

B. Triangular Interval Type-2 Fuzzy Set (TIT2FS) [3]

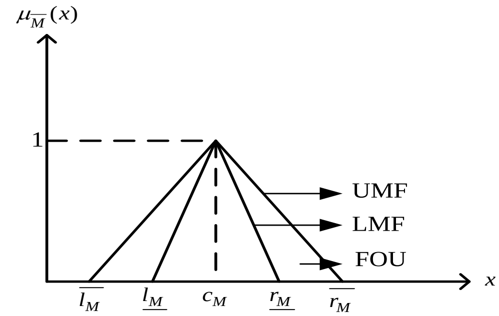

Fig. 1. TIT2FS.

The membership function (MFs) are developed using triangular fuzzy number in IT2FS called TIT2FS. In IT2FS, upper and lower MFs represented by a triangular fuzzy number $\overline{M}=\left\langle[\underline{l_M},\overline{l_M}],c_M,[\underline{r_M},\overline{r_M}]\right\rangle$ called TIT2FS and are defined by

$$LMF_{\overline{M}}(x)=\begin{cases}\dfrac{x-\overline{l_M}}{c_M-\overline{l_M}} & , \quad \overline{l_M}\le x<c_M\\ 1 & , \quad x=c_M\\ \dfrac{x-\underline{r_M}}{c_M-\underline{r_M}} & , \quad c_M\le x<\underline{r_M}\\ 0 & , \quad otherwise\end{cases} \tag{1}$$

$$UMF_{\overline{M}}(x)=\begin{cases}\dfrac{x-\underline{l_M}}{c_M-\underline{l_M}} & , \quad \underline{l_M}\le x<c_M\\ 1 & , \quad x=c_M\\ \dfrac{x-\overline{r_M}}{c_M-\overline{r_M}} & , \quad c_M\le x<\overline{r_M}\\ 0 & , \quad otherwise\end{cases} \tag{2}$$

Where $\underline{l_M},\overline{l_M},c_M,\underline{r_M},\overline{r_M}$ are the measuring points on TIT2FS satisfying $0\le\underline{l_M}\le\overline{l_M}\le c_M\le\underline{r_M}\le\overline{r_M}\le 1$. If we consider X as a set of real numbers, a TIT2FS in X is called TIT2FN. The FOU is the area between lower and upper membership functions in figure 1. If $\underline{l_M}=\overline{l_M},\underline{r_M}=\overline{r_M}$, then $UMF_{\overline{M}}(x)$ = $LMF_{\overline{M}}(x)$ for all the values of x in X, then the TIT2FS will become Type-1 case. Here FOU is the footprint of Uncertainty.

C. Ranking formula for TIT2FN [3]

Let $\overline{M}=\langle[A,B],C,[D,E]\rangle$ where $A=\underline{l_M},B=\overline{l_M},C=c_M,D=\underline{r_M},E=\overline{r_M}$ be the TIT2FN. The ranking value is defined by

$$Rank(\overline{M})=\left(\frac{A+E}{2}+1\right)\times\frac{A+B+D+E+4C}{8} \tag{3}$$

D. Yager Triangular Norms [7]

$\underset{Y}{\otimes}$ is Yager product (T Norm) and $\underset{Y}{\oplus}$ is a Yager sum (T conorm) and are defined as follows.

$$r\underset{Y}{\otimes}s=\max\left(1-[(1-r)^{\eta}+(1-s)^{\eta}]^{\frac{1}{\eta}},0\right),\eta>0,\ for\ all\ r,s\in[0,1]^{2} \tag{4}$$

$$r\underset{Y}{\oplus}s=\min\left(\left(r^{\eta}+s^{\eta}\right)^{\frac{1}{\eta}},1\right),\eta>0,\ for\ all\ r,s\in[0,1]^{2} \tag{5}$$

E. Triangular Interval Type-2 Fuzzy Yager Weighted Arithmetic (TIT2FYWA) Operator [3]

Consider a set of TIT2FNs and the operator $TIT2FYWA_{\varepsilon}:\Omega^{n}\to\Omega$ is defined by $TIT2FYWA_{\varepsilon}\left\langle\overline{M_1},\overline{M_2},...,\overline{M_n}\right\rangle=\varepsilon_1\underset{Y}{\bullet}\overline{M_1}\underset{Y}{\oplus}\varepsilon_2\underset{Y}{\bullet}\overline{M_2}\underset{Y}{\oplus}...\underset{Y}{\oplus}\varepsilon_n\underset{Y}{\bullet}\overline{M_n}$

and its weight vector is $\varepsilon=\left(\varepsilon_1,\varepsilon_2,...,\varepsilon_n\right)^{T}$ and the sum of the weight vectors is equal to 1, when $\varepsilon=\left(1/n,1/n,...,1/n\right)^{T}$, triangular interval type-2 fuzzy Yager weighted arithmetic operator will become triangular interval type-2 fuzzy Yager arithmetic averaging operator of dimension n and is defined by

$$TIT2FYAA_{\varepsilon}(\overline{M_1},\overline{M_2},...,\overline{M_n})=\frac{1}{n}\underset{Y}{\bullet}\left(\overline{M_1}\underset{Y}{\oplus}\overline{M_2},...,\underset{Y}{\oplus}\overline{M_n}\right). \tag{6}$$

F. Triangular Interval Type-2 Fuzzy Yager Weighted Geometric (TIT2FYWG) Operator [3]

Let $\overline{M}=\left([\underline{l_{M_p}},\overline{l_{M_p}}],c_{M_p},[\underline{r_{M_p}},\overline{r_{M_p}}]\right)$, $p=1,2,...,n$ be a set of TIT2FNs. Triangular *Interval Type-2 fuzzy Yager Weighted Geometric Mean Operator (TIT2FYWA)*, TIT2FYWG:$\varepsilon^{n}\to\varepsilon$ is $\text{TIT2FYWG}_{\varepsilon}\left(\overline{M_1},\overline{M_2},...,\overline{M_n}\right)=\overline{M_1}^{\bullet_Y\varepsilon_1}\underset{Y}{\otimes}\overline{M_2}^{\bullet_Y\varepsilon_2}\underset{Y}{\otimes}...\underset{Y}{\otimes}\overline{M_n}^{\bullet_Y\varepsilon_n}$ .Here also sum of all weight vectors is equal to 1, when $\varepsilon=\left(1/n,1/n,...,1/n\right)^{T}$, triangular interval type-2 fuzzy Yager weighted arithmetic operator will become triangular interval type-2 fuzzy Yager geometric averaging operator of dimension n and is defined by

$$TIT2FYGA_{\varepsilon}(\overline{M_1},\overline{M_2},...,\overline{M_2})=\frac{1}{n}\left(\overline{M_1}\underset{Y}{\otimes}\overline{M_2}\underset{Y}{\otimes},...,\underset{Y}{\otimes}\overline{M_n}\right)^{\hat{Y}1/n}. \tag{7}$$

IV. PROPOSED OPERATIONAL LAWS

Let $\overline{M},\overline{M_1},\overline{M_2}$ be three TIT2FNs and $\eta>0$, then we define their operational laws as follows.

A. Addition

Consider, $A_1=\underset{p=1}{\overset{2}{sum}}\left(\underline{l_{M_p}}\right),B_1=\underset{p=1}{\overset{2}{sum}}\left(\overline{l_{M_p}}\right),C_1=\underset{p=1}{\overset{2}{sum}}\left(c_{M_p}\right),$

$D_1=\underset{p=1}{\overset{2}{sum}}\left(\underline{r_{M_p}}\right),E_1=\underset{p=1}{\overset{2}{sum}}\left(\overline{r_{M_p}}\right)$

$$\overline{M_1}\underset{Y}{\oplus}\overline{M_2}=\left(\left[\min\left(A_1^{\frac{1}{\eta}},1\right),\min\left(B_1^{\frac{1}{\eta}},1\right)\right],\ \min\left(C_1^{\frac{1}{\eta}},1\right),\left[\min\left(D_1^{\frac{1}{\eta}},1\right),\min\left(E_1^{\frac{1}{\eta}},1\right)\right]\right) \tag{8}$$

B. Multiplication

Consider,

$A_2=\underset{p=1}{\overset{2}{sum}}\left(1-\underline{l_{M_p}}\right)^{\eta},B_2=\underset{p=1}{\overset{2}{sum}}\left(1-\overline{l_{M_p}}\right)^{\eta},$

$C_2=\underset{p=1}{\overset{2}{sum}}\left(1-c_{M_p}\right)^{\eta},$

$D_2=\underset{p=1}{\overset{2}{sum}}\left(1-\underline{r_{M_p}}\right)^{\eta},E_2=\underset{p=1}{\overset{2}{sum}}\left(1-\overline{r_{M_p}}\right)^{\eta}.$

$$\overline{M_1}\underset{Y}{\otimes}\overline{M_2}=\left\{\left[\max\left(1-A_2^{\frac{1}{\eta}},0\right),\max\left(1-B_2^{\frac{1}{\eta}},0\right)\right],\ \max\left(1-C_2^{\frac{1}{\eta}},0\right),\left[\max\left(1-D_2^{\frac{1}{\eta}},0\right),\ \max\left(1-E_2^{\frac{1}{\eta}},0\right)\right]\right\} \tag{9}$$

C. Multiplication by an ordinary number

Consider, $A=\underline{l_M},B=\overline{l_M},C=c_M,D=\underline{r_M},E=\overline{r_M}$

$$k\underset{Y}{\oplus}\overline{M}=\left\{\left[\min\left\langle A^{\frac{k}{\eta}},1\right\rangle,\min\left\langle B^{\frac{k}{\eta}},1\right\rangle\right],\ \min\left\langle C^{\frac{k}{\eta}},1\right\rangle,\left[\min\left\langle D^{\frac{k}{\eta}},1\right\rangle,\min\left\langle E^{\frac{k}{\eta}},1\right\rangle\right]\right\} \tag{10}$$

D. Power

Consider

$A_3=1-\underline{l_M},B_3=1-\overline{l_M},C_3=1-c_M,\ D_3=1-\underline{r_M},E_3=1-\overline{r_M}$

$$\overline{M}^{\hat{Y}k}=\left\{\left[\max\left(1-\left[A_3^{\eta}\right]^{k/\eta},0\right),\max\left(1-\left[B_3^{\eta}\right]^{k/\eta},0\right)\right],\right.$$

$$\max\left(1-\left[C_3^{\eta}\right]^{k/\eta},0\right),$$

$$\left[\max\left(1-\left[D_3{}^{\eta}\right]^{k/\eta},0\right),\max\left(1-\left[E_3{}^{\eta}\right]^{k/\eta},0\right)\right]\Bigg\} \quad (11)$$

V. Proposed Theorems

Here the mathematical properties of aggregation properties for TIT2FN using TIT2FYWG and TIT2FYWA operators are proved and they are playing an important role in image processing.

Consider a collection of TIT2FNs $\overline{M}=\left([\underline{l_{M_p}},\overline{l_{M_p}}],c_{M_p},[\underline{r_{M_p}},\overline{r_{M_p}}]\right),\quad p=1,2,...,n$

Where $0\le \underline{l_M}\le \overline{l_M}\le c_M\le \underline{r_M}\le \overline{r_M}\le 1$

A. Theorem

The aggregation value of these fuzzy numbers using TIT2FYWG operator is again a TIT2FN and

$$TIT2FYWG_{\varepsilon}\left\langle \overline{M_1},\overline{M_2},...,\overline{M_n}\right\rangle=\Bigg\{\Bigg[\max\left(1-\left[A_n{}^{\eta}\right]^{\varepsilon_p/\eta},0\right),$$

$$\max\left(1-\left[B_n{}^{\eta}\right]^{\varepsilon_p/\eta},0\right)\Bigg],\ \max\left(1-\left[C_n{}^{\eta}\right]^{\varepsilon_p/\eta},0\right),$$

$$\left[\max\left(1-\left[D_n{}^{\eta}\right]^{\varepsilon_p/\eta},0\right),\ \max\left(1-\left[E_n{}^{\eta}\right]^{\varepsilon_p/\eta},0\right)\right]\Bigg\} \quad (12)$$

Where the weight vector is $\varepsilon=\left(\varepsilon_1,\varepsilon_2,...,\varepsilon_n\right)^T,\varepsilon_n\ge 0$,

the sum of the weight vectors is equal to 1.

Proof:

Here use mathematical induction method.

Case (i): For $n=2$.

Consider, $P_1=\left(1-\underline{l_{M_1}}\right)^{\eta},Q_1=\left(1-\overline{l_{M_1}}\right)^{\eta},R_1=\left(1-c_{M_1}\right)^{\eta}$

$S_1=\left(1-\underline{r_{M_1}}\right)^{\eta},T_1=\left(1-\overline{r_{M_1}}\right)^{\eta}$

Using Yager power operation

$$\overline{M_1}^{\overset{\bullet}{Y}k}=\Bigg\{\Bigg[\max\left(1-P_1{}^{\frac{k}{\eta}},0\right),\max\left(1-Q_1{}^{\frac{k}{\eta}},0\right)\Bigg],$$

$$\max\left(1-R_1{}^{\frac{k}{\eta}},0\right),\Bigg[\max\left(1-S_1{}^{\frac{k}{\eta}},0\right),\max\left(1-T_1{}^{\frac{k}{\eta}},0\right)\Bigg]\Bigg\}.$$

Consider, $P_2=\left(1-\underline{l_{M_1}}\right)^{\eta},\ Q_2=\left(1-\overline{l_{M_1}}\right)^{\eta},R_2=\left(1-c_{M_1}\right)^{\eta}$,

$S_2=\left(1-\underline{r_{M_1}}\right)^{\eta},\ T_2=\left(1-\overline{r_{M_1}}\right)^{\eta}$

$$\overline{M_2}^{\overset{\bullet}{Y}k}=\Bigg\{\Bigg[\max\left(1-P_2{}^{\frac{k}{\eta}},0\right),\max\left(1-Q_2{}^{\frac{k}{\eta}},0\right)\Bigg],$$

$$\max\left(1-R_2{}^{\frac{k}{\eta}},0\right),\Bigg[\max\left(1-S_2{}^{\frac{k}{\eta}},0\right),\max\left(1-T_2{}^{\frac{k}{\eta}},0\right)\Bigg]\Bigg\}.$$

$$TIT2FYWG_{\varepsilon}\left(\overline{M_1},\overline{M_2}\right)=\overline{M_1}^{\overset{\bullet}{Y}\varepsilon_1}\underset{Y}{\otimes}\overline{M_2}^{\overset{\bullet}{Y}\varepsilon_2}$$

$$=\Bigg\{\Bigg[\max\left(1-\left[\underset{p=1}{\overset{2}{sum}}\left(1-\max\left(1-\left(1-A_2\right)\frac{\varepsilon_p}{\eta},0\right)\right)\right],0\right),$$

$$\max\left(1-\left[\underset{p=1}{\overset{2}{sum}}\left(1-\max\left(1-\left(1-B_2\right)\frac{\varepsilon_p}{\eta},0\right)\right)\right],0\right)\Bigg],$$

$$\max\left(1-\left[\underset{p=1}{\overset{2}{sum}}\left(1-\max\left(1-\left(1-C_2\right)\frac{\varepsilon_p}{\eta},0\right)\right)\right],0\right),$$

$$\Bigg[\max\left(1-\left[\underset{p=1}{\overset{2}{sum}}\left(1-\max\left(1-\left(1-D_2\right)\frac{\varepsilon_p}{\eta},0\right)\right)\right],0\right),$$

$$\max\left(1-\left[\underset{p=1}{\overset{2}{sum}}\left(1-\max\left(1-\left(1-E_2\right)\frac{\varepsilon_p}{\eta},0\right)\right)\right],0\right)\Bigg]\Bigg\}.$$

$A_4=\left(1-\underline{l_{M_p}}\right)^{\eta},B_4=\left(1-\overline{l_{M_p}}\right)^{\eta},C_4=\left(1-c_{M_p}\right)^{\eta}$,

$D_4=\left(1-\underline{r_{M_p}}\right)^{\eta},E_4=\left(1-\overline{r_{M_p}}\right)^{\eta}$

$$=\Bigg\{\Bigg[\max\left(1-\left(\underset{p=1}{\overset{2}{sum}}\left(1-\max\left(1-A_4{}^{\frac{\varepsilon_p}{\eta}},0\right)\right)\right),0\right),$$

$$\max\left(1-\left(\underset{p=1}{\overset{2}{sum}}\left(1-\max\left(1-B_4{}^{\frac{\varepsilon_p}{\eta}},0\right)\right)\right),0\right)\Bigg],$$

$$\max\left(1-\left(\underset{p=1}{\overset{2}{sum}}\left(1-\max\left(1-C_4^{\frac{\varepsilon_p}{\eta}},0\right)\right)\right),0\right)\Bigg),$$

$$\left[\max\left(1-\left(\underset{p=1}{\overset{2}{sum}}\left(1-\max\left(1-D_4^{\frac{\varepsilon_p}{\eta}},0\right)\right)\right),0\right)\right),$$

$$\max\left(1-\left(\underset{p=1}{\overset{2}{sum}}\left(1-\max\left(1-E_4^{\frac{\varepsilon_p}{\eta}},0\right)\right)\right),0\right)\Bigg)\Bigg]\Bigg\}.$$

$$=\left\{\left[\max\left[1-A_2^{\frac{\varepsilon_p}{\eta}},0\right],\max\left[1-B_2^{\frac{\varepsilon_p}{\eta}},0\right]\right],\max\left[1-C_2^{\frac{\varepsilon_p}{\eta}},0\right],\right.$$

$$\left.\left[\max\left[1-D_2^{\frac{\varepsilon_p}{\eta}},0\right],\max\left[1-E_2^{\frac{\varepsilon_p}{\eta}},0\right]\right]\right\}$$

For $n=k,$

$$A_k=\underset{p=1}{\overset{k}{sum}}\left(1-\underline{l_{M_p}}\right)^{\eta},B_k=\underset{p=1}{\overset{k}{sum}}\left(1-\overline{l_{M_p}}\right)^{\eta},C_k=\underset{p=1}{\overset{k}{sum}}\left(1-c_{M_p}\right)^{\eta},$$

$$D_k=\underset{p=1}{\overset{k}{sum}}\left(1-\underline{r_{M_p}}\right)^{\eta},E_k=\underset{p=1}{\overset{k}{sum}}\left(1-\overline{r_{M_p}}\right)^{\eta}.$$

$$TIT2FYWG_{\varepsilon}\left\langle\overline{M_1},\overline{M_2},...,\overline{M_k}\right\rangle$$

$$=\left\{\left[\max\left(1-\left[\underset{p=1}{\overset{k}{sum}}\left(1-\max\left(1-\left[A_k\right]^{\frac{\omega_p}{\eta}},0\right)\right)\right],0\right),\right.\right.$$

$$\left.\max\left(1-\left[\underset{p=1}{\overset{k}{sum}}\left(1-\max\left(1-B_k^{\frac{\omega_p}{\eta}},0\right)\right)\right],0\right)\right],$$

$$\max\left(1-\left[\underset{p=1}{\overset{k}{sum}}\left(1-\max\left(1-C_k^{\frac{\omega_p}{\eta}},0\right)\right)\right],0\right),$$

$$\left[\max\left(1-\left[\underset{p=1}{\overset{k}{sum}}\left(1-\max\left(1-D_k^{\frac{\varepsilon_p}{\eta}},0\right)\right)\right],0\right),\right.$$

$$\left.\left.\max\left(1-\left[\underset{p=1}{\overset{k}{sum}}\left(1-\max\left(1-E_k^{\frac{\varepsilon_p}{\eta}},0\right)\right)\right],0\right)\right]\right\}.$$

$$=\left\{\left[\max\left[1-A_k^{\frac{\varepsilon_p}{\eta}},0\right],\max\left[1-B_k^{\frac{\varepsilon_p}{\eta}},0\right]\right],\right.$$

$$\left.\max\left[1-C_k^{\frac{\varepsilon_p}{\eta}},0\right],\left[\max\left[1-D_k^{\frac{\varepsilon_p}{\eta}},0\right],\max\left[1-E_k^{\frac{\varepsilon_p}{\eta}},0\right]\right]\right\}.$$

For $n=k+1,$

$$TIT2FYWG_{\varepsilon}\left\langle\overline{M_1},\overline{M_2},...,\overline{M_k}\right\rangle\underset{Y}{\otimes}TIT2FYWG_{\varepsilon}\left\langle\overline{M_{k+1}}\right\rangle$$

$$=\left\{\left[\max\left(1-\left[\underset{p=1}{\overset{k}{sum}}\left(1-\max\left(1-A_k^{\frac{\varepsilon_p}{\eta}},0\right)\right)\right],0\right),\right.\right.$$

$$\left.\max\left(1-\left[\underset{p=1}{\overset{k}{sum}}\left(1-\max\left(1-B_k^{\frac{\varepsilon_p}{\eta}},0\right)\right)\right],0\right)\right],$$

$$\max\left(1-\left[\underset{p=1}{\overset{k}{sum}}\left(1-\max\left(1-C_k^{\frac{\varepsilon_p}{\eta}},0\right)\right)\right],0\right),$$

$$\left[\max\left(1-\left[\underset{p=1}{\overset{k}{sum}}\left(1-\max\left(1-D_k^{\frac{\varepsilon_p}{\eta}},0\right)\right)\right],0\right),\right.$$

$$\left.\left.\max\left(1-\left[\underset{p=1}{\overset{k}{sum}}\left(1-\max\left(1-E_k^{\frac{\varepsilon_p}{\eta}},0\right)\right)\right],0\right)\right]\right\}.$$

$$\underset{Y}{\otimes}\left\{\left[\max\left(1-P_{k+1}^{\frac{\omega_{k+1}}{\eta}},0\right),\max\left(1-Q_{k+1}^{\frac{\varepsilon_{k+1}}{\eta}},0\right)\right],\right.$$

$$\max\left(1-R_{k+1}^{\frac{\varepsilon_{k+1}}{\eta}},0\right),$$

$$\left.\left[\max\left(1-S_{k+1}^{\frac{\varepsilon_{k+1}}{\eta}},0\right),\max\left(1-T_{k+1}^{\frac{\varepsilon_{k+1}}{\eta}},0\right)\right]\right\}.$$

$$=\left\{\left[\max\left(1-\left[\underset{p=1}{\overset{k+1}{sum}}\left(1-\max\left(1-A_k^{\frac{\varepsilon_p}{\eta}},0\right)\right)\right],0\right),\right.\right.$$

$$\max\left(1-\left[\underset{p=1}{\overset{k+1}{sum}}\left(1-\max\left(1-B_k^{\frac{\varepsilon_p}{\eta}},0\right)\right)\right],0\right)\Bigg],$$

$$\max\left(1-\left[\underset{p=1}{\overset{k+1}{sum}}\left(1-\max\left(1-C_k^{\frac{\varepsilon_p}{\eta}},0\right)\right)\right],0\right),$$

$$\Bigg[\max\left(1-\left[\underset{p=1}{\overset{k+1}{sum}}\left(1-\max\left(1-D_k^{\frac{\varepsilon_p}{\eta}},0\right)\right)\right],0\right),$$

$$\max\left(1-\left[\underset{p=1}{\overset{k+1}{sum}}\left(1-\max\left(1-E_k^{\frac{\varepsilon_p}{\eta}},0\right)\right)\right],0\right)\Bigg]\Bigg\}.$$

$$=\left\{\left[\max\left[1-A_{k+1}^{\frac{\varepsilon_p}{\eta}},0\right],\max\left[1-B_{k+1}^{\frac{\varepsilon_p}{\eta}},0\right]\right],\right.$$

$$\left.\max\left[1-C_{k+1}^{\frac{\varepsilon_p}{\eta}},0\right],\left[\max\left[1-D_{k+1}^{\frac{\omega_p}{\eta}},0\right],\max\left[1-E_{k+1}^{\frac{\varepsilon_p}{\eta}},0\right]\right]\right\}.$$

Hence the result holds for all the values of n.

B. Theorem(Idempotency)

If $\overline{M_p}=\overline{M}$ for all the values of p then $TIT2FYWG_{\varepsilon}\left\langle\overline{M_1},\overline{M_2},...,\overline{M_n}\right\rangle=\overline{M}.$ (13)

Proof:

By theorem A,

$$TIT2FYWG_{\varepsilon}\left\langle\overline{M_1},\overline{M_2},...,\overline{M_n}\right\rangle$$

$$=\left\{\left[\max\left(1-A_n^{\frac{\varepsilon_p}{\eta}},0\right),\max\left(1-B_n^{\frac{\omega_p}{\eta}},0\right)\right],\right.$$

$$\left.\max\left(1-C_n^{\frac{\varepsilon_p}{\eta}},0\right),\left[\max\left(1-D_n^{\frac{\varepsilon_p}{\eta}},0\right),\max\left(1-E_n^{\frac{\varepsilon_p}{\eta}},0\right)\right]\right\}.$$

$$TIT2FYWG_{\varepsilon}\left\langle\overline{M_1},\overline{M_2},...,\overline{M_n}\right\rangle$$

$$=\left\{\left[\max\left(1-A_4^{\underset{p=1}{\overset{k+1}{sum}}\left(\frac{\varepsilon_p}{\eta}\right)},0\right),\max\left(1-B_4^{\underset{p=1}{\overset{k+1}{sum}}\left(\frac{\omega_p}{\eta}\right)},0\right)\right],\right.$$

$$\left.\max\left(1-C_4^{\underset{p=1}{\overset{k+1}{sum}}\left(\frac{\varepsilon_p}{\eta}\right)},0\right),\left[\max\left(1-D_4^{\underset{p=1}{\overset{k+1}{sum}}\left(\frac{\varepsilon_p}{\eta}\right)},0\right),\max\left(1-E_4^{\underset{p=1}{\overset{k+1}{sum}}\left(\frac{\varepsilon_p}{\eta}\right)},0\right)\right]\right\}$$

$$=\left\{\left[\max\left(1-A_5^{\frac{1}{\eta}},0\right),\max\left(1-B_5^{\frac{1}{\eta}},0\right)\right],\right.$$

$$\left.\max\left(1-C_5^{\frac{1}{\eta}},0\right),\left[\max\left(1-D_5^{\frac{1}{\eta}},0\right),\max\left(1-E_5^{\frac{1}{\eta}},0\right)\right]\right\}.$$

$$=\left\{\left[A_3,B_3,C_3,D_3,E_3\right]\right\}=\left\{\left[A,B\right],C,\left[D,E\right]\right\}=\overline{M}$$

C. Theorem(Boundary)

Let

$$\overline{M}^{+}=\left\{\left[\max_{p=1}^{n}\left(\underline{l_{M_p}}\right),\max_{p=1}^{n}\left(\overline{l_{M_p}}\right)\right],\max_{p=1}^{n}c_{M_p},\left[\max_{p=1}^{n}\left(\underline{r_{M_p}}\right),\max_{p=1}^{n}\left(\overline{r_{M_p}}\right)\right]\right\}$$

$$\overline{M}^{-}=\left\{\left[\min_{p=1}^{n}\left(\underline{l_{M_p}}\right),\min_{p=1}^{n}\left(\overline{l_{M_p}}\right)\right],\min_{p=1}^{n}c_{M_p},\left[\min_{p=1}^{n}\left(\underline{r_{M_p}}\right),\min_{j=p}^{n}\left(\overline{r_{M_p}}\right)\right]\right\}$$

Then $\overline{M}^{-}\le TIT2FYWG_{\varepsilon}\left\langle\overline{M_1},\overline{M_2},...,\overline{M_n}\right\rangle\le\overline{M}^{+}$ (14)

Proof:

Since,

$$\min_{p=1}^{n}\left(\underline{l_{M_p}}\right)\le\underline{l_{M_p}}\le\max_{p=1}^{n}\left(\underline{l_{M_p}}\right),\ \min_{p=1}^{n}\left(\overline{l_{M_p}}\right)\le\overline{l_{M_p}}\le\max_{p=1}^{n}\left(\overline{l_{M_p}}\right),$$

$$\min_{p=1}^{n}\left(c_{M_p}\right)\le c_{M_p}\le\max_{p=1}^{n}\left(c_{M_p}\right),$$

$$\min_{p=1}^{n}\left(\underline{r_{M_p}}\right)\le\underline{r_{M_p}}\le\max_{p=1}^{n}\left(\underline{r_{M_p}}\right),\ \min_{p=1}^{n}\left(\overline{r_{M_p}}\right)\le\overline{r_{M_p}}\le\max_{j=1}^{n}\left(\overline{r_{M_p}}\right).$$

we have,

$$1-\max_{p=1}^{n}\left(\underline{l_{M_p}}\right)\le\underline{l_{M_p}}\le1-\min_{p=1}^{n}\left(\underline{l_{M_p}}\right)$$

$$\Rightarrow\min\left(\left[\underset{p=1}{\overset{n}{sum}}\left(1-\max\left(\underline{l_{M_p}}\right)\right)^{\eta}\right]^{\frac{1}{\eta}},1\right)\le\min\left(\left[\underset{p=1}{\overset{n}{sum}}\left(1-\left(\underline{l_{M_p}}\right)\right)^{\eta}\right]^{\frac{1}{\eta}},1\right)$$

$$\le\min\left(\left[\underset{p=1}{\overset{n}{sum}}\left(1-\min\left(\underline{l_{M_p}}\right)\right)^{\eta}\right]^{\frac{1}{\eta}},1\right).$$

$$\Rightarrow \min\left(\left[\left(1-\max\left(\underline{l_{Mp}}\right)\right)^{\eta}\right]^{\overset{n}{\underset{p=1}{prod}}\left(\frac{1}{\eta}\right)},1\right) \le \min\left(\left[\left(1-\underline{l_{M_p}}\right)^{\gamma}\right]^{\overset{n}{\underset{p=1}{prod}}\left(\frac{1}{\eta}\right)},1\right)$$

$$\le \min\left(\left[\left(1-\min\left(\underline{l_{M_p}}\right)\right)^{\eta}\right]^{\overset{n}{\underset{p=1}{prod}}\left(\frac{1}{\eta}\right)},1\right)$$

$$\Rightarrow \min\left(\left[\left(1-\max\left(\underline{l_{M_p}}\right)\right)\right],1\right) \le \min\left(\left[\left(1-\underline{l_{M_p}}\right)\right],1\right)$$

$$\le \min\left(\left[\left(1-\min\left(\underline{l_{M_p}}\right)\right)\right],1\right)$$

$$\Rightarrow \min\left(\min\left(\underline{l_{M_p}}\right),1\right) \le \min\left(1-\underset{p=1}{\overset{n}{sum}}\left(\underline{l_{M_p}}\right),1\right) \le \min\left(\max\left(\underline{l_{M_p}}\right),1\right)$$

$$\Rightarrow \min\left(\underline{l_{M_p}}\right) \le \min\left(1-\underset{p=1}{\overset{n}{sum}}\left(\underline{l_{M_p}}\right),1\right) \le \max\left(\underline{l_{M_p}}\right)$$

Similarly we have,

$$\min\left(\overline{l_{M_p}}\right) \le \min\left(1-\underset{p=1}{\overset{n}{sum}}\left(\overline{l_{M_p}}\right),1\right) \le \max\left(\overline{l_{M_p}}\right),$$

$$\min\left(c_{M_p}\right) \le \min\left(1-\underset{p=1}{\overset{n}{sum}}\left(c_{M_p}\right),1\right) \le \max\left(c_{M_p}\right)$$

$$\min\left(\underline{r_{M_p}}\right) \le \min\left(1-\underset{p=1}{\overset{n}{sum}}\left(\underline{r_{M_p}}\right),1\right) \le \max\left(\underline{r_{M_p}}\right),$$

$$\min\left(\overline{r_{M_p}}\right) \le \min\left(1-\underset{p=1}{\overset{n}{sum}}\left(\overline{l_{M_p}}\right),1\right) \le \max\left(\overline{r_{M_p}}\right)$$

By using the ranking value formula for TIT2FN and using the arithmetic average ranking value,

$$R\left(\overline{M}\right)=\left(\frac{\underline{l_{M_p}}+\overline{r_{M_p}}}{2}+1\right)\times\frac{\underline{l_{M_p}}+\overline{l_{M_p}}+\underline{r_{M_p}}+\overline{r_{M_p}}+4c_M}{8}$$

$$\le\left(\frac{\max\left(\underline{l_{M_p}}\right)+\max\left(\overline{r_{M_p}}\right)}{2}+1\right)\times\left(\underset{p=1}{\overset{n}{\max}}\left(\underline{l_{M_p}}\right)+\underset{p=1}{\overset{n}{\max}}\left(\overline{l_{M_p}}\right)+\underset{p=1}{\overset{n}{\max}}\left(\underline{r_{M_p}}\right)\right)$$

$$+\underset{p=1}{\overset{n}{\max}}\left(\overline{r_{M_p}}\right)+4\underset{p=1}{\overset{n}{\max}}\left(c_{M_p}\right)\Big)\times 8^{-1}=R\left(\overline{M}^{+}\right)$$

Hence the result.

D. Theorem

If $t>0$ for all the values of p then

$$TIT2FYWG_{\varepsilon}\left(\overline{M_1}^{\bullet t}_{Y},\overline{M_2}^{\bullet t}_{Y},...,\overline{M_n}^{\bullet t}_{Y}\right)$$

$$=TIT2FYWG_{\varepsilon}\left(\overline{M_1},\overline{M_2},...,\overline{M_n}\right)^{\bullet t}_{Y} \tag{15}$$

Proof:

$$\overline{M}^{\bullet t}_{Y}=\left\{\left[\max\left(1-A_5^{\frac{t}{\eta}},0\right),\max\left(1-B_5^{\frac{t}{\eta}},0\right)\right],\max\left(1-C_5^{\frac{t}{\eta}},0\right),\right.$$

$$\left.\left[\max\left(1-D_5^{\frac{t}{\eta}},0\right),\max\left(1-E_5^{\frac{t}{\eta}},0\right)\right]\right\}.$$

$$=\left\{\left[\max\left(1-\left[\underset{p=1}{\overset{k}{sum}}\left(1-\max\left(1-A_k^{\frac{\varepsilon_p}{\eta}},0\right)\right)\right],0\right),\right.\right.$$

$$\left.\max\left(1-\left[\underset{p=1}{\overset{k}{sum}}\left(1-\max\left(1-B_k^{\frac{\varepsilon_p}{\eta}},0\right)\right)\right],0\right)\right],$$

$$\max\left(1-\left[\underset{p=1}{\overset{k}{sum}}\left(1-\max\left(1-C_k^{\frac{\varepsilon_p}{\eta}},0\right)\right)\right],0\right),$$

$$\left[\max\left(1-\left[\underset{p=1}{\overset{k}{sum}}\left(1-\max\left(1-D_k^{\frac{\varepsilon_p}{\eta}},0\right)\right)\right],0\right),\right.$$

$$\left.\left.\max\left(1-\left[\underset{p=1}{\overset{k}{sum}}\left(1-\max\left(1-E_k^{\frac{\varepsilon_p}{\eta}},0\right)\right)\right],0\right)\right]\right\}.$$

$$TIT2FYWG_{\varepsilon}\left\langle\overline{M_1},\overline{M_2},...,\overline{M_n}\right\rangle$$

$$=\overline{M_1}^{\bullet\varepsilon_1}_{Y}\underset{Y}{\otimes}\overline{M_2}^{\bullet\varepsilon_2}_{Y}\underset{Y}{\otimes}...\underset{Y}{\otimes}\overline{M_n}^{\bullet\varepsilon_n}_{Y}$$

$$=\left\{\left[\max\left(1-\left[\underset{p=1}{\overset{n}{sum}}\left(1-\max\left(1-\left(1-A_2\right)^{\frac{\varepsilon_p}{\eta}},0\right)\right)\right],0\right),\right.\right.$$

$$\left.\max\left(1-\left[\underset{p=1}{\overset{n}{sum}}\left(1-\max\left(1-\left(1-B_2\right)^{\frac{\varepsilon_p}{\eta}},0\right)\right)\right],0\right)\right]$$

$$\max\left(1-\left[\underset{p=1}{\overset{n}{sum}}\left(1-\max\left(1-\left(1-C_2\right)^{\frac{\varepsilon_p}{\eta}},0\right)\right)\right],0\right),$$

$$\left[\max\left(1-\left[\underset{p=1}{\overset{n}{sum}}\left(1-\max\left(1-\left(1-D_2\right)^{\frac{\varepsilon_p}{\eta}},0\right)\right)\right],0\right),\right.$$

$$\left.\left.\max\left(1-\left[\underset{p=1}{\overset{n}{sum}}\left(1-\max\left(1-\left(1-E_2\right)^{\frac{\varepsilon_p}{\eta}},0\right)\right)\right],0\right)\right]\right\}.$$

$$=\left\{\left[\max\left(1-A_n^{\frac{t\varepsilon_p}{\eta}},0\right),\max\left(1-B_n^{\frac{t\varepsilon_p}{\eta}},0\right)\right],\right.$$

$$\left.\max\left(1-C_n^{\frac{t\varepsilon_p}{\eta}},0\right),\left[\max\left(1-D_n^{\frac{t\varepsilon_p}{\eta}},0\right),\max\left(1-E_n^{\frac{t\varepsilon_p}{\eta}},0\right)\right]\right\}. \tag{16}$$

Also since,

$$TIT2FYWG_{\varepsilon}\left(\overline{M_1},\overline{M_2},\ldots,\overline{M_n}\right)^{\bullet_Y t}$$

$$=\left\{\left[\max\left(1-A_n^{\frac{t\varepsilon_p}{\eta}},0\right),\max\left(1-B_n^{\frac{t\varepsilon_p}{\eta}},0\right)\right],\right.$$

$$\left.\max\left(1-C_n^{\frac{t\varepsilon_p}{\eta}},0\right),\left[\max\left(1-D_n^{\frac{t\varepsilon_p}{\eta}},0\right),\max\left(1-E_n^{\frac{t\varepsilon_p}{\eta}},0\right)\right]\right\}^{\bullet_Y t}.$$

$$=\left\{\left[\max\left(\left(1-A_n^{\frac{t}{\eta}}\right)^{\varepsilon_p},0\right),\max\left(\left(1-B_n^{\frac{t}{\eta}}\right)^{\varepsilon_p},0\right)\right],\max\left(\left(1-C_n^{\frac{t}{\eta}}\right)^{\varepsilon_p},0\right),\right.$$

$$\left.\left[\max\left(\left(1-D_n^{\frac{t}{\eta}}\right)^{\varepsilon_p},0\right),\max\left(\left(1-E_n^{\frac{t}{\eta}}\right)^{\varepsilon_p},0\right)\right]\right\}.$$

$$=\left\{\left[\max\left(\left(\left(1-A_n^{\frac{t\varepsilon_p}{\eta}}\right)\right),0\right),\max\left(\left(\left(1-B_n^{\frac{t\varepsilon_p}{\eta}}\right)\right),0\right)\right],\right.$$

$$\left.\max\left(\left(\left(1-C_n^{\frac{t\varepsilon_p}{\eta}}\right)\right),0\right),\left[\max\left(\left(\left(1-D_n^{\frac{t\varepsilon_p}{\eta}}\right)\right),0\right),\max\left(\left(\left(1-E_n^{\frac{t\varepsilon_p}{\eta}}\right)\right),0\right)\right]\right\} \tag{17}$$

Since (16) = (17), hence the result.

E. Theorem(Stability)

If $t>0$, $\overline{M_{n+1}}=\left(\left[\underline{l_{M_{n+1}}},\overline{l_{M_{n+1}}}\right],c_{M_{n+1}},\left[\underline{r_{M_{n+1}}},\overline{r_{M_{n+1}}}\right]\right)$ then

$$TIT2FYWG_{\varepsilon}\left(\overline{M_1}^{\bullet_Y t}\underset{Y}{\otimes}\overline{M_{n+1}},\overline{M_2}^{\bullet_Y t}\underset{Y}{\otimes}\overline{M_{n+1}},\ldots,\overline{M_n}^{\bullet_Y t}\underset{Y}{\otimes}\overline{M_{n+1}}\right)$$

$$=TIT2FYWG_{\varepsilon}\left(\overline{M_1},\overline{M_2},\ldots,\overline{M_n}\right)^{\bullet_Y t}\underset{Y}{\otimes}\overline{M_{n+1}} \tag{18}$$

Proof:

$$TIT2FYWG_{\varepsilon}\left(\overline{M_1}^{\bullet_Y t}\underset{Y}{\otimes}\overline{M_{n+1}},\overline{M_2}^{\bullet_Y t}\underset{Y}{\otimes}\overline{M_{n+1}},\ldots,\overline{M_n}^{\bullet_Y t}\underset{Y}{\otimes}\overline{M_{n+1}}\right)$$

$$=\left\{\left[\max\left(1-A_n^{\frac{t\varepsilon_p}{\eta}},0\right)\underset{Y}{\otimes}\overline{M_{n+1}},\max\left(1-B_n^{\frac{t\varepsilon_p}{\eta}},0\right)\underset{Y}{\otimes}\overline{M_{n+1}}\right],\right.$$

$$\max\left(1-C_n^{\frac{t\varepsilon_p}{\eta}},0\right)\underset{Y}{\otimes}\overline{M_{n+1}},$$

$$\left.\left[\max\left(1-D_n^{\frac{t\varepsilon_p}{\eta}},0\right)\underset{Y}{\otimes}\overline{M_{n+1}},\max\left(1-E_n^{\frac{t\varepsilon_p}{\eta}},0\right)\underset{Y}{\otimes}\overline{M_{n+1}}\right]\right\}$$

$$=\left\{\left[\max\left(1-\left[\underset{p=1}{\overset{n}{sum}}\left(1-\max\left(1-\left[\underset{q=\{p,n+1\}}{sum}\left(1-\underline{l_{M_q}}\right)^{\eta}\right]^{\frac{t}{\eta}},0\right)^{\varepsilon_p}\right)\right],0\right),\right.\right.$$

$$\left.\max\left(1-\left[\underset{p=1}{\overset{n}{sum}}\left(1-\max\left(1-\left[\underset{q=\{p,n+1\}}{sum}\left(1-\overline{l_{M_q}}\right)^{\eta}\right]^{\frac{t}{\eta}},0\right)^{\varepsilon_p}\right)\right],0\right)\right],$$

$$\max\left(1-\left[\underset{p=1}{\overset{n}{sum}}\left(1-\max\left(1-\left[\underset{q=\{p,n+1\}}{sum}\left(1-c_{M_q}\right)^{\eta}\right]^{\frac{t}{\eta}},0\right)^{\varepsilon_p}\right)\right],0\right),$$

$$\left[\max\left(1-\left[\underset{p=1}{\overset{n}{sum}}\left(1-\max\left(1-\left[\underset{q=\{p,n+1\}}{sum}\left(1-\underline{r_{M_q}}\right)^{\eta}\right]^{\frac{t}{\eta}},0\right)^{\varepsilon_p}\right)\right],0\right),\right.$$

$$\left.\left.\max\left(1-\left[\underset{p=1}{\overset{n}{sum}}\left(1-\max\left(1-\left[\underset{q=\{p,n+1\}}{sum}\left(1-\overline{r_{M_q}}\right)^{\eta}\right]^{\frac{t}{\eta}},0\right)^{\varepsilon_p}\right)\right],0\right)\right]\right\}.$$

$$=\left\{\left[\max\left(\left(1-A_n^{\frac{\varepsilon_p}{\eta}}\right)+\left(\left[\left(1-\underline{l_{M+1}}\right)^{\eta}\right]^{\frac{1}{\eta}}\right)^{\underset{p=1}{\overset{n}{sum}}\varepsilon_p},0\right),\right.\right.$$

$$\max\left(\left(1-B_n^{\frac{\varepsilon_p}{\eta}}\right)+\left(\left[\left(1-\overline{l_{M+1}}\right)^{\eta}\right]^{\frac{1}{\eta}}\right)^{\overset{n}{\underset{p=1}{sum}}\varepsilon_p},0\right)\Bigg]\Bigg],$$

$$\max\left(\left(1-C_n^{\frac{\varepsilon_p}{\eta}}\right)+\left(\left[\left(1-c_{M+1}\right)^{\eta}\right]^{\frac{1}{\eta}}\right)^{\overset{n}{\underset{p=1}{sum}}\varepsilon_p},0\right),$$

$$\left[\max\left(\left(1-D_n^{\frac{\varepsilon_p}{\eta}}\right)+\left(\left[\left(1-\underline{r_{M+1}}\right)^{\eta}\right]^{\frac{1}{\eta}}\right)^{\overset{n}{\underset{p=1}{sum}}\varepsilon_p},0\right),\right.$$

$$\left.\max\left(\left(1-E_n^{\frac{\varepsilon_p}{\eta}}\right)+\left(\left[\left(1-\overline{r_{M+1}}\right)^{\eta}\right]^{\frac{1}{\eta}}\right)^{\overset{n}{\underset{p=1}{sum}}\varepsilon_p},0\right)\Bigg]\Bigg]\right\}. \tag{19}$$

Based on the theorem A and the operational law,

$$\text{TIT2FYWG}_{\varepsilon}\left\langle\overline{M_1},\overline{M_2},...,\overline{M_n}\right\rangle^{\bullet t}_{Y}\underset{Y}{\otimes}\overline{M_{n+1}}$$

$$\left\{\left[\max\left(1-A_n^{\frac{t\varepsilon_p}{\eta}},0\right),\max\left(1-B_n^{\frac{t\varepsilon_p}{\eta}},0\right)\right],\right.$$

$$\left.\max\left(1-C_n^{\frac{t\varepsilon_p}{\eta}},0\right),\left[\max\left(1-D_n^{\frac{t\varepsilon_p}{\eta}},0\right),\ \max\left(1-E_n^{\frac{t\varepsilon_p}{\eta}},0\right)\right]\right\}.$$

$$\underset{Y}{\otimes}\left\langle\left[\underline{l_{M_{n+1}}},\overline{l_{M_{n+1}}}\right],c_{M_{n+1}},\left[\underline{r_{M_{n+1}}},\overline{r_{M_{n+1}}}\right]\right\rangle$$

$$=\left\{\left[\max\left(\left(1-A_n^{\frac{\varepsilon_p}{\eta}}\right)+\left(\left[\left(1-\underline{l_{M+1}}\right)^{\eta}\right]^{\frac{1}{\eta}}\right),0\right),\right.\right.$$

$$\max\left(\left(1-B_n^{\frac{\varepsilon_p}{\eta}}\right)+\left(\left[\left(1-\overline{l_{M+1}}\right)^{\eta}\right]^{\frac{1}{\eta}}\right),0\right)\Bigg],$$

$$\max\left(\left(1-C_n^{\frac{\varepsilon_p}{\eta}}\right)+\left(\left[\left(1-c_{M+1}\right)^{\eta}\right]^{\frac{1}{\eta}}\right),0\right),$$

$$\left[\max\left(\left(1-D_n^{\frac{\varepsilon_p}{\eta}}\right)+\left(\left[\left(1-\underline{r_{M+1}}\right)^{\eta}\right]^{\frac{1}{\eta}}\right),0\right),\right.$$

$$\left.\left.\max\left(\left(1-E_n^{\frac{\varepsilon_p}{\eta}}\right)+\left(\left[\left(1-\overline{r_{M+1}}\right)^{\eta}\right]^{\frac{1}{\eta}}\right),0\right)\right]\right\} \tag{20}$$

Here, (19) = (20).

Hence the result.

F. Theorem(Image Contrast)

For given arguments $\overline{M}_p, p=1,2,...,n$ and the parameter $\eta\in(1,+\infty)$ then TIT2FYWG operator is monotonically non-decreasing (MND) with respect to the parameter.

Proof:

To prove the operator is MND with respect to the parameter, we have to prove the same for every reference point function is MND w.r.t the parameter.

Since $0\le \underline{l_M}\le \overline{l_M}\le c_M\le \underline{r_M}\le \overline{r_M}\le 1$, $\max\left(1-A_5^{\frac{k}{\eta}},0\right)>0.$

And it is true for all the reference points.

Hence the result.

Note: The above theorems also can be proved by using TIT2FYWA operator.

VI. Theory of Image Processing and Role of Yager Norms

The advantage of considering Yager triangular norms is having maximum and minimum operators which will be much useful in Image Processing while filtering.

A. Associativity of Yager T Norms [8]

Each fuzzy norm should be satisfied the associativity property to compute the norm for more than two values using continual manner as follows.

Consider the associativity property for Yager T Norm (YTN)

$$YTN\left[TN\left(\overline{M}_1,\overline{M}_2\right),\overline{M}_3\right]$$

$$=\max\left\{1-\left[\left[1-T\left(\overline{M}_1,\overline{M}_2\right)\right]^{\eta},\left(1-\overline{M}_3\right)^{\eta}\right]^{1/\eta},0\right\}$$

$$=\max\left\{1-\left\{\left[1-\left[\max\left\{1-\left[\left[\left(1-\overline{M}_1\right)^{\eta}+\left(1-\overline{M}_2\right)^{\eta}\right]^{1/\eta}\right]^{\eta},0\right\}\right]\right]+\left(1-\overline{M}_3\right)^{\eta}\right\}^{1/\eta},0\right\}$$

$$=\max\left\{\left[\left\{1-1+\left\{\max\left[1-\left(\left(1-\overline{M}_1\right)^{\eta}+\left(1-\overline{M}_2\right)^{\eta}\right)^{1/\eta}\right]^{\eta},0\right\}+\left(1-\overline{M}_3\right)^{\eta}\right\}^{1/\eta}\right],0\right\}$$

$$= \max\left\{\left[\left\{\max\left\{\left[1-\left(\left(1-\overline{M}_1\right)^{\eta}+\left(1-\overline{M}_2\right)^{\eta}\right)^{1/\eta}\right]^{\eta},0\right\}+\left(1-\overline{M}_3\right)^{\eta}\right\}^{1/\eta}\right],0\right\}$$

$$= \max\left\{\left(\left[1-\left(\left(1-\overline{M}_1\right)^{\eta}+\left(1-\overline{M}_2\right)^{\eta}\right)^{1/\eta}\right]^{\eta}+\left(1-\overline{M}_3\right)^{\eta}\right)^{1/\eta},0\right\}$$

$$= \max\left\{\left(\left[1-\left(1-\overline{M}_1\right)^{\eta}+\left(\left(1-\overline{M}_2\right)^{\eta}+\left(1-\overline{M}_3\right)^{\gamma}\right)^{1/\eta}\right]^{\eta}\right)^{1/\eta},0\right\}$$

$$= YTN\left[\overline{M}_1, TN\left(\overline{M}_2,\overline{M}_3\right)\right]. \tag{21}$$

Similarly for Yager T conorm (YTCN).

Here we can consider $\overline{M}_1$ and $\overline{M}_2$ as the Interval Type-2 Triangular Fuzzy Number. If $\max\left(\overline{M}_1,\overline{M}_2\right)=1$ then (YTN) will become $\min\left(\overline{M}_1,\overline{M}_2\right)$. If $\min\left(\overline{M}_1,\overline{M}_2\right)=0$ then (YTCN) will become $\max\left(\overline{M}_1,\overline{M}_2\right)$. In the same manner we can have for YTCN. Here the definition of YTN accords the inference around the effectiveness of the norm and γ, its complimentary parameter. Generally, it allows tuning between the norms.

If η approaches 0, then YTN will be $\min\left(\overline{M}_1,\overline{M}_2\right)$ only when $\max\left(\overline{M}_1,\overline{M}_2\right)=1$ i.e., their drastic product. If η approaches 1, then YTN becomes $\max\left[(\overline{M}_1+\overline{M}_2-1),0\right]$.

B. Role of Associativity of Yager T Norm in Image Processing

Using Fuzzy Set approach, we can generalize a binary morphology into MFM. Morphological operations are the basic tools to modify the image I_1 over the structural aspect of I_2. To study the structure of I_1, size and shape of the I_2 are chosen accordingly.

C. Morphological Operations [24]

(i). Erosion (Maximum)

(ii). Dilation (Minimum)

(iii). Opening and Closing (Idempotency)

D. Role of T-Norms in Image Processing [17]

For constructing FM, we use Conjunctions and Implications. Among these two, we used conjunctions (t-norms) here and from the below, the representation of mathematical properties in image processing has been explained.

1) Commutativity:

The result of IDS application on two successive points P and Q is the same as applying on them in inverse order, since the value of flapped points is the sum of values of all data diluted on that point and therefore the operator is commutative.

2) Monotonicity:

If the brightness of P is less than or equal to Q then all the data points in brightness of P is less than or equal to brightness with respect to the corresponding data points of brightness of Q.

Therefore for any point n, the brightness appeared from P is $n+aP$, where a is proportional to inverse of distance. Similarly, the brightness appears from Q is $n+aQ$ Since $a>0$, the brightness of n appeared from P is less than or equal to that of from Q.

3) Associativity:

Assume that P, Q and R are the sources of light going to affect to the point n by IDS.

For every source, IDS increases the brightness with respect to the distance regardless of other sources.

On the point n, the order of applying IDS does not affect the distance.

Sum of effects of P, Q and R is the value of n. Therefore, the operator is associative.

4) Idempotency:

This property and its generalization is used for the morphological operation opening and closing.

5) Neutrality of 0:

Consider a pyramid of height 0, sum of this with others does not influence them. Therefore, 0 is the neutral element.

E. Morphological Gradient (MG) [24]

It is useful to detect an edge and act as a first approximation to a morphological segmentation. MG is the discrepancy between

a) dilation and erosion

b) dilation and the original image

c) original image and its erosion

F. T-Norm and Image Compression(IC) [17]

IC is based on Fuzzy Relational (FR) Equations and it is a grayscale image of size $C\times D$ as a FR $\Re\in F(A,B)$ where, $A=\{a_1,a_2,...,a_C\}$, $B=\{b_1,b_2,...,b_D\}$ enclosed the depth range of each pixel into $[0,1]$.

$CS=\left\{CS^{(R)},CS^{(G)},CS^{(B)}\right\}\subset F(A,B)$ represent the color image on RGB Color Space (CS). Here $CS^{(R)}$, $CS^{(G)}$ and $CS^{(B)}$ are the Red, Green and Blue color spaces.

For clarity, gray scale image (GSI) will be considered. The GSI $\Re\in F(A,B)$ is restrict into $\wp\in F(I\times J)$, through a max t-norm FR equations with composition $\wp = \max_{b \in B} \left\{ V_j(b) TN \max_{a \in A} \left[U_i(a) TN \square\ (a,b) \right] \right\}$, where TN is a continuous t-norm, $U_i \in U = \{U_1, U_2, ..., U_I\} \subset \mathrm{F}(A)$ and $V_i \in V = \{V_1, V_2, ..., V_J\} \subset \mathrm{F}(B)$ are the coders.

The shape of the FSs of coders are the triangular line, preferable for IC. We can adjust the compression rate of IC by the sum of FSs consist in U and V ad is defined by $\zeta = \frac{IJ}{AB}$. Here IJ and AB the compressed and original image coefficients respectively. By adjusting the parameter η, YTN will all the continuous T-norm where as Zadeh's and major t norms cannot do the same. Though Frank t-norm can do the same, due to the computational complexity, we prefer Yager's t norm for image processing.

G. Role of T2FS [24]

Here the components of an image processing and the role of T2FS is correlated.

1) Image Contrast Enrichment:

The most common image enrichment method is histogram equalization. Since an image has an imprecise pixel grey values, it may not produce acceptable results in IP. To handle the ambiguity of the gray values, Fuzzy methods have been suggested by many researchers.

By adjusting the membership values, the contrast of the image is increased by contrast intensification operator and it transforms the higher MVs to much higher and lower MVs to much lower in a nonlinear aspect. Since this aspect considers whole image, global histogram fails to produce satisfactory results.

Though the fuzzy methods deals ambiguity well and produced proper enrichment, it fails in some case and hence T2FS has been considered for this purpose since it deals more uncertainties.

2) Image Segmentation:

Region boundaries of an image may not have a fine growth, therefore fuzzy decision is used to check whether the pixel exists to a region and T2FS may be applied to get better threshold images.

3) Clustering:

The images have different regions with different pitch, clustering collects the similar pixels in a group with membership value 1 and collects different pixels in different group with membership value 0. But in fuzzy clustering the pixel associate to different number of groups and hence the MVs are not 0.

4) Edge Detection:

Since most of the images have poor brightness, the proper decision cannot be taken in checking the existence of an edge in an image. Edges may be enriched before carrying out the edge detection. In taking off the edge due to ambiguity, fuzzy method may be useful and may not find better edges. At this junction T2FS is useful as it handles more uncertainties.

5) Morphology:

Which is a non-linear image processing technique and is used to shape the image features. Here also T2FS plays an important role to get better results.

VII. Application of Image Proessing

Fig. 2. Shows that the Architecture has been proposed for edge detection on DICOM image using triangular norms

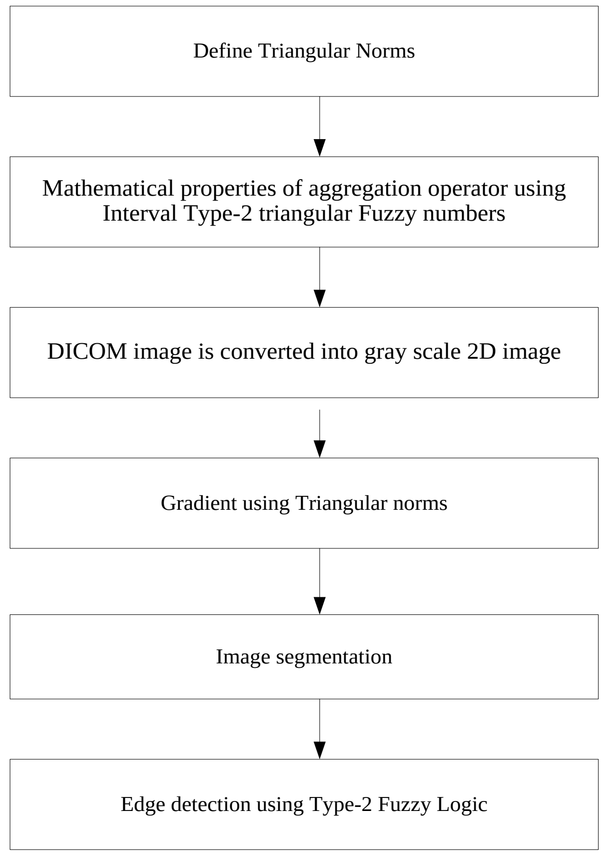

Fig. 2. Architecture for Edge Detection.

Using MATLAB 2015a, triangular norms has been applied in medical image processing from a patient DICOM image. In this case 3D image is converted to 2D image.

In Fig. 3, the image is collected from our experimental data set from a patient DICOM image in the Fig. 7. From this Fig. 7. the clear image Fig. 8. Has been considered for the experiment.

Size of the image = 512 x 517.

Mean of the image = 28.83.

Standard deviation = 60.79

Mean absolute deviation = 40.03.

To identify the gradient of the image by dilation-erosion, triangular norms are used.

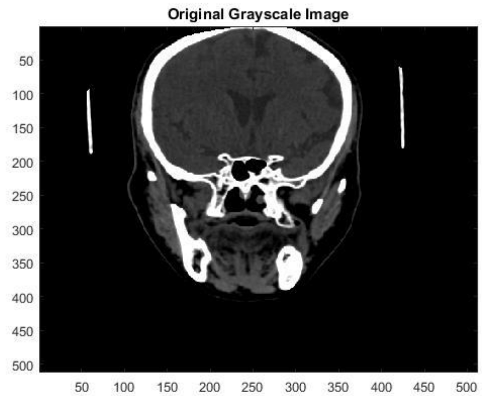

Fig. 3. Original gray scale DICOM image.

Structuring elements are used in gradient value.

Image Erosion is

0.9961	1.0000	0
1.0000	0.9961	0
1.0000	1.0000	1.0000

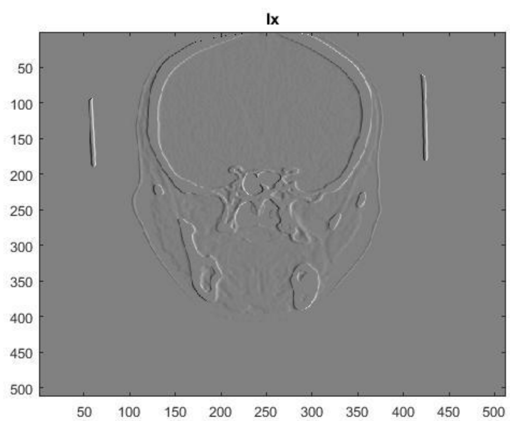

Fig. 4. Gradient through x axis.

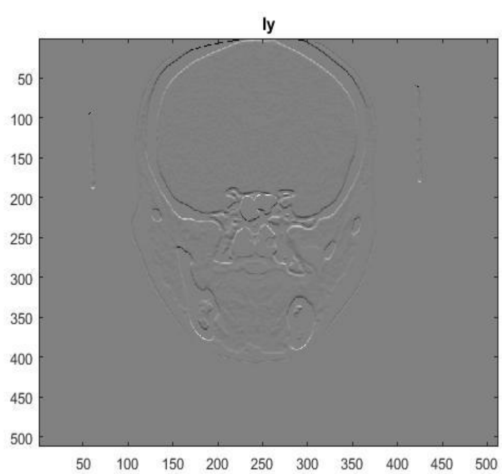

Fig. 5. Gradient through y axis.

Gradient through y axis.The below figures are the output of the image processing application in edge detection through triangular norms by MATLAB 2015 a.

Fig. 4. is the gradient through x axis and Fig. 5. is the gradient through the y axis.

The figures reveals that the image gradient to identify the region uniformly.

Fig. 6. is the output of the edge detection through T2 fuzzy by our experimental output using MATLAB 2015a

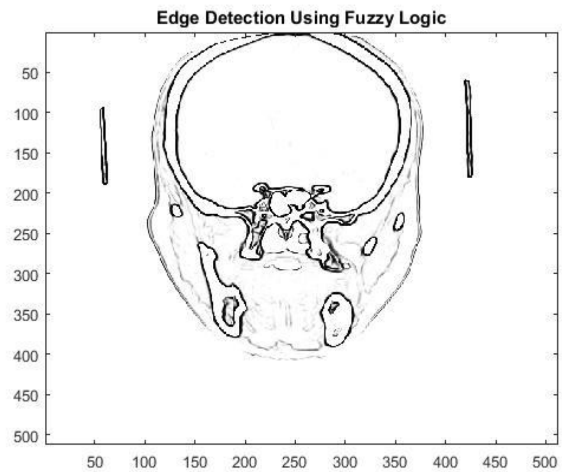

Fig. 6. Edge detection image.

Fig. 6. shows that the edges of the object through FIS and equating the pixel on both direction. If the edge is block then pixel is not 0.

Edge detection plays a vital role in image identification. It is observed that, fuzzy logic edge detector helps in reducing the memory for saving medical images.

VIII. Conclusion

In this paper, operational laws of addition, multiplication, power and multiplication by an arbitrary number using Yager triangular norms for TIT2FN are derived. Also some properties of aggregation operation using Fuzzy Yager Weighted Geometric operators have been proved. Since Yager aggregation operator contains minimum and maximum operator, it will be act as a morphological filters in medical image processing. Detailed representation of the mathematical properties in image processing is presented. Also, the gradient of the DICOM image of MRI scan of a patient using Triangular norms is found and done edge detection using MATLAB in T2 fuzzy logic. The future work is planned to apply T2 Fuzzy logic in edge extraction on medical image in 3D models.

Data Availability statement

The DICOM data used to support the findings of this study are available from the corresponding author upon request.

Conflict of interest

The authors declare that they have no conflict of interest.

Supplementary Materials

The data set in Fig. 7. is the montage of the images in a single file and is from a patient MRI. This MRI which is in the 3D form is converted to 2D form (DICOM) using MATLAB2015a. The 3D format consists of 25 DICOM file formats; the montage of the images is obtained as a single frame. Out of these 25 DICOM images a clear full image is chosen as in Fig. 8. Using Dilation corrosion method, the gradient is identified. The edge detection is performed through triangular norms using MATLAB 2015a.

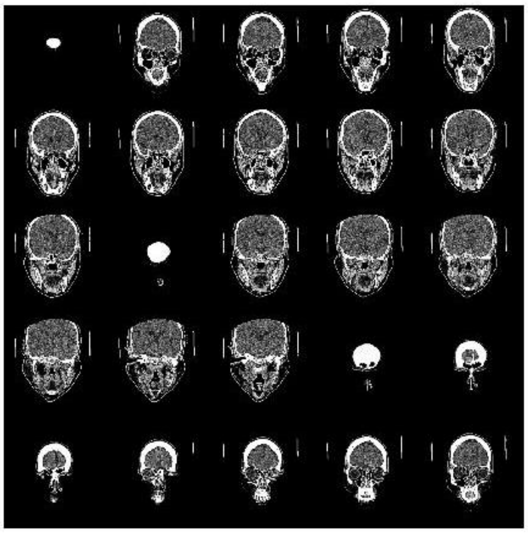

Fig. 7. Montage of the images.

Fig. 8. Clear image from montage.

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A Type-2 Fuzzy in Image Extraction for DICOM Image

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***Abstract*—Eradication of a desired portion of an image is a very important role in image processing and is also called feature extraction. This is mainly concern about reducing the number of possessions required to portray a large set of data and also reduce memory space requirement and power of data processing. Perfectly optimized feature extraction is an essential process for an effective design construction. Though there are many tools are available for extracting a feature, Type-2 Fuzzy Logic plays a vital role in producing good results. In this paper, weighted arithmetic operator is proposed using Yager triangular norms and proved the properties of the triangular norms using proposed operator. Also, the paper relates the properties to feature extraction. Also Brain has been extracted from patient MRI DICOM image using MATLAB based on Type-2 Fuzzy setting.**

***Keywords*—Feature extraction; MRI image; type-2 fuzzy; MATLAB; triangular norms; mathematical properties**

I. Introduction

Image processing is a mapping from image points to a new value by dealing a single point from original image and thereafter it will continue for group of points. Feature extraction is nothing but shape eradication which analyzes the contest of low level data to a known design of a desired shape Also it indicates verdict their location, direction and size. Minor particulars of the image such as dots and lines are called low level features whereas high level features are assembled on the elite of low level features to catch objects and bigger shapes in the image [1-4]. This shows the important steps involved in feature extraction.

In image processing number of points and their range makes a specific effect and the main components are grayscale, resolution, color, dynamic range and storage. Image processing and feature extraction is equitable to the illumination of the matching point in the location and generally its value is determined from the output. Image is a matrix of picture elements in square shape and is described by $M \times M$ n-bit picture elements i.e., pixels. Here M is the number of points through the axes and n manages the number of values for brightness. Adopting n bits provides a range of $2n$ values, ordering from 0 to $2n-1$ which is the brightness level and is generally presented as white and black. Smaller value of n represents the decrement in image contrast [5, 9-10]. This part expresses the color contrast of the image.

Feature extraction is also meant for increasing pixel brightness to find the desired part of the image. In extracting a feature invariance property is to be strictly followed as this process should not vary according to the specified conditions it also implies the reliability and stability of the shape which is extracted. Hence it is necessary to choose the technology for feature extraction where there is a control over the parameters [11-12]. Hence choosing the technology for image extraction is very important to get better results.

The passion histogram exhibits how particular brightness levels are involved in an image and contrast of the image is calculated by the range of brightness levels. This technology plans the number of pixels with a specific level of brightness contrary to the level of brightness. All these contrary things can be dealt by fuzzy logic as it deals with uncertainty well. Generally images have uncertainty in the desired part and clear shape of it. Type-2 fuzzy logic plays an efficient role as it deals with upper and lower membership functions and foot print of uncertainty (the area between upper and lower membership functions) and its length represents uncertainty level of the image or part which is to be extracted [13]. This is the role of fuzzy logic under type-1 and type-2 fuzzy environments.

These technologies are the mathematical systems which provide an outstanding direction for better understanding about the process. Mathematica, Mathcad, MATLAB and Maple are some of the popular tools. Where in MATLAB, by giving instructions at the mandate line, the procedure will be operated and the outputs can be displayed as surfaces, graphs and images. Hence MATLAB under Type-2 fuzzy setting will provide a desired result. In this work, using Type-2 Fuzzy MATLAB, brain is extracted from MRI image. Even though the convenient measures the performance for the credential design analysis are not recognizable and difficult to derive, it helps to design a system which improves its performance measures for training data set and forecast the performance of the system for testing data set [1,14].

Feature extraction of the image to an observable level is an essential key in advance of Content Based Image Retrieval systems. From texture, color and structure of the image, the low level optical features of an image can be separated and can be used while recovery to correlate concern image and other images in the database [12, 18]. This is the stage of acquiring information about image objects which is to be analyzed. On the basis of both quantitative and qualitative reasoning, the features may be determined where the ideas might be hypothetical from the expert which is modified into quantitative values. Color image is one of the universally used features due to its stability, efficiency and computational simplicity [12, 15-16]. Hence color images have potential of clarity of the features which are to be extracted.

A color image based on mathematics is called a digital image and it consists of color information for every picture element. It is also a binary image which has only two possible values for every pixel and is reserved as a single bit 0 or 1. To design a binary image, a threshold intensity value needs to be selected. Pixels with greater intensity than the threshold value are switch to 0 (black) whereas when the intensity value less than the threshold level are switch to 1 (White) and hence the image is converted as a binary image. Gray scale images are having a range of darkness without possible color and used as a fewer information needs to be contributed for each and every pixel. The calculation of mathematical captions could contribute more information about the parameters of morphology but they are not interpreted easily [17-20, 30]. Therefore the procedure for morphology is supposed to be taken care for getting a good feature.

Medical images are generally uncertain in nature. Though there are many methodologies are available to extract the feature from the image, Fuzzy logic (Type-1) helps to extract the feature in an efficient manner but the membership functions are crisp which lies between 0 and 1. MATLAB on Type-2 fuzzy setting provides an optimized result as it handles more uncertainties based on the Footprint of uncertainty, the area between upper and lower membership functions.

From the previous proposed solutions image extraction has not been done using interval type-2 fuzzy logic for feature extraction of the brain from patient MRI. This is the shortcoming of the previous studies and the motivation of the present study. Throughout the paper type-2 fuzzy has been considered as interval type-2 fuzzy environment.

In this paper, the mathematical properties of triangular norms has been proved as it represents the essential qualities of image processing and brain is extracted from patient MRI which is taken from our experimental data using MATLAB and provided the 3D image of an extracted brain with the key components such as DICOM image of a patient MRI, interval type-2 mat lab coding for feature extraction.

II. Review of Literature

The authors of, [1] proposed the idea of linguistic variable and its application in the field of approximate reasoning. The researcher in [2] proposed different classes of fuzzy operators. The researcher in [3] proposed novel aggregations operators under probabilistic fuzzy environment. The researcher in [4] reviewed aggregation connectives under fuzzy environment. The researcher in [5] introduced theory of t-norms and inference methods under fuzzy setting. The researcher in [6] designed fuzzy systems and derived aggregation operators. The researcher in [7] discussed about imprecise reasoning for interval based data with the support of fuzzy and rough sets. The researcher in [8] proposed aggregation operators in detail and applied them video querying.

The researcher in [9] introduced fuzzy image processing using Dubois and Prade aggregation operators. The researcher in [10] proposed a computer based system on hand written records to hold forensic studies. The researcher in [11] proposed means with weight using triangular co norms. The researcher in [12] introduced elementary minimum and maximum operational laws for fuzzy numbers. The researcher in [13], the author aggregated the information collected using aggregation operations. The researcher in [14] applied fuzzy relational equations for Lossy image compression and reconstruction. The researcher in [15] proposed OWA operators with imprecise weights based on type-1 fuzzy. The researcher in [16] analyzed aggregation functions. The researcher in [17] introduced exact computations of protracted logical operations based on uncertain truth values. The researcher in [18] examined and compared different approaches over edge detection.

The researcher in [19] applied morphological operators in image analysis on uninorms. The researcher in [20] proposed novel aggregation operators for the method of active learning. The researcher in [21] explained and derived aggregation operators in detail. The researcher in [22] proposed triangular interval type-2 aggregation operators using Frank triangular norms and applied them in a decision making method. The researcher in [23] proposed fuzzy metric spaces. The researcher in [24], the author applied idea of fuzzy methodology in medical image processing. The researcher in [25] examined collection of information and the related aggregation operators. The researcher in [26] reviewed recent year applications of image processing under type-2 fuzzy.

In [27] the imitation of edge detection of an image using MATLAB with fuzzy logic is done. The researcher in [28] reviewed the role of type-2 fuzzy in the field of Bio medicine. The researcher in [29] proposed edge detection method for a digital image using fuzzy logic. The researcher in [30] proposed a methodology for edge detection for a DICOM image using aggregation operators under type-2 fuzzy. The researcher in [31] implemented a methodology for image fusion using intuitionistic fuzzy logic. The researcher in [32] reviewed about fuzzy controllers to sustain the stability of the system using type-2 fuzzy. The researcher in [33] analyzed surface of the material on curve features of digital images using fuzzy logic. The researcher in [34] proposed 3D version of brain visualization using machine learning. The researcher in [35] proposed block processing and edge detection on DICOM image. The researcher in [36] introduced denoising of the image using LU decomposition method and feature extraction using GLCM.

From this review it is found that there is no contribution of research for image extraction from a DICOM image using interval Type-2 fuzzy logic. This is the motivation of the present work.

III. Basic Definitions

The following basic concepts are given for the better understanding of the paper.

A. *Aggregation Operator [22]*

Let $(M_\alpha)_{\alpha \in [0,1]}$ be a group of aggregation operators (AOs) which is non-decreasing. If A is an AO then

$$M_A : \bigcup_{n \in N} [0,1]^n \to [0,1].$$

Triangular Interval Type-2 Fuzzy Set (TIT2FS) [22]

The membership function (MFs) are developed using triangular fuzzy number in IT2FS called TIT2FS. In IT2FS, upper and lower MFs represented by a triangular fuzzy number $\overline{M} = \left\langle [\underline{l_M}, \overline{l_M}], c_M, [\underline{r_M}, \overline{r_M}] \right\rangle$ called TIT2FS and are defined by

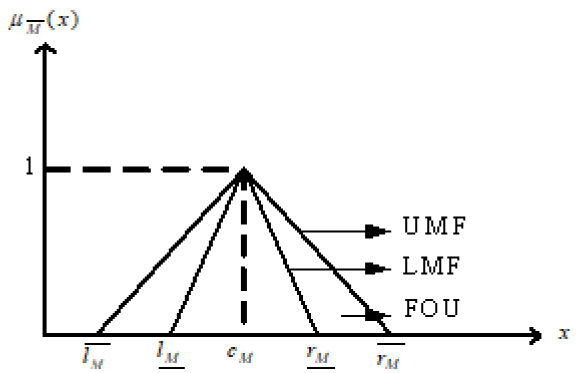

Fig. 1. TIT2FS.

$$LMF_{\overline{M}}(x) = \begin{cases} \dfrac{x - \overline{l_M}}{c_M - \overline{l_M}} & , \quad \overline{l_M} \le x < c_M \\ 1 & , \quad x = c_M \\ \dfrac{x - \underline{r_M}}{c_M - \underline{r_M}} & , \quad c_M \le x < \underline{r_M} \\ 0 & , \quad otherwise \end{cases} \tag{1}$$

$$UMF_{\overline{M}}(x) = \begin{cases} \dfrac{x - \underline{l_M}}{c_M - \underline{l_M}} & , \quad \underline{l_M} \le x < c_M \\ 1 & , \quad x = c_M \\ \dfrac{x - \overline{r_M}}{c_M - \overline{r_M}} & , \quad c_M \le x < \overline{r_M} \\ 0 & , \quad otherwise \end{cases} \tag{2}$$

where $\underline{l_M}, \overline{l_M}, c_M, \underline{r_M}, \overline{r_M}$ are the measuring points on TIT2FS satisfying $0 \le \underline{l_M} \le \overline{l_M} \le c_M \le \underline{r_M} \le \overline{r_M} \le 1$. If we consider X as a set of real numbers, a TIT2FS in X is called TIT2FN. The FOU is the area between lower and upper membership functions in figure 1. If $\underline{l_M} = \overline{l_M}, \underline{r_M} = \overline{r_M}$, then $UMF_{\overline{M}}(x)$ = $LMF_{\overline{M}}(x)$ for all the values of x in X, then the TIT2FS will become Type-1 case. Here FOU is the footprint of Uncertainty.

B. *Ranking Formula for TIT2FN [22]*

Let $\overline{M} = \langle [A, B], C, [D, E] \rangle$

where $A = \underline{l_M}, B = \overline{l_M}, C = c_M, D = \underline{r_M}, E = \overline{r_M}$ be the TIT2FN. The ranking value is defined by

$$Rank(\overline{M}) = \left(\frac{A+E}{2} + 1 \right) \times \frac{A+B+D+E+4C}{8} \tag{3}$$

C. *Yager Triangular Norms [5]*

$\underset{Y}{\otimes}$ is Yager product (T Norm) and $\underset{Y}{\oplus}$ is a Yager sum (T conorm) and are defined as follows.

$$r \underset{Y}{\otimes} s = \max\left(1 - [(1-r)^\eta + (1-s)^\eta]^{\frac{1}{\eta}}, 0 \right), \eta > 0, \; for\, all \; r, s \in [0,1]^2$$

(4)

$$r \underset{Y}{\oplus} s = \min\left(\left(r^\eta + s^\eta \right)^{\frac{1}{\eta}}, 1 \right), \eta > 0, \; for\, all \; r, s \in [0,1]^2 \tag{5}$$

D. *Triangular Interval Type-2 Fuzzy Yager Weighted Arithmetic (TIT2FYWA) Operator [22]*

Consider a set of TIT2FNs and the operator $TIT2FYWA_\varepsilon : \Omega^n \to \Omega$ is defined by $TIT2FYWA_\varepsilon \left\langle \overline{M_1}, \overline{M_2}, ..., \overline{M_n} \right\rangle = \varepsilon_1 \underset{Y}{\bullet} \overline{M_1} \underset{Y}{\oplus} \varepsilon_2 \underset{Y}{\bullet} \overline{M_2} \underset{Y}{\oplus} ... \underset{Y}{\oplus} \varepsilon_n \underset{Y}{\bullet} \overline{M_n}$

and its weight vector is $\varepsilon = (\varepsilon_1, \varepsilon_2, ..., \varepsilon_n)^T$ and the sum of the weight vectors is equal to 1, when $\varepsilon = \left(\frac{1}{n}, \frac{1}{n}, ..., \frac{1}{n} \right)^T$, triangular interval type-2 fuzzy Yager weighted arithmetic operator will become triangular interval type-2 fuzzy Yager arithmetic averaging operator of dimension n and is defined by.

$$TIT2FYAA_\varepsilon(\overline{M_1}, \overline{M_2}, ..., \overline{M_n}) = \frac{1}{n} \underset{Y}{\bullet} \left(\overline{M_1} \underset{Y}{\oplus} \overline{M_2}, ..., \underset{Y}{\oplus} \overline{M_n} \right) \tag{6}$$

E. *Triangular Interval Type-2 Fuzzy Yager Weighted Geometric (TIT2FYWG) Operator [22]*

Let $\overline{M} = \left\langle [\underline{l_{M_p}}, \overline{l_{M_p}}], c_{M_p}, [\underline{r_{M_p}}, \overline{r_{M_p}}] \right\rangle$, $p = 1, 2, ..., n$ be a set of TIT2FNs. Triangular Interval Type-2 fuzzy Yager Weighted Geometric Mean Operator (TIT2FYWA), $TIT2FYWG: \varepsilon^n \to \varepsilon$ is $TIT2FYWG_\varepsilon \left(\overline{M_1}, \overline{M_2}, ..., \overline{M_n} \right) = \overline{M_1}^{\bullet_Y \varepsilon_1} \underset{Y}{\otimes} \overline{M_2}^{\bullet_Y \varepsilon_2} \underset{Y}{\otimes} ... \underset{Y}{\otimes} \overline{M_n}^{\bullet_Y \varepsilon_n}$ .Here also sum of all weight vectors is equal to 1, when $\varepsilon = \left(\frac{1}{n}, \frac{1}{n}, ..., \frac{1}{n} \right)^T$, triangular interval type-2 fuzzy Yager weighted arithmetic operator will become triangular interval type-2 fuzzy Yager geometric averaging operator of dimension n and is defined by $TIT2FYGA_{\varepsilon}(\overline{M_1}, \overline{M_2}, \ldots, \overline{M_2}) =$

$$\frac{1}{n}\left(\overline{M_1} \underset{Y}{\otimes} \overline{M_2} \underset{Y}{\otimes}, \ldots, \underset{Y}{\otimes} \overline{M_n}\right)^{\dot{Y}\, 1/n} \tag{7}$$

F. Feature Extraction [18]

The task of feature extraction converts affluent content of images into different content features and a process of producing features to be used in the classification and selection effort. It is the second accent of classification of the image and helps to reduce the number of features supplied to the categorization task. After the selection of desired feature, that will be used in the task and failing are discarded. Even though a reduction is desirable in dimensionality, increment in inaccuracy has to be along with selective power of classifiers. This technique is useful when the size of the image is large and extracted feature is helpful for quick completion of the task like matching of the images and recovery.

G. Class of Features [19]

An operation of more than one dimension which designates few perceptible properties of the image/ body is called a feature and it measures some important components of the object. The following are the different features.

1) General features: Texture, color and shape are the purpose free features and called general features. According to the conceptual level, further they are divided into the following.

- Pixel-level features, which are estimated at every pixel such as location and color.
- Local Features, features estimated over the decision of class of the image line on image bisection and detection of edge.
- Global Features, features steered over the complete image impartially.

2) Domain explicit features: These features are application dependent such as finger prints, human faces and visionary features which are generally a fusion of low level features for a particular domain.

H. Color Feature [19]

Color feature is the visual feature and is extensively used in image recovery. Stability, efficiency, simplicity in implementation and computationally as well and low storage capacity are the advantages of color features.

I. Morphological Operators [19]

Morphological operators convert the original image into another one by interacting with the other images of the structural elements like absolute size and shape and it provides a systematic approach for characterization in many applications like object segmentation, edge detection and suppression of noise. The goal of Fuzzy mathematical morphology is to develop the binary morphological operators to gray level images.

J. Mathematical Morphology (MM) [19]

MM is a group of operations. Erosion, dilation, opening and closing are the essential morphological operations on the image pixels. To segregate bright and dark structures than the neighboring features, opening and closing transforms can be used. General methodology has been introduced for fuzzy dilation, erosion, opening and closing.

Dilation is the mechanism of increasing the maximum value in the window thereafter the brightness of the image will be increased and the image objects will be extended as well by modifying the value from 0 to 1. Erosion is the reverse process of dilation, here the image becomes darker than the original and deals the image by converting the pixel value from 1 to 0.

For getting stabilized gray scale images, morphological openings and closings are playing a vital role in image processing. These mathematical morphological operations are similar to the design of set theory and its operations as well. At this junction, fuzzy set theory plays an efficient role in mathematical morphology as the images are uncertain in nature.

K. Morphological Caption [19]

Area, volume, perimeter, gray levels, density, maximum, minimum average, standard deviation of major and minor axes, location, unusualness, point of restriction and centroid are the morphological descriptors or captions in image processing.

L. Projections [19]

Generally in image processing, there are two types projections are available on binary images such as horizontal and vertical projections. This process scan from left side of each line and records the pixel changes from 0 to 1and again to 0 where the number of changes is independent of the pixels. Stability can be expected even in noisy condition in this process. After obtaining associated components, progress of the pixel values from 0 to 1 or vice versa, need to be checked horizontally whereas background area have less progress or transitions.

If the allotted amount of changes of each row lies between two thresholds such as low and high then that row will be considered as a desired area. Next vertical transitions will be considered to find the exact location of the feature which is to be extracted by inquiring the length and height of the feature and their ratio and adequate number of pixels in that area [13].

IV. Proposed Operational Laws

Let $\overline{F}, \overline{F}_1, \overline{F}_2$ be three TIT2FNs and $\chi > 0$, then we define their operational laws as follows.

A. Addition Consider

$$U_1 = \underset{t=1}{\overset{2}{aot}}\left(\underline{l_{F_t}}\right), V_1 = \underset{t=1}{\overset{2}{aot}}\left(\overline{l_{F_t}}\right), W_1 = \underset{t=1}{\overset{2}{aot}}\left(c_{M_t}\right), X_1 = \underset{t=1}{\overset{2}{aot}}\left(\underline{r_{F_t}}\right), Y_1 = \underset{t=1}{\overset{2}{aot}}\left(\overline{r_{F_t}}\right).$$

where aot= sum of the terms

$$\overline{F_1} \oplus \overline{F_2} = \left\{ \left[\min\left(a_1^{\frac{1}{\chi}}, 1 \right), \min\left(b_1^{\frac{1}{\chi}}, 1 \right) \right], \min\left(c_1^{\frac{1}{\chi}}, 1 \right), \right.$$

$$\left. \left[\min\left(d_1^{\frac{1}{\chi}}, 1 \right), \min\left(e_1^{\frac{1}{\chi}}, 1 \right) \right] \right\} \quad (8)$$

B. Multiplication Consider

$U_2 = \underset{t=1}{\overset{2}{aot}} \left(\underline{l_{F_t}} \right)^{\chi}, V_2 = \underset{t=1}{\overset{2}{aot}} \left(\overline{l_{F_t}} \right)^{\chi}, \; W_2 = \underset{t=1}{\overset{2}{aot}} \left(c_{F_t} \right)^{\chi},$

$X_2 = \underset{t=1}{\overset{2}{aot}} \left(\underline{r_{F_t}} \right)^{\chi}, Y_2 = \underset{t=1}{\overset{2}{aot}} \left(\overline{r_{F_t}} \right)^{\chi}.$

$$\overline{F_1} \oplus \overline{F_2} = \left\{ \left[\min\left(U_2^{\frac{1}{\chi}}, 1 \right), \min\left(V_2^{\frac{1}{\chi}}, 1 \right) \right], \min\left(W_2^{\frac{1}{\chi}}, 1 \right), \right.$$

$$\left. \left[\min\left(X_2^{\frac{1}{\chi}}, 1 \right), \; \min\left(Y_2^{\frac{1}{\chi}}, 1 \right) \right] \right\} \quad (9)$$

C. Multiplication by an Ordinary Number Consider

$U = \underline{l_F}, V = \overline{l_F}, W = c_F, X = \underline{r_F}, Y = \overline{r_F}$

$$g \bullet \overline{F} = \left\{ \left[\min\left\langle U^{\frac{g}{\chi}}, 1 \right\rangle, \min\left\langle V^{\frac{g}{\chi}}, 1 \right\rangle \right], \min\left\langle W^{\frac{g}{\chi}}, 1 \right\rangle, \right.$$

$$\left. \left[\min\left\langle X^{\frac{g}{\chi}}, 1 \right\rangle, \min\left\langle Y^{\frac{g}{\chi}}, 1 \right\rangle \right] \right\}. \quad (10)$$

D. Power Consider

$U_3 = \underline{l_F}, V_3 = \overline{l_F}, W_3 = c_F, \; X_3 = \underline{r_F}, Y_3 = \overline{r_F}$

$$\overline{F}^{g} = \left\{ \left[\max\left(1 - \left[a_3^{\chi} \right]^{g/\chi}, 0 \right), \max\left(1 - \left[b_3^{\chi} \right]^{g/\chi}, 0 \right) \right], \right.$$

$$\max\left(1 - \left[c_3^{\chi} \right]^{g/\chi}, 0 \right),$$

$$\left. \left[\max\left(1 - \left[d_3^{\chi} \right]^{g/\chi}, 0 \right), \max\left(1 - \left[e_3^{\chi} \right]^{g/\chi}, 0 \right) \right] \right\} \quad (11)$$

V. Proposed Theorems

Here the mathematical properties of aggregation properties are proved for TIT2FN using TIT2WA operator which represents the essential qualities of the image processing such as idempotent ability, stability and image contrast.

Let TIT2FNs $\overline{F} = \left([\underline{l_{F_t}}, \overline{l_{F_t}}], c_{F_t}, [\underline{r_{F_t}}, \overline{r_{F_t}}] \right), t = 1, 2, ..., n$, where $0 \le \underline{l_F} \le \overline{l_F} \le c_F \le \underline{r_F} \le \overline{r_F} \le 1$ be a collection of TIT2FNs.

A. Theorem

The aggregation value of these fuzzy numbers using TIT2FYWG operator is again a TIT2FN and

$$TIT2WA_{\rho} \left\langle \overline{F_1}, \overline{F_2}, ..., \overline{F_n} \right\rangle = \left\{ \left[\min\left(\left[U_n^{\chi} \right]^{\rho_t/\chi}, 1 \right), \min\left(\left[V_n^{\eta} \right]^{\rho_t/\chi}, 1 \right) \right], \right.$$

$$\left. \min\left(\left[W_n^{\eta} \right]^{\rho_t/\chi}, 1 \right), \left[\min\left(\left[X_n^{\eta} \right]^{\rho_t/\chi}, 1 \right), \; \min\left(\left[Y_n^{\eta} \right]^{\rho_t/\chi}, 1 \right) \right] \right\} \quad (12)$$

Where the weight vector is $\rho = \left(\rho_1, \rho_2, ..., \rho_n \right)^T, \rho_n \ge 0$, the sum of the weight vectors is equal to 1.

Proof:

The aggregation value of these fuzzy numbers using TIT2FYWG operator is again a TIT2FN and

$$TIT2WA_{\rho} \left\langle \overline{F_1}, \overline{F_2}, ..., \overline{F_n} \right\rangle = \left\{ \left[\min\left(\left[U_n^{\chi} \right]^{\rho_t/\chi}, 1 \right), \min\left(\left[V_n^{\eta} \right]^{\rho_t/\chi}, 1 \right) \right], \right.$$

$$\left. \min\left(\left[W_n^{\eta} \right]^{\rho_t/\chi}, 1 \right), \left[\min\left(\left[X_n^{\eta} \right]^{\rho_t/\chi}, 1 \right), \; \min\left(\left[Y_n^{\eta} \right]^{\rho_t/\chi}, 1 \right) \right] \right\},$$

Where the weight vector is $\rho = \left(\rho_1, \rho_2, ..., \rho_n \right)^T, \rho_n \ge 0$, the sum of the weight vectors is equal to 1.

Here use mathematical induction method.

Case (i): For $n = 2$

Consider,

$U_1 = \left(\underline{l_{F_1}} \right)^{\chi}, V_1 = \left(\overline{l_{F_1}} \right)^{\chi}, W_1 = \left(c_{F_1} \right)^{\chi} \quad X_1 = \left(\underline{r_{F_1}} \right)^{\chi}, Y_1 = \left(\overline{r_{F_1}} \right)^{\chi}$

Using multiplication operation

$$g \bullet \overline{F_1} = \left\{ \left[\min\left\langle U_1^{\frac{g}{\chi}}, 1 \right\rangle, \min\left\langle V_1^{\frac{g}{\chi}}, 1 \right\rangle \right], \min\left\langle W_1^{\frac{g}{\chi}}, 1 \right\rangle, . \right.$$

$$\left. \left[\min\left\langle X_1^{\frac{g}{\chi}}, 1 \right\rangle, \min\left\langle Y_1^{\frac{g}{\chi}}, 1 \right\rangle \right] \right\}$$

Consider,

$U_2 = \left(\underline{l_{F_2}} \right)^{\chi}, V_2 = \left(\overline{l_{F_2}} \right)^{\chi}, W_2 = \left(c_{F_2} \right)^{\chi}, X_2 = \left(\underline{r_{F_2}} \right)^{\chi}, Y_2 = \left(\overline{r_{F_2}} \right)^{\chi}$

$$g\bullet\overline{F_2}=\left\{\left[\min\left\langle U_2^{\frac{g}{\chi}},1\right\rangle,\min\left\langle V_2^{\frac{g}{\chi}},1\right\rangle\right],\min\left\langle W_2^{\frac{g}{\chi}},1\right\rangle,\right.$$

$$\left.\left[\min\left\langle X_2^{\frac{g}{\chi}},1\right\rangle,\min\left\langle Y_2^{\frac{g}{\chi}},1\right\rangle\right]\right\}.$$

$$r\underset{Y}{\oplus}s=\min\left(\left(r^{\eta}+s^{\eta}\right)^{\frac{1}{\eta}},1\right),\eta>0,\ for\,all\ r,s\in[0,1]^2$$

$$TIT2FWA_{\rho}\left(\overline{F_1},\overline{F_2}\right)=\rho_1\bullet\overline{F_1}\oplus\rho_1\bullet\overline{F_2}$$

$$=\left\{\left[\min\left(\left[\mathop{aot}_{t=1}^{2}\left(\min\left(\left(U_2\right)^{\frac{\rho_t}{\chi}},1\right)\right)\right],1\right),\ \min\left(\left[\mathop{aot}_{t=1}^{2}\left(\min\left(\left(V_2\right)^{\frac{\rho_t}{\chi}},1\right)\right)\right],1\right)\right],\right.$$

$$\min\left(\left[\mathop{aot}_{p=1}^{2}\left(\min\left(\left(W_2\right)^{\frac{\rho_t}{\chi}},1\right)\right)\right],1\right),$$

$$\left.\left[\min\left(\left[\mathop{aot}_{t=1}^{2}\left(\min\left(\left(X_2\right)^{\frac{\rho_t}{\chi}},1\right)\right)\right],1\right),\min\left(\left[\mathop{aot}_{1=1}^{2}\left(\min\left(\left(Y_2\right)^{\frac{\rho_t}{\chi}},1\right)\right)\right],1\right)\right]\right\}.$$

Consider,

$$U_4=\left(1-\underline{l_{M_p}}\right)^{\chi},V_4=\left(1-\overline{l_{M_t}}\right)^{\chi},W_4=\left(1-c_{M_t}\right)^{\chi},$$

$$X_4=\left(1-\underline{r_{M_t}}\right)^{\chi},Y_4=\left(1-\overline{r_{M_t}}\right)^{\chi}.$$

$$=\left\{\left[\min\left(\left(\mathop{aot}_{t=1}^{2}\left(\min\left(U_4^{\frac{\rho_t}{\chi}},1\right)\right)\right),1\right),\min\left(\left(\mathop{aot}_{t=1}^{2}\left(\min\left(1-V_4^{\frac{\rho_t}{\chi}},1\right)\right)\right),1\right)\right],\right.$$

$$\min\left(\left(\mathop{aot}_{t=1}^{2}\left(\min\left(W_4^{\frac{\rho_t}{\chi}},1\right)\right)\right),1\right),$$

$$\left.\left[\min\left(\left(\mathop{aot}_{t=1}^{2}\left(\min\left(X_4^{\frac{\rho_t}{\chi}},1\right)\right)\right),1\right),\min\left(\left(\mathop{aot}_{p=1}^{2}\left(\min\left(Y_4^{\frac{\rho_t}{\chi}},1\right)\right)\right),1\right)\right]\right\}.$$

$$=\left\{\left[\min\left[U_2^{\frac{\rho_t}{\chi}},1\right],\min\left[V_2^{\frac{\rho_t}{\chi}},1\right]\right],\ \min\left[W_2^{\frac{\rho_t}{\chi}},1\right],\right.$$

$$\left.\left[\min\left[X_2^{\frac{\rho_t}{\chi}},1\right],\min\left[Y_2^{\frac{\rho_t}{\chi}},1\right]\right]\right\}$$

For $n=k,$

$$U_k=\mathop{aot}_{t=1}^{k}\left(1-\underline{l_{F_t}}\right)^{\chi},V_k=\mathop{aot}_{t=1}^{k}\left(1-\overline{l_{F_t}}\right)^{\chi},W_k=\mathop{aot}_{t=1}^{k}\left(1-c_{F_t}\right)^{\chi},$$

$$X_k=\mathop{aot}_{t=1}^{k}\left(1-\underline{r_{F_t}}\right)^{\chi},Y_k=\mathop{aot}_{t=1}^{k}\left(1-\overline{r_{F_t}}\right)^{\chi}$$

$$TIT2WA_{\rho}\left\langle\overline{F_1},\overline{F_2},...,\overline{F_k}\right\rangle$$

$$=\left\{\left[\min\left(\left[\mathop{aot}_{t=1}^{k}\left(\min\left(\left[U_k^{\frac{\rho_t}{\chi}},1\right]\right)\right)\right],1\right),\min\left(\left[\mathop{aot}_{t=1}^{k}\left(\min\left(V_k^{\frac{\rho_t}{\chi}},1\right)\right)\right],1\right)\right],\right.$$

$$\min\left(\left[\mathop{aot}_{t=1}^{k}\left(\min\left(W_k^{\frac{\rho_t}{\chi}},1\right)\right)\right],1\right),$$

$$\left.\left[\min\left(\left[\mathop{aot}_{t=1}^{k}\left(\min\left(X_k^{\frac{\rho_t}{\chi}},1\right)\right)\right],1\right),\ \min\left(\left[\mathop{aot}_{t=1}^{k}\left(\min\left(Y_k^{\frac{\rho_t}{\chi}},1\right)\right)\right],1\right)\right]\right\}.$$

$$=\left\{\left[\min\left[U_k^{\frac{\rho_t}{\chi}},1\right],\min\left[V_k^{\frac{\rho_t}{\chi}},1\right]\right],\ \min\left[W_k^{\frac{\rho_t}{\chi}},1\right],\right.$$

$$\left.\left[\max\left[X_k^{\frac{\rho_t}{\chi}},1\right],\max\left[Y_k^{\frac{\rho_t}{\chi}},1\right]\right]\right\}.$$

For $n=k+1,$

$$TIT2WA_{\rho}\left\langle\overline{F_1},\overline{F_2},...,\overline{F_k}\right\rangle\oplus\left\langle\overline{F_{k+1}}\right\rangle$$

$$=\left\{\left[\min\left(\left[\mathop{aot}_{t=1}^{k}\left(\min\left(U_k^{\frac{\rho_t}{\chi}},1\right)\right)\right],1\right),\min\left(\left[\mathop{aot}_{t=1}^{k}\left(\min\left(V_k^{\frac{\rho_t}{\chi}},1\right)\right)\right],1\right)\right],\right.$$

$$\min\left(\left[\mathop{aot}_{t=1}^{k}\left(\min\left(W_k^{\frac{\rho_t}{\chi}},1\right)\right)\right],1\right),$$

$$\left.\left[\min\left(\left[\mathop{aot}_{t=1}^{k}\left(\min\left(X_k^{\frac{\rho_t}{\chi}},1\right)\right)\right],1\right),\min\left(\left[\mathop{sum}_{p=1}^{k}\left(\min\left(Y_k^{\frac{\rho_t}{\chi}},1\right)\right)\right],1\right)\right]\right\}.$$

$$\oplus\left\{\left[\min\left(U_{k+1}^{\frac{\rho_{k+1}}{\chi}},1\right),\min\left(V_{k+1}^{\frac{\rho_{k+1}}{\chi}},1\right)\right],\min\left(W_{k+1}^{\frac{\rho_{k+1}}{\chi}},1\right),\right.$$

$$\left[\max\left(X_{k+1}^{\frac{\rho_{k+1}}{\chi}},1\right),\max\left(Y_{k+1}^{\frac{\rho_{k+1}}{\chi}},1\right)\right]\Bigg\}.$$

$$=\left\{\left[\min\left(\left[\operatorname*{aot}_{t=1}^{k+1}\left(\min\left(U_k^{\frac{\rho_t}{\chi}},1\right)\right)\right],1\right),\min\left(\left[\operatorname*{aot}_{t=1}^{k+1}\left(\min\left(V_k^{\frac{\rho_t}{\chi}},1\right)\right)\right],1\right)\right],\right.$$

$$\min\left(\left[\operatorname*{aot}_{t=1}^{k+1}\left(\min\left(W_k^{\frac{\rho_t}{\chi}},1\right)\right)\right],1\right),$$

$$\left.\left[\min\left(\left[\operatorname*{aot}_{t=1}^{k+1}\left(\min\left(X_k^{\frac{\rho_t}{\chi}},1\right)\right)\right],1\right),\min\left(\left[\operatorname*{aot}_{t=1}^{k+1}\left(\min\left(Y_k^{\frac{\rho_t}{\chi}},1\right)\right)\right],1\right)\right]\right\}.$$

$$=\left\{\left[\min\left[U_{k+1}^{\frac{\rho_t}{\chi}},1\right],\min\left[V_{k+1}^{\frac{\rho_t}{\chi}},1\right]\right],\ \min\left[W_{k+1}^{\frac{\rho_t}{\chi}},1\right],\right.$$

$$\left.\left[\min\left[X_{k+1}^{\frac{\rho_t}{\chi}},1\right],\min\left[Y_{k+1}^{\frac{\rho_t}{\chi}},1\right]\right]\right\}.$$

Hence the result holds for all the values of n.

B. Theorem (Idempotency)

If $\overline{F_t}=\overline{F}$ for all the values of t then

$$TIT2WA_\rho\left\langle\overline{F_1},\overline{F_2},...,\overline{F_n}\right\rangle=\overline{F}. \tag{13}$$

Proof:

By theorem A,

$$TIT2WA_\rho\left\langle\overline{F_1},\overline{F_2},...,\overline{F_n}\right\rangle=\left\{\left[\min\left(U_n^{\frac{\rho_t}{\chi}},1\right),\min\left(V_n^{\frac{\rho_t}{\chi}},1\right)\right],\ \min\left(W_n^{\frac{\rho_t}{\chi}},1\right),\right.$$

$$\left.\left[\min\left(X_n^{\frac{\rho_t}{\chi}},1\right),\min\left(Y_n^{\frac{\rho_t}{\chi}},1\right)\right]\right\}$$

$$TIT2WA_\rho\left\langle\overline{F_1},\overline{F_2},...,\overline{F_n}\right\rangle$$

$$=\left\{\left[\min\left(U_4^{\operatorname*{aot}_{t=1}^{k+1}\left(\frac{\rho_t}{\chi}\right)},1\right),\min\left(V_4^{\operatorname*{aot}_{t=1}^{k+1}\left(\frac{\rho_t}{\chi}\right)},1\right)\right],\ \min\left(W_4^{\operatorname*{aot}_{t=1}^{k+1}\left(\frac{\rho_t}{\chi}\right)},1\right),\right.$$

$$\left.\left[\min\left(X_4^{\operatorname*{aot}_{t=1}^{k+1}\left(\frac{\rho_t}{\chi}\right)},1\right),\min\left(Y_4^{\operatorname*{aot}_{t=1}^{k+1}\left(\frac{\rho_t}{\chi}\right)},1\right)\right]\right\}$$

$$=\left\{\left[\min\left(U_5^{\frac{1}{\chi}},1\right),\min\left(V_5^{\frac{1}{\chi}},1\right)\right],\min\left(W_5^{\frac{1}{\chi}},1\right),\right.$$

$$\left.\left[\min\left(X_5^{\frac{1}{\chi}},1\right),\min\left(Y_5^{\frac{1}{\chi}},1\right)\right]\right\}$$

$$=\left\{\left[U_3,V_3,W_3,X_3,Y_3\right]\right\}=\left\{\left[U,V\right],W,\left[X,Y\right]\right\}=\overline{F}$$

C. Theorem

If $f>0$ for all the values of p then

$$TIT2WA_\rho\left(f\bullet\overline{F_1},f\bullet\overline{F_2},...,f\bullet\overline{F_n}\right)=f\bullet TIT2WA_\rho\left(\overline{F_1},\overline{F_2},...,\overline{F_n}\right) \tag{14}$$

Proof: Using,

$$f\bullet\overline{F}=\left\{\left[\min\left(U_5^{\frac{f}{\chi}},1\right),\min\left(V_5^{\frac{f}{\chi}},1\right)\right],\min\left(W_5^{\frac{f}{\chi}},1\right),\right.$$

$$\left.\left[\min\left(X_5^{\frac{f}{\chi}},1\right),\min\left(Y_5^{\frac{f}{\chi}},1\right)\right]\right\}$$

$$TIT2WA_\rho\left(f\bullet\overline{F_1},f\bullet\overline{F_2},...,f\bullet\overline{F_n}\right)$$

$$=\left\{\left[\min\left(\left[\operatorname*{aot}_{t=1}^{n}\left(\min\left((U_2)^{\frac{\rho_t}{\chi}},1\right)\right)\right],1\right),\ \min\left(\left[\operatorname*{aot}_{t=1}^{n}\left(\min\left((V_2)^{\frac{\rho_t}{\chi}},1\right)\right)\right],1\right)\right],\right.$$

$$\min\left(\left[\operatorname*{aot}_{t=1}^{n}\left(\min\left((W_2)^{\frac{\rho_t}{\chi}},1\right)\right)\right],1\right),$$

$$\left.\left[\min\left(\left[\operatorname*{aot}_{t=1}^{n}\left(\min\left((X_2)^{\frac{\rho_t}{\chi}},1\right)\right)\right],1\right),\ \min\left(\left[\operatorname*{aot}_{t=1}^{n}\left(\min\left((Y_2)^{\frac{\rho_t}{\chi}},1\right)\right)\right],1\right),\right]\right\}.$$

$$=\left\{\left[\min\left(U_n^{\frac{f\rho_t}{\chi}},1\right),\ \min\left(V_n^{\frac{f\rho_t}{\chi}},1\right)\right],\ \min\left(W_n^{\frac{f\rho_t}{\chi}},1\right),\right.$$

$$\left.\left[\min\left(X_n^{\frac{f\rho_t}{\chi}},1\right),\min\left(Y_n^{\frac{f\rho_t}{\chi}},1\right)\right]\right\} \tag{15}$$

Now consider,

$$f \bullet TIT2WA_{\rho}\left(\overline{F_1},\overline{F_2},\ldots,\overline{F_n}\right)$$

$$= f \bullet \Bigg\{\Bigg[\min\left(U_n^{\frac{\rho_t}{\chi}},1\right),\min\left(V_n^{\frac{\rho_t}{\chi}},1\right)\Bigg],\min\left(W_n^{\frac{\rho_t}{\chi}},1\right),$$

$$\Bigg[\min\left(X_n^{\frac{\rho_t}{\chi}},1\right),\min\left(Y_n^{\frac{\rho_t}{\chi}},1\right)\Bigg]\Bigg\}$$

$$=\Bigg\{\Bigg[\min\left(\left(U_n^{\frac{f}{\chi}}\right)^{\rho_t},1\right),\min\left(\left(V_n^{\frac{f}{\chi}}\right)^{\rho_t},1\right)\Bigg],\min\left(\left(W_n^{\frac{f}{\chi}}\right)^{\rho_t},1\right),$$

$$\Bigg[\min\left(\left(X_n^{\frac{f}{\chi}}\right)^{\rho_t},1\right),\min\left(\left(Y_n^{\frac{f}{\chi}}\right)^{\rho_t},1\right)\Bigg]\Bigg\}$$

$$=\Bigg\{\Bigg[\min\left(\left(U_n^{\frac{f\rho_t}{\chi}}\right),1\right),\min\left(\left(V_n^{\frac{f\rho_t}{\chi}}\right),1\right)\Bigg],\min\left(\left(W_n^{\frac{f\rho_t}{\chi}}\right),1\right),$$

$$\Bigg[\min\left(\left(X_n^{\frac{f\rho_t}{\chi}}\right),1\right),\min\left(\left(Y_n^{\frac{f\rho_t}{\chi}}\right),1\right)\Bigg]\Bigg\} \qquad (16)$$

Since (15) = (16), hence the result.

D. *Theorem (Stability)*

If $t>0$ $\overline{F_{n+1}} = \left(\left[\underline{l_{F_{n+1}}},\overline{l_{F_{n+1}}}\right],c_{F_{n+1}},\left[\underline{r_{F_{n+1}}},\overline{r_{F_{n+1}}}\right]\right)$ then

$$TIT2WA_{\rho}\left(f\bullet\overline{F_1}\oplus\overline{F_{n+1}},\ f\bullet\overline{F_2}\oplus\overline{F_{n+1}},\ldots,\ f\bullet\overline{F_n}\oplus\overline{F_{n+1}}\right) \qquad (17)$$

Proof:

$$TIT2WA_{\rho}\left(f\bullet\overline{F_1}\oplus\overline{F_{n+1}},\ f\bullet\overline{F_2}\oplus\overline{F_{n+1}},\ldots,\ f\bullet\overline{F_n}\oplus\overline{F_{n+1}}\right)$$

$$=\Bigg\{\Bigg[\min\left(U_n^{\frac{f\rho_t}{\chi}},1\right)\oplus\overline{F_{n+1}},\ \min\left(V_n^{\frac{f\rho_t}{\chi}},1\right)\oplus\overline{F_{n+1}}\Bigg],$$

$$\min\left(W_n^{\frac{f\rho_t}{\chi}},1\right)\oplus\overline{F_{n+1}},$$

$$\Bigg[\min\left(X_n^{\frac{f\rho_t}{\chi}},1\right)\oplus\overline{F_{n+1}},\min\left(Y_n^{\frac{f\rho_t}{\chi}},1\right)\oplus\overline{F_{n+1}}\Bigg]\Bigg\}$$

$$=\Bigg\{\Bigg[\min\left(\left[\mathop{aot}_{t=1}^{n}\left(\min\left(\left[\mathop{aot}_{r=\{t,n+1\}}\left(\underline{l_{F_t}}\right)^{\chi}\right]^{\frac{f}{\chi}},1\right)^{\rho_t}\right)\right],1\right),$$

$$\min\left(\left[\mathop{aot}_{t=1}^{n}\left(\min\left(\left[\mathop{aot}_{r=\{t,n+1\}}\left(\overline{l_{F_t}}\right)^{\chi}\right]^{\frac{f}{\chi}},1\right)^{\rho_t}\right)\right],1\right)\Bigg]$$

$$\min\left(\left[\mathop{aot}_{t=1}^{n}\left(\min\left(\left[\mathop{aot}_{r=\{t,n+1\}}\left(c_{F_t}\right)^{\chi}\right]^{\frac{f}{\chi}},1\right)^{\rho_t}\right)\right],1\right),$$

$$\Bigg[\min\left(\left[\mathop{aot}_{t=1}^{n}\left(\min\left(\left[\mathop{aot}_{r=\{t,n+1\}}\left(\underline{r_{F_t}}\right)^{\chi}\right]^{\frac{f}{\chi}},1\right)^{\rho_t}\right)\right],1\right),$$

$$\min\left(\left[\mathop{aot}_{t=1}^{n}\left(\min\left(\left[\mathop{aot}_{r=\{t,n+1\}}\left(\overline{r_{F_t}}\right)^{\chi}\right]^{\frac{f}{\chi}},1\right)^{\rho_t}\right)\right],1\right)\Bigg]\Bigg\}$$

$$=\Bigg\{\Bigg[\min\left(\left(U_n^{\frac{\rho_t}{\chi}}\right)+\left(\left[\left(\underline{l_{F+1}}\right)^{\chi}\right]^{\frac{1}{\chi}}\right)^{\mathop{aot}_{t=1}^{n}\rho_t},1\right),$$

$$\min\left(\left(V_n^{\frac{\rho_t}{\chi}}\right)+\left(\left[\left(\overline{l_{F+1}}\right)^{\chi}\right]^{\frac{1}{\chi}}\right)^{\mathop{aot}_{t=1}^{n}\rho_t},1\right)\Bigg],$$

$$\min\left(\left(W_n^{\frac{\varepsilon_p}{\eta}}\right)+\left(\left[\left(c_{F+1}\right)^{\chi}\right]^{\frac{1}{\chi}}\right)^{\mathop{aot}_{t=1}^{n}\rho_t},1\right),$$

$$\Bigg[\min\left(\left(X_n^{\frac{\rho_t}{\chi}}\right)+\left(\left[\left(\underline{r_{F+1}}\right)^{\chi}\right]^{\frac{1}{\chi}}\right)^{\mathop{aot}_{t=1}^{n}\rho_t},1\right),$$

$$\min\left(\left(Y_n^{\frac{\rho_t}{\chi}}\right)+\left(\left[\left(\overline{r_{F+1}}\right)^{\chi}\right]^{\frac{1}{\chi}}\right)^{\underset{t=1}{\overset{n}{aot}}\rho_t},1\right)\Bigg]\Bigg]\Bigg\} \tag{18}$$

$$f \bullet TIT2WA_{\rho}\left(\overline{F_1},\overline{F_2},...,\overline{F_n}\right)\oplus\overline{F_{n+1}}$$

$$=\left\{\left[\left[\min\left(U_n^{\frac{f\rho_t}{\chi}},1\right),\min\left(V_n^{\frac{f\rho_t}{\chi}},1\right)\right],\min\left(W_n^{\frac{f\rho_t}{\chi}},1\right),\right.\right.$$

$$\left.\left.\left[\min\left(X_n^{\frac{f\rho_t}{\chi}},1\right),\ \min\left(Y_n^{\frac{f\rho_t}{\chi}},1\right)\right]\right]\right\}.$$

$$\oplus\left\langle\left[\underline{l_{F_{n+1}}},\overline{l_{F_{n+1}}}\right],c_{F_{n+1}},\left[\underline{r_{F_{n+1}}},\overline{r_{F_{n+1}}}\right]\right\rangle$$

$$=\left\{\left[\min\left(\left(U_n^{\frac{\rho_t}{\chi}}\right)+\left(\left[\left(\underline{l_{F+1}}\right)^{\chi}\right]^{\frac{1}{\chi}}\right),1\right),\min\left(\left(U_n^{\frac{\rho_t}{\chi}}\right)+\left(\left[\left(\overline{l_{F+1}}\right)^{\chi}\right]^{\frac{1}{\chi}}\right),1\right)\right],\right.$$

$$\min\left(\left(W_n^{\frac{\rho_t}{\chi}}\right)+\left(\left[\left(1-c_{F+1}\right)^{\chi}\right]^{\frac{1}{\chi}}\right),1\right),$$

$$\left.\left[\min\left(\left(X_n^{\frac{\rho_t}{\chi}}\right)+\left(\left[\left(\underline{r_{F+1}}\right)^{\chi}\right]^{\frac{1}{\chi}}\right),1\right),\ \min\left(\left(Y_n^{\frac{\rho_t}{\chi}}\right)+\left(\left[\left(\overline{r_{F+1}}\right)^{\chi}\right]^{\frac{1}{\chi}}\right),1\right)\right]\right\}. \tag{19}$$

Here, (18) = (19) Hence the result.

E. Theorem (Image Contrast)

For given arguments $\overline{F}_t, t = 1, 2, ..., n$ and the parameter $\chi \in (1, +\infty)$ then TIT2WA operator is monotonically non-decreasing (MND) with respect to the parameter.

Proof:

To prove the operator is MND with respect to the parameter, we have to prove the same for every reference point function is MND w.r.t the parameter.

Since $0 \le \underline{l_F} \le \overline{l_F} \le c_F \le \underline{r_F} \le \overline{r_F} \le 1$, $\min\left(U_5^{\frac{k}{\chi}},1\right) > 0.$

And it is true for all the reference points. Hence the result.

$$\max\left(\left(1-C_n^{\frac{\varepsilon_p}{\eta}}\right)+\left(\left[\left(1-c_{M+1}\right)^{\eta}\right]^{\frac{1}{\eta}}\right)^{\underset{p=1}{\overset{n}{sum}}\varepsilon_p},0\right), \tag{20}$$

Based on the theorem A and the operational law,

$$\text{TIT2FYWG}_{\varepsilon}\left\langle\overline{M_1},\overline{M_2},...,\overline{M_n}\right\rangle_Y^{\bullet t}\underset{Y}{\otimes}\overline{M_{n+1}}$$

IV. Application of Type-2 Fuzzy Logic for Feature Extraction

Fig. 2 is the proposed algorithm for Brain extraction from a DICOM image using triangular norms.

MRI of the patient DICOM image has been considered for this application, Fig. 3. Using MATLAB 2015a, brain has been extracted from MRI. The image is taken from our empirical data and its description is as follows

Size of the image	: 512 x 512
Mean of the image	: 242 x 4
Standard deviation	: 50.31
Mean absolute deviation	: 22.43

The below figures are the output of the image processing application in edge detection through triangular norms by MATLAB 2015 a.

Fig. 4 is the gradient through x axis and Fig. 5 is the gradient through the y axis.

The figures reveals that the image gradient to identify the region uniformly.

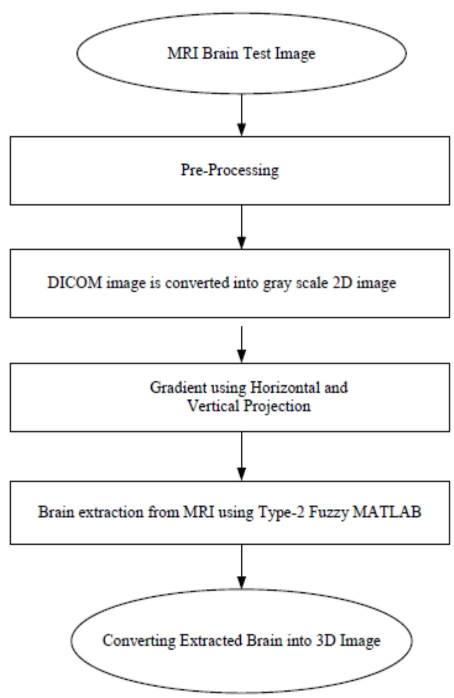

Fig. 2. Architecture of Brain Extraction.

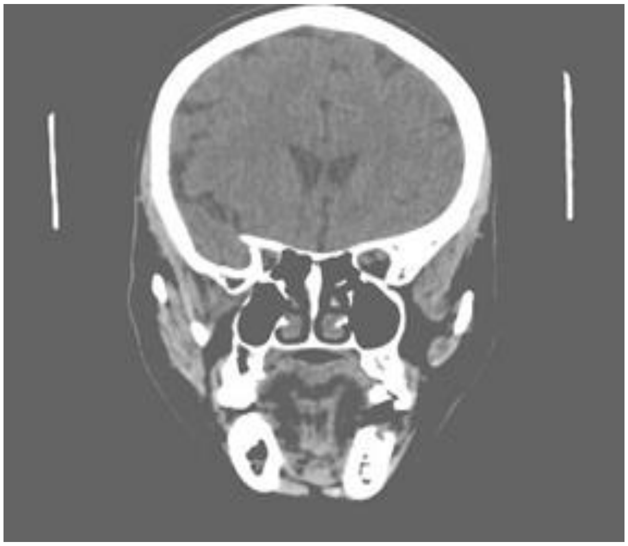

Fig. 3. Original Gray Scale DICOM Image.

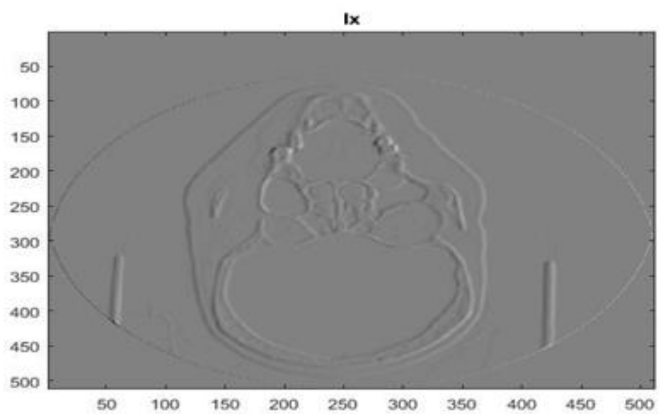

Fig. 4. Gradient through X-Axis.

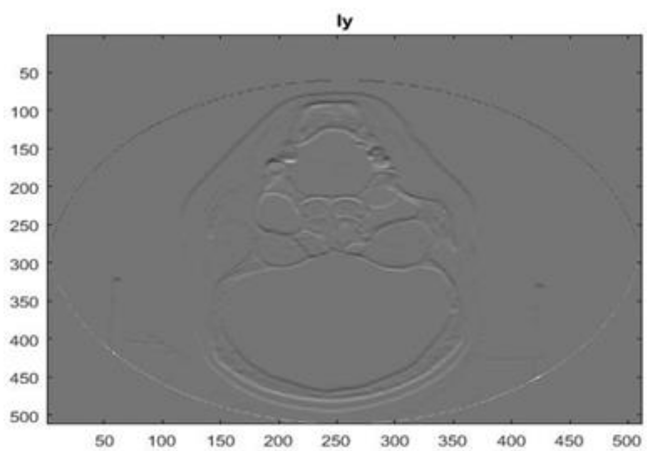

Fig. 5. Gradient through Y-axis.

Fig. 6. Extracted Color Image.

Fig. 7. Extracted Feature.

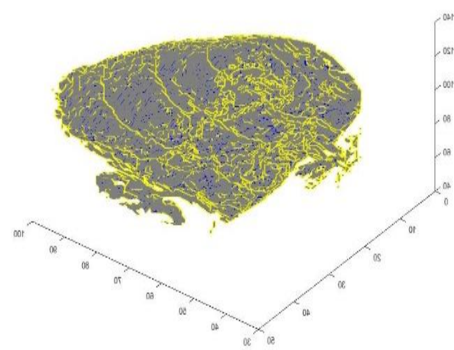

Fig. 8. 3D image of the Extracted Brain Image.

Fig. 6 is the color image output of the Brain extraction using Type-2 Fuzzy based MATLAB 2015a and Fig. 7 is the extracted brain from the DICOM image.

Fig. 8 is the 3D version of the extracted Brain image.

Feature extraction is an essential part of the image processing. In this work, brain has been extracted from patient MRI using MATLAB 2015a. It is examined that, fuzzy logic feature extractor helps to reduce the dimensionality of the image.

Matlab coding based on Interval type-2 fuzzy logic unable to handle non-membership and indeterminacy of the feature which is to be extracted and it is the limitation of the present study.

VI. Discussion

The coding of MATLAB 2015 a under interval type-2 fuzzy has not been used to extract a feature from the image. In Literature review, previous studies have been reviewed on feature extraction and there is no contribution of work for brain extraction using the applied MATLAB coding and 3D version of the extracted color image. This shows the novelty of the proposed work.

VII. Conclusion

Extracting a desired feature plays a key role of image processing. This transforms the pieces from high dimensional space to low dimensional and to decrease the degree of the dimensional. Hence it helps to reduce the dimensionality of the image. In this paper, the proved mathematical properties are related to image processing especially feature extraction such as stability and image contrast and brain has been extracted from patient MRI in a better way and produced 3D image of the extracted brain using interval Type-2 fuzzy MATLAB and it is very helpful for dimensionality reduction while saving the data of the image. Using this technology, extra growth of the cells can be detected and diagnosed if any. In future this work would be extended to intuitionistic and neutrosophic environments.

Data Availability statement

The DICOM data used to support the findings of this study are available from the corresponding author upon request.

Conflict of interest

The authors declare that they have no conflict of interest.

Supplementary Materials

The data set in Fig. 9 is the montage of the images in a single file and is from a patient MRI. This MRI which is in the 3D form is converted to 2D form (DICOM) using MATLAB2015a. The 3D format consists of 25 DICOM file formats; the montage of the images is obtained as a single frame. Out of these 25 DICOM images a clear full image is chosen as in Fig. 10. Using Dilation and erosion methods, the gradient is identified. The edge detection is performed through triangular norms using MATLAB 2015a.

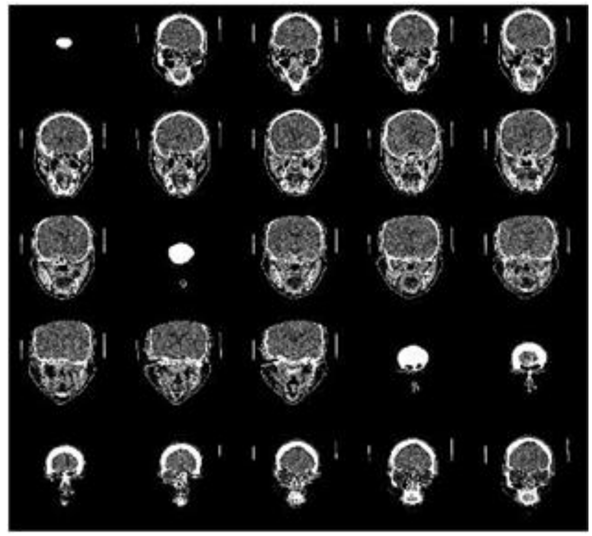

Fig. 9. Montage of the Images.

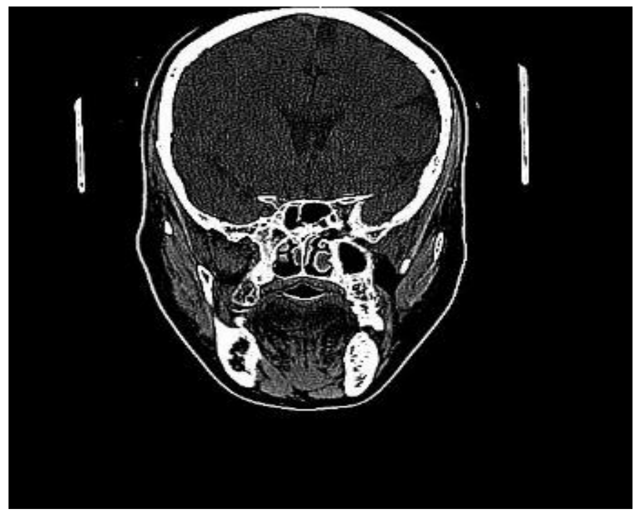

Fig. 10. Clear Image from Montage.

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Review on Type-2 Fuzzy in Biomedicine

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ABSTRACT

Application of physiological and biological ethics to clinical practice is called medical science or Bio-medicine. This branch includes biochemistry, molecular biology, biological engineering neuro science, immunology, pathology and other life science applied to medicine. In this paper, a review has been done for creating a new path and motivation in this field for the new researchers as an application of fuzzy logic in life science areas. Since medical field has uncertainty in nature this topic will be very useful for the future research.

Keywords : *Type-2 Fuzzy, Biological Systems, Biomedicine, Image Processing, Chemical Engineering, Fuzzy Logic Controller*

INTRODUCTION

Biological systems are generally too complicated as they are so complex since developing automated systems are not truthful effort always. An explicit model for biological systems may not prevail or may be very difficult to design. Since human minds perform from rough data, condensed relevant information and carrying out crisp solutions, fuzzy logic may be considered as an optimal tool[1]. Logic and mechanism of Fuzzy logic approach does not have unperturbed boundaries such as human logic and it is not like sure or binary logic. Fuzzy logic control system is one of the most common application with this concept. This system do not demand complete knowledge based model like PID control system. With the knowledge of medical experts and their experience with the imprecise data Fuzzy systems can be designed[2,3].

This model has been used for an automatic control of drug delivery in surgical environment. There is a difficulty of identifying abdominal organs in anticipating the structure of the organ in clinical training, teaching, diagnosis and retrieval of medical image. Fuzzy logic inference system can conquer these problems with the use of automatic identification from a set of slices of CT image. A precise central segmentation may be useful in analyzing microscope images for detecting pathology. This can be done with the use of semi-supervised training fuzzy logic engine[4, 5, 6].

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Adaptive nonlinear predictive control can be used to control glucose concentration during fasting subject to type-1 diabetes. Where the controller employs a section model which represents the gluco supervisory system and cover sub models to represent digestion of vaccines and medication regulated short-coming insulin Lispro and interior absorption with the use of Bayesian parameter calculation for determining time varying parameter changes[7].

Diagnosis of disease associates various levels of imprecision and uncertainty which is essential to medicine. In general precise description of disease individuals uses linguistic terms which is also imprecise and vague and hence fuzzy logic can be used for an optimal result. Segmentation of medical image is a complicated and challenging task due to inherent nature of images. For example brain has a specific complex structure and its exact segmentation is very crucial for identifying edema, tumors and dangerous tissues for applying proper therapy. For the early detection of unusual changes in organs and tissues can be diagnosed by the diagnostic image technique called Magnetic

Resonance Imaging (MRI)[8,9].

Control algorithm combines the expert's knowledge about the treatment of any disease can be treated by using Mamdani-Type fuzzy logic controllers to control the blood glucose level. Fuzzy logic is used to handle uncertainties using natural language and hence it is an approach of qualitative computation. Since impreciseness exists in the field of medicine and huge data in bioinformatics, fuzzy logic is recommended to handle the situation for getting a desired solution. Bioinformatics is also a knowledge based computer analysis of biological data and contains the details stored in the genetic code and empirical results from different sources. Here also impreciseness will occur and hence fuzzy logic will be very useful[10, 11].

MR images have a good comparison resolution for various tissues and have an advantage of automatic tomography for brain studies. Thus majority of research concerns about MR images. Threshold determination is very difficult for brain images as the allocation of tissue intensities are complex and hence logic is used for brain segmentation[9-15, 23-30].

Even in chemical engineering fuzzy logic plays a vital role to handle the system uncertainties while the process of changing chemicals into valuable forms. In all the above cases the role of fuzzy logic has been explained. While getting more uncertainties in those mentioned cases, type-1 fuzzy cannot give the appropriate result as the consequences may have uncertainties[16-22, 31-33]. At this junction, Type-2 fuzzy logic can be used for its adaptivity and stability. In the following chapters, literature study and the role fuzzy in different field have been reviewed.

Fundamental Concepts

Role of Fuzzy Logic

In Medicine[1]

The difficulty of medical process makes conventional quantitative methods of analysis incorrect. Incomplete information, impreciseness and conflict nature are natural. In the field of medicine impreciseness can be classified by the following sources.

- Patient information
- Patient's medical history which is usually highly subjective and uncertain
- Uncertainty of the physical examination where the boundary between normal and pathological condition is uncertain
- Mistakes in the laboratory results due to patient's lack of support.
- Incomplete information given by the patient like understated/exaggerated

Fuzzy logic can be applied in the following experiments:

- To analyze the reaction to the treatment for alcohol dependence
- To evaluate diabetic neuropathy and early symptoms
- To measure the volume of the brain tissue
- To enhance decision making in radiation therapy
- To stabilize hypertension during unconscious stage due t anesthesia
- To diagnose breast cancer
- To estimate significant estimates of usage of drug
- Also fuzzy logic plays an essential role in clinical support systems.

In Bioinformatics[11]

Bioinformatics is an automatic analysis of biological data which includes the information saved in genetic code and results of the experiment from different sources, scientific literature and patient statistics. This branch incorporates computer science, principles of chemical and physical thing, biology, methodologies of modelling huge sets of biological data, cloning, training approach of bio-automatic systems etc. Molecular biology is presently employs project of uncertain data collection. DNA microarrays are the high methodologies with rapidly huge amount of data and it is difficult to apply traditional approaches whereas fuzzy logic deals this problem very easily as it handle multiple membership functions.

In Chemical Engineering[22]

This branch handles with the physical science and life science application such as biochemistry, biology and micro biology. Naturally uncertainty occurs in the mentioned areas and obviously fuzzy logic can handle the impreciseness and can produce the desired result.

In Image processing[33]

In Image segmentation, Edge detection, feature extraction fuzzy logic plays a vital role whereas Type-2 fuzzy sets can deal with more uncertainties as there is a chance of having the inference may be uncertain.

Review on application of Fuzzy Logic in Bio Medicine

The authors, analyzed the usage of fuzzy logic control and auditing in medical sciences with the possible future diffusion[1]. Presented about fuzzy pharmacology with theoretical and applications aspect[2]. Proposed a combined method for automatic diagnosing abdominal organs from a sequence of CT image portions[3]. Used Bayesian parameter calculation to decide model parameters where there is a fluctuation in time[4].

Proposed a novel methodology for filtering framework called two-component adaptive vector filters which enables processing cDNA micro array images[5]. Have done a segmentation on cell nuclei using fuzzy logic engine with fuzzy rules under semi supervised training[6]. Used fuzzy c-means clustering for image segmentation as MR images always have noise due to performance of the operator, environment and equipment and analyzed the robustness of the proposed method[7]. Classified multi class cancer using fuzzy support vector machine and binary decision tree with the choice of gene[8].

Extracted generic feature using fuzzy c-means clustering[9]. Presented a general view of the applications of fuzzy logic in medicine and bioinformatics and presented geometrical perception of fuzzy sets in a fuzzy hypercube[10]. Presented a control algorithm subject to type-1 diabetes mellitus and this algorithm connect the expert's knowledge of the treatment for the disease[11]. Presented classification system using technique of pattern recognition with ARTMAP classifiers to produce a numerical vector representation of a sequence of protein and finding the nature of the sequence into number of given families. And they proved that the proposed system able to classify the protein sequence with an accuracy of 93%[12].

Applied interval type-2 fuzzy logic system to help radiologists to identify micro categorization in mammograms for Brest cancer[13]. Have done an electron tomographic data sets segmentation using the principles of fuzzy set theory[14]. Surveyed about the process of fuzzy expert systems in medical area such as the risk of coronary heart disease, prostate cancer, degree of child anemia, determining the level of anemia with iron deficiency, examination of periodontal dental disease, decision on drug dose etc which will helpful for the physicians[15].

Discussed physical fuzzy confidence curves for the natural unusual activity of falling and used modelling and monitoring human activity[16]. Presented a way of diagnosing thyroid cancer disease using fuzzy-neural networks[17]. Investigated more details on the application of fuzzy logic in chemical engineering[18]. Applied a system of telecardiology to help practitioner doctor when clinical data of patient suspect heart failures[19]. Developed a diagnostic alarm for clinical purpose based on fuzzy logic to detect diagnostic events when anesthesia is given[20].

Presented cost effective method for feature selection for diabetes diagnosis using genetic algorithms and fuzzy logic[21]. Used fuzzy entropy measure along with similarity classifier for feature selection[22]. Presented a novel approach by combined data mining and fuzzy logic for heart disease diagnosis[23]. Used weighted fuzzy rules to predict the risk level of heart disease as a clinical decision support system[24].

Used fuzzy c-means algorithm for image segmentation on MR Brain image[25]. Presented an automatic method based on fuzzy connectedness for extracting an object by segmenting jaw tissues and process of morphology for various views of pseudo orthopantomographic[26]. Discussed the importance of fuzzy logic in medical field. Presented the common idea for fuzzy logic publications with the applications in different fields of biology[28].

Detected breast cancer using fuzzy c means approach[29]. Have done a performance analysis of derived rule base multivariate type-2 self-organizing fuzzy logic controller employed to anesthesia[30]. Presented a work on automatic topic spotting in biomedical literature[31]. Generalized triangular fuzzy numbers are applied in medical decision making[32]. Interval Type-2 Fuzzy has been Inference System and adaptive filter on raw tumor MRI edge detection[33].

CONCLUSION

Field of Biomedicine includes all the main areas like biological and life science as well. According to the review it is found that fuzzy logic plays an effective role in all the areas and there is not enough research on application of Type-2 Fuzzy in those areas. Hence it is concluded that this review process will give a motivation for the new researchers to do their research on Type-2 fuzzy in the mentioned areas.

Ethical approval

Compliance with ethical Standards

The article does not contain any studies with human participants or animal performed by any of the authors.

Source of Funding - Self

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A Review on Type-2 Fuzzy Controller on Control System

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Abstract--- In day today life there are many systems have been used and control process plays a vital role in all the system. Control process makes the system effective and helps for getting desired reaction by controlling the output. In this paper, basic concepts, procedure, stability analysis of the control system and review on applications of control system in real world applications using interval type-2 fuzzy logic controller are presented.

Keywords--- Type 2 Fuzzy Controller, Stability Analysis, Fuzzy Logic.

I. Introduction

A control system is described as a controller which regulates an operation of a system to get the response according to the commanded action. In early years, technical systems made by electrical or mechanical hardware peripherals to perform the most control functions whereas for modern system modelling, fixed processors executes the control function. This kind of well-designed fixed or embedded controllers can afford distinguished system performance under various operating situation. The aim of the controller is to act a system from its initial condition to an aspiring state and maintains the same state. Where one get signal error defined by the difference between the original state and the desired state. A control system that endeavor to retain the output signal at a constant level for elongated periods of time is termed as a regulator. Here the fascinated output value is called the set point. Also control system pursuits to record an input signal which changes periodically. There are two types of control systems namely open loop control (uncontrolled system) and closed loop control (controlled system). Here closed loop control systems may also called as feedback control systems (FCS). FCSs estimate the parameter of the system being controlled and determine the signal of the control actuator but open loop controller fails to use feedback i.e., the output is controlled in feedback systems whereas the output is not controlled in an open loop system. In all the real world applications feedback controllers have been used to get an optimized output. Closed loop or feedback systems are complex but the most useful one in industrial applications where one can get a stable output at a desired value. Proportional Integral Derivative (PID) controller is the example for feedback controller which is accepted globally for its simplicity, good stability and fast response. While dealing the real world problems, many of the problems undergone uncertain situations.

These type of uncertain situations can be dealt by Fuzzy Logic Controllers (FLCs) (Soni and Singh, 2013). Fuzzy logic is employed in many control application with great success. Here one can utilize the human ability and sense to design the controller by using fuzzy IF-THEN rules, whereas in PID controller, objective function to be formed in explicit terms. Two cases of FLCs have been applied in control system namely Type-1 and Type-2 based on Mamdani or Takagi-Sugeno fuzzy models.

In Type-1 Fuzzy Logic system (T1FLS), the membership functions contain no imprecise information, hence the control problem cannot be handled directly when the plant is subject to uncertain parameters. But Type-2 fuzzy logic system handles uncertainty in a potential manner in bringing out the stability analysis for a nonlinear plant and it is a collection of infinite number of Type-2 FLSs. Especially uncertainties can be handled effectively by considering an Interval Type-2 FLS (IT2FLS) along with upper and lower membership functions. An interval between these two is called Foot Print of Uncertainty (FOU) and the length of the interval represent the uncertainty level of the problem. Therefore, Interval Type-2 fuzzy model based control system has got the high priority in control system (Lam and Seneviratne, 2008).

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II. Basic Concepts

Aspects of a System

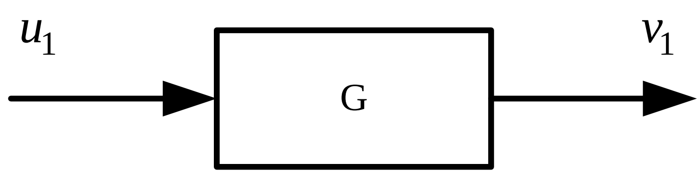
For getting a desired and optimized output, a system needs to be controlled. A system has one or more input as well as outputs, where inputs are riding by actuators and outputs are measured by sensors. System behavior may be simplex or complex.

Linear System

A system is called linear when the output of the system is proportional to its input. Here, changes in input affects the output according to small or large changes.

Also, the final response induced by two or more stimuli is the sum of the responses caused by individual stimulus called superposition property.

This kind of systems have static linearity as well as Sinusoidal Fidelity. Hence the linear control system satisfies homogeneity and additively property.

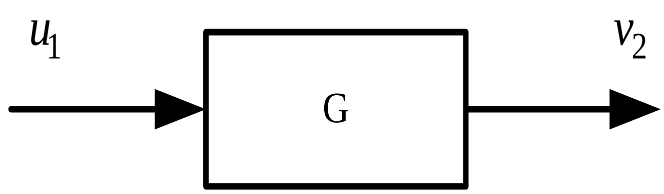


Block Diagram

Where the input u_1 is proportional to the output v_1

Nonlinear System

A system is called nonlinear when the output does not directly vary with respect to the input. Here the superposition property is not applied. Nonlinear systems do not have static linearity as well as Sinusoidal Fidelity.



Block Diagram

Where the input u_1 is not proportional to the output v_2 .

Control System

A system of tools or set of gadgets that operates, regulates or directs the action or behavior of other gadgets to obtain a desired results is called control system.

For example, automation immensely requires control of gadgets. Recently, control systems plays a vital role in modern technology for the development as well as advancement. Morally one's day today life is concerned more or less by few control system. For example, refrigerator, air conditioner, geezer etc., all are control systems. These are also used in industries for producing more output. An ethic of control theory has been applied in engineering and non-engineering domains.

Control Systems with Fuzzy Logic

A system does not have authentic models for the reason of uncertainty and absence of pure knowledge. Measurements of uncertainty do not naturally have stochastic explosion or noise designs. Considering the above points, researchers are motivated to use fuzzy logic in control systems to handle uncertainty. A control system works with fuzzy logic is called fuzzy logic control system or fuzzy logic controller which contributes the conversion of linguistic control approach using IF-THEN rules.

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Premises of Fuzzy Logic Control Structure

To design a fuzzy logic controller, the following points to be taken care.

1. The input, output and state variables should be accessible for the measurement and controlling aspiration.
2. Knowledge body which has linguistic rules and a set of input-output data from which rules can be excerpted, supposed to be existed.
3. There must be a solution.
4. A good enough solution supposed to be expected rather than an optimized one.
5. Fuzzy Logic Controller must be modelled with in sufficient range of rigor.
6. Issues regarding stability and optimality must be taken care while designing the controller.

Why Choosing Interval Type-2 Fuzzy Logic Controller

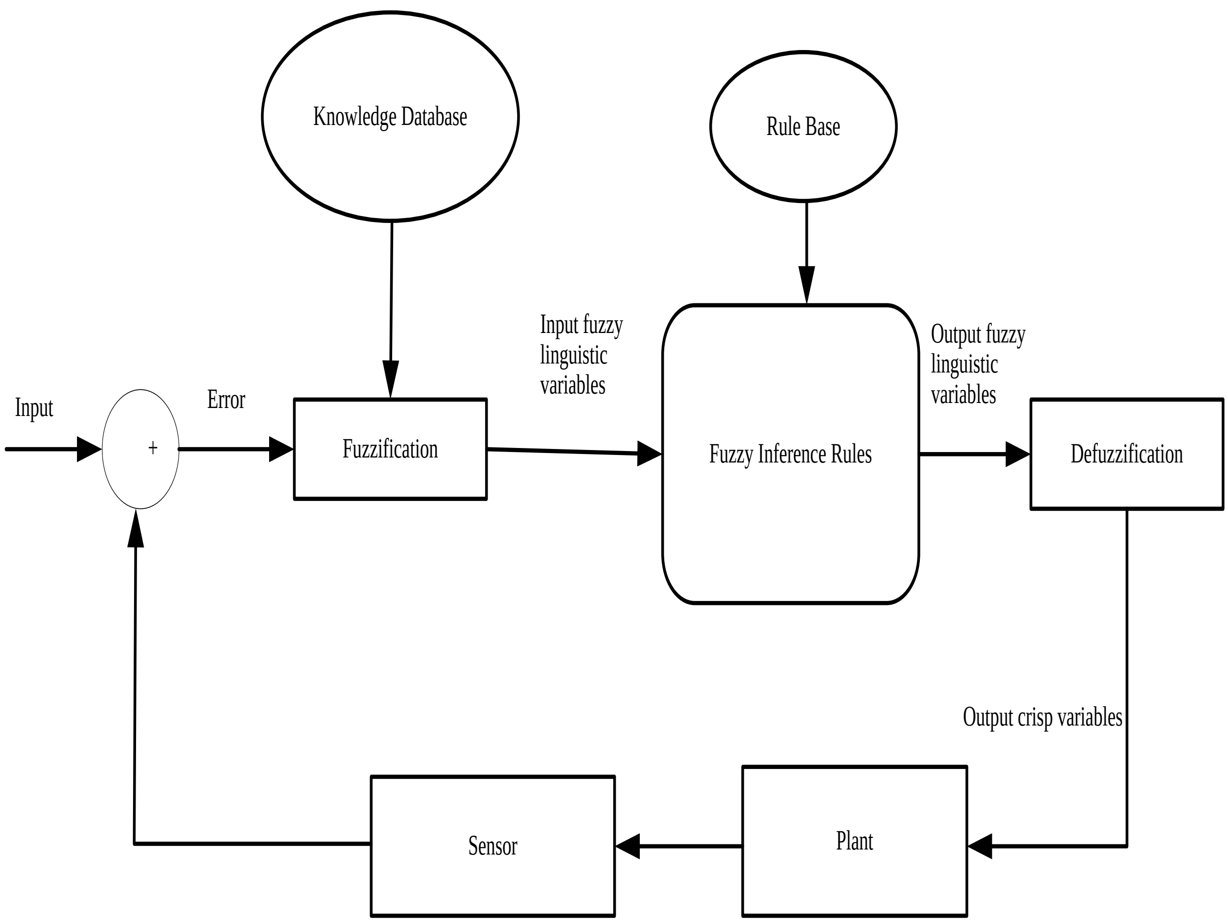
An interval of continuous real numbers defined by two portraits respectively attained by an increasing and decreasing component of the membership functions is called fuzzy interval. A continuous number from this interval takes an element in every alpha cut.

Type-2 Fuzzy Logic Systems have advantages of Type-1 FLSs as well as captures high level of uncertainties, producing complex input-output functions and better results as well. Here the computational complexity is more and theoretical analysis is difficult.

An interval type-2 fuzzy sets reduce all the difficulties discussed above and hence it has been applied in many of the real world problems like designing and control fields.

Using primary and secondary membership functions uncertainties can be dealt effectively in easy manner and stability of the system also can be retained.

Architecture of Fuzzy Logic Controller



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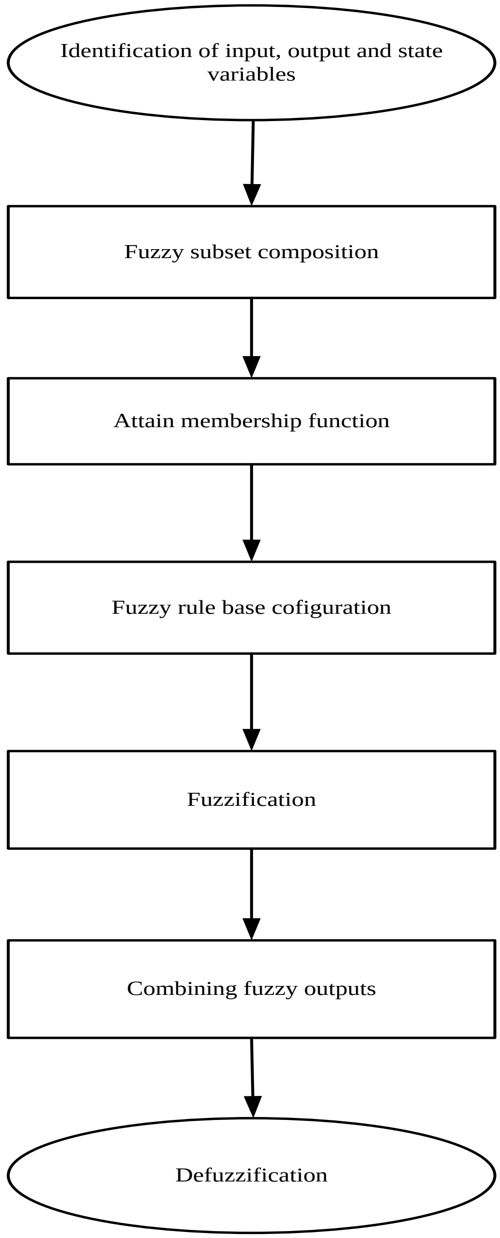
Process of Fuzzy Logic Control System



In the above architecture and process Interval Type-2 Fuzzy Logic can be used.

Stability of the Control System

Stability is an essential factor for the control system. If the output of the system is under control then the system is known as a stable one. Otherwise it is unstable. Usually stable system gives a constrained output for a given constrained input. There are three types of systems based on stability namely absolutely stable (stability exist for every dimension of the system component values), conditionally stable (stable only for a certain dimension ofthe system component values) and marginally stable (only by producing an output signal along with consistent amplitude and frequency of vibrations for bounded input).

Application of Control System

Control system has been applied to enrich the production, adaptability and safety in many of the real world fields like chemical systems, agriculture, power plant, environmental control, quality control, food mechanism, pharmaceutical fabrication.

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Application Areas of Fuzzy Logic

Automatic control, simplified control of robots, Camera-aiming for the program of sports events, optimized planning, Prediction system, Medicine technology, Automatic motor control, Stable control of car efficiently and so on.

Review on Interval Type-2 Fuzzy Logic Controller and its Applications

(Lam et al. 2013) found a model for IT2 control under imprecise criteria for nonlinear systems. (Kobersi et al. 2013) interpreted simulation concept as a mechanism for validating an action and testing of energy of heating systems. (Alam et al. 2013) constructed a model based on FL technique and gave the applications of an intelligent traffic lights control system. (Souverville et al. 2015) applied fuzzy logic in improving image determination using Gaussian membership functions

(Soni and Singh, 2015) presented a tuning technique of PID controllers.(Sagu and Ayygari, 2016) compared the real time preparatory and consequences of the simulation for IT2FL systems and linear quadratic regulator (LQR) of three-tank hybrid system in level control. (Tai et al. 2016) surveyed the application of IT2FLCs in recent years. (Chao et al. 2017) presented the compatibility between traditional PID and fuzzy PID controllers.(Liu et al. 2017) examined the model to find the point of intersection of an underwater glider under analysis of hydrodynamics of fuzzy PID controllers.

(Han and Hamasaki, 2018) introduced the path for the partial impreciseness associated with the control input of a FLC. (Jebelli et al. 2018) explained about the model and control along with the potential of perception and detection of an autonomous under water vehicle using fuzzy logic. (Izzuddin et al. 2018) applied open loop fuzzy logic control system in irrigation system using Mamdani control system. (Senapati et al. 2018) applied FLC in controlling the speed of a smart car. From the review, one can understand the importance and effectiveness of IT2FLC in an uncertain situation exist in the control system.

III. Conclusion

Control system is an interrelationship of components designing a system configuration which will produce a desired response of the system. In this types of systems, basics of control system, fuzzy control system, application of control system as well as fuzzy control system, advantages of using interval type-2 fuzzy controller and the works done so far using fuzzy and interval type-2 fuzzy controller has been reviewed. This paper will be very useful for the new researchers in the field of control systems to get their new direction of the research.

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