Interval-Valued Neutrosophic Bonferroni Mean Operators and the Application in the Selection of Renewable Energy

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ABSTRACT

Renewable energy selection, which is a multi-criteria decision-making (MCDM) problem, is crucial for the sustainable development of economy. Criteria are interdependent in the selection problem of renewable energy. Moreover, fuzzy and uncertain information exist during the selection processes, and information can be comprehensively reflected by interval-valued neutrosophic sets. This chapter aims to construct selection approaches for renewable energy considering the interrelationships among criteria. To do that, Bonferroni mean (BM) and geometric BM (GBM) are employed. Firstly, the interval-valued neutrosophic BM (IVNBM) and the interval-valued neutrosophic GBM (IVNGBM) are propsoed as extensions of BM and GBM, respectively. Then, to take into consideration the relative importance of each element, the interval-valued neutrosophic weighted BM (IVNWBM) and the interval-valued neutrosophic weighted GBM (IVNWGBM) are further defined. Subsequently, the novel MCDM approaches for the selection of renewable energy, which are in view of the interrelationships among elements, are explored based on the IVNWBM and IVNWGBM operator. Furthermore, the applicability of the proposed approaches is demonstrated by a numerical example about the selection of renewable energy. In addition, the influence of the parameters is investigated and discussed. Finally, a comparative analysis composed of two cases verifies the feasibility of the proposed MCDM approaches.

KEYWORDS: multi-criteria decision-making; interval-valued neutrosophic set; weighted Bonferroni mean; weighted geometric Bonferroni mean; renewable energy selection

1. INTRODUCTION

Renewable energy has been replacing traditional non-renewable energy owing to the limitation of the latter and environmental protection. Renewable energy is energy that can be circularly regenerated in nature. It mainly includes solar energy, wind energy, biomass energy, tidal energy and ocean thermal energy, just name a few. Many researchers have been studying the selection problem of renewable energy (Mardani, Jusoh, Zavadskas, Cavallaro, & Khalifah, 2015; Troldborg, Heslop, & Hough, 2014). Some of them pointed out that the selection of renewable energy is a multi-criteria decision-making (MCDM) problem (Cristóbal, 2011; Yazdani-Chamzini, Fouladgar, Zavadskas, & Moini, 2013). Experts assess renewable energy with regard to several criteria including power of energy, investment ratio and emissions of carbon dioxide (CO_2) avoided per year and so forth. The most proper renewable energy can be selected on the basis of the assessment information provided by experts. Because it becomes difficult for decision-makers to identify an optimal alternative that maximizes all decision criteria, a multi-objective approach is required to examine tradeoffs considering each criterion. Kaya and Kahraman (Kaya & Kahraman, 2010) proposed a modified fuzzy VIKOR methodology to make a multi-criteria selection among alternative renewable energy options and production sites for Istanbul area using an integrated VIKOR-AHP methodology.

Fuzziness and uncertainty may exist in the assessment information due to the complexity and limitation of human cognition and sometimes the criteria are interdependent. For example, an expert may be uncertain about the upper bound of the power of an individual renewable energy. However, fuzzy and uncertain information do not be fully utilized in extant approaches of the selection of renewable energy. Especially, the interrelationships among criteria are not considered in the extant approaches. Therefore, novel MCDM approaches are required. In this paper, we propose selection approaches for renewable energy considering the interrelationships among criteria. To do that, Bonferroni mean (BM) and geometric BM (GBM) are employed. To take into consideration the relative importance of each element, we further define the interval-valued neutrosophic weighted BM (IVNWBM) and the interval-valued neutrosophic weighted GBM (IVNWGBM). Subsequently, the novel MCDM approaches for the selection of renewable energy, which are in view of the interrelationships among elements, are explored based on the IVNWBM and IVNWGBM operator.

The remainder of this paper is organized as follows. In Section 2, we review the applications of MCDM in renewable energy selection. What's more, neutrosophic set (NS) and BM are reviewed in this section. In Section 3, the definition and some properties of IVNGBM and IVNWGBM are investigated, based on which, novel MCDM approaches for the selection of renewable energy with interval-valued neutrosophic numbers (IVNNs) are presented. In order to demonstrate the application and verify the feasibility of the proposed MCDM approaches, a numerical example and a comparative analysis are conducted and discussed in Section 4. In addition, it also discusses the influence of parameters in IVNGBM and IVNWGBM on the proposed MCDM approaches. Finally, Section 5 concludes this paper and suggests several directions for future research.

2. LITERATURE REVIEW

Since FS was proposed by Zadeh (L. A. Zadeh, 1965) in 1965, it has become a vital tool to construct MCDM approaches (Aghdaie, Zolfani, & Zavadskas, 2013; Bellman & Zadeh, 1970; Yager, 1977). After that, many researchers have been devoting themselves to handling with the imprecise, incomplete and uncertain information and have put forward numerous extensions of FS (Cao, Zhou, & Wang, 2016; H.-g. Peng & Wang, 2016; Turksen, 1986; Lotfi Asker Zadeh, 1968). Particularly, Florentin Smarandache (Smarandache, 1998, 1999) introduced the neutrosophic logic and the NS.

2.1 NEUTROSOPHIC SET (NS)

NS makes use of the functions of truth, indeterminacy and falsity to depict the fuzzy information. And the values of these three functions lie in $]0^-, 1^+[$, the non-standard unit interval (Rivieccio, 2008), which is the extension to the standard interval $[0, 1]$ of IFS. The indeterminacy factor here is impervious to truth and falsity values while the incorporated uncertainty in IFS rests with the degrees of belongingness and non-belongingness (Majumdar & Samanta, 2014). Nevertheless, it is difficult to apply NS in realistic problems. Hence, Wang *et al*. (H. B. Wang, Smarandache, Zhang, & Sunderraman, 2010) defined the single-valued neutrosophic set (SVNS) and Ye (Ye, 2014) put forward the notion of the simplified neutrosophic set (SNS), which are both instances of NS. In addition, manifold MCDM approaches have been developed under single-valued neutrosophic environments and simplified neutrosophic environments (Ji, Wang, & Zhang, 2016; Liu & Wang, 2014; J. J. Peng, Wang, Wang, Zhang, & Chen, 2016; J. J. Peng, Wang, Zhang, & Chen, 2014; Şahin & Liu, 2016; Wu, Wang, Peng, & Chen, 2016; Ye, 2013).

In the light of that it is more practicable to utilize interval numbers to describe the degrees of truth, falsity and indeterminacy about a certain statement in some circumstances rather than exact numbers, Wang *et al*. (H. B. Wang, Smarandache, Zhang, & Sunderraman, 2005) put forward the concept of the interval-valued neutrosophic set (IVNS) and presented the set-theoretic operators of IVNS. Other than NSs, the degrees of truth, indeterminacy and falsity of IVNSs are interval numbers. Up to now, plenty of MCDM approaches utilizing IVNS have been put forward (Chi & Liu, 2013; Şahin & Karabacak, 2015; Z. Tian, Zhang, Wang, Wang, & Chen, 2016; H. Zhang, Ji, Wang, & Chen, 2015; H. Zhang, Wang, & Chen, 2016) and IVNSs have been applied in addressing practical problems (H. Ma, Hu, Li, & Zhang, 2016). Furthermore, studies about other extensions of NSs have been investigated (Z. P. Tian, Wang, Zhang, &

Wang, 2016; Hong Yu Zhang, Ji, Wang, & Chen, 2016), like multi-valued neutrosophic sets (Ji, Zhang, & Wang, 2016; J.-j. Peng, Wang, Wu, Wang, & Chen, 2015; J. Peng, Wang, & Yang, 2017), single valued trapezoidal neutrosophic sets (Liang, Wang, & Li, 2016), n-valued refined neutrosophic sets (Smarandache, 2013) and neutrosophic linguistic sets (Y. X. Ma, Wang, Wang, & Wu, 2016; Z. P. Tian, Wang, Wang, & Zhang, 2016a, 2016b; J. Q. Wang, Yang, & Li, 2016).

The score function and accuracy function of IVNNs have been given as well as the comparative method of two IVNNs, which make it practical.

Definition 1 (Şahin, 2014). Let $A=\langle[\inf T_A,\sup T_A],[\inf I_A,\sup I_A],[\inf F_A,\sup F_A]\rangle$ be an IVNN, a score function L of A can be defined by

$$L(A)=\frac{2+\inf T_A+\sup T_A-2\inf I_A-2\sup I_A-\inf F_A-\sup F_A}{4} \tag{1}$$

where $L(A)\in[-1,1]$.

Definition 2 (Şahin, 2014). Let $A=\langle[\inf T_A,\sup T_A],[\inf I_A,\sup I_A],[\inf F_A,\sup F_A]\rangle$ be an IVNN, an accuracy function N of A can be defined by

$$\begin{aligned}N(A)=\frac{1}{2}\big[&\inf T_A+\sup T_A-\inf I_A\times(1-\inf T_A)-\sup I_A\times(1-\sup T_A)\\&-\inf F_A\times(1-\inf I_A)-\sup F_A\times(1-\sup I_A)\big]\end{aligned}, \tag{2}$$

where $N(A)\in[-1,1]$.

Definition 3 (Şahin, 2014). Suppose that $A=\langle[\inf T_A,\sup T_A],[\inf I_A,\sup I_A],[\inf F_A,\sup F_A]\rangle$ and $B=\langle[\inf T_B,\ \sup T_B],[\inf I_B,\sup I_B],[\inf F_B,\sup F_B]\rangle$ be two IVNNs. The comparative method of A and B can be defined as follows:

(i). When $L(A)>L(B)$, $A\succ B$; and

(ii). When $L(A)=L(B)$ and $N(A)>N(B)$, $A\succ B$.

Definition 4 (H. Y. Zhang, Wang, & Chen, 2014). Let $A=\langle[\inf T_A,\sup T_A],[\inf I_A,\sup I_A],[\inf F_A,\sup F_A]\rangle$ and $B=\langle[\inf T_B,\sup T_B],\ [\inf I_B,\sup I_B],[\inf F_B,\sup F_B]\rangle$ be any two IVNNs and $\lambda>0$. The operations are defined as follows:

(1) $A+B=\langle[\inf T_A+\inf T_B-\inf T_A\cdot\inf T_B,\sup T_A+\sup T_B-\sup T_A\cdot\sup T_B],$
$[\inf I_A\cdot\inf I_B,\sup I_A\cdot\sup I_B],[\inf F_A\cdot\inf F_B,\sup F_A\cdot\sup F_B]\rangle$;

(2) $A\cdot B=\langle[\inf T_A\cdot\inf T_B,\sup T_A\cdot\sup T_B],[\inf I_A+\inf I_B-\inf I_A\cdot\inf I_B,$
$\sup I_A+\sup I_B-\sup I_A\cdot\sup I_B],[\inf F_A+\inf F_B-\inf F_A\cdot\inf F_B,$
$\sup F_A+\sup F_B-\sup F_A\cdot\sup F_B]\rangle$;

(3) $\lambda A=\langle[1-(1-\inf T_A)^\lambda,1-(1-\sup T_A)^\lambda],[(\inf I_A)^\lambda,(\sup I_A)^\lambda],[(\inf F_A)^\lambda,(\sup F_A)^\lambda]\rangle$;

(4) $A^\lambda=\langle[(\inf T_A)^\lambda,(\sup T_A)^\lambda],[1-(1-\inf I_A)^\lambda,1-(1-\sup I_A)^\lambda],[1-(1-\inf F_A)^\lambda,1-(1-\sup F_A)^\lambda]\rangle$; and

(5) $neg(A)=\langle[\inf F_A,\sup F_A],[1-\sup I_A,1-\inf I_A],[\inf T_A,\sup T_A]\rangle$.

2.2 Multi-criteria decision-making (MCDM)

The applications of the extensions of FSs have attracted considerable researchers' attention (Joshi & Kumar, 2012; J. J. Peng, Wang, Wang, Yang, & Chen, 2015; Shinoj & Sunil, 2012; J. Q. Wang, Han, & Zhang, 2014; J. Q. Wang, Wu, Wang, Zhang, & Chen, 2016; X.-Z. Wang et al., 2015), not excepting the researchers in energy. Wang *et al*. (B. Wang, Nistor, Murty, & Wei, 2014) using the TOPSIS (the

Technique for Order Preference by Similarity to Ideal Solution) approach, one of the branches of MCDM models, assessed the efficiency of hydropower generation in Canada. Khalili-Damghani *et al*. (Khalili-Damghani, M., Santos-Arteaga, & Mohtasham, 2015) proposed a dynamic multi-stage approach to evaluate the efficiency of cotton production energy consumption by utilizing data envelopment analysis, a tool of MCDM. Additionally, critical reviews of MCDM approaches have been done to survey MCDM models, techniques and their empirical applications in various fields (Ananda & Herath, 2009; Govindan, Rajendran, Sarkis, & Murugesan, 2015; Ho, Xu, & Dey, 2010).

As an important tool in constructing MCDM approaches, the aggregation operator captures widespread attention and some researches about the aggregation operator have been done under interval-valued neutrosophic environments. Zhang *et al*. (H. Y. Zhang et al., 2014) proposed the interval-valued neutrosophic weighted average (IVNWA) operator and the interval-valued neutrosophic weighted geometric average (IVNWG) operator. Based on these two aggregation operators, Ye (Ye, 2014) defined the ordered weighted average operator and the ordered weighted geometric averaging operator for IVNSs.

All the aggregation operators mentioned above suppose that the elements integrated are mutually independent. In theory, the criteria in a MCDM problem should satisfy the requirement of independence. However, in some realistic MCDM problems like the selection of renewable energy, the criteria are correlative, in which the aggregation operators illustrated above become inapplicable. For instance, power, investment ratio, operation and maintenance cost and operating hours are four of the criteria in the selection of renewable energy and they are not independent. In the example, as known to all, investment ratio may be affected by power, and operation and maintenance cost may be bound up with operating hours. In order to overcome this deficiency and take into account the interrelationships among criteria, the Bonferroni mean (BM) is introduced.

2.3 Bonferroni mean (BM)

BM, firstly put forward by Bonferroni in Ref. (Bonferroni, 1950), has been extended to several kinds of FSs. For instance, Xu and Yager (Xu & Yager, 2011) defined the intuitionistic fuzzy BM (IFBM) and the intuitionistic fuzzy weighted BM (IFWBM) according to previous studies about BM and the weighted BM (WBM). Moreover, Xia *et al*. (Xia, Xu, & Zhu, 2012) investigated the generalized BM, which is proposed by Beliakov (Beliakov, James, Mordelová, Rückschlossová, & Yager, 2010), under intuitionistic fuzzy environments and developed the generalized WBM and the generalized intuitionistic fuzzy WBM. Furthermore, Zhou and He (Zhou & He, 2012) pointed out some drawbacks of WBM. To conquer these drawbacks, they proposed a novel WBM operator, which is called the normal WBM (NWBM). Based on BM, Xia *et al*. (Xia, Xu, & Zhu, 2013) defined geometric BM (GBM) and introduced the intuitionistic fuzzy GBM (IFGBM) and the weighted IFGBM (WIFGBM). And they also discussed some properties of IFGBM. On the basis of GBM in Ref. (Xia et al., 2013), Zhu *et al*. (Zhu & Xu, 2013) explored the GBM under hesitant fuzzy environments and put forward the hesitant fuzzy GBM (HFGBM) and the hesitant fuzzy Choquet GBM (HFCGBM). In addition, Liu and Wang (Liu & Wang, 2014) extended NWBM to aggregate single-valued neutrosophic numbers (SVNNs) and defined the single-valued neutrosophic BM (SVNBM) and the single-valued neutrosophic NWBM. Besides, many other extensions of BM have been developed (Z. P. Tian, Wang, Wang, & Chen, 2015; Z. P. Tian, Wang, Zhang, Chen, & Wang, 2015) and applied to tackle practical problems (Hong Yu Zhang, Ji, Wang, & Chen, 2017).

IVNSs can more comprehensively express fuzzy and uncertain information during the processes of selecting renewable energy than other extensions of NSs like SVNSs. Moreover, criteria may be correlative in the selection problems of renewable energy. For solving such problems in selecting renewable energy, we intend to introduce BM. Nevertheless, to the best of our knowledge, BM has not been studied under interval-valued neutrosophic environments. To overcome this deficiency, in the first place, we propose the interval-valued neutrosophic BM (IVNBM) and the interval-valued neutrosophic GBM (IVNGBM). Considering that IVNBM and IVNGBM do not take into account the relative importance of each element, the interval-valued neutrosophic WBM (IVNWBM) and the interval-valued neutrosophic weighted GBM (IVNWGBM) are also put forward in this study. Additionally, novel MCDM approaches for the selection of renewable energy are constructed based on the proposed aggregation operators.

3. MCDM APPROACHES FOR THE SELECTION OF RENEWABLE ENERGY

In this section, based on SVNBM in Ref. (Liu & Wang, 2014), the definition of IVNBM and IVNGBM are put forward based on previous studies about IVIFBM and SVNBM. However, IVNBM and IVNGBM do not take into consideration the relative importance of each IVNN. IVNWBM and IVNWGBM are proposed in order to conquer this disadvantage. In addition, some properties of IVNBM and IVNGBM are investigated. Based on the proposed aggregation operators, novel MCDM approaches for the selection of renewable energy are constructed and the procedures are discussed in this section.

3.1 IVNBM

Definition 5. Let p, $q \geq 0$ and $x_i = \left\langle \left[T_i^-, T_i^+\right], \left[I_i^-, I_i^+\right], \left[F_i^-, F_i^+\right] \right\rangle$ $(i = 1, 2, \cdots, n)$ be a collection of IVNNs. IVNBM can be defined as:

$$IVNBM^{p,q}\left(x_1, x_2, \cdots, x_n\right) = \left(\frac{1}{n(n-1)} \underset{\substack{i,j=1 \\ i \neq j}}{\overset{n}{\oplus}} \left(x_i^p \otimes x_j^q \right) \right)^{\frac{1}{p+q}}. \tag{3}$$

Theorem 1. Let p, $q \geq 0$ and $x_i = \left\langle \left[T_i^-, T_i^+\right], \left[I_i^-, I_i^+\right], \left[F_i^-, F_i^+\right] \right\rangle$ $(i = 1, 2, \cdots, n)$ be a collection of IVNNs. The aggregated value by IVNBM in (3) is also an IVNN, and

$$\begin{aligned}
&IVNBM^{p,q}\left(x_1, x_2, \cdots, x_n\right) \\
&= \Bigg\langle \left[\left(1 - \prod_{\substack{i,j=1 \\ i \neq j}}^{n} \left(1 - \left(T_i^-\right)^p \left(T_j^-\right)^q \right)^{\frac{1}{n(n-1)}} \right)^{\frac{1}{p+q}}, \left(1 - \prod_{\substack{i,j=1 \\ i \neq j}}^{n} \left(1 - \left(T_i^+\right)^p \left(T_j^+\right)^q \right)^{\frac{1}{n(n-1)}} \right)^{\frac{1}{p+q}} \right], \\
&\left[1 - \left(1 - \prod_{\substack{i,j=1 \\ i \neq j}}^{n} \left(1 - \left(1 - I_i^-\right)^p \left(1 - I_j^-\right)^q \right)^{\frac{1}{n(n-1)}} \right)^{\frac{1}{p+q}}, 1 - \left(1 - \prod_{\substack{i,j=1 \\ i \neq j}}^{n} \left(1 - \left(1 - I_i^+\right)^p \left(1 - I_j^+\right)^q \right)^{\frac{1}{n(n-1)}} \right)^{\frac{1}{p+q}} \right], \\
&\left[1 - \left(1 - \prod_{\substack{i,j=1 \\ i \neq j}}^{n} \left(1 - \left(1 - F_i^-\right)^p \left(1 - F_j^-\right)^q \right)^{\frac{1}{n(n-1)}} \right)^{\frac{1}{p+q}}, 1 - \left(1 - \prod_{\substack{i,j=1 \\ i \neq j}}^{n} \left(1 - \left(1 - F_i^+\right)^p \left(1 - F_j^+\right)^q \right)^{\frac{1}{n(n-1)}} \right)^{\frac{1}{p+q}} \right] \Bigg\rangle.
\end{aligned} \tag{4}$$

Proof.

According to the operations (2) and (4) in Definition 4, we have $x_i^p = \left\langle \left[\left(T_i^-\right)^p, \left(T_i^+\right)^p \right], \left[1 - \left(1 - I_i^-\right)^p, 1 - \left(1 - I_i^+\right)^p \right], \left[1 - \left(1 - F_i^-\right)^p, 1 - \left(1 - F_i^+\right)^p \right] \right\rangle$, $x_j^q = \left\langle \left[\left(T_j^-\right)^p, \left(T_j^+\right)^p \right], \left[1 - \left(1 - I_j^-\right)^p, 1 - \left(1 - I_j^+\right)^p \right], \left[1 - \left(1 - F_j^-\right)^p, 1 - \left(1 - F_j^+\right)^p \right] \right\rangle$ and $x_i^p \otimes x_j^q = \left\langle \left[\left(T_i^-\right)^p \cdot \left(T_j^-\right)^q, \left(T_i^+\right)^p \cdot \left(T_j^+\right)^q \right], \left[1 - \left(1 - I_i^-\right)^p \left(1 - I_j^-\right)^q, 1 - \left(1 - I_i^+\right)^p \left(1 - I_j^+\right)^q \right], \left[1 - \left(1 - F_i^-\right)^p \left(1 - F_j^-\right)^q, 1 - \left(1 - F_i^+\right)^p \left(1 - F_j^+\right)^q \right] \right\rangle$. Let $a_{ij} = \left\langle \left[T_{ij}^-, T_{ij}^+\right], \left[I_{ij}^-, I_{ij}^+\right], \left[F_{ij}^-, F_{ij}^+\right] \right\rangle = \left(w_i x_i\right)^p \otimes \left(w_j x_j\right)^q$, $IVNWBM^{p,q}(x_1, x_2, \cdots, x_n) = \left(\frac{1}{n(n-1)} \underset{\substack{i,j=1 \\ i \neq j}}{\overset{n}{\oplus}} \left(\left(w_i x_i\right)^p \otimes \left(w_j x_j\right)^q \right) \right)^{\frac{1}{p+q}} = \left(\frac{1}{n(n-1)} \underset{\substack{i,j=1 \\ i \neq j}}{\overset{n}{\oplus}} \left(\alpha_{ij} \right) \right)^{\frac{1}{p+q}}$. According to the operational laws (1) and (3) in Definition 4, $\frac{1}{n(n-1)} \underset{\substack{i,j=1 \\ i \neq j}}{\overset{n}{\oplus}} \left(\alpha_{ij} \right) =$

$$\left\langle\left[1-\prod_{\substack{i,j=1\\i\neq j}}^{n}\left(1-T_{ij}^{-}\right)^{\frac{1}{n(n-1)}},1-\prod_{\substack{i,j=1\\i\neq j}}^{n}\left(1-T_{ij}^{+}\right)^{\frac{1}{n(n-1)}}\right],\left[\prod_{\substack{i,j=1\\i\neq j}}^{n}\left(I_{ij}^{-}\right)^{\frac{1}{n(n-1)}},\prod_{\substack{i,j=1\\i\neq j}}^{n}\left(I_{ij}^{+}\right)^{\frac{1}{n(n-1)}}\right],\left[\prod_{\substack{i,j=1\\i\neq j}}^{n}\left(F_{ij}^{-}\right)^{\frac{1}{n(n-1)}},\prod_{\substack{i,j=1\\i\neq j}}^{n}\left(F_{ij}^{+}\right)^{\frac{1}{n(n-1)}}\right]\right\rangle .$$

Therefore, $IVNWBM^{p,q}\left(x_1,x_2,\cdots,x_n\right)=\left(\frac{1}{n(n-1)}\underset{\substack{i,j=1\\i\neq j}}{\overset{n}{\oplus}}\left(\alpha_{ij}\right)\right)^{\frac{1}{p+q}}=\left\langle\left[\left(1-\prod_{\substack{i,j=1\\i\neq j}}^{n}\left(1-T_{ij}^{-}\right)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{p+q}},\right.\right.$

$$\left.\left(1-\prod_{\substack{i,j=1\\i\neq j}}^{n}\left(1-T_{ij}^{+}\right)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{p+q}}\right],\left[1-\left(1-\prod_{\substack{i,j=1\\i\neq j}}^{n}\left(I_{ij}^{-}\right)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{p+q}},1-\left(1-\prod_{\substack{i,j=1\\i\neq j}}^{n}\left(I_{ij}^{+}\right)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{p+q}}\right],\left[1-\left(1-\prod_{\substack{i,j=1\\i\neq j}}^{n}\left(F_{ij}^{-}\right)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{p+q}},\right.$$

$$\left.\left.1-\left(1-\prod_{\substack{i,j=1\\i\neq j}}^{n}\left(F_{ij}^{+}\right)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{p+q}}\right]\right\rangle=\left\langle\left[\left(1-\left(\prod_{\substack{i,j=1\\i\neq j}}^{n}\left(1-\left(T_i^{-}\right)^{p}\left(T_j^{-}\right)^{q}\right)\right)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{p+q}},\left(1-\left(\prod_{\substack{i,j=1\\i\neq j}}^{n}\left(1-\left(T_i^{+}\right)^{p}\left(T_j^{+}\right)^{q}\right)\right)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{p+q}}\right],\right.$$

$$\left[1-\left(1-\left(\prod_{\substack{i,j=1\\i\neq j}}^{n}\left(1-\left(1-I_i^{-}\right)^{p}\left(1-I_j^{-}\right)^{q}\right)\right)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{p+q}},1-\left(1-\left(\prod_{\substack{i,j=1\\i\neq j}}^{n}\left(1-\left(1-I_i^{+}\right)^{p}\left(1-I_j^{+}\right)^{q}\right)\right)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{p+q}}\right],$$

$$\left.\left[1-\left(1-\left(\prod_{\substack{i,j=1\\i\neq j}}^{n}\left(1-\left(1-F_i^{-}\right)^{p}\left(1-F_j^{-}\right)^{q}\right)\right)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{p+q}},1-\left(1-\left(\prod_{\substack{i,j=1\\i\neq j}}^{n}\left(1-\left(1-F_i^{+}\right)^{p}\left(1-F_j^{+}\right)^{q}\right)\right)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{p+q}}\right]\right\rangle.$$

Furthermore, the following inequalities are true:

$$0\leq\left(1-\left(\prod_{\substack{i,j=1\\i\neq j}}^{n}\left(1-\left(T_i^{-}\right)^{p}\left(T_j^{-}\right)^{q}\right)\right)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{p+q}}\leq 1 \quad , \quad 0\leq\left(1-\left(\prod_{\substack{i,j=1\\i\neq j}}^{n}\left(1-\left(T_i^{+}\right)^{p}\left(T_j^{+}\right)^{q}\right)\right)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{p+q}}\leq 1 \quad ,$$

$$0\leq 1-\left(1-\left(\prod_{\substack{i,j=1\\i\neq j}}^{n}\left(1-\left(1-I_i^{-}\right)^{p}\left(1-I_j^{-}\right)^{q}\right)\right)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{p+q}}\leq 1,\ 0\leq 1-\left(1-\left(\prod_{\substack{i,j=1\\i\neq j}}^{n}\left(1-\left(1-I_i^{+}\right)^{p}\left(1-I_j^{+}\right)^{q}\right)\right)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{p+q}}\leq 1,$$

$$0\leq 1-\left(1-\left(\prod_{\substack{i,j=1\\i\neq j}}^{n}\left(1-\left(1-F_i^{-}\right)^{p}\left(1-F_j^{-}\right)^{q}\right)\right)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{p+q}}\leq 1 \text{ and } 0\leq 1-\left(1-\left(\prod_{\substack{i,j=1\\i\neq j}}^{n}\left(1-\left(1-F_i^{+}\right)^{p}\left(1-F_j^{+}\right)^{q}\right)\right)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{p+q}}\leq 1$$

which meets the requirements of an IVNN.

Therefore, Theorem 1 holds.

In the following part, we investigate some properties of IVNBM:

(1) When $x_i = \langle [1,1],[0,0],[0,0] \rangle$ $(i = 1,2,\cdots,n)$, $IVNBM^{p,q}(x_1, x_2, \cdots, x_n) = \langle [1,1],[0,0],[0,0] \rangle$.

(2) When $x_i = \langle [0,0],[1,1].[1,1] \rangle$ $(i = 1,2,\cdots,n)$, $IVNBM^{p,q}(x_1, x_2, \cdots, x_n) = \langle [0,0],[1,1],[1,1] \rangle$.

(3) (**Idempotency**) When all IVNNs x_i $(i = 1,2,\cdots,n)$ are equal, i.e., $x_i = x$ for all i,

$$IVNBM^{p,q}(x_1, x_2, \cdots, x_n) = x. \tag{5}$$

Proof. Since $x_i = x$ for all i, we can obtain that $IVNBM^{p,q}(x_1, x_2, \cdots, x_n) = \left(\frac{1}{n(n-1)} \underset{\substack{i,j=1 \\ i \neq j}}{\overset{n}{\oplus}} \left(x_i^p \otimes x_j^q \right) \right)^{\frac{1}{p+q}} =$ $\left(\frac{1}{n(n-1)} \underset{\substack{i,j=1 \\ i \neq j}}{\overset{n}{\oplus}} \left(x^p \otimes x^q \right) \right)^{\frac{1}{p+q}} = \left(\frac{1}{n(n-1)} \underset{\substack{i,j=1 \\ i \neq j}}{\overset{n}{\oplus}} \left(x^{p+q} \right) \right)^{\frac{1}{p+q}} = x$.

(4) (**Monotonicity**) Let $x_i = \left\langle \left[T_{x_i}^-, T_{x_i}^+ \right], \left[I_{x_i}^-, I_{x_i}^+ \right], \left[F_{x_i}^-, F_{x_i}^+ \right] \right\rangle$ $(i = 1,2,\cdots,n)$ and $y_i = \left\langle \left[T_{y_i}^-, T_{y_i}^+ \right], \left[I_{y_i}^-, I_{y_i}^+ \right], \right.$ $\left. \left[F_{y_i}^-, F_{y_i}^+ \right] \right\rangle$ $(i = 1,2,\cdots,n)$ be two collections of IVNNs. When $T_{x_i}^- \le T_{y_i}^-$, $T_{x_i}^+ \le T_{y_i}^+$, $I_{x_i}^- \ge I_{y_i}^-$, $I_{x_i}^+ \ge I_{y_i}^+$, $F_{x_i}^- \ge F_{y_i}^-$ and $F_{x_i}^+ \ge F_{y_i}^+$ for all i,

$$IVNBM^{p,q}(x_1, x_2, \cdots, x_n) \le IVNBM^{p,q}(y_1, y_2, \cdots, y_n). \tag{6}$$

(5) (**Commutativity**) Let $\left(x_1, x_2, \cdots, x_n \right)$ be any permutation of $(x_1, x_2, \cdots, x_n)$, $IVNBM^{p,q}\left(x_1, x_2, \cdots, x_n \right) = IVNBM^{p,q}(x_1, x_2, \cdots, x_n)$.

(6) (**Boundedness**) Let $x_i = \left\langle \left[T_i^-, T_i^+ \right], \left[I_i^-, I_i^+ \right], \left[F_i^-, F_i^+ \right] \right\rangle$ $(i = 1,2,\cdots,n)$ be a collection of IVNNs, and $x^- = \left\langle \left[\min_i \left\{ T_i^- \right\}, \min_i \left\{ T_i^+ \right\} \right], \left[\max_i \left\{ I_i^- \right\}, \max_i \left\{ I_i^+ \right\} \right], \left[\max_i \left\{ F_i^- \right\}, \max_i \left\{ F_i^+ \right\} \right] \right\rangle$, $x^+ = \left\langle \left[\max_i \left\{ T_i^- \right\}, \max_i \left\{ T_i^+ \right\} \right], \right.$ $\left. \left[\min_i \left\{ I_i^- \right\}, \min_i \left\{ I_i^+ \right\} \right], \left[\min_i \left\{ F_i^- \right\}, \min_i \left\{ F_i^+ \right\} \right] \right\rangle$. We can obtain that $x^- \le IVNBM^{p,q}(x_1, x_2, \cdots, x_n) \le x^+$.

Proof. Since $x_i \ge x^-$, according to Equation (5) and Inequality (6), we have $IVNBM^{p,q}\left(x_1, x_2, \cdots, x_n \right) \ge$ $IVNBM^{p,q}\left(x^-, x^-, \cdots, x^- \right) = x^-$. Likewise, we can obtain that $IVNBM^{p,q}\left(x_1, x_2, \cdots, x_n \right) \le$ $IVNBM^{p,q}\left(x^+, x^+, \cdots, x^+ \right) = x^+$. Then, $x^- \le IVNBM^{p,q}(x_1, x_2, \cdots, x_n) \le x^+$.

In the following part, we discuss some special cases of IVNBM.

1. When $q \to 0$, from Equation (3) and (4), IVNBM reduces to the generalized interval-valued neutrosophic average (GIVNA) operator as follows:

$$\lim_{q\to 0} IVNBM^{p,q}(x_1,x_2,\cdots,x_n)=\lim_{q\to 0}\left(\frac{1}{n(n-1)}\underset{\substack{i,j=1\\ i\neq j}}{\overset{n}{\oplus}}\left(x_i^p\otimes x_j^q\right)\right)^{\frac{1}{p+q}}=\left(\frac{1}{n}\overset{n}{\underset{i=1}{\oplus}}\left(x_i^p\right)\right)^{\frac{1}{p}}$$

$$=\left(\left[\left(1-\prod_{i=1}^{n}\left(1-\left(T_i^-\right)^p\right)^{\frac{1}{n}}\right)^{\frac{1}{p}},\left(1-\prod_{i=1}^{n}\left(1-\left(T_i^+\right)^p\right)^{\frac{1}{n}}\right)^{\frac{1}{p}}\right],\right.$$

$$\left[1-\left(1-\prod_{i=1}^{n}\left(1-\left(1-I_i^-\right)^p\right)^{\frac{1}{n}}\right)^{\frac{1}{p}},1-\left(1-\prod_{i=1}^{n}\left(1-\left(1-I_i^+\right)^p\right)^{\frac{1}{n}}\right)^{\frac{1}{p}}\right],$$

$$\left.\left[1-\left(1-\prod_{i=1}^{n}\left(1-\left(1-F_i^-\right)^p\right)^{\frac{1}{n}}\right)^{\frac{1}{p}},1-\left(1-\prod_{i=1}^{n}\left(1-\left(1-F_i^+\right)^p\right)^{\frac{1}{n}}\right)^{\frac{1}{p}}\right]\right)$$

$$=IVNBM^{p,0}(x_1,x_2,\cdots,x_n).$$

2. When $p=2$ and $q\to 0$, IVNBM reduces to the interval-valued neutrosophic square average (IVNSA) operator as follows:

$$IVNBM^{2,0}(x_1,x_2,\cdots,x_n)=\left(\frac{1}{n}\overset{n}{\underset{i=1}{\oplus}}\left(x_i^2\right)\right)^{\frac{1}{2}}$$

$$=\left(\left[\left(1-\prod_{i=1}^{n}\left(1-\left(T_i^-\right)^2\right)^{\frac{1}{n}}\right)^{\frac{1}{2}},\left(1-\prod_{i=1}^{n}\left(1-\left(T_i^+\right)^2\right)^{\frac{1}{n}}\right)^{\frac{1}{2}}\right],\right.$$

$$\left[1-\left(1-\prod_{i=1}^{n}\left(1-\left(1-I_i^-\right)^2\right)^{\frac{1}{n}}\right)^{\frac{1}{2}},1-\left(1-\prod_{i=1}^{n}\left(1-\left(1-I_i^+\right)^2\right)^{\frac{1}{n}}\right)^{\frac{1}{2}}\right],$$

$$\left.\left[1-\left(1-\prod_{i=1}^{n}\left(1-\left(1-F_i^-\right)^2\right)^{\frac{1}{n}}\right)^{\frac{1}{2}},1-\left(1-\prod_{i=1}^{n}\left(1-\left(1-F_i^+\right)^2\right)^{\frac{1}{n}}\right)^{\frac{1}{2}}\right]\right).$$

3. When $p=1$ and $q\to 0$, IVNBM reduces to the interval-valued neutrosophic average (IVNA) operator as follows:

$$IVNBM^{1,0}(x_1,x_2,\cdots,x_n)=\frac{1}{n}\overset{n}{\underset{i=1}{\oplus}}(x_i)$$

$$=\left(\left[1-\prod_{i=1}^{n}\left(1-\left(T_i^-\right)\right)^{\frac{1}{n}},1-\prod_{i=1}^{n}\left(1-\left(T_i^+\right)\right)^{\frac{1}{n}}\right],\left[\prod_{i=1}^{n}\left(I_i^-\right)^{\frac{1}{n}},\prod_{i=1}^{n}\left(I_i^+\right)^{\frac{1}{n}}\right],\left[\prod_{i=1}^{n}\left(F_i^-\right)^{\frac{1}{n}},\prod_{i=1}^{n}\left(F_i^+\right)^{\frac{1}{n}}\right]\right).$$

4. When $p=q=1$, IVNBM reduces to the interval-valued neutrosophic interrelated average (IVNIA) operator as follows:

$$IVNBM^{1,1}\left(x_1, x_2, \cdots, x_n\right) = \left(\frac{1}{n(n-1)} \underset{\substack{i,j=1 \\ i\neq j}}{\overset{n}{\oplus}} \left(x_i \otimes x_j\right)\right)^{\frac{1}{2}}$$

$$= \left(\left[\left(1-\prod_{\substack{i,j=1\\ i\neq j}}^{n}\left(1-\left(T_i^-\right)\left(T_j^-\right)\right)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{2}}, \left(1-\prod_{\substack{i,j=1\\ i\neq j}}^{n}\left(1-\left(T_i^+\right)\left(T_j^+\right)\right)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{2}}\right],\right.$$

$$\left[1-\left(1-\prod_{\substack{i,j=1\\ i\neq j}}^{n}\left(1-\left(1-I_i^-\right)\left(1-I_j^-\right)\right)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{2}}, 1-\left(1-\prod_{\substack{i,j=1\\ i\neq j}}^{n}\left(1-\left(1-I_i^+\right)\left(1-I_j^+\right)\right)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{2}}\right],$$

$$\left.\left[1-\left(1-\prod_{\substack{i,j=1\\ i\neq j}}^{n}\left(1-\left(1-F_i^-\right)\left(1-F_j^-\right)\right)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{2}}, 1-\left(1-\prod_{\substack{i,j=1\\ i\neq j}}^{n}\left(1-\left(1-F_i^+\right)\left(1-F_j^+\right)\right)^{\frac{1}{n(n-1)}}\right)^{\frac{}{2}}\right]\right).$$

3.2 IVNWBM

Definition 6. Let p, $q \geq 0$ and $x_i = \left\langle \left[T_i^-, T_i^+\right], \left[I_i^-, I_i^+\right], \left[F_i^-, F_i^+\right] \right\rangle$ $\left(i = 1, 2, \cdots, n\right)$ be a collection of IVNNs. $w = \left(w_1, w_2, \cdots, w_n\right)^T$ is the weight vector of x_i $\left(i = 1, 2, \cdots, n\right)$ where $w_i > 0$ $\left(i = 1, 2, \cdots, n\right)$ and $\sum_{i=1}^{n} w_i = 1$. IVNWBM can be defined as:

$$IVNWBM^{p,q}(x_1, x_2, \cdots, x_n) = \left(\frac{1}{n(n-1)} \underset{\substack{i,j=1 \\ i\neq j}}{\overset{n}{\oplus}} \left(\left(w_i x_i\right)^p \otimes \left(w_j x_j\right)^q\right)\right)^{\frac{1}{p+q}}. \tag{7}$$

Theorem 2. Let p, $q \geq 0$ and $x_i = \left\langle \left[T_i^-, T_i^+\right], \left[I_i^-, I_i^+\right], \left[F_i^-, F_i^+\right] \right\rangle$ $\left(i = 1, 2, \cdots, n\right)$ be a collection of IVNNs. $w = \left(w_1, w_2, \cdots, w_n\right)^T$ is the weight vector of x_i $\left(i = 1, 2, \cdots, n\right)$ where $w_i > 0$ $\left(i = 1, 2, \cdots, n\right)$ and $\sum_{i=1}^{n} w_i = 1$. The aggregated value by IVNWBM in Equation (7) is also an IVNN, and

$$IVNWBM^{p,q}\left(x_1,x_2,\cdots,x_n\right)$$
$$=\left(\left[\left(1-\prod_{\substack{i,j=1\\ i\neq j}}^{n}\left(1-\left(1-\left(1-T_i^-\right)^{w_i}\right)^p\left(1-\left(1-T_j^-\right)^{w_j}\right)^q\right)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{p+q}},\right.\right.$$
$$\left.\left(1-\prod_{\substack{i,j=1\\ i\neq j}}^{n}\left(1-\left(1-\left(1-T_i^+\right)^{w_i}\right)^p\left(1-\left(1-T_j^+\right)^{w_j}\right)^q\right)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{p+q}}\right],$$
$$\left[1-\left(1-\prod_{\substack{i,j=1\\ i\neq j}}^{n}\left(1-\left(1-\left(I_i^-\right)^{w_i}\right)^p\left(1-\left(I_j^-\right)^{w_j}\right)^q\right)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{p+q}},\right.$$
$$\left.1-\left(1-\prod_{\substack{i,j=1\\ i\neq j}}^{n}\left(1-\left(1-\left(I_i^+\right)^{w_i}\right)^p\left(1-\left(I_j^+\right)^{w_j}\right)^q\right)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{p+q}}\right],$$
$$\left[1-\left(1-\prod_{\substack{i,j=1\\ i\neq j}}^{n}\left(1-\left(1-\left(F_i^-\right)^{w_i}\right)^p\left(1-\left(F_j^-\right)^{w_j}\right)^q\right)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{p+q}},\right. \tag{8}$$
$$\left.\left.1-\left(1-\prod_{\substack{i,j=1\\ i\neq j}}^{n}\left(1-\left(1-\left(F_i^+\right)^{w_i}\right)^p\left(1-\left(F_j^+\right)^{w_j}\right)^q\right)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{p+q}}\right]\right).$$

Proof is given in appendix.

3.3 IVNGBM

Definition 7. Let p, $q\geq 0$ and $x_i=\left\langle\left[T_i^-,T_i^+\right],\left[I_i^-,I_i^+\right],\left[F_i^-,F_i^+\right]\right\rangle$ $\left(i=1,2,\cdots,n\right)$ be a collection of IVNNs. IVNGBM can be defined as:

$$IVNGBM^{p,q}(x_1,x_2,\cdots,x_n)=\frac{1}{p+q}\underset{\substack{i,j=1\\ i\neq j}}{\overset{n}{\otimes}}\left(px_i\oplus qx_j\right)^{\frac{1}{n(n-1)}}. \tag{9}$$

Theorem 3. Let p, $q\geq 0$ and $x_i=\left\langle\left[T_i^-,T_i^+\right],\left[I_i^-,I_i^+\right],\left[F_i^-,F_i^+\right]\right\rangle$ $\left(i=1,2,\cdots,n\right)$ be a collection of IVNNs, then the aggregated value by IVNGBM in Equation (9) is also an IVNN, and

$$IVNGBM^{p,q}(x_1,x_2,\cdots,x_n)$$
$$=\left\langle\left[1-\left(1-\prod_{\substack{i,j=1\\i\neq j}}^{n}\left(1-(1-T_i^-)^p(1-T_j^-)^q\right)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{p+q}},1-\left(1-\prod_{\substack{i,j=1\\i\neq j}}^{n}\left(1-(1-T_i^+)^p(1-T_j^+)^q\right)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{p+q}}\right],\right.$$
$$\left[\left(1-\prod_{\substack{i,j=1\\i\neq j}}^{n}\left(1-\left(I_i^-\right)^p\left(I_j^-\right)^q\right)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{p+q}},\left(1-\prod_{\substack{i,j=1\\i\neq j}}^{n}\left(1-\left(I_i^+\right)^p\left(I_j^+\right)^q\right)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{p+q}}\right], \tag{10}$$
$$\left.\left[\left(1-\prod_{\substack{i,j=1\\i\neq j}}^{n}\left(1-\left(F_i^-\right)^p\left(F_j^-\right)^q\right)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{p+q}},\left(1-\prod_{\substack{i,j=1\\i\neq j}}^{n}\left(1-\left(F_i^+\right)^p\left(F_j^+\right)^q\right)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{p+q}}\right]\right\rangle.$$

Proof is given in appendix.

In the following part, we investigate some properties of IVNGBM:

(1) When $x_i=\langle[1,1],[0,0],[0,0]\rangle$ $(i=1,2,\cdots,n)$, $IVNGBM^{p,q}(x_1,x_2,\cdots,x_n)=\langle[1,1],[0,0],[0,0]\rangle$.

(2) When $x_i=\langle[0,0],[1,1],[1,1]\rangle$ $(i=1,2,\cdots,n)$, $IVNGBM^{p,q}(x_1,x_2,\cdots,x_n)=\langle[0,0],[1,1],[1,1]\rangle$.

(3) (**Idempotency**) When all IVNNs x_i $(i=1,2,\cdots,n)$ are equal, i.e., $x_i=x$ for all i,

$$IVNGBM^{p,q}(x_1,x_2,\cdots,x_n)=x. \tag{11}$$

Proof. Since $x_i=x$ for all i, we can obtain that $IVNGBM^{p,q}(x_1,x_2,\cdots,x_n)=\frac{1}{p+q}\underset{\substack{i,j=1\\i\neq j}}{\overset{n}{\otimes}}\left(px_i\oplus qx_j\right)^{\frac{1}{n(n-1)}}=\frac{1}{p+q}\left(\left((p+q)x\right)^{\frac{1}{n(n-1)}}\right)^{n(n-1)}=x$.

(4) (**Monotonicity**) Let $x_i=\left\langle\left[T_{x_i}^-,T_{x_i}^+\right],\left[I_{x_i}^-,I_{x_i}^+\right],\left[F_{x_i}^-,F_{x_i}^+\right]\right\rangle$ $(i=1,2,\cdots,n)$ and $y_i=\left\langle\left[T_{y_i}^-,T_{y_i}^+\right],\left[I_{y_i}^-,I_{y_i}^+\right],\right.$ $\left.\left[F_{y_i}^-,F_{y_i}^+\right]\right\rangle$ $(i=1,2,\cdots,n)$ be two collections of IVNNs. If $T_{x_i}^-\le T_{y_i}^-$, $T_{x_i}^+\le T_{y_i}^+$, $I_{x_i}^-\ge I_{y_i}^-$, $I_{x_i}^+\ge I_{y_i}^+$, $F_{x_i}^-\ge F_{y_i}^-$ and $F_{x_i}^+\ge F_{y_i}^+$ for all i, we can obtain that

$$IVNGBM^{p,q}(x_1,x_2,\cdots,x_n)\le IVNGBM^{p,q}(y_1,y_2,\cdots,y_n). \tag{12}$$

(5) (**Commutativity**) Let $\left(x_1,x_2,\cdots,x_n\right)$ be any permutation of $(x_1,x_2,\cdots,x_n)$, then $IVNGBM^{p,q}\left(x_1,x_2,\cdots,x_n\right)=IVNGBM^{p,q}(x_1,x_2,\cdots,x_n)$.

(6) (**Boundedness**) Let $x_i=\left\langle\left[T_i^-,T_i^+\right],\left[I_i^-,I_i^+\right],\left[F_i^-,F_i^+\right]\right\rangle$ $(i=1,2,\cdots,n)$ be a collection of IVNNs, $x^-=\left\langle\left[\min_i\{T_i^-\},\min_i\{T_i^+\}\right],\left[\max_i\{I_i^-\},\max_i\{I_i^+\}\right],\left[\max_i\{F_i^-\},\max_i\{F_i^+\}\right]\right\rangle$ and $x^+=\left\langle\left[\max_i\{T_i^-\},\right.\right.$ $\left.\max_i\{T_i^+\}\right],\left[\min_i\{I_i^-\},\min_i\{I_i^+\}\right],\left.\left[\min_i\{F_i^-\},\min_i\{F_i^+\}\right]\right\rangle$, it is true that $x^-\le IVNGBM^{p,q}(x_1,x_2,\cdots,x_n)\le x^+$.

Proof. Since $x_i\ge x^-$, according to Equation (11) and inequality (12), we have $IVNGBM^{p,q}(x_1,x_2,\cdots,x_n)$ $\ge IVNGBM^{p,q}\left(x^-,x^-,\cdots,x^-\right)=x^-$. Likewise, we can obtain $IVNGBM^{p,q}(x_1,x_2,\cdots,x_n)\le$ $IVNGBM^{p,q}\left(x^+,x^+,\cdots,x^+\right)=x^+$. Therefore, $x^-\le IVNGBM^{p,q}(x_1,x_2,\cdots,x_n)\le x^+$.

In the following part, we discuss some special cases of IVNGBM.

1. When $q \to 0$, from Equation (9) and (10), IVNGBM reduces to the generalized interval-valued neutrosophic geometric average (GIVNGA) operator as follows:

$$\lim_{q\to 0} IVNGBM^{p,q}(x_1,x_2,\cdots,x_n)=\lim_{q\to 0}\frac{1}{p+q}\underset{\substack{i,j=1\\ i\neq j}}{\overset{n}{\otimes}}\left(px_i\oplus qx_j\right)^{\frac{1}{n(n-1)}}=\frac{1}{p}\overset{n}{\underset{i=1}{\otimes}}\left(px_i\right)^{\frac{1}{n}}$$

$$=\left\langle\left[1-\left(1-\prod_{i=1}^{n}\left(1-(1-T_i^-)^p\right)^{\frac{1}{n}}\right)^{\frac{1}{p}},1-\left(1-\prod_{i=1}^{n}\left(1-(1-T_i^+)^p\right)^{\frac{1}{n}}\right)^{\frac{1}{p}}\right],\right.$$

$$\left[\left(1-\prod_{i=1}^{n}\left(1-\left(I_i^-\right)^p\right)^{\frac{1}{n}}\right)^{\frac{1}{p}},\left(1-\prod_{i=1}^{n}\left(1-\left(I_i^+\right)^p\right)^{\frac{1}{n}}\right)^{\frac{1}{p}}\right],\left[\left(1-\prod_{i=1}^{n}\left(1-\left(F_i^-\right)^p\right)^{\frac{1}{n}}\right)^{\frac{1}{p}},\right.$$

$$\left.\left.\left(1-\prod_{i=1}^{n}\left(1-\left(F_i^+\right)^p\right)^{\frac{1}{n}}\right)^{\frac{1}{p}}\right]\right\rangle.$$

2. When $p=2$ and $q\to 0$, IVNBM reduces to the interval-valued neutrosophic square geometric average (IVNSGA) operator as follows:

$$IVNBM^{2,0}(x_1,x_2,\cdots,x_n)=\frac{1}{2}\underset{\substack{i,j=1\\ i\neq j}}{\overset{n}{\otimes}}\left(2x_i\right)^{\frac{1}{n(n-1)}}=\frac{1}{2}\overset{n}{\underset{i=1}{\otimes}}\left(2x_i\right)^{\frac{1}{n}}$$

$$=\left\langle\left[1-\left(1-\prod_{i=1}^{n}\left(1-(1-T_i^-)^2\right)^{\frac{1}{n}}\right)^{\frac{1}{2}},1-\left(1-\prod_{i=1}^{n}\left(1-(1-T_i^+)^2\right)^{\frac{1}{n}}\right)^{\frac{1}{2}}\right],\right.$$

$$\left[\left(1-\prod_{i=1}^{n}\left(1-\left(I_i^-\right)^2\right)^{\frac{1}{n}}\right)^{\frac{1}{2}},\left(1-\prod_{i=1}^{n}\left(1-\left(I_i^+\right)^2\right)^{\frac{1}{n}}\right)^{\frac{1}{2}}\right],\left[\left(1-\prod_{i=1}^{n}\left(1-\left(F_i^-\right)^2\right)^{\frac{1}{n}}\right)^{\frac{1}{2}},\right.$$

$$\left.\left.\left(1-\prod_{i=1}^{n}\left(1-\left(F_i^+\right)^2\right)^{\frac{1}{n}}\right)^{\frac{1}{2}}\right]\right\rangle.$$

3. When $p=1$ and $q\to 0$, IVNGBM reduces to the interval-valued neutrosophic geometric average (IVNGA) operator as follows:

$$IVNGBM^{1,0}(x_1,x_2,\cdots,x_n)=\overset{n}{\underset{i=1}{\otimes}}\left(x_i\right)^{\frac{1}{n}}$$

$$=\left\langle\left[\prod_{i=1}^{n}\left(T_i^-\right)^{\frac{1}{n}},\prod_{i=1}^{n}\left(T_i^+\right)^{\frac{1}{n}}\right],\left[\left(1-\prod_{i=1}^{n}\left(1-I_i^-\right)^{\frac{1}{n}}\right),\left(1-\prod_{i=1}^{n}\left(1-I_i^+\right)^{\frac{1}{n}}\right)\right],\right.$$

$$\left.\left[\left(1-\prod_{i=1}^{n}\left(1-F_i^-\right)^{\frac{1}{n}}\right),\left(1-\prod_{i=1}^{n}\left(1-F_i^+\right)^{\frac{1}{n}}\right)\right]\right\rangle.$$

4. When $p=q=1$, IVNGBM reduces to the interval-valued neutrosophic interrelated square geometric average (IVNISGA) operator as follows:

$$IVNGBM^{1,1}\left(x_1,x_2,\cdots,x_n\right)=\frac{1}{2}\underset{\substack{i,j=1\\i\neq j}}{\overset{n}{\otimes}}\left(x_i\oplus x_j\right)^{\frac{1}{n(n-1)}}$$

$$=\left\langle\left[1-\left(1-\prod_{\substack{i,j=1\\i\neq j}}^{n}\left(1-(1-T_i^-)(1-T_j^-)\right)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{2}},1-\left(1-\prod_{\substack{i,j=1\\i\neq j}}^{n}\left(1-(1-T_i^+)(1-T_j^+)\right)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{2}}\right],\right.$$

$$\left[\left(1-\prod_{\substack{i,j=1\\i\neq j}}^{n}\left(1-\left(I_i^-\right)\left(I_j^-\right)\right)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{2}},\left(1-\prod_{\substack{i,j=1\\i\neq j}}^{n}\left(1-\left(I_i^+\right)\left(I_j^+\right)\right)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{2}}\right],$$

$$\left.\left[\left(1-\prod_{\substack{i,j=1\\i\neq j}}^{n}\left(1-\left(F_i^-\right)\left(F_j^-\right)\right)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{2}},\left(1-\prod_{\substack{i,j=1\\i\neq j}}^{n}\left(1-\left(F_i^+\right)\left(F_j^+\right)\right)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{2}}\right]\right\rangle.$$

3.4 IVNWGBM

Definition 8. Let p , $q\geq 0$ and $x_i=\left\langle\left[T_i^-,T_i^+\right],\left[I_i^-,I_i^+\right],\left[F_i^-,F_i^+\right]\right\rangle$ $\left(i=1,2,\cdots,n\right)$ be a collection of IVNNs. $w=\left(w_1,w_2,\cdots,w_n\right)^T$ is the weight vector of x_i $\left(i=1,2,\cdots,n\right)$ where $w_i>0$ $\left(i=1,2,\cdots,n\right)$ and $\sum_{i=1}^{n}w_i=1$. IVNWGBM can be defined as:

$$IVNWGBM^{p,q}\left(x_1,x_2,\cdots,x_n\right)=\frac{1}{p+q}\left(\underset{\substack{i,j=1\\i\neq j}}{\overset{n}{\otimes}}\left(px_i^{w_i}\oplus qx_j^{w_j}\right)^{\frac{1}{n(n-1)}}\right). \tag{13}$$

Theorem 4. Let p , $q\geq 0$ and $x_i=\left\langle\left[T_i^-,T_i^+\right],\left[I_i^-,I_i^+\right],\left[F_i^-,F_i^+\right]\right\rangle$ $\left(i=1,2,\cdots,n\right)$ be a collection of IVNNs. $w=\left(w_1,w_2,\cdots,w_n\right)^T$ is the weight vector of x_i $\left(i=1,2,\cdots,n\right)$ where $w_i>0$ $\left(i=1,2,\cdots,n\right)$ and $\sum_{i=1}^{n}w_i=1$. The aggregated value by IVNWGBM in (13) is also an IVNN, and

$$
\begin{aligned}
&IVNWGBM^{p,q}\left(x_1,x_2,\cdots,x_n\right)\\
&=\Bigg\langle\Bigg[1-\left(1-\prod_{\substack{i,j=1\\ i\neq j}}^{n}\left(1-\left(1-\left(T_i^-\right)^{w_i}\right)^p\left(1-\left(T_j^-\right)^{w_j}\right)^q\right)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{p+q}},\\
&\quad 1-\left(1-\prod_{\substack{i,j=1\\ i\neq j}}^{n}\left(1-\left(1-\left(T_i^+\right)^{w_i}\right)^p\left(1-\left(T_j^+\right)^{w_j}\right)^q\right)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{p+q}}\Bigg],\\
&\quad\Bigg[\left(1-\prod_{\substack{i,j=1\\ i\neq j}}^{n}\left(1-\left(1-\left(1-I_i^-\right)^{w_i}\right)^p\left(1-\left(1-I_j^-\right)^{w_j}\right)^q\right)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{p+q}},\\
&\quad\left(1-\prod_{\substack{i,j=1\\ i\neq j}}^{n}\left(1-\left(1-\left(1-I_i^+\right)^{w_i}\right)^p\left(1-\left(1-I_j^+\right)^{w_j}\right)^q\right)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{p+q}}\Bigg],\\
&\quad\Bigg[\left(1-\prod_{\substack{i,j=1\\ i\neq j}}^{n}\left(1-\left(1-\left(1-F_i^-\right)^{w_i}\right)^p\left(1-\left(1-F_j^-\right)^{w_j}\right)^q\right)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{p+q}},\\
&\quad\left(1-\prod_{\substack{i,j=1\\ i\neq j}}^{n}\left(1-\left(1-\left(1-F_i^+\right)^{w_i}\right)^p\left(1-\left(1-F_j^+\right)^{w_j}\right)^q\right)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{p+q}}\Bigg]\Bigg\rangle.
\end{aligned}
\tag{14}
$$

Proof is given in appendix.

3.5 PROCEDURES OF THE PROPOSED APPROACHES

Here we present our novel MCDM approaches for the selection of renewable energy based on the WBM (or the WGBM) for IVNNs.

Assume there are m alternatives $A=\{A_1,\ A_2,\ \cdots,\ A_m\}$ and n criteria $C=\{C_1,\ C_2,\ \cdots,\ C_n\}$, whose subjective weight vector provided by the decision maker is $w=(w_1,w_2,\cdots,w_n)$, where $w_j\geq 0$ ($j=1,2,\cdots,n$) and $\sum_{j=1}^{n}w_j=1$. Let $U=\left(a_{ij}\right)_{m\times n}$ be the interval-valued neutrosophic decision matrix, where $a_{ij}=\left\langle T_{a_{ij}},I_{a_{ij}},F_{a_{ij}}\right\rangle$ is an evaluation value, denoted by IVNN, where $T_{a_{ij}}=\left\lfloor \inf T_{a_{ij}},\sup T_{a_{ij}}\right\rfloor$ indicates the truth-membership function that the alternative A_i satisfies the criterion C_j, $I_{a_{ij}}=\left\lfloor \inf I_{a_{ij}},\sup I_{a_{ij}}\right\rfloor$ indicates the indeterminacy-membership function that the alternative A_i satisfies the criterion C_j and $F_{a_{ij}}=\left\lfloor \inf F_{a_{ij}},\sup F_{a_{ij}}\right\rfloor$ indicates the falsity-membership function that the alternative A_i satisfies the criterion C_j.

In the following part, the proposed MCDM approach to rank and select the most desirable alternative(s) is based upon IVNWBM (or IVNWGBM) and its procedures are as follows:

Step 1: Normalize the decision matrix.

Criteria can be divided into two types: benefit criterion and cost criterion. The bigger the value of an alternative under a benefit criterion is, the better the attribute will be; conversely, the smaller the value of an alternative under a cost criterion is, the better the alternative is.

To unify all criteria, the decision matrix needs to be normalized, and the normalized decision matrix $N = \left(b_{ij}\right)^{n\times m}$ can be obtained by:

$$b_{ij} = \begin{cases} a_{ij} & \text{if } C_j \text{ is a benefit criterion} \\ neg\left(a_{ij}\right) & \text{if } C_j \text{ is a cost criterion} \end{cases}. \tag{15}$$

Step 2: Calculate the overall performance value r_i $\left(i = 1, 2, \cdots, m\right)$ of alternative A_i.

The overall performance value r_i can be computed by making use of IVNWBM or IVNWGBM.

Step 3: Calculate the score value s_i of the collective IVNN r_i $\left(i = 1, 2, \cdots, m\right)$.

According to the score function of IVNN defined in Definition 1, we can obtain the score value s_i of each collective IVNN r_i utilizing Equation (1).

Step 4: Calculate the accuracy value a_i of the collective IVNN r_i $\left(i = 1, 2, \cdots, m\right)$.

According to the score function of IVNN defined in Definition 2, we can get the accuracy value a_i of each collective IVNN r_i utilizing Equation (2).

Step 5: Rank the alternatives according to the comparative method of IVNNs.

According to the comparative method defined in Definition 3, we can derive the final ranking of alternatives.

4. EXAMPLE AND COMPARATIVE ANALYSIS

4.1 NUMERICAL EXAMPLE

In this subsection, a numerical example for the MCDM problem with IVNNs is used to demonstrate the applicability of the proposed decision-making approaches.

The following example about the selection of renewable energy is adapted from Ref. (Yazdani-Chamzini et al., 2013).

A government intends to select one kind of renewable energy to use for the sustainable development of local economy. After preliminary selection, there are three kinds of renewable energy: (1) solar energy (A_1); (2) wind energy (A_2); (3) hydraulic energy (A_3). These three kinds of renewable energy are assessed by experts with respect to seven criteria: (1) power (C_1); (2) investment ratio (C_2); and (3) implementation period (C_3); (4) operating hours (C_4); (5) useful life (C_5); (6) operation and maintenance costs (C_6); (7) emissions of CO_2 avoided per year (C_7). The criteria of C_1, C_4, C_5 and C_6 are benefit ones while the rest three criteria are cost ones. Moreover, these seven criteria are correlative. The weight vector of the criteria is calculated by Yazdani-Chamzini (Yazdani-Chamzini et al., 2013) as $w = (0.319, 0.09, 0.026, 0.116, 0.134, 0.042, 0.273)$. In order to reflect the reality more accurately and obtain more fuzzy and uncertain information, we transform the evaluation values provided by experts into IVNNs, as shown in Table 1.

Table1: **The evaluation information**

	A_1	A_2	A_3
C_1	$\langle[0.7,0.8],[0.3,0.4],[0.4,0.5]\rangle$	$\langle[0.7,0.9],[0.2,0.4],[0.4,0.6]\rangle$	$\langle[0.7,0.9],[0.2,0.3],[0.4,0.5]\rangle$
C_2	$\langle[0.2,0.3],[0.8,0.9],[0.6,0.7]\rangle$	$\langle[0.2,0.3],[0.6,0.7],[0.6,0.7]\rangle$	$\langle[0.3,0.6],[0.3,0.5],[0.8,0.9]\rangle$
C_3	$\langle[0.3,0.4],[0.6,0.9],[0.7,0.8]\rangle$	$\langle[0.3,0.4],[0.6,0.7],[0.5,0.6]\rangle$	$\langle[0.4,0.5],[0.6,0.8],[0.7,0.9]\rangle$

C_4	$\langle[0.6,0.8],[0.1,0.2],[0.3,0.4]\rangle$	$\langle[0.8,0.9],[0.1,0.3],[0.3,0.4]\rangle$	$\langle[0.8,0.9],[0.3,0.4],[0.1,0.2]\rangle$
C_5	$\langle[0.8,0.9],[0.1,0.2],[0.2,0.3]\rangle$	$\langle[0.8,0.9],[0.3,0.5],[0.4,0.6]\rangle$	$\langle[0.8,0.9],[0.4,0.5],[0.3,0.4]\rangle$
C_6	$\langle[0.8,0.9],[0.5,0.6],[0.1,0.2]\rangle$	$\langle[0.5,0.8],[0.1,0.2],[0.3,0.4]\rangle$	$\langle[0.8,1],[0.1,0.3],[0.1,0.2]\rangle$
C_7	$\langle[0.2,0.3],[0.8,0.9],[0.9,1]\rangle$	$\langle[0.2,0.4],[0.5,0.7],[0.8,0.9]\rangle$	$\langle[0.1,0.2],[0.7,0.9],[0.7,0.8]\rangle$

Assume $p=q=1$, we firstly utilize IVNWBM to solve the above MCDM problem about the selection of renewable energy, and the procedure is shown as follows:

Step 1: Normalize the decision matrix.

Since the criteria of C_1, C_4, C_5 and C_6 are benefit ones while the criteria C_2, C_3, and C_7 are cost ones, the decision matrix can be normalized utilizing Equation (15), and the normalized decision information are shown in Table 2.

Table2: Normalized evaluation information

	A_1	A_2	A_3
C_1	$\langle[0.7,0.8],[0.3,0.4],[0.4,0.5]\rangle$	$\langle[0.7,0.9],[0.2,0.4],[0.4,0.6]\rangle$	$\langle[0.7,0.9],[0.2,0.3],[0.4,0.5]\rangle$
C_2	$\langle[0.6,0.7],[0.1,0.2],[0.2,0.3]\rangle$	$\langle[0.6,0.7],[0.3,0.4],[0.2,0.3]\rangle$	$\langle[0.8,0.9],[0.5,0.7],[0.3,0.6]\rangle$
C_3	$\langle[0.7,0.8],[0.1,0.4],[0.3,0.4]\rangle$	$\langle[0.5,0.6],[0.3,0.4],[0.3,0.4]\rangle$	$\langle[0.7,0.9],[0.2,0.4],[0.4,0.5]\rangle$
C_4	$\langle[0.6,0.8],[0.1,0.2],[0.3,0.4]\rangle$	$\langle[0.8,0.9],[0.1,0.3],[0.3,0.4]\rangle$	$\langle[0.8,0.9],[0.3,0.4],[0.1,0.2]\rangle$
C_5	$\langle[0.8,0.9],[0.1,0.2],[0.2,0.3]\rangle$	$\langle[0.8,0.9],[0.3,0.5],[0.4,0.6]\rangle$	$\langle[0.8,0.9],[0.4,0.5],[0.3,0.4]\rangle$
C_6	$\langle[0.8,0.9],[0.5,0.6],[0.1,0.2]\rangle$	$\langle[0.5,0.8],[0.1,0.2],[0.3,0.4]\rangle$	$\langle[0.8,1],[0.1,0.3],[0.1,0.2]\rangle$
C_7	$\langle[0.9,1],[0.1,0.2],[0.2,0.3]\rangle$	$\langle[0.8,0.9],[0.3,0.5],[0.2,0.4]\rangle$	$\langle[0.7,0.8],[0.1,0.3],[0.1,0.2]\rangle$

Step 2: Calculate the collective overall value r_i $(i=1,2,\cdots,m)$ of alternative A_i.

Utilizing Equation (8), the collective matrix formed by the collective overall value r_i $(i=1,2,\cdots,m)$ is

$$C=\begin{pmatrix}\langle[0.1708,0.2791],[0.7824,0.8406],[0.8340,0.8723]\rangle\\ \langle[0.1598,0.2327],[0.8208,0.8873],[0.8527,0.9028]\rangle\\ \langle[0.1668,0.3710],[0.8225,0.8801],[0.8179,0.8682]\rangle\end{pmatrix}.$$

Step 3: Calculate the score value s_i of the collective IVNN r_i $(i=1,2,\cdots,m)$.

Utilizing Equation (1), the score vector can be obtained as $s=[-0.6256,-0.6948,-0.6384]$.

Step 4: Calculate the accuracy value a_i of the collective IVNN r_i $(i=1,2,\cdots,m)$.

Utilizing Equation (2), the accuracy vector can be calculated as $a=[-0.0521,-0.0339,0.0114]$.

Step 5: Rank the alternatives according to the comparative method of IVNNs.

Based on the above steps, the final order $A_1 \succ A_3 \succ A_2$ is obtained. Obviously, among the four alternatives, A_1 is the best one and A_2 is the worst one.

Then, we utilize IVNWGBM to solve the above MCDM problem, and the ranking result is obtained: $A_1 \succ A_3 \succ A_2$. It is evident that the best alternative is A_1 and the worst one is A_2.

4.2 The influence of parameters

As discussed in Ref. (Zhu & Xu, 2013), the collective IVNN for a certain alternative with IVNWBM or IVNWGBM is monotonically increasing with increasing *p* (or *q*) and is symmetric about $p = q$. In order to demonstrate the influence of the parameters *p* and *q* on the final ranking order of this numerical example, we calculate the ranking results of alternatives using different values of these two parameters. All referred values of *p* and *q* can be divided into three categories. In the first category, the value of *p* is smaller than that of *q*, the values of *p* and *q* are equal in the second category, whilst the value of *p* is bigger than that of *q* in the third category. The significant pairs of *p* and *q* and the respective final ranking results of two proposed approaches are shown in Table 3 and Table 4, respectively. When the difference between the values of *p* and *q* is big enough, the ranking result will stay stable. In Tables 1 and 2, we obtain the ranking results when the difference between the values of *p* and *q* varies to represent the influence of *p* and *q*.

Table3: Ranking results of the approach using IVNWBM with different *p* and *q*

p*, *q	**Score value** s_i	**Ranking result**
$p = 0.001, q = 1$	$s_1 = -0.5042, s_2 = -0.6683, s_3 = -0.5220$	$A_1 \succ A_3 \succ A_2$
$p = 0.1, q = 1$	$s_1 = -0.5934, s_2 = -0.6839, s_3 = -0.6031$	$A_1 \succ A_3 \succ A_2$
$p = 1, q = 2$	$s_1 = -0.5725, s_2 = -0.6568, s_3 = -0.5817$	$A_1 \succ A_3 \succ A_2$
$p = 1, q = 5$	$s_1 = -0.4235, s_2 = -0.5546, s_3 = -0.4203$	$A_3 \succ A_1 \succ A_2$
$p = 1, q = 10$	$s_1 = -0.2920, s_2 = -0.4662, s_3 = -0.2914$	$A_3 \succ A_1 \succ A_2$
$p = 0.1, q = 0.1$	$s_1 = -0.6936, s_2 = -0.7453, s_3 = -0.7014$	$A_1 \succ A_3 \succ A_2$
$p = 1, q = 1$	$s_1 = -0.6256, s_2 = -0.6948, s_3 = -0.6384$	$A_1 \succ A_3 \succ A_2$
$p = 4, q = 4$	$s_1 = -0.4684, s_2 = -0.5689, s_3 = -0.4528$	$A_3 \succ A_1 \succ A_2$
$p = 10, q = 10$	$s_1 = -0.3688, s_2 = -0.5228, s_3 = -0.3336$	$A_3 \succ A_1 \succ A_2$
$p = 0.1, q = 0$	$s_1 = -0.4697, s_2 = -0.7182, s_3 = -0.5034$	$A_1 \succ A_3 \succ A_2$
$p = 0.5, q = 0$	$s_1 = -0.4480, s_2 = -0.6971, s_3 = -0.4812$	$A_1 \succ A_3 \succ A_2$
$p = 1, q = 0$	$s_1 = -0.4192, s_2 = -0.6681, s_3 = -0.4489$	$A_1 \succ A_3 \succ A_2$
$p = 5, q = 0$	$s_1 = -0.2640, s_2 = -0.5035, s_3 = -0.2597$	$A_3 \succ A_1 \succ A_2$

As displayed in Table 3, with changeable values of *p* and *q*, the ranking result of alternatives may be slightly different. Furthermore, all score values shown in Table 1 obtained by the proposed approach using IVNWBM are smaller than 0. In addition, two different ranking results exist when the value of *p* is smaller than that of *q*. A_2 is the worst alternative in both of these two different ranking results. The best alternative is A_1 when the value of *q* is smaller than 2 while the best one is A_3 in the when the values of *p* and *q* are not smaller than 4. Two different ranking results, which are same with the ranking results in the first category, exist in the second category. When the values of *p* and *q* are smaller than 1, the best alternative is A_1 and the worst one is A_3. In the third category, A_1 is the best alternative and A_2 is the worst one when the value of *p* is not bigger than 1. There is another ranking result whose best alternative is A_3 and the worst one is A_2 in the third category.

Table4: Ranking results of the approach using the IVNWGBM with different *p* and *q*

p*, *q	**Score value** s_i	**Ranking result**
$p=0.001$, $q=1$	$s_1=0.6016$, $s_2=0.5426$, $s_3=0.5710$	$A_1 \succ A_3 \succ A_2$
$p=0.1$, $q=1$	$s_1=0.6696$, $s_2=0.6142$, $s_3=0.6397$	$A_1 \succ A_3 \succ A_2$
$p=1$, $q=2$	$s_1=0.9800$, $s_2=0.9690$, $s_3=0.9751$	$A_1 \succ A_3 \succ A_2$
$p=1$, $q=6$	$s_1=0.9996$, $s_2=0.9992$, $s_3=0.9997$,	$A_3 \succ A_1 \succ A_2$
$p=0.1$, $q=0.1$	$s_1=-0.3768$, $s_2=-0.4375$, $s_3=-0.4027$	$A_1 \succ A_3 \succ A_2$
$p=0.5$, $q=0.5$	$s_1=0.6495$, $s_2=0.5892$, $s_3=0.6135$	$A_1 \succ A_3 \succ A_2$
$p=1$, $q=1$	$s_1=0.9254$, $s_2=0.8985$, $s_3=0.9096$	$A_1 \succ A_3 \succ A_2$
$p=0.1$, $q=0$	$s_1=-0.6694$, $s_2=-0.7020$, $s_3=-0.6843$	$A_1 \succ A_3 \succ A_2$
$p=0.5$, $q=0$	$s_1=0.1470$, $s_2=0.0805$, $s_3=0.1093$	$A_1 \succ A_3 \succ A_2$
$p=1$, $q=0$	$s_1=0.5977$, $s_2=0.5418$, $s_3=0.5672$	$A_1 \succ A_3 \succ A_2$
$p=2$, $q=0$	$s_1=0.8862$, $s_2=0.8614$, $s_3=0.8792$	$A_1 \succ A_3 \succ A_2$
$p=5$, $q=0$	$s_1=0.9946$, $s_2=0.9922$, $s_3=0.9954$	$A_3 \succ A_1 \succ A_2$

As noted in Table 4, like what's shown in Table 1, when the values of *p* and *q* vary, there may be slight differences in the ranking results of alternatives. In addition, when the value of *p* equals to that of *q*, the ranking result are same. The best alternative is A_1 and the worst one is A_2. Two different ranking results exist in the first category. When the values of *p* and *q* are bigger than 6, the best alternative is A_3 and the worst one is A_2. Another ranking result in the first category is same as the ranking result in the second category and that in the third category when the value of *p* is smaller than 2. In the third category, there is another ranking result whose best alternative is A_3 and the worst one is A_4.

Moreover, all score values presented in Table 4 obtained by the proposed approach using IVNWGBM are bigger than those in Table 1 when the values of *p* and *q* are constant.

According to Tables 3 and 4, we can conclude that as the values of *p* and *q* change, the ranking results obtained by a certain approach may be different. The reason for this difference is discussed. The values of these two parameters, which are determined according to the subjective preference of decision maker, can reflect his risk preference. And it is obvious that the ranking result of alternatives may be distinct when the decision maker's risk preference varies. Therefore, the difference mentioned above, which also exists in the extant studies about BM, is reasonable. In practical, if the values of *p* and *q* are known or can be obtained by regression analysis with decision maker's available data, it is considerable to utilize the proposed approaches. Otherwise, the proposed approaches are not suitable since their ranking results may be inaccurate and volatile.

4.3 COMPARATIVE ANALYSIS

For the sake of validating the feasibility of the proposed decision-making approaches, a comparative study is conducted. The study includes two cases. The first case compares the proposed approaches with approaches proposed by Liu and Wang (Liu & Wang, 2014) and Şahin (Şahin, 2014) under single-valued neutrosophic environments. The second case compares the proposed approaches with two approaches proposed by Şahin (Şahin, 2014) and two approaches proposed by Zhang *et al*. (H. Y. Zhang et al., 2014) under interval-valued neutrosophic environments. Since the extant MCDM selection approaches (Cristóbal, 2011; Yazdani-Chamzini et al., 2013) for renewable energy cannot deal with IVNNs, the proposed approaches are not compare with approaches in Ref. (Cristóbal, 2011; Yazdani-Chamzini et al., 2013). The detail of the study is described in the following of this subsection.

Case 1: The comparative analysis under single-valued neutrosophic environments.

This case is based upon the same numerical example of MCDM problem with SVNNs in Ref. (Şahin, 2014). The ranking results of the proposed approaches are compared with that of the approaches in Refs.

(Liu & Wang, 2014; Şahin, 2014). The approaches in Ref. (Şahin, 2014) are constructed on the basis of the proposed single-valued neutrosophic weighted operators and score function. Two single-valued neutrosophic weighted operators are developed by Şahin (Şahin, 2014) including the single-valued neutrosophic weighted average (SVNWA) operator and the single-valued neutrosophic weighted geometric average (SVNWGA) operator. The approach in Ref. (Liu & Wang, 2014) utilizes the proposed single-valued neutrosophic normalized WBM (SVNNWBM) operator and the score function to rank alternatives. The ranking results of the proposed approaches and the approaches in Refs. (Liu & Wang, 2014; Şahin, 2014) are listed in Table 5.

Table5: Ranking results under single-valued neutrosophic environments

Approach	The ranking result	The best alternative(s)	The worst alternative(s)
Approach using SVNWA in Ref. (Şahin, 2014)	$A_4 \succ A_2 \succ A_3 \succ A_1$	A_4	A_1
Approach using SVNWGA in Ref. (Şahin, 2014)	$A_4 \succ A_2 \succ A_3 \succ A_1$	A_4	A_1
Approach using SVNNWBM in Ref. (Liu & Wang, 2014)	$A_4 \succ A_2 \succ A_3 \succ A_1$	A_4	A_1
The proposed approach using IVNWBM	$A_4 \succ A_2 \succ A_3 \succ A_1$	A_4	A_1
The proposed approach using IVNWGBM	$A_2 \succ A_3 \succ A_4 \succ A_1$	A_2	A_1

From Table 5, same ranking result is obtained by the approaches proposed by Şahin (Şahin, 2014) and Liu and Wang (Liu & Wang, 2014) and the proposed approach using IVNWBM. The best alternative of these approaches is A_4 and the worst one is A_1. A different ranking result is obtained by the proposed approach using IVNWGBM. The best alternative of this proposed approach is A_2 and the worst one is A_1.

In this case study, the ranking results of the approach using SVNWA in Ref. (Şahin, 2014), the approach in Ref. (Liu & Wang, 2014) and the proposed approach using IVNWBM are the same. And the same rankings of these three approaches illustrates that the proposed approach using IVNWBM can be effectively utilized to solve MCDM problems under single-valued neutrosophic environments. Different ranking results are obtained by the approach using SVNWGA in Ref. (Şahin, 2014) and the proposed approach using IVNWGBM. The reason is provided as follows. The proposed approach using IVNWGBM takes into account the interrelationships among criteria while the approach using SVNWGA in Ref. (Şahin, 2014) assumes that the criteria are independent. It is rational that the ranking results of these two approaches are different. We also explain why the ranking results of two proposed approaches are different. The proposed approach utilizing IVNWBM obtains a pessimistic result, while the proposed approach using IVNWGBM calculates an optimistic one. Therefore, the ranking results of the two proposed approaches may be different.

In general, the proposed approaches can be used to tackle MCDM problems with SVNSs while the extant SVNS approaches cannot address MCDM problems with IVNSs. From this perspective, the proposed approaches are flexible ones.

Case 2: The comparative analysis with extant interval-valued neutrosophic MCDM approaches.

This case is based upon the same numerical example of MCDM problem with IVNNs presented in Ref. (Şahin, 2014). The ranking results of the proposed approaches are compared with those of the MCDM approaches in Refs. (H. Y. Zhang et al., 2014) and (Şahin, 2014). Two approaches in Ref. (Şahin, 2014) make use of the IVNWA and IVNWG operators respectively to obtain the integrated value of each alternative considering all criteria. Two approaches proposed by Zhang *et al.* (H. Y. Zhang et al., 2014) utilize the novel IVNWA and IVNWG operators which are developed based on improved operations for

IVNSs. Additionally, the score value and the accuracy value are calculated to get the ranking list of alternatives. The ranking results of the proposed approaches and the approaches in Refs. (Şahin, 2014; H. Y. Zhang et al., 2014) are listed in Table 6.

Table6: Ranking results under interval-valued neutrosophic environments

Approach	The ranking result	The best alternative(s)	The worst alternative(s)
Approach using IVNWA in Ref. (Şahin, 2014)	$A_4 \succ A_1 \succ A_2 \succ A_3$	A_4	A_3
Approach using IVNWG in Ref. (Şahin, 2014)	$A_1 \succ A_4 \succ A_2 \succ A_3$	A_1	A_3
Approach using IVNWA in Ref. (H. Y. Zhang et al., 2014)	$A_4 \succ A_1 \succ A_2 \succ A_3$	A_4	A_3
Approach using IVNWG in Ref. (H. Y. Zhang et al., 2014)	$A_1 \succ A_4 \succ A_2 \succ A_3$	A_1	A_3
The proposed approach using IVNWBM	$A_4 \succ A_1 \succ A_3 \succ A_2$	A_4	A_2
The proposed approach using IVNWGBM	$A_1 \succ A_3 \succ A_4 \succ A_2$	A_1	A_2

As shown in Table 6, the best alternative of the proposed approach using IVNWGBM and approach using IVNWG in Refs. (Şahin, 2014; H. Y. Zhang et al., 2014) is A_1 while that of the other three approaches is A_4. Moreover, two proposed approaches get the same worst alternative which is different from that obtained by the four approaches in Refs. (Şahin, 2014; H. Y. Zhang et al., 2014). The worst alternative in the proposed approaches is A_2 while that of the four approaches in Refs. (Şahin, 2014; H. Y. Zhang et al., 2014) is A_3.

The reasons why inconsistencies exist in Table 6 are provided. Firstly, the operations and comparative method in approaches in Ref. (H. Y. Zhang et al., 2014) overcome the deficiencies of those in approaches proposed by Şahin (Şahin, 2014). The ranking results of approaches in Refs. (Şahin, 2014; H. Y. Zhang et al., 2014) may be different when using same aggregation operator. From Table 6, the ranking results obtained by the approaches in Refs. (Şahin, 2014; H. Y. Zhang et al., 2014) are same when using same aggregation operator. The reason is that the differences between approaches in Refs. (Şahin, 2014; H. Y. Zhang et al., 2014) with same aggregation operator do not influence the ranking result in this study. Nevertheless, different ranking results may be obtained by the approaches in Refs. (Şahin, 2014; H. Y. Zhang et al., 2014) using same aggregation operator when the decision matrix changes. Secondly, the approaches in Ref. (Şahin, 2014) assume that criteria are independent while the proposed approaches take into account the interrelationships among criteria. What's more, the operations and comparative method utilized in the proposed approaches are different from those in the approaches in Ref. (Şahin, 2014). Therefore, it is reasonable that different ranking results can be obtained by the ranking results of the proposed approaches and the approaches in Ref. (Şahin, 2014). Thirdly, the two proposed approaches investigate the interrelationships among criteria while the two approaches in Ref. (H. Y. Zhang et al., 2014) assume criteria independent. However, criteria are usually correlative in practical MCDM problems like the selection of renewable energy. Thus, the ranking results obtained by the proposed approaches are in accord with decision makers' preferences than those obtained by the two approaches in Ref. (H. Y. Zhang et al., 2014). Fourthly, similar to what's presented in Case 1, IVNWBM can be thought as a more pessimistic operator while IVNWGBM can be thought as a more optimistic one. Thus, difference may exist in the ranking results of the two proposed approaches. In addition, it is not necessary to say which proposed approach is the best. Utilizing which approach to obtain the ranking result relies on the preference of decision maker, for instance, if a decision maker has a pessimistic nature, it may be more appropriate to utilize the proposed approach utilizing IVNWBM.

Generally speaking, the proposed approaches can be used to solve MCDM problems under single-valued neutrosophic environments and interval-valued neutrosophic environments. In addition, the proposed approaches take into consideration the interrelationships among criteria, which make them more suitable in dealing with practical MCDM problems under interval-valued neutrosophic environments than the extant approaches.

5. CONCLUSION AND FUTURE RESEARCH

In practice, the fuzziness and uncertainty often exist in the decision information provided by decision makers when selecting renewable energy, and IVNSs can depict the information. Moreover, the criteria may be interdependent in the problems of selecting renewable energy. BM is a valid tool to consider the interrelationships among criteria. Therefore, in this study, we extended BM and GBM to interval-valued neutrosophic environments, and defined IVNBM and IVNGBM. Some properties of these two operators were discussed. As IVNBM and IVNGBM do not take the relative importance of each integrated element into account, IVNWBM and IVNWGBM were proposed in this study. As well, two approaches applying IVNWBM and IVNWGBM respectively were presented to solve selection problems of renewable energy under interval-value neutrosophic environments. In addition, a numerical example about the selection of renewable energy is used to demonstrate the application of the proposed approaches. And the influence of parameters on final rankings is discussed. Subsequently, we verify the feasibility of the proposed approaches by comparing with other existing MCDM approaches.

The contributions of this paper are concluded as follows: firstly, this paper established novel approaches for the selection of renewable energy. Secondly, BM and GBM were extended into interval-valued neutrosophic environments. This theoretical extension can provide support for future other application researches. Thirdly, the proposed approaches reduce the loss of information during the processes of selecting renewable energy by utilizing IVNSs to deal with fuzzy and uncertain information. Fourthly, the proposed approaches take into consideration of the interrelationships among criteria, and the ranking results obtained by the proposed approaches are closer to decision makers' preferences than extant approaches. The feasibility and effectiveness have been proved by the comparative analysis.

Two promising directions are provided for future research. First, it is significant to apply IVNWBM and the IVMWGBM to solve problems in various other fields, such as purchasing decision-making, commodity recommendation and medical diagnosis. Second, the priority levels of criteria are different. It is worth of further study to construct a MCDM approach which considers the priority of criteria on the basis of this paper.

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APPENDIX. PROOF OF THEOREMS

Proof of Theorem 2.

According to the operations (3) and (4) in Definition 4, we have $w_i x_i = \left\langle \left[1-\left(1-T_i^-\right)^{w_i}, 1-\left(1-T_i^+\right)^{w_i}\right], \left[\left(I_i^-\right)^{w_i}, \left(I_i^+\right)^{w_i}\right], \left[\left(F_i^-\right)^{w_i}, \left(F_i^+\right)^{w_i}\right]\right\rangle$, $w_j x_j = \left\langle \left[1-\left(1-T_j^-\right)^{w_j}, 1-\left(1-T_j^+\right)^{w_j}\right], \left[\left(I_j^-\right)^{w_j}, \left(I_j^+\right)^{w_j}\right], \left[\left(F_j^-\right)^{w_j}, \left(F_j^+\right)^{w_j}\right]\right\rangle$, $\left(w_i x_i\right)^p = \left\langle \left[\left(1-\left(1-T_i^-\right)^{w_i}\right)^p, \left(1-\left(1-T_i^+\right)^{w_i}\right)^p\right], \left[1-\left(1-\left(I_i^-\right)^{w_i}\right)^p, 1-\left(1-\left(I_i^+\right)^{w_i}\right)^p\right], \left[1-\left(1-\left(F_i^-\right)^{w_i}\right)^p, 1-\left(1-\left(F_i^+\right)^{w_i}\right)^p\right]\right\rangle$ and $\left(w_j x_j\right)^{q} = \left\langle \left[\left(1-\left(1-T_j^-\right)^{w_j}\right)^q, \left(1-\left(1-T_j^+\right)^{w_j}\right)^q\right],\right.$

$\left[1-\left(1-\left(I_j^-\right)^{w_j}\right)^q, 1-\left(1-\left(I_j^+\right)^{w_j}\right)^q\right], \left[1-\left(1-\left(F_j^-\right)^{w_j}\right)^q, 1-\left(1-\left(F_j^+\right)^{w_j}\right)^q\right]\right\rangle$. Then, $\left(w_i x_i\right)^p \otimes \left(w_j x_j\right)^q =$ $\left\langle\left[\left(1-\left(1-T_i^-\right)^{w_i}\right)^p \cdot \left(1-\left(1-T_j^-\right)^{w_j}\right)^q, \left(1-\left(1-T_i^+\right)^{w_i}\right)^p \cdot \left(1-\left(1-T_j^+\right)^{w_j}\right)^q\right], \left[1-\left(1-\left(I_i^-\right)^{w_i}\right)^p \left(1-\left(I_j^-\right)^{w_j}\right)^q,\right.\right.$ $\left.1-\left(1-\left(I_i^+\right)^{w_i}\right)^p \left(1-\left(I_j^+\right)^{w_j}\right)^q\right], \left[1-\left(1-\left(F_i^-\right)^{w_i}\right)^p \left(1-\left(F_j^-\right)^{w_j}\right)^q, 1-\left(1-\left(F_i^+\right)^{w_i}\right)^p \left(1-\left(F_j^+\right)^{w_j}\right)^q\right]\right\rangle$. Let $\alpha_{ij} = \left\langle\left[T_{ij}^-, T_{ij}^+\right], \left[I_{ij}^-, I_{ij}^+\right], \left[F_{ij}^-, F_{ij}^+\right]\right\rangle = \left(w_i x_i\right)^p \otimes \left(w_j x_j\right)^q$, we can obtain that $IVNWBM^{p,q}\left(x_1, x_2, \cdots, x_n\right) =$ $\left(\frac{1}{n(n-1)} \underset{\substack{i,j=1 \\ i \neq j}}{\overset{n}{\oplus}} \left(\left(w_i x_i\right)^p \otimes \left(w_j x_j\right)^q\right)\right)^{\frac{1}{p+q}} = \left(\frac{1}{n(n-1)} \underset{\substack{i,j=1 \\ i \neq j}}{\overset{n}{\oplus}} \left(\alpha_{ij}\right)\right)^{\frac{1}{p+q}}$. By the operational laws (1) and (3) in Definition 4, $\frac{1}{n(n-1)} \underset{\substack{i,j=1 \\ i \neq j}}{\overset{n}{\oplus}} \left(\alpha_{ij}\right) = \left\langle\left[1-\prod_{\substack{i,j=1 \\ i \neq j}}^n \left(1-T_{ij}^-\right)^{\frac{1}{n(n-1)}}, 1-\prod_{\substack{i,j=1 \\ i \neq j}}^n \left(1-T_{ij}^+\right)^{\frac{1}{n(n-1)}}\right], \left[\prod_{\substack{i,j=1 \\ i \neq j}}^n \left(I_{ij}^-\right)^{\frac{1}{n(n-1)}}, \prod_{\substack{i,j=1 \\ i \neq j}}^n \left(I_{ij}^+\right)^{\frac{1}{n(n-1)}}\right],\right.$ $\left.\left[\prod_{\substack{i,j=1 \\ i \neq j}}^n \left(F_{ij}^-\right)^{\frac{1}{n(n-1)}}, \prod_{\substack{i,j=1 \\ i \neq j}}^n \left(F_{ij}^+\right)^{\frac{1}{n(n-1)}}\right]\right\rangle$. Therefore, $IVNWBM^{p,q}\left(x_1, x_2, \cdots, x_n\right) = \left(\frac{1}{n(n-1)} \underset{\substack{i,j=1 \\ i \neq j}}{\overset{n}{\oplus}} \left(\alpha_{ij}\right)\right)^{\frac{1}{p+q}} =$

$$\left\langle\left[\left(1-\prod_{\substack{i,j=1 \\ i \neq j}}^n \left(1-T_{ij}^-\right)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{p+q}}, \left(1-\prod_{\substack{i,j=1 \\ i \neq j}}^n \left(1-T_{ij}^+\right)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{p+q}}\right], \left[1-\left(1-\prod_{\substack{i,j=1 \\ i \neq j}}^n \left(I_{ij}^-\right)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{p+q}}, 1-\left(1-\prod_{\substack{i,j=1 \\ i \neq j}}^n \left(I_{ij}^+\right)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{p+q}}\right],\right.$$

$$\left.\left[1-\left(1-\prod_{\substack{i,j=1 \\ i \neq j}}^n \left(F_{ij}^-\right)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{p+q}}, 1-\left(1-\prod_{\substack{i,j=1 \\ i \neq j}}^n \left(F_{ij}^+\right)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{p+q}}\right]\right\rangle = \left\langle\left[\left(1-\prod_{\substack{i,j=1 \\ i \neq j}}^n \left(1-\left(1-\left(1-T_i^-\right)^{w_i}\right)^p \left(1-\left(1-T_j^-\right)^{w_j}\right)^q\right)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{p+q}},\right.\right.$$

$$\left.\left(1-\prod_{\substack{i,j=1 \\ i \neq j}}^n \left(1-\left(1-\left(1-T_i^+\right)^{w_i}\right)^p \left(1-\left(1-T_j^+\right)^{w_j}\right)^q\right)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{p+q}}\right], \left[1-\left(1-\prod_{\substack{i,j=1 \\ i \neq j}}^n \left(1-\left(1-\left(I_i^-\right)^{w_i}\right)^p \left(1-\left(I_j^-\right)^{w_j}\right)^q\right)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{p+q}},\right.$$

$$\left.1-\left(1-\prod_{\substack{i,j=1 \\ i \neq j}}^n \left(1-\left(1-\left(I_i^+\right)^{w_i}\right)^p \left(1-\left(I_j^+\right)^{w_j}\right)^q\right)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{p+q}}\right], \left[1-\left(1-\prod_{\substack{i,j=1 \\ i \neq j}}^n \left(1-\left(1-\left(F_i^-\right)^{w_i}\right)^p \left(1-\left(F_j^-\right)^{w_j}\right)^q\right)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{p+q}},\right.$$

$$\left.\left.1-\left(1-\prod_{\substack{i,j=1 \\ i \neq j}}^n \left(1-\left(1-\left(F_i^+\right)^{w_i}\right)^p \left(1-\left(F_j^+\right)^{w_j}\right)^q\right)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{p+q}}\right]\right\rangle.$$

In addition, the following inequalities are right:

$$0 \leq \left(1-\left(1-\prod_{\substack{i,j=1 \\ i \neq j}}^n \left(1-\left(1-T_i^-\right)^{w_i}\right)^p \left(1-\left(1-T_j^-\right)^{w_j}\right)^q\right)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{p+q}} \leq 1, \quad 0 \leq \left(1-\left(1-\prod_{\substack{i,j=1 \\ i \neq j}}^n \left(1-\left(1-T_i^+\right)^{w_i}\right)^p \left(1-\left(1-T_j^+\right)^{w_j}\right)^q\right)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{p+q}} \leq 1,$$

$$0 \le 1-\left(1-\prod_{\substack{i,j=1\\ i\ne j}}^{n}\left(1-\left(1-\left(I_i^-\right)^{w_i}\right)^p\left(1-\left(I_j^-\right)^{w_j}\right)^q\right)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{p+q}} \le 1$$

$$0 \le 1-\left(1-\prod_{\substack{i,j=1\\ i\ne j}}^{n}\left(1-\left(1-\left(I_i^+\right)^{w_i}\right)^p\left(1-\left(I_j^+\right)^{w_j}\right)^q\right)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{p+q}} \le 1$$

$$0 \le 1-\left(1-\prod_{\substack{i,j=1\\ i\ne j}}^{n}\left(1-\left(1-\left(F_i^-\right)^{w_i}\right)^p\left(1-\left(F_j^-\right)^{w_j}\right)^q\right)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{p+q}} \le 1$$ and

$0 \le 1-\left(1-\prod_{\substack{i,j=1\\ i\ne j}}^{n}\left(1-\left(1-\left(F_i^+\right)^{w_i}\right)^p\left(1-\left(F_j^+\right)^{w_j}\right)^q\right)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{p+q}} \le 1$ which meets the requirements of an IVNN.

Hence, Theorem 2 is true.

Proof of Theorem 3.

According to the operations (1) and (3) in Definition 4, we have $px_i = \left\langle\left[1-\left(1-T_i^-\right)^p, 1-\left(1-T_i^+\right)^p\right], \left[(I_i^-)^p, (I_i^+)^p\right], \left[(F_i^-)^p, (F_i^+)^p\right]\right\rangle$ and $qx_j = \left\langle\left[1-\left(1-T_j^-\right)^q, 1-\left(1-T_j^+\right)^q\right], \left[(I_j^-)^q, (I_j^+)^q\right], \left[(F_j^-)^q, (F_j^+)^q\right]\right\rangle$, then $px_i \oplus qx_j = \left\langle\left[1-\left(1-T_i^-\right)^p\left(1-T_j^-\right)^q, 1-\left(1-T_i^+\right)^p\left(1-T_j^+\right)^q\right], \left[\left(I_i^-\right)^p\left(I_j^-\right)^q, \left(I_i^+\right)^p\left(I_j^+\right)^q\right], \left[\left(F_i^-\right)^p\left(F_j^-\right)^q, \left(F_i^+\right)^p\left(F_j^+\right)^q\right]\right\rangle$. Let $\alpha_{ij} = \left\langle\left[T_{ij}^-, T_{ij}^+\right], \left[I_{ij}^-, I_{ij}^+\right], \left[F_{ij}^-, F_{ij}^+\right]\right\rangle = px_i \oplus qx_j = \left\langle\left[1-(1-T_i^-)^p(1-T_j^-)^q, 1-\left(1-T_i^+\right)^p\left(1-T_j^+\right)^q\right], \left[\left(I_i^-\right)^p\left(I_j^-\right)^q, \left(I_i^+\right)^p\left(I_j^+\right)^q\right], \left[\left(F_i^-\right)^p\left(F_j^-\right)^q, \left(F_i^+\right)^p\left(F_j^+\right)^q\right]\right\rangle$, then $IVNGBM^{p,q}(x_1, x_2, \cdots, x_n) = \frac{1}{p+q}\left(\underset{\substack{i,j=1\\ i\ne j}}{\overset{n}{\otimes}}(px_i \oplus qx_j)^{\frac{1}{n(n-1)}}\right) = \frac{1}{p+q}\left(\underset{\substack{i,j=1\\ i\ne j}}{\overset{n}{\otimes}}(\alpha_{ij})^{\frac{1}{n(n-1)}}\right)$.

According to the operation (4) in Definition 4, $\left(\alpha_{ij}\right)^{\frac{1}{n(n-1)}} = \left\langle\left[\left(T_{ij}^-\right)^{\frac{1}{n(n-1)}}, \left(T_{ij}^+\right)^{\frac{1}{n(n-1)}}\right], \left[1-\left(1-I_{ij}^-\right)^{\frac{1}{n(n-1)}}, 1-\left(1-I_{ij}^+\right)^{\frac{1}{n(n-1)}}\right], \left[1-\left(1-F_{ij}^-\right)^{\frac{1}{n(n-1)}}, 1-\left(1-F_{ij}^+\right)^{\frac{1}{n(n-1)}}\right]\right\rangle$. And according to the operation (2) in Definition 4, $\underset{\substack{i,j=1\\ i\ne j}}{\overset{n}{\otimes}}\left(\alpha_{ij}\right)^{\frac{1}{n(n-1)}} = \left\langle\left[\prod_{\substack{i,j=1\\ i\ne j}}^{n}\left(T_{ij}^-\right)^{\frac{1}{n(n-1)}}, \prod_{\substack{i,j=1\\ i\ne j}}^{n}\left(T_{ij}^+\right)^{\frac{1}{n(n-1)}}\right], \left[1-\prod_{\substack{i,j=1\\ i\ne j}}^{n}\left(1-I_{ij}^-\right)^{\frac{1}{n(n-1)}}, 1-\prod_{\substack{i,j=1\\ i\ne j}}^{n}\left(1-I_{ij}^+\right)^{\frac{1}{n(n-1)}}\right], \left[1-\prod_{\substack{i,j=1\\ i\ne j}}^{n}\left(1-F_{ij}^-\right)^{\frac{1}{n(n-1)}}, 1-\prod_{\substack{i,j=1\\ i\ne j}}^{n}\left(1-F_{ij}^+\right)^{\frac{1}{n(n-1)}}\right]\right\rangle$. According to the operation (4) in Definition 4, $\frac{1}{p+q}\underset{\substack{i,j=1\\ i\ne j}}{\overset{n}{\otimes}}\left(\alpha_{ij}\right)^{\frac{1}{n(n-1)}} =$

$$\left\langle\left[1-\left(1-\prod_{\substack{i,j=1\\ i\ne j}}^{n}\left(T_{ij}^-\right)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{p+q}}, 1-\left(1-\prod_{\substack{i,j=1\\ i\ne j}}^{n}\left(T_{ij}^+\right)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{p+q}}\right], \left[\left(1-\prod_{\substack{i,j=1\\ i\ne j}}^{n}\left(1-I_{ij}^-\right)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{p+q}}, \left(1-\prod_{\substack{i,j=1\\ i\ne j}}^{n}\left(1-I_{ij}^+\right)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{p+q}}\right],\right.$$

$$\left[\left(1-\prod_{\substack{i,j=1\\i\neq j}}^{n}\left(1-F_{ij}^{-}\right)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{p+q}},\left(1-\prod_{\substack{i,j=1\\i\neq j}}^{n}\left(1-F_{ij}^{+}\right)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{p+q}}\right]\right\rangle=\left\langle\left[1-\left(1-\prod_{\substack{i,j=1\\i\neq j}}^{n}\left(1-(1-T_i^{-})^{p}(1-T_j^{-})^{q}\right)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{p+q}},\right.\right.$$

$$\left.1-\left(1-\prod_{\substack{i,j=1\\i\neq j}}^{n}\left(1-(1-T_i^{+})^{p}(1-T_j^{+})^{q}\right)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{p+q}}\right],\left[\left(1-\prod_{\substack{i,j=1\\i\neq j}}^{n}\left(1-\left(I_i^{-}\right)^{p}\left(I_j^{-}\right)^{q}\right)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{p+q}},\right.$$

$$\left.\left(1-\prod_{\substack{i,j=1\\i\neq j}}^{n}\left(1-\left(I_i^{+}\right)^{p}\left(I_j^{+}\right)^{q}\right)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{p+q}}\right],\left.\left[\left(1-\prod_{\substack{i,j=1\\i\neq j}}^{n}\left(1-\left(F_i^{-}\right)^{p}\left(F_j^{-}\right)^{q}\right)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{p+q}},\left(1-\prod_{\substack{i,j=1\\i\neq j}}^{n}\left(1-\left(F_i^{+}\right)^{p}\left(F_j^{+}\right)^{q}\right)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{p+q}}\right]\right\rangle.$$

Moreover, the following inequalities hold:

$$0\leq 1-\left(1-\prod_{\substack{i,j=1\\i\neq j}}^{n}\left(1-\left(1-T_i^{-}\right)^{p}\left(1-T_j^{-}\right)^{q}\right)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{p+q}}\leq 1\ ,\quad 0\leq 1-\left(1-\prod_{\substack{i,j=1\\i\neq j}}^{n}\left(1-(1-T_i^{+})^{p}(1-T_j^{+})^{q}\right)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{p+q}}\leq 1\ ,$$

$$0\leq\left(1-\prod_{\substack{i,j=1\\i\neq j}}^{n}\left(1-\left(I_i^{-}\right)^{p}\left(I_j^{-}\right)^{q}\right)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{p+q}}\leq 1\ ,\quad 0\leq\left(1-\prod_{\substack{i,j=1\\i\neq j}}^{n}\left(1-\left(I_i^{+}\right)^{p}\left(I_j^{+}\right)^{q}\right)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{p+q}}\leq 1\ ,$$

$0\leq\left(1-\prod_{\substack{i,j=1\\i\neq j}}^{n}\left(1-\left(F_i^{-}\right)^{p}\left(F_j^{-}\right)^{q}\right)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{p+q}}\leq 1$ and $0\leq\left(1-\prod_{\substack{i,j=1\\i\neq j}}^{n}\left(1-\left(F_i^{+}\right)^{p}\left(F_j^{+}\right)^{q}\right)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{p+q}}\leq 1$, which meets the requirements of an IVNN.

Therefore, Theorem 3 holds.

Proof of Theorem 4.

By the operation (4) in Definition 4, we have $x_i^{w_i}=\left\langle\left[\left(T_i^{-}\right)^{w_i},\left(T_i^{+}\right)^{w_i}\right],\left[1-\left(1-I_i^{-}\right)^{w_i},1-\left(1-I_i^{+}\right)^{w_i}\right],\left[1-\left(1-F_i^{-}\right)^{w_i},1-\left(1-F_i^{+}\right)^{w_i}\right]\right\rangle$ and $x_j^{w_j}=\left\langle\left[\left(T_j^{-}\right)^{w_j},\left(T_j^{+}\right)^{w_j}\right],\left[1-\left(1-I_j^{-}\right)^{w_j},1-\left(1-I_j^{+}\right)^{w_j}\right],\left[1-\left(1-F_j^{-}\right)^{w_j},1-\left(1-F_j^{+}\right)^{w_j}\right]\right\rangle$. And according to the operation (4) in Definition 4, $px_i^{w_i}=$ $\left\langle\left[1-\left(1-\left(T_i^{-}\right)^{w_i}\right)^{p},1-\left(1-\left(T_i^{+}\right)^{w_i}\right)^{p}\right],\left[\left(1-\left(1-I_i^{-}\right)^{w_i}\right)^{p},\left(1-\left(1-I_i^{+}\right)^{w_i}\right)^{p}\right],\left[\left(1-\left(1-F_i^{-}\right)^{w_i}\right)^{p},\left(1-\left(1-F_i^{+}\right)^{w_i}\right)^{p}\right]\right\rangle$ and $qx_j^{w_j}=\left\langle\left[1-\left(1-\left(T_j^{-}\right)^{w_j}\right)^{q},1-\left(1-\left(T_j^{+}\right)^{w_j}\right)^{q}\right],\left[\left(1-\left(1-I_j^{-}\right)^{w_j}\right)^{q},\left(1-\left(1-I_j^{+}\right)^{w_j}\right)^{q}\right],\left[\left(1-\left(1-F_j^{-}\right)^{w_j}\right)^{q},\left(1-\left(1-F_j^{+}\right)^{w_j}\right)^{q}\right]\right\rangle$. Then, $px_i^{w_i}+qx_j^{w_j}=\left\langle\left[1-\left(1-\left(T_i^{-}\right)^{w_i}\right)^{p}\left(1-\left(T_j^{-}\right)^{w_j}\right)^{q},1-\left(1-\left(T_i^{+}\right)^{w_i}\right)^{p}\left(1-\left(T_j^{+}\right)^{w_j}\right)^{q}\right],\right.$ $\left[\left(1-\left(1-I_i^{-}\right)^{w_i}\right)^{p}\left(1-\left(1-I_j^{-}\right)^{w_j}\right)^{q},\left(1-\left(1-I_i^{+}\right)^{w_i}\right)^{p}\left(1-\left(1-I_j^{+}\right)^{w_j}\right)^{q}\right],\left[\left(1-\left(1-F_i^{-}\right)^{w_i}\right)^{p}\left(1-\left(1-F_j^{-}\right)^{w_j}\right)^{q},\right.$

$\left(1-\left(1-F_i^+\right)^{w_i}\right)^p\left(1-\left(1-F_j^+\right)^{w_j}\right)^q\Big]\Big\rangle$. Let $\alpha_{ij}=\left\langle\left[T_{ij}^-,T_{ij}^+\right],\left[I_{ij}^-,I_{ij}^+\right],\left[F_{ij}^-,F_{ij}^+\right]\right\rangle=px_i^{w_i}\oplus qx_j^{w_j}=$ $\Big\langle\Big[1-\left(1-\left(T_i^-\right)^{w_i}\right)^p\left(1-\left(T_j^-\right)^{w_j}\right)^q,1-\left(1-\left(T_i^+\right)^{w_i}\right)^p\left(1-\left(T_j^+\right)^{w_j}\right)^q\Big],\Big[\left(1-\left(1-I_i^-\right)^{w_i}\right)^p\left(1-\left(1-I_j^-\right)^{w_j}\right)^q,$ $\left(1-\left(1-I_i^+\right)^{w_i}\right)^p\left(1-\left(1-I_j^+\right)^{w_j}\right)^q\Big],\Big[\left(1-\left(1-F_i^-\right)^{w_i}\right)^p\left(1-\left(1-F_j^-\right)^{w_j}\right)^q,\left(1-\left(1-F_i^+\right)^{w_i}\right)^p\left(1-\left(1-F_j^+\right)^{w_j}\right)^q\Big]\Big\rangle$, according to the operation (4) in Definition 4, $(\alpha_{ij})^{\frac{1}{n(n-1)}}=\Big\langle\Big[\left(T_{ij}^-\right)^{\frac{1}{n(n-1)}},\left(T_{ij}^+\right)^{\frac{1}{n(n-1)}}\Big],\Big[1-\left(1-I_{ij}^-\right)^{\frac{1}{n(n-1)}},$ $1-\left(1-I_{ij}^+\right)^{\frac{1}{n(n-1)}}\Big],\Big[1-\left(1-F_{ij}^-\right)^{\frac{1}{n(n-1)}},1-\left(1-F_{ij}^+\right)^{\frac{1}{n(n-1)}}\Big]\Big\rangle$. And according to the operation (2) in Definition 4,

$$\underset{\substack{i,j=1\\i\neq j}}{\overset{n}{\otimes}}\left(\alpha_{ij}\right)^{\frac{1}{n(n-1)}}=\left\langle\left[\prod_{\substack{i,j=1\\i\neq j}}^{n}\left(T_{ij}^-\right)^{\frac{1}{n(n-1)}},\prod_{\substack{i,j=1\\i\neq j}}^{n}\left(T_{ij}^+\right)^{\frac{1}{n(n-1)}}\right],\left[1-\prod_{\substack{i,j=1\\i\neq j}}^{n}\left(1-I_{ij}^-\right)^{\frac{1}{n(n-1)}},1-\prod_{\substack{i,j=1\\i\neq j}}^{n}\left(1-I_{ij}^+\right)^{\frac{1}{n(n-1)}}\right],\left[1-\prod_{\substack{i,j=1\\i\neq j}}^{n}\left(1-F_{ij}^-\right)^{\frac{1}{n(n-1)}},\right.\right.$$

$$\left.\left.1-\prod_{\substack{i,j=1\\i\neq j}}^{n}\left(1-F_{ij}^+\right)^{\frac{1}{n(n-1)}}\right]\right\rangle.$$

Then, $IVNWGBM^{p,q}\left(x_1,x_2,\cdots,x_n\right)=\frac{1}{p+q}\left(\underset{\substack{i,j=1\\i\neq j}}{\overset{n}{\otimes}}\left(\alpha_{ij}\right)^{\frac{1}{n(n-1)}}\right)=$

$$\left\langle\left[1-\left(1-\prod_{\substack{i,j=1\\i\neq j}}^{n}\left(T_{ij}^-\right)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{p+q}},1-\left(1-\prod_{\substack{i,j=1\\i\neq j}}^{n}\left(T_{ij}^+\right)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{p+q}}\right],\left[\left(1-\prod_{\substack{i,j=1\\i\neq j}}^{n}\left(1-I_{ij}^-\right)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{p+q}},\left(1-\prod_{\substack{i,j=1\\i\neq j}}^{n}\left(1-I_{ij}^+\right)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{p+q}}\right],\right.$$

$$\left.\left[\left(1-\prod_{\substack{i,j=1\\i\neq j}}^{n}\left(1-F_{ij}^-\right)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{p+q}},\left(1-\prod_{\substack{i,j=1\\i\neq j}}^{n}\left(1-F_{ij}^+\right)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{p+q}}\right]\right\rangle=\left\langle\left[1-\left(1-\prod_{\substack{i,j=1\\i\neq j}}^{n}\left(1-\left(1-\left(T_i^-\right)^{w_i}\right)^p\left(1-\left(T_j^-\right)^{w_j}\right)^q\right)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{p+q}},\right.\right.$$

$$\left.1-\left(1-\prod_{\substack{i,j=1\\i\neq j}}^{n}\left(1-\left(1-\left(T_i^+\right)^{w_i}\right)^p\left(1-\left(T_j^+\right)^{w_j}\right)^q\right)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{p+q}}\right],\left[\left(1-\prod_{\substack{i,j=1\\i\neq j}}^{n}\left(1-\left(1-\left(1-I_i^-\right)^{w_i}\right)^p\left(1-\left(1-I_j^-\right)^{w_j}\right)^q\right)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{p+q}},\right.$$

$$\left.\left(1-\prod_{\substack{i,j=1\\i\neq j}}^{n}\left(1-\left(1-\left(1-I_i^+\right)^{w_i}\right)^p\left(1-\left(1-I_j^+\right)^{w_j}\right)^q\right)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{p+q}}\right],\left[\left(1-\prod_{\substack{i,j=1\\i\neq j}}^{n}\left(1-\left(1-\left(1-F_i^-\right)^{w_i}\right)^p\left(1-\left(1-F_j^-\right)^{w_j}\right)^q\right)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{p+q}},\right.$$

$$\left.\left.\left(1-\prod_{\substack{i,j=1\\i\neq j}}^{n}\left(1-\left(1-\left(1-F_i^+\right)^{w_i}\right)^p\left(1-\left(1-F_j^+\right)^{w_j}\right)^q\right)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{p+q}}\right]\right\rangle.$$

Additionally, the following inequalities are proved to be true:

$$0\leq1-\left(1-\prod_{\substack{i,j=1\\i\neq j}}^{n}\left(1-\left(1-\left(T_i^-\right)^{w_i}\right)^p\left(1-\left(T_j^-\right)^{w_j}\right)^q\right)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{p+q}}\leq1$$

$$0 \le 1-\left(1-\prod_{\substack{i,j=1\\ i\neq j}}^{n}\left(1-\left(1-\left(T_i^+\right)^{w_i}\right)^p\left(1-\left(T_j^+\right)^{w_j}\right)^q\right)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{p+q}} \le 1$$

$$0 \le \left(1-\prod_{\substack{i,j=1\\ i\neq j}}^{n}\left(1-\left(1-\left(1-I_i^-\right)^{w_i}\right)^p\left(1-\left(1-I_j^-\right)^{w_j}\right)^q\right)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{p+q}} \le 1$$

$$0 \le \left(1-\prod_{\substack{i,j=1\\ i\neq j}}^{n}\left(1-\left(1-\left(1-I_i^+\right)^{w_i}\right)^p\left(1-\left(1-I_j^+\right)^{w_j}\right)^q\right)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{p+q}} \le 1, \quad 0 \le \left(1-\prod_{\substack{i,j=1\\ i\neq j}}^{n}\left(1-\left(1-\left(1-F_i^-\right)^{w_i}\right)^p\left(1-\left(1-F_j^-\right)^{w_j}\right)^q\right)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{p+q}} \le 1$$

and $0 \le \left(1-\prod_{\substack{i,j=1\\ i\neq j}}^{n}\left(1-\left(1-\left(1-F_i^+\right)^{w_i}\right)^p\left(1-\left(1-F_j^+\right)^{w_j}\right)^q\right)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{p+q}} \le 1$, which meets the requirements of an IVNN.

Hence, Theorem 4 is true.

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