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# Interval-Valued Neutrosophic Graph Structures

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**Abstract.** In this research article, we introduce certain notions of interval-valued neutrosophic graph structures. We elaborate the concepts of interval-valued neutrosophic graph structures with examples. Moreover, we discuss the concept of $\varphi$-complement of an interval-valued neutrosophic graph structure. Finally, we present some related properties, including $\varphi$-complement, totally-self complementary and totally-strong-self complementary, of interval-valued neutrosophic graph structures.

**AMS (MOS) Subject Classification Codes: 35S29, 40S70, 25U09**
**Key Words:** Graph structure, Interval-valued neutrosophic graph structure, $\varphi$-complement.

## 1. Introduction

Zadeh [33] introduced interval-valued fuzzy set theory which is an extension of fuzzy set theory [32]. Membership degrees in an interval-valued fuzzy set are intervals rather than numbers and uncertainty is reflected by length of interval membership degree. Zhan et al. [35, 36] applied the concept of interval-valued fuzzy sets to algebraic structures. For representing vagueness and uncertainty Atanassov [10] proposed an extension of fuzzy sets by adding a new component, called intuitionistic fuzzy sets. The concept of intuitionistic fuzzy sets is more meaningful and inventive due to the presence of degree of truth, indeterminacy and falsity membership. The intuitionistic fuzzy sets have more describing possibilities as compared to fuzzy sets. The hesitation margin of an intuitionistic fuzzy set is its uncertainty by default and sum of truth-membership degree and falsity-membership degree does not exceeds unity. In many phenomenons like information fusion, uncertainty and indeterminacy is doubtlessly quantified. Smarandache [24, 25] proposed the idea of neutrosophic sets, he mingled tricomponent logic, non-standard analysis, and philosophy. "It is a branch of philosophy which studies the origin, nature and scope of neutralities as well as their interactions with different ideational spectra". For convenient and advantageous usage of neutrosophic sets in science and engineering, Wang et al. [29] proposed the notion of single-valued neutrosophic(SVNS)sets, whose three independent components have values in standard unit interval $[0, 1]$. Neutrosophic set theory being a generalization of fuzzy set theory and intuitionistic fuzzy set theory, is more practical, advantageous and applicable in various fields, including medical diagnosis, control theory, topology, decision-making problems and in many more real-life problems. Wang et al. [30] proposed the notion of interval-valued neutrosophic sets, which is more precise and flexible than the single-valued neutrosophic sets. An interval-valued neutrosophic set is a generalization of the notion of single-valued neutrosophic set, in which three independent components $(t, i, f)$ are intervals which are subsets of standard unit interval $[0, 1]$.

On the basis of Zadeh's fuzzy relations [35] Kaufmann proposed fuzzy graph [18]. Rosenfeld [22] discussed fuzzy analogue of various graph-theoretic ideas. Later on, Bhattacharya [11] gave some remarks on fuzzy graphs, and some operations on fuzzy graphs were introduced by Mordeson and Peng [19]. Mordeson and Nair presented a valuable contribution on fuzzy graphs as well as fuzzy hypergraphs in [20]. Mathew and Sunitha [26, 27] discussed arcs and strong cycles in fuzzy graphs. On the other hand, Dinesh and Ramakrishnan [14] defined fuzzy graph structures and discussed its properties. Akram and Akmal [7] proposed the notion of bipolar fuzzy graph structures. Akram et al. [1, 2, 3, 4] have introduced several concepts on interval-valued fuzzy graphs and interval-valued neutrosophic graphs. Akram and Shahzadi [8] introduced the notion of neutrosophic soft graphs with applications. Recently, Akram and Nasir [5, 6] considered interval-valued neutrosophic graphs. In this research article, we introduce certain notions of interval-valued neutrosophic graph structures. We elaborate the concepts of interval-valued neutrosophic graph structures with examples. Moreover, we discuss the concept of $\varphi$-complement of an interval-valued neutrosophic graph structure.

## 2. Interval-Valued Neutrosophic Graph Structures

Sampathkumar [23] introduced the graph structure which is a generalization of undirected graph and is quite useful in studying some structures, including, graphs, signed graphs, labelled graphs and edge colored graphs.

**Definition 2.1.** [23] A *graph structure* $G = (U, R_1, \ldots, R_t)$ consists of a non-empty set $U$ together with relations $R_1, R_2, \ldots, R_t$ on $U$ which are mutually disjoint such that each $R_j$, $1 \leq j \leq t$, is symmetric and irreflexive.

One can represent a graph structure $G = (U, R_1, \ldots, R_t)$ in the plane just like a graph where each edge is labelled as $R_j$, $1 \leq j \leq t$.

**Example 2.2.** Let $U = \{r_1, r_2, r_3, r_4, r_5\}$. Let $R_1 = \{(r_1, r_2), (r_3, r_4), (r_1, r_4)\}$, $R_2 = \{(r_1, r_3), (r_1, r_5)\}$, $R_3 = \{(r_2, r_3), (r_4, r_5)\}$ be mutually disjoint, symmetric and irreflexive relations on set $U$. Thus $G = (U, R_1, R_2, R_3)$ is a graph structure and is represented in plane as a graph where each edge is labelled as $R_1$, $R_2$ or $R_3$.

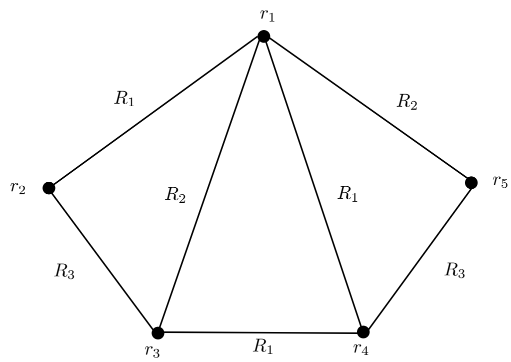


FIGURE 1. Graph structure $G = (U, R_1, R_2, R_3)$

**Definition 2.3.** [30, 31] The *interval-valued neutrosophic set* $I$ on set $U$ is defined by

$$I = \{(r, [t^-(r), t^+(r)], [i^-(r), i^+(r)], [f^-(r), f^+(r)]) : r \in U\},$$

where, $t^-$, $t^+$, $i^-$, $i^+$, $f^-$, and $f^+$ are functions from $U$ to $[0, 1]$ such that: $t^-(r) \leq t^+(r)$, $i^-(r) \leq i^+(r)$ and $f^-(r) \leq f^+(r)$ for all $r \in U$.

**Definition 2.4.** $\check{G}_{iv} = (I, I_1, I_2, \ldots, I_t)$ is called an *interval-valued neutrosophic graph structure(IVNGS)* of graph structure $G = (U, R_1, R_2, \ldots, R_t)$ if
$I = \{(r, [t^-(r), t^+(r)], [i^-(r), i^+(r)], [f^-(r), f^+(r)]) : r \in U\}$
and $I_j = \{ ((r, s), [t_j^-(r, s), t_j^+(r, s)], [i_j^-(r, s), i_j^+(r, s)], [f_j^-(r, s), f_j^+(r, s)]) : (r, s) \in R_j \}$ are interval-valued neutrosophic(IVN) sets on $U$ and $R_j$, respectively, such that:

(1) $t_j^-(r, s) \leq \min\{t^-(r), t^-(s)\}$, $\quad t_j^+(r, s) \leq \min\{t^+(r), t^+(s)\}$,
(2) $i_j^-(r, s) \leq \min\{i^-(r), i^-(s)\}$, $\quad i_j^+(r, s) \leq \min\{i^+(r), i^+(s)\}$,
(3) $f_j^-(r, s) \leq \min\{f^-(r), f^-(s)\}$, $\quad f_j^+(r, s) \leq \min\{f^+(r), f^+(s)\}$,

where, $t_j^-$, $t_j^+$, $i_j^-$, $i_j^+$, $f_j^-$, and $f_j^+$ are functions from $R_j$ to $[0, 1]$ such that $t_j^-(r, s) \leq t_j^+(r, s)$, $i_j^-(r, s) \leq i_j^+(r, s)$ and $f_j^-(r, s) \leq f_j^+(r, s)$ for all $(r, s) \in R_j$.
In this paper we will use $rs$ in place of ordered pair $(r, s)$ which represents an edge between vertices $r$ and $s$.

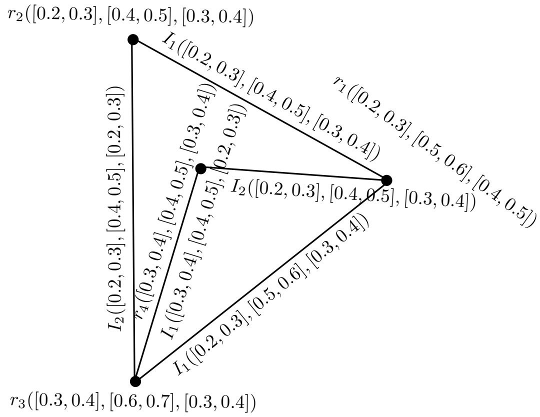


FIGURE 2. Interval-valued neutrosophic graph structure

**Example 2.5.** Consider the graph structure $G = (U, R_1, R_2)$ such that $U = \{r_1, r_2, r_3, r_4\}$, $R_1 = \{r_1r_3, r_1r_2, r_3r_4\}$, $R_2 = \{r_1r_4, r_2r_3\}$. By defining interval-valued neutrosophic sets $I$, $I_1$ and $I_2$ on $U$, $R_1$ and $R_2$, respectively, we draw an IVNGS as shown in Fig. 2.

**Definition 2.6.** Let $\check{G}_{iv} = (I, I_1, I_2, \ldots, I_t)$ be an IVNGS of graph structure(GS) $G = (U, R_1, R_2, \ldots, R_t)$. If $\check{H}_{iv} = (I', I'_1, I'_2, \ldots, I'_t)$ is an IVNGS of $G$ such that

$$t'^-(r) \leq t^-(r), i'^-(r) \leq i^-(r), f'^-(r) \leq f^-(r),$$
$$t'^+(r) \leq t^+(r), i'^+(r) \leq i^+(r), f'^+(r) \leq f^+(r)\ ,$$

$$t'^-_j(rs) \leq t^-_j(rs),\ i'^-_j(rs) \leq i^-_j(rs),\ f'^-_j(rs) \leq f^-_j(rs),$$
$$t'^+_j(rs) \leq t^+_j(rs),\ i'^+_j(rs) \leq i^+_j(rs),\ f'^+_j(rs) \leq f^+_j(rs),$$
$$\text{for all } r \in U \text{ and } rs \in R_j,\ j = 1, 2, \ldots, t.$$

Then $\check{H}_{iv}$ is called an *interval-valued neutrosophic(IVN) subgraph-structure* of IVNGS $\check{G}_{iv}$.

**Example 2.7.** Consider an IVNGS $\check{H}_{iv} = (I', I'_1, I'_2)$ of graph structure $G = (U, R_1, R_2)$ as illustrated in Fig. 3. Through direct calculations, it is shown that $\check{H}_{iv}$ is an IVN subgraph-structure of IVNGS $\check{G}_{iv}$ shown in Fig. 2.

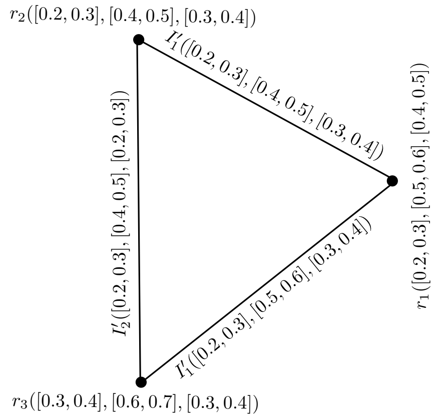


FIGURE 4. IVN induced subgraph-structure

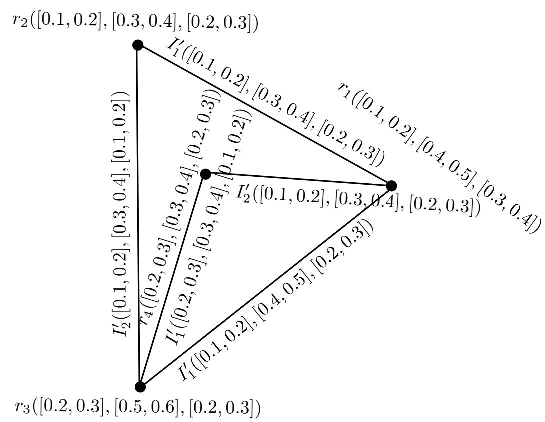


FIGURE 3. Interval-valued neutrosophic subgraph-structure

**Definition 2.8.** An IVNGS $\check{H}_{iv} = (I', I_1', I_2', \ldots, I_t')$ is called an *IVN induced subgraph-structure* of IVNGS $\check{G}_{iv}$ by $Q \subseteq U$ if

$$t'^-(r) = t^-(r), i'^-(r) = i^-(r), f'^-(r) = f^-(r),$$
$$t'^+(r) = t^+(r), i'^+(r) = i^+(r), f'^+(r) = f^+(r),$$
$$t_j'^-(rs) = t_j^-(rs), i_j'^-(rs) = i_j^-(rs), f_j'^-(rs) = f_j^-(rs), t_j'^+(rs) = t_j^+(rs),$$
$$i_j'^+(rs) = i_j^+(rs), f_j'^+(rs) = f_j^+(rs), \text{ for all } r, s \in Q, j = 1, 2, \ldots, t.$$

**Example 2.9.** An IVNGS $\check{H}_{iv} = (I', I_1', I_2')$ of graph structure $G = (U, R_1, R_2)$ shown in Fig. 4 is an IVN induced subgraph-structure of IVNGS $\check{G}_{iv} = (I, I_1, I_2)$ represented in Fig. 2.

**Definition 2.10.** An IVNGS $\check{H}_{iv} = (I', I_1', I_2', \ldots, I_t')$ is called *IVN spanning subgraph-structure* of IVNGS $\check{G}_{iv} = (I, I_1, I_2, \ldots, I_t)$ if $I' = I$ and

$$t_j'^-(rs) \le t_j^-(rs),\ i_j'^-(rs) \le i_j^-(rs),\ f_j'^-(rs) \le f_j^-(rs),$$

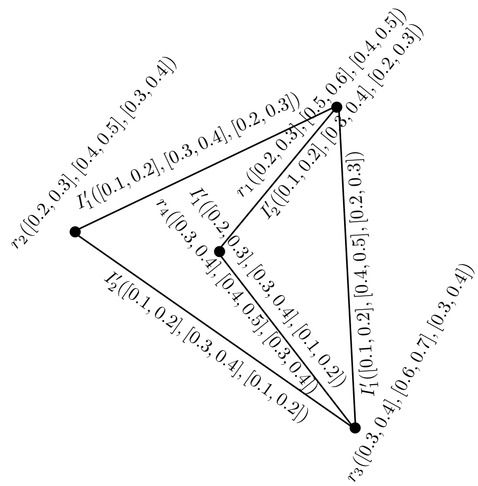


FIGURE 5. IVN spanning subgraph-structure

$$t'^{+}_{j}(rs) \leq t^{+}_{j}(rs),\ i'^{+}_{j}(rs) \leq i^{+}_{j}(rs),\ f'^{+}_{j}(rs) \leq f^{+}_{j}(rs), j = 1, 2, \ldots, t.$$

**Example 2.11.** An IVNGS $\check{H}_{iv} = (I', I'_1, I'_2)$ shown in Fig. 5 is an IVN spanning subgraph-structure of IVNGS $\check{G}_{iv} = (I, I_1, I_2)$ represented in Fig. 2.

**Definition 2.12.** Let $\check{G}_{iv} = (I, I_1, I_2, \ldots, I_t)$ be an IVNGS. Then edge $rs \in I_j$ is called an *IVN $I_j$-edge* or in short an *$I_j$-edge* if

$$t^{-}_{j}(rs) > 0 \text{ or } i^{-}_{j}(rs) > 0 \text{ or } f^{-}_{j}(rs) > 0 \text{ or } t^{+}_{j}(rs) > 0 \text{ or } i^{+}_{j}(rs) > 0 \text{ or } f^{+}_{j}(rs) > 0$$

or all of conditions are satisfied. Hence support of $I_j$ is defined as;

$$supp(I_j) = \{rs \in I_j : t^{-}_{j}(rs) > 0\} \cup \{rs \in I_j : i^{-}_{j}(rs) > 0\} \cup \{rs \in I_j : f^{-}_{j}(rs) > 0\} \cup \{rs \in I_j : t^{+}_{j}(rs) > 0\} \cup \{rs \in I_j : i^{+}_{j}(rs) > 0\} \cup \{rs \in I_j : f^{+}_{j}(rs) > 0\}, j = 1, 2, \ldots, t.$$

**Definition 2.13.** An *$I_j$-path* in an IVNGS $\check{G}_{iv} = (I, I_1, I_2, \ldots, I_t)$ is a sequence $r_1, r_2, \ldots, r_t$ of distinct vertices (except $r_t = r_1$) in $U$ such that $r_{j-1}r_j$ is an IVN $I_j$-edge for all $j = 2, 3, \ldots, t$.

**Definition 2.14.** An IVNGS $\check{G}_{iv} = (I, I_1, I_2, \ldots, I_t)$ is *$I_j$-strong* for any $j \in \{1, 2, \ldots, t\}$ if

$$t^{-}_{j}(rs) = \min\{t^{-}(r), t^{-}(s)\},\ i^{-}_{j}(rs) = \min\{i^{-}(r), i^{-}(s)\},$$
$$f^{-}_{j}(rs) = \min\{f^{-}(r), f^{-}(s)\},\ t^{+}_{j}(rs) = \min\{t^{+}(r), t^{+}(s)\},$$
$$i^{+}_{j}(rs) = \min\{i^{+}(r), i^{+}(s)\},\ f^{+}_{j}(rs) = \min\{f^{+}(r), f^{+}(s)\},$$

for all $rs \in supp(I_j)$. If $\check{G}_{iv}$ is $I_j$-strong for all $j \in \{1, 2, \ldots, t\}$, then $\check{G}_{iv}$ is called a *strong IVNGS*.

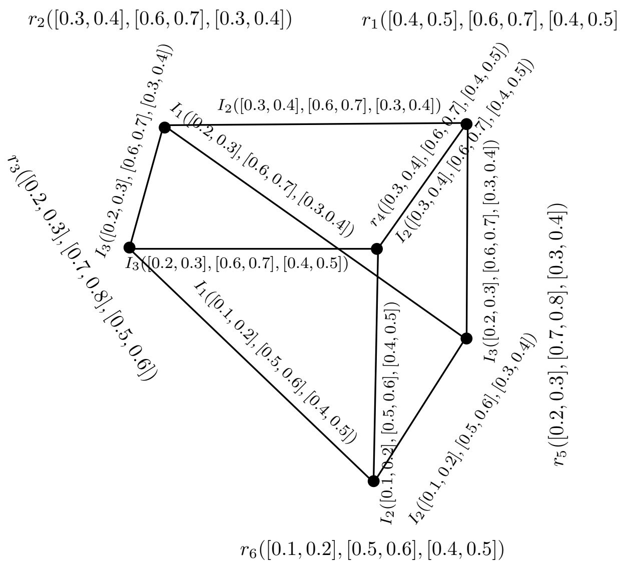


FIGURE 6. Strong IVNGS

**Example 2.15.** Consider an IVNGS $\check{G}_{iv} = (I, I_1, I_2, I_3)$ as shown in Fig. 6. $\check{G}_{iv}$ is a strong IVNGS, since it is $I_1$, $I_2$ and $I_3$ strong.

**Definition 2.16.** An IVNGS $\check{G}_{iv} = (I, I_1, I_2, \ldots, I_t)$ is called complete IVNGS, if

(1) $\check{G}_{iv}$ is a strong IVNGS.
(2) $Supp(I_j) \neq \emptyset$, for all $j = 1, 2, \ldots, t$.
(3) For all $r, s \in U$, $rs$ is an $I_j - edge$ for some j.

**Example 2.17.** Let $\check{G}_{iv} = (I, I_1, I_2, I_3)$ be an IVNGS of graph structure $G = (U, R_1, R_2, R_3)$ and it is shown in Fig.7. Where, $U = \{r_1, r_2, r_3, r_4, r_5, r_6\}$, $R_1 = \{r_1r_6, r_1r_2, r_2r_4, r_2r_5, r_2r_6,$
$r_4r_5\}$, $R_2 = \{r_4r_3, r_5r_6, r_1r_4\}$, $R_3 = \{r_1r_5, r_5r_3, r_2r_3, r_1r_3, r_4r_6\}$. By direct calculations, we can show that $\check{G}_{iv}$ is a strong IVNGS. Moreover, $supp(I_1) \neq \emptyset$, $supp(I_2) \neq \emptyset$, $supp(I_3) \neq \emptyset$ and each pair $r_jr_k$ of nodes in $U$, is either an $I_1-$ edge or $I_2-$edge or $I_3-$ edge. Hence $\check{G}_{iv}$ is a complete IVNGS, that is, $I_1I_2I_3-$complete IVNGS.

**Definition 2.18.** Let $\check{G}_{iv} = (I, I_1, I_2, \ldots, I_t)$ be an IVNGS. The *truth strength* $[t^-.P_{I_j},\ t^+.P_{I_j}]$, *indeterminacy strength* $[i^-.P_{I_j},\ i^+.P_{I_j}]$ and *falsity strength* $[f^-.P_{I_j},\ f^+.P_{I_j}]$ of an $I_j$-path, $P_{I_j} = r_1, r_2, \ldots, r_n$ are defined as:

$$[t^-.P_{I_j},\ t^+.P_{I_j}] = [\bigwedge_{k=2}^{n}[t^-_{I_j}(r_{k-1}r_k)],\ \ \bigwedge_{k=2}^{n}[t^+_{I_j}(r_{k-1}r_k)]],$$

$$[i^-.P_{I_j},\ i^+.P_{I_j}] = [\bigwedge_{k=2}^{n}[i^-_{I_j}(r_{k-1}r_k)],\ \ \bigwedge_{k=2}^{n}[i^+_{I_j}(r_{k-1}r_k)]],$$

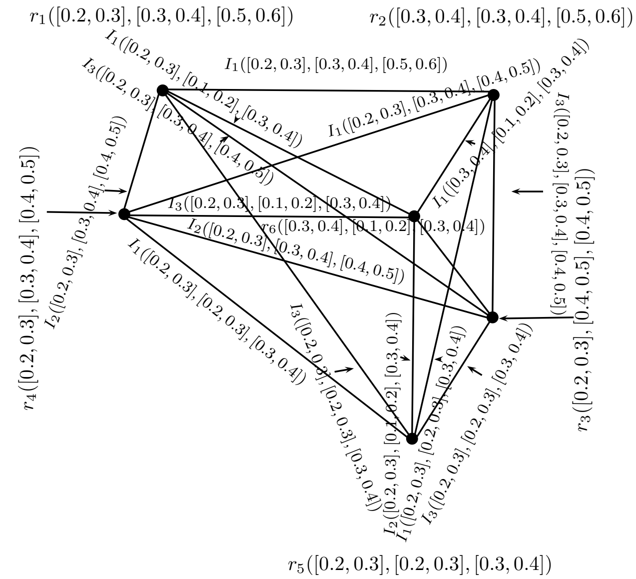


FIGURE 7. Complete IVNGS

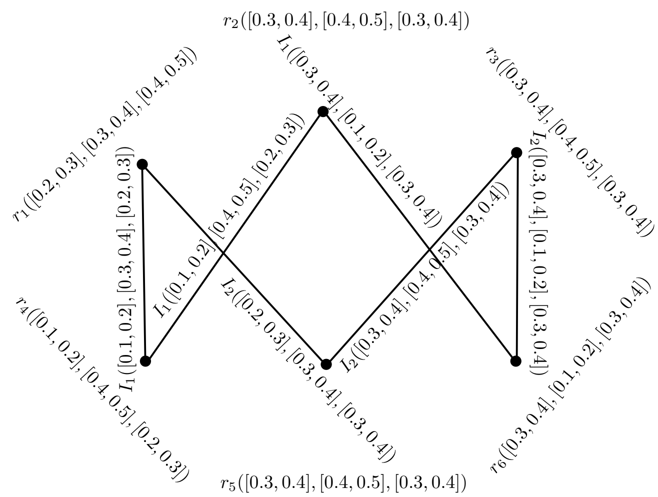


FIGURE 8. IVNGS $\check{G}_{iv} = (I, I_1, I_2)$

$$[f^-.P_{I_j},\ \ f^+.P_{I_j}] = [\bigwedge_{k=2}^{n}[f^-_{I_j}(r_{k-1}r_k)],\ \ \bigwedge_{k=2}^{n}[f^+_{I_j}(r_{k-1}r_k)]].$$

**Example 2.19.** Consider an IVNGS $\check{G}_{iv} = (I, I_1, I_2)$ of Graph structure $G = (U, R_1, R_2)$ as shown in Fig. 8. For $I_2$-path $P_{I_2} = r_1, r_5, r_3, r_6$, $[t^-.P_{I_2},\ t^+.P_{I_2}]$ = [0.2, 0.3], $[i^-.P_{I_2},\ i^+.P_{I_2}]$ = [0.1, 0.2] and $[f^-.P_{I_2},\ f^+.P_{I_2}]$ = [0.3, 0.4].

**Definition 2.20.** Let $\check{G}_{iv} = (I, I_1, I_2, \ldots, I_t)$ be an IVNGS. Then

- $I_j$- *Truth strength of connectedness* between two nodes $r$ and $s$ is defined by:
  $[t^{-\infty}_{I_j}(rs), t^{+\infty}_{I_j}(rs)] = [\bigvee\limits_{i\geq 1}\{t^{-i}_{I_j}(rs)\}, \bigvee\limits_{i\geq 1}\{t^{+i}_{I_j}(rs)\}]$ such that
  $[t^{-i}_{I_j}(rs), t^{+i}_{I_j}(rs)] = [(t^{-(i-1)}_{I_j} \circ t^{-(1)}_{I_j})(rs), (t^{+(i-1)}_{I_j} \circ t^{+(1)}_{I_j})(rs)]$ for $i \geq 2$
  and $[t^{-2}_{I_j}(rs), t^{+2}_{I_j}(rs)] = [(t^{-1}_{I_j} \circ t^{-1}_{I_j})(rs), (t^{+1}_{I_j} \circ t^{+1}_{I_j})(rs)]$
  $= [\bigvee\limits_{y}(t^{-1}_{I_j}(ry) \wedge t^{-1}_{I_j}(ys)), \bigvee\limits_{y}(t^{+1}_{I_j}(ry) \wedge t^{+1}_{I_j}(ys))]$.
- $I_j$- *Indeterminacy strength of connectedness* between two nodes $r$ and $s$ is defined by:
  $[i^{-\infty}_{I_j}(rs), i^{+\infty}_{I_j}(rs)] = [\bigvee\limits_{i\geq 1}\{i^{-i}_{I_j}(rs)\}, \bigvee\limits_{i\geq 1}\{i^{+i}_{I_j}(rs)\}]$ such that
  $[i^{-i}_{I_j}(rs), i^{+i}_{I_j}(rs)] = [(i^{-(i-1)}_{I_j} \circ i^{-(1)}_{I_j})(rs), (i^{+(i-1)}_{I_j} \circ i^{+(1)}_{I_j})(rs)]$ for $i \geq 2$
  and $[i^{-2}_{I_j}(rs), i^{+2}_{I_j}(rs)] = [(i^{-1}_{I_j} \circ i^{-1}_{I_j})(rs), (i^{+1}_{I_j} \circ i^{+1}_{I_j})(rs)]$
  $= [\bigvee\limits_{y}(i^{-1}_{I_j}(ry) \wedge i^{-1}_{I_j}(ys)), \bigvee\limits_{y}(i^{+1}_{I_j}(ry) \wedge i^{+1}_{I_j}(ys))]$.
- $I_j$- *Falsity strength of connectedness* between two nodes $r$ and $s$ is defined by:
  $[f^{-\infty}_{I_j}(rs), f^{+\infty}_{I_j}(rs)] = [\bigvee\limits_{i\geq 1}\{f^{-i}_{I_j}(rs)\}, \bigvee\limits_{i\geq 1}\{f^{+i}_{I_j}(rs)\}]$ such that
  $[f^{-i}_{I_j}(rs), f^{+i}_{I_j}(rs)] = [(f^{-(i-1)}_{I_j} \circ f^{-(1)}_{I_j})(rs), (f^{+(i-1)}_{I_j} \circ f^{+(1)}_{I_j})(rs)]$ for $i \geq 2$ and $[f^{-2}_{I_j}(rs), f^{+2}_{I_j}(rs)] = [(f^{-1}_{I_j} \circ f^{-1}_{I_j})(rs), (f^{+1}_{I_j} \circ f^{+1}_{I_j})(rs)]$
  $= [\bigvee\limits_{y}(f^{-1}_{I_j}(ry) \wedge f^{-1}_{I_j}(ys)), \bigvee\limits_{y}(f^{+1}_{I_j}(ry) \wedge f^{+1}_{I_j}(ys))]$.

**Definition 2.21.** An IVNGS $\check{G}_{iv} = (I, I_1, I_2, \ldots, I_t)$ is called an $I_j$-*cycle* if $(supp(I),$ $supp(I_1)$,$supp(I_2), \ldots, supp(I_t))$ is an $I_j - cycle$.

**Definition 2.22.** An IVNGS $\check{G}_{iv} = (I, I_1, I_2, \ldots, I_t)$ is an IVN fuzzy $I_j$-cycle (for some j) if $\check{G}_{iv}$ is an $I_j$-cycle and no unique $I_j$-edge $rs$ exists in $\check{G}_{iv}$ such that:
$[t^-_{I_j}(rs), t^+_{I_j}(rs)] = [\min\{t^-_{I_j}(uv) : uv \in I_j = supp(I_j)\}$,
$\min\{t^+_{I_j}(uv) : uv \in I_j = supp(I_j)\}]$ or
$[i^-_{I_j}(rs), i^+_{I_j}(rs)] = [\min\{i^-_{I_j}(uv) : uv \in I_j = supp(I_j)\}$,
$\min\{i^+_{I_j}(uv) : uv \in I_j = supp(I_j)\}]$ or
$[f^-_{I_j}(rs), f^+_{I_j}(rs)] = [\min\{f^-_{I_j}(uv) : uv \in I_j = supp(I_j)\}$,
$\min\{f^+_{I_j}(uv) : uv \in I_j = supp(I_j)\}]$.

**Example 2.23.** Consider an IVNGS $\check{G}_{iv} = (I, I_1, I_2)$ of graph structure $G = (U, R_1, R_2)$ as shown in Fig. 9.

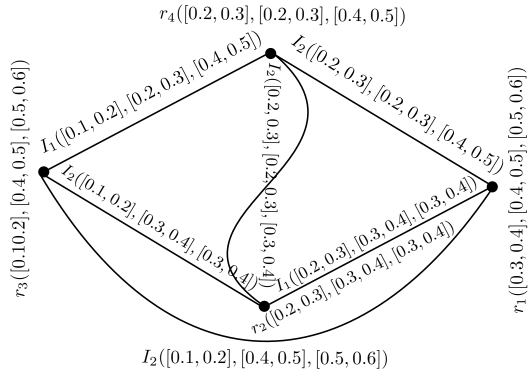


FIGURE 9. Interval-valued neutrosophic fuzzy $I_2$-cycle

This IVNGS $\check{G}_{iv}$ is an $I_2$-cycle, that is, $r_1 - r_4 - r_2 - r_3 - r_1$ and no unique $I_2$-edge $rs$ exists in $\check{G}_{iv}$ satisfying following condition:
$[t^-_{I_2}(rs), t^+_{I_2}(rs)] = [\min\{t^-_{I_2}(uv) : uv \in I_2 = supp(I_2)\},$
$\min\{t^+_{I_2}(uv) : uv \in I_2 = supp(I_2)\}]$ or
$[i^-_{I_2}(rs), i^+_{I_2}(rs)] = [\min\{i^-_{I_2}(uv) : uv \in I_2 = supp(I_2)\},$
$\min\{i^+_{I_2}(uv) : uv \in I_2 = supp(I_2)\}]$ or
$[f^-_{I_2}(rs), f^+_{I_2}(rs)] = [\min\{f^-_{I_2}(uv) : uv \in I_2 = supp(I_2)\},$
$\min\{f^+_{I_2}(uv) : uv \in I_2 = supp(I_2)\}]$.

**Definition 2.24.** Let $\check{G}_{iv} = (I, I_1, I_2, \ldots, I_t)$ be an IVNGS and '$r$' be a vertex of $\check{G}_{iv}$. If $(I', I'_1, I'_2, \ldots, I'_t)$ is an IVN subgraph-structure of $\check{G}_{iv}$ induced by $U \setminus \{r\}$ such that for all $u \neq r, v \neq r$

$$t^-_{I'}(r) = i^-_{I'}(r) = f^-_{I'}(r) = t^-_{I'_j}(ru) = i^-_{I'_j}(ru) = f^-_{I'_j}(ru) = 0,$$
$$t^+_{I'}(r) = i^+_{I'}(r) = f^+_{I'}(r) = t^+_{I'_j}(ru) = i^+_{I'_j}(ru) = f^+_{I'_j}(ru) = 0,$$
$$[t^-_{I'}(u), t^+_{I'}(u)] = [t^-_{I}(u), t^+_{I}(u)], \quad [i^-_{I'}(u), i^+_{I'}(u)] = [i^-_{I}(u), i^+_{I}(u)],$$
$$[f^-_{I'}(u), f^+_{I'}(u)] = [f^-_{I}(u), f^+_{I}(u)],$$
$$[t^-_{I'_j}(uv), t^+_{I'_j}(uv)] = [t^-_{I_j}(uv), t^+_{I_j}(uv)], \quad [i^-_{I'_j}(uv), i^+_{I'_j}(uv)] = [i^-_{I_j}(uv), i^+_{I_j}(uv)],$$
$$[f^-_{I'_j}(uv), f^+_{I'_j}(uv)] = [f^-_{I_j}(uv), f^+_{I_j}(uv)].$$

for all edges $ru, uv \in \check{G}_{iv}$, then vertex $r$ is an *IVN fuzzy $I_j$ cut-vertex*, if

(1) $t^{-\infty}_{I_j}(uv) > t^{-\infty}_{I'_j}(uv), t^{+\infty}_{I_j}(uv) > t^{+\infty}_{I'_j}(uv), [t^{-\infty}_{I_j}(uv), t^{+\infty}_{I_j}(uv)] \cap [t^{-\infty}_{I'_j}(uv), t^{+\infty}_{I'_j}(uv)] = \emptyset$
(2) $i^{-\infty}_{I_j}(uv) > i^{-\infty}_{I'_j}(uv), i^{+\infty}_{I_j}(uv) > i^{+\infty}_{I'_j}(uv), [i^{-\infty}_{I_j}(uv), i^{+\infty}_{I_j}(uv)] \cap [i^{-\infty}_{I'_j}(uv), i^{+\infty}_{I'_j}(uv)] = \emptyset$
(3) $f^{-\infty}_{I_j}(uv) > f^{-\infty}_{I'_j}(uv), f^{+\infty}_{I_j}(uv) > f^{+\infty}_{I'_j}(uv), [f^{-\infty}_{I_j}(uv), f^{+\infty}_{I_j}(uv)] \cap [f^{-\infty}_{I'_j}(uv), f^{+\infty}_{I'_j}(uv)] = \emptyset$

for some $u, v \in U \setminus \{r\}$. Note that vertex $r$ is an

- *IVN fuzzy $I_j - t$ cut-vertex*, if $t^{-\infty}_{I_j}(uv) > t^{-\infty}_{I'_j}(uv)$, $t^{+\infty}_{I_j}(uv) > t^{+\infty}_{I'_j}(uv)$, $[t^{-\infty}_{I_j}(uv), t^{+\infty}_{I_j}(uv)] \cap [t^{-\infty}_{I'_j}(uv), t^{+\infty}_{I'_j}(uv)] = \emptyset$
- *IVN fuzzy $I_j - i$ cut-vertex*, if $i^{-\infty}_{I_j}(uv) > i^{-\infty}_{I'_j}(uv)$, $i^{+\infty}_{I_j}(uv) > i^{+\infty}_{I'_j}(uv)$, $[i^{-\infty}_{I_j}(uv), i^{+\infty}_{I_j}(uv)] \cap [i^{-\infty}_{I'_j}(uv), i^{+\infty}_{I'_j}(uv)] = \emptyset$
- *IVN fuzzy $I_j - f$ cut-vertex*, if $f^{-\infty}_{I_j}(uv) > f^{-\infty}_{I'_j}(uv)$, $f^{+\infty}_{I_j}(uv) > f^{+\infty}_{I'_j}(uv)$, $[f^{-\infty}_{I_j}(uv), f^{+\infty}_{I_j}(uv)] \cap [f^{-\infty}_{I'_j}(uv), f^{+\infty}_{I'_j}(uv)] = \emptyset$

**Example 2.25.** Consider an IVNGS $\check{G}_{iv} = (I, I_1, I_2)$ of graph structure $G = (U, R_1, R_2)$ as represented in Fig. 10.

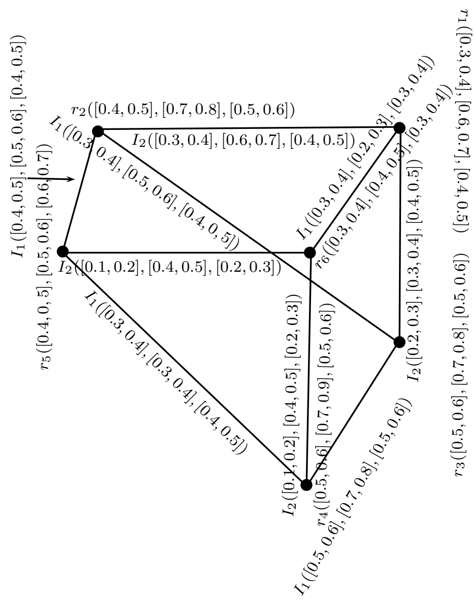


FIGURE 10. IVNGS $\check{G}_{iv} = (I, I_1, I_2)$

$\check{H}_{iv} = (I', I'_1, I'_2)$ is an IVN subgraph-structure of IVNGS $\check{G}_{iv}$, which is obtained by deleting vertex $r_2$ and shown in Fig. 11.

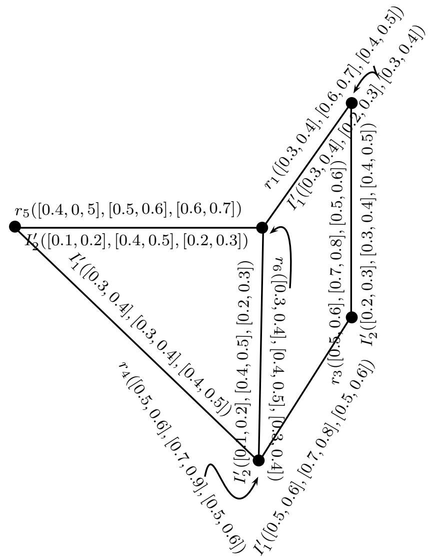


FIGURE 11. IVNGS $\check{H}_{iv} = (I', I'_1, I'_2)$

The vertex $r_2$ is an IVN fuzzy $I_1 - i$ cut-vertex. Since
$i^{-\infty}_{I'_1}(r_4r_5) = 0.3$, $i^{-\infty}_{I_1}(r_4r_5) = 0.5$, $i^{+\infty}_{I'_1}(r_4r_5) = 0.4$, $i^{+\infty}_{I_1}(r_4r_5) = 0.6$.
Clearly $i^{-\infty}_{I_1}(r_4r_5) = 0.5 > 0.3 = i^{-\infty}_{I'_1}(r_4r_5)$, $i^{+\infty}_{I_1}(r_4r_5) = 0.6 > 0.4 = i^{+\infty}_{I'_1}(r_4r_5)$,
$[i^{-\infty}_{I_1}(r_4r_5), i^{+\infty}_{I_1}(r_4r_5)] \cap [i^{-\infty}_{I'_1}(r_4r_5), i^{+\infty}_{I'_1}(r_4r_5)] = [0.5, 0.6] \cap [0.3, 0.4] = \emptyset$.

**Definition 2.26.** Let $\check{G}_{iv} = (I, I_1, I_2, \ldots, I_t)$ be an IVNGS and $rs$ be an $I_j - edge$. If $(I', I'_1, I'_2, \ldots, I'_t)$ is an IVN fuzzy spanning subgraph-structure of $\check{G}_{iv}$, such that

$$t^-_{I'_j}(rs) = i^-_{I'_j}(rs) = f^-_{I'_j}(rs) = 0, \quad t^+_{I'_j}(rs) = i^+_{I'_j}(rs) = f^+_{I'_j}(rs) = 0,$$
$$[t^-_{I'_j}(wx), t^+_{I'_j}(wx)] = [t^-_{I_j}(wx), t^+_{I_j}(wx)], \quad [i^-_{I'_j}(wx), i^+_{I'_j}(wx)] = [i^-_{I_j}(wx), i^+_{I_j}(wx)],$$
$$[f^-_{I'_j}(wx), f^+_{I'_j}(wx)] = [f^-_{I_j}(wx), f^+_{I_j}(wx)],$$

for all edges $wx \neq rs$, then edge $rs$ is an IVN fuzzy $I_j$-bridge if

(1) $t^{-\infty}_{I_j}(uv) > t^{-\infty}_{I'_j}(uv)$, $t^{+\infty}_{I_j}(uv) > t^{+\infty}_{I'_j}(uv)$, $[t^{-\infty}_{I_j}(uv), t^{+\infty}_{I_j}(uv)] \cap [t^{-\infty}_{I'_j}(uv), t^{+\infty}_{I'_j}(uv)] = \emptyset$
(2) $i^{-\infty}_{I_j}(uv) > i^{-\infty}_{I'_j}(uv)$, $i^{+\infty}_{I_j}(uv) > i^{+\infty}_{I'_j}(uv)$, $[i^{-\infty}_{I_j}(uv), i^{+\infty}_{I_j}(uv)] \cap [i^{-\infty}_{I'_j}(uv), i^{+\infty}_{I'_j}(uv)] = \emptyset$
(3) $f^{-\infty}_{I_j}(uv) > f^{-\infty}_{I'_j}(uv)$, $f^{+\infty}_{I_j}(uv) > f^{+\infty}_{I'_j}(uv)$, $[f^{-\infty}_{I_j}(uv), f^{+\infty}_{I_j}(uv)] \cap [f^{-\infty}_{I'_j}(uv), f^{+\infty}_{I'_j}(uv)] = \emptyset$

for some $u, v \in U$. Note that edge $rs$ is an

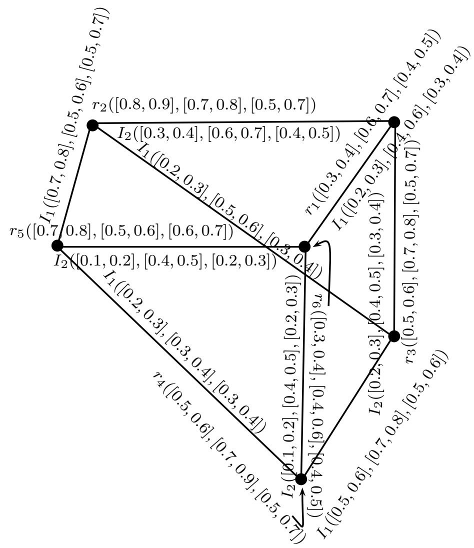


FIGURE 12. IVNGS $\check{G}_{iv} = (I, I_1, I_2)$

- ***IVN fuzzy*** $I_j - t$ ***bridge***, if $t^{-\infty}_{I_j}(uv) > t^{-\infty}_{I'_j}(uv)$, $t^{+\infty}_{I_j}(uv) > t^{+\infty}_{I'_j}(uv)$, $[t^{-\infty}_{I_j}(uv), t^{+\infty}_{I_j}(uv)] \cap [t^{-\infty}_{I'_j}(uv), t^{+\infty}_{I'_j}(uv)] = \emptyset$
- ***IVN fuzzy*** $I_j - i$ ***bridge***, if $i^{-\infty}_{I_j}(uv) > i^{-\infty}_{I'_j}(uv)$, $i^{+\infty}_{I_j}(uv) > i^{+\infty}_{I'_j}(uv)$, $[i^{-\infty}_{I_j}(uv), i^{+\infty}_{I_j}(uv)] \cap [i^{-\infty}_{I'_j}(uv), i^{+\infty}_{I'_j}(uv)] = \emptyset$
- ***IVN fuzzy*** $I_j - f$ ***bridge***, if $f^{-\infty}_{I_j}(uv) > f^{-\infty}_{I'_j}(uv)$, $f^{+\infty}_{I_j}(uv) > f^{+\infty}_{I'_j}(uv)$, $[f^{-\infty}_{I_j}(uv), f^{+\infty}_{I_j}(uv)] \cap [f^{-\infty}_{I'_j}(uv), f^{+\infty}_{I'_j}(uv)] = \emptyset$

**Example 2.27.** Consider an IVNGS $\check{G}_{iv} = (I, I_1, I_2)$ of graph structure $G = (U, R_1, R_2)$ as shown in Fig. 12. $\check{H}_{iv} = (I', I'_1, I'_2)$ is an IVN spanning subgraph-structure of IVNGS $\check{G}_{iv}$ obtained by deleting an $I_1$-edge $r_2r_5$ and shown in Fig. 13. The edge $r_2r_5$ is an IVN fuzzy $I_1 - bridge$ since

- $t^{-\infty}_{I'_1}(r_2r_5) = 0.2$, $t^{-\infty}_{I_1}(r_2r_5) = 0.7$, $t^{+\infty}_{I'_1}(r_2r_5) = 0.3$, $t^{+\infty}_{I_1}(r_2r_5) = 0.8$. $t^{-\infty}_{I_1}(r_2r_5) = 0.7 > 0.2 = t^{-\infty}_{I'_1}(r_2r_5)$, $t^{+\infty}_{I_1}(r_2r_5) = 0.8 > 0.3 = t^{+\infty}_{I'_1}(r_2r_5)$, $[t^{-\infty}_{I_1}(r_2r_5), t^{+\infty}_{I_1}(r_2r_5)] \cap [t^{-\infty}_{I'_1}(r_2r_5), t^{+\infty}_{I'_1}(r_2r_5)]$= [0.7, 0.8] $\cap$ [0.2, 0.3] = $\emptyset$.
- $i^{-\infty}_{I'_1}(r_2r_5) = 0.3$, $i^{-\infty}_{I_1}(r_2r_5) = 0.5$, $i^{+\infty}_{I'_1}(r_2r_5) = 0.4$, $i^{+\infty}_{I_1}(r_2r_5) = 0.6$. $i^{-\infty}_{I_1}(r_2r_5) = 0.5 > 0.3 = i^{-\infty}_{I'_1}(r_2r_5)$, $i^{+\infty}_{I_1}(r_2r_5) = 0.6 > 0.4 = i^{+\infty}_{I'_1}(r_2r_5)$,
  $[i^{-\infty}_{I_1}(r_2r_5), i^{+\infty}_{I_1}(r_2r_5)] \cap [i^{-\infty}_{I'_1}(r_2r_5), i^{+\infty}_{I'_1}(r_2r_5)]$= [0.5, 0.6] $\cap$ [0.3, 0.4] = $\emptyset$.

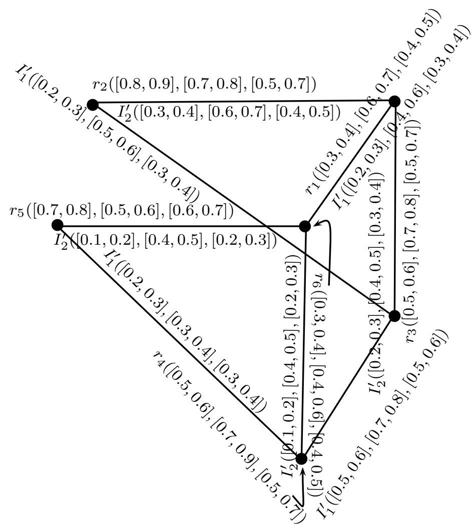

FIGURE 13. IVNGS $\check{H}_{iv} = (I', I_1', I_2')$

- $f^{-\infty}_{I_1'}(r_2r_5) = 0.3,\ f^{-\infty}_{I_1}(r_2r_5) = 0.5,\ f^{+\infty}_{I_1'}(r_2r_5) = 0.4,\ f^{+\infty}_{I_1}(r_2r_5) = 0.7$. $f^{-\infty}_{I_1}(r_2r_5) = 0.5 > 0.3 = f^{-\infty}_{I_1'}(r_2r_5)$, $f^{+\infty}_{I_1}(r_2r_5) = 0.7 > 0.4 = f^{+\infty}_{I_1'}(r_2r_5)$, $[f^{-\infty}_{I_1}(r_2r_5), f^{+\infty}_{I_1}(r_2r_5)] \cap [f^{-\infty}_{I_1'}(r_2r_5), f^{+\infty}_{I_1'}(r_2r_5)] = [0.5, 0.7] \cap [0.3, 0.4] = \emptyset$.

**Definition 2.28.** An IVNGS $\check{G}_{iv} = (I, I_1, I_2, \ldots, I_t)$ is an $I_j$*-tree* if $(supp(I), supp(I_1), supp(I_2), \ldots, supp(I_t))$ is an $I_j - tree$. Alternatively, $\check{G}_{iv}$ is an $I_j$-tree, if $\check{G}_{iv}$ has a subgraph induced by $supp(I_j)$ that forms a tree.

**Definition 2.29.** An IVNGS $\check{G}_{iv} = (I, I_1, I_2, \ldots, I_t)$ is an *IVN fuzzy* $I_j$*-tree* if $\check{G}_{iv}$ has an IVN fuzzy spanning subgraph-structure $\check{H}_{iv} = (I'', I_1'', I_2'', \ldots, I_t'')$ such that for all $I_j$-edges $rs$ not in $\check{H}_{iv}$,
$\check{H}_{iv}$ is an $I_j''$-tree and

(1) $t^-_{I_j}(rs) < t^{-\infty}_{I_j''}(rs),\ t^+_{I_j}(rs) < t^{+\infty}_{I_j''}(rs), [t^-_{I_j}(rs),\ t^+_{I_j}(rs)] \cap [t^{-\infty}_{I_j''}(rs), t^{+\infty}_{I_j''}(rs)] = \emptyset$
(2) $i^-_{I_j}(rs) < i^{-\infty}_{I_j''}(rs),\ i^+_{I_j}(rs) < i^{+\infty}_{I_j''}(rs), [i^-_{I_j}(rs),\ i^+_{I_j}(rs)] \cap [i^{-\infty}_{I_j''}(rs), i^{+\infty}_{I_j''}(rs)] = \emptyset$
(3) $f^-_{I_j}(rs) < f^{-\infty}_{I_j''}(rs),\ f^+_{I_j}(rs) < f^{+\infty}_{I_j''}(rs),\ [f^-_{I_j}(rs), f^+_{I_j}(rs)] \cap [f^{-\infty}_{I_j''}(rs), f^{+\infty}_{I_j''}(rs)] = \emptyset$

In particular,

- $\check{G}_{iv}$ is an *IVN fuzzy* $I_j - t$ *tree* if $t^-_{I_j}(rs) < t^{-\infty}_{I_j''}(rs),\ t^+_{I_j}(rs) < t^{+\infty}_{I_j''}(rs)$, $[t^-_{I_j}(rs), t^+_{I_j}(rs)] \cap [t^{-\infty}_{I_j''}(rs), t^{+\infty}_{I_j''}(rs)] = \emptyset$

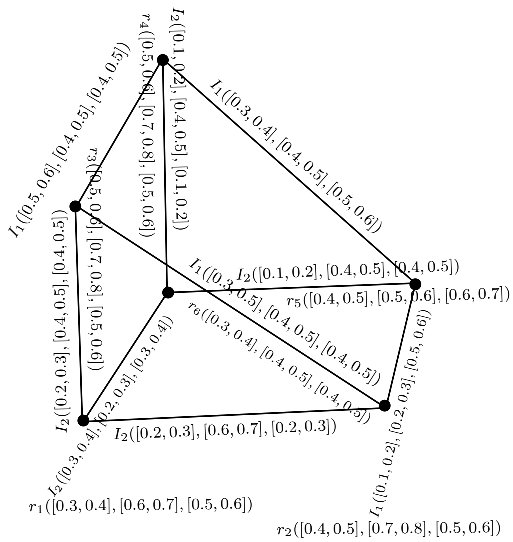


FIGURE 14. $\check{G}_{iv} = (I, I_1, I_2)$

- $\check{G}_{iv}$ is an *IVN fuzzy* $I_j - i$ *tree* if $i^-_{I_j}(rs) < i^{-\infty}_{I''_j}(rs)$, $i^+_{I_j}(rs) < i^{+\infty}_{I''_j}(rs)$, $[i^-_{I_j}(rs), i^+_{I_j}(rs)] \cap [i^{-\infty}_{I''_j}(rs), i^{+\infty}_{I''_j}(rs)] = \emptyset$
- $\check{G}_{iv}$ is an *IVN fuzzy* $I_j - f$ *tree* if $f^-_{I_j}(rs) < f^{-\infty}_{I''_j}(rs)$, $f^+_{I_j}(rs) < f^{+\infty}_{I''_j}(rs)$, $[f^-_{I_j}(rs), f^+_{I_j}(rs)] \cap [f^{-\infty}_{I''_j}(uv), f^{+\infty}_{I''_j}(uv)] = \emptyset$

**Example 2.30.** Consider an IVNGS $\check{G}_{iv} = (I, I_1, I_2)$ of graph structure $G = (U, R_1, R_2)$ as shown in Fig. 14. This IVNGS is $I_2$-tree, not $I_1$-tree. But it is IVN fuzzy $I_1 - t$ tree, since it has an IVN fuzzy spanning subgraph-structure $\check{H}_{iv} = (I'', I''_1, I''_2)$ as an $I''_1$-tree, which is obtained by deleting $I_1$-edge $r_2r_5$ from $\check{G}_{iv}$ and shown in Fig. 15. By direct calculations, we found that
$t^{-\infty}_{I''_1}(r_2r_5) = 0.3,\ t^{+\infty}_{I''_1}(r_2r_5) = 0.5,\ t^-_{I_1}(r_2r_5) = 0.1,\ t^+_{I_1}(r_2r_5) = 0.2,$
$t^-_{I_1}(r_2r_5) = 0.1 < 0.3 = t^{-\infty}_{I''_1}(r_2r_5),\ t^+_{I_1}(r_2r_5) = 0.2 < 0.5 = t^{+\infty}_{I''_1}(r_2r_5),$
$[t^{-\infty}_{I''_1}(r_2r_5), t^{+\infty}_{I''_1}(r_2r_5)] \cap [t^-_{I_1}(r_2r_5), t^+_{I_1}(r_2r_5)] = [0.3, 0.5] \cap [0.1, 0.2] = \emptyset.$

**Definition 2.31.** An IVNGS $\check{G}_{iv1} = (I_1, I_{11}, I_{12}, \ldots, I_{1t})$ of graph structure $G_1 = (U_1, R_{11}, R_{12}, \ldots, R_{1t})$ is isomorphic to IVNGS $\check{G}_{iv2} = (I_2, I_{21}, I_{22}, \ldots, I_{2t})$ of graph structure $G_2 = (U_2, R_{21}, R_{22}, \ldots, R_{2t})$, if there is a pair $(f, \varphi)$, where $f : U_1 \to U_2$ is bijection and $\varphi$ is a permutation on set $\{1, 2, \ldots, t\}$ such that:

$$[t^-_{I_1}(r), t^+_{I_1}(r)] = [t^-_{I_2}(f(r)), t^+_{I_2}(f(r))],\ \ [i^-_{I_1}(r), i^+_{I_1}(r)] = [i^-_{I_2}(f(r)), i^+_{I_2}(f(r))],$$
$$[f^-_{I_1}(r), f^+_{I_1}(r)] = [f^-_{I_2}(f(r)), f^+_{I_2}(f(r))],$$
$$[t^-_{I_{1j}}(rs), t^+_{I_{1j}}(rs)] = [t^-_{I_{2\varphi(j)}}(f(r)f(s)), t^+_{I_{2\varphi(j)}}(f(r)f(s))],$$
$$[i^-_{I_{1j}}(rs), i^+_{I_{1j}}(rs)] = [i^-_{I_{2\varphi(j)}}(f(r)f(s)), i^+_{I_{2\varphi(j)}}(f(r)f(s))],$$
$$[f^-_{I_{1j}}(rs), f^+_{I_{1j}}(rs)] = [f^-_{I_{2\varphi(j)}}(f(r)f(s)), f^+_{I_{2\varphi(j)}}(f(r)f(s))],$$

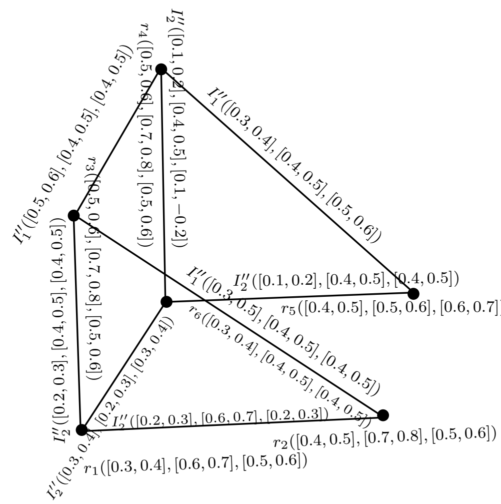

FIGURE 15. $\check{H}_{iv} = (I'', I''_1, I''_2)$

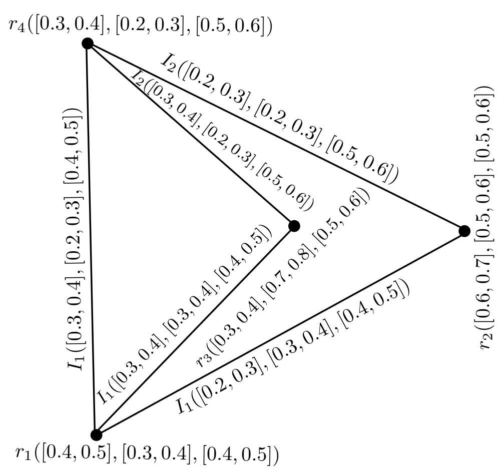


FIGURE 16. IVNGS $\check{G}_{iv1} = (I, I_1, I_2)$

for all $r \in U_1$, $rs \in I_{1j}$, $j \in \{1, 2, \ldots, t\}$.

**Example 2.32.** Let $\check{G}_{iv1} = (I, I_1, I_2)$ and $\check{G}_{iv2} = (I', I'_1, I'_2)$ be two IVNGSs of two GSs $G_1 = (U, R_1, R_2)$ and $G_2 = (U', R'_1, R'_2)$ as shown in Fig. 16 and Fig. 17, respectively.

$\check{G}_{iv1}$ and $\check{G}_{iv2}$ are isomorphic under $(f, \varphi)$, where $f : U \to U'$ is bijection and $\varphi$ is permutation on set $\{1, 2\}$ defined as $\varphi(1) = 2$, $\varphi(2) = 1$, such that:

$$[t^-_I(r_i), t^+_I(r_i)] = [t^-_{I'}(f(r_i)), t^+_{I'}(f(r_i))],$$

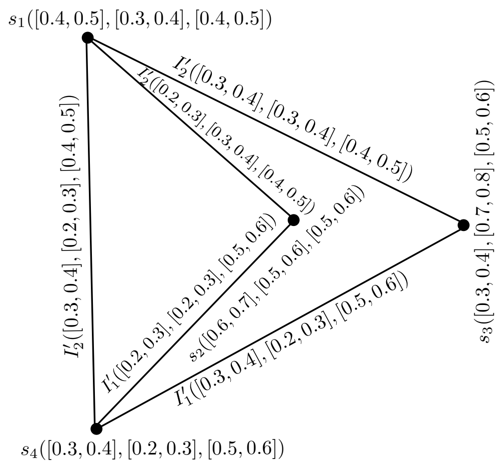


FIGURE 17. IVNGS $\check{G}_{iv2}=(I',I_1',I_2')$

$$[i_I^-(r_i),\, i_I^+(r_i)] = [i_{I'}^-(f(r_i)),\, i_{I'}^+(f(r_i))],$$
$$[f_I^-(r_i),\, f_I^+(r_i)] = [f_{I'}^-(f(r_i)),\, f_{I'}^+(f(r_i))],$$
$$[t_{I_j}^-(r_ir_k),\, t_{I_j}^+(r_ir_k)] = [t_{I_{\varphi(j)}}^-(f(r_i)f(r_k)),\, t_{I_{\varphi(j)}}^+(f(r_i)f(r_k))],$$
$$[i_{I_j}^-(r_ir_k),\, i_{I_j}^+(r_ir_k)] = [i_{I_{\varphi(j)}}^-(f(r_i)f(r_k)),\, i_{I_{\varphi(j)}}^+(f(r_i)f(r_k))],$$
$$[f_{I_j}^-(r_ir_k),\, f_{I_j}^+(r_ir_k)] = [f_{I_{\varphi(j)}}^-(f(r_i)f(r_k)),\, f_{I_{\varphi(j)}}^+(f(r_i)f(r_k))],$$

for all $r_i \in U$, $r_ir_k \in I_j$, $j \in \{1,2\}$ and $i,k \in \{1,2,3,4\}$.

**Definition 2.33.** An IVNGS $\check{G}_{iv1}=(I_1,I_{11},I_{12},\ldots,I_{1t})$ of graph structure $G_1 = (U_1,R_{11},R_{12},\ldots,R_{1t})$ is identical to IVNGS $\check{G}_{iv2}=(I_2,I_{21},I_{22},\ldots,I_{2t})$ of graph structure $G_2=(U_2,R_{21},R_{22},\ldots,R_{2t})$ if $f:U_1\to U_2$ is a bijection, such that

$$[t_{I_1}^-(r),\, t_{I_1}^+(r)] = [t_{I_2}^-(f(r)),\, t_{I_2}^+(f(r))],\quad [i_{I_1}^-(r),\, i_{I_1}^+(r)] = [i_{I_2}^-(f(r)),\, i_{I_2}^+(f(r))],$$
$$[f_{I_1}^-(r),\, f_{I_1}^+(r)] = [f_{I_2}^-(f(r)),\, f_{I_2}^+(f(r))],$$
$$[t_{I_{1j}}^-(rs),\, t_{I_{1j}}^+(rs)] = [t_{I_{2j}}^-(f(r)f(s)),\, t_{I_{2j}}^+(f(r)f(s))],$$
$$[i_{I_{1j}}^-(rs),\, i_{I_{1j}}^+(rs)] = [i_{I_{2j}}^-(f(r)f(s)),\, i_{I_{2j}}^+(f(r)f(s))],$$
$$[f_{I_{1j}}^-(rs),\, f_{I_{1j}}^+(rs)] = [f_{I_{2j}}^-(f(r)f(s)),\, f_{I_{2j}}^+(f(r)f(s))],$$

for all $r \in U_1$, $rs \in U_{1j}$, $j \in \{1,2,\ldots,t\}$.

**Example 2.34.** Let $\check{G}_{iv1}=(I,I_1,I_2)$ and $\check{G}_{iv2}=(I',I_1',I_2')$ be two IVNGSs of the graph structures $G_1=(U,R_1,R_2)$ and $G_2=(U',R_1',R_2')$, respectively as shown in Fig. 18 and Fig. 19, respectively.

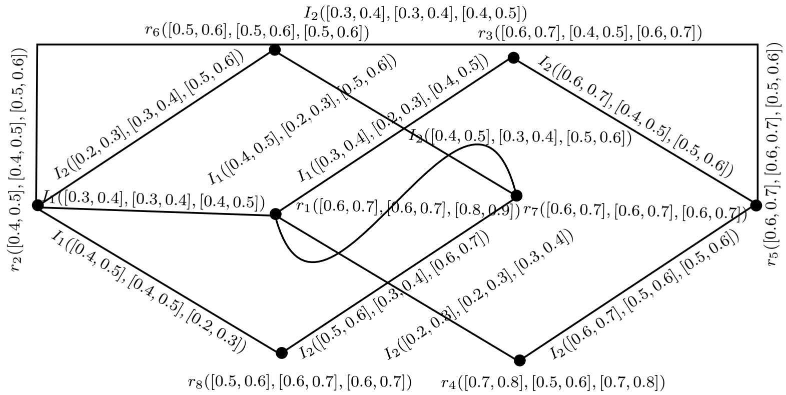


FIGURE 18. IVNGS $\check{G}_{iv1}$

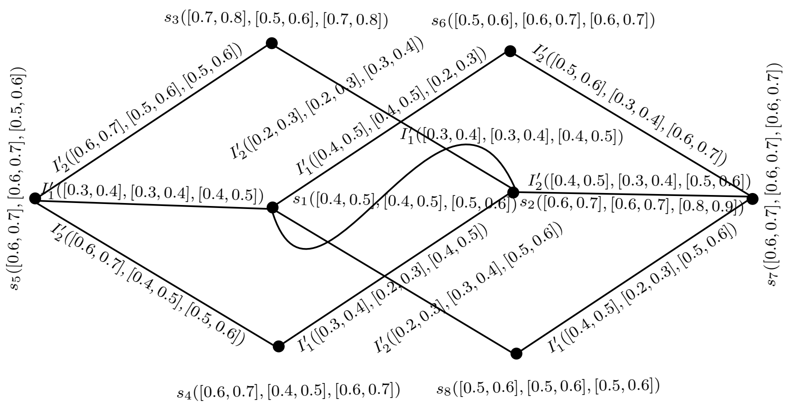


FIGURE 19. IVNGS $\check{G}_{iv2}$

IVNGS $\check{G}_{iv1}$ is identical to $\check{G}_{iv2}$ under $f : U \to U'$ defined as :
$f(r_1) = s_2$, $f(r_2) = s_1$, $f(r_3) = s_4$, $f(r_4) = s_3$, $f(r_5) = s_5$, $f(r_6) = s_8$, $f(r_7) = s_7$, $f(r_8) = s_6$. Moreover,

$$[t_I^-(r_i), t_I^+(r_i)] = [t_{I'}^-(f(r_i)), t_{I'}^+(f(r_i))],$$
$$[i_I^-(r_i), i_I^+(r_i)] = [i_{I'}^-(f(r_i)), i_{I'}^+(f(r_i))],$$
$$[f_I^-(r_i), f_I^+(r_i)] = [f_{I'}^-(f(r_i)), f_{I'}^+(f(r_i))],$$
$$[t_{I_j}^-(r_ir_k), t_{I_j}^+(r_ir_k)] = [t_{I'_j}^-(f(r_i)f(r_k)), t_{I'_j}^+(f(r_i)f(r_k))],$$
$$[i_{I_j}^-(r_ir_k), i_{I_j}^+(r_ir_k)] = [i_{I'_j}^-(f(r_i)f(r_k)), i_{I'_j}^+(f(r_i)f(r_k))],$$
$$[f_{I_j}^-(r_ir_k), f_{I_j}^+(r_ir_k)] = [f_{I'_j}^-(f(r_i)f(r_k)), f_{I'_j}^+(f(r_i)f(r_k))],$$

for all $r_i \in U$, $r_ir_k \in R_j$, $j \in \{1,2\}$, $i,k \in \{1,2,\ldots,8\}$.

**Definition 2.35.** Let $\check{G}_{iv} = (I, I_1, I_2, \ldots, I_t)$ be an IVNGS and $\varphi$ be a permutation on $\{I_1, I_2, \ldots, I_t\}$ and also on the set $\{1, 2, \ldots, t\}$, that is, $\varphi(I_j) = I_l$ iff $\varphi(j) = l$ for all $j$. If $rs \in I_j$ and

$$[t^-_{I^\varphi_j}(rs), t^+_{I^\varphi_j}(rs)] = [t^-_I(r) \wedge t^-_I(s) - \bigvee_{l \neq j} t^-_{\varphi(I_l)}(rs),$$
$$t^+_I(r) \wedge t^+_I(s) - \bigvee_{l \neq j} t^+_{\varphi(I_l)}(rs)], [i^-_{I^\varphi_j}(rs), i^+_{I^\varphi_j}(rs)] =$$
$$[i^-_I(r) \wedge i^-_I(s) - \bigvee_{l \neq j} i^-_{\varphi(I_l)}(rs), i^+_I(r) \wedge i^+_I(s) - \bigvee_{l \neq j} i^+_{\varphi(I_l)}(rs)],$$
$$[f^-_{I^\varphi_j}(rs), f^+_{I^\varphi_j}(rs)] =$$
$$[f^-_I(r) \wedge f^-_I(s) - \bigvee_{l \neq j} f^-_{\varphi(I_l)}(rs), f^+_I(r) \wedge f^+_I(s) - \bigvee_{l \neq j} f^+_{\varphi(I_l)}(rs)],$$

$j = 1, 2, \ldots, t$, then $rs \in I^\varphi_u$, where $u$ is selected, such that

- $t^-_{I^\varphi_u}(rs) \geq t^-_{I^\varphi_j}(rs)$, $t^+_{I^\varphi_u}(rs) \geq t^+_{I^\varphi_j}(rs)$, $[t^-_{I^\varphi_u}(rs), t^+_{I^\varphi_u}(rs)] \cap [\, t^-_{I^\varphi_j}(rs), t^+_{I^\varphi_j}(rs)] = \emptyset$
- $i^-_{I^\varphi_u}(rs) \geq i^-_{I^\varphi_j}(rs)$, $i^+_{I^\varphi_u}(rs) \geq i^+_{I^\varphi_j}(rs)$, $[i^-_{I^\varphi_u}(rs), i^+_{I^\varphi_u}(rs)] \cap [\, i^-_{I^\varphi_j}(rs), i^+_{I^\varphi_j}(rs)] = \emptyset$
- $f^-_{I^\varphi_u}(rs) \geq f^-_{I^\varphi_j}(rs)$, $f^+_{I^\varphi_u}(rs) \geq f^+_{I^\varphi_j}(rs)$, $[f^-_{I^\varphi_u}(rs), f^+_{I^\varphi_u}(rs)] \cap [\, f^-_{I^\varphi_j}(rs), f^+_{I^\varphi_j}(rs)] = \emptyset$

for all $j$. Then IVNGS $(I, I^\varphi_1, I^\varphi_2, \ldots, I^\varphi_t)$ is said to be $\varphi$-complement of IVNGS $\check{G}_{iv}$ and denoted by $\check{G}^{\varphi c}_{iv}$.

**Example 2.36.** Let $I = \{(r_1, [0.4, 0.5], [0.4, 0.5], [0.7, 0.8]), (r_2, [0.6, 0.7], [0.6, 0.7], [0.4, 0.5]), (r_3, [0.8, 0.9], [0.5, 0.6], [0.3, 0.4])\}, I_1 = \{(r_1r_3, [0.4, 0.5], [0.4, 0.5], [0.3, 0.4])\}$, $I_2 = \{(r_2r_3, [0.6, 0.7], [0.4, 0.5], [0.3, 0.4])\}, I_3 = \{(r_1r_2, [0.4, 0.5], [0.3, 0.4], [0.4, 0.5])\}$ be IVN subsets of $U = \{r_1, r_2, r_3\}$, $R_1 = \{r_1r_3\}$, $R_2 = \{r_2r_3\}$, $R_3 = \{r_1r_2\}$, respectively. Obviously, $\check{G}_{iv} = (I, I_1, I_2, I_3)$ is an IVNGS of GS $G = (U, R_1, R_2, R_3)$ as shown in Fig. 20

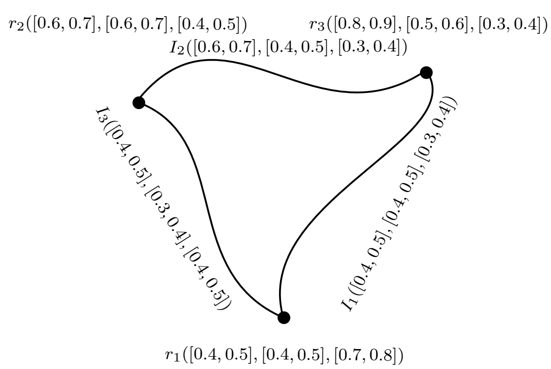


FIGURE 20. $\check{G}_{iv} = (I, I_1, I_2, I_3)$

Simple calculations of edges $r_1r_3, r_2r_3, r_1r_2 \in I_1, I_2, I_3$, respectively, show that $r_1r_3 \in I_3^{\varphi}$, $r_2r_3 \in I_1^{\varphi}$, $r_1r_2 \in I_2^{\varphi}$. So, $\check{G}_{iv}^{\varphi c}$ =$(I, I_1^{\varphi}, I_2^{\varphi}, I_3^{\varphi})$ is $\varphi$-complement of IVNGS $\check{G}_{iv}$ as shown in Fig. 21.

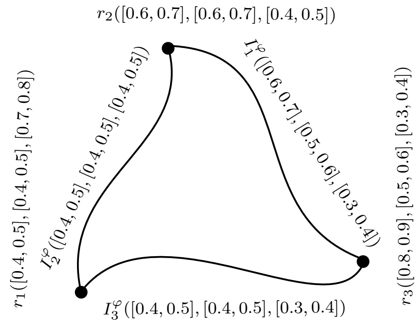


FIGURE 21. $\check{G}_{iv} = (I, I_1^{\varphi}, I_2^{\varphi}, I_3^{\varphi})$

**Proposition 2.37.** *$\varphi$-complement of an IVNGS $\check{G}_{iv} = (I, I_1, I_2, \ldots, I_t)$ is a strong IVNGS. Moreover, if $\varphi(j) = u$, where $j, u \in \{1, 2, \ldots, t\}$, then all $I_u$-edges in IVNGS $(I, I_1, I_2, \ldots, I_t)$ become $I_j^{\varphi}$-edges in $(I, I_1^{\varphi}, I_2^{\varphi}, \ldots, I_t^{\varphi})$.*

*Proof.* By definition of $\varphi$-complement,

$$[t^-_{I_j^{\varphi}}(rs),\ t^+_{I_j^{\varphi}}(rs)] = [t^-_I(r) \wedge t^-_I(s) - \bigvee_{l\neq j} t^-_{\varphi(I_l)}(rs),\ t^+_I(r) \wedge t^+_I(s) - \bigvee_{l\neq j} t^+_{\varphi(I_l)}(rs)] \quad (2.\ 1)$$

$$[i^-_{I_j^{\varphi}}(rs),\ i^+_{I_j^{\varphi}}(rs)] = [i^-_I(r) \wedge i^-_I(s) - \bigvee_{l\neq j} i^-_{\varphi(I_l)}(rs),\ i^+_I(r) \wedge i^+_I(s) - \bigvee_{l\neq j} i^+_{\varphi(I_l)}(rs)] \quad (2.\ 2)$$

$$[f^-_{I_j^{\varphi}}(rs),\ f^+_{I_j^{\varphi}}(rs)] = [f^-_I(r) \wedge f^-_I(s) - \bigvee_{l\neq j} f^-_{\varphi(I_l)}(rs),\ f^+_I(r) \wedge f^+_I(s) - \bigvee_{l\neq j} f^+_{\varphi(I_l)}(rs)] \quad (2.\ 3)$$

for $j \in \{1, 2, \ldots, t\}$. For expression of truth membership value:
As $t^-_I(r) \wedge t^-_I(s) \geq 0$, $t^+_I(r) \wedge t^+_I(s) \geq 0$ and $\bigvee\limits_{l\neq j} t^-_{\varphi(I_l)}(rs) \geq 0$, $\bigvee\limits_{l\neq j} t^+_{\varphi(I_l)}(rs) \geq 0$. Since $t^-_{I_j}(rs) \leq t^-_I(r) \wedge t^-_I(s)$, $t^+_{I_j}(rs) \leq t^+_I(r) \wedge t^+_I(s)$, for all $I_j$. This implies $\bigvee\limits_{l\neq j} t^-_{\varphi(I_l)}(rs) \leq t^-_I(r) \wedge t^-_I(s)$ and $\bigvee\limits_{l\neq j} t^+_{\varphi(I_l)}(rs) \leq t^+_I(r) \wedge t^+_I(s)$. It shows that $t^-_I(r) \wedge t^-_I(r) - \bigvee\limits_{l\neq j} t^-_{\varphi(I_l)}(rs) \geq 0$, $t^+_I(r) - \bigvee\limits_{l\neq j} t^+_{\varphi(I_l)}(rs) \geq 0$. Hence $t^-_{I_j^{\varphi}}(rs) \geq 0$ and $t^+_{I_j^{\varphi}}(rs) \geq 0$, for all $j$. Furthermore, $t^-_{I_j^{\varphi}}(rs)$ and $t^+_{I_j^{\varphi}}(rs)$ obtain maximum value when $\bigvee\limits_{l\neq j} t^-_{\varphi(I_l)}(rs)$ and $\bigvee\limits_{l\neq j} t^+_{\varphi(I_l)}(rs)$ are zero. Obviously, when $\varphi(I_j) = I_u$ and $rs$ is an $I_u$-edge then $\bigvee\limits_{l\neq j} t^-_{\varphi(I_l)}(rs)$ and $\bigvee\limits_{l\neq j} t^+_{\varphi(I_l)}(rs)$ acquire zero value. Hence

$$[t^-_{I_j^{\varphi}}(rs),\ t^+_{I_j^{\varphi}}(rs)] = [t^-_I(r) \wedge t^-_I(s),\ t^+_I(r) \wedge t^+_I(s)],\ \ for\ \ (rs) \in I_u,\ \varphi(I_j) = I_u \quad (2.\ 4)$$

For expression of indeterminacy membership value:
As $i_I^-(r) \wedge i_I^-(s) \geq 0$, $i_I^+(r) \wedge i_I^+(s) \geq 0$ and $\bigvee_{l \neq j} i^-_{\varphi(I_l)}(rs) \geq 0$, $\bigvee_{l \neq j} i^+_{\varphi(I_l)}(rs) \geq 0$. Since $i^-_{I_j}(rs) \leq i_I^-(r) \wedge i_I^-(s)$, $i^+_{I_j}(rs) \leq i_I^+(r) \wedge i_I^+(s)$, for all $I_j$. This implies $\bigvee_{l \neq j} i^-_{\varphi(I_l)}(rs) \leq i_I^-(r) \wedge i_I^-(s)$ and $\bigvee_{l \neq j} i^+_{\varphi(I_l)}(rs) \leq i_I^+(r) \wedge i_I^+(s)$. It shows that $i_I^-(r) \wedge i_I^-(r) - \bigvee_{l \neq j} i^-_{\varphi(I_l)}(rs) \geq 0$, $i_I^+(r) - \bigvee_{l \neq j} i^+_{\varphi(I_l)}(rs) \geq 0$. Hence $i^-_{I_j^\varphi}(rs) \geq 0$ and $i^+_{I_j^\varphi}(rs) \geq 0$, for all $j$. Furthermore, $i^-_{I_j^\varphi}(rs)$ and $i^+_{I_j^\varphi}(rs)$ achieve maximum value when $\bigvee_{l \neq j} i^-_{\varphi(I_l)}(rs)$ and $\bigvee_{l \neq j} i^+_{\varphi(I_l)}(rs)$ are zero. Obviously, when $\varphi(I_j) = I_u$ and $rs$ is an $I_u$-edge then $\bigvee_{l \neq j} i^-_{\varphi(I_l)}(rs)$ and $\bigvee_{l \neq j} i^+_{\varphi(I_l)}(rs)$ get zero value. Hence

$$[i^-_{I_j^\varphi}(rs),\ i^+_{I_j^\varphi}(rs)] = [i_I^-(r) \wedge i_I^-(s),\ i_I^+(r) \wedge i_I^+(s)],\ \ for\ \ (rs) \in I_u,\ \varphi(I_j) = I_u \quad (2.5)$$

For expression of falsity membership value:
As $f_I^-(r) \wedge f_I^-(s) \geq 0$, $f_I^+(r) \wedge f_I^+(s) \geq 0$ and $\bigvee_{l \neq j} f^-_{\varphi(I_l)}(rs) \geq 0$, $\bigvee_{l \neq j} f^+_{\varphi(I_l)}(rs) \geq 0$. Since $f^-_{I_j}(rs) \leq f_I^-(r) \wedge f_I^-(s)$, $f^+_{I_j}(rs) \leq f_I^+(r) \wedge f_I^+(s)$, for all $I_j$. This implies $\bigvee_{l \neq j} f^-_{\varphi(I_l)}(rs) \leq f_I^-(r) \wedge f_I^-(s)$ and $\bigvee_{l \neq j} f^+_{\varphi(I_l)}(rs) \leq f_I^+(r) \wedge f_I^+(s)$. It shows that $f_I^-(r) \wedge f_I^-(r) - \bigvee_{l \neq j} f^-_{\varphi(I_l)}(rs) \geq 0$, $f_I^+(r) - \bigvee_{l \neq j} f^+_{\varphi(I_l)}(rs) \geq 0$. Hence $f^-_{I_j^\varphi}(rs) \geq 0$ and $f^+_{I_j^\varphi}(rs) \geq 0$, for all $j$. Furthermore, $f^-_{I_j^\varphi}(rs)$ and $f^+_{I_j^\varphi}(rs)$ obtain maximum value when $\bigvee_{l \neq j} f^-_{\varphi(I_l)}(rs)$ and $\bigvee_{l \neq j} f^+_{\varphi(I_l)}(rs)$ are zero. Obviously, when $\varphi(I_j) = I_u$ and $rs$ is an $I_u$-edge then $\bigvee_{l \neq j} f^-_{\varphi(I_l)}(rs)$ and $\bigvee_{l \neq j} f^+_{\varphi(I_l)}(rs)$ acquire zero value. Hence

$$[f^-_{I_j^\varphi}(rs),\ f^+_{I_j^\varphi}(rs)] = [f_I^-(r) \wedge f_I^-(s),\ f_I^+(r) \wedge f_I^+(s)],\ \ for\ \ (rs) \in I_u,\ \varphi(I_j) = I_u \quad (2.6)$$

From expressions (2.4), (2.5) and (2.6), it is clear that

$$t_j^-(rs) = \min\{t^-(r), t^-(d)\},\ i_j^-(rs) = \min\{i^-(r), i^-(s)\},$$
$$f_j^-(rs) = \min\{f^-(-), f^-(s)\},\ t_j^+(rs) = \min\{t^+(r), t^+(s)\},$$
$$i_j^+(rs) = \min\{i^+(r), i^+(s)\},\ f_j^+(rs) = \min\{f^+(r), f^+(s)\},$$

Hence $\check{G}_{iv}$ is a strong $IVNGS$ and all $I_u$-edges in IVNGS $(I, I_1, I_2, \ldots, I_t)$ become $I_j^\varphi$-edges in $(I, I_1^\varphi, I_2^\varphi, \ldots, I_t^\varphi)$. □

**Definition 2.38.** Let $\check{G}_{iv} = (I, I_1, I_2, \ldots, I_t)$ be an IVNGS and $\varphi$ be a permutation on $\{1, 2, \ldots, t\}$. Then

(i) $\check{G}_{iv}$ is self complementary IVNGS if $\check{G}_{iv}$ is isomorphic to $\check{G}_{iv}^{\varphi c}$.
(ii) $\check{G}_{iv}$ is strong-self complementary IVNGS if $\check{G}_{iv}$ is identical to $\check{G}_{iv}^{\varphi c}$.

**Definition 2.39.** Let $\check{G}_{iv} = (I, I_1, I_2, \ldots, I_t)$ be an IVNGS. Then

(i) $\check{G}_{iv}$ is totally-self complementary IVNGS if $\check{G}_{iv}$ is isomorphic to $\check{G}_{iv}^{\varphi c}$, for all permutations $\varphi$ on $\{1, 2, \ldots, t\}$.

(ii) $\check{G}_{iv}$ is totally-strong-self complementary IVNGS if $\check{G}_{iv}$ is identical to $\check{G}_{iv}^{\varphi c}$, for all permutations $\varphi$ on $\{1, 2, \dots, t\}$.

**Example 2.40.** An IVNGS $\check{G}_{iv} = (I, I_1, I_2, I_3)$ shown in Fig. 22 is identical to $\varphi-$complement for all permutations $\varphi$ on set $\{1, 2, 3\}$. Hence it is totally-strong-self complementary IVNGS.

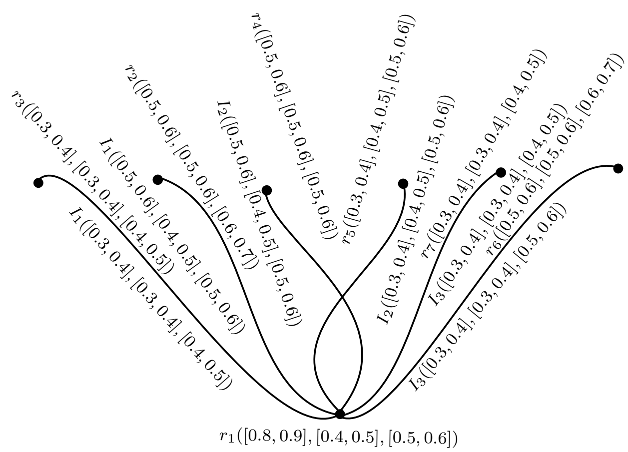


FIGURE 22. Totally-strong-self complementary IVNGS

**Theorem 2.41.** *An IVNGS is totally-self complementary if and only if it is a strong IVNGS.*

*Proof.* Consider a strong IVNGS $\check{G}_{iv}$ and permutation $\varphi$ on $\{$1,2, ..., t$\}$. By proposition 2.37, $\varphi$-complement of IVNGS $\check{G}_{iv} = (I, I_1, I_2, \dots, I_t)$ is a strong IVNGS. Moreover, if $\varphi^{-1}(u) = j$, where $j, u \in \{1, 2, \dots, t\}$, then all $I_u$-edges in IVNGS $(I, I_1, I_2, \dots, I_t)$ become $I_j^{\varphi}$-edges in $(I, I_1^{\varphi}, I_2^{\varphi}, \dots, I_t^{\varphi})$, this leads

$$t^-_{I_u}(rs) = t^-_I(r) \wedge t^-_I(s) = t^-_{I_j^\varphi}(rs),\ i^-_{I_u}(rs) = i^-_I(r) \wedge i^-_I(s) = i^-_{I_j^\varphi}(rs),$$
$$f^-_{I_u}(rs) = f^-_I(r) \wedge f^-_I(s) = f^-_{I_j^\varphi}(rs),\ t^+_{I_u}(rs) = t^+_I(r) \wedge t^+_I(s) = t^+_{I_j^\varphi}(rs),$$
$$i^+_{I_u}(rs) = i^+_I(r) \wedge i^+_I(s) = i^+_{I_j^\varphi}(rs),\ f^+_{I_u}(rs) = f^+_I(r) \wedge f^+_I(s) = f^+_{I_j^\varphi}(rs).$$

Therefore, under $f : U \to U$ (identity mapping), $\check{G}_{iv}$ and $\check{G}_{iv}^{\varphi}$ are isomorphic such that:

$$t^-_I(r) = t^-_I(f(r)),\ i^-_I(r) = i^-_I(f(r)),\ f^-_I(r) = f^-_I(f(r)),$$
$$t^+_I(r) = t^+_I(f(r)),\ i^+_I(r) = i^+_I(f(r)),\ f^+_I(r) = f^+_I(f(r)).$$

$$t^-_{I_u}(rs) = t^-_{I_j^\varphi}(f(r)f(s)) = t^-_{I_j^\varphi}(rs),\quad t^+_{I_u}(rs) = t^+_{I_j^\varphi}(f(r)f(s)) = t^+_{I_j^\varphi}(rs),$$

$$i^-_{I_u}(rs) = i^-_{I_j^\varphi}(f(r)f(s)) = i^-_{I_j^\varphi}(rs),\quad i^+_{I_u}(rs) = i^+_{I_j^\varphi}(f(r)f(s)) = i^+_{I_j^\varphi}(rs),$$

$$f^-_{I_u}(rs) = f^-_{I_j^\varphi}(f(r)f(s)) = f^-_{I_j^\varphi}(rs),\quad f^+_{I_u}(rs) = f^+_{I_j^\varphi}(f(r)f(s)) = f^+_{I_j^\varphi}(rs),$$

for all $rs \in I_u$, for $\varphi^{-1}(u) = j$; $j, u = 1, 2, \ldots, t$.
This holds for every permutation $\varphi$ on $\{1, 2, \ldots, t\}$. Hence $\check{G}_{iv}$ is totally-self complementary IVNGS. Conversely, let $\check{G}_{iv}$ is isomorphic to $\check{G}_{iv}^{\varphi}$ for each permutation $\varphi$ on $\{1, 2, \ldots, t\}$. Moreover, according to the definitions of isomorphism of IVNGSs and $\varphi$-complement of an IVNGS

$$t^-_{I_u}(rs) = t^-_{I^{\varphi}_j}(f(r)f(s)) = t^-_I(f(r)) \wedge t^-_I(f(s)) = t^-_I(r) \wedge t^-_I(s),$$
$$t^+_{I_u}(rs) = t^+_{I^{\varphi}_j}(f(r)f(s)) = t^+_I(f(r)) \wedge t^+_I(f(s)) = t^+_I(r) \wedge t^+_I(s),$$

$$i^-_{I_u}(rs) = i^-_{I^{\varphi}_j}(f(r)f(s)) = i^-_I(f(r)) \wedge i^-_I(f(s)) = i^-_I(r) \wedge i^-_I(s),$$
$$i^+_{I_u}(rs) = i^+_{I^{\varphi}_j}(f(r)f(s)) = i^+_I(f(r)) \wedge i^+_I(f(s)) = i^+_I(r) \wedge i^+_I(s),$$

$$f^-_{I_u}(rs) = f^-_{I^{\varphi}_j}(f(r)f(s)) = f^-_I(f(r)) \wedge f^-_I(f(s)) = f^-_I(r) \wedge f^-_I(s),$$
$$f^+_{I_u}(rs) = f^+_{I^{\varphi}_j}(f(r)f(s)) = f^+_I(f(r)) \wedge f^+_I(f(s)) = f^+_I(r) \wedge f^+_I(s),$$

for all $rs \in I_u$, $u = 1, 2, \ldots, t$. Hence $\check{G}_{iv}$ is a strong IVNGS. □

**Remark.** Every self complementary IVNGS is totally-self complementary.

**Theorem 2.42.** *If $G = (U, R_1, R_2, \ldots, R_t)$ is a totally-strong-self complementary graph structure and*
*$I = ([t^-_I, t^+_I], [i^-_I, i^+_I], [f^-_I, f^+_I])$ is an IVN subset of $U$ where $t^-_I, i^-_I, f^-_I, t^+_I, i^+_I, f^+_I$ are constant functions, then every strong IVNGS of $G$ with IVN vertex set $I$ is a totally-strong-self complementary IVNGS.*

*Proof.* Let $a$, $a' \in [0, 1]$, $b$, $b' \in [0, 1]$ and $c$, $c' \in [0, 1]$ be six constants and

$$t^-_I(r) = a,\ i^-_I(r) = b,\ f^-_I(r) = c,\ t^+_I(r) = a',\ i^+_I(r) = b',\ f^+_I(r) = c',$$
$$\text{for all } r \in U.$$

Since $G$ is a totally-strong-self complementary GS, so for every permutation $\varphi^{-1}$ on $\{1, 2, \ldots, t\}$ there is a bijection $f : U \to U$, such that for every $I_u-$edge $(rs)$, (f(r)f(s)) [an $I_j$-edge in $G$ ] is an $I_u$-edge in ${G_s}^{\varphi^{-1}c}$. Thus for every $I_u$-edge $(rs)$, (f(r)f(s)) [an $I_j$-edge in $\check{G}_{iv}$ ] is an $I^{\varphi}_u$-edge in ${\check{G}_{iv}}^{\varphi^{-1}c}$.
Moreover, $\check{G}_{iv}$ is a strong IVNGS, so

$$t^-_I(r) = a = t^-_I(f(r)),\ i^-_I(r) = b = i^-_I(f(r)),\ f^-_I(r) = c = f^-_I(f(r)),$$
$$t^+_I(r) = a' = t^+_I(f(r)),\ i^+_I(r) = b' = i^+_I(f(r)),\ f^+_I(r) = c' = f^+_I(f(r)),$$

for all $r \in U$, and

$$t^-_{I_u}(rs) = t^-_I(r) \wedge t^-_I(s) = t^-_I(f(r)) \wedge t^-_I(f(s)) = t^-_{I^{\varphi}_j}(f(r)f(s)),$$
$$i^-_{I_u}(rs) = i^-_I(r) \wedge i^-_I(s) = i^-_I(f(r)) \wedge i^-_I(f(s)) = i^-_{I^{\varphi}_j}(f(r)f(s)),$$
$$f^-_{I_u}(rs) = f^-_I(r) \wedge i^-_I(s) = f^-_I(f(r)) \wedge f^-_I(f(s)) = f^-_{I^{\varphi}_j}(f(r)f(s)),$$
$$t^+_{I_u}(rs) = t^+_I(r) \wedge t^+_I(s) = t^+_I(f(r)) \wedge t^+_I(f(s)) = t^+_{I^{\varphi}_j}(f(r)f(s)),$$
$$i^+_{I_u}(rs) = i^+_I(r) \wedge i^+_I(s) = i^+_I(f(r)) \wedge i^+_I(f(s)) = i^+_{I^{\varphi}_j}(f(r)f(s)),$$
$$f^+_{I_u}(rs) = f^+_I(r) \wedge i^+_I(s) = f^+_I(f(r)) \wedge f^+_I(f(s)) = f^+_{I^{\varphi}_j}(f(r)f(s)),$$

for all $rs \in I_j$, $j = 1, 2, \ldots, t$.
This shows $\check{G}_{iv}$ is a strong-self complementary IVNGS. This satisfies for each permutation $\varphi$ and $\varphi^{-1}$ on set $\{1, 2, \ldots, t\}$, thus $\check{G}_{iv}$ is a totally-strong-self complementary IVNGS. This completes the proof. □

**Remark.** Converse of theorem 2.42 may not true, for example a IVNGS depicted in Fig. 22 is totally-strong-self complementary IVNGS, it is also strong IVNGS with a totally-strong-self complementary underlying graph structure but $t_I^-$, $i_I^-$, $f_I^-$,$t_I^+$, $i_I^+$, $f_I^+$ are not the constant functions.

## 3. Conclusions

Interval-valued fuzzy set theory has numerous applications in various fields of science and technology, including, fuzzy control, artificial intelligence, operations research and decision-making. An interval-valued neutrosophic graph constitutes a generalization of the notion interval-valued fuzzy graph. In this research paper, we have introduced the notion of interval-valued neutrosophic graph structures and discussed many relevant notions with appropriate examples. We have also discussed some interesting properties of these notions.
**Conflict of interest.** The authors declare that they have no conflict of interest.
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