ARTICLE IN PRESS

ISA Transactions xxx (xxxx) xxx

Contents lists available at ScienceDirect

ISA Transactions

journal homepage: www.elsevier.com/locate/isatrans

Research article

Linear quadratic regulator problem governed by granular neutrosophic fractional differential equations[☆]

Nguyen Thi Kim Son [a], Nguyen Phuong Dong [b], Hoang Viet Long [c,d,*], Le Hoang Son [e], Alireza Khastan [f]

[a] Faculty of Natural Science, Hanoi Metropolitan University, Hanoi, Viet Nam
[b] Department of Mathematics, Hanoi Pedagogical University 2, Vinh Phuc, Viet Nam
[c] Division of Computational Mathematics and Engineering, Institute for Computational Science, Ton Duc Thang University, Ho Chi Minh City, Viet Nam
[d] Faculty of Mathematics and Statistics, Ton Duc Thang University, Ho Chi Minh City, Viet Nam
[e] VNU Information Technology Institute, Vietnam National University, Hanoi, Viet Nam
[f] Department of Mathematics, Institute for Advanced Studies in Basic Sciences, Zanjan, Iran

HIGHLIGHTS

- LQR problem governed by neutrosophic fractional differential equation.
- Neutrosophic Riemann–Liouville and Caputo derivatives under granular computing.
- Numerical neutrosophic solutions of some fractional telegraph equations.
- Key applications to DC motor model and one-link robot manipulator model.

ARTICLE INFO

Article history:
Received 3 March 2019
Received in revised form 17 July 2019
Accepted 1 August 2019
Available online xxxx

Keywords:
Neutrosophic numbers
Horizontal membership functions
Neutrosophic fractional derivatives
Neutrosophic fractional differential equations
Neutrosophic fractional linear quadratic regulator problem

ABSTRACT

Quadratic cost functions estimation in the linear optimal control systems governed by differential equations (DEs) or partial differential equations (PDEs) has a well-established discipline in mathematics with many interfaces to science and engineering. During its development, the impact of uncertain phenomena to objective function and the complexity of the systems to be controlled have also increased significantly. Many engineering problems like magnetohydromechanical, electromagnetical and signal analysis for the transmission and propagation of electrical signals under uncertain environment can be dealt with. In this paper, we study the optimal control problem with operating a fractional DEs and PDEs at minimum quadratic objective function in the framework of neutrosophic environment and granular computing. However, there has been no studies appeared on the neutrosophic calculus of fractional order. Hence, we will introduce some derivatives of fractional order, including the neutrosophic Riemann–Liouville fractional derivatives and neutrosophic Caputo fractional derivatives. Next, we propose a new setting of two important problems in engineering. In the first problem, we investigate the numerical and exact solutions of some neutrosophic fractional DEs and neutrosophic telegraph PDEs. In the second problem, we study the optimality conditions together with the simulation of states of a linear quadratic optimal control problem governed by neutrosophic fractional DEs and PDEs. Some key applications to DC motor model and one-link robot manipulator model are investigated to prove the effectiveness and correctness of the proposed method.

1. Introduction

1.1. Neutrosophic sets — An extension of introduced sets

Neutrosophic analysis ultimately based on neutrosophic set (NS) and neutrosophic logic has been recognized as a natural extension of classical analysis, fuzzy analysis and intuitionistic fuzzy analysis. The distinctions of NS with previous sets are the degree

☆ This research is funded by Vietnam National Foundation for Science and Technology Development (NAFOSTED) under grant number 101.02-2018.311.
∗ Corresponding author at: Division of Computational Mathematics and Engineering, Institute for Computational Science, Ton Duc Thang University, Ho Chi Minh City, Viet Nam.
E-mail addresses: ntkson@daihocthudo.edu.vn (N.T.K. Son), nguyenphuongdong@hpu2.edu.vn (N.P. Dong), hoangvietlong@tdtu.edu.vn (H.V. Long), sonlh@vnu.edu.vn (L.H. Son), khastan@gmail.com (A. Khastan).

https://doi.org/10.1016/j.isatra.2019.08.006
0019-0578/

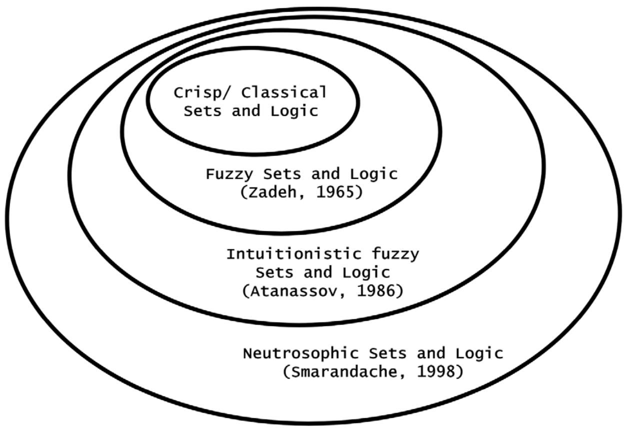

Fig. 1. The distinctions between neutrosophic sets with previous introduced sets.

of indeterminacy/neutrality as the independent component and relationship as shown in Fig. 1.

Smarandache has laid the first mathematical foundations on neutrosophic sets and measure [1,2], neutrosophic probability and statistic [3], neutrosophic calculus and precalculus [4], etc. The original basic studies of Smarandache have opened up a new trend of uncertain theoretical research with various applications to engineering. To promote his achievements, many scientists have been working hard to bring this theory reaching superior position with many achievements. For examples, we can mention researches on linear programming with applications to various problems [5], state feedback design for single input–single output neutrosophic linear systems [6], soft computing [7], fuzzy clustering [8], neutrosophic optimization technique [9,10]. Especially, neutrosophic theory has been successfully applied to decision making processing [11] and other applications [12].

Alongside with the history of formation and development of neutrosophic theory, Smarandache [13] classified the neutrosophic structure into two different types: the (t, i, f)-neutrosophic structures (based on the components t = truth, i = numerical indeterminacy, f = falsehood) and I-neutrosophic algebraic structures (based on neutrosophic numbers of the form $a + bI$, where $a, b \in \mathbb{R}$, I = literal indeterminacy). The neutrosophic differential calculus and neutrosophic dynamic systems based on (t, i, f)-neutrosophic structures were studied in [7]. In which, the authors used the parametric form of single-valued neutrosophic sets to define the neutrosophic differentiability of integer order via granular difference. The precalculus and calculus of neutrosophic numbers of the form $a+bI$ were first introduced in [4]. In this setting, differential calculus and integral calculus are built with integer order and based on the set analysis. However, the neutrosophic fractional calculus has not been studied yet.

1.2. Fuzzy fractional calculus with applications

Fractional calculus is an important branch of mathematical analysis with a long history of studying and applications. This theory aims to study some different possibilities in defining the order of power of differential operator in the case this order is not integer. Alongside with the development of fractional calculus, fractional DEs, fractional PDEs and fractional order control have been also dramatically studied by the high applicability, see for instance [14,15]. In the last decade, fractional differential calculus in uncertain environment has obtained a significant achievement. Fuzzy differential calculus was introduced by Dubois and Prade [16]. Later, fuzzy DEs of integer order was studied in [17], and after that, it has attracted a large amount of research over the past three decades including extensive works for fuzzy PDEs [18,19]. Fuzzy fractional calculus extends the concept of fuzzy derivative to non-integer order. Fuzzy fractional DEs without using fuzzy fractional derivatives was first studied in [20] for Cauchy problem. Then Allahviranloo et al. [21] extended this concept to fractional DEs with Riemann–Liouville derivatives. Mazandarani and Kamyad [22] studied numerical solutions of fractional initial problem under Caputo gH-differentiability. In the last decade, fuzzy analysis of fractional order has amazingly developed in various applications including fuzzy fractional PDEs [23]. For optimal problems, the combination of fuzzy theory and fractional calculus theory in optimal control has also obtained many significant achievements. Shahri et al. [24] proposed a new procedure called augmented Lagrangian particle swarm optimization using fractional order velocity to enhance the performance and convergence rate of a desired controller. Sharma et al. [25] proposed a scheme implemented two-layered fractional order fuzzy logic controller for robotic manipulator. Alinezhad and Allahviranloo [14] presented an extension to determine the best possible fuzzy control which satisfies the related fuzzy fractional dynamic systems and minimizes the fuzzy performance index. Jafari et al. [26] used interval type-2 Fuzzy logic systems to approximate an underlying relationship in a fractional-order nonlinear system. Long [19] combined two most common types of uncertainty into one fractional PDEs, namely fuzzy random PDEs. For further references on fractional calculus of fuzzy-valued functions with various applications, readers are kindly referred to [27–30].

1.3. Intuitionistic fuzzy fractional calculus with applications

The difficulty in defining the difference operator in the set of fuzzy intuitionistic sets has limited further research on fuzzy intuitionistic derivatives, fuzzy intuitionistic integral and beyond that is the study of fuzzy intuitionistic DEs and fuzzy intuitionistic optimal control problem. Modal [31] studied the differentiability for intuitionistic fuzzy-valued function by generalized Hukuhara difference. This approach has a limitation that leads to multiplicity of solutions with different geometrical representations. Furthermore, the switching points of differentiability may requires us to solve many complicated cases, i.e., the fuzzy solutions depend on the natural of uncertainty. As far as we know, there is no result on the Atanassov's intuitionistic fuzzy dynamics system of fractional order, even in some aspects the fuzzy intuitionistic set is considered as a very significant extension of the Zadeh's fuzzy set.

1.4. Neutrosophic fractional calculus with applications

Smarandache proposed the concept of neutrosophic sets in 1998, and this new born theory has been widely accepted and studied by many scientists. However, it seems to be an imbalance between the study of neutrosophic algebraic structure (neutrosophic approximate reasoning, neutrosophic decision making, etc.) compared to the study of neutrosophic analysis (neutrosophic dynamic systems, neutrosophic optimal control problems, etc.). The idea of integrating neutrosophic uncertainty into the dynamic systems was initiated by Smarandache in Section 2.21 [13], that from the classical point of view, it is a chaotic extension. Instead of fixed points in classical dynamic systems, one deals with fixed regions, as approximate values of the neutrosophic variables. In general, all neutrosophic dynamic system's components are interacted in a certain degree, repelling in another degree, and neutral (no interaction) in a different degree. For example, let us consider a simple model of one-link robot manipulator where the motion of robot's arm is controlled by a DC motor via a gear (see Example 6.2. for more details). Using Lagrange's equation, we can derive the dynamics of robot manipulator as follows

$$M\ell^2 \, {}^{gr}_{0^+}\mathrm{D}^{2\beta}\theta(t) = Mg\ell\sin\theta(t) + \rho K_{mo}\left(\frac{u(t)}{R} - \frac{\rho K_b \, {}^{gr}_{0^+}\mathrm{D}^{\beta}\theta(t)}{R}\right),$$

where the reasonable parameters of robot are given in Table 6. Here, due to the influence of environment factors such as the height, temperature, humidity, air pressure and the errors arising from measurements, the gravitational acceleration g cannot be fixed value, it should be neutrosophic-valued $g = 9.7 + 0.3I$.

On the basis of the aforesaid motivation, in this paper, we introduce some basic definitions and results for neutrosophic fractional calculus related to I-neutrosophic structures and investigate some applications to neutrosophic fractional DEs, neutrosophic optimal control governed by fractional DEs, fractional PDEs. Let us recall that the fractional calculus induced from fractional order operators that is a powerful mathematical tool for describing long memory and hereditary properties of different phenomena and process, see [24,27]. Experimental results prove that fractional computing are appropriate to represent many industrial automation better than integer order, see [32]. Thus, the extension of neutrosophic differential calculus to fractional order will extend the applicable scope of optimal control problem to many important engineering problems.

1.5. The contributions and novelties

The contribution in this paper is threefold:

(1) Firstly, we introduce fractional calculus of neutrosophic-valued functions. We will define new concepts of neutrosophic fractional derivatives in both Riemann–Liouville and Caputo types. Neutrosophic fractional integral will be also introduced with its fundamental properties. We note that the notions of the differential operator on the space of neutrosophic sets is not easy to define because of the semi-linear algebraic structure of the base space. Moreover, the well-defined property of the differential operator for neutrosophic-valued functions often requires very complicated hypothesis. Generalized Hukuhara difference [33] can be a candidate for differential calculus on the neutrosophic space but the multi-trajectory with different long-term geometric behavior related to switching points will cause the complexity for the multi-solution of neutrosophic DEs. Thanks to granular representation concept and relative-distance-measure variables introduced in [34,35], we manage in establishing the fractional calculus of neutrosophic functions. Our results extend and develop the previous works [36–38] to neutrosophic environment.

(2) Secondly, we investigate new setting of fractional order equations with uncertainty, namely neutrosophic fractional DEs and neutrosophic fractional PDEs. Under granular differentiability, the neutrosophic DEs can be transformed equivalently into the set of deterministic DEs with granular parameters and they preserve the qualitative properties of the neutrosophic solution. One of the most advantages of the proposed method is that we can proceed both analytical methods and numerical methods on deterministic DEs and then the correspondence neutrosophic solutions can be retrieved by inverse transformation. This is very useful for practical applications in the real world, where we always have to solve the solutions of modeling problems affected by uncertainty. To demonstrate the efficiency of the theoretical method, we utilize Adomian decomposition method and Matlab Toolbox to solve approximate solution of neutrosophic fractional telegraph equations.

(3) Thirdly, our main goal is to study Linear Quadratic Regulator (LQR) problem for a linear time-invariant neutrosophic fractional differential system (22). The aim of LQR problem is to find a control input $u(t)$ that steers the state variable $x(t)$ of system (22) from an initial state $x(t_0) = x_0$ to the original at the time $t = t_f$ and minimizes the performance index (23). To this end, we establish some sufficient conditions for (x, u) to be an optimal pair and apply to some important problems such as DC motor model and one-link robot manipulator model.

The paper is organized as follows: Section 2 presents some preliminaries on fractional calculus, space of neutrosophic numbers. Moreover, we introduce the concepts of neutrosophic-valued functions of several variables and their granular partial derivatives. The definitions and related properties of granular Riemann–Liouville derivative, granular Caputo derivative and granular integrals are introduced in Section 3, while Section 4 is devoted to present neutrosophic granular partial fractional derivatives. In addition, some theorems regarding to the relationship between these concepts are also given. In Section 5, under the granular differentiability, the Cauchy problems for neutrosophic fractional DEs and neutrosophic fractional PDEs are investigated. And then, in order to illustrate the theoretical results, we consider some realistic models such as fractional damped single degree of freedom spring mass model and neutrosophic space–time fractional telegraph equations and calculate their approximate solutions by numerical methods and Matlab program. Next, based on the new definitions and theorem presented in the previous sections, the solution to the Neutrosophic Fractional Linear Quadratic Regulator problem is given in Section 6. Additionally, two application examples are given to illustrate obtained result. The brief of the obtained results are explained and discussed in Section 7. Finally, conclusions and future works are discussed in Section 8.

2. Preliminaries

For the convenience of the readers, we will recall some preliminaries on fractional calculus. For more details, see [15] and the references therein. Some common notations used throughout this paper will be summarized in Table 1.

2.1. Fractional derivative and integral of real functions

For convenience to the readers, we briefly recall some notions of fractional real analysis. For more details, see [15] and the references therein.

The right-sided and left-sided fractional integrals $\mathcal{I}^{\beta}_{a+}f(t)$, $\mathcal{I}^{\beta}_{b-}f(t)$ of order $\beta > 0$ of a function $f \in L^1([a, b], \mathbb{R})$ are defined by

$$\mathcal{I}^{\beta}_{a+}f(t) := \frac{1}{\Gamma(\beta)}\int_a^t (t-s)^{\beta-1}f(s)ds,\ \mathcal{I}^{\beta}_{b-}f(t) := \frac{1}{\Gamma(\beta)}\int_t^b (s-t)^{\beta-1}f(s)ds,$$

respectively. Here, $\Gamma(\beta)$ is the gamma function $\Gamma(\beta) = \int_0^\infty t^{\beta-1} e^{-t}dt$.

The right-sided and left-sided Riemann–Liouville fractional derivatives ${}^{RL}_{a^+}\mathcal{D}^{\beta}f(t)$, ${}^{RL}_{b^-}\mathcal{D}^{\beta}f(t)$ of order $\beta > 0$ of a function f are defined by

$${}^{RL}_{a^+}\mathcal{D}^{\beta}f(t) := \frac{1}{\Gamma(n-\beta)}\left(\frac{d}{dt}\right)^n \int_a^t (t-s)^{n-\beta-1}f(s)ds,$$

and

$${}^{RL}_{b^-}\mathcal{D}^{\beta}f(t) := \frac{1}{\Gamma(n-\beta)}\left(-\frac{d}{dt}\right)^n \int_t^b (s-t)^{n-\beta-1}f(s)ds,$$

Table 1
The mathematical notations.

Notation	Description	Location
$\mathcal{I}^{\beta}_{b-}$, $\mathcal{I}^{\beta}_{a+}$	Left-sided, Right-sided fractional integrals	Section 2.1
${}^{RL}_{b-}\mathcal{D}^{\beta}$, ${}^{RL}_{a+}\mathcal{D}^{\beta}$	Left-sided, Right-sided Riemann–Liouville derivatives	Definition 2.1
${}^{C}_{b-}\mathrm{D}^{\beta}$, ${}^{C}_{a+}\mathrm{D}^{\beta}$	Left-sided, Right-sided Caputo derivatives	Section 2.1
$\mathcal{L}(u)$	Horizontal membership function of u	Definition 2.1
d_{gr}/dt	Granular derivative	Definition 2.6
$\partial_{gr}/\partial x_i$	Granular partial derivative w.r.t. variable x_i	Definition 2.9
${}^{gr}\mathcal{I}^{\beta}_{b-}$, ${}^{gr}\mathcal{I}^{\beta}_{a+}$	Left-sided, Right-sided neutrosophic granular integrals	Definition 3.1
${}^{gr}_{b-}\mathcal{D}^{\beta}$, ${}^{gr}_{a+}\mathcal{D}^{\beta}$	Left-sided, Right-sided gr-Riemann–Liouville derivatives	Definition 3.2
${}^{gr}_{b-}\mathrm{D}^{\beta}$, ${}^{gr}_{a+}\mathrm{D}^{\beta}$	Left-sided, Right-sided gr-Caputo derivatives	Definition 3.3
${}^{gr}\mathcal{I}^{\sigma_k}_{b_k^-}$, ${}^{gr}\mathcal{I}^{\sigma_k}_{a_k^+}$	Left-sided, Right-sided gr-partial integrals w.r.t. x_k	Definition 4.1
${}^{gr}_{b_k^-}\mathrm{D}^{\beta}_{x_k}$, ${}^{gr}_{a_k^+}\mathrm{D}^{\beta}_{x_k}$	Left-sided, Right-sided gr-Caputo partial derivatives w.r.t. x_k	Definition 4.2

respectively, where $n-1<\beta<n$, $n\in\mathbb{N}$ and the function f has absolutely continuous derivatives up to order $(n-1)$.

The right-sided and left-sided Caputo fractional derivatives ${}^{C}_{a+}\mathrm{D}^{\beta}f(t)$, ${}^{C}_{b-}\mathrm{D}^{\beta}f(t)$ of order $\beta>0$ of a function $f\in C^n([a,b],\mathbb{R})$ are defined by

$$ {}^{C}_{a+}\mathrm{D}^{\beta}f(t) := \frac{1}{\Gamma(n-\beta)}\int_a^t (t-s)^{n-\beta-1}f^{(n)}(s)ds, $$

and

$$ {}^{C}_{b-}\mathrm{D}^{\beta}f(t) := \frac{(-1)^n}{\Gamma(n-\beta)}\int_t^b (s-t)^{n-\beta-1}f^{(n)}(s)ds, $$

respectively, where $n-1<\beta<n$, $n\in\mathbb{N}$.

Remark 2.1.

(i) The Caputo fractional derivatives of a constant function C is equal to 0.

(ii) The Riemann–Liouville fractional derivatives of order $\beta\in(0,1)$ of a constant function C is given by

$$ {}^{RL}_{a+}\mathcal{D}^{\beta}C = \frac{C}{\Gamma(1-\beta)}(t-a)^{-\beta} \quad \text{and} $$

$$ {}^{RL}_{b-}\mathcal{D}^{\beta}C = \frac{C}{\Gamma(1-\beta)}(b-t)^{-\beta}. $$

2.2. Space of neutrosophic numbers

The set of all neutrosophic numbers, denoted by $\mathcal{E}$, consists of elements of the form $u=a+bI$, where $a\in\mathbb{R}$, $b\in\mathbb{R}^+$ and I is the indeterminacy [4].

Note that the indeterminacy I can be presented as a possible changeable range $\left[\underline{I},\overline{I}\right]$. Then, the neutrosophic number $u=a+bI$ can be specified as an interval representation $\left[a+b\underline{I},a+b\overline{I}\right]$. In particular case, if either $b=0$ or $\underline{I}=\overline{I}$, that means $bI=0$, then $u=a$ is a real number. The basic arithmetic operations on the set $\mathcal{E}$ such as addition, multiplication, scalar multiplication and division can be referred to [4].

Definition 2.1 (*gr-Representation*)**.** For $u=a+bI\in\mathcal{E}$, the number u can be rewritten in the following horizontal membership function form

$$ \begin{aligned} u^{gr} &: [0,1] \to [a+b\underline{I}, a+b\overline{I}] \\ &\mu \mapsto u^{gr}(\mu) = a+b\underline{I}+b\,\mathrm{diam}[I]\mu \end{aligned} $$

in which $\mathrm{diam}[I]=\overline{I}-\underline{I}$, the notion "gr" stands for the granule of information included in $[a+b\underline{I},a+b\overline{I}]$ and $\mu\in[0,1]$ is called the relative distance measure variable.

Let us denote by $\mathcal{L}(u)\triangleq u^{gr}(\mu)$ the horizontal membership function of an element $u\in\mathcal{E}$. In addition, we can see that the interval representation of $u\in\mathcal{E}$ can be obtained by using the following inverse transformation

$$ \mathcal{L}^{-1}(u^{gr}(\mu)) = \left[\min_{\mu\in[0,1]} u^{gr}(\mu), \max_{\mu\in[0,1]} u^{gr}(\mu)\right]. \tag{1} $$

For two neutrosophic numbers u_1 and u_2, we have $u_1=u_2$ if and only if $\mathcal{L}(u_1)=\mathcal{L}(u_2)$. Then, we define

$$ \mathcal{L}(u_1\otimes u_2)\triangleq\mathcal{L}(u_1)*\mathcal{L}(u_2), $$

where the notions "$\otimes$", "$*$" are used to present the arithmetic operations such as addition, subtraction, multiplication or division, in $\mathcal{E}$ and $\mathbb{R}$, respectively. Especially, the difference in this sense, denoted by $\ominus^{gr}$, is called granular difference (or gr-difference for short).

Definition 2.2 (*gr-Metric*)**.** Let u_1, u_2 be two neutrosophic numbers. Then, the gr-distance is defined as follows

$$ \rho^{gr}(z_1,z_2) = \max_{\mu_1,\mu_2}\left|z_1^{gr}(\mu_1)-z_2^{gr}(\mu_2)\right|. $$

We can see that the space $(\mathcal{E},\rho^{gr})$ is a complete metric space.

2.3. Neutrosophic matrices

Definition 2.3. Matrix $A=(a_{ij})_{m\times n}$ is called a neutrosophic matrix of order $m\times n$ if all the entries a_{ij} are neutrosophic numbers. Especially, if $m=n$ then the matrix A is called square neutrosophic matrix of order n. Furthermore, based on the horizontal membership function approach, the granular representation of A is given by $A_{gr}(\mu_A)=(a_{ij}^{gr}(\mu_{ij}))_{m\times n}$, where $\mu_A:=\{\mu_{ij}\in[0,1]: i=\overline{1,m},\ j=\overline{1,n}\}$.

Remark 2.2. Based on arithmetic operations in $\mathcal{E}$, we can perform the neutrosophic matrix operations, e.g., matrix addition–subtraction, scalar multiplication, matrix transpose, matrix inverse, and so on.

Definition 2.4. The inverse matrix A^{-1} and the transpose matrix A^{T} of a neutrosophic matrix A are neutrosophic matrices such that $\mathcal{L}\left(A^{-1}\right)=(\mathcal{L}(A))^{-1}$ and $\mathcal{L}\left(A^{\mathrm{T}}\right)=(\mathcal{L}(A))^{\mathrm{T}}$, respectively.

Definition 2.5. Let $A=(a_{ij})_{n\times n}$ be a square neutrosophic matrix of order n. We call a number $\lambda_i\in\mathcal{E}$ is an eigenvalue of A if and only if

$$ \det\left(\lambda_i^{gr}(\mu_i)\mathrm{I}_n - A_{gr}(\mu_A)\right)=0, $$

where $\det(\cdot)$ and I_n represent the determinant and the $n\times n$ identity matrix, respectively.

Please cite this article as: N.T.K. Son, N.P. Dong, H.V. Long et al., Linear quadratic regulator problem governed by granular neutrosophic fractional differential equations. ISA Transactions (2019), https://doi.org/10.1016/j.isatra.2019.08.006.

Example 2.1. Let $I = [0, 1]$ and consider a neutrosophic matrix $A = \begin{pmatrix} -1 & 1 \\ 0 & -3+I \end{pmatrix}$. The granular representation of the matrix A, given by $\begin{pmatrix} -1 & 1 \\ 0 & -3+\mu \end{pmatrix}$, has the characteristic equation $(\lambda^{gr}(\mu))^2 + (4-\mu)\lambda^{gr}(\mu) + (3-\mu) = 0$, for each $\mu \in [0, 1]$. Then, we have the eigenvalues of $A^{gr}(\mu)$ is $\lambda_1^{gr}(\mu) = -1,\ \lambda_2^{gr}(\mu) = -3+\mu$, corresponding to the eigenvalues $\lambda_1 = -1\ \lambda_2 = -3+I$.

2.4. Neutrosophic-valued functions and their calculus properties

A function $f : [a, b] \subset \mathbb{R} \to \mathcal{E}$ is said to be a neutrosophic-valued function or $\mathcal{E}$-valued function. If the $\mathcal{E}$-valued function f includes n distinct neutrosophic numbers $u_1, u_2, \ldots, u_n$, then the horizontal membership function of f at $t \in [a, b]$, denoted by $\mathcal{L}(f(t)) \triangleq f^{gr}(t, \mu_f)$, can be given as

$$f^{gr} : [a, b] \times [0, 1] \times \cdots \times [0, 1] \to \mathbb{R},$$

where $\mu_f \triangleq (\mu_1, \mu_2, \ldots, \mu_n)$.

Definition 2.6 (*The Differentiability*)**.** Let $f : (a, b) \subset \mathbb{R} \to \mathcal{E}$ be a neutrosophic-valued function and $t_0 \in (a, b)$. Then, we say that f is granular differentiable (gr-differentiable) at the point t_0 if there exists an element $\frac{d_{gr}f(t_0)}{dt} \in \mathcal{E}$ such that the limit

$$\lim_{h\to 0} \frac{f(t_0+h) \ominus^{gr} f(t_0)}{h} = \frac{d_{gr}f(t_0)}{dt}, \quad (2)$$

holds for h sufficiently near 0. Then, we call the value $\frac{d_{gr}f(t_0)}{dt}$ the granular derivative (gr-derivative) of function f at the point t_0.

As a result, the function f is said to be gr-differentiable on the interval (a, b) if and only if the gr-derivative $\frac{d_{gr}f(t)}{dt}$ exists for all $t \in (a, b)$. Then, the mapping $t \mapsto \frac{d_{gr}f(t)}{dt}$ is called the gr-derivative of f on (a, b) and denoted by $\frac{d_{gr}f}{dt}$ or f'_{gr}.

Remark 2.3. In Definition 2.6, if the certain domain $(a, b) \subset \mathbb{R}$ is replaced by the domain $\mathcal{E}_1 \subseteq \mathcal{E}$, then the conclusion still holds. Here, we note that the division on the left side of the formula (2) can be known as the division of neutrosophic numbers.

Next, we give a necessary and sufficient condition for the granular differentiability of a neutrosophic-valued function.

Proposition 2.1. *Let $f : (a, b) \subset \mathbb{R} \to \mathcal{E}$ be an $\mathcal{E}$-valued function and $t_0 \in (a, b)$. The function f is gr-differentiable at the point t_0 if and only if its horizontal membership function is differentiable at t_0. Then, we have*

$$\mathcal{L}\left(\frac{d_{gr}f(t_0)}{dt}\right) = \frac{\partial f^{gr}(t_0, \mu_f)}{\partial t}.$$

Definition 2.7. Assume that $\Phi : [a, b] \to \mathcal{E}$ is a continuous $\mathcal{E}$-valued function and its horizontal membership function $\mathcal{L}(\Phi(t)) := \Phi(t, \overline{\mu})$ is integrable on $[a, b]$, i.e., there exists a number $\mathcal{I}(\overline{\mu}) \in \mathbb{R}$ such that $\mathcal{I}(\overline{\mu}) = \int_a^b \Phi(t, \overline{\mu})dt$. Then, the neutrosophic number $\mathcal{I}$, obtained by the transformation $\mathcal{I} := \mathcal{L}^{-1}(\mathcal{I}(\overline{\mu}))$, is said to be the granular integral (gr-integral) of function Φ on $[a, b]$ and denoted by $\mathcal{I} = \int_a^b \Phi(t)dt$.

Remark 2.4. By analogous arguments as in Proposition 2.1, we can also prove that the granular integrability of neutrosophic-valued function f and the integrability of its horizontal membership function are equivalent.

Theorem 2.1 (*Newton–Leibniz's Formula*)**.** *Assume that $\phi : [a, b] \subseteq \mathbb{R} \to \mathcal{E}$ is a neutrosophic-valued function that is gr-differentiable on $[a, b]$ and the function $\Phi(t) := \phi'_{gr}(t)$ is continuous on this interval. Then, Φ is gr-integrable and*

$$\int_a^b \Phi(t)dt = \phi(b) \ominus^{gr} \phi(a).$$

Example 2.2. Let $\phi : [0, 2\pi] \to \mathcal{E}$ be given by $\phi(t) = z_1 e^{-t} + z_2 \cos 2t$, where $z_1 = 4+I$, $z_2 = -6+I \in \mathcal{E}$. Then, the horizontal membership function of $\phi(t)$ is

$$\phi^{gr}(t, \mu_1, \mu_2) = (4+\mu_1)\, e^{-t} + (-6+\mu_2)\cos 2t.$$

We see that $\phi(t)$ is gr-differentiable on $[0, 2\pi]$ and its derivative, denoted by $\Phi(t) = \phi'_{gr}(t)$, is a continuous function on $[0, 2\pi]$ and we have

$$\Phi^{gr}(t, \mu_1, \mu_2) = \frac{\partial \phi^{gr}(t, \mu_1, \mu_2)}{\partial t} = (-4-\mu_1)\, e^{-t} + (12-2\mu_2)\sin 2t,$$

where $\mu_1, \mu_2 \in [0, 1]$. Then, the gr-integral $\mathcal{I}$ of $\Phi(t)$ on $[0, 2\pi]$ is given by

$$\mathcal{I} = \mathcal{L}^{-1}\left(\int_0^{2\pi} [(-4-\mu_1)\, e^{-t} + (12-2\mu_2)\sin 2t]dt\right) = 5(e^{-2\pi}-1) + \left(1-e^{-2\pi}\right) I.$$

On the other hand, we can see that all assumptions of Theorem 2.1 are fulfilled and thus, it follows that

$$\int_0^{2\pi} \Phi(t)dt = \phi(7) \ominus^{gr} \phi(0) = 5(e^{-2\pi}-1) + \left(1-e^{-2\pi}\right) I.$$

2.5. Neutrosophic-valued functions of several variables and their granular partial derivative

This section presents the concepts of neutrosophic-valued functions of several variables, their granular partial derivative and the granular chain rule. Definitions and theorems corresponding to these notions are given as follows.

Definition 2.8. A mapping $f : \mathcal{E}^n \to \mathcal{E}$, defined by $(x_1, \ldots, x_n) \mapsto f(x_1, \ldots, x_n)$, is called a neutrosophic-valued functions of several variables. And, the granular representation is $f^{gr}\left(x_1^{gr}(\mu_1), \ldots, x_n^{gr}(\mu_n); \mu_f\right)$ for all $\mu_i, \mu_f \in [0, 1]$.

Definition 2.9. Let $f : \mathcal{E}^n \to \mathcal{E}$. Then, the function f is said to be granular partial differentiable with respect to x_i if there exists an element $\frac{\partial_{gr} f(x_1, \ldots, x_i, \ldots, x_n)}{\partial x_i} \in \mathcal{E}$ such that the limit

$$\lim_{\tilde{h}\to\tilde{0}} \frac{1}{\tilde{h}}\left(f(x_1, \ldots, x_i+\tilde{h}, \ldots, x_n) \ominus^{gr} f(x_1, \ldots, x_i, \ldots, x_n)\right) = \frac{\partial_{gr} f(x_1, \ldots, x_i, \ldots, x_n)}{\partial x_i},$$

holds for all $\tilde{h}$ near $\tilde{0}$.

Remark 2.5. We consider some particular cases of neutrosophic-valued (several) variables functions that will be used throughout this work.

- If $n = 1$, then the function $f : \mathcal{E} \to \mathcal{E}$, given by $x \mapsto f(x)$, is said to be a neutrosophic-valued function. Its granular (partial) derivative w.r.t. variable x is given by $\frac{\partial_{gr} f(x)}{\partial x}$.
- If $n = 2$, then the function $f : \mathcal{E}^2 \to \mathcal{E}$, given by $(x_1, x_2) \mapsto f(x_1, x_2)$, is said to be a neutrosophic-valued two variables function. Then, its granular partial derivatives w.r.t. variable x_1 and variable x_2 are given respectively by $\frac{\partial_{gr} f(x_1, x_2)}{\partial x_1}$, $\frac{\partial_{gr} f(x_1, x_2)}{\partial x_2}$.

Theorem 2.2. *A neutrosophic-valued several variables function $f : \mathcal{E}^n \to \mathcal{E}$ is granular partial differential with respect to x_i if and only if its horizontal membership function is differentiable with respect to $\mathcal{L}(x_i)$. Moreover,*

$$\mathcal{L}\left(\frac{\partial_{gr} f\,(x_1, \ldots, x_i, \ldots, x_n)}{\partial x_i}\right) = \frac{\partial f^{gr}\left(x_1^{gr}(\mu_1), \ldots, x_n^{gr}(\mu_n); \mu_f\right)}{\partial x_i^{gr}(\mu_i)}.$$

Definition 2.10 (*gr-Derivative of Composite Function*)**.** Assume that

(i) the function $f : [a, b] \subset \mathbb{R} \to \mathcal{E}_1 \subseteq \mathcal{E}$ is gr-differentiable at the point $t_0 \in [a, b]$;
(ii) the function $g : \mathcal{E}_1 \to \mathcal{E}$ is granular partial differentiable w.r.t the variable $f(t)$ at the point t_0.

Then, the composite function of the functions f and g at the point $t_0 \in [a, b]$ is denoted by $(g \circ f)(t)$ or $g\,(f(t))$ and it is said to be granular differentiable at the point t_0 if there exists an element $\frac{d_{gr}(g(f(t_0)))}{dt} \in \mathcal{E}$ such that the following limit exists

$$\lim_{h \to 0} \frac{g\,(f(t_0 + h)) \ominus^{gr} g\,(f(t_0))}{h} = \frac{d_{gr}(g(f(t_0)))}{dt}$$

for all h sufficiently near 0.

Theorem 2.3 (*Granular Chain Rule*)**.** *Assume that*

(i) The function $f : [a, b] \subset \mathbb{R} \to \mathcal{E}_1 \subseteq \mathcal{E}$ is gr-differentiable at the point $t_0 \in [a, b]$.
(ii) The function $g : \mathcal{E}_1 \to \mathcal{E}$ is granular partial differentiable with respect to variable $f(t)$ at the point t_0.

Then, granular derivative of the composite function $(g \circ f)$ at the point t_0 is given by

$$\frac{d_{gr}(g(f(t_0)))}{dt} = \frac{\partial_{gr} g(f(t_0))}{\partial f(t)} \cdot \frac{d_{gr} f(t_0)}{dt}.$$

Proof. Using the condition (i), it follows that for all $h > 0$ sufficiently near 0, the neutrosophic number $f'_{gr}(t_0)$ satisfies $\lim_{h \to 0} \frac{1}{h}\,[f(t_0 + \delta) \ominus^{gr} f(t_0)] = f'_{gr}(t_0)$, or equivalently, there exists an element $\delta(h) \in \mathcal{E}$ depending on h such that $\delta(h)$ approaches zero neutrosophic number $\tilde{0}$ as $h \to 0$ and $\frac{1}{h}\,[f(t_0 + h) \ominus^{gr} f(t_0)] = \delta(h) + f'_{gr}(t_0)$, which follows that $f(t_0 + h) \ominus^{gr} f(t_0) = \left[\delta(h) + f'_{gr}(t_0)\right] h$. Thus, the element $g(f(t_0 + h))$ can be rewritten as follows $g(f(t_0 + h)) = g\left(f(t_0) + \left[\delta(h) + f'_{gr}(t_0)\right] h\right)$.

Similarly, due to the assumption (ii), we can consider an element $\frac{\partial_{gr} g(f(t_0))}{\partial f(t)} \in \mathcal{E}$ as the granular partial derivative of function g with respect to variable $f(t)$ at the point t_0 and there exists an element $\omega(\tilde{h}) \in \mathcal{E}$ depending on $\tilde{h} \in \mathcal{E}$ such that $\frac{1}{\tilde{h}}\left[g(f(t_0) + \tilde{h}) \ominus^{gr} g(f(t_0))\right] = \omega(\tilde{h}) + \frac{\partial_{gr} g(f(t_0))}{\partial f(t)}$, or equivalent to $g(f(t_0) + \tilde{h}) \ominus^{gr} g(f(t_0)) = \left[\omega(\tilde{h}) + \frac{\partial_{gr} g(f(t_0))}{\partial f(t)}\right] \tilde{h}$, where $\omega(\tilde{h})$ is as small as $\tilde{h} \to \tilde{0}$. Moreover, we can see that $\left[\delta(h) + f'_{gr}(t_0)\right] h$ approaches $\tilde{0}$ as $h \to 0$ and so, by choosing $\tilde{h} = \left[\delta(h) + f'_{gr}(t_0)\right] h$, this formula becomes $g(f(t_0) + \tilde{h}) \ominus^{gr} g(f(t_0)) = \left[\omega(\tilde{h}) + \frac{\partial_{gr} g(f(t_0))}{\partial f(t)}\right] \left[\delta(h) + f'_{gr}(t_0)\right] h$, or equivalent to, $\frac{1}{h}\left[g(f(t_0) + \tilde{h}) \ominus^{gr} g(f(t_0))\right] = \left[\omega(\tilde{h}) + \frac{\partial_{gr} g(f(t_0))}{\partial f(t)}\right] \left[\delta(h) + f'_{gr}(t_0)\right]$.

Next, by letting $h \to 0$, we immediately obtain $\lim_{h \to 0} \frac{1}{h}\left[g(f(t_0) + \tilde{h}) \ominus^{gr} g(f(t_0))\right] = \frac{\partial_{gr} g(f(t_0))}{\partial f(t)} f'_{gr}(t_0)$. Therefore, the proof is completed. □

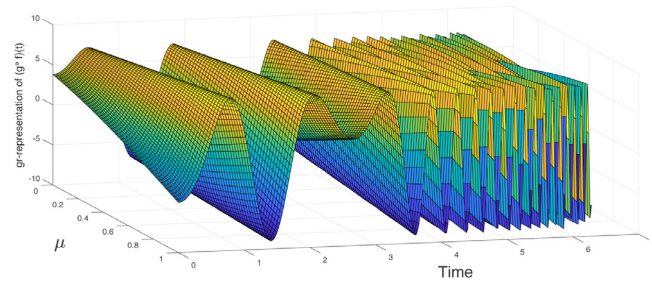

Fig. 2. The granular representation of the composite function $(g \circ f)(t)$ on the interval $[0, 2\pi]$.

Corollary 2.1. *Consider a neutrosophic-valued function $\tilde{f} : [a, b] \times C([a, b], \mathcal{E}) \to \mathcal{E}$. Then, we have the granular derivative of function $\tilde{f}$ is given by*

$$\frac{d_{gr}}{dt}\left(\tilde{f}(t, \tilde{x}(t))\right) = \frac{\partial_{gr} \tilde{f}(t, \tilde{x}(t))}{\partial t} + \frac{\partial_{gr} \tilde{f}(t, \tilde{x}(t))}{\partial \tilde{x}(t)} \frac{d_{gr} \tilde{x}(t)}{dt},$$

where $\frac{\partial_{gr} \tilde{f}(t, \tilde{x}(t))}{\partial \tilde{x}(t)}$ is denoted for the granular partial derivative w.r.t. variable $\tilde{x}(t)$.

Example 2.3. Consider neutrosophic-valued functions $f(t) = t^3 - 4t^2 + \tilde{a}$, $t \in [0, 2\pi]$ and $g : \mathcal{E} \to \mathcal{E}$, $g(x) = \tilde{b} \cos x$, where $\tilde{a} = 1 + I$, $\tilde{b} = 7 + 2I$ are neutrosophic numbers with indeterminacy $I = [0, 1]$. Then, the composite function $(g \circ f)(t)$ is defined as

$$(g \circ f)(t) = \tilde{b} \cdot \cos\left(t^3 - 4t^2 + \tilde{a}\right).$$

Since the respective granular representations of $\tilde{a}$ and $\tilde{b}$ are given by $\mathcal{L}(\tilde{a}) = 1 + \mu$ and $\mathcal{L}(\tilde{b}) = 7 + 2\mu$ for each $\mu \in [0, 1]$, we have

$$\begin{aligned}\mathcal{L}\left((g \circ f)(t)\right) &= (7 + 2\mu)\cos(\mathcal{L}(f(t))) \\ &= (7 + 2\mu)\cos(t^3 - 4t^2 + 1 + \mu),\end{aligned}$$

and its graphical representation is given in Fig. 2.

As a result of Proposition 2.1, it is easy to see that $\mathcal{L}\left(\frac{d_{gr} f(t)}{dt}\right) = 3t^2 - 8t$, and therefore, based on Theorems 2.2 and 2.3, we have

$$\begin{aligned}\mathcal{L}\left(\frac{\partial_{gr} g(f(t))}{\partial f(t)}\right) &= -(7 + 2\mu)\sin(\mathcal{L}(f(t))) \\ &= -(7 + 2\mu)\sin(t^3 - 4t^2 + 1 + \mu). \\ \mathcal{L}\left(\frac{d_{gr}(g \circ f)(t)}{dt}\right) &= \mathcal{L}\left(\frac{\partial_{gr} g(f(t))}{\partial f(t)}\right)\mathcal{L}\left(\frac{d_{gr} f(t)}{dt}\right) \\ &= -(7 + 2\mu)\sin(t^3 - 4t^2 + 1 + \mu)(3t^2 - 8t).\end{aligned}$$

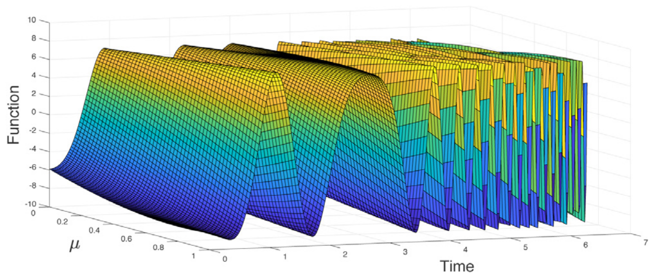

Fig. 3. The granular representation of the function $\frac{\partial_{gr} g(f(t))}{\partial f(t)}$ on the interval $[0, 2\pi]$.

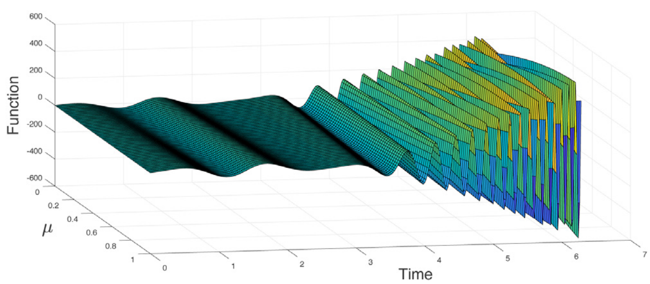

Fig. 4. The granular representation of the function $\frac{d_{gr}(g \circ f)(t)}{dt}$ on the interval $[0, 2\pi]$.

Hence, using the transformation (1), we obtain

$$\begin{aligned}\frac{d_{gr}(g \circ f)(t)}{dt} &= \mathcal{L}^{-1}\left(-(7+2\mu)\sin(t^3-4t^2+1+\mu)(3t^2-8t)\right)\\ &= \tilde{b}\,(8t-3t^2)\sin(t^3-4t^2+\tilde{a}).\end{aligned}$$

The graphical representations of functions $\frac{\partial_{gr} g(f(t))}{\partial f(t)}$ and $\frac{d_{gr}(g\circ f)(t)}{dt}$ on the interval $[0, 2\pi]$ are shown in Figs. 3 and 4, respectively.

3. Neutrosophic granular fractional integrals and fractional derivatives

In this section, we present the notions of granular fractional integral, granular Riemann–Liouville, and granular Caputo fractional derivatives of neutrosophic-valued functions. Some related properties of the granular Riemann–Liouville and granular Caputo fractional derivatives are also mentioned.

Definition 3.1. Let $f : [a, b] \subset \mathbb{R} \to \mathcal{E}$. Then, the right-sided and left-sided neutrosophic granular fractional integrals of order $\beta \in (0, 1)$ of f are defined by

$$\begin{aligned}{}^{gr}\mathcal{I}^{\beta}_{a+}f(t) &= \frac{1}{\Gamma(\beta)}\int_a^t (t-s)^{\beta-1}f(s)ds,\\ {}^{gr}\mathcal{I}^{\beta}_{b-}f(t) &= \frac{1}{\Gamma(\beta)}\int_t^b (s-t)^{\beta-1}f(s)ds.\end{aligned}$$

Remark 3.1. Thanks to Definition 2.7, we obtain

(i) The horizontal membership function of the right-sided neutrosophic granular fractional integrals of order $\beta \in (0, 1)$ is given as follows

$$\begin{aligned}\mathcal{L}\left({}^{gr}\mathcal{I}^{\beta}_{a+}f(t)\right) &= \frac{1}{\Gamma(\beta)}\int_a^t \mathcal{L}\left((t-s)^{\beta-1}f(s)\right)ds\\ &= \frac{1}{\Gamma(\beta)}\int_a^t (t-s)^{\beta-1}\mathcal{L}\left(f(s)\right)ds,\end{aligned}$$

that means $\mathcal{L}\left({}^{gr}\mathcal{I}^{\beta}_{a+}f(t)\right) = \mathcal{I}^{\beta}_{a+}\mathcal{L}\left(f(t)\right)$.

(ii) The horizontal membership function of the left-sided neutrosophic granular fractional integrals of order $\beta \in (0, 1)$ is given as follows

$$\begin{aligned}\mathcal{L}\left({}^{gr}\mathcal{I}^{\beta}_{b-}f(t)\right) &= \frac{1}{\Gamma(\beta)}\int_t^b \mathcal{L}\left((s-t)^{\beta-1}f(s)\right)ds\\ &= \frac{1}{\Gamma(\beta)}\int_t^b (s-t)^{\beta-1}\mathcal{L}\left(f(s)\right)ds,\end{aligned}$$

that means $\mathcal{L}\left({}^{gr}\mathcal{I}^{\beta}_{b-}f(t)\right) = \mathcal{I}^{\beta}_{b-}\mathcal{L}\left(f(t)\right)$.

Definition 3.2. Let $f : [a, b] \subset \mathbb{R} \to \mathcal{E}$. Then, the right-sided and left-sided neutrosophic granular Riemann–Liouville fractional derivatives of order $\beta \in (0, 1)$ of the function f are defined as follows

$$\begin{aligned}{}^{gr}_{a+}\mathcal{D}^{\beta}f(t) &= \frac{1}{\Gamma(1-\beta)}\frac{d_{gr}}{dt}\left(\int_a^t (t-s)^{-\beta}f(s)ds\right),\\ {}^{gr}_{b-}\mathcal{D}^{\beta}f(t) &= -\frac{1}{\Gamma(1-\beta)}\frac{d_{gr}}{dt}\left(\int_t^b (s-t)^{-\beta}f(s)ds\right).\end{aligned}$$

Remark 3.2. The horizontal membership function of the neutrosophic granular Riemann–Liouville fractional derivatives of order $\beta \in (0, 1)$ are given as follows

(i) $\mathcal{L}\left({}^{gr}_{a+}\mathcal{D}^{\beta}f(t)\right) = {}^{RL}_{a+}\mathcal{D}^{\beta}\mathcal{L}\left(f(t)\right)$.
(ii) $\mathcal{L}\left({}^{gr}_{b-}\mathcal{D}^{\beta}f(t)\right) = {}^{RL}_{b-}\mathcal{D}^{\beta}\mathcal{L}\left(f(t)\right)$.

Indeed, we have

$$\begin{aligned}\mathcal{L}\left({}^{gr}_{a+}\mathcal{D}^{\beta}f(t)\right) &= \mathcal{L}\left(\frac{1}{\Gamma(1-\beta)}\frac{d_{gr}}{dt}\left(\int_a^t (t-s)^{-\beta}f(s)ds\right)\right)\\ &= \frac{1}{\Gamma(1-\beta)}\mathcal{L}\left(\frac{d_{gr}}{dt}\left(\int_a^t (t-s)^{-\beta}f(s)ds\right)\right)\\ &= \frac{1}{\Gamma(1-\beta)}\frac{d}{dt}\left(\int_a^t (t-s)^{-\beta}\mathcal{L}\left(f(s)\right)ds\right)\\ &= {}^{RL}_{a+}\mathcal{D}^{\beta}\mathcal{L}\left(f(t)\right).\end{aligned}$$

Thus, the assertion (i) holds. Similarly, we also have the rest of proof.

Definition 3.3. Let $f : [a, b] \subset \mathbb{R} \to \mathcal{E}$ be a granular differentiable function. Then, the right-sided and left-sided neutrosophic granular Caputo fractional derivatives of order $\beta \in (0, 1)$ of the function f are defined as follows

$$\begin{aligned}{}^{gr}_{a+}\mathrm{D}^{\beta}f(t) &= \frac{1}{\Gamma(1-\beta)}\int_a^t (t-s)^{-\beta}\frac{d_{gr}f(s)}{ds}ds\\ &= {}^{gr}\mathcal{I}^{1-\beta}_{a+}\left(\frac{d_{gr}f(t)}{dt}\right),\\ {}^{gr}_{b-}\mathrm{D}^{\beta}f(t) &= -\frac{1}{\Gamma(1-\beta)}\int_t^b (s-t)^{-\beta}\frac{d_{gr}f(s)}{ds}ds\\ &= -{}^{gr}\mathcal{I}^{1-\beta}_{b-}\left(\frac{d_{gr}f(t)}{dt}\right).\end{aligned}$$

As a consequence, we can see that if $f(t)$ is a constant function then we have

$$ {}^{gr}_{a+}\mathrm{D}^{\beta}f(t) = {}^{gr}_{b-}\mathrm{D}^{\beta}f(t) = \hat{0}, \tag{3}$$

It should be noted that the equality (3) is one of characteristic properties of granular Caputo fractional derivative, which does not hold for the case of granular Riemann–Liouville fractional derivative.

Please cite this article as: N.T.K. Son, N.P. Dong, H.V. Long et al., Linear quadratic regulator problem governed by granular neutrosophic fractional differential equations. ISA Transactions (2019), https://doi.org/10.1016/j.isatra.2019.08.006.

Remark 3.3. By similar arguments as in Remarks 3.1 and 3.2, we can conclude that

(i) $\mathcal{L}\left({}^{gr}_{a+}\mathrm{D}^{\beta}f(t)\right) = {}^{C}_{a+}\mathrm{D}^{\beta}\mathcal{L}\left(f(t)\right)$.
(ii) $\mathcal{L}\left({}^{gr}_{b-}\mathrm{D}^{\beta}f(t)\right) = {}^{C}_{b-}\mathrm{D}^{\beta}\mathcal{L}\left(f(t)\right)$.

Proposition 3.1. *The right-sided and left-sided neutrosophic granular Caputo fractional derivatives of order $\beta \in (0, 1)$ are linear operators, i.e., for all gr-differentiable neutrosophic-valued functions $f, g : [a, b] \to \mathcal{E}$ and $\lambda_1, \lambda_2 \in \mathbb{R}$, we have*

(i) ${}^{gr}_{a+}\mathrm{D}^{\beta}\left[\lambda_1 f(t) + \lambda_2 g(t)\right] = \lambda_1\, {}^{gr}_{a+}\mathrm{D}^{\beta}f(t) + \lambda_2\, {}^{gr}_{a+}\mathrm{D}^{\beta}g(t)$.
(ii) ${}^{gr}_{b-}\mathrm{D}^{\beta}\left[\lambda_1 f(t) + \lambda_2 g(t)\right] = \lambda_1\, {}^{gr}_{b-}\mathrm{D}^{\beta}f(t) + \lambda_2\, {}^{gr}_{b-}\mathrm{D}^{\beta}g(t)$.

The next theorem presents the relation between neutrosophic granular Riemann–Liouville fractional derivatives and neutrosophic granular Caputo fractional derivatives

Theorem 3.1. *Let $f : [a, b] \subset \mathbb{R} \to \mathcal{E}$ be a granular differentiable function. Then, the relation between the neutrosophic granular Riemann–Liouville fractional derivatives and the neutrosophic granular Caputo fractional derivatives of order $\beta \in (0, 1)$ of f can be characterized by following equalities*

(i) ${}^{gr}_{a+}\mathrm{D}^{\beta}f(t) = {}^{gr}_{a+}\mathcal{D}^{\beta}f(t) \ominus^{gr} \dfrac{(t-a)^{-\beta}}{\Gamma(1-\beta)}f(a)$.
(ii) ${}^{gr}_{b-}\mathrm{D}^{\beta}f(t) = {}^{gr}_{b-}\mathcal{D}^{\beta}f(t) \ominus^{gr} \dfrac{(b-t)^{-\beta}}{\Gamma(1-\beta)}f(b)$.

Proof. We will prove the first assertion, while the second will be proved similarly. Indeed, it is sufficient to prove that

$$\begin{aligned}\mathcal{L}\left({}^{gr}_{a+}\mathrm{D}^{\beta}f(t)\right) &= \mathcal{L}\left({}^{gr}_{a+}\mathcal{D}^{\beta}f(t) \ominus^{gr} \frac{(t-a)^{-\beta}}{\Gamma(1-\beta)}f(a)\right)\\ &= \mathcal{L}\left({}^{gr}_{a+}\mathcal{D}^{\beta}f(t)\right) - \frac{(t-a)^{-\beta}}{\Gamma(1-\beta)}\mathcal{L}\left(f(a)\right).\end{aligned}$$

Employ the formula (2.4.8) in [39], we have

$${}^{C}_{a+}\mathrm{D}^{\beta}\mathcal{L}\left(f(t)\right) = {}^{RL}_{a+}\mathcal{D}^{\beta}\mathcal{L}\left(f(t)\right) - \frac{(t-a)^{-\beta}}{\Gamma(1-\beta)}\mathcal{L}\left(f(a)\right).$$

On the other hand, from Remark 3.2(i) and Remark 3.3(i), we can see that

$$\mathcal{L}\left({}^{gr}_{a+}\mathcal{D}^{\beta}f(t)\right) = {}^{RL}_{a+}\mathcal{D}^{\beta}\mathcal{L}\left(f(t)\right)$$
$$\mathcal{L}\left({}^{gr}_{a+}\mathrm{D}^{\beta}f(t)\right) = {}^{C}_{a+}\mathrm{D}^{\beta}\mathcal{L}\left(f(t)\right)$$

Therefore, we deduce that

$$\mathcal{L}\left({}^{gr}_{a+}\mathrm{D}^{\beta}f(t)\right) = \mathcal{L}\left({}^{gr}_{a+}\mathcal{D}^{\beta}f(t) \ominus^{gr} \frac{(t-a)^{-\beta}}{\Gamma(1-\beta)}f(a)\right).$$

Hence, the proof is complete. □

Corollary 3.1. *As a consequence of Theorem 3.1, we directly obtain that*

(i) If $f(a) = 0$ *then* ${}^{gr}_{a+}\mathrm{D}^{\beta}f(t) = {}^{gr}_{a+}\mathcal{D}^{\beta}f(t)$.
(ii) If $f(b) = 0$ *then* ${}^{gr}_{b-}\mathrm{D}^{\beta}f(t) = {}^{gr}_{b-}\mathcal{D}^{\beta}f(t)$.

Theorem 3.2. *Assume that*

(i) $f, g : [a, b] \subset \mathbb{R} \to \mathcal{E}$ *are granular differentiable functions.*
(ii) $f(a) = g(b) = 0$.

Then, the following equality holds

$$\int_a^b \left({}^{gr}_{a+}\mathrm{D}^{\beta}f(t)\right)g(t)dt = \int_a^b f(t)\left({}^{gr}_{b-}\mathrm{D}^{\beta}g(t)\right)dt. \tag{4}$$

Proof. By using Definition 2.7 and Remark 3.2(i), the left side of (4) becomes

$$\mathcal{L}\left(\int_a^b \left({}^{gr}_{a+}\mathrm{D}^{\beta}f(t)\right)g(t)dt\right) = \int_a^b {}^{C}_{a+}\mathrm{D}^{\beta}\mathcal{L}\left(f(t)\right)\mathcal{L}\left(g(t)\right)dt.$$

In addition, due to the granular differentiability of f and g, we imply that the real-valued functions $\mathcal{L}(f(t))$, $\mathcal{L}(g(t))$ are differentiable on $[a, b]$, and therefore, based Section 2.6 in [15], if $\mathcal{L}(f(a)) = \mathcal{L}(g(b)) = 0$ then the right hand side of above equality equals

$$\int_a^b {}^{C}_{a+}\mathrm{D}^{\beta}\mathcal{L}\left(f(t)\right)\mathcal{L}\left(g(t)\right)dt = \int_a^b \mathcal{L}\left(f(t)\right)\, {}^{C}_{b-}\mathrm{D}^{\beta}\mathcal{L}\left(g(t)\right)dt,$$

that means

$$\mathcal{L}\left(\int_a^b \left({}^{gr}_{a+}\mathrm{D}^{\beta}f(t)\right)g(t)dt\right) = \mathcal{L}\left(\int_a^b f(t)\left({}^{gr}_{b-}\mathrm{D}^{\beta}g(t)\right)dt\right),$$

or equivalent to

$$\int_a^b \left({}^{gr}_{a+}\mathrm{D}^{\beta}f(t)\right)g(t)dt = \int_a^b f(t)\left({}^{gr}_{b-}\mathrm{D}^{\beta}g(t)\right)dt. \quad \square$$

4. Neutrosophic granular partial integrals and neutrosophic granular partial derivatives

This section presents the concepts of granular partial integrals and derivatives of order $\sigma \in (0, 1)^n$ for a neutrosophic-valued several variables functions $f(z_1, z_2, \ldots, z_n)$. Such operations of fractional integration and fractional differentiation are natural generalizations of the corresponding fractional integration and fractional differentiation of single variable neutrosophic-valued functions.

For $z = (z_1, z_2, \ldots, z_n) \in \mathcal{E}^n$, $\sigma = (\sigma_1, \sigma_2, \ldots, \sigma_n) \in (0, 1)^n$, $a = (a_1, a_2, \ldots, a_n) \in \mathbb{R}^n$ and $b = (b_1, b_2, \ldots, b_n) \in \mathbb{R}^n$, the following notations will be used throughout this paper: $\Gamma(\sigma) := \Gamma(\sigma_1)\Gamma(\sigma_2)\cdots\Gamma(\sigma_n)$ and $[a, b] = [a_1, b_1] \times \cdots \times [a_n, b_n] \subset \mathbb{R}^n$.

Definition 4.1. Let $f : [a, b] \subset \mathbb{R}^n \to \mathcal{E}$. Then, the right-sided and left-sided granular partial fractional integrals of order $\sigma_k \in (0, 1)$ w.r.t. the kth variable z_k are defined by

$${}^{gr}\mathcal{I}^{\sigma_k}_{a_k^+}f(z) := \frac{1}{\Gamma(\sigma_k)}\int_{a_k}^{z_k} \frac{f(z_1, \ldots, \tau_k, \ldots, z_n)}{(z_k - \tau_k)^{1-\sigma_k}}d\tau_k,$$
$${}^{gr}\mathcal{I}^{\sigma_k}_{b_k^-}f(z) := \frac{1}{\Gamma(\sigma_k)}\int_{z_k}^{b_k} \frac{f(z_1, \ldots, \tau_k, \ldots, z_n)}{(\tau_k - z_k)^{1-\sigma_k}}d\tau_k.$$

As a corollary, we can define the granular mixed fractional integrals of order $\sigma \in (0, 1)^n$.

Corollary 4.1. *Let $f : [a, b] \subset \mathbb{R}^n \to \mathcal{E}$. Then, the right-sided and left-sided granular mixed fractional integrals of order $\sigma \in (0, 1)^n$ are defined by*

$$\begin{aligned}{}^{gr}\mathcal{I}^{\sigma}_{a+}f(z) &= \left({}^{gr}\mathcal{I}^{\sigma_1}_{a_1^+}\, {}^{gr}\mathcal{I}^{\sigma_2}_{a_2^+} \cdots {}^{gr}\mathcal{I}^{\sigma_n}_{a_n^+}\right)f(z)\\ &= \frac{1}{\Gamma(\sigma)}\int_{a_1}^{z_1}\int_{a_2}^{z_2}\cdots\int_{a_n}^{z_n} \frac{f(\tau)}{(z-\tau)^{1-\sigma}}d\tau,\\ {}^{gr}\mathcal{I}^{\sigma}_{b-}f(z) &= \left({}^{gr}\mathcal{I}^{\sigma_1}_{b_1^-}\, {}^{gr}\mathcal{I}^{\sigma_2}_{b_2^-} \cdots {}^{gr}\mathcal{I}^{\sigma_n}_{b_n^-}\right)f(z)\\ &= \frac{1}{\Gamma(\sigma)}\int_{z_1}^{b_1}\int_{z_2}^{b_2}\cdots\int_{z_n}^{b_n} \frac{f(\tau)}{(\tau-z)^{1-\sigma}}d\tau,\end{aligned}$$

where $\tau = (\tau_1, \tau_2, \ldots, \tau_n)$ and $d\tau = d\tau_1 d\tau_2 \ldots d\tau_n$.

Definition 4.2. Let $f : [a, b] \subset \mathbb{R}^n \to \mathcal{E}$ be a granular partial differentiable function. Then, the right-sided and left-sided granular Caputo partial fractional derivatives of order $\sigma_k \in (0, 1)$ with respect to the kth variable z_k are defined by

$$
\begin{aligned}
{}^{gr}_{a_k^+}\mathrm{D}^{\beta}_{z_k} f(z) &= \frac{1}{\Gamma(1-\sigma_k)} \int_{a_k}^{z_k} (z_k - \tau_k)^{-\sigma_k} \\
&\quad \times \frac{\partial_{gr} f(z_1, \ldots, \tau_k, \ldots, z_n)}{\partial \tau_k} d\tau_k, \\
{}^{gr}_{b_k^-}\mathrm{D}^{\beta}_{z_k} f(z) &= -\frac{1}{\Gamma(1-\sigma_k)} \int_{z_k}^{b_k} (\tau_k - z_k)^{-\sigma_k} \\
&\quad \times \frac{\partial_{gr} f(z_1, \ldots, \tau_k, \ldots, z_n)}{\partial \tau_k} d\tau_k.
\end{aligned}
$$

Remark 4.1. As a consequence of Theorem 2.2 and Remarks 3.1 and 3.3, the following relations are fulfilled:

(i) $\mathcal{L}\left({}^{gr}\mathcal{I}^{\sigma_k}_{a_k^+} f(z)\right) = \mathcal{I}^{\sigma_k}_{a_k^+} \mathcal{L}(f(z))$,

(ii) $\mathcal{L}\left({}^{gr}\mathcal{I}^{\sigma_k}_{b_k^-} f(z)\right) = \mathcal{I}^{\sigma_k}_{b_k^-} \mathcal{L}(f(z))$,

(iii) $\mathcal{L}\left({}^{gr}_{a_k^+}\mathrm{D}^{\beta}_{z_k} f(z)\right) = {}^{C}_{a_k^+}\mathrm{D}^{\beta}_{z_k} \mathcal{L}(f(z))$,

(iv) $\mathcal{L}\left({}^{gr}_{b_k^-}\mathrm{D}^{\beta}_{z_k} f(z)\right) = {}^{C}_{b_k^-}\mathrm{D}^{\beta}_{z_k} \mathcal{L}(f(z))$.

5. Neutrosophic fractional DEs under granular differentiability

In this section, based on the concepts of neutrosophic granular Caputo derivatives, we investigate the initial problem for following neutrosophic fractional DEs

$$
\begin{cases}
{}^{gr}_{*}\mathrm{D}^{\beta} x(t) &= f(t, x(t)) \\
x(t_0) &= x_0,
\end{cases} \tag{5}
$$

where the notion ${}^{gr}_{*}\mathrm{D}^{\beta} x(t)$ denotes for the right-sided or left-sided granular Caputo derivatives of order $\beta \in (0, 1)$ of the state vector $x : [t_0, t_f] \to \mathcal{E}^n$, $x_0 \in \mathcal{E}$ is the initial condition and f is a neutrosophic-valued function that will be specified later.

Thanks to Remark 3.2, the Cauchy problem (5) can be rewritten in following granular form

$$
\begin{cases}
{}^{C}_{*}\mathrm{D}^{\beta} x^{gr}(t, \mu_1) &= f(t, x^{gr}(t, \mu_1), \mu_f), \\
x^{gr}(t_0, \mu_1) &= x_0^{gr}(\mu_0),
\end{cases} \tag{6}
$$

for all $\mu_0, \mu_1, \mu_f \in [0, 1]$.

Here, we can see that by the use of horizontal membership function approach, the Cauchy problem (5) for neutrosophic fractional DEs is transformed into Cauchy problems for a set of real-valued fractional DEs, which are called granular fractional DEs. It is well-known that under this approach, the solution sets of Cauchy problems for both types of fractional DEs are equivalent, i.e., if the Cauchy problem (5) does not have any solution then the Cauchy problem (6) also does not. Conversely, if $x^{gr}(t, \mu)$ is a solution of the problem (6) then the neutrosophic-valued function $\mathcal{L}^{-1}(x^{gr}(t, \mu))$ is a solution of the problem (5). Moreover, it should be noted that some important results such the well-posed property or the unique existence of solution to Cauchy problem (5) also correspond to those of Cauchy problem (6). We propose the following procedure to solve numerical or analysis neutrosophic solutions of problem (5).

Remark 5.1. The procedure to solve the Cauchy problem for neutrosophic fractional DEs under granular fractional differentiability can be given as follows:

Step 1. Convert the considered Cauchy problem (5) into the corresponding granular form (6);

Step 2. Employ analytic or numerical methods to obtain the exact or approximate solution of granular fractional DEs (6);

Step 3. Use the inverse transformation (1) to convert the obtained granular solution into changeable range form;

Step 4. Sketch the graphical representation of the neutrosophic solution by DE or PDE Toolbox — Matlab.

Now, in order to illustrate the useful of proposed method, we will consider some Cauchy problems for neutrosophic fractional DEs. Additionally, based on numerical methods mentioned in [15,39,40], we will investigate the exact or approximate solutions of considered problems.

Example 5.1. Consider the Cauchy problem for following neutrosophic fractional DE

$$
\begin{cases}
{}^{gr}_{0^+}\mathrm{D}^{1/2} x(t) &= \tilde{\lambda} t^{\beta} x(t), \\
x(0) &= x_0,
\end{cases} \tag{7}
$$

where the parameter $\tilde{\lambda} = 2+I$ and the initial condition $x_0 = 5+2I$ are uncertain quantities with the indeterminacy $I = [0, 0.5]$ and $\beta \in \mathbb{R}$, $\beta > -\frac{1}{2}$.

Based on the horizontal membership function approach, we can transform the fractional system (7) into the following granular form

$$
\begin{cases}
{}^{C}_{0^+}\mathrm{D}^{1/2} x^{gr}(t, \mu_1) &= (2+\mu_2) t^{\beta} x^{gr}(t, \mu_1), \\
x^{gr}(0, \mu_1) &= 5 + 2\mu,
\end{cases} \tag{8}
$$

for all $\mu_1, \mu_2 \in [0, 1]$.

Then, based on the method of reduction to Volterra integral equations introduced in [39], we can give a closed-form solution of the system (8) as follows

$$
x^{gr}_{\beta}(t, \mu_1) = (5+2\mu)\mathrm{E}_{\frac{1}{2}, 2\beta+1, 2\beta}\left[(2+\mu_2) t^{\beta+\frac{1}{2}}\right],
$$

that means $x_{\beta}(t) = x_0 \cdot \mathrm{E}_{\frac{1}{2}, 2\beta+1, 2\beta}\left(\tilde{\lambda} t^{\beta+\frac{1}{2}}\right)$, $t \geq 0$ is the solution of Cauchy problem (7).

In particular, if $\beta = 0$ then the problem (7) is equivalent to the Cauchy problem for Malthusian equation [41] that describes a simplest model of the population dynamics of a bacteria species. Here, the parameter $\tilde{\lambda}$ is the rate of growth of the population and x_0 is the initial population. Since the fast growth speed of the bacteria species and limitation of measure equipment, the parameter $\tilde{\lambda}$ and initial population x_0 may not be measured exactly. So, in this model, we present the parameter $\tilde{\lambda}$ and initial condition x_0 as neutrosophic numbers. Then, as a consequence, the solution of population model is given by

$$
x(t) = x_0 \cdot \mathrm{E}_{\frac{1}{2}, 1, 0}\left(\tilde{\lambda}\sqrt{t}\right) = x_0 \cdot \mathrm{E}_{\frac{1}{2}}\left(\tilde{\lambda}\sqrt{t}\right),
$$

and by using Matlab program 'fde12' to evaluate the Mittag-Leffler function $\mathrm{E}_{\frac{1}{2}}(\cdot)$, the graphical representation of approximate solution $x(t)$ is shown in Fig. 5.

Example 5.2. Consider the following neutrosophic fractional DE

$$
{}^{gr}_{0^+}\mathrm{D}^{3/2} x(t) = \lambda \, {}^{gr}_{0^+}\mathrm{D}^{1/2} x(t), \qquad t > 0, \tag{9}
$$

with the initial conditions

$$
x(0) = 1 + 0.3I, \quad \frac{d_{gr} x(0)}{dt} = 0.5 + 0.1I, \tag{10}
$$

where $\lambda \in \mathbb{R}$ and $x_0, x_1 \in \mathcal{E}$ are initial conditions with the indeterminacy $I = [0, 1]$. Under the horizontal membership function approach, the Cauchy problem for the fractional DE (9) can be transformed into following granular form

$$
\begin{cases}
{}^{C}_{0^+}\mathrm{D}^{3/2} x^{gr}(t, \mu) &= \lambda \, {}^{C}_{0^+}\mathrm{D}^{1/2} x^{gr}(t, \mu), \\
x^{gr}(0, \mu) &= 1 + 0.3\mu, \\
\dfrac{\partial x(0, \mu)}{\partial t} &= 0.5 + 0.1\mu,
\end{cases} \tag{11}
$$

for each $\mu \in [0, 1]$.

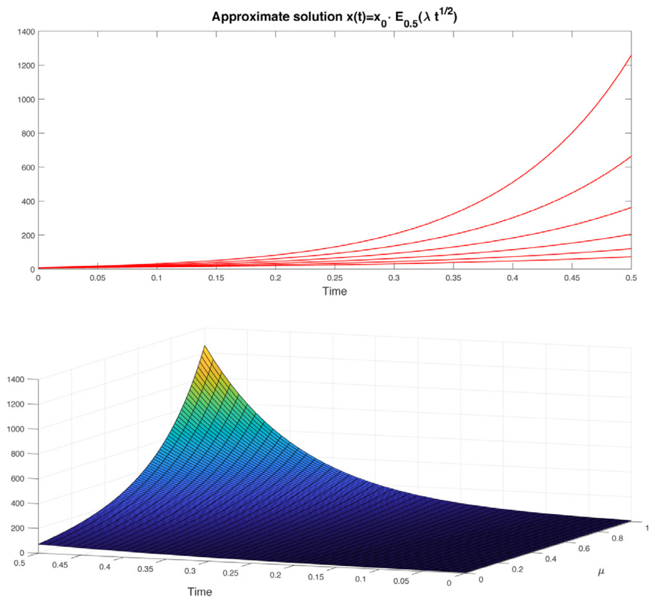

Fig. 5. Approximate solution of the Cauchy problem (7) for $t \in \left[0, \frac{1}{2}\right]$.

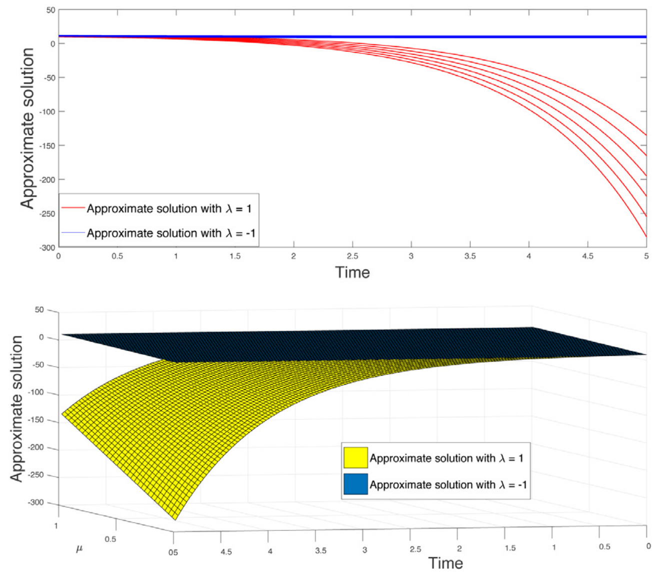

Fig. 6. The approximate solution of the NFDE (9) with $t \in [0, 5]$ with $\lambda = 1$ and $\lambda = -1$.

Adapting to Laplace transform method used in Section 5.3 of [39], approximate solution of the fractional DE (11) has following general form

$$\begin{aligned} x^{gr}(t,\mu) &= C_1(\mu)\left[E_1(\lambda t) - \lambda t E_{1,2}(\lambda t)\right] + C_2(\mu) t E_{1,2}(\lambda t) \\ &= C_1(\mu) + \frac{C_2(\mu)}{\lambda}\left(e^{\lambda t} - 1\right), \end{aligned} \tag{12}$$

where $E_1(z) = e^z$ and $E_{1,2}(z) = \dfrac{e^z - 1}{z}$ are Mittag-Leffler functions for each $z \in \mathbb{C}$. Next, by substituting the initial condition (10) into (12), we immediately obtain

$$x^{gr}(t,\mu) = (1 + 0.3\mu) + \frac{1}{\lambda}(0.5 + 0.1\mu)\left(e^{\lambda t} - 1\right), \qquad \mu \in [0,1].$$

Therefore, using inverse transformation (1), the approximate solution of fractional DE (9) subject to initial conditions (10) is

$$x(t) = 1 + 0.3I + \frac{1}{\lambda}(0.5 + 0.1I)\left(e^{\lambda t} - 1\right),$$

and its graphical representation is given in Fig. 6.

Example 5.3. In this example, we consider following fractionally damped single degree of freedom spring mass system whose equation of motion is modeled by

$$m\frac{d_{gr}^2 x(t)}{dt^2} + \rho\, {}^{gr}_{0^+}D^{\beta} x(t) + kx(t) = f(t), \tag{13}$$

with the initial conditions

$$x(0) = x_0, \qquad \frac{d_{gr}x(0)}{dt} = v_0,$$

where ${}^{gr}_{0^+}D^{\beta}x(t)$ is granular Caputo fractional derivative of order $\beta \in (0,1)$ of the displacement function $x(t)$. Here, the parameter $\beta = \frac{1}{2}$, known as the memory parameter, describes the frequency dependence of the damping materials quite satisfactorily in the crisp fractional dynamic systems and the parameters m, ρ and k represent for the mass, damping and stiffness coefficients. In order to control the mechanical system, we apply an external force $f := f(t)$. Finally, the initial displacement and initial velocity are uncertainties given by $x_0 = -0.1 + I$, $v_0 = -0.1 + I$, respectively, with the indeterminacy $I = [0, 0.2]$ (see Fig. 7).

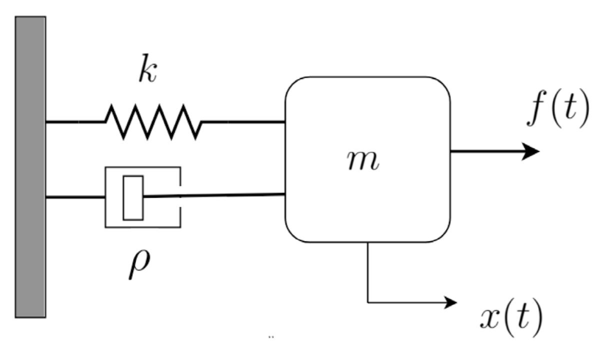

Fig. 7. A fractionally damped single degree of freedom spring mass system.

Then, based on the horizontal membership function approach, we can rewrite the Cauchy problem for fractional differential system (13) as follows

$$\begin{cases} \dfrac{\partial^2 x^{gr}(t,\mu)}{\partial t^2} &= -\dfrac{\rho}{m}\, {}^{C}_{0^+}D^{1/2} x^{gr}(t,\mu) - \dfrac{k}{m} x^{gr}(t,\mu) \\ & \quad + \dfrac{1}{m} f^{gr}(t,\mu), \\ x^{gr}(0,\mu) &= -0.1 + 0.2\mu, \\ \dfrac{\partial x^{gr}(0,\mu)}{\partial t} &= -0.1 + 0.2\mu, \end{cases} \tag{14}$$

for each $\mu \in [0,1]$. Based on Homotopy Perturbation Method introduced in [42], we obtain the general form of approximate solution of the problem (14) is given as follows

$$\begin{aligned} x^{gr}(t,\mu) &= (0.2\mu - 0.1) + \left[\frac{f(t)}{m} - \frac{k}{m}(0.2\mu - 0.1)\right] \\ &\quad \times \sum_{n=0}^{\infty} \frac{(-1)^n}{n!}\left(\frac{k}{m}\right)^n t^{2(n+1)} E^n_{\frac{3}{2},\frac{n}{2}+3}\left(-\frac{\rho}{m} t^{\frac{3}{2}}\right), \end{aligned} \tag{15}$$

where $E^n_{\frac{3}{2},\frac{n}{2}+3}(\cdot)$ is the generalized Mittag-Leffler function defined in Section 1.9 of [39] and the values of parameters k, ρ and m are given in Table 2.

Table 2
Parameter's value.

k	stiffness coefficient	9 N/m
ρ	damping coefficient	1.2 N.s/m
m	mass	1 kg
$f(t)$	external applied force	1.6 N

Thus, the formula (15) becomes

$$x^{gr}(t,\mu) = (0.2\mu - 0.1) + (2.5 - 1.8\mu) \times \sum_{n=0}^{\infty} \frac{(-9)^n}{n!} t^{2(n+1)} \mathrm{E}^n_{\frac{3}{2},\frac{n}{2}+3}\left(-1.2t^{\frac{3}{2}}\right).$$

Finally, by using the transformation (1), we obtain the approximate solution of Cauchy problem for fractional DE (13) is equal to

$$x(t) = (-0.1 + 0.2I) + (0.7 + 9I) \times \sum_{n=0}^{\infty} \frac{(-9)^n}{n!} t^{2(n+1)} \mathrm{E}^n_{\frac{3}{2},\frac{n}{2}+3}\left(-1.2t^{\frac{3}{2}}\right),$$

with the indeterminacy $I = [0, 0.2]$.

Example 5.4. Telegraph equations can be used in modeling reaction–diffusion problems and in signal analysis for the transmission and propagation of electrical signals. In this example, we consider following neutrosophic space–time-fractional telegraph equation

$$ {}^{gr}_{0^+}\mathrm{D}^{\beta}_{x} u(t,x) = \frac{\partial^2_{gr} u(t,x)}{\partial t^2} + \frac{\partial_{gr} u(t,x)}{\partial t} + u(t,x), \tag{16}$$

subject to the initial and boundary conditions

$$\begin{cases} u(t,0) = c_1 e^{-t}, & t \geq 0, \\ \dfrac{\partial_{gr} u(t,0)}{\partial t} = c_1 e^{-t}, & t \geq 0, \\ u(0,x) = c_2 e^{x}, & 0 < x < 2, \end{cases} \tag{17}$$

where ${}^{gr}_{0^+}\mathrm{D}^{\beta}_{t} u(t,x)$ denotes for the right-sided granular Caputo partial fractional derivative of order $\beta \in (0, 2]$ w.r.t. the state x and $c_1 = 7 + 2I$, $c_2 = 1 + I$ are neutrosophic numbers with the indeterminacy $I = [0, 1]$.

By using granular transformation presented in Definition 2.1, the Cauchy problem for the neutrosophic space–time-fractional telegraph equation (16) with the initial and boundary conditions (17) can be written as follows

$$\begin{cases} {}^{C}_{0^+}\mathrm{D}^{\beta}_{x} u^{gr}(t,x,\mu) &= \dfrac{\partial^2 u^{gr}(t,x,\mu)}{\partial t^2} \\ & \quad + \dfrac{\partial u^{gr}(t,x,\mu)}{\partial t} + u^{gr}(t,x,\mu), \\ u^{gr}(t,0,\mu) &= (7+2\mu_1)e^{-t}, \\ \dfrac{\partial u^{gr}(t,0,\mu)}{\partial x} &= (7+2\mu_1)e^{-t}, \\ u^{gr}(0,x,\mu) &= (1+\mu_2)e^{x}, \end{cases} \tag{18}$$

for all $\mu_1, \mu_2, \mu \in [0, 1]$.

Next, we will apply Adomian's decomposition method [43] to find the approximate solution of the Cauchy problem (18). Firstly, we assume that the approximate solution is given in following series form

$$u^{gr}(t,x,\mu) = \sum_{k=0}^{\infty} u^{gr}_n(t,x,\mu), \ \ \mu \in [0,1].$$

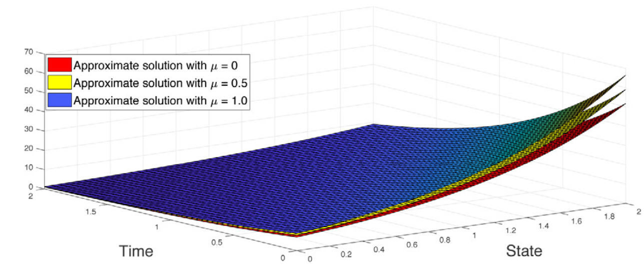

Fig. 8. The approximate solution $u(t, x)$ for $\beta = 2$ in the domain $[0, 2] \times [0, 2]$.

Then, according to Adomian's decomposition method used in [43,44], we obtain some first components of the above decomposition series

$$u^{gr}_0(t,x,\mu) = (1+x)(7+2\mu_1)e^{-t}$$
$$u^{gr}_1(t,x,\mu) = \left(\frac{x^{\beta}}{\Gamma(\beta+1)} + \frac{x^{\beta+1}}{\Gamma(\beta+2)}\right)(7+2\mu_1)e^{-t}$$
$$u^{gr}_2(t,x,\mu) = \left(\frac{x^{2\beta}}{\Gamma(2\beta+1)} + \frac{x^{2\beta+1}}{\Gamma(2\beta+2)}\right)(7+2\mu_1)e^{-t}$$
$$u^{gr}_3(t,x,\mu) = \left(\frac{x^{3\beta}}{\Gamma(3\beta+1)} + \frac{x^{3\beta+1}}{\Gamma(3\beta+2)}\right)(7+2\mu_1)e^{-t}$$
$$\cdots\cdots\cdots$$

And thanks to this manner, the rest of components of the decomposition series are also determined. Hence, the approximate solution $u^{gr}(t, x, \mu)$ is given by

$$u^{gr}(t,x,\mu) = (7+2\mu_1)e^{-t}\left[1 + x + \frac{x^{\beta}}{\Gamma(\beta+1)} + \frac{x^{\beta+1}}{\Gamma(\beta+2)} + \frac{x^{2\beta}}{\Gamma(2\beta+1)} + \frac{x^{2\beta+1}}{\Gamma(2\beta+2)} + \cdots\right],$$

that means the solution of the problem (16)–(17) is equal to

$$u(t,x) = \sum_{k=0}^{\infty} c_1 e^{-t}\left(\frac{x^{k\beta}}{\Gamma(k\beta+1)} + \frac{x^{k\beta+1}}{\Gamma(k\beta+2)}\right).$$

Here, we can see that there is no linearization or perturbation and the closed form of solution is obtainable by adding more terms to the decomposition series. Especially, if $\beta = 2$ then the solution $u(t, x)$ becomes

$$u(t,x) = c_1 e^{-t}\left[1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \cdots\right],$$

that is equivalent to the exact solution $u(t, x) = c_1 e^{x-t}$.

Finally, by using Matlab program, the graphical representations of the approximate solution $u(t, x)$ with different values of parameters β are shown in Figs. 8–10.

Example 5.5. Consider the following non-homogeneous neutrosophic space-fractional telegraph equation

$$ {}^{gr}_{0^+}\mathrm{D}^{\beta}_{x} u(t,x) = \frac{\partial^2_{gr} u(t,x)}{\partial t^2} + \frac{\partial_{gr} u(t,x)}{\partial t} + u(t,x) + g(t,x), \tag{19}$$

subject to the initial and boundary conditions

$$\begin{matrix} u(t,0) = \bar{c}t, & t \geq 0, \\ \dfrac{\partial_{gr} u(t,0)}{\partial x} = 0, & t \geq 0, \\ u(0,x) = \bar{c}x^2, & 0 < x < 1, \end{matrix} \tag{20}$$

where ${}^{gr}_{0^+}\mathrm{D}^{\beta}_{t} u(t,x)$ denotes for the right-sided granular Caputo partial fractional derivative of order $\beta \in (0, 2]$ with respect to

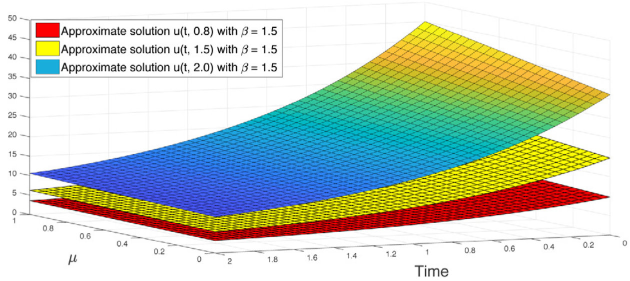

Fig. 9. The approximate solution $u(t,x)$ for $\beta=\frac{3}{2}$, $n=200$ and some different values of x.

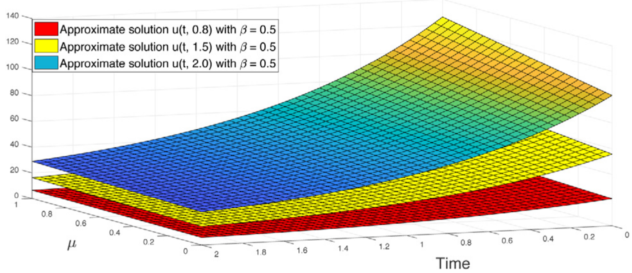

Fig. 10. The approximate solution $u(t,x)$ for $\beta=\frac{1}{2}$, $n=200$ and some different values of x.

state variable x, $g(t,x)=-x^2-t+1$ and $\bar{c}=5+I$ are neutrosophic numbers with the indeterminacy $I=[0,1]$.

By similar arguments as in Example 5.4, the Cauchy problem for the non-homogeneous neutrosophic space-fractional telegraph equation (19) subject to the initial and boundary conditions (20) can be given as

$$\begin{cases} {}^{C}_{0+}\mathrm{D}^{\beta}_{x} u^{gr}(t,x,\mu) &= \dfrac{\partial^2 u^{gr}(t,x,\mu)}{\partial t^2}+\dfrac{\partial u^{gr}(t,x,\mu)}{\partial t} \\ & \quad + u^{gr}(t,x,\mu)-x^2-t+1, \\ u^{gr}(t,0,\mu) &= (5+\mu_1)t, \\ \dfrac{\partial u^{gr}(t,0,\mu)}{\partial x} &= 0, \\ u^{gr}(0,x,\mu) &= (5+\mu_1)x^2, \end{cases} \tag{21}$$

for all $\mu,\ \mu_1\in[0,1]$.

In order to find the approximate solution of the problem (21), we will use the Homotopy Perturbation Method [42,45]. Firstly, for an embedding parameter $p\in[0,1]$, we construct the following homotopy

$$\begin{aligned} &\left({}^{C}_{0+}\mathrm{D}^{\beta}_{x} u^{gr}(t,x,\mu)-{}^{C}_{0+}\mathrm{D}^{\beta}_{x} u^{gr}_0(t,x,\mu)\right) \\ &\quad = \left(\frac{\partial^2 u^{gr}(t,x,\mu)}{\partial t^2}+\frac{\partial u^{gr}(t,x,\mu)}{\partial t}+u^{gr}(t,x,\mu)\right. \\ &\qquad \left. +1-x^2-t-{}^{C}_{0+}\mathrm{D}^{\beta}_{x} u^{gr}_0(t,x,\mu)\right). \end{aligned}$$

And then it follows that the approximate solution of the problem (21) can be obtained from the limit of a power series expansion of p as $p\to1$, that is

$$u^{gr}(t,x,\mu)=\lim_{p\to1}u^{gr}(t,x,\mu;p)=\sum_{k=0}^{\infty}p^k u^{gr}_k(t,x,\mu),$$

where $u^{gr}_k(t,x,\mu)$ $(i=0,1,2,\dots)$ are functions that need to be determined. Here, by applying homotopy perturbation method, one may have

$$\begin{aligned} u^{gr}_0(t,x,\mu) &= \bar{c}^{gr}(\mu_1)t \\ u^{gr}_1(t,x,\mu) &= \left(\bar{c}^{gr}(\mu_1)+1\right)\frac{x^{\beta}}{\Gamma(\beta+1)}+\left(\bar{c}^{gr}(\mu_1)-1\right)\frac{tx^{\beta}}{\Gamma(\beta+1)} \\ &\quad -\frac{2x^{\beta+2}}{\Gamma(\beta+3)}, \\ u^{gr}_2(t,x,\mu) &= \frac{2x^{2\beta}}{\Gamma(2\beta+1)}+\left(\bar{c}^{gr}(\mu_1)-1\right)\frac{tx^{2\beta}}{\Gamma(2\beta+1)} \\ &\quad -\frac{2x^{2\beta+2}}{\Gamma(2\beta+3)}, \\ u^{gr}_3(t,x,\mu) &= \left(\bar{c}^{gr}(\mu_1)+1\right)\frac{x^{3\beta}}{\Gamma(3\beta+1)} \\ &\quad +\left(\bar{c}^{gr}(\mu_1)-1\right)\frac{tx^{3\beta}}{\Gamma(3\beta+1)}-\frac{2x^{3\beta+2}}{\Gamma(3\beta+3)}, \\ &\quad \cdots\cdots\cdots \end{aligned}$$

Continuing this procedure, the general form of the approximate solution $u^{gr}(t,x,\mu)$ is

$$\begin{aligned} u^{gr}(t,x,\mu) &= \bar{c}^{gr}(\mu_1)\left[t+\frac{x^{\beta}}{\Gamma(\beta+1)}+\frac{tx^{\beta}}{\Gamma(\beta+1)}\right. \\ &\quad \left.+\frac{tx^{2\beta}}{\Gamma(2\beta+1)}+\frac{x^{3\beta}}{\Gamma(3\beta+1)}+\frac{tx^{3\beta}}{\Gamma(3\beta+1)}+\cdots\right] \\ &\quad +\left[\frac{x^{\beta}}{\Gamma(\beta+1)}-\frac{tx^{\beta}}{\Gamma(\beta+1)}-\frac{2x^{\beta+2}}{\Gamma(\beta+3)}\right. \\ &\quad +\frac{2x^{2\beta}}{\Gamma(2\beta+1)}-\frac{tx^{2\beta}}{\Gamma(2\beta+1)}-\frac{2x^{2\beta+2}}{\Gamma(2\beta+3)} \\ &\quad \left.+\frac{x^{3\beta}}{\Gamma(3\beta+1)}-\frac{tx^{3\beta}}{\Gamma(3\beta+1)}-\frac{2x^{3\beta+2}}{\Gamma(3\beta+3)}+\cdots\right], \end{aligned}$$

for each $\mu,\ \mu_1\in[0,1]$. Therefore, we obtain the formula of the approximate solution of the Cauchy problem (19)–(20) as follows.

$$\begin{aligned} u(t,x) &= \bar{c}\left[t+\frac{x^{\beta}}{\Gamma(\beta+1)}+\frac{tx^{\beta}}{\Gamma(\beta+1)}+\frac{tx^{2\beta}}{\Gamma(2\beta+1)}\right. \\ &\quad \left.+\frac{x^{3\beta}}{\Gamma(3\beta+1)}+\frac{tx^{3\beta}}{\Gamma(3\beta+1)}+\cdots\right] \\ &\quad +\left[\frac{x^{\beta}}{\Gamma(\beta+1)}-\frac{tx^{\beta}}{\Gamma(\beta+1)}-\frac{2x^{\beta+2}}{\Gamma(\beta+3)}\right. \\ &\quad +\frac{2x^{2\beta}}{\Gamma(2\beta+1)}-\frac{tx^{2\beta}}{\Gamma(2\beta+1)}-\frac{2x^{2\beta+2}}{\Gamma(2\beta+3)} \\ &\quad \left.+\frac{x^{3\beta}}{\Gamma(3\beta+1)}-\frac{tx^{3\beta}}{\Gamma(3\beta+1)}-\frac{2x^{3\beta+2}}{\Gamma(3\beta+3)}+\cdots\right]. \end{aligned}$$

Finally, by using Matlab program, the graphical representations of the approximate solution $u(t,x)$ with different values of parameters β are shown in Figs. 11–13.

The above figures illustrated for the fact that for $\beta=1.5$ or $\beta=1.95$, the space-fractional telegraph equation (19) is of hyperbolic type while in the case $\beta=0.5$, it is of parabolic type.

6. Neutrosophic fractional linear quadratic regulator problem

6.1. *Linear quadratic regulator problem for a LTI fractional differential system*

Consider a linear time-invariant (LTI) neutrosophic fractional differential system

$$\begin{cases} {}^{gr}_{t_0^+}\mathrm{D}^{\beta}x(t) &= Ax(t)+Bu(t) \\ x(t_0) &= x_0, \end{cases} \tag{22}$$

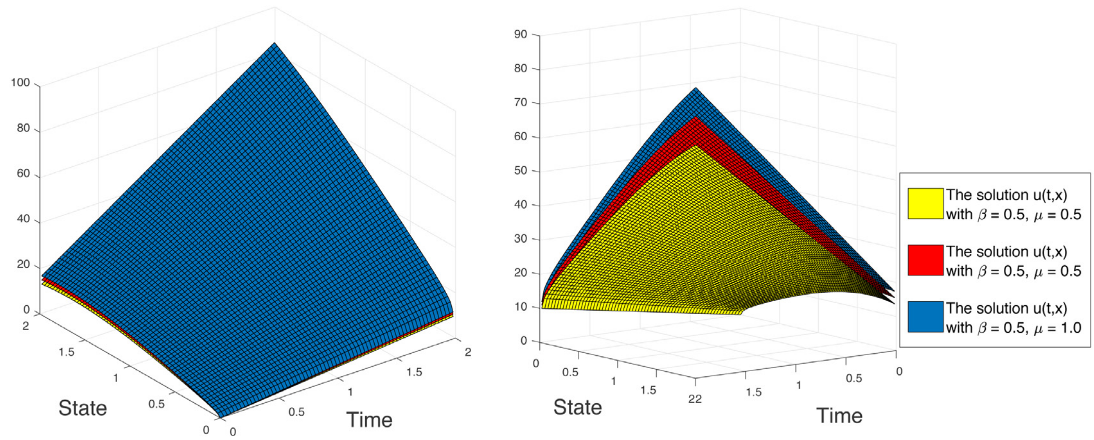

Fig. 11. The approximate solution $u(t, x)$ on the domain $[0, 2] \times [0, 2]$ with $\beta = 0.5$.

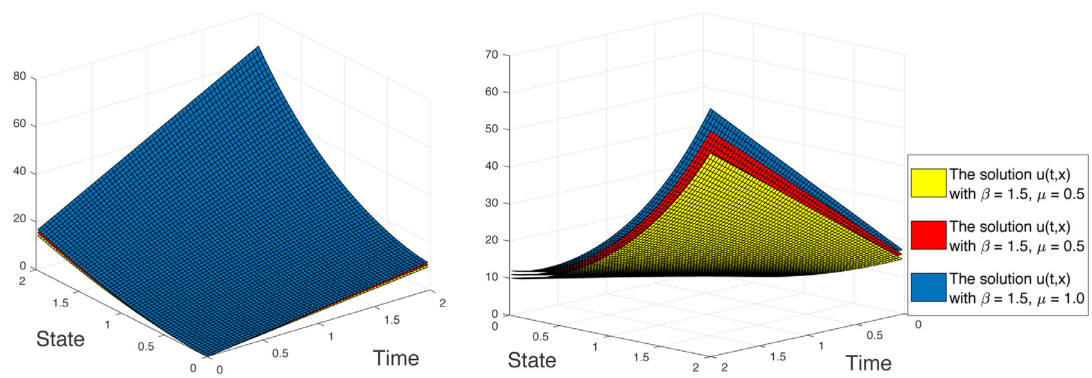

Fig. 12. The approximate solution $u(t, x)$ on the domain $[0, 2] \times [0, 2]$ with $\beta = 1.5$.

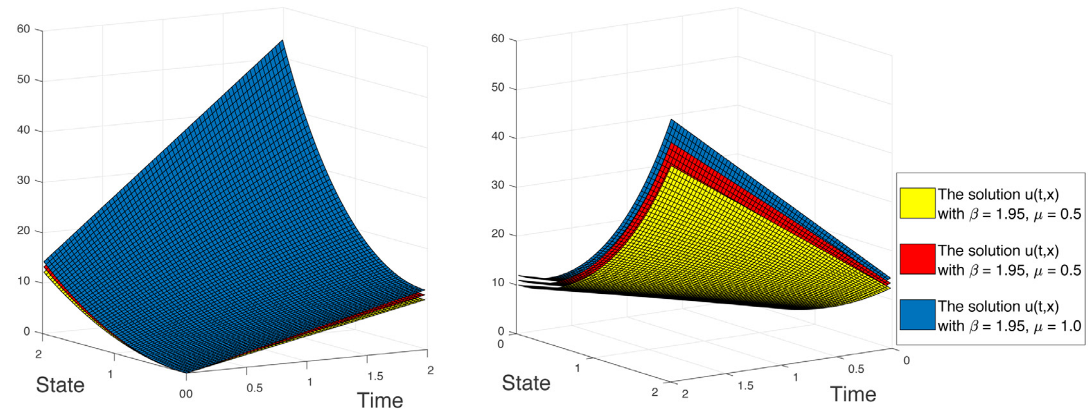

Fig. 13. The approximate solution $u(t, x)$ on the domain $[0, 2] \times [0, 2]$ with $\beta = 1.95$.

where the state vector $x : [t_0, t_f] \to \mathcal{E}^n$ is a granular Caputo fractional differentiable function; A and B are neutrosophic matrices with appropriate dimensions, $u : [t_0, t_f] \to \mathcal{E}^m$ is the control input vector and $x_0 \in \mathcal{E}^n$ is the neutrosophic initial condition.

The aim of neutrosophic fractional linear quadratic regulator (NFLQR) problem is to find a control input $u(t)$ that steers the state $x(t)$ of the system (22) from an initial state $x(t_0) = x_0$ to the origin at the time $t = t_f$ and minimizes following performance index

$$J(x, u) = \frac{1}{2}[x(t_f)]^{\mathrm{T}} P x(t_f) + \frac{1}{2}\int_{t_0}^{t_f} \left\{ [x(t)]^{\mathrm{T}} Q x(t) + [u(t)]^{\mathrm{T}} R u(t) \right\} dt, \tag{23}$$

where P, $Q \in \mathrm{Mat}_{n\times n}(\mathbb{R})$ are positive semi-definite symmetric matrices and $R \in \mathrm{Mat}_{m\times m}(\mathbb{R})$ is a positive definite symmetric matrix.

Definition 6.1.

(i) A pair $(\overline{x}, \overline{u})$ is said to be an admissible pair if it satisfies the linear time-invariant neutrosophic fractional system (22).
(ii) A pair $(\overline{x}, \overline{u})$ is said to be an optimal pair if it is an admissible pair and minimizes the performance index (23).

The following theorem presents a standard to determine the optimal pair $(\overline{x}, \overline{u})$.

Theorem 6.1. *A pair $(\overline{x},\overline{u})$ is an optimal pair if the following conditions are fulfilled.*

$$\begin{cases} {}^{gr}_{t_f^-}D^{\beta}\overline{\lambda}(t) &= Q\overline{x}(t)+A^{T}\overline{\lambda}(t)\\ {}^{gr}_{t_0^+}D^{\beta}x(t) &= Ax(t)+Bu(t)\\ \overline{u}(t) &= -R^{-1}B^{T}\overline{\lambda}(t)\\ P\overline{x}(t_f) &= 0\\ \overline{\lambda}(t_f) &= 0\\ \overline{x}(t_0) &= x_0. \end{cases} \quad (24)$$

Proof. Assume that $(\overline{x},\overline{u})$ is an optimal pair of the NFLQR problem. For this proof, let us define a function $\lambda:[t_0,t_f]\subset\mathbb{R}\to\mathcal{E}^n$ as the neutrosophic Lagrange multipliers vector corresponding to the problem. Next, we denote by $\omega(x(t_f))=\frac{1}{2}[x(t_f)]^{T}Px(t_f)$ and $\Phi[x,u](t)=\frac{1}{2}\int_{t_0}^{t_f}[x(\tau)]^{T}Qx(\tau)+[u(\tau)]^{T}Ru(\tau)d\tau$.

By the use of Newton–Leibniz's formula (Theorem 2.1), we have $\omega(x(t_f))=\omega(x(t_0))+\int_{t_0}^{t_f}\omega'_{gr}(x(\tau))d\tau$. Then, the performance index $J(x,u)$ becomes

$$J(x,u)=\omega(x(t_0))+\int_{t_0}^{t_f}\left[\Phi[x,u](\tau)+\omega'_{gr}(x(\tau))\right]d\tau. \quad (25)$$

Due to the fact that $\omega(x(t_0))$ is known as a neutrosophic number that will not affect to the minimization, it follows that the minimization (25) depends on only the second term of right-hand side. Then, as a consequence of Theorem 2.3, the performance index (25) can be deformed as follows

$$\begin{aligned}\mathcal{J}(x,u)&=\int_{t_0}^{t_f}\Phi[x,u](\tau)+\left[\frac{\partial_{gr}\omega(x(\tau))}{\partial x(\tau)}\right]^{T}x'_{gr}(\tau)\\ &\quad+\lambda^{T}(\tau)\left[Ax(\tau)+Bu(\tau)\ominus^{gr}{}^{gr}_{t_0^+}D^{\beta}x(t)\right]d\tau\\ &=\int_{t_0}^{t_f}\mathcal{Q}\left(x(\tau),u(\tau),x'_{gr}(\tau),\lambda(\tau),{}^{gr}_{t_0^+}D^{\beta}x(\tau)\right)d\tau.\end{aligned}$$

Here, for simplicity in representation, denote $\mathcal{Q}[x,u](t):=\mathcal{Q}\left(x(t),u(t),x'_{gr}(t),\lambda(t),{}^{gr}_{t_0^+}D^{\beta}x(t)\right)$ and $\mathcal{Q}^{gr}[x,u](t,\mu_0)$ by the respective granular representation of $\mathcal{Q}[x,u](t)$ with $\mu_0\in[0,1]$.

Next, for any $\varepsilon>0$, let us consider the following formation

$$\begin{cases} x(t) &= \overline{x}(t)+\varepsilon x(t)\\ u(t) &= \overline{u}(t)+\varepsilon u(t)\\ x'_{gr}(t) &= \overline{x}'_{gr}(t)+\varepsilon x'_{gr}(t)\\ \lambda(t) &= \overline{\lambda}(t)+\varepsilon\lambda(t)\\ {}^{gr}_{t_0^+}D^{\beta}x(t) &= {}^{gr}_{t_0^+}D^{\beta}\overline{x}(t)+\varepsilon\,{}^{gr}_{t_0^+}D^{\beta}x(t). \end{cases}$$

Based on this formation, we can see that if the pair $(\overline{x},\overline{u})$ is optimal, i.e., it minimizes the functional (23) then the increment of $\mathcal{J}(x,u)$ must be always non-negative, that is

$$\Delta\mathcal{J}=\mathcal{J}(x,u)\ominus^{gr}\mathcal{J}(\overline{x},\overline{u})\geq 0, \quad (26)$$

or equivalent to

$$\Delta\mathcal{J}=\int_{t_0}^{t_f}\left\{\mathcal{Q}[\overline{x}+\varepsilon x,\overline{u}+\varepsilon u](t)\ominus^{gr}\mathcal{Q}[\overline{x},\overline{u}](t)\right\}d\tau\geq 0,$$

where $\mathcal{Q}[\overline{x}+\varepsilon x,\overline{u}+\varepsilon u](t)$ is known as

$$\mathcal{Q}\left(\overline{x}(t)+\varepsilon x(t),\overline{u}(t)+\varepsilon u(t),\overline{x}'_{gr}(t)+\varepsilon x'_{gr}(t),\overline{\lambda}(t)+\varepsilon\lambda(t),\right.$$
$$\left.{}^{gr}_{t_0^+}D^{\beta}\overline{x}(t)+\varepsilon\,{}^{gr}_{t_0^+}D^{\beta}x(t)\right).$$

Then, by using the horizontal membership function approach, the inequality (26) becomes

$$\mathcal{L}(\Delta\mathcal{J})\geq 0\iff\mathcal{L}(\mathcal{J}(x,u))-\mathcal{L}(\mathcal{J}(\overline{x},\overline{u}))\geq 0,$$

in which

$$\begin{aligned}\mathcal{L}(\mathcal{J}(\overline{x},\overline{u}))=\int_{t_0}^{t_f}&\mathcal{Q}^{gr}\Big(\overline{x}^{gr}(\tau,\overline{\mu}_1),\overline{u}^{gr}(\tau,\overline{\mu}_2),(\overline{x}'_{gr})^{gr}(\tau,\overline{\mu}_1),\\ &\overline{\lambda}^{gr}(\tau,\overline{\mu}_3),{}^{C}_{t_0^+}D^{\beta}\overline{x}^{gr}(\tau,\overline{\mu}_1);\mu_0\Big)d\tau\\ \mathcal{L}(\mathcal{J}(x,u))=\int_{t_0}^{t_f}&\mathcal{Q}^{gr}\Big(\overline{x}^{gr}(\tau,\overline{\mu}_1)+\varepsilon x^{gr}(\tau,\mu_1),\overline{u}^{gr}(\tau,\overline{\mu}_2)\\ &+\varepsilon u^{gr}(\tau,\mu_2),(\overline{x}'_{gr})^{gr}(\tau,\overline{\mu}_1)+\varepsilon(x'_{gr})^{gr}(\tau,\mu_1),\\ &\overline{\lambda}^{gr}(\tau,\overline{\mu}_3)+\varepsilon\lambda^{gr}(\tau,\mu_3),{}^{C}_{t_0^+}D^{\beta}\overline{x}^{gr}(\tau,\overline{\mu}_1)\\ &+\varepsilon\,{}^{C}_{t_0^+}D^{\beta}x^{gr}(\tau,\mu_1);\mu_0\Big)d\tau,\end{aligned}$$

where $(\overline{x}'_{gr})^{gr}(\tau,\overline{\mu}_1)=\dfrac{\partial\overline{x}^{gr}(t,\overline{\mu}_1)}{\partial t}$, $(x'_{gr})^{gr}(\tau,\mu_1)=\dfrac{\partial x^{gr}(t,\mu_1)}{\partial t}$ and $\mu_i,\overline{\mu}_i\in[0,1]$ $(i=\overline{0,3})$.

For the minimization of $\mathcal{L}(\Delta\mathcal{J})$, note that the first order changes of $\mathcal{L}(\Delta\mathcal{J})$ with respect to the variables $\overline{x}^{gr}(\tau,\overline{\mu}_1)$, $\overline{u}^{gr}(\tau,\overline{\mu}_2)$, $\overline{\lambda}^{gr}(\tau,\overline{\mu}_3)$, ${}^{C}_{t_0^+}D^{\beta}\overline{x}^{gr}(\tau,\overline{\mu}_1)$ and $\overline{x}(t_f,\overline{\mu}_1)$ need to be zero. Then, by applying Theorem 2.2, we obtain

$$\begin{aligned}\int_{t_0}^{t_f}\Bigg\{&\left(\frac{\partial\mathcal{Q}^{gr}[\overline{x},\overline{u}](\tau,\mu_0)}{\partial\overline{x}^{gr}(\tau,\overline{\mu}_1)}\right)^{T}\varepsilon x^{gr}(\tau,\mu_1)\\ &+\left(\frac{\partial\mathcal{Q}^{gr}[\overline{x},\overline{u}](\tau,\mu_0)}{\partial\overline{u}^{gr}(\tau,\overline{\mu}_2)}\right)^{T}\varepsilon u^{gr}(\tau,\mu_2)\\ &+\left(\frac{\partial\mathcal{Q}^{gr}[\overline{x},\overline{u}](\tau,\mu_0)}{\partial\overline{\lambda}^{gr}(\tau,\overline{\mu}_3)}\right)^{T}\varepsilon\lambda^{gr}(\tau,\mu_3)\\ &+\left(\frac{\partial\mathcal{Q}^{gr}[\overline{x},\overline{u}](\tau,\mu_0)}{\partial\,{}^{C}_{t_0^+}D^{\beta}\overline{x}^{gr}(\tau,\overline{\mu}_1)}\right)^{T}\varepsilon\,{}^{C}_{t_0^+}D^{\beta}x^{gr}(\tau,\mu_1)\\ &+\left(\frac{\partial\mathcal{Q}^{gr}[\overline{x},\overline{u}](\tau,\mu_0)}{\partial(\overline{x}'_{gr})^{gr}(\tau,\overline{\mu}_1)}\right)^{T}\varepsilon(x'_{gr})^{gr}(\tau,\mu_1)\Bigg\}d\tau=0.\end{aligned}$$

Next, by integrating by parts, we immediately get that

$$\begin{aligned}\int_{t_0}^{t_f}\Bigg\{&\Bigg[\left(\frac{\partial\mathcal{Q}^{gr}[\overline{x},\overline{u}](\tau,\mu_0)}{\partial\overline{x}^{gr}(\tau,\overline{\mu}_1)}\right)^{T}\\ &-\frac{d}{d\tau}\left(\frac{\partial\mathcal{Q}^{gr}[\overline{x},\overline{u}](\tau,\mu_0)}{\partial(\overline{x}'_{gr})^{gr}(\tau,\overline{\mu}_1)}\right)^{T}\Bigg]\varepsilon x^{gr}(\tau,\mu_1)\\ &+\left(\frac{\partial\mathcal{Q}^{gr}[\overline{x},\overline{u}](\tau,\mu_0)}{\partial\overline{u}^{gr}(\tau,\overline{\mu}_2)}\right)^{T}\varepsilon u^{gr}(\tau,\mu_2)\\ &+\left(\frac{\partial\mathcal{Q}^{gr}[\overline{x},\overline{u}](\tau,\mu_0)}{\partial\,{}^{C}_{t_0^+}D^{\beta}\overline{x}^{gr}(\tau,\overline{\mu}_1)}\right)^{T}\varepsilon\,{}^{C}_{t_0^+}D^{\beta}x^{gr}(\tau,\mu_1)\\ &+\left(\frac{\partial\mathcal{Q}^{gr}[\overline{x},\overline{u}](\tau,\mu_0)}{\partial\overline{\lambda}^{gr}(\tau,\overline{\mu}_3)}\right)^{T}\varepsilon\lambda^{gr}(\tau,\mu_3)\Bigg\}d\tau\\ &+\left(\frac{\partial\mathcal{Q}^{gr}[\overline{x},\overline{u}](\tau,\mu_0)}{\partial(\overline{x}'_{gr})^{gr}(\tau,\overline{\mu}_1)}\right)^{T}\varepsilon x^{gr}(\tau,\mu_1)\Big|_{\tau=t_f}=0,\end{aligned}$$

where by the setting of $\mathcal{Q}[\overline{x},\overline{u}](t)$, the terms of above equality can be rewritten as follows

$$\frac{\partial \mathcal{Q}^{gr}[\overline{x},\overline{u}](\tau,\mu_0)}{\partial \overline{x}^{gr}(\tau,\overline{\mu}_1)} = \frac{\partial \Phi^{gr}[\overline{x},\overline{u}](\tau,\mu_\Phi)}{\partial \overline{x}^{gr}(\tau,\overline{\mu}_1)} + A^{\mathrm{T}}\overline{\lambda}^{gr}(\tau,\overline{\mu}_3) = Q\overline{x}^{gr}(\tau,\overline{\mu}_1) + A^{\mathrm{T}}\overline{\lambda}^{gr}(\tau,\overline{\mu}_3)$$

$$\frac{\partial \mathcal{Q}^{gr}[\overline{x},\overline{u}](\tau,\mu_0)}{\partial \overline{u}^{gr}(\tau,\overline{\mu}_2)} = \frac{\partial \Phi^{gr}[\overline{x},\overline{u}](\tau,\mu_\Phi)}{\partial \overline{u}^{gr}(\tau,\overline{\mu}_2)} + B^{\mathrm{T}}\overline{\lambda}^{gr}(\tau,\overline{\mu}_3) = R\overline{u}^{gr}(\tau,\overline{\mu}_2) + B^{\mathrm{T}}\overline{\lambda}^{gr}(\tau,\overline{\mu}_3)$$

$$\frac{\partial \mathcal{Q}^{gr}[\overline{x},\overline{u}](\tau,\mu_0)}{\partial \overline{\lambda}^{gr}(\tau,\overline{\mu}_3)} = A\overline{x}^{gr}(\tau,\overline{\mu}_1) + B\overline{u}^{gr}(\tau,\overline{\mu}_2) - {}^{C}_{t_0^+}\mathrm{D}^{\beta}\overline{x}^{gr}(\tau,\overline{\mu}_1)$$

$$\frac{\partial \mathcal{Q}^{gr}[\overline{x},\overline{u}](\tau,\mu_0)}{\partial\, {}^{C}_{t_0^+}\mathrm{D}^{\beta}\overline{x}^{gr}(\tau,\overline{\mu}_1)} = -\overline{\lambda}^{gr}(\tau,\overline{\mu}_3)$$

$$\frac{\partial \mathcal{Q}^{gr}[\overline{x},\overline{u}](\tau,\mu_0)}{\partial (\overline{x}'_{gr})^{gr}(\tau,\overline{\mu}_1)} = \mathcal{L}\left(\frac{\partial_{gr}\omega(x(\tau))}{\partial x(\tau)}\right) = P\overline{x}^{gr}(\tau,\overline{\mu}_1).$$

Then, we have

$$\mathcal{L}^{\mathrm{T}}\left(\frac{\partial_{gr}\omega(x(\tau))}{\partial x(\tau)}\right)\varepsilon x^{gr}(\tau,\mu_1)\Big|_{\tau=t_f} + \int_{t_0}^{t_f}\Bigg\{\left[\left(\overline{x}^{gr}(\tau,\overline{\mu}_1)\right)^{\mathrm{T}}Q + \left(\overline{\lambda}^{gr}(\tau,\overline{\mu}_3)\right)^{\mathrm{T}}A\right]\varepsilon x^{gr}(\tau,\mu_1) + \left[\left(\overline{u}^{gr}(\tau,\overline{\mu}_2)\right)^{\mathrm{T}}R + \left(\overline{\lambda}^{gr}(\tau,\overline{\mu}_3)\right)^{\mathrm{T}}B\right]\varepsilon u^{gr}(\tau,\mu_2) - \left(\overline{\lambda}^{gr}(\tau,\overline{\mu}_3)\right)^{\mathrm{T}}\varepsilon\, {}^{C}_{t_0^+}\mathrm{D}^{\beta}x^{gr}(\tau,\mu_1) + \left(A\overline{x}^{gr}(\tau,\overline{\mu}_1) + B\overline{u}^{gr}(\tau,\overline{\mu}_2) - {}^{gr}_{t_0^+}\mathrm{D}^{\beta}\overline{x}^{gr}(\tau,\overline{\mu}_1)\right)^{\mathrm{T}}\varepsilon\lambda^{gr}(\tau,\mu_3)\Bigg\}d\tau = 0. \tag{27}$$

Since the fact that $\overline{x}^{gr}(t_0,\mu_1) = x_0^{gr}(\mu_1)$ is known for all $\mu_1 \in [0,1]$, it implies $\varepsilon x^{gr}(t_0,\mu_1) = 0$. Thus, as a corollary of Theorem 3.2, the following integral equality holds

$$\int_{t_0}^{t_f}\left(\overline{\lambda}^{gr}(\tau,\overline{\mu}_3)\right)^{\mathrm{T}}\varepsilon\, {}^{C}_{t_0^+}\mathrm{D}^{\beta}x^{gr}(\tau,\mu_1)d\tau = \int_{t_0}^{t_f}\left({}^{C}_{t_f^-}\mathrm{D}^{\beta}\overline{\lambda}^{gr}(\tau,\overline{\mu}_3)\right)^{\mathrm{T}}\varepsilon x^{gr}(\tau,\mu_1)d\tau, \tag{28}$$

provided that $\overline{\lambda}^{gr}(t_f,\overline{\mu}_3) = 0$ for all $\mu_3 \in [0,1]$. Next, by substituting the expression (28) into (27), it follows that the following integral equality holds for all small variations $\varepsilon x^{gr}(\tau,\mu_1)$, $\varepsilon u^{gr}(\tau,\mu_2)$, $\varepsilon\lambda^{gr}(\tau,\mu_3)$ and $\varepsilon x^{gr}(t_f,\mu_1)$:

$$\mathcal{L}^{\mathrm{T}}\left(\frac{\partial_{gr}\omega(x(\tau))}{\partial x(\tau)}\right)\varepsilon x^{gr}(\tau,\mu_1)\Big|_{\tau=t_f} + \int_{t_0}^{t_f}\Bigg\{\left[\left(\overline{u}^{gr}(\tau,\overline{\mu}_2)\right)^{\mathrm{T}}R + \left(\overline{\lambda}^{gr}(\tau,\overline{\mu}_3)\right)^{\mathrm{T}}B\right]\varepsilon u^{gr}(\tau,\mu_2) + \left[\left(\overline{x}^{gr}(\tau,\overline{\mu}_1)\right)^{\mathrm{T}}Q + \left(\overline{\lambda}^{gr}(\tau,\overline{\mu}_3)\right)^{\mathrm{T}}A - \left({}^{C}_{t_f^-}\mathrm{D}^{\beta}\overline{\lambda}^{gr}(\tau,\overline{\mu}_3)\right)^{\mathrm{T}}\right]\varepsilon x^{gr}(\tau,\mu_1) + \left(A\overline{x}^{gr}(\tau,\overline{\mu}_1) + B\overline{u}^{gr}(\tau,\overline{\mu}_2) - {}^{C}_{t_0^+}\mathrm{D}^{\beta}\overline{x}^{gr}(\tau,\overline{\mu}_1)\right)^{\mathrm{T}} \times \varepsilon\lambda^{gr}(\tau,\mu_3)\Bigg\}d\tau = 0. \tag{29}$$

As a consequence, the equality (29) yields

$$\begin{cases} {}^{C}_{t_0^+}\mathrm{D}^{\beta}\overline{x}^{gr}(\tau,\overline{\mu}_1) = A\overline{x}^{gr}(\tau,\overline{\mu}_1) + B\overline{u}^{gr}(\tau,\overline{\mu}_2) \\ {}^{C}_{t_f^-}\mathrm{D}^{\beta}\overline{\lambda}^{gr}(\tau,\overline{\mu}_3) = Q\overline{x}^{gr}(\tau,\overline{\mu}_1) + A^{\mathrm{T}}\overline{\lambda}^{gr}(\tau,\overline{\mu}_3) \\ R\,\overline{u}^{gr}(\tau,\overline{\mu}_2) + B^{\mathrm{T}}\overline{\lambda}^{gr}(\tau,\overline{\mu}_3) = 0 \\ P\,\overline{x}^{gr}(t_f,\overline{\mu}_1) = 0 \\ \overline{\lambda}^{gr}(t_f,\overline{\mu}_3) = 0 \\ \overline{x}^{gr}(t_0,\overline{\mu}_1) = x_0^{gr}(\overline{\mu}_1), \end{cases} \tag{30}$$

for all $\overline{\mu}_1, \overline{\mu}_2, \overline{\mu}_3 \in [0,1]$. By inverse transformation (1), the system (30) is equivalent to (24). Therefore, the proof is completed. □

6.2. Numerical example

Remark 6.1. From the proof of Theorem 6.1, we implement the procedure to solve the fractional LQR problem for a neutrosophic LTI fractional differential system under granular Caputo fractional differentiability as follows:

Step 1. Convert the neutrosophic fractional LQR problem (22)–(23) into corresponding granular form;

Step 2. Employ Matlab function "LQR" to solve matrix S from the associated Riccati equation (32)

Step 3. Find out the granular form of the optimal control input $u(t)$ from (30);

Step 4. Use the transformation (1) to convert the obtained control input into changeable range form;

Step 5. Simulate the orientation of the state vector by Matlab function "FDE12".

To illustrate our obtained result, let us consider following examples.

Example 6.1. In this example, we consider following Neutrosophic Fractional Linear Quadratic Regulator problem

$$\min\ \mathcal{J}(x,u) = \frac{1}{2}x_2^2(t) + \frac{1}{2}\int_0^{\infty}\left[x_1^2(\tau) + \rho u^2(\tau)\right]d\tau$$

s.t.

$$\begin{cases} {}^{gr}_{0^+}\mathrm{D}^{\frac{1}{2}}x(t) = \begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix}x(t) + \begin{bmatrix}0\\1\end{bmatrix}u(t), \qquad t > 0, \\ x(0) = \begin{bmatrix}1\\1\end{bmatrix}, \end{cases} \tag{31}$$

where ${}^{gr}_{0^+}\mathrm{D}^{\frac{1}{2}}x(t)$ is granular Caputo fractional derivative of order $\beta = \frac{1}{2}$ of the state vector $x(t) = \left[x_1(t) \quad x_2(t)\right]^{\mathrm{T}}$, $u(t)$ is the control input and $\rho = 9 + I$ is the neutrosophic number with the indeterminacy $I = [0,1]$.

According to Section 6.1, let us define the matrices

$$A = \begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix}, \qquad B = \begin{bmatrix}0\\1\end{bmatrix}, \qquad P = \begin{bmatrix}0 & 0\\ 0 & 1\end{bmatrix},$$

$$Q = \begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}, \qquad R = \rho.$$

Here, the Lagrange multiplier $\lambda(t)$ can be rewritten in the form $\lambda(t) = Sx(t)$, where S is a positive definite symmetric matrix satisfying following Riccati equation

$$SA + A^{\mathrm{T}}S + Q - SBR^{-1}B^{\mathrm{T}}S = 0. \tag{32}$$

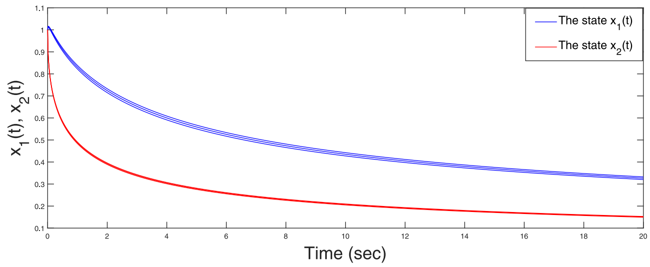

Fig. 14. The approximate solution of the problem (33) with $\mu \in \{0, 0.5, 1.0\}$.

Indeed, if we denote $S = \begin{bmatrix} a & b \\ b & c \end{bmatrix}$ then the Riccati equations becomes

$$\begin{bmatrix} a & b \\ b & c \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} a & b \\ b & c \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} a & b \\ b & c \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix}\rho^{-1}\begin{bmatrix} 0 & 1 \end{bmatrix}\begin{bmatrix} a & b \\ b & c \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

Its solution is given by $a = \sqrt[4]{4\rho}$, $b = \sqrt{\rho}$, $c = \sqrt[4]{4\rho^3}$, that means

$$S = \begin{bmatrix} \sqrt[4]{4\rho} & \sqrt{\rho} \\ \sqrt{\rho} & \sqrt[4]{4\rho^3}. \end{bmatrix}$$

Therefore, we obtain the corresponding optimal control $\overline{u}(t)$ of the problem (31) as follows

$$\overline{u}(t) = -\rho^{-1}\begin{bmatrix} 0 & 1 \end{bmatrix}\begin{bmatrix} \sqrt[4]{4\rho} & \sqrt{\rho} \\ \sqrt{\rho} & \sqrt[4]{4\rho^3} \end{bmatrix}\overline{x}(t) = -\begin{bmatrix} \frac{1}{\sqrt{\rho}} & \frac{\sqrt{2}}{\sqrt[4]{\rho}} \end{bmatrix}\overline{x}(t).$$

For the optimal control $\overline{u}(t)$, the LTI fractional differential system of (31) becomes

$$\begin{bmatrix} {}^{gr}_{0^+}\mathrm{D}^{\frac{1}{2}}x_1(t) {}^{gr}_{0^+}\mathrm{D}^{\frac{1}{2}}x_2(t) \end{bmatrix} = \begin{bmatrix} -\frac{\sqrt{2}}{\sqrt[4]{\rho}} & 1 \\ 0 & -\frac{\sqrt{2}}{\sqrt[4]{\rho}} \end{bmatrix}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}, \qquad t > 0. \tag{33}$$

As a consequence of Example 5.1, the approximate solution of the fractional differential system (22) with the initial condition $\begin{bmatrix} x_1(0) & x_2(0) \end{bmatrix} = \begin{bmatrix} 1 & 1 \end{bmatrix}$ is given by

$$\begin{cases} x_1(t) = \mathrm{E}_{\frac{1}{2}}\left(-\frac{\sqrt{2}}{\sqrt[4]{\rho}}t^{\frac{1}{2}}\right) + \int_0^t (t-\tau)^{-\frac{1}{2}}\mathrm{E}_{\frac{1}{2},\frac{1}{2}}\left(-\frac{\sqrt{2}}{\sqrt[4]{\rho}}(t-\tau)^{\frac{1}{2}}\right) \\ \qquad \times \mathrm{E}_{\frac{1}{2}}\left(-\frac{\sqrt{2}}{\sqrt[4]{\rho}}\tau^{\frac{1}{2}}\right)d\tau, \\ x_2(t) = \mathrm{E}_{\frac{1}{2}}\left(-\frac{\sqrt{2}}{\sqrt[4]{\rho}}t^{\frac{1}{2}}\right). \end{cases}$$

Its approximate solution can be obtained by using Matlab programs 'fde12' and the plot of approximate solution is shown in Fig. 14.

Example 6.2. A DC motor model consists of a permanent magnet and a rotor made of wires, as shown in Fig. 15.

The notations and coefficients are given in Table 3

By applying Kirchhoff's voltage law to the electrical part and Newton's second law to the mechanical part, we obtain the following state equations

$$\begin{bmatrix} {}^{gr}_{0^+}\mathrm{D}^{\beta}\theta(t) \\ {}^{gr}_{0^+}\mathrm{D}^{\beta}\omega(t) \\ {}^{gr}_{0^+}\mathrm{D}^{\beta}i(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & -\frac{D_{mo}}{J_{mo}} & \frac{K}{J_{mo}} \\ 0 & -\frac{K}{L} & -\frac{R}{L} \end{bmatrix}\begin{bmatrix} \theta(t) \\ \omega(t) \\ i(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \frac{1}{L} \end{bmatrix}v(t).$$

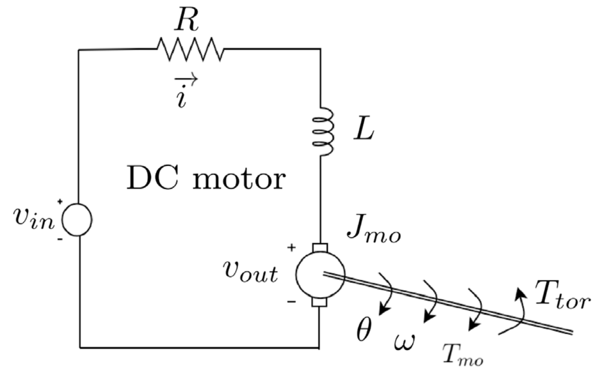

Fig. 15. The DC motor model.

Table 3
Some notations and coefficients.

R	the resistance of rotor	1.1 Ω
L	the inductance of rotor	0.05 H
T_{tor}	the load torque	0
J_{mo}	the inertia of rotor	0.3 kg·m^2
D_{mo}	the friction coefficient	[0.1, 0.2] N·m/Rad
K	the proportional coefficient	0.75
v	the input voltage	variable
θ	the angular displacement of motor	variable
ω	the angular velocity of motor	variable
i	the current in motor	variable

In addition, the output equation is given by

$$\theta(t) = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}\begin{bmatrix} \theta(t) \\ \omega(t) \\ i(t) \end{bmatrix}.$$

Next, by using the parameters in Table 3, we obtain following granular LTI fractional differential system

$$\begin{bmatrix} {}^{gr}_{0^+}\mathrm{D}^{\beta}\theta^{gr}(t,\mu_1) \\ {}^{gr}_{0^+}\mathrm{D}^{\beta}\omega^{gr}(t,\mu_2) \\ {}^{gr}_{0^+}\mathrm{D}^{\beta}i^{gr}(t,\mu_3) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & -\frac{2}{3}+\frac{\mu}{3} & 2.5 \\ 0 & -15 & -22 \end{bmatrix}\begin{bmatrix} \theta^{gr}(t,\mu_1) \\ \omega^{gr}(t,\mu_2) \\ i^{gr}(t,\mu_3) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 20 \end{bmatrix}v^{gr}(t,\mu_4),$$

for each $\mu, \mu_1, \mu_2, \mu_3, \mu_4 \in [0, 1]$.

For simplicity, let us denote

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & -\frac{2}{3}+\frac{1}{3}I & 2.5 \\ 0 & -15 & -22 \end{bmatrix}, \qquad B = \begin{bmatrix} 0 \\ 0 \\ 20 \end{bmatrix},$$

$$C = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}, \qquad x(t) = \begin{bmatrix} \theta(t) \\ \omega(t) \\ i(t) \end{bmatrix},$$

where the indeterminacy $I = [0, 1]$.

In a standard FLQR problem, the goal is to move the state vector to origin. However, in some practical applications, the goal is to move the output to some constant. For instance, in the DC motor model, our goal is to move the motor to $\theta_e = 10$ while minimizing the performance index

$$\mathcal{J} = \int_0^{\infty}\left\{9\left(\theta(\tau) \ominus^{gr} \theta_e\right)^2 + v^2(\tau)\right\}d\tau.$$

Table 4
The optimal solution of the FLQR problem (34) for $\mu \in [0, 1]$.

μ	The multipliers $\overline{\lambda}^{gr}(t,\mu)$	The control $\delta\overline{v}^{gr}(t,\mu)$
0.0	$\overline{\lambda}_{0.0} = \begin{bmatrix} 6.0406 & 1.4429 & 0.15 \\ 1.4429 & 0.4368 & 0.0471 \\ 0.15 & 0.0471 & 0.0051 \end{bmatrix} \delta\overline{x}_{0.0}$	$\delta\overline{v}_{0.0} = \begin{bmatrix} 3 & 0.9429 & 0.1024 \end{bmatrix} \delta\overline{x}_{0.0}$
0.25	$\overline{\lambda}_{0.25} = \begin{bmatrix} 5.9804 & 1.4453 & 0.15 \\ 1.4453 & 0.4462 & 0.0481 \\ 0.15 & 0.0481 & 0.0052 \end{bmatrix} \delta\overline{x}_{0.25}$	$\delta\overline{v}_{0.25} = \begin{bmatrix} 3 & 0.9625 & 0.1065 \end{bmatrix} \delta\overline{x}_{0.25}$
0.5	$\overline{\lambda}_{0.5} = \begin{bmatrix} 5.9219 & 1.4478 & 0.15 \\ 1.4478 & 0.456 & 0.0491 \\ 0.15 & 0.0491 & 0.0053 \end{bmatrix} \delta\overline{x}_{0.5}$	$\delta\overline{v}_{0.5} = \begin{bmatrix} 3 & 0.9827 & 0.1024 \end{bmatrix} \delta\overline{x}_{0.5}$
0.75	$\overline{\lambda}_{0.75} = \begin{bmatrix} 5.8649 & 1.4504 & 0.15 \\ 1.4504 & 0.4661 & 0.0502 \\ 0.15 & 0.0502 & 0.0054 \end{bmatrix} \delta\overline{x}_{0.75}$	$\delta\overline{v}_{0.75} = \begin{bmatrix} 3 & 1.0035 & 0.1087 \end{bmatrix} \delta\overline{x}_{0.75}$
1.0	$\overline{\lambda}_{1.0} = \begin{bmatrix} 5.8096 & 1.4531 & 0.15 \\ 1.4531 & 0.4766 & 0.0513 \\ 0.15 & 0.0513 & 0.0055 \end{bmatrix} \delta\overline{x}_{1.0}$	$\delta\overline{v}_{1.0} = \begin{bmatrix} 3 & 1.0251 & 0.1109 \end{bmatrix} \delta\overline{x}_{1.0}$

To solve this problem, we first need to find the corresponding desired state and input

$$\lim_{t\to\infty} x(t) = x_e, \qquad \lim_{t\to\infty} v(t) = v_e,$$

that achieve the final state θ_e. It is well-known that if there is no state x_e and input v_e then the goal of $\lim_{t\to\infty}\theta(t) = \theta_e$ is not achievable. Additionally, it is obvious that x_e and v_e satisfy the state and output equations, that is

$$\begin{cases} {}^{gr}_{0^+}\mathrm{D}^{\beta} x_e &= Ax_e + Bv_e \\ \theta_e &= Cx_e \end{cases} \Leftrightarrow \begin{bmatrix} A & B \\ C & 0 \end{bmatrix} \begin{bmatrix} x_e \\ v_e \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 10 \end{bmatrix},$$

which means that

$$\begin{bmatrix} \theta_e \\ \omega_e \\ i_e \\ v_e \end{bmatrix} = \begin{bmatrix} A & B \\ C & 0 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 10 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & -\frac{2}{3}+\frac{1}{3}U & 2.5 & 0 \\ 0 & -15 & -22 & 20 \\ 1 & 0 & 0 & 0 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 10 \end{bmatrix}.$$

Now, we will use Matlab software to find the value of $\begin{bmatrix} \theta(t) & \omega(t) & i(t) & v(t) \end{bmatrix}^{\mathrm{T}}$, which is shown in the following program

```
syms I;
Abs = [0,1,0,0;0,−2/3+I/3,2.5,0;0,−15,−22,20;1,0,
  0,0];
Bbs = [0,0,0,10];
C = inv(Abs)
X = C*(Bbs)'
```

Then, we define new variables as $\delta v(t) = v(t) - v_e$ and $\delta x(t) = x(t) - x_e = \begin{bmatrix} \theta(t)-\theta_e \\ \omega(t)-\omega_e \\ i(t)-i_e \end{bmatrix}$, by using the equality (3) and Proposition 3.1, we derive the state and output equations for $\delta x(t)$, $\delta v(t)$ and $\delta\theta(t)$ as follows

$$\begin{aligned} {}^{gr}_{0^+}\mathrm{D}^{\beta}\delta x(t) &= {}^{gr}_{0^+}\mathrm{D}^{\beta}\left[x(t)-x_e\right] = {}^{gr}_{0^+}\mathrm{D}^{\beta}x(t) \\ &= A\left[\delta x(t)+x_e\right] + B\left[\delta v^{gr}(t)+v_e\right] \\ &= \left[A\delta x(t)+B\delta v(t)\right] + \left[Ax_e+Bv_e\right] \\ &= A\delta x(t)+B\delta v(t), \end{aligned}$$

$$\delta\theta(t) = Cx(t) - Cx_e = C\delta x(t).$$

Therefore, the given problem is equivalent to following FLQR problem

$$\min \quad \mathcal{J} = \int_0^{\infty} \left\{ 9\left[\delta\theta(\tau)\right]^2 + \left[\delta v(\tau)+v_e\right]^2 \right\} d\tau, \tag{34}$$

s.t.

$$\begin{cases} {}^{gr}_{0^+}\mathrm{D}^{\beta}\delta x(t) &= A\delta x(t) + B\delta v(t), \\ \delta\theta(t) &= C\delta x(t). \end{cases}$$

Note that after finding the optimal control $\delta v^*(t)$, we can take $\overline{v}(t) = \delta\overline{v}(t) + v_e$, which is the desired optimal control of the original problem.

According to Section 6.1, we have that

$$A_\mu = \begin{bmatrix} 0 & 1 & 0 \\ 0 & -\frac{2}{3}+\frac{\mu}{3} & 2.5 \\ 0 & -15 & -22 \end{bmatrix}, \qquad B = \begin{bmatrix} 0 \\ 0 \\ 20 \end{bmatrix},$$

$$Q = \begin{bmatrix} 9 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad R = 1,$$

for each $\mu \in [0, 1]$. In addition, for simplicity in representation, we denote $\overline{\lambda}_\mu := \overline{\lambda}^{gr}(t,\mu)$, $\delta\overline{v}_\mu := \delta\overline{v}^{gr}(t,\mu)$, $\delta\overline{\theta}_\mu := \delta\overline{\theta}^{gr}(t,\mu)$, $\delta\overline{\omega}_\mu := \delta\overline{\omega}^{gr}(t,\mu)$, $\delta\overline{i}_\mu := \delta\overline{i}^{gr}(t,\mu)$, $\delta\overline{x}_\mu := \begin{bmatrix} \delta\overline{\theta}_\mu & \delta\overline{\omega}_\mu & \delta\overline{i}_\mu \end{bmatrix}^{\mathrm{T}}$.

Then, by using Matlab program 'LQR' for some values of $\mu \in [0, 1]$, we can calculate the solution of the LQR problem (34) as illustrated in Table 4.

Hence, the optimal control of the original problem is given in Table 5.

Example 6.3. In this example, we consider a simple model of one-link robot manipulator where the motion of robot's arm is controlled by a DC motor via a gear (see Fig. 16). Here, we assume that

- The motor moment of inertia is negligible compared with that of the robot's arm and the arm can be known as a point mass M attached to the end of a rod of length ℓ;
- The gear train has no backlash and all connecting shafts are rigid;
- The ratio between radii of motor gear and arm gear is $1 : \rho$.

Table 5
The optimal control $v^*(t)$ of the NFLQR problem for indeterminacy $I = [0, 1]$.

μ	The optimal control $\overline{v}^{gr}(t, \mu)$
0.0	$\overline{v}^{gr}(t, 0.0) = \begin{bmatrix} 3 & 0.9429 & 0.1024 \end{bmatrix} \begin{bmatrix} \overline{\theta}^{gr}(t, 0.0) \\ \overline{\omega}^{gr}(t, 0.0) \\ \overline{i}^{gr}(t, 0.0) \end{bmatrix} - 30$
0.25	$\overline{v}^{gr}(t, 0.25) = \begin{bmatrix} 3 & 0.9625 & 0.1065 \end{bmatrix} \begin{bmatrix} \overline{\theta}^{gr}(t, 0.25) \\ \overline{\omega}^{gr}(t, 0.25) \\ \overline{i}^{gr}(t, 0.25) \end{bmatrix} - 30$
0.5	$\overline{v}^{gr}(t, 0.5) = \begin{bmatrix} 3 & 0.9827 & 0.1024 \end{bmatrix} \begin{bmatrix} \overline{\theta}^{gr}(t, 0.5) \\ \overline{\omega}^{gr}(t, 0.5) \\ \overline{i}^{gr}(t, 0.5) \end{bmatrix} - 30$
0.75	$\overline{v}^{gr}(t, 0.75) = \begin{bmatrix} 3 & 1.0035 & 0.1087 \end{bmatrix} \begin{bmatrix} \overline{\theta}^{gr}(t, 0.75) \\ \overline{\omega}^{gr}(t, 0.75) \\ \overline{i}^{gr}(t, 0.75) \end{bmatrix} - 30$
1.0	$\overline{v}^{gr}(t, 1.0) = \begin{bmatrix} 3 & 1.0251 & 0.1109 \end{bmatrix} \begin{bmatrix} \overline{\theta}^{gr}(t, 1.0) \\ \overline{\omega}^{gr}(t, 1.0) \\ \overline{i}^{gr}(t, 1.0) \end{bmatrix} - 30$

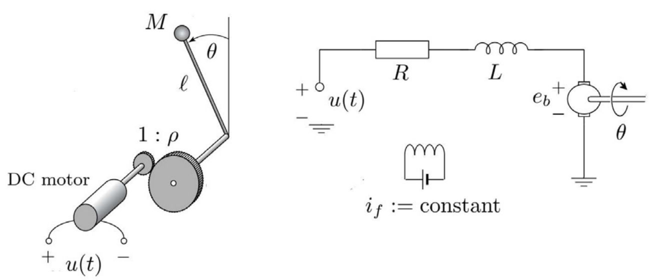

Fig. 16. One-link robot manipulator model.

Table 6
Parameters for the robot.

M	the mass	1 kg
R	the resistance of motor	1 Ω
ℓ	the length of massless rod	1 m
K_b	the back emf constant	0.15 V·sec/rad
K_{mo}	the motor-torque constant	0.1 N·m/A
g	the gravitational acceleration	[9.7, 10] m/s^2
ρ	the ratio of radii of motor & arm gear	10

By using Lagrange's equation, we can derive the dynamics of robot manipulator as follows

$$M\ell^2 \, {}^{gr}_{0^+}\mathrm{D}^{2\beta}\theta(t) = Mg\ell \sin\theta(t) + \rho K_{mo}\left(\frac{u(t)}{R} - \frac{\rho K_b \, {}^{gr}_{0^+}\mathrm{D}^{\beta}\theta(t)}{R}\right), \quad (35)$$

where the reasonable parameters for the robot are given in Table 6.

Next, we define new state variables and output variable as $x_1(t) = \theta(t)$, $x_2(t) = {}^{gr}_{0^+}\mathrm{D}^{\beta}\theta(t)$, $y(t) = x_1(t)$. With the above parameters and mechanical equation (35), we can construct the fractional state–space model of the one-link robot manipulator as follows

$$\begin{bmatrix} {}^{gr}_{0^+}\mathrm{D}^{\beta}x_1(t) \\ {}^{gr}_{0^+}\mathrm{D}^{\beta}x_2(t) \end{bmatrix} = \begin{bmatrix} x_2(t) \\ \frac{g}{\ell}\sin x_1(t) - \frac{\rho^2 K_b K_{mo}}{M\ell^2 R}x_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{\rho K_{mo}}{M\ell^2 R} \end{bmatrix} u(t)$$
$$= \begin{bmatrix} x_2(t) \\ (9.7 + 0.3I)\sin x_1(t) \ominus^{gr} 1.5x_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t) \quad (36)$$

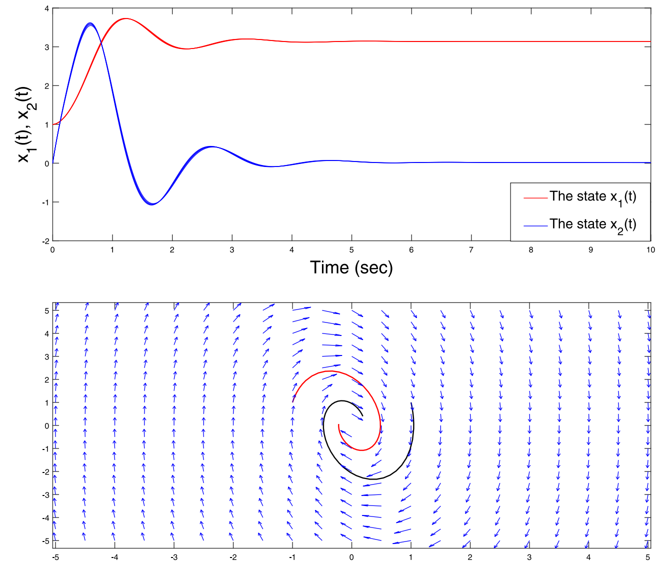

Fig. 17. The plot of (x_1, x_2) versus time and phrase portrait of the uncontrolled system.

$$y(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

subject to the initial condition

$$x(0) = \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad (37)$$

where ${}^{gr}_{0^+}\mathrm{D}^{\beta}x_1(t)$, ${}^{gr}_{0^+}\mathrm{D}^{\beta}x_2(t)$ stand for the right-sided granular Caputo fractional derivatives of order $\beta = 0.5$ and $u(t)$ is the input control function. Here, due to the influence of environment factors such as the height, temperature, humidity or air pressure, etc., and the errors when measuring or calculating, the gravitational acceleration g cannot be exactly measured. Thus, in this example, we consider the parameter g as a neutrosophic number $g = 9.7 + 0.3I$ with the indeterminacy $I = [0, 1]$.

In particular case, we consider the open-loop nonlinear fractional differential system, i.e., the control input $u \equiv 0$. By using Matlab, we simulate the histories of state trajectories of the open-loop nonlinear fractional differential system (36) with the initial condition (37). The results are shown in Fig. 17.

It can be seen that in the case of no controller, the state vector of the mechanical system is stable. However, the component $x_1(t) = \theta(t)$ is convergent to a non-zero state as time increases. Thus, if we need to steer the state of this system to a desired state, e.g. null state, in shortest time then there is a need to design an optimal controller and consider a fractional optimal control problem for considered mechanical system.

Next, we denote $A = \begin{bmatrix} 0 & 1 \\ 9.7 + 0.3U & -1.5 \end{bmatrix}$, $x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$, $B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. Now, for the goal of investigating the fractional optimal control problem for nonlinear fractional DEs system (36)–(37), by using the linearized method, we obtain the corresponding linearized model of the fractional differential system (36), whose granular representation can be given as follows

$${}^{C}_{0^+}\mathrm{D}^{\beta}x^{gr}(t, \mu) = \begin{bmatrix} 0 & 1 \\ 9.7 + 0.3\mu & -1.5 \end{bmatrix} x^{gr}(t, \mu) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u^{gr}(t, \mu),$$
$$y^{gr}(t, \mu) = \begin{bmatrix} 1 & 0 \end{bmatrix} x^{gr}(t, \mu),$$

where $x^{gr}(t, \mu) = \begin{bmatrix} x_1^{gr}(t, \mu) & x_2^{gr}(t, \mu) \end{bmatrix}^{\mathrm{T}}$ and $\mu \in [0, 1]$.

Table 7
The optimal solution of the FLQR problem (38) for $\mu \in [0, 1]$.

μ	The matrix S	The control input $\overline{u}^{gr}(t,\mu)$
0	$S = \begin{bmatrix} 77.1057 & 19.4514 \\ 19.4514 & 4.9150 \end{bmatrix}$	$\overline{u}^{gr}(t,0) = \begin{bmatrix} 19.4514 & 4.9150 \end{bmatrix} \overline{x}^{gr}(t,0)$
$\frac{1}{3}$	$S = \begin{bmatrix} 78.1995 & 19.6509 \\ 19.6509 & 4.9461 \end{bmatrix}$	$\overline{u}^{gr}(t,\frac{1}{3}) = \begin{bmatrix} 19.6509 & 4.9461 \end{bmatrix} \overline{x}^{gr}(t,\frac{1}{3})$
$\frac{2}{3}$	$S = \begin{bmatrix} 79.2980 & 19.8504 \\ 19.8504 & 4.9769 \end{bmatrix}$	$\overline{u}^{gr}(t,\frac{2}{3}) = \begin{bmatrix} 19.8504 & 4.9769 \end{bmatrix} \overline{x}^{gr}(t,\frac{2}{3})$
1	$S = \begin{bmatrix} 80.4013 & 20.0499 \\ 20.0499 & 5.0077 \end{bmatrix}$	$\overline{u}^{gr}(t,1) = \begin{bmatrix} 20.0499 & 5.0077 \end{bmatrix} \overline{x}^{gr}(t,1)$

Our goal is to find an optimal control input $u(t) = -Kx(t)$, where the gain $K = R^{-1}B^{\mathrm{T}}S$, such that it minimizes the following performance index

$$\mathcal{J}(x,u) = \int_0^{\infty} \left[x_1^2(\tau) + u^2(\tau)\right] d\tau \tag{38}$$

subject to

$${}^{gr}_{0^+}\mathrm{D}^{\beta}x(t) = Ax(t) + Bu(t).$$

Here, we have that

$$P = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad Q = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad R = 1.$$

Then, the following associated Riccati equation

$$SA + A^{\mathrm{T}}S + Q - SBR^{-1}B^{\mathrm{T}}S = 0$$

has the positive definite symmetric solution $S \in \mathrm{Mat}_{2\times 2}(\mathbb{R})$, and then, by employing the formula $u(t) = -R^{-1}B^{\mathrm{T}}Sx(t)$, we immediately obtain the form of the optimal control input $\overline{u}(t)$. In Table 7, by using Matlab program 'LQR', we can give the some results of the matrix S and $\overline{u}^{gr}(t,\mu)$ corresponding to some values of $\mu \in [0, 1]$.

Finally, by applying the above optimal controller $u(t)$ to the nonlinear model, we can show the graphical representations of the state variables $x_1(t)$, $x_2(t)$ versus time of the closed-loop nonlinear system in Fig. 18. Thanks to this optimal controller, the state of considered mechanical system, in particular the component $x_1(t) = \theta(t)$, is transferred into the null state in a shortest time.

Moreover, based on the idea of Example 6.2, we also consider an extended optimal control problem for this mechanical system, where the state of the system will be transferred into an arbitrary desired state in a shortest time.

7. Analysis and discussions

In [14], Alinezhad and Allahviranloo presented an extension of fractional optimal control problem related fuzzy fractional dynamic systems. The authors employed generalized Hukahara (gH) differentiability to set the description of problem in two forms corresponding with two types of differentiability. This idea can be extended to fractional optimal control problem in neutrosophic environment. However, it is well-known that gH-differentiability depends on gH-difference $\ominus_{gH}$ defined as

$a = b \ominus_{gH} c$ if and only if (1) $b = a + c$ or (2) $c = b + (-1)a$.

Hence, the approach based on gH-differentiability has some following shortcomings. First, the gH-difference does not always

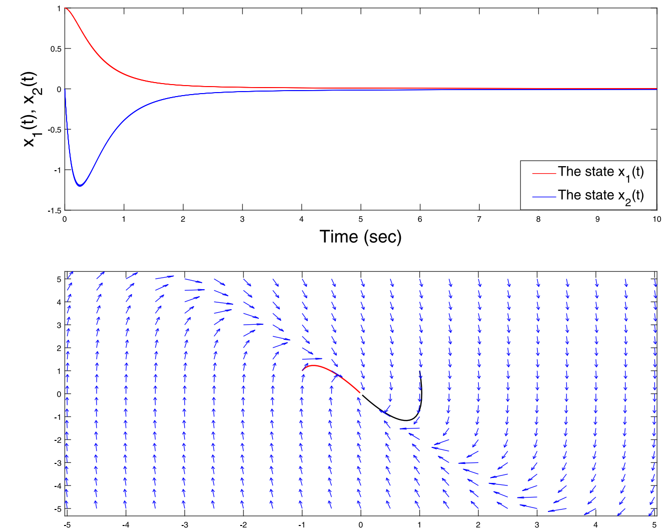

Fig. 18. The plot of (x_1, x_2) versus time and phrase portrait of the nonlinear closed-loop system.

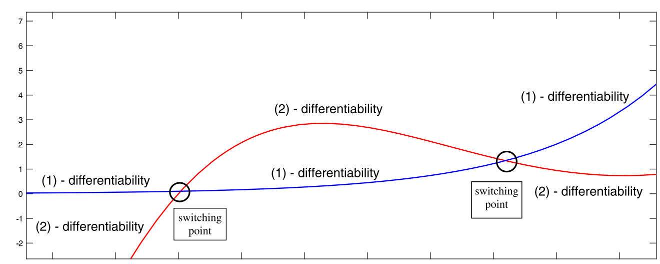

Fig. 19. The switching points of gH-differentiability.

exist for every fuzzy number b and c. So, if we define gH-differentiability for neutrosophic functions, it requires some complicated existence conditions. Furthermore, under gH-differentiability the solutions of following equations are not the same, see [46]

$${}^{gH}_{t_0^+}\mathrm{D}^{\beta}x(t) = Ax(t) + Bu(t)$$
$${}^{gH}_{t_0^+}\mathrm{D}^{\beta}x(t) + (-1)Ax(t) = Bu(t)$$
$${}^{gH}_{t_0^+}\mathrm{D}^{\beta}x(t) + (-1)Bu(t) = Ax(t)$$
$${}^{gH}_{t_0^+}\mathrm{D}^{\beta}x(t) + (-1)Ax(t) + (-1)Bu(t) = 0.$$

This shortcoming phenomenon is called "unnatural behavior in modeling". Additionally, the switching points of gH-differentiability often makes engineers dividing the problem into many cases when they want to apply an numerical or an analysis method to solve problem. The same appeared in [31] where the author extended gH-differentiability technique to intuitionistic fuzzy environment (see Fig. 19).

Ye and Cui [6] first introduced neutrosophic number into single input–single output linear systems. They established the state feedback design method for achieving a desired closed-loop state equation. The numerical simulation stated that the designed state feedback can perform its effectiveness and robustness in neutrosophic environment. We notice that [6] considered an neutrosophic linear systems related to a neutrosophic dynamic systems of integer order. Furthermore, no neutrosophic differentiability of functional relationship was considered. This seems not perfect yet because the neutrosophic-valued input variables lead to the neutrosophic-valued output variable. Thus differential calculus of plant state, closed-loop state equations and output variables of systems must be taken into account in neutrosophic environment, that is neutrosophic differential calculus.

In this paper, by representation neutrosophic number with respect to horizontal membership function of relative distance measure variable the differentiability of neutrosophic function is built via granular difference, fractional DEs and fractional PDEs are considered in new setting. Under new differentiability, the proposed approach not only overcomes previous limitations but also owns some following benefits:

1. We can conveniently define derivative and integral of neutrosophic function to ARBITRARY ORDER;
2. There does not exist SWITCHING POINT in neutrosophic derivatives;
3. Avoiding MULTIPLICITY neutrosophic solutions, there is only one neutrosophic solutions correspondence with one granular representation;
4. We can apply numerical method or analysis method to neutrosophic DEs and neutrosophic PDEs in a convenient way.

8. Conclusions

A new class of linear quadratic regulator problems for a class of controlled systems modeled by neutrosophic fractional DEs and granular derivatives has been introduced and the major contributions can be illustrated and reviewed as follows:

(1) New notions of Riemann–Liouville and Caputo derivatives for neutrosophic-valued functions were defined via relative distance measure and granular computing. The proposed techniques avoid multiplicity of solutions caused by switching points or doubling property when we solve numerical solutions of neutrosophic DEs and Neutrosophic PDEs.

(2) We proposed using horizontal membership function, that the neutrosophic equations can be transformed as the combination of classical equations with parameters. Thus, some numerical solution algorithm for neutrosophic fractional DEs can be developed and demonstrated by neutrosophic fractional damped single degree of freedom spring mass system and neutrosophic fractional telegraph model.

(3) The optimal control of linear quadratic function driven by neutrosophic fractional telegraph PDEs have been investigated. Our study is unique until now on neutrosophic DEs with non-integer order.

It is interesting that, fractional telegraph equations have found many applications to digital signal processing, image restoration or describing sound propagation in rigid tubes, etc. Thus our research may be useful for some further real world applications in the neutrosophic environments. Besides, some related open issues need to be further studied:

- To research on calculus of some extensions and modified form of neutrosophic set such as Plithogenic set-valued functions with applications to real-life uncertain dynamic systems modeling and simulation, not limit to fundamental research;

- To study the considered problem under some new settings such as neutrosophic optimal problem under generalized Hukuhara differentiability, neutrosophic linear quadratic regulator problem under Fréchet differentiability. Furthermore, the comparison on the advantages and disadvantages of different approaches when engineers apply to solve real-world problems;

- In general, neutrosophic techniques relate to nonparametric statistics. For the future work, it is interesting if we can integrate probability models into neutrosophic environment.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Acknowledgments

The authors are greatly indebted to Editor-in-Chief, Associate Editor, and the anonymous referees for their helpful comments and valuable suggestions, that greatly improve the quality and clarity of the paper. We also would like to thank Professor Florentin Smarandache for his valuable discussion and guidance at the begining of this work. This research is funded by Vietnam National Foundation for Science and Technology Development (NAFOSTED) under grant number 101.02-2018.311.

References

[1] Smarandache F. A unifying field in logics: neutrosophic logic. Neutrosophy, neutrosophic set, neutrosophic probability. Rehoboth: American Research Press; 1998.
[2] Smarandache F. Introduction to neutrosophic measure, neutrosophic integral, and neutrosophic probability. Craiova: Sitech & Education Publisher; 2013.
[3] Smarandache F. Introduction to neutrosophic statistics. Sitech & Education Publishing; 2014.
[4] Smarandache F. Neutrosophic precalculus and neutrosophic calculus. Brussels: Europa-Nova; 2015.
[5] Abdel-Basset M, Gunasekaran M, Mohamed M, Smarandache F. A novel method for solving the fully neutrosophic linear programming problems. Neural Comput Appl 2019;31(5):1595–605.
[6] Ye J, Cui W. Neutrosophic state feedback design method for SISO neutrosophic linear systems. Cog Syst Res 2018;52:1056–65.
[7] Son NTK, Dong NP, Son LH, Long HV. Towards granular calculus of single-valued neutrosophic functions under granular computing. Multimedia Tools Appl 2019. http://dx.doi.org/10.1007/s11042-019-7388-8.
[8] Long HV, Ali M, Son LH, Khan M, Tu DN. A novel approach for fuzzy clustering based on neutrosophic association matrix. Comput Ind Eng 2019;127:687–97.
[9] Sarkar M, Roy TK. Neutrosophic optimization technique and its application on structural design. J Ultra Sci Phys Sci 2016;28(6):309–21.
[10] Sarkar M, Ghosh S, Roy TK. Multi-objective welded beam optimization using neutrosophic optimization technique: A comparative study. J Ultra Sci Phys Sci 2017;29(6):243–63.
[11] Fan CX, Fan E, Hu K. New form of single-valued neutrosophic uncertain linguistic variables aggregation operators for decision-making. Cog Syst Res 2018;52:1045–55.
[12] Dey A, Broumi S, Son LH, Bakali A, Talea M, Smarandache F. A new algorithm for finding minimum spanning trees with undirected neutrosophic graphs. Granular Comput 2019;4(1):63–9.
[13] Smarandache F. Symbolic neutrosophic theory. Brussels, Belgium: Europa-Nova; 2015.
[14] Alinezhad M, Allahviranloo T. On the solution of fuzzy fractional optimal control problems with the Caputo derivative. Inf Sci 2017;421:218–36.
[15] Samko SG, Kilbas AA, Marichev OL. Fractional integrals and derivatives, theory and applications. Switzerland, Philadelphia, Pa. USA: Gordon and Breach Science Publishers; 1993.
[16] Dubois D, Prade H. Towards fuzzy differential calculus. part 3: differentiation. Fuzzy Sets Syst 1982;8:225–33.
[17] Kaleva O. Fuzzy Differential equations. Fuzzy Sets Syst 1987;24(3):301–17.
[18] Long HV, Son NTK, Tam HTT. The solvability of fuzzy fractional partial differential equations under Caputo gH-differentiability. Fuzzy Sets Syst 2017;309:35–63.
[19] Long HV. On fuzzy random fractional partial interger-differential equations under Caputo generalized Hukuhara diferentiability. Comput Appl Math 2018;37(3):2738–65.
[20] Agarwal RP, Lakshmikantham V, Nieto JJ. On the concept of solution for fractional differential equations with uncertainty. Nonlinear Anal 2010;72:2859–62.
[21] Allahviranloo T, Salahshour S, Abbasbandy S. Explicit solutions of fractional differential equations with uncertainty. Soft Comput 2012;16:297–302.
[22] Mazandarani M, Kamyad AV. Modified fractional Euler method for solving fuzzy fractional initial value problem. Commun Nonlinear Sci Numer Simul 2013;18:12–21.

Please cite this article as: N.T.K. Son, N.P. Dong, H.V. Long et al., Linear quadratic regulator problem governed by granular neutrosophic fractional differential equations. ISA Transactions (2019), https://doi.org/10.1016/j.isatra.2019.08.006.

[23] Son NTK. A foundation on semigroups of operators defined on the set of triangular fuzzy numbers and its application to fuzzy fractional evolution equations. Fuzzy Sets Syst 2018;347:1–28.

[24] Shahri ESA, Alfi A, Tenreiro Machado JA. Fractional fixed-structure H_∞ controller design using augmented Lagrangian particle swarm optimization with fractional order velocity. Appl Soft Comput 2019;77:688–95.

[25] Sharma R, Gaur P, Mittal AP. Design of two-layered fractional order fuzzy logic controllers applied to robotic manipulator with variable payload. Appl Soft Comput 2016;47:565–76.

[26] Jafari AA, Mohammadi SMA, Naseriyeh MH. Adaptive type-2 fuzzy backstepping control of uncertain fractional-order nonlinear systems with unknown dead-zone. Appl Math Model 2019;69:506–32.

[27] Chehlabi M, Allahviranloo T. Concreted solutions to fuzzy linear fractional differential equations. Appl Soft Comput 2016;44:108–16.

[28] Moezi SA, Zakeri E, Eghtesad M. Optimal adaptive interval type-2 fuzzy fractional-order backstepping sliding mode control method for some classes of nonlinear systems, ISA Trans [in press]. http://dx.doi.org/10.1016/j.isatra.2019.03.006.

[29] Sakthivel R, Raajananthini K, Kwon OM, Mohanapriya S. Estimation and disturbance rejection performance for fractional order fuzzy systems. ISA Trans. http://dx.doi.org/10.1016/j.isatra.2019.02.005.

[30] Zakeri E, Alireza Moezi S, Eghtesad M. Optimal interval type-2 fuzzy fractional order super twisting algorithm: A second order sliding mode controller for fully-actuated and under-actuated nonlinear systems. ISA Trans 2019;85:13–32.

[31] Mondal SP. Interval valued intuitionistic fuzzy number and its application in differential equation. J Intell Fuzzy Syst 2018;34(1):677–87.

[32] Efe MO. Fractional order systems in industrial automation - a survey. IEEE Tran Ind Inform 2011;7(4):1551–3203.

[33] Stefanini L. A generalization of hukuhara difference and division for interval and fuzzy arithmetic. Fuzzy Sets Syst 2010;161:1564–84.

[34] Piegat A, Landowski M. Aggregation of inconsistent expert opinions with use of horizontal intuitionistic membership functions. In: Novel Developments in Uncertainty Representation and Processing. Springer; 2016, p. 215–23.

[35] Piegat A, Landowski M. Solving different practical granular problems under the same system of equations. Granul Comput 2018;(3):39–48.

[36] Mazandarani M, Pariz N, Kamyad AV. Granular differentiability of fuzzy-number-valued functions. IEEE Tran Fuzzy Syst 2018;26(1):310–23.

[37] Mazandarani M, Pariz N. Sub-optimal control of fuzzy linear dynamical systems under granular differentiability concept. ISA Trans 2018;76:1–17.

[38] Najariyan M, Zhao Y. Fuzzy Fractional quadratic regulator problem under granular fuzzy fractional derivatives. IEEE Tran Fuzzy Syst 2018;26(4):2273–88.

[39] Kilbas AA, Srivastava HM, Trujillo JJ. Theory and applications of fractional differential equations, 204 of North-Holland mathematics studies. Amsterdam, The Netherlands: Elsevier; 2006.

[40] Garrappa R. Numerical solution of fractional differential equations: a survey and a software tutorial. Mathematics 2018;6(2).

[41] Zăk SH. Systems and control. School of Electrical and Computer Engineering Purdue University, Oxford Uni. Press; 2003.

[42] Chakraverty S, Tapaswini S, Behera D. Fuzzy arbitrary order system: fuzzy fractional differential equations and applications. Wiley; 2016.

[43] Adomian G. Solving frontier problems of physics: the decomposition method. Boston: Kluwer Academic Publishers; 1994.

[44] Momani S. Analytic and approximate solutions of the space and time fractional telegraph equations. Appl Math Comput 2005;170:1126–34.

[45] Abdulaziz O, Hashim I, Momani S. Application of homotopy perturbation method to fractional IVPs. J Comput Appl Math 2018;216:574–84.

[46] Bede B, Rudas IJ, Bencsik AL. First order linear fuzzy differential equations under generalized differentiability. Inf Sci 2007;177:1648–62.

Please cite this article as: N.T.K. Son, N.P. Dong, H.V. Long et al., Linear quadratic regulator problem governed by granular neutrosophic fractional differential equations. ISA Transactions (2019), https://doi.org/10.1016/j.isatra.2019.08.006.