symmetry

MDPI

Article

Linguistic Neutrosophic Generalized Partitioned Bonferroni Mean Operators and Their Application to Multi-Attribute Group Decision Making

Yumei Wang and Peide Liu *

School of Management Science and Engineering, Shandong University of Finance and Economics, Jinan 250014, China; wangyumei@mail.sdufe.edu.cn

* Corresponding Author: peide.liu@gmail.com; Tel.: +86-531-82222188

Received: 10 April 2018; Accepted: 26 April 2018; Published: 14 May 2018

Abstract: To solve the problems related to inhomogeneous connections among the attributes, we introduce a novel multiple attribute group decision-making (MAGDM) method based on the introduced linguistic neutrosophic generalized weighted partitioned Bonferroni mean operator (*LNGWPBM*) for linguistic neutrosophic numbers (LNNs). First of all, inspired by the merits of the generalized partitioned Bonferroni mean (GPBM) operator and LNNs, we combine the GPBM operator and LNNs to propose the linguistic neutrosophic GPBM (*LNGPBM*) operator, which supposes that the relationships are heterogeneous among the attributes in MAGDM. Then, we discuss its desirable properties and some special cases. In addition, aimed at the different importance of each attribute, the weighted form of the *LNGPBM* operator is investigated, which we call the *LNGWPBM* operator. Then, we discuss some of its desirable properties and special examples accordingly. In the end, we propose a novel MAGDM method on the basis of the introduced *LNGWPBM* operator, and illustrate its validity and merit by comparing it with the existing methods.

Keywords: *LNGPBM* operator; *LNGWPBM* operator; Linguistic neutrosophic sets; generalized partitioned Bonferroni mean operator; multiple attribute group decision-making (MAGDM)

1. Introduction

The goal of the multiple attribute group decision-making (MAGDM) method is to select the optimal scheme from finite alternatives. First of all, decision makers (DMs) evaluate each alternative under the different attributes. Then, based on the DMs' evaluation information, the alternatives are ranked in a certain way. As a research hotspot in recent decades, the MAGDM theory and methods have widely been used in all walks of life, such as supplier selection [1–3], medical diagnosis, clustering analysis, pattern recognition, and so on [4–11]. When evaluating alternatives, DMs used to evaluate alternatives by crisp numbers, but sometimes it is hard to use precise numbers because the surrounding environment has too much redundant data or interfering information. As a result, DMs have difficulty fully understanding the object of evaluation and exploiting exact information. As an example, when we evaluate people's morality or vehicle performance, we can easily use linguistic term such as good, fair, or poor, or fuzzy concepts such as slightly, obviously, or mightily, to give evaluation results. For this reason, Zadeh [12] put forward the concept of linguistic variables (LVs) in 1975. Later, Herrera and Herrera-Viedma [5,6] proposed a linguistic assessments consensus model and further developed the steps of linguistic decision analysis. Subsequently, it has become an area of wide concern, and resulted in several in-depth studies, especially in MAGDM [8,11,13–15]. In addition, for the reason of fuzziness, Atanassov [16] introduced the intuitionistic fuzzy set (IFS) on the basis of the fuzzy set developed by Zadeh [17]. IFS can embody the degrees of satisfaction and dissatisfaction to judge alternatives, synchronously, and has been studied by large numbers of scholars in many fields [1,2,9,10,18–23]. However, intuitionistic fuzzy numbers (IFNs) use the two real numbers of the interval [0,1] to represent membership degree and non-membership degree, which is not adequate or sufficient to quantify DMs' opinions. Hence, Chen et al. [24] used LVs to express the degrees of satisfaction and dissatisfaction instead of the real numbers of the interval [0,1], and proposed the linguistic intuitionistic fuzzy number (LIFN). LIFNs contain the advantages of both linguistic term sets and IFNs, so that it can address vague or imprecise information more accurately than LVs and IFNs. Since the birth of LIFNs, some scholars have proposed some improved aggregation operators and have applied them to MAGDM problems [10,25–28].

With the further development of fuzzy theory, Fang and Ye [29] noted while LIFNs can deal with vague or imprecise information more accurately than LVs and IFNs, it can only express incomplete information rather than indeterminate or inconsistent information. Since the indeterminacy of LIFN $I_A(x)$ is reckoned by $1 - T_A(x) - F_A(x)$ in default, evaluating the indeterminate or inconsistent information, i.e., $I_A(x) < 1 - T_A(x) - F_A(x)$ or $I_A(x) > 1 - T_A(x) - F_A(x)$, is beyond the scope of the LIFN. Hence, a new form of information expression needs to be found. Fortunately, the neutrosophic sets (NSs) developed by Smarandache [30] are able to quantify the indeterminacy clearly, which is independent of truth-membership and false-membership, but NSs are not easy to apply to the MAGDM. So, some stretched form of NS was proposed for solving MAGDM, such as single-valued neutrosophic sets (SVNSs) [31], interval neutrosophic sets (INSs) [32], simplified neutrosophic sets (SNSs) [33], and so on. Meanwhile, they have attracted a lot of research, especially related to MAGDM [34–41]. Due to the characteristic of SNSs that use three crisp numbers of the interval [0,1] to depict truth-membership, indeterminacy-membership, and false-membership, motivated by the narrow scope of the LIFN, Fang and Ye [29] put forward the concept of linguistic neutrosophic numbers (LNNs) by combining linguistic terms and a simplified neutrosophic number (SNN). LNNs use LVs in the predefined linguistic term set to express the truth-membership, indeterminacy-membership, and falsity-membership of SNNs. So, LNNs are more appropriate to depict qualitative information than SNNs, and are also an extension of the LIFNs, obviously. Therefore, in this paper, we tend to study the MAGDM problems with LNNs.

In MAGDM, the key step is how to select the optimal alternative according to the existing information. Usually, we adopt the traditional evaluation methods or the information aggregation operators. The common traditional evaluation methods include TOPSIS [7,9], VIKOR [19], ELECTRE [42], TODIM [20,43], PROMETHE [18], etc., and they can only give the priorities in order regarding alternatives. However, the information aggregation operators first integrate DMs' evaluation information into a comprehensive value, and then rank the alternatives. In other words, they not only give the prioritization orders of alternatives, they also give each alternative an integrated assessment value, so that the information aggregation operators are more workable than the traditional evaluation approaches in solving MAGDM problems. Hence, our study is concentrated on how to use information aggregation operators to solve the MAGDM problems with LNNs. In addition, in real MAGDM problems, there are often homogeneous connections among the attributes. Using a common example, quality is related to customer satisfaction when picking goods on the Internet. In order to solve this MAGDM problems where the attributes are interrelated, many related results have been achieved as a result, especially information aggregation operators such as the Bonferroni mean (BM) operator [23,44], the Maclaurin symmetric mean (MSM) operator [45], the Hamy mean operator [46], the generalized MSM operator [47], and so forth. However, the heterogeneous connections among the attributes may also exist in real MAGDM problems. For instance, in order to choose a car, we may consider the following attributes: the basic requirements (G_1), the physical property (G_2), the brand influence (G_3), and the user appraisal (G_4), where the attribute G_1 is associated with the attribute G_2, and the attribute G_3 is associated with the attribute G_4, but the attributes G_1 and G_2 are independent of the attributes G_3 and G_4. So, the four attributes can be sorted into two clusters, P_1 and P_2, namely $P_1 = \{G_1, G_2\}$ and $P_2 = \{G_3, G_4\}$ meeting the condition where P_1 and P_2 have no relationship. To solve this issue, Dutta and Guha [48] proposed the partition Bonferroni mean (*PBM*) operator, where all attributes are sorted into several clusters, and the members have an inherent connection in the same clusters, but independence in different clusters. Subsequently, Banerjee et al. [4] extended the *PBM* operator to the general form that was called the generalized partitioned Bonferroni mean (GPBM) operator, which further clarified the heterogeneous relationship and individually processed the elements that did not belong to any cluster of correlated elements, so the GPBM operator can model the average of the respective satisfaction of the independent and dependent input arguments. Besides, the GPBM operator can be translated into the BM operator, arithmetic mean operator, and *PBM* operator, so the GPBM operator is a wider range of applications for solving MAGDM problems with related attributes. Therefore, in this paper, we are further focused on how to combine the GPBM operator with LNNs to address the MAGDM problems with heterogeneous relationships among attributes. Inspired by the aforementioned ideas, we aim at:

(1) establishing a linguistic neutrosophic GPBM (*LNGPBM*) operator and the weighted form of the *LNGPBM* operator (the form of shorthand is *LNGWPBM*).
(2) discussing their properties and particular cases.
(3) proposing a novel MAGDM method in light of the proposed *LNGWPBM* operator to address the MAGDM problems with LNNs and the heterogeneous relationships among its attributes.
(4) showing the validity and merit of the developed method.

The arrangement of this paper is as follows. In Section 2, we briefly retrospect some elementary knowledge, including the definitions, operational rules, and comparison method of the LNNs. We also review some definitions and characteristics of the *PBM* operator and GPBM operator. In Section 3, we construct the *LNGPBM* operator and *LNGWPBM* operator for LNNs, including their characteristics and some special cases. In Section 4, we propose a novel MAGDM method based on the proposed *LNGWPBM* operator to address the MAGDM problems where heterogeneous connections exist among the attributes. In Section 5, we give a practical application related to the selection of green suppliers to show the validity and the generality of the MAGDM method, and compare the experimental results of the proposed MAGDM method with the ones of Fang and Ye's MAGDM method [29] and Liang et al.'s MAGDM method [7]. Section 6 presents the conclusions.

2. Preliminaries

To understand this article much better, this section intends to retrospect some elementary knowledge, including the definitions, operational rules, and comparison method of the LNNs, *PBM* operator, and generalized *PBM* operator.

2.1. Linguistic Neutrosophic Set (LNS)

Definition 1 [29]. *Let Z be the universe of discourse, and z be a generic element in Z, and let $L = (l_0, l_1, \cdots, l_s)$ be a linguistic term set. A LNS X in Z is represented by:*

$$X = \left\{ \left(z, l_{T_X(z)}, l_{I_X(z)}, l_{F_X(z)} \right) \middle| z \in Z \right\} \tag{1}$$

where T_X, I_X, and F_X denote the truth-membership function, indeterminacy-membership function, and falsity-membership function of z in the set X, respectively, and $T_X, I_X, F_X : Z \to [0, s]$ with s is an even number.

In [29], Fang and Ye called the pair $(l_\alpha, l_\beta, l_\gamma)$ an LNN, which meets $\alpha, \beta, \gamma : Z \to [0, s]$, and s is an even number.

Definition 2 [29]. *Let $z = (l_\alpha, l_\beta, l_\gamma)$ be an optional LNN in L, where the score function $C(z)$ of the LNN z is defined as shown:*

$$C(z) = \frac{2s + \alpha - \beta - \gamma}{3s} \tag{2}$$

where $\alpha, \beta, \gamma \in [0, s]$ *and* $C(z) \in [0, 1]$.

Definition 3 [29]. *Let* $z = (l_\alpha, l_\beta, l_\gamma)$ *be an optional LNN in L, where the accuracy function* $A(z)$ *of the LNN z is defined as shown:*

$$A(z) = \frac{\alpha - \gamma}{s} \tag{3}$$

where $\alpha, \beta, \gamma \in [0, s]$ *and* $A(z) \in [-1, 1]$.

Definition 4 [29]. *Let* $z_1 = (l_{\alpha_1}, l_{\beta_1}, l_{\gamma_1})$ *and* $z_2 = (l_{\alpha_2}, l_{\beta_2}, l_{\gamma_2})$ *be two optional LNNs in L. Then, the order between* z_1 *and* z_2 *is given by the following rules:*

(1) *If* $C(z_1) > C(z_2)$, *then* $z_1 > z_2$;
(2) *If* $C(z_1) = C(z_2)$, *then*

If $A(z_1) > A(z_2)$, *then* $z_1 > z_2$;

If $A(z_1) = A(z_2)$, *then* $z_1 = z_2$.

Example 1. *Suppose* $L = (l_0, l_1, \cdots, l_6)$ *is a linguistic term set, and* $z_1 = (l_6, l_2, l_3)$ *and* $z_2 = (l_4, l_1, l_1)$ *are two LNNs in L. Then, we can calculate the values of their score functions and accuracy functions as* $C(z_1) = 0.7222$, $C(z_2) = 0.7778$, $A(z_1) = 0.5$, and $A(z_2) = 0.5$. *According to Definition 4, it is easy to find that* $z_1 < z_2$.

Definition 5 [29]. *Let* $L = (l_0, l_1, \cdots, l_s)$ *be a linguistic term set, and* $z_1 = (l_{\alpha_1}, l_{\beta_1}, l_{\gamma_1})$ *and* $z_2 = (l_{\alpha_2}, l_{\beta_2}, l_{\gamma_2})$ *be two haphazard LNNs in L. The basic operational laws between the two LNNs are shown as below:*

$$z_1 \oplus z_2 = \left(l_{\alpha_1+\alpha_2-\alpha_1\alpha_2/s}, l_{\beta_1\beta_2/s}, l_{\gamma_1\gamma_2/s}\right), \tag{4}$$

$$z_1 \otimes z_2 = \left(l_{\alpha_1\alpha_2/s}, l_{\beta_1+\beta_2-\beta_1\beta_2/s}, l_{\gamma_1+\gamma_2-\gamma_1\gamma_2/s}\right), \tag{5}$$

$$\theta z_1 = \left(l_{s-s(1-\alpha_1/s)^\theta}, l_{s(\beta_1/s)^\theta}, l_{s(\gamma_1/s)^\theta}\right), \text{ where } \theta > 0, \tag{6}$$

$$z_1{}^\theta = \left(l_{s(\alpha_1/s)^\theta}, l_{s-s(1-\beta_1/s)^\theta}, l_{s-s(1-\gamma_1/s)^\theta}\right), \text{ where } \theta > 0 \tag{7}$$

It is easy to prove the following operational properties of the LNNs, according to Definition 5. Let $z_1 = (l_{\alpha_1}, l_{\beta_1}, l_{\gamma_1})$ and $z_2 = (l_{\alpha_2}, l_{\beta_2}, l_{\gamma_2})$ be any two LNNs in L. Then:

$$z_1 \oplus z_2 = z_2 \oplus z_1, \tag{8}$$

$$z_1 \otimes z_2 = z_2 \otimes z_1, \tag{9}$$

$$\theta(z_1 \oplus z_2) = \theta z_1 \oplus \theta z_2, \text{ where } \theta > 0, \tag{10}$$

$$\theta_1 z_1 \oplus \theta_2 z_1 = (\theta_1 + \theta_2) z_1, \text{ where } \theta_1, \theta_2 > 0, \tag{11}$$

$$z_1{}^{\theta_1} \otimes z_1{}^{\theta_2} = z_1{}^{\theta_1+\theta_2}, \text{ where } \theta_1, \theta_2 > 0, \tag{12}$$

$$z_1^\theta \otimes z_2^\theta = (z_1 \otimes z_2)^\theta, \text{ where } \theta > 0 \tag{13}$$

2.2. Generalized Partitioned Bonferroni Mean Operators

Definition 6 [48]. *Suppose the non-negative real set* $A = \{a_1, a_2, \cdots, a_n\}$ *is divided into t clusters* $P_1, P_2, \cdots, P_t$, *which satisfies* $P_x \cap P_y = \varnothing$, $x \neq y$ *and* $\bigcup_{r=1}^{t} P_r = A$. *Then, the partitioned Bonferroni mean (PBM) operator is defined as follows:*

$$PBM^{p,q}(a_1, a_2, \cdots, a_n) = \frac{1}{t}\left(\sum_{r=1}^{t}\left(\frac{1}{h_r}\sum_{i=1}^{h_r} a_i^p\left(\frac{1}{h_r-1}\sum_{\substack{j=1\\ j\neq i}}^{h_r} a_j^q\right)\right)^{\frac{1}{p+q}}\right) \tag{14}$$

where $p, q \geq 0$ and $p + q > 0$, h_r indicates the number of elements in partition P_r and $\sum\limits_{r=1}^{t} h_r = n$.

The *PBM* operator is used to integrate the input arguments of the different clusters, which satisfies that the data has inherent connections in the same clusters, but independence in different clusters. However, sometimes, some of the input arguments have nothing to do with any other argument, that is, it does not exist in any cluster. We can part these arguments and deal with them individually. Hence, we sort the input arguments into two groups: F_1 contains the relevant arguments, and F_2 contains the input arguments that are irrelevant to any argument. These easily derive $F_1 \cap F_2 = \varnothing$ and $|F_1| + |F_2| = n$ where $|F_1|$ and $|F_2|$ denote the numbers of arguments in F_1 and F_2, respectively. According to the upper description, we suppose that the arguments of F_1 are divided into t partitions $P_1, P_2, \cdots, P_t$ on the basis of the interrelationship pattern [4]. To address this issue, the *PBM* operator is modified, and the GPBM operator is proposed, as shown in the following.

Definition 7 [4]. *Suppose that the non-negative real set $A = \{a_1, a_2, \cdots, a_n\}$ is sorted into two groups: F_1 and F_2. In F_1, the elements are divided into t clusters $P_1, P_2, \cdots, P_t$, which satisfies $P_x \cap P_y = \varnothing$, $x \neq y$ and $\bigcup\limits_{r=1}^{t} P_r = F_1$; in F_2, the elements are irrelevant to any element. Then, the GPBM operator is defined as follows:*

$$GPBM^{p,q}(a_1, a_2, \cdots, a_n) = \left(\frac{n-|F_2|}{n}\left(\frac{1}{t}\sum_{r=1}^{t}\left(\frac{1}{h_r}\sum_{i=1}^{h_r} a_i^p\left(\frac{1}{h_r-1}\sum_{\substack{j=1\\ j\neq i}}^{h_r} a_j^q\right)\right)^{\frac{p}{p+q}}\right) + \frac{|F_2|}{n}\left(\frac{1}{|F_2|}\sum_{i=1}^{|F_2|} a_i^p\right)\right)^{\frac{1}{p}} \tag{15}$$

where $p, q \geq 0$ and $p + q > 0$, $|F_2|$ denotes the number of elements in F_2, h_r indicates the number of elements in cluster P_r and $\sum\limits_{r=1}^{t} h_r = n - |F_2|$.

Remark 1. *If $|F_2| = 0$, we consider the first sum, and if $|F_2| = n$, we consider the last sum. At the same time, we have made the convention $\frac{0}{0} = 0$ (we only need to define $\frac{0}{0}$; its conventional real value is not important here).*

The interpretation of the GPBM operator is detailed by Banerjee et al. in [4], and the GPBM operator has the following characteristics: idempotency, monotonicity, and boundedness [4].

Based on the characteristics of F_2, there are some special cases of GPBM operator, which are described as follows [4]:

(1) When $|F_2| = 0$, all elements belong to the group F_1 and are divided into t clusters.

$$GPBM^{p,q}(a_1, a_2, \cdots, a_n) = \frac{1}{t}\sum_{r=1}^{t}\left(\frac{1}{h_r}\sum_{i=1}^{h_r} a_i^p\left(\frac{1}{h_r-1}\sum_{\substack{j=1\\ j\neq i}}^{h_r} a_j^q\right)\right)^{\frac{1}{p+q}} = PBM^{p,q}(a_1, a_2, \cdots, a_n)$$

It is simplified as the *PBM* operator described in Formula (15).

(2) When $|F_2|$ = 0 and $t = 1$, all elements belong to the same cluster.

$$GPBM^{p,q}(a_1,a_2,\cdots,a_n) = \left(\frac{1}{h_r}\sum_{i=1}^{h_r} a_i^p\left(\frac{1}{h_r-1}\sum_{\substack{j=1\\ j\neq i}}^{h_r} a_j^q\right)\right)^{\frac{1}{p+q}} = \left(\frac{1}{n}\sum_{i=1}^{n} a_i^p\left(\frac{1}{n-1}\sum_{\substack{j=1\\ j\neq i}}^{n} a_j^q\right)\right)^{\frac{1}{p+q}} = BM^{p,q}(a_1,a_2,\cdots,a_n)$$

It becomes the *BM* operator [44].

(3) When $|F_2|$ = n, all elements are independent.

$$GPBM^{p,q}(a_1,a_2,\cdots,a_n) = \left(\frac{1}{|F_2|}\sum_{i=1}^{|F_2|} a_i^p\right)^{\frac{1}{p}} = \left(\frac{1}{n}\sum_{i=1}^{n} a_i^p\right)^{\frac{1}{p}}$$

It is simplified as the power root arithmetic mean operator [4].

3. The Linguistic Neutrosophic GPBM Operators

In this section, we will construct the *LNGPBM* operator from the GPBM operator and LNNs. Moreover, with respect to the different weights of different attributes in real life, we will propose the corresponding weighted operators, and call it the *LNGWPBM* operator. They are defined as follows.

3.1. The LNGPBM Operator

Definition 8. *Let $z_1, z_2, \cdots$ and z_n be LNNs, which are sorted into two groups: F_1 and F_2. In F_1, the elements are divided into t clusters $P_1, P_2, \cdots, P_t$, which satisfies $P_x \cap P_y = \varnothing$, $x \neq y$ and $\bigcup_{r=1}^{t} P_r = F_1$; in F_2, the elements are irrelevant to any element. The LNGPBM operator of the LNNs $z_1, z_2, \cdots$ and z_n is defined as follows:*

$$LNGPBM^{p,q}(z_1,z_2,\cdots,z_n) = \left(\frac{n-|F_2|}{n}\left(\frac{1}{t}\bigoplus_{r=1}^{t}\left(\frac{1}{h_r}\bigoplus_{i=1}^{h_r} z_i^p \otimes \left(\frac{1}{h_r-1}\bigoplus_{\substack{j=1\\ j\neq i}}^{h_r} z_j^q\right)\right)^{\frac{p}{p+q}}\right) \oplus \frac{|F_2|}{n}\left(\frac{1}{|F_2|}\bigoplus_{i=1}^{|F_2|} z_i^p\right)\right)^{\frac{1}{p}} \quad (16)$$

where $z_i = (l_{\alpha_i}, l_{\beta_i}, l_{\gamma_i})$ and $\alpha_i, \beta_i, \gamma_i \in [0,s]$ $(i = 1,2,\cdots,n)$; $p,q \geq 0$ and $p+q > 0$; $|F_2|$ denotes the number of elements in F_2, h_r indicates the number of elements in cluster P_r and $\sum_{r=1}^{t} h_r = n - |F_2|$.

Theorem 1. *Let $z_1, z_2, \cdots$ and z_n be LNNs, where $z_i = (l_{\alpha_i}, l_{\beta_i}, l_{\gamma_i})$ and $\alpha_i, \beta_i, \gamma_i \in [0,s]$ $(i = 1,2,\cdots,n)$. The synthesized result of the LNGPBM operator of the LNNs $z_1, z_2, \cdots$ and z_n is still a LNN, which is shown as follows:*

$$LNGPBM^{p,q}(z_1,z_2,\cdots,z_n)=\left(\begin{array}{l} l_{s\left(1-\left(\left(\left(\prod\limits_{r=1}^{t}\left(1-(1-H_\alpha)^{\frac{p}{p+q}}\right)\right)^{\frac{1}{t}}\right)^{\frac{n-|F_2|}{n}}\right)\left(\left(\left(\prod\limits_{i=1}^{|F_2|}\left(1-(\alpha_i/s)^p\right)\right)^{\frac{1}{|F_2|}}\right)^{\frac{|F_2|}{n}}\right)^{\frac{1}{p}}}, \\ l_{s-s\left(1-\left(\left(\prod\limits_{r=1}^{t}\left(1-(1-H_\beta)^{\frac{p}{p+q}}\right)\right)^{\frac{1}{t}}\right)^{\frac{n-|F_2|}{n}}\left(\left(\prod\limits_{i=1}^{|F_2|}\left(1-(1-\beta_i/s)^p\right)\right)^{\frac{1}{|F_2|}}\right)^{\frac{|F_2|}{n}}\right)^{\frac{1}{p}}}, l_{s-s\left(1-\left(\left(\prod\limits_{r=1}^{t}\left(1-(1-H_\gamma)^{\frac{p}{p+q}}\right)\right)^{\frac{1}{t}}\right)^{\frac{n-|F_2|}{n}}\left(\left(\prod\limits_{i=1}^{|F_2|}\left(1-(1-\gamma_i/s)^p\right)\right)^{\frac{1}{|F_2|}}\right)^{\frac{|F_2|}{n}}\right)^{\frac{1}{p}}} \end{array}\right) \quad (17)$$

where $H_\alpha = \left(\prod\limits_{i=1}^{h_r}\left(1-(\alpha_i/s)^p\left(1-\left(\prod\limits_{\substack{j=1\\ j\neq i}}^{h_r}\left(1-(\alpha_j/s)^q\right)\right)^{\frac{1}{h_r-1}}\right)\right)\right)^{\frac{1}{h_r}}$, $H_\beta = \left(\prod\limits_{i=1}^{h_r}\left(1-(1-\beta_i/s)^p\left(1-\left(\prod\limits_{\substack{j=1\\ j\neq i}}^{h_r}\left(1-(1-\beta_j/s)^q\right)\right)^{\frac{1}{h_r-1}}\right)\right)\right)^{\frac{1}{h_r}}$ and $H_\gamma = \left(\prod\limits_{i=1}^{h_r}\left(1-(1-\gamma_i/s)^p\left(1-\left(\prod\limits_{\substack{j=1\\ j\neq i}}^{h_r}\left(1-(1-\gamma_j/s)^q\right)\right)^{\frac{1}{h_r-1}}\right)\right)\right)^{\frac{1}{h_r}}$.

Proof. According to Formula (16), first of all, we can part two steps: the processing of F_1 and F_2, and then combine them to prove.

(i) The processing of F_1:

Based on the operational rules of LNNs, we can get $z_j^q = \left(l_{s(\alpha_j/s)^q}, l_{s-s(1-\beta_j/s)^q}, l_{s-s(1-\gamma_j/s)^q}\right)$ and

$$\mathop{\oplus}\limits_{\substack{j=1\\ j\neq i}}^{h_r} z_j^q = \left(l_{s-s\prod\limits_{\substack{j=1\\ j\neq i}}^{h_r}\left(1-(\alpha_j/s)^q\right)}, l_{s\prod\limits_{\substack{j=1\\ j\neq i}}^{h_r}\left(1-(1-\beta_j/s)^q\right)}, l_{s\prod\limits_{\substack{j=1\\ j\neq i}}^{h_r}\left(1-(1-\gamma_j/s)^q\right)}\right).$$

Then, we can calculate the average satisfaction of the elements in P_r except z_i:

$$\frac{1}{h_r-1}\mathop{\oplus}\limits_{\substack{j=1\\ j\neq i}}^{h_r} z_j^q = \left(l_{s-s\left(\prod\limits_{\substack{j=1\\ j\neq i}}^{h_r}\left(1-(\alpha_j/s)^q\right)\right)^{\frac{1}{h_r-1}}}, l_{s\left(\prod\limits_{\substack{j=1\\ j\neq i}}^{h_r}\left(1-(1-\beta_j/s)^q\right)\right)^{\frac{1}{h_r-1}}}, l_{s\left(\prod\limits_{\substack{j=1\\ j\neq i}}^{h_r}\left(1-(1-\gamma_j/s)^q\right)\right)^{\frac{1}{h_r-1}}}\right)$$

and the conjunction of the satisfaction of element z_i with the average satisfaction of the rest of elements in P_r:

$$z_i^p \otimes \left(\frac{1}{h_r-1} \underset{\substack{j=1\\ j\neq i}}{\overset{h_r}{\oplus}} z_j^q \right) = \left(l_{s(\alpha_i/s)^p \left(1-\left(\prod\limits_{\substack{j=1\\ j\neq i}}^{h_r} \left(1-(\alpha_j/s)^q\right)\right)^{\frac{1}{h_r-1}}\right)}, l_{s-s\left((1-\beta_i/s)^p\right)\left(1-\left(\prod\limits_{\substack{j=1\\ j\neq i}}^{h_r} \left(1-(1-\beta_j/s)^q\right)\right)^{\frac{1}{h_r-1}}\right)}, l_{s-s\left((1-\gamma_i/s)^p\right)\left(1-\left(\prod\limits_{\substack{j=1\\ j\neq i}}^{h_r} \left(1-(1-\gamma_j/s)^q\right)\right)^{\frac{1}{h_r-1}}\right)} \right)$$

Then, the satisfaction of the interrelated elements of P_r is:

$$\frac{1}{h_r} \overset{h_r}{\underset{i=1}{\oplus}} z_i^p \otimes \left(\frac{1}{h_r-1} \underset{\substack{j=1\\ j\neq i}}{\overset{h_r}{\oplus}} z_j^q \right) = \left(l_{s-s\left(\prod\limits_{i=1}^{h_r}\left(1-(\alpha_i/s)^p\left(1-\left(\prod\limits_{\substack{j=1\\ j\neq i}}^{h_r} \left(1-(\alpha_j/s)^q\right)\right)^{\frac{1}{h_r-1}}\right)\right)\right)^{\frac{1}{h_r}}}, \right.$$

$$\left. l_{s\left(\prod\limits_{i=1}^{h_r}\left(1-(1-\beta_i/s)^p\left(1-\left(\prod\limits_{\substack{j=1\\ j\neq i}}^{h_r} \left(1-(1-\beta_j/s)^q\right)\right)^{\frac{1}{h_r-1}}\right)\right)\right)^{\frac{1}{h_r}}}, l_{s\left(\prod\limits_{i=1}^{h_r}\left(1-(1-\gamma_i/s)^p\left(1-\left(\prod\limits_{\substack{j=1\\ j\neq i}}^{h_r} \left(1-(1-\gamma_j/s)^q\right)\right)^{\frac{1}{h_r-1}}\right)\right)\right)^{\frac{1}{h_r}}} \right)$$

So, the average satisfaction of all of the elements of the t clusters is:

$$A = \frac{1}{t} \overset{t}{\underset{r=1}{\oplus}} \left(\frac{1}{h_r} \overset{h_r}{\underset{i=1}{\oplus}} z_i^p \otimes \left(\frac{1}{h_r-1} \underset{\substack{j=1\\ j\neq i}}{\overset{h_r}{\oplus}} z_j^q \right) \right)^{\frac{p}{p+q}} = \left(l_{s-s\left(\prod\limits_{r=1}^{t}\left(1-\left(1-\left(\prod\limits_{i=1}^{h_r}\left(1-(\alpha_i/s)^p\left(1-\left(\prod\limits_{\substack{j=1\\ j\neq i}}^{h_r} \left(1-(\alpha_j/s)^q\right)\right)^{\frac{1}{h_r-1}}\right)\right)\right)^{\frac{1}{h_r}}\right)^{\frac{p}{p+q}}\right)\right)^{\frac{1}{t}}}, \right.$$

$$\left. l_{s\left(\prod\limits_{r=1}^{t}\left(1-\left(1-\left(\prod\limits_{i=1}^{h_r}\left(1-(1-\beta_i/s)^p\left(1-\left(\prod\limits_{\substack{j=1\\ j\neq i}}^{h_r} \left(1-(1-\beta_j/s)^q\right)\right)^{\frac{1}{h_r-1}}\right)\right)\right)^{\frac{1}{h_r}}\right)^{\frac{p}{p+q}}\right)\right)^{\frac{1}{t}}}, l_{s\left(\prod\limits_{r=1}^{t}\left(1-\left(1-\left(\prod\limits_{i=1}^{h_r}\left(1-(1-\gamma_i/s)^p\left(1-\left(\prod\limits_{\substack{j=1\\ j\neq i}}^{h_r} \left(1-(1-\gamma_j/s)^q\right)\right)^{\frac{1}{h_r-1}}\right)\right)\right)^{\frac{1}{h_r}}\right)^{\frac{p}{p+q}}\right)\right)^{\frac{1}{t}}} \right)$$

We suppose $H_\alpha = \left(\prod_{i=1}^{h_r}\left(1-(\alpha_i/s)^p\left(1-\left(\prod_{\substack{j=1\\ j\neq i}}^{h_r}\left(1-(\alpha_j/s)^q\right)\right)^{\frac{1}{h_r-1}}\right)\right)\right)^{\frac{1}{h_r}}$,

$H_\beta = \left(\prod_{i=1}^{h_r}\left(1-(1-\beta_i/s)^p\left(1-\left(\prod_{\substack{j=1\\ j\neq i}}^{h_r}\left(1-(1-\beta_j/s)^q\right)\right)^{\frac{1}{h_r-1}}\right)\right)\right)^{\frac{1}{h_r}}$ and $H_\gamma = \left(\prod_{i=1}^{h_r}\left(1-(1-\gamma_i/s)^p\left(1-\left(\prod_{\substack{j=1\\ j\neq i}}^{h_r}\left(1-(1-\gamma_j/s)^q\right)\right)^{\frac{1}{h_r-1}}\right)\right)\right)^{\frac{1}{h_r}}$, then the upper formula can be rewritten as:

$$A = \frac{1}{t}\bigoplus_{r=1}^{t}\left(\frac{1}{h_r}\bigoplus_{i=1}^{h_r} z_i^p \otimes \left(\frac{1}{h_r-1}\bigoplus_{\substack{j=1\\ j\neq i}}^{h_r} z_j^q\right)\right)^{\frac{p}{p+q}} = \left(l_{s-s\left(\prod_{r=1}^{t}\left(1-(1-H_\alpha)^{\frac{p}{p+q}}\right)\right)^{\frac{1}{t}}}, l_{s\left(\prod_{r=1}^{t}\left(1-(1-H_\beta)^{\frac{p}{p+q}}\right)\right)^{\frac{1}{t}}}, l_{s\left(\prod_{r=1}^{t}\left(1-(1-H_\gamma)^{\frac{p}{p+q}}\right)\right)^{\frac{1}{t}}}\right)$$

(ii) The processing of F_2:

The average satisfaction of all the elements that are irrelevant to any element is:

$$B = \frac{1}{|F_2|}\bigoplus_{i=1}^{|F_2|} z_i^p = \left(l_{s-s\left(\prod_{i=1}^{|F_2|}\left(1-(\alpha_i/s)^p\right)\right)^{\frac{1}{|F_2|}}}, l_{s\left(\prod_{i=1}^{|F_2|}\left(1-(1-\beta_i/s)^p\right)\right)^{\frac{1}{|F_2|}}}, l_{s\left(\prod_{i=1}^{|F_2|}\left(1-(1-\gamma_i/s)^p\right)\right)^{\frac{1}{|F_2|}}}\right)$$

Finally, we can compute the average satisfaction of the elements $z_1, z_2, \cdots$ and z_n:

$$\left(\frac{n-|F_2|}{n}A \oplus \frac{|F_2|}{n}B\right)^{\frac{1}{p}} = \left(\begin{array}{l} l_{s\left(1-\left(\left(\left(\prod_{r=1}^{t}\left(1-(1-H_\alpha)^{\frac{p}{p+q}}\right)\right)^{\frac{1}{t}}\right)^{\frac{n-|F_2|}{n}}\right)\left(\left(\left(\prod_{i=1}^{|F_2|}\left(1-(\alpha_i/s)^p\right)\right)^{\frac{1}{|F_2|}}\right)^{\frac{|F_2|}{n}}\right)\right)^{\frac{1}{p}}}, l_{s-s\left(1-\left(\left(\prod_{r=1}^{t}\left(1-(1-H_\beta)^{\frac{p}{p+q}}\right)\right)^{\frac{1}{t}}\right)^{\frac{n-|F_2|}{n}}\left(\left(\prod_{i=1}^{|F_2|}\left(1-(1-\beta_i/s)^p\right)\right)^{\frac{1}{|F_2|}}\right)^{\frac{|F_2|}{n}}\right)^{\frac{1}{p}}}, \\ l_{s-s\left(1-\left(\left(\prod_{r=1}^{t}\left(1-(1-H_\gamma)^{\frac{p}{p+q}}\right)\right)^{\frac{1}{t}}\right)^{\frac{n-|F_2|}{n}}\left(\left(\prod_{i=1}^{|F_2|}\left(1-(1-\gamma_i/s)^p\right)\right)^{\frac{1}{|F_2|}}\right)^{\frac{|F_2|}{n}}\right)^{\frac{1}{p}}} \end{array}\right)$$

That proves that Formula (17) is kept. Then, we prove that the aggregated result of Formula (17) is a LNN. It is easy to prove the following inequalities:

$$0 \leq s\left(1-\left(\left(\left(\prod_{r=1}^{t}\left(1-(1-H_\alpha)^{\frac{p}{p+q}}\right)\right)^{\frac{1}{t}}\right)^{\frac{n-|F_2|}{n}}\right)\left(\left(\left(\prod_{i=1}^{|F_2|}\left(1-(\alpha_i/s)^p\right)\right)^{\frac{1}{|F_2|}}\right)^{\frac{|F_2|}{n}}\right)\right)^{\frac{1}{p}} \leq s,$$

$$0 \le s - s\left(1 - \left(\left(\prod_{r=1}^{t}\left(1-(1-H_\beta)^{\frac{p}{p+q}}\right)\right)^{\frac{1}{t}}\right)^{\frac{n-|F_2|}{n}}\left(\left(\prod_{i=1}^{|F_2|}\left(1-(1-\beta_i/s)^p\right)\right)^{\frac{1}{|F_2|}}\right)^{\frac{|F_2|}{n}}\right)^{\frac{1}{p}} \le s,$$

and:

$$0 \le s - s\left(1 - \left(\left(\prod_{r=1}^{t}\left(1-(1-H_\gamma)^{\frac{p}{p+q}}\right)\right)^{\frac{1}{t}}\right)^{\frac{n-|F_2|}{n}}\left(\left(\prod_{i=1}^{|F_2|}\left(1-(1-\gamma_i/s)^p\right)\right)^{\frac{1}{|F_2|}}\right)^{\frac{|F_2|}{n}}\right)^{\frac{1}{p}} \le s.$$

Firstly, we prove $0 \le H_\alpha \le 1$, $0 \le H_\beta \le 1$ and $0 \le H_\gamma \le 1$.

Since $\alpha_j, \beta_j, \gamma_j \in [0, s]$ and $q \ge 0$, we can get $0 \le 1 - (\alpha_j/s)^q \le 1$, $0 \le 1 - (1-\beta_j/s)^q \le 1$ and $0 \le 1 - (1-\gamma_j/s)^q \le 1$. Owing to $h_r > 0$, the following inequalities are established:

$$0 \le 1 - \left(\prod_{\substack{j=1 \\ j \ne i}}^{h_r}\left(1-(\alpha_j/s)^q\right)\right)^{\frac{1}{h_r-1}} \le 1, 0 \le 1 - \left(\prod_{\substack{j=1 \\ j \ne i}}^{h_r}\left(1-(1-\beta_j/s)^q\right)\right)^{\frac{1}{h_r-1}} \le 1, \text{ and } 0 \le 1 - \left(\prod_{\substack{j=1 \\ j \ne i}}^{h_r}\left(1-(1-\gamma_j/s)^q\right)\right)^{\frac{1}{h_r-1}} \le 1.$$

According to $p \ge 0$, it is easy to obtain the below inequality: $0 \le H_\alpha \le 1$, $0 \le H_\beta \le 1$ and $0 \le H_\gamma \le 1$.

In addition, because $p+q > 0$, $t > 0$, and $|F_2| > 0$, we can get the following inequalities:

$$0 \le \left(\left(\prod_{r=1}^{t}\left(1-(1-H_\alpha)^{\frac{p}{p+q}}\right)\right)^{\frac{1}{t}}\right)^{\frac{n-|F_2|}{n}} \le 1, 0 \le \left(\left(\prod_{r=1}^{t}\left(1-(1-H_\beta)^{\frac{p}{p+q}}\right)\right)^{\frac{1}{t}}\right)^{\frac{n-|F_2|}{n}} \le 1, \text{and} 0 \le \left(\left(\prod_{r=1}^{t}\left(1-(1-H_\gamma)^{\frac{p}{p+q}}\right)\right)^{\frac{1}{t}}\right)^{\frac{n-|F_2|}{n}} \le 1.$$

$$0 \le \left(\left(\prod_{i=1}^{|F_2|}\left(1-(\alpha_i/s)^p\right)\right)^{\frac{1}{|F_2|}}\right)^{\frac{|F_2|}{n}} \le 1, 0 \le \left(\left(\prod_{i=1}^{|F_2|}\left(1-(1-\beta_i/s)^p\right)\right)^{\frac{1}{|F_2|}}\right)^{\frac{|F_2|}{n}} \le 1, \text{and} 0 \le \left(\left(\prod_{i=1}^{|F_2|}\left(1-(1-\gamma_i/s)^p\right)\right)^{\frac{1}{|F_2|}}\right)^{\frac{|F_2|}{n}} \le 1.$$

Besides, on the basis of the upper inequalities, we can get:

$$0 \le \left(1 - \left(\left(\left(\prod_{r=1}^{t}\left(1-(1-H_\alpha)^{\frac{p}{p+q}}\right)\right)^{\frac{1}{t}}\right)^{\frac{n-|F_2|}{n}}\right)\left(\left(\left(\prod_{i=1}^{|F_2|}\left(1-(\alpha_i/s)^p\right)\right)^{\frac{1}{|F_2|}}\right)^{\frac{|F_2|}{n}}\right)\right)^{\frac{1}{p}} \le 1,$$

$$0 \le \left(1 - \left(\left(\prod_{r=1}^{t}\left(1-(1-H_\beta)^{\frac{p}{p+q}}\right)\right)^{\frac{1}{t}}\right)^{\frac{n-|F_2|}{n}}\left(\left(\prod_{i=1}^{|F_2|}\left(1-(1-\beta_i/s)^p\right)\right)^{\frac{1}{|F_2|}}\right)^{\frac{|F_2|}{n}}\right)^{\frac{1}{p}} \le 1, \text{ and}$$

$$0 \le \left(1 - \left(\left(\prod_{r=1}^{t}\left(1-(1-H_\gamma)^{\frac{p}{p+q}}\right)\right)^{\frac{1}{t}}\right)^{\frac{n-|F_2|}{n}}\left(\left(\prod_{i=1}^{|F_2|}\left(1-(1-\gamma_i/s)^p\right)\right)^{\frac{1}{|F_2|}}\right)^{\frac{|F_2|}{n}}\right)^{\frac{1}{p}} \le 1.$$

which can derive directly:

$$0 \le s\left(1-\left(\left(\left(\prod_{r=1}^{t}\left(1-(1-H_\alpha)^{\frac{p}{p+q}}\right)\right)^{\frac{1}{t}}\right)^{\frac{n-|F_2|}{n}}\right)\left(\left(\left(\prod_{i=1}^{|F_2|}\left(1-(\alpha_i/s)^p\right)\right)^{\frac{1}{|F_2|}}\right)^{\frac{|F_2|}{n}}\right)\right)^{\frac{1}{p}} \le s,$$

$$0 \le s-s\left(1-\left(\left(\prod_{r=1}^{t}\left(1-(1-H_\beta)^{\frac{p}{p+q}}\right)\right)^{\frac{1}{t}}\right)^{\frac{n-|F_2|}{n}}\left(\left(\prod_{i=1}^{|F_2|}\left(1-(1-\beta_i/s)^p\right)\right)^{\frac{1}{|F_2|}}\right)^{\frac{|F_2|}{n}}\right)^{\frac{1}{p}} \le s,$$

$$\text{and } 0 \le s-s\left(1-\left(\left(\prod_{r=1}^{t}\left(1-(1-H_\gamma)^{\frac{p}{p+q}}\right)\right)^{\frac{1}{t}}\right)^{\frac{n-|F_2|}{n}}\left(\left(\prod_{i=1}^{|F_2|}\left(1-(1-\gamma_i/s)^p\right)\right)^{\frac{1}{|F_2|}}\right)^{\frac{|F_2|}{n}}\right)^{\frac{1}{p}} \le s.$$

Therefore, Theorem 1 is kept if some of the partitions only contain one element. □

In the following, we will demonstrate the desired properties of the proposed *LNGPBM* operator:

(1) **Idempotency**: If $z_1, z_2, \cdots$ and z_n are LNNs meeting the condition $z_i = \left(l_{\alpha_i}, l_{\beta_i}, l_{\gamma_i}\right) = z = \left(l_\alpha, l_\beta, l_\gamma\right)$ $(i = 1, 2, \cdots, n)$; then, $LNGPBM^{p,q}(z_1, z_2, \cdots, z_n) = z$.

Proof. Since $z_i = \left(l_{\alpha_i}, l_{\beta_i}, l_{\gamma_i}\right) = z = \left(l_\alpha, l_\beta, l_\gamma\right)$, we can get:

$$H_\alpha = \left(\prod_{i=1}^{h_r}\left(1-(\alpha_i/s)^p\left(1-\left(\prod_{\substack{j=1\\ j\neq i}}^{h_r}\left(1-(\alpha_j/s)^q\right)\right)^{\frac{1}{h_r-1}}\right)\right)\right)^{\frac{1}{h_r}} = \left(\prod_{i=1}^{h_r}\left(1-(\alpha/s)^p\left(1-\left(\prod_{\substack{j=1\\ j\neq i}}^{h_r}\left(1-(\alpha/s)^q\right)\right)^{\frac{1}{h_r-1}}\right)\right)\right)^{\frac{1}{h_r}}.$$

$$= \left(\prod_{i=1}^{h_r}\left(1-(\alpha/s)^p\left(1-\left(\left(1-(\alpha/s)^q\right)^{h_r-1}\right)^{\frac{1}{h_r-1}}\right)\right)\right)^{\frac{1}{h_r}} = \left(\prod_{i=1}^{h_r}\left(1-(\alpha/s)^p(\alpha/s)^q\right)\right)^{\frac{1}{h_r}} = 1-(\alpha/s)^{p+q}$$

In the same way, we can obtain $H_\beta = 1-(1-\beta/s)^{p+q}$ and $H_\gamma = 1-(1-\gamma/s)^{p+q}$.
According to Theorem 1, we can obtain:

$$LNGPBM^{p,q}(z_1, z_2, \cdots, z_n) = \left(l_{s\left(1-\left(\left(\left(\prod_{r=1}^{t}\left(1-(1-H_\alpha)^{\frac{p}{p+q}}\right)\right)^{\frac{1}{t}}\right)^{\frac{n-|F_2|}{n}}\right)\left(\left(\left(\prod_{i=1}^{|F_2|}\left(1-(\alpha_i/s)^p\right)\right)^{\frac{1}{|F_2|}}\right)^{\frac{|F_2|}{n}}\right)\right)^{\frac{1}{p}}},\right.$$
$$\left. l_{s-s\left(1-\left(\left(\prod_{r=1}^{t}\left(1-(1-H_\beta)^{\frac{p}{p+q}}\right)\right)^{\frac{1}{t}}\right)^{\frac{n-|F_2|}{n}}\left(\left(\prod_{i=1}^{|F_2|}\left(1-(1-\beta_i/s)^p\right)\right)^{\frac{1}{|F_2|}}\right)^{\frac{|F_2|}{n}}\right)^{\frac{1}{p}}}, l_{s-s\left(1-\left(\left(\prod_{r=1}^{t}\left(1-(1-H_\gamma)^{\frac{p}{p+q}}\right)\right)^{\frac{1}{t}}\right)^{\frac{n-|F_2|}{n}}\left(\left(\prod_{i=1}^{|F_2|}\left(1-(1-\gamma_i/s)^p\right)\right)^{\frac{1}{|F_2|}}\right)^{\frac{|F_2|}{n}}\right)^{\frac{1}{p}}}\right)$$

$$
\begin{aligned}
&= \Bigg(l_{s\left(1-\left(\left(\left(\prod_{r=1}^{t}\left(1-\left(1-\left(1-(\alpha/s)^{p+q}\right)\right)^{\frac{p}{p+q}}\right)\right)^{\frac{1}{t}}\right)^{\frac{n-|F_2|}{n}}\right)\left(\left(\left(\prod_{i=1}^{|F_2|}\left(1-(\alpha/s)^{p}\right)\right)^{\frac{1}{|F_2|}}\right)^{\frac{|F_2|}{n}}\right)\right)^{\frac{1}{p}}}, \\
&\quad l_{s-s\left(1-\left(\left(\prod_{r=1}^{t}\left(1-\left(1-\left(1-(1-\beta/s)^{p+q}\right)\right)^{\frac{p}{p+q}}\right)\right)^{\frac{1}{t}}\right)^{\frac{n-|F_2|}{n}}\left(\left(\prod_{i=1}^{|F_2|}\left(1-(1-\beta/s)^{p}\right)\right)^{\frac{1}{|F_2|}}\right)^{\frac{|F_2|}{n}}\right)^{\frac{1}{p}}}, l_{s-s\left(1-\left(\left(\prod_{r=1}^{t}\left(1-\left(1-\left(1-(1-\gamma/s)^{p+q}\right)\right)^{\frac{p}{p+q}}\right)\right)^{\frac{1}{t}}\right)^{\frac{n-|F_2|}{n}}\left(\left(\prod_{i=1}^{|F_2|}\left(1-(1-\gamma/s)^{p}\right)\right)^{\frac{1}{|F_2|}}\right)^{\frac{|F_2|}{n}}\right)^{\frac{1}{p}}} \Bigg) \\
&= \Bigg(l_{s\left(1-\left(\left(\left(\prod_{r=1}^{t}\left(1-(\alpha/s)^{p}\right)\right)^{\frac{1}{t}}\right)^{\frac{n-|F_2|}{n}}\right)\left(\left(1-(\alpha/s)^{p}\right)^{\frac{|F_2|}{n}}\right)\right)^{\frac{1}{p}}}, l_{s-s\left(1-\left(\left(\prod_{r=1}^{t}\left(1-(1-\beta/s)^{p}\right)\right)^{\frac{1}{t}}\right)^{\frac{n-|F_2|}{n}}\left(\left(1-(1-\beta/s)^{p}\right)^{\frac{|F_2|}{n}}\right)\right)^{\frac{1}{p}}}, l_{s-s\left(1-\left(\left(\prod_{r=1}^{t}\left(1-(1-\gamma/s)^{p}\right)\right)^{\frac{1}{t}}\right)^{\frac{n-|F_2|}{n}}\left(\left(1-(1-\gamma/s)^{p}\right)^{\frac{|F_2|}{n}}\right)\right)^{\frac{1}{p}}} \Bigg) \\
&= \Bigg(l_{s\left(1-\left(\left(1-(\alpha/s)^{p}\right)^{\frac{n-|F_2|}{n}}\right)\left(\left(1-(\alpha/s)^{p}\right)^{\frac{|F_2|}{n}}\right)\right)^{\frac{1}{p}}}, l_{s-s\left(1-\left(\left(1-(1-\beta/s)^{p}\right)^{\frac{n-|F_2|}{n}}\right)\left(\left(1-(1-\beta/s)^{p}\right)^{\frac{|F_2|}{n}}\right)\right)^{\frac{1}{p}}}, l_{s-s\left(1-\left(\left(1-(1-\gamma/s)^{p}\right)^{\frac{n-|F_2|}{n}}\right)\left(\left(1-(1-\gamma/s)^{p}\right)^{\frac{|F_2|}{n}}\right)\right)^{\frac{1}{p}}} \Bigg) \\
&= \left(l_{s\left(1-\left(1-(\alpha/s)^{p}\right)\right)^{\frac{1}{p}}}, l_{s-s\left(1-\left(1-(1-\beta/s)^{p}\right)\right)^{\frac{1}{p}}}, l_{s-s\left(1-\left(1-(1-\gamma/s)^{p}\right)\right)^{\frac{1}{p}}} \right) = \left(l_{s\left((\alpha/s)^{p}\right)^{\frac{1}{p}}}, l_{s-s\left((1-\beta/s)^{p}\right)^{\frac{1}{p}}}, l_{s-s\left((1-\gamma/s)^{p}\right)^{\frac{1}{p}}} \right). \\
&= (l_\alpha, l_\beta, l_\gamma)
\end{aligned}
$$

□

(2) **Monotonicity**: If $z_i = \left(l_{\alpha_i}, l_{\beta_i}, l_{\gamma_i}\right)$ $(i = 1, 2, \cdots, n)$ and $y_i = \left(l_{\delta_i}, l_{\eta_i}, l_{\sigma_i}\right)$ $(i = 1, 2, \cdots, n)$ are any two sets of LNNs; they satisfy the condition $\alpha_i \geq \delta_i$, $\beta_i \leq \eta_i$ and $\gamma_i \leq \sigma_i$, then $LNGPBM^{p,q}(z_1, z_2, \cdots, z_n) \geq LNGPBM^{p,q}(y_1, y_2, \cdots, y_n)$.

Proof. Suppose that $LNGPBM^{p,q}(z_1, z_2, \cdots, z_n) = z = \left(l_\alpha, l_\beta, l_\gamma\right)$ and $LNGPBM^{p,q}(y_1, y_2, \cdots, y_n) = y = \left(l_\delta, l_\eta, l_\sigma\right)$, then:

$$\alpha = s\left(1-\left(\left(\left(\prod_{r=1}^{t}\left(1-(1-H_\alpha)^{\frac{p}{p+q}}\right)\right)^{\frac{1}{t}}\right)^{\frac{n-|F_2|}{n}}\right)\left(\left(\left(\prod_{i=1}^{|F_2|}\left(1-(\alpha_i/s)^{p}\right)\right)^{\frac{1}{|F_2|}}\right)^{\frac{|F_2|}{n}}\right)\right)^{\frac{1}{p}},$$

$$\delta = s\left(1-\left(\left(\left(\prod_{r=1}^{t}\left(1-(1-H_\delta)^{\frac{p}{p+q}}\right)\right)^{\frac{1}{t}}\right)^{\frac{n-|F_2|}{n}}\right)\left(\left(\left(\prod_{i=1}^{|F_2|}\left(1-(\delta_i/s)^{p}\right)\right)^{\frac{1}{|F_2|}}\right)^{\frac{|F_2|}{n}}\right)\right)^{\frac{1}{p}},$$

$$\beta = s-s\left(1-\left(\left(\prod_{r=1}^{t}\left(1-(1-H_\beta)^{\frac{p}{p+q}}\right)\right)^{\frac{1}{t}}\right)^{\frac{n-|F_2|}{n}}\left(\left(\prod_{i=1}^{|F_2|}\left(1-(1-\beta_i/s)^{p}\right)\right)^{\frac{1}{|F_2|}}\right)^{\frac{|F_2|}{n}}\right)^{\frac{1}{p}},$$

$$\eta = s-s\left(1-\left(\left(\prod_{r=1}^{t}\left(1-(1-H_\eta)^{\frac{p}{p+q}}\right)\right)^{\frac{1}{t}}\right)^{\frac{n-|F_2|}{n}}\left(\left(\prod_{i=1}^{|F_2|}\left(1-(1-\eta_i/s)^{p}\right)\right)^{\frac{1}{|F_2|}}\right)^{\frac{|F_2|}{n}}\right)^{\frac{1}{p}},$$

$$\gamma = s-s\left(1-\left(\left(\prod_{r=1}^{t}\left(1-(1-H_\gamma)^{\frac{p}{p+q}}\right)\right)^{\frac{1}{t}}\right)^{\frac{n-|F_2|}{n}}\left(\left(\prod_{i=1}^{|F_2|}\left(1-(1-\gamma_i/s)^{p}\right)\right)^{\frac{1}{|F_2|}}\right)^{\frac{|F_2|}{n}}\right)^{\frac{1}{p}},$$

$$\sigma = s - s\left(1 - \left(\left(\prod_{r=1}^{t}\left(1-(1-H_{\sigma})^{\frac{p}{p+q}}\right)\right)^{\frac{1}{t}}\right)^{\frac{n-|F_2|}{n}}\left(\left(\prod_{i=1}^{|F_2|}\left(1-(1-\sigma_i/s)^{p}\right)\right)^{\frac{1}{|F_2|}}\right)^{\frac{|F_2|}{n}}\right)^{\frac{1}{p}}.$$

In order to prove this property, we need to compute their score function values $C(z)$ and $C(y)$, and their accuracy values $A(z)$ and $A(y)$ to compare their synthesized result, i.e., $z \geq y$. Firstly, on the basis of the condition $\alpha_i \geq \delta_i$, $\beta_i \leq \eta_i$, and $\gamma_i \leq \sigma_i$, we can get the compared result of their truth-membership degrees, indeterminacy-membership degrees, and falsity-membership degrees, respectively.

(i) The comparison of the truth-membership degrees:

Based on $\alpha_i \geq \delta_i$, we can get:

$$\begin{aligned}
&H_{\alpha} = \left(\prod_{i=1}^{h_r}\left(1-(\alpha_i/s)^{p}\left(1-\left(\prod_{\substack{j=1\\ j\neq i}}^{h_r}\left(1-(\alpha_j/s)^{q}\right)\right)^{\frac{1}{h_r-1}}\right)\right)\right)^{\frac{1}{h_r}} \leq \left(\prod_{i=1}^{h_r}\left(1-(\delta_i/s)^{p}\left(1-\left(\prod_{\substack{j=1\\ j\neq i}}^{h_r}\left(1-(\delta_j/s)^{q}\right)\right)^{\frac{1}{h_r-1}}\right)\right)\right)^{\frac{1}{h_r}} = H_{\delta} \\
&\Rightarrow \left(\left(\prod_{r=1}^{t}\left(1-(1-H_{\alpha})^{\frac{p}{p+q}}\right)\right)^{\frac{1}{t}}\right)^{\frac{n-|F_2|}{n}} \leq \left(\left(\prod_{r=1}^{t}\left(1-(1-H_{\delta})^{\frac{p}{p+q}}\right)\right)^{\frac{1}{t}}\right)^{\frac{n-|F_2|}{n}} \\
&\text{and } \left(\left(\prod_{i=1}^{|F_2|}\left(1-(\alpha_i/s)^{p}\right)\right)^{\frac{1}{|F_2|}}\right)^{\frac{|F_2|}{n}} \leq \left(\left(\prod_{i=1}^{|F_2|}\left(1-(\delta_i/s)^{p}\right)\right)^{\frac{1}{|F_2|}}\right)^{\frac{|F_2|}{n}}
\end{aligned}.$$

In accordance with the upper two inequalities, we have:

$$s\left(1-\left(\left(\left(\prod_{r=1}^{t}\left(1-(1-H_{\alpha})^{\frac{p}{p+q}}\right)\right)^{\frac{1}{t}}\right)^{\frac{n-|F_2|}{n}}\right)\left(\left(\left(\prod_{i=1}^{|F_2|}(1-(\alpha_i/s)^{p})\right)^{\frac{1}{|F_2|}}\right)^{\frac{|F_2|}{n}}\right)\right)^{\frac{1}{p}} \geq s\left(1-\left(\left(\left(\prod_{r=1}^{t}\left(1-(1-H_{\delta})^{\frac{p}{p+q}}\right)\right)^{\frac{1}{t}}\right)^{\frac{n-|F_2|}{n}}\right)\left(\left(\left(\prod_{i=1}^{|F_2|}(1-(\delta_i/s)^{p})\right)^{\frac{1}{|F_2|}}\right)^{\frac{|F_2|}{n}}\right)\right)^{\frac{1}{p}}$$

That is, $\alpha \geq \delta$.

(ii) The comparision of indeterminacy-membership degrees and falsity-membership degrees, respectively:

Based on $\beta_i \leq \eta_i$ and $\gamma_i \leq \sigma_i$, we can also obtain $\beta \leq \eta$ and $\gamma \leq \sigma$; this process is similar to the process of the truth-membership degrees.

Thus, it can be obtained that $C(z) = \frac{2s+\alpha-\beta-\gamma}{3s} \geq \frac{2s+\delta-\eta-\sigma}{3s} = C(y)$. In the following, we discuss two cases.

(i) If $C(z) > C(y)$, then $z > y$, according to Definition 2.
(ii) If $C(z) = C(y)$, then $(\alpha-\gamma)-\beta = (\delta-\sigma)-\eta$. Since $\alpha-\gamma \geq \delta-\sigma$ in the light of $\alpha \geq \delta$ and $\gamma \leq \sigma$, now we assume $\alpha-\gamma > \delta-\sigma$, then $\beta > \eta$, which is in contradiction with the previous proof $\beta \leq \eta$. So, we can conclude that $\alpha-\gamma = \delta-\sigma$. That is, $A(z) = \frac{\alpha-\gamma}{s} = \frac{\delta-\sigma}{s} = A(y)$, which testifies $z = y$.

In conclusion, the synthesized result $z \geq y$, which explains:

$$LNGPBM^{p,q}(z_1, z_2, \cdots, z_n) \geq LNGPBM^{p,q}(y_1, y_2, \cdots, y_n)$$

□

(3) **Boundedness**: Let $z_i = \left(l_{\alpha_i}, l_{\beta_i}, l_{\gamma_i}\right)$ $(i = 1, 2, \cdots, n)$ be an arbitrary set of LNNs, then:

$$\min_i z_i \leq LNGPBM^{p,q}(z_1, z_2, \cdots, z_n) \leq \max_i z_i$$

Proof. Since $z_i \geq \underset{i}{min} z_i$, according to the monotonicity and idempotency of the proposed *LNGPBM* operator, we can obtain the following result:

$$LNGPBM^{p,q}(z_1, z_2, \cdots, z_n) \geq LNGPBM^{p,q}\left(\underset{i}{min} z_i, \underset{i}{min} z_i, \cdots, \underset{i}{min} z_i\right) = \underset{i}{min} z_i$$

Similarly, we can obtain the corresponding result for $\underset{i}{max} z_i$:

$$LNGPBM^{p,q}(z_1, z_2, \cdots, z_n) \leq LNGPBM^{p,q}\left(\underset{i}{max} z_i, \underset{i}{max} z_i, \cdots, \underset{i}{max} z_i\right) = \underset{i}{max} z_i$$

Therefore, $\underset{i}{min} z_i \leq LNGPBM^{p,q}(z_1, z_2, \cdots, z_n) \leq \underset{i}{max} z_i$. $\square$

Based on the character of F_2, some special cases are discussed about the *LNGPBM* operator, and shown in the following.

(1) When $|F_2| = 0$, all arguments belong to the group F_1, and are divided into t clusters; then, the proposed *LNGPBM* operator is simplified as the following form:

$$LNGPBM^{p,q}(z_1, z_2, \cdots, z_n) = \frac{1}{t} \overset{t}{\underset{r=1}{\oplus}} \left(\frac{1}{h_r} \overset{h_r}{\underset{i=1}{\oplus}} z_i^p \otimes \left(\frac{1}{h_r - 1} \overset{h_r}{\underset{\substack{j=1 \\ j \neq i}}{\oplus}} z_j^q \right) \right)^{\frac{1}{p+q}} = LNPBM^{p,q}(z_1, z_2, \cdots, z_n)$$

The *LNPBM* is called the linguistic neutrosophic *PBM* operator.

(2) When $|F_2| = 0$ and $t = 1$, all arguments belong to the same cluster, i.e., $h_r = n$; then, the proposed *LNGPBM* operator becomes the following form:

$$LNGPBM^{p,q}(z_1, z_2, \cdots, z_n) = \left(\frac{1}{h_r} \overset{h_r}{\underset{i=1}{\oplus}} z_i^p \otimes \left(\frac{1}{h_r - 1} \overset{h_r}{\underset{\substack{j=1 \\ j \neq i}}{\oplus}} z_j^q \right) \right)^{\frac{1}{p+q}} = \left(\frac{1}{n} \overset{n}{\underset{i=1}{\oplus}} z_i^p \otimes \left(\frac{1}{n - 1} \overset{n}{\underset{\substack{j=1 \\ j \neq i}}{\oplus}} z_j^q \right) \right)^{\frac{1}{p+q}} = LNBM^{p,q}(z_1, z_2, \cdots, z_n)$$

The *LNBM* is called the linguistic neutrosophic *BM* operator.

(3) When $|F_2| = n$, there is no element in group F_1 and all elements are independent; then, the proposed *LNGPBM* operator reduces to the following form:

$$LNGPBM^{p,q}(z_1, z_2, \cdots, z_n) = \left(\frac{1}{|F_2|} \overset{|F_2|}{\underset{i=1}{\oplus}} z_i^p \right)^{\frac{1}{p}} = \left(\frac{1}{n} \overset{n}{\underset{i=1}{\oplus}} z_i^p \right)^{\frac{1}{p}} = LNPRAM^p(z_1, z_2, \cdots, z_n)$$

The *LNPRAM* is called the linguistic neutrosophic power root arithmetic mean operator.

Moreover, we can also get some special cases by distributing different values to the parameters p and q.

(1) When $q \to 0$, the proposed *LNGPBM* operator becomes the *LNPRAM* operator, which was described in the previous discussion. Since there is no inner connection in group F_1, all of the elements are independent.

(2) When $p = 1$ and $q \to 0$, the proposed *LNGPBM* operator reduces to the linguistic neutrosophic arithmetic mean (*LNAM*) operator, which is shown as follows:

$$LNGPBM^{p=1,q\to 0}(z_1, z_2, \cdots, z_n) = \left(\frac{1}{n}\underset{i=1}{\overset{n}{\oplus}} z_i^p\right)^{\frac{1}{p}} = \frac{1}{n}\underset{i=1}{\overset{n}{\oplus}} z_i = LNAM(z_1, z_2, \cdots, z_n)$$

(3) When $p = 2$ and $q \to 0$, the proposed *LNGPBM* operator is transformed into the linguistic neutrosophic square root arithmetic mean (*LNSRAM*) operator, which is shown as follows:

$$LNGPBM^{p=2,q\to 0}(z_1, z_2, \cdots, z_n) = \left(\frac{1}{n}\underset{i=1}{\overset{n}{\oplus}} z_i^2\right)^{\frac{1}{2}} = LNSRAM(z_1, z_2, \cdots, z_n).$$

(4) When $p = q = 1$, the proposed *LNGPBM* operator is simplified as the simplest form of the *LNGPBM* operator, which is shown as follows:

$$LNGPBM^{p=1,q=1}(z_1, z_2, \cdots, z_n) = \frac{n-|F_2|}{n}\left(\frac{1}{t}\underset{r=1}{\overset{t}{\oplus}}\left(\frac{1}{h_r}\underset{i=1}{\overset{h_r}{\oplus}} z_i \otimes \left(\frac{1}{h_r-1}\underset{\substack{j=1\\ j\neq i}}{\overset{h_r}{\oplus}} z_j\right)\right)\right)^{\frac{1}{2}} \oplus \frac{|F_2|}{n}\left(\frac{1}{|F_2|}\underset{i=1}{\overset{|F_2|}{\oplus}} z_i\right)$$

It is often used to simplify the calculation in a problem.

3.2. The LNGWPBM Operator

In Definitions 8, we assume that all the input arguments have the same position. However, in many realistic decision-makings, every input argument may have different importance. Accordingly, we give different values to the weights of input arguments, and propose the weighted form of the *LNGPBM* operator. Let the weight of input argument $z_i = \left(l_{\alpha_i}, l_{\beta_i}, l_{\gamma_i}\right)$ $(i = 1, 2, \cdots, n)$ be ω_i, where $\omega_i \in [0, 1]$ and $\sum\limits_{i=1}^{n} \omega_i = 1$. The weighted form of the *LNGPBM* operator is shown in the following.

Definition 9. *Let $z_1, z_2, \cdots$ and z_n be LNNs that are sorted into two groups: F_1 and F_2. In F_1, the elements are divided into t clusters $P_1, P_2, \cdots, P_t$, which satisfy $P_x \cap P_y = \varnothing$, $x \neq y$ and $\underset{r=1}{\overset{t}{\cup}} P_r = F_1$; in F_2, the elements are irrelevant to any element. The weighted form of the LNGPBM operator of the LNNs $z_1, z_2, \cdots$ and z_n is defined as follows:*

$$LNGWPBM^{p,q}(z_1, z_2, \cdots, z_n) = \left(\frac{n-|F_2|}{n}\left(\frac{1}{t}\underset{r=1}{\overset{t}{\oplus}}\left(\frac{1}{\sum\limits_{i=1}^{h_r}\omega_i}\underset{i=1}{\overset{h_r}{\oplus}} \omega_i z_i^p \otimes \left(\frac{1}{\sum\limits_{\substack{j=1\\ j\neq i}}^{h_r}\omega_j}\underset{\substack{j=1\\ j\neq i}}{\overset{h_r}{\oplus}} \omega_j z_j^q\right)\right)\right)^{\frac{p}{p+q}} \oplus \frac{|F_2|}{n}\left(\frac{1}{\sum\limits_{i=1}^{|F_2|}\omega_i}\underset{i=1}{\overset{|F_2|}{\oplus}} \omega_i z_i^p\right)\right)^{\frac{1}{p}} \tag{18}$$

where $z_i = \left(l_{\alpha_i}, l_{\beta_i}, l_{\gamma_i}\right)$ and $\alpha_i, \beta_i, \gamma_i \in [0, s]$ $(i = 1, 2, \cdots, n)$; ω_i is the weight of input argument z_i meeting $\omega_i \in [0, 1]$ and $\sum\limits_{i=1}^{n} \omega_i = 1$; $p, q \geq 0$ and $p + q > 0$; $|F_2|$ denotes the number of elements in F_2; h_r indicates the number of elements in partition P_r; and $\sum\limits_{r=1}^{t} h_r = n - |F_2|$. Then, we call it a linguistic neutrosophic generalized weighted PBM (LNGWPBM) operator.

Theorem 2. *Let $z_1, z_2, \cdots$ and z_n be LNNs, where $z_i = \left(l_{\alpha_i}, l_{\beta_i}, l_{\gamma_i}\right)$ and $\alpha_i, \beta_i, \gamma_i \in [0, s]$ $(i = 1, 2, \cdots, n)$, and let the weight of input argument z_i be ω_i, where $\omega_i \in [0, 1]$ and $\sum\limits_{i=1}^{n} \omega_i = 1$. Then, the synthesized result of the LNGWPBM operator of the LNNs $z_1, z_2, \cdots$ and z_n is still a LNN, which is shown as follows:*

$$LNGWPBM^{p,q}(z_1, z_2, \cdots, z_n) = \left(l_{s\left(1-\left(\left(\left(\prod\limits_{r=1}^{t}\left(1-(1-K_\alpha)^{\frac{p}{p+q}}\right)\right)^{\frac{1}{t}}\right)^{\frac{n-|F_2|}{n}}\right)\left(\left(\left(\prod\limits_{i=1}^{|F_2|}\left(1-(\alpha_i/s)^p\right)^{\omega_i}\right)^{\frac{1}{\sum_{i=1}^{|F_2|}\omega_i}}\right)^{\frac{|F_2|}{n}}\right)\right)^{\frac{1}{p}}}, l_{s-s\left(1-\left(\left(\prod\limits_{r=1}^{t}\left(1-\left(1-K_\beta\right)^{\frac{p}{p+q}}\right)\right)^{\frac{1}{t}}\right)^{\frac{n-|F_2|}{n}}\left(\left(\prod\limits_{i=1}^{|F_2|}\left(1-(1-\beta_i/s)^p\right)^{\omega_i}\right)^{\frac{1}{\sum_{i=1}^{|F_2|}\omega_i}}\right)^{\frac{|F_2|}{n}}\right)^{\frac{1}{p}}}, l_{s-s\left(1-\left(\left(\prod\limits_{r=1}^{t}\left(1-(1-K_\gamma)^{\frac{p}{p+q}}\right)\right)^{\frac{1}{t}}\right)^{\frac{n-|F_2|}{n}}\left(\left(\prod\limits_{i=1}^{|F_2|}\left(1-(1-\gamma_i/s)^p\right)^{\omega_i}\right)^{\frac{1}{\sum_{i=1}^{|F_2|}\omega_i}}\right)^{\frac{|F_2|}{n}}\right)^{\frac{1}{p}}} \right) \tag{19}$$

where $$K_\alpha = \left(\prod_{i=1}^{h_r}\left(1-\left(1-\left(1-(\alpha_i/s)^p\right)^{\omega_i}\right)\left(1-\left(\prod_{\substack{j=1\\ j\neq i}}^{h_r}\left(1-(\alpha_j/s)^q\right)^{\omega_j}\right)^{\frac{1}{\sum\limits_{\substack{j=1\\ j\neq i}}^{h_r}\omega_j}}\right)\right)\right)^{\frac{1}{\sum_{i=1}^{h_r}\omega_i}},$$

$$K_\beta = \left(\prod_{i=1}^{h_r}\left(1-\left(1-\left(1-(1-\beta_i/s)^p\right)^{\omega_i}\right)\left(1-\left(\prod_{\substack{j=1\\ j\neq i}}^{h_r}\left(1-(1-\beta_j/s)^q\right)^{\omega_j}\right)^{\frac{1}{\sum\limits_{\substack{j=1\\ j\neq i}}^{h_r}\omega_j}}\right)\right)\right)^{\frac{1}{\sum_{i=1}^{h_r}\omega_i}}, \text{ and}$$

$$K_\gamma = \left(\prod_{i=1}^{h_r}\left(1-\left(1-\left(1-(1-\gamma_i/s)^p\right)^{\omega_i}\right)\left(1-\left(\prod_{\substack{j=1\\ j\neq i}}^{h_r}\left(1-(1-\gamma_j/s)^q\right)^{\omega_j}\right)^{\frac{1}{\sum\limits_{\substack{j=1\\ j\neq i}}^{h_r}\omega_j}}\right)\right)\right)^{\frac{1}{\sum_{i=1}^{h_r}\omega_i}}.$$

Proof. Along the lines of Theorem 1, we also process the groups F_1 and F_2 separately, and then combine them to prove.

(i) The processing of F_1:

Firstly, we successively use Formulas (7), (6), and (4) to get the following formula:

$$\bigoplus_{\substack{j=1\\ j\neq i}}^{h_r}\omega_j z_j^q=\left(l_{s-s\prod\limits_{\substack{j=1\\ j\neq i}}^{h_r}\left(1-\left(\alpha_j/s\right)^q\right)^{\omega_j}},\ l_{s\prod\limits_{\substack{j=1\\ j\neq i}}^{h_r}\left(1-\left(1-\beta_j/s\right)^q\right)^{\omega_j}},\ l_{s\prod\limits_{\substack{j=1\\ j\neq i}}^{h_r}\left(1-\left(1-\gamma_j/s\right)^q\right)^{\omega_j}}\right),$$

Then, we have:

$$\frac{1}{\sum\limits_{\substack{j=1\\ j\neq i}}^{h_r}\omega_j}\bigoplus_{\substack{j=1\\ j\neq i}}^{h_r}\omega_j z_j^q=\left(l_{s-s\left(\prod\limits_{\substack{j=1\\ j\neq i}}^{h_r}\left(1-\left(\alpha_j/s\right)^q\right)^{\omega_j}\right)^{\frac{1}{\sum_{j=1, j\neq i}^{h_r}\omega_j}}},\ l_{s\left(\prod\limits_{\substack{j=1\\ j\neq i}}^{h_r}\left(1-\left(1-\beta_j/s\right)^q\right)^{\omega_j}\right)^{\frac{1}{\sum_{j=1, j\neq i}^{h_r}\omega_j}}},\ l_{s\left(\prod\limits_{\substack{j=1\\ j\neq i}}^{h_r}\left(1-\left(1-\gamma_j/s\right)^q\right)^{\omega_j}\right)^{\frac{1}{\sum_{j=1, j\neq i}^{h_r}\omega_j}}}\right).$$

Since $\omega_i z_i^p=\left(l_{s-s\left(1-(\alpha_i/s)^p\right)^{\omega_i}},\ l_{s\left(1-(1-\beta_i/s)^p\right)^{\omega_i}},\ l_{s\left(1-(1-\gamma_i/s)^p\right)^{\omega_i}}\right)$, we can get:

$$\omega_i z_i^p\otimes\left(\frac{1}{\sum\limits_{\substack{j=1\\ j\neq i}}^{h_r}\omega_j}\bigoplus_{\substack{j=1\\ j\neq i}}^{h_r}\omega_j z_j^q\right)=$$

$$\left(l_{s\left(1-\left(1-(\alpha_i/s)^p\right)^{\omega_i}\right)\left(1-\left(\prod\limits_{\substack{j=1\\ j\neq i}}^{h_r}\left(1-\left(\alpha_j/s\right)^q\right)^{\omega_j}\right)^{\frac{1}{\sum_{j=1, j\neq i}^{h_r}\omega_j}}\right)},\ l_{s-s\left(1-\left(1-(1-\beta_i/s)^p\right)^{\omega_i}\right)\left(1-\left(\prod\limits_{\substack{j=1\\ j\neq i}}^{h_r}\left(1-\left(1-\beta_j/s\right)^q\right)^{\omega_j}\right)^{\frac{1}{\sum_{j=1, j\neq i}^{h_r}\omega_j}}\right)},\ l_{s-s\left(1-\left(1-(1-\gamma_i/s)^p\right)^{\omega_i}\right)\left(1-\left(\prod\limits_{\substack{j=1\\ j\neq i}}^{h_r}\left(1-\left(1-\gamma_j/s\right)^q\right)^{\omega_j}\right)^{\frac{1}{\sum_{j=1, j\neq i}^{h_r}\omega_j}}\right)}\right)$$

$$\Rightarrow\bigoplus_{i=1}^{h_r}\omega_i z_i^p\otimes\left(\frac{1}{\sum\limits_{\substack{j=1\\ j\neq i}}^{h_r}\omega_j}\bigoplus_{\substack{j=1\\ j\neq i}}^{h_r}\omega_j z_j^q\right)=$$

$$\left(l_{s\left(1-\prod\limits_{i=1}^{h_r}\left(1-\left(1-\left(1-(\alpha_i/s)^p\right)^{\omega_i}\right)\left(1-\left(\prod\limits_{\substack{j=1\\ j\neq i}}^{h_r}\left(1-\left(\alpha_j/s\right)^q\right)^{\omega_j}\right)^{\frac{1}{\sum_{j=1, j\neq i}^{h_r}\omega_j}}\right)\right)\right)},\ l_{s\prod\limits_{i=1}^{h_r}\left(1-\left(1-\left(1-(1-\beta_i/s)^p\right)^{\omega_i}\right)\left(1-\left(\prod\limits_{\substack{j=1\\ j\neq i}}^{h_r}\left(1-\left(1-\beta_j/s\right)^q\right)^{\omega_j}\right)^{\frac{1}{\sum_{j=1, j\neq i}^{h_r}\omega_j}}\right)\right)},\ l_{s\prod\limits_{i=1}^{h_r}\left(1-\left(1-\left(1-(1-\gamma_i/s)^p\right)^{\omega_i}\right)\left(1-\left(\prod\limits_{\substack{j=1\\ j\neq i}}^{h_r}\left(1-\left(1-\gamma_j/s\right)^q\right)^{\omega_j}\right)^{\frac{1}{\sum_{j=1, j\neq i}^{h_r}\omega_j}}\right)\right)}\right)$$

Hence, the following equation is established in the light of the upper:

$$\overline{H}=\frac{1}{\sum\limits_{i=1}^{h_r}\omega_i}\bigoplus_{i=1}^{h_r}\omega_i z_i^p\otimes\left(\frac{1}{\sum\limits_{\substack{j=1\\ j\neq i}}^{h_r}\omega_j}\bigoplus_{\substack{j=1\\ j\neq i}}^{h_r}\omega_j z_j^q\right)=\left(l_{s-s\left(\prod\limits_{i=1}^{h_r}\left(1-\left(1-\left(1-(\alpha_i/s)^p\right)^{\omega_i}\right)\left(1-\left(\prod\limits_{\substack{j=1\\ j\neq i}}^{h_r}\left(1-\left(\alpha_j/s\right)^q\right)^{\omega_j}\right)^{\frac{1}{\sum_{j=1, j\neq i}^{h_r}\omega_j}}\right)\right)\right)^{\frac{1}{\sum_{i=1}^{h_r}\omega_i}}},\right.$$

$$l_{s\left(\prod_{i=1}^{h_r}\left(1-\left(1-(1-(1-\beta_i/s)^p)^{\omega_i}\right)\left(1-\left(\prod_{\substack{j=1\\ j\neq i}}^{h_r}\left(1-(1-\beta_j/s)^q\right)^{\omega_j}\right)^{\frac{1}{\sum_{j=1,j\neq i}^{h_r}\omega_j}}\right)\right)\right)^{\frac{1}{\sum_{i=1}^{h_r}\omega_i}}}, l_{s\left(\prod_{i=1}^{h_r}\left(1-\left(1-(1-(1-\gamma_i/s)^p)^{\omega_i}\right)\left(1-\left(\prod_{\substack{j=1\\ j\neq i}}^{h_r}\left(1-(1-\gamma_j/s)^q\right)^{\omega_j}\right)^{\frac{1}{\sum_{j=1,j\neq i}^{h_r}\omega_j}}\right)\right)\right)^{\frac{1}{\sum_{i=1}^{h_r}\omega_i}}}\Bigg)$$

Since the expression $\overline{H}$ is too long, we suppose:

$$K_\alpha = \left(\prod_{i=1}^{h_r}\left(1-\left(1-(1-(\alpha_i/s)^p)^{\omega_i}\right)\left(1-\left(\prod_{\substack{j=1\\ j\neq i}}^{h_r}\left(1-(\alpha_j/s)^q\right)^{\omega_j}\right)^{\frac{1}{\sum_{j=1,j\neq i}^{h_r}\omega_j}}\right)\right)\right)^{\frac{1}{\sum_{i=1}^{h_r}\omega_i}},$$

$$K_\beta = \left(\prod_{i=1}^{h_r}\left(1-\left(1-(1-(1-\beta_i/s)^p)^{\omega_i}\right)\left(1-\left(\prod_{\substack{j=1\\ j\neq i}}^{h_r}\left(1-(1-\beta_j/s)^q\right)^{\omega_j}\right)^{\frac{1}{\sum_{j=1,j\neq i}^{h_r}\omega_j}}\right)\right)\right)^{\frac{1}{\sum_{i=1}^{h_r}\omega_i}},$$

$$K_\gamma = \left(\prod_{i=1}^{h_r}\left(1-\left(1-(1-(1-\gamma_i/s)^p)^{\omega_i}\right)\left(1-\left(\prod_{\substack{j=1\\ j\neq i}}^{h_r}\left(1-(1-\gamma_j/s)^q\right)^{\omega_j}\right)^{\frac{1}{\sum_{j=1,j\neq i}^{h_r}\omega_j}}\right)\right)\right)^{\frac{1}{\sum_{i=1}^{h_r}\omega_i}},$$

Then, the expression $\overline{H}$ can be written as $\overline{H} = \left(l_{s\times(1-K_\alpha)}, l_{s\times K_\beta}, l_{s\times K_\gamma}\right)$. Next, we can get the below expression:

$$A' = \frac{1}{t}\bigoplus_{r=1}^{t}\left(\left(\frac{1}{\sum_{i=1}^{h_r}\omega_i}\bigoplus_{i=1}^{h_r}\omega_i z_i^p \otimes\left(\frac{1}{\sum_{\substack{j=1\\ j\neq i}}^{h_r}\omega_j}\bigoplus_{\substack{j=1\\ j\neq i}}^{h_r}\omega_j z_j^q\right)\right)\right)^{\frac{p}{p+q}} = \frac{1}{t}\bigoplus_{r=1}^{t}\left(\overline{H}\right)^{\frac{p}{p+q}} = \left(l_{s-s\left(\prod_{r=1}^{t}\left(1-(1-K_\alpha)^{\frac{p}{p+q}}\right)\right)^{\frac{1}{t}}}, l_{s\left(\prod_{r=1}^{t}\left(1-(1-K_\beta)^{\frac{p}{p+q}}\right)\right)^{\frac{1}{t}}}, l_{s\left(\prod_{r=1}^{t}\left(1-(1-K_\gamma)^{\frac{p}{p+q}}\right)\right)^{\frac{1}{t}}}\right)$$

(ii) The processing of F_2:

Based on the operational laws of LNNs, it is easy to obtain:

$$B' = \frac{1}{\sum_{i=1}^{|F_2|}\omega_i}\bigoplus_{i=1}^{|F_2|}\omega_i z_i^p = \left(l_{s-s\left(\prod_{i=1}^{|F_2|}\left(1-(\alpha_i/s)^p\right)^{\omega_i}\right)^{\frac{1}{\sum_{i=1}^{|F_2|}\omega_i}}}, l_{s\left(\prod_{i=1}^{|F_2|}\left(1-(1-\beta_i/s)^p\right)^{\omega_i}\right)^{\frac{1}{\sum_{i=1}^{|F_2|}\omega_i}}}, l_{s\left(\prod_{i=1}^{|F_2|}\left(1-(1-\gamma_i/s)^p\right)^{\omega_i}\right)^{\frac{1}{\sum_{i=1}^{|F_2|}\omega_i}}}\right)$$

Finally, we compute the synthesized result of the *LNGWPBM* operator:

$$\left(\frac{n-|F_2|}{n}A' \oplus \frac{|F_2|}{n}B'\right)^{\frac{1}{p}} = \left(l_{s\left(1-\left(\left(\left(\prod_{r=1}^{t}\left(1-(1-K_\alpha)^{\frac{p}{p+q}}\right)\right)^{\frac{1}{t}}\right)^{\frac{n-|F_2|}{n}}\right)\left(\left(\left(\prod_{i=1}^{|F_2|}\left(1-(\alpha_i/s)^p\right)^{\omega_i}\right)^{\frac{1}{\sum_{i=1}^{|F_2|}\omega_i}}\right)^{\frac{|F_2|}{n}}\right)\right)^{\frac{1}{p}}}, \right.$$

$$\left. l_{s-s\left(1-\left(\left(\prod_{r=1}^{t}\left(1-(1-K_\beta)^{\frac{p}{p+q}}\right)\right)^{\frac{1}{t}}\right)^{\frac{n-|F_2|}{n}}\left(\left(\prod_{i=1}^{|F_2|}\left(1-(1-\beta_i/s)^p\right)^{\omega_i}\right)^{\frac{1}{\sum_{i=1}^{|F_2|}\omega_i}}\right)^{\frac{|F_2|}{n}}\right)^{\frac{1}{p}}}, l_{s-s\left(1-\left(\left(\prod_{r=1}^{t}\left(1-(1-K_\gamma)^{\frac{p}{p+q}}\right)\right)^{\frac{1}{t}}\right)^{\frac{n-|F_2|}{n}}\left(\left(\prod_{i=1}^{|F_2|}\left(1-(1-\gamma_i/s)^p\right)^{\omega_i}\right)^{\frac{1}{\sum_{i=1}^{|F_2|}\omega_i}}\right)^{\frac{|F_2|}{n}}\right)^{\frac{1}{p}}} \right)$$

That proves that Formula (19) is kept. Then, we prove that the aggregated result of Formula (19) is an LNN. It is easy to prove the following inequalities:

$$0 \le s\left(1-\left(\left(\left(\prod_{r=1}^{t}\left(1-(1-K_\alpha)^{\frac{p}{p+q}}\right)\right)^{\frac{1}{t}}\right)^{\frac{n-|F_2|}{n}}\right)\left(\left(\left(\prod_{i=1}^{|F_2|}\left(1-(\alpha_i/s)^p\right)^{\omega_i}\right)^{\frac{1}{\sum_{i=1}^{|F_2|}\omega_i}}\right)^{\frac{|F_2|}{n}}\right)\right)^{\frac{1}{p}} \le s,$$

$$0 \le s-s\left(1-\left(\left(\prod_{r=1}^{t}\left(1-(1-K_\beta)^{\frac{p}{p+q}}\right)\right)^{\frac{1}{t}}\right)^{\frac{n-|F_2|}{n}}\left(\left(\prod_{i=1}^{|F_2|}\left(1-(1-\beta_i/s)^p\right)^{\omega_i}\right)^{\frac{1}{\sum_{i=1}^{|F_2|}\omega_i}}\right)^{\frac{|F_2|}{n}}\right)^{\frac{1}{p}} \le s,$$

and $$0 \le s-s\left(1-\left(\left(\prod_{r=1}^{t}\left(1-(1-K_\gamma)^{\frac{p}{p+q}}\right)\right)^{\frac{1}{t}}\right)^{\frac{n-|F_2|}{n}}\left(\left(\prod_{i=1}^{|F_2|}\left(1-(1-\gamma_i/s)^p\right)^{\omega_i}\right)^{\frac{1}{\sum_{i=1}^{|F_2|}\omega_i}}\right)^{\frac{|F_2|}{n}}\right)^{\frac{1}{p}} \le s.$$

Firstly, we prove $0 \le K_\alpha \le 1$, $0 \le K_\beta \le 1$, and $0 \le H_\gamma \le 1$.

Based on the previous conditions such as $\alpha_j \in [0, s]$, $p \ge 0$, $q \ge 0$, $\omega_i \in [0, 1]$, and so on, we can get $0 \le \left(1-(\alpha_i/s)^p\right)^{\omega_i} \le 1$ and $0 \le \left(\prod_{\substack{j=1 \\ j \ne i}}^{h_r}\left(1-\left(\alpha_j/s\right)^q\right)^{\omega_j}\right)^{\frac{1}{\sum_{\substack{j=1 \\ j \ne i}}^{h_r}\omega_j}} \le 1$, which can deduce the following inequality:

$$0 \leq \left(1-\left(1-(\alpha_i/s)^p\right)^{\omega_i}\right)\left(1-\left(\prod_{\substack{j=1\\ j\neq i}}^{h_r}\left(1-(\alpha_j/s)^q\right)^{\omega_j}\right)^{\frac{1}{\sum\limits_{\substack{j=1\\ j\neq i}}^{h_r}\omega_j}}\right) \leq 1.$$

So, we can easily obtain $0 \leq \left(\prod_{i=1}^{h_r}\left(1-\left(1-\left(1-(\alpha_i/s)^p\right)^{\omega_i}\right)\left(1-\left(\prod_{\substack{j=1\\ j\neq i}}^{h_r}\left(1-(\alpha_j/s)^q\right)^{\omega_j}\right)^{\frac{1}{\sum\limits_{\substack{j=1\\ j\neq i}}^{h_r}\omega_j}}\right)\right)\right)^{\frac{1}{\sum_{i=1}^{h_r}\omega_i}} \leq 1$, i.e., $0 \leq K_\alpha \leq 1$.

Similarly, we also have $0 \leq K_\beta \leq 1$ and $0 \leq H_\gamma \leq 1$.

Next, we put the first to prove $0 \leq s\left(1-\left(\left(\left(\prod_{r=1}^{t}\left(1-(1-K_\alpha)^{\frac{p}{p+q}}\right)\right)^{\frac{1}{t}}\right)^{\frac{n-|F_2|}{n}}\right)\left(\left(\left(\prod_{i=1}^{|F_2|}\left(1-(\alpha_i/s)^p\right)^{\omega_i}\right)^{\frac{1}{\sum_{i=1}^{|F_2|}\omega_i}}\right)^{\frac{|F_2|}{n}}\right)\right)^{\frac{1}{p}} \leq s$.

According to $0 \leq K_\alpha \leq 1$, $p+q>0$ and $t>0$, we can illustrate $0 \leq \left(\prod_{r=1}^{t}\left(1-(1-K_\alpha)^{\frac{p}{p+q}}\right)\right)^{\frac{1}{t}} \leq 1$ and $0 \leq \left(\prod_{i=1}^{|F_2|}\left(1-(\alpha_i/s)^p\right)^{\omega_i}\right)^{\frac{1}{\sum_{i=1}^{|F_2|}\omega_i}} \leq 1$, which can deduce the following inequality:

$$0 \leq \left(\left(\left(\prod_{r=1}^{t}\left(1-(1-K_\alpha)^{\frac{p}{p+q}}\right)\right)^{\frac{1}{t}}\right)^{\frac{n-|F_2|}{n}}\right)\left(\left(\left(\prod_{i=1}^{|F_2|}\left(1-(\alpha_i/s)^p\right)^{\omega_i}\right)^{\frac{1}{\sum_{i=1}^{|F_2|}\omega_i}}\right)^{\frac{|F_2|}{n}}\right) \leq 1$$

Then, we find that $0 \leq s\left(1-\left(\left(\left(\prod_{r=1}^{t}\left(1-(1-K_\alpha)^{\frac{p}{p+q}}\right)\right)^{\frac{1}{t}}\right)^{\frac{n-|F_2|}{n}}\right)\left(\left(\left(\prod_{i=1}^{|F_2|}\left(1-(\alpha_i/s)^p\right)^{\omega_i}\right)^{\frac{1}{\sum_{i=1}^{|F_2|}\omega_i}}\right)^{\frac{|F_2|}{n}}\right)\right)^{\frac{1}{p}} \leq s$.

Likewise, we can illustrate $0 \leq s-s\left(1-\left(\left(\prod_{r=1}^{t}\left(1-(1-K_\beta)^{\frac{p}{p+q}}\right)\right)^{\frac{1}{t}}\right)^{\frac{n-|F_2|}{n}}\left(\left(\prod_{i=1}^{|F_2|}\left(1-(1-\beta_i/s)^p\right)^{\omega_i}\right)^{\frac{1}{\sum_{i=1}^{|F_2|}\omega_i}}\right)^{\frac{|F_2|}{n}}\right)^{\frac{1}{p}} \leq s$ and $0 \leq s-s\left(1-\left(\left(\prod_{r=1}^{t}\left(1-(1-K_\gamma)^{\frac{p}{p+q}}\right)\right)^{\frac{1}{t}}\right)^{\frac{n-|F_2|}{n}}\left(\left(\prod_{i=1}^{|F_2|}\left(1-(1-\gamma_i/s)^p\right)^{\omega_i}\right)^{\frac{1}{\sum_{i=1}^{|F_2|}\omega_i}}\right)^{\frac{|F_2|}{n}}\right)^{\frac{1}{p}} \leq s$.

Therefore, Theorem 2 is kept. □

In the following, we demonstrate the desired properties of the proposed *LNGWPBM* operator:

(1) **Monotonicity**: If $z_i = \left(l_{\alpha_i}, l_{\beta_i}, l_{\gamma_i}\right)$ $(i=1,2,\cdots,n)$ and $y_i = \left(l_{\delta_i}, l_{\eta_i}, l_{\sigma_i}\right)$ $(i=1,2,\cdots,n)$ are any two sets of LNNs, they satisfy the conditions $\alpha_i \geq \delta_i$, $\beta_i \leq \eta_i$ and $\gamma_i \leq \sigma_i$, then:

$$LNGWPBM^{p,q}(z_1, z_2, \cdots, z_n) \geq LNGWPBM^{p,q}(y_1, y_2, \cdots, y_n).$$

Proof. Similar to the monotonicity property of the *LNGPBM* operator, we also suppose that $LNGWPBM^{p,q}(z_1, z_2, \cdots, z_n) = z = \left(l_\alpha, l_\beta, l_\gamma\right)$ and $LNGWPBM^{p,q}(y_1, y_2, \cdots, y_n) = y = \left(l_\delta, l_\eta, l_\sigma\right)$.

Then:

$$\alpha = s\left(1-\left(\left(\left(\prod_{r=1}^{t}\left(1-(1-K_\alpha)^{\frac{p}{p+q}}\right)\right)^{\frac{1}{t}}\right)^{\frac{n-|F_2|}{n}}\right)\left(\left(\left(\prod_{i=1}^{|F_2|}\left(1-(\alpha_i/s)^p\right)^{\omega_i}\right)^{\frac{1}{\sum_{i=1}^{|F_2|}\omega_i}}\right)^{\frac{|F_2|}{n}}\right)\right)^{\frac{1}{p}}$$

$$\delta = s\left(1-\left(\left(\left(\prod_{r=1}^{t}\left(1-(1-K_\delta)^{\frac{p}{p+q}}\right)\right)^{\frac{1}{t}}\right)^{\frac{n-|F_2|}{n}}\right)\left(\left(\left(\prod_{i=1}^{|F_2|}\left(1-(\delta_i/s)^p\right)^{\omega_i}\right)^{\frac{1}{\sum_{i=1}^{|F_2|}\omega_i}}\right)^{\frac{|F_2|}{n}}\right)\right)^{\frac{1}{p}},$$

$$\beta = s-s\left(1-\left(\left(\prod_{r=1}^{t}\left(1-(1-K_\beta)^{\frac{p}{p+q}}\right)\right)^{\frac{1}{t}}\right)^{\frac{n-|F_2|}{n}}\left(\left(\prod_{i=1}^{|F_2|}\left(1-(1-\beta_i/s)^p\right)^{\omega_i}\right)^{\frac{1}{\sum_{i=1}^{|F_2|}\omega_i}}\right)^{\frac{|F_2|}{n}}\right)^{\frac{1}{p}},$$

$$\eta = s-s\left(1-\left(\left(\prod_{r=1}^{t}\left(1-(1-K_\eta)^{\frac{p}{p+q}}\right)\right)^{\frac{1}{t}}\right)^{\frac{n-|F_2|}{n}}\left(\left(\prod_{i=1}^{|F_2|}\left(1-(1-\eta_i/s)^p\right)^{\omega_i}\right)^{\frac{1}{\sum_{i=1}^{|F_2|}\omega_i}}\right)^{\frac{|F_2|}{n}}\right)^{\frac{1}{p}},$$

$$\gamma = s-s\left(1-\left(\left(\prod_{r=1}^{t}\left(1-(1-K_\gamma)^{\frac{p}{p+q}}\right)\right)^{\frac{1}{t}}\right)^{\frac{n-|F_2|}{n}}\left(\left(\prod_{i=1}^{|F_2|}\left(1-(1-\gamma_i/s)^p\right)^{\omega_i}\right)^{\frac{1}{\sum_{i=1}^{|F_2|}\omega_i}}\right)^{\frac{|F_2|}{n}}\right)^{\frac{1}{p}}, \text{ and}$$

$$\sigma = s-s\left(1-\left(\left(\prod_{r=1}^{t}\left(1-(1-K_\sigma)^{\frac{p}{p+q}}\right)\right)^{\frac{1}{t}}\right)^{\frac{n-|F_2|}{n}}\left(\left(\prod_{i=1}^{|F_2|}\left(1-(1-\sigma_i/s)^p\right)^{\omega_i}\right)^{\frac{1}{\sum_{i=1}^{|F_2|}\omega_i}}\right)^{\frac{|F_2|}{n}}\right)^{\frac{1}{p}}.$$

In order to prove this property, we need to compute their score function values $C(z)$ and $C(y)$, and their accuracy values $A(z)$ and $A(y)$ to compare their synthesized result, i.e., $z \geq y$. Firstly, on the basis of the condition $\alpha_i \geq \delta_i$, $\beta_i \leq \eta_i$ and $\gamma_i \leq \sigma_i$, we can get the compared result of their truth-membership degrees, indeterminacy-membership degrees, and falsity-membership degrees, respectively.

(i) The comparison of the truth-membership degrees:

Based on $\alpha_i \geq \delta_i$, we can get:

$$K_\alpha = \left(\prod_{i=1}^{h_r} \left(1 - \left(1 - \left(1 - (\alpha_i/s)^p \right)^{\omega_i} \right) \left(1 - \left(\prod_{\substack{j=1 \\ j \neq i}}^{h_r} \left(1 - (\alpha_j/s)^q \right)^{\omega_j} \right)^{\frac{1}{\sum\limits_{\substack{j=1 \\ j \neq i}}^{h_r} \omega_j}} \right) \right) \right)^{\frac{1}{\sum\limits_{i=1}^{h_r} \omega_i}}$$

$$\leq \left(\prod_{i=1}^{h_r} \left(1 - \left(1 - \left(1 - (\delta_i/s)^p \right)^{\omega_i} \right) \left(1 - \left(\prod_{\substack{j=1 \\ j \neq i}}^{h_r} \left(1 - (\delta_j/s)^q \right)^{\omega_j} \right)^{\frac{1}{\sum\limits_{\substack{j=1 \\ j \neq i}}^{h_r} \omega_j}} \right) \right) \right)^{\frac{1}{\sum\limits_{i=1}^{h_r} \omega_i}} = K_\delta$$

$$\Rightarrow \left(\left(\prod_{r=1}^{t} \left(1 - (1 - K_\alpha)^{\frac{p}{p+q}} \right) \right)^{\frac{1}{t}} \right)^{\frac{n-|F_2|}{n}} \leq \left(\left(\prod_{r=1}^{t} \left(1 - (1 - K_\delta)^{\frac{p}{p+q}} \right) \right)^{\frac{1}{t}} \right)^{\frac{n-|F_2|}{n}}$$

$$\text{and} \left(\left(\prod_{i=1}^{|F_2|} \left(1 - (\alpha_i/s)^p \right)^{\omega_i} \right)^{\frac{1}{\sum\limits_{i=1}^{|F_2|} \omega_i}} \right)^{\frac{|F_2|}{n}} \leq \left(\left(\prod_{i=1}^{|F_2|} \left(1 - (\delta_i/s)^p \right)^{\omega_i} \right)^{\frac{1}{\sum\limits_{i=1}^{|F_2|} \omega_i}} \right)^{\frac{|F_2|}{n}}.$$

In accordance with the upper two inequalities, we have:

$$s \left(1 - \left(\left(\left(\prod_{r=1}^{t} \left(1 - (1 - K_\alpha)^{\frac{p}{p+q}} \right) \right)^{\frac{1}{t}} \right)^{\frac{n-|F_2|}{n}} \right) \left(\left(\left(\prod_{i=1}^{|F_2|} \left(1 - (\alpha_i/s)^p \right)^{\omega_i} \right)^{\frac{1}{\sum\limits_{i=1}^{|F_2|} \omega_i}} \right)^{\frac{|F_2|}{n}} \right) \right)^{\frac{1}{p}} \geq s \left(1 - \left(\left(\left(\prod_{r=1}^{t} \left(1 - (1 - K_\delta)^{\frac{p}{p+q}} \right) \right)^{\frac{1}{t}} \right)^{\frac{n-|F_2|}{n}} \right) \left(\left(\left(\prod_{i=1}^{|F_2|} \left(1 - (\delta_i/s)^p \right)^{\omega_i} \right)^{\frac{1}{\sum\limits_{i=1}^{|F_2|} \omega_i}} \right)^{\frac{|F_2|}{n}} \right) \right)^{\frac{1}{p}}$$

That is, $\alpha \geq \delta$.

(ii) The comparison of indeterminacy-membership degrees and falsity-membership degrees, respectively:

Based on $\beta_i \leq \eta_i$ and $\gamma_i \leq \sigma_i$, we can also obtain $\beta \leq \eta$ and $\gamma \leq \sigma$, which is similar to the process of the truth-membership degrees.

Thus, it can be obtained that $C(z) = \frac{2s+\alpha-\beta-\gamma}{3s} \geq \frac{2s+\delta-\eta-\sigma}{3s} = C(y)$. In the following, we discuss two cases.

(i) If $C(z) > C(y)$, then $z > y$ according to Definition 2.
(ii) If $C(z) = C(y)$, then $(\alpha - \gamma) - \beta = (\delta - \sigma) - \eta$. Since $\alpha - \gamma \geq \delta - \sigma$ in the light of $\alpha \geq \delta$ and $\gamma \leq \sigma$, now we assume $\alpha - \gamma > \delta - \sigma$, then $\beta > \eta$, which is in contradiction with the previous proof $\beta \leq \eta$. So, we can conclude that $\alpha - \gamma = \delta - \sigma$. That is $A(z) = \frac{\alpha-\gamma}{s} = \frac{\delta-\sigma}{s} = A(y)$, which testifies $z = y$.

In conclusion, the synthesized result is $z \geq y$, which explains $LNGWPBM^{p,q}(z_1, z_2, \cdots, z_n) \geq LNGWPBM^{p,q}(y_1, y_2, \cdots, y_n)$. □

(2) **Boundedness**: Let $z_i = \left(l_{\alpha_i}, l_{\beta_i}, l_{\gamma_i} \right)$ $(i = 1, 2, \cdots, n)$ be any set of LNNs, then:

$$LNGWPBM^{p,q}\left(\min_i z_i, \min_i z_i, \cdots, \min_i z_i \right) \leq LNGWPBM^{p,q}(z_1, z_2, \cdots, z_n) \leq LNGWPBM^{p,q}\left(\max_i z_i, \max_i z_i, \cdots, \max_i z_i \right).$$

Based on the monotonicity property of the *LNGWPBM* operator, it is easy to prove, and the detailed process is omitted here.

Based on the character of F_2, some special cases are discussed about the *LNGPBM* operator, as shown in the following.

(1) When $|F_2| = 0$, all of the arguments belong to the group F_1, and are divided into t partitions; then, the proposed *LNGWPBM* operator is simplified as the linguistic neutrosophic weighted *PBM* (*LNWPBM*) operator:

$$LNGWPBM^{p,q}(z_1, z_2, \cdots, z_n) = \frac{1}{t}\overset{t}{\underset{r=1}{\oplus}}\left(\frac{1}{\sum_{i=1}^{h_r}\omega_i}\overset{h_r}{\underset{i=1}{\oplus}}\omega_i z_i^p \otimes \left(\frac{1}{\sum_{\substack{j=1\\ j\neq i}}^{h_r}\omega_j}\overset{h_r}{\underset{\substack{j=1\\ j\neq i}}{\oplus}}\omega_j z_j^q\right)\right)^{\frac{1}{p+q}} = LNWPBM^{p,q}(z_1, z_2, \cdots, z_n)$$

(2) When $|F_2| = 0$ and $t = 1$, all of the arguments belong to the same partition, i.e., h_r= n; then, the proposed *LNGWPBM* operator is translated into the linguistic neutrosophic normalized weighted *BM* (*LNNWBM*) operator:

$$LNGWPBM^{p,q}(z_1, z_2, \cdots, z_n) = \left(\frac{1}{\sum_{i=1}^{h_r}\omega_i}\overset{h_r}{\underset{i=1}{\oplus}}\omega_i z_i^p \otimes \left(\frac{1}{\sum_{\substack{j=1\\ j\neq i}}^{h_r}\omega_j}\overset{h_r}{\underset{\substack{j=1\\ j\neq i}}{\oplus}}\omega_j z_j^q\right)\right)^{\frac{1}{p+q}} = \left(\frac{1}{\sum_{i=1}^{n}\omega_i}\overset{n}{\underset{i=1}{\oplus}}\omega_i z_i^p \otimes \left(\frac{1}{\sum_{\substack{j=1\\ j\neq i}}^{n}\omega_j}\overset{n}{\underset{\substack{j=1\\ j\neq i}}{\oplus}}\omega_j z_j^q\right)\right)^{\frac{1}{p+q}}$$

$$= \left(\overset{n}{\underset{i=1}{\oplus}}\omega_i z_i^p \otimes \left(\frac{1}{1-\omega_i}\overset{n}{\underset{\substack{j=1\\ j\neq i}}{\oplus}}\omega_j z_j^q\right)\right)^{\frac{1}{p+q}} = LNNWBM^{p,q}(z_1, z_2, \cdots, z_n).$$

(3) When $|F_2| = n$, there is no element in group F_1 and all of the elements are independent; then, the proposed *LNGWPBM* operator reduces to the linguistic neutrosophic power root weighted mean (*LNPRWM*) operator:

$$LNGWPBM^{p,q}(z_1, z_2, \cdots, z_n) = \left(\frac{1}{\sum_{i=1}^{|F_2|}\omega_i}\overset{|F_2|}{\underset{i=1}{\oplus}}\omega_i z_i^p\right)^{\frac{1}{p}} = \left(\frac{1}{\sum_{i=1}^{n}\omega_i}\overset{n}{\underset{i=1}{\oplus}}\omega_i z_i^p\right)^{\frac{1}{p}} = \left(\overset{n}{\underset{i=1}{\oplus}}\omega_i z_i^p\right)^{\frac{1}{p}} = LNPRWM^p(z_1, z_2, \cdots, z_n)$$

Moreover, we can also get some special cases by distributing different values to the parameters p and q.

(1) When $q \to 0$, the *LNGWPBM* operator is translated into the *LNPRWM* operator, as described in the previous discussion. Since there are no inner connections in group F_1, all of the elements are unrelated.
(2) When $p = 1$ and $q \to 0$, the *LNGWPBM* operator becomes the *LNNWAA* operator, as defined by Fang and Ye [29]:

$$LNGWPBM^{p=1,q\to 0}(z_1, z_2, \cdots, z_n) = \left(\overset{n}{\underset{i=1}{\oplus}}\omega_i z_i^p\right)^{\frac{1}{p}} = \overset{n}{\underset{i=1}{\oplus}}\omega_i z_i = LNNWAA(z_1, z_2, \cdots, z_n).$$

(3) When $p = 2$ and $q \to 0$, the *LNGWPBM* operator is transformed into the linguistic neutrosophic square root weighted mean (*LNSRWM*) operator, which is shown as follows:

$$LNGWPBM^{p=2,q\to 0}(z_1, z_2, \cdots, z_n) = \left(\overset{n}{\underset{i=1}{\oplus}}\omega_i z_i^p\right)^{\frac{1}{p}} = \left(\overset{n}{\underset{i=1}{\oplus}}\omega_i z_i^2\right)^{\frac{1}{2}} = LNSRWA(z_1, z_2, \cdots, z_n).$$

(4) When $p = q = 1$, the *LNGWPBM* operator is simplified as the simplest form of the *LNGWPBM* operator, which is shown as follows:

$$LNGWPBM^{p=1,q=1}(z_1, z_2, \cdots, z_n) = \left(\frac{n-|F_2|}{n} \left(\frac{1}{t} \overset{t}{\underset{r=1}{\oplus}} \left(\frac{1}{\sum_{i=1}^{h_r} \omega_i} \overset{h_r}{\underset{i=1}{\oplus}} \omega_i z_i \otimes \left(\frac{1}{\sum_{\substack{j=1 \\ j \neq i}}^{h_r} \omega_j} \overset{h_r}{\underset{\substack{j=1 \\ j \neq i}}{\oplus}} \omega_j z_j \right) \right) \right)^{\frac{1}{2}} \oplus \frac{|F_2|}{n} \left(\frac{1}{\sum_{i=1}^{|F_2|} \omega_i} \overset{|F_2|}{\underset{i=1}{\oplus}} \omega_i z_i \right) \right)$$

It is often used to simplify the calculation in a problem with different weights.

4. A Novel MAGDM Method by the Introduced *LNGWPBM* Operator

In this section, we develop a novel MAGDM method based on the proposed *LNGWPBM* operator to address the kind of problems where the attributes are sorted into two groups: one group contains several clusters where the attributes are relevant in same cluster, but independent in different clusters, and another contains the attributes that are irrelevant to any other attribute. Firstly, we put this kind of problem in a nutshell. Then, we detail the procedures of the proposed method to solve the above problems.

Suppose $X = \{X_1, X_2, \cdots, X_m\}$ is a set of alternatives, and $G = \{G_1, G_2, \cdots, G_n\}$ is a set of attributes, ω_j is the weight of the attribute $G_j (j = 1, 2, \cdots, n)$, where $0 \leq \omega_j \leq 1$ $(j = 1, 2, \cdots, n)$, $\sum_{j=1}^{n} \omega_j = 1$. Experts D_k $(k = 1, 2, \cdots, d)$ can use the LNNs to judge the alternative X_i for attribute G_j and denote it as $z_{ij}^k = \left(l_{\alpha_{ij}^k}, l_{\beta_{ij}^k}, l_{\gamma_{ij}^k} \right)$ in a linguistic term set $L = (l_0, l_1, \cdots, l_s)$, which meets $\alpha_{ij}^k, \beta_{ij}^k, \gamma_{ij}^k \in [0, s]$, and s is an even number. The experts' weight vector is $\pi = (\pi_1, \pi_2, \cdots, \pi_d)^T$ satisfying with $0 \leq \pi_k \leq 1$ $(k = 1, 2, \cdots, d)$, $\sum_{k=1}^{d} \pi_k = 1$. Thus, we form the evaluation values given by expert D_k into a decision matrix $Z^k = \left[z_{ij}^k \right]_{m \times n}$ $(k = 1, 2, \cdots, d)$.

We further hypothesize that the set of attributes $G = \{G_1, G_2, \cdots, G_n\}$ is sorted into two groups: F_1 and F_2. In F_1, the attributes are divided into t clusters $P_1, P_2, \cdots, P_t$, which satisfies $P_x \cap P_y = \varnothing, x \neq y$ and $\overset{t}{\underset{r=1}{\cup}} P_r = F_1$. It means that the group F_1 contains several clusters, where the attributes are relevant in same cluster, but independent in different clusters; in F_2, the attributes are irrelevant to any attribute. Afterwards, we decide the priority of alternatives according to the information provided above.

The procedures of the proposed method are designed as follows.

Step 1. Normalize the LNNs.

Since the attributes generally fall into two types, the corresponding attribute values have the two types. In order to achieve normalization, we generally transform the cost attribute values into benefit attribute values. First of all, we assume that $Y^k = \left[y_{ij}^k \right]_{m \times n}$ is the normalized matrix of $Z^k = \left[z_{ij}^k \right]_{m \times n}$, where $y_{ij}^k = \left(l_{\delta_{ij}^k}, l_{\eta_{ij}^k}, l_{\sigma_{ij}^k} \right)$, $1 \leq i \leq m$, $1 \leq j \leq n$, and $1 \leq k \leq d$. Then, the standardizing method is described in the following [7]:

(1) For benefit attribute values:

$$y_{ij}^k = z_{ij}^k = \left(l_{\alpha_{ij}^k}, l_{\beta_{ij}^k}, l_{\gamma_{ij}^k} \right) \tag{20}$$

(2) For cost attribute values:

$$y_{ij}^k = z_{ij}^k = \left(l_{s-\alpha_{ij}^k}, l_{s-\beta_{ij}^k}, l_{s-\gamma_{ij}^k}\right) \tag{21}$$

Step 2. Calculate the collective decision information by the *LNGWPBM* operator fixed with $|F_2| = 0$ and $t = 1$ (i.e., the *LNNWBM* operator discussed in Section 3.2), because there is no need to divide the experts into different clusters. Then, we can get the unfolding form:

$$\begin{aligned} y_{ij} &= \left(l_{\delta_{ij}}, l_{\eta_{ij}}, l_{\sigma_{ij}}\right) = LNNWBM^{p,q}\left(y_{ij}^1, y_{ij}^2, \cdots, y_{ij}^k\right) = \left(\mathop{\oplus}_{k=1}^{d} \pi_k \left(y_{ij}^k\right)^p \otimes \left(\frac{1}{1-\pi_k} \mathop{\oplus}_{\substack{h=1 \\ h \neq k}}^{d} \pi_h \left(y_{ij}^h\right)^q\right)\right)^{\frac{1}{p+q}} \\ &= \left(l_{s\left(1-\prod_{k=1}^{d}\left(1-\left(1-\left(\delta_{ij}^k/s\right)^p\right)^{\pi_k}\right)\left(1-\left(\prod_{\substack{h=1 \\ h \neq k}}^{d}\left(1-\left(\delta_{ij}^h/s\right)^q\right)^{\pi_h}\right)^{\frac{1}{1-\pi_k}}\right)\right)^{\frac{1}{p+q}}}, l_{s-s\left(1-\prod_{k=1}^{d}\left(1-\left(1-\left(1-\eta_{ij}^k/s\right)^p\right)^{\pi_k}\right)\left(1-\left(\prod_{\substack{h=1 \\ h \neq k}}^{d}\left(1-\left(1-\eta_{ij}^h/s\right)^q\right)^{\pi_h}\right)^{\frac{1}{1-\pi_k}}\right)\right)^{\frac{1}{p+q}}}, l_{s-s\left(1-\prod_{k=1}^{d}\left(1-\left(1-\left(1-\sigma_{ij}^k/s\right)^p\right)^{\pi_k}\right)\left(1-\left(\prod_{\substack{h=1 \\ h \neq k}}^{d}\left(1-\left(1-\sigma_{ij}^h/s\right)^q\right)^{\pi_h}\right)^{\frac{1}{1-\pi_k}}\right)\right)^{\frac{1}{p+q}}} \right) \end{aligned} \tag{22}$$

where $1 \leq i \leq m$, $1 \leq j \leq n$, $p, q \geq 0$, and $p + q > 0$.

Step 3. Compute the comprehensive value of each alternative based on the *LNGWPBM* operator; the unfolding form is detailed in the following:

$$\begin{aligned} y_i &= \left(l_{\delta_i}, l_{\eta_i}, l_{\sigma_i}\right) = LNGWPBM^{p,q}(y_{i1}, y_{i2}, \cdots, y_{in}) = \left(\frac{n-|F_2|}{n}\left(\frac{1}{t}\mathop{\oplus}_{r=1}^{t}\left(\frac{1}{\sum_{j=1}^{h_r}\omega_j}\mathop{\oplus}_{j=1}^{h_r}\omega_j y_{ij}^p \otimes \left(\frac{1}{\sum_{\substack{g=1 \\ g\neq j}}^{h_r}\omega_g}\mathop{\oplus}_{\substack{g=1 \\ g\neq j}}^{h_r}\omega_g y_{ig}^q\right)\right)\right)^{\frac{p}{p+q}} \oplus \frac{|F_2|}{n}\left(\frac{1}{\sum_{j=1}^{|F_2|}\omega_j}\mathop{\oplus}_{j=1}^{|F_2|}\omega_j y_{ij}^p\right)\right)^{\frac{1}{p}} \\ &= \left(l_{s\left(1-\left(\left(\left(\prod_{r=1}^{t}\left(1-(1-K_\alpha)^{\frac{p}{p+q}}\right)\right)^{\frac{1}{t}}\right)^{\frac{n-|F_2|}{n}}\right)\left(\left(\left(\prod_{j=1}^{|F_2|}\left(1-\left(\alpha_{ij}/s\right)^p\right)^{\omega_j}\right)^{\frac{1}{\sum_{j=1}^{|F_2|}\omega_j}}\right)^{\frac{|F_2|}{n}}\right)\right)^{\frac{1}{p}}}, l_{s-s\left(1-\left(\left(\prod_{r=1}^{t}\left(1-(1-K_\beta)^{\frac{p}{p+q}}\right)\right)^{\frac{1}{t}}\right)^{\frac{n-|F_2|}{n}}\left(\left(\prod_{j=1}^{|F_2|}\left(1-\left(1-\beta_{ij}/s\right)^p\right)^{\omega_j}\right)^{\frac{1}{\sum_{j=1}^{|F_2|}\omega_j}}\right)^{\frac{|F_2|}{n}}\right)^{\frac{1}{p}}}, l_{s-s\left(1-\left(\left(\prod_{r=1}^{t}\left(1-(1-K_\gamma)^{\frac{p}{p+q}}\right)\right)^{\frac{1}{t}}\right)^{\frac{n-|F_2|}{n}}\left(\left(\prod_{j=1}^{|F_2|}\left(1-\left(1-\gamma_{ij}/s\right)^p\right)^{\omega_j}\right)^{\frac{1}{\sum_{j=1}^{|F_2|}\omega_j}}\right)^{\frac{|F_2|}{n}}\right)^{\frac{1}{p}}} \right) \\ K_\alpha &= \left(\prod_{j=1}^{h_r}\left(1-\left(1-\left(1-\left(\alpha_{ij}/s\right)^p\right)^{\omega_j}\right)\left(1-\left(\prod_{\substack{g=1 \\ g\neq j}}^{h_r}\left(1-\left(\alpha_{ig}/s\right)^q\right)^{\omega_g}\right)^{\frac{1}{\sum_{\substack{g=1 \\ g\neq j}}^{h_r}\omega_g}}\right)\right)\right)^{\frac{1}{\sum_{j=1}^{h_r}\omega_j}}, \\ K_\beta &= \left(\prod_{j=1}^{h_r}\left(1-\left(1-\left(1-\left(1-\beta_{ij}/s\right)^p\right)^{\omega_j}\right)\left(1-\left(\prod_{\substack{g=1 \\ g\neq j}}^{h_r}\left(1-\left(1-\beta_{ig}/s\right)^q\right)^{\omega_g}\right)^{\frac{1}{\sum_{\substack{g=1 \\ g\neq j}}^{h_r}\omega_g}}\right)\right)\right)^{\frac{1}{\sum_{j=1}^{h_r}\omega_j}}, \\ K_\gamma &= \left(\prod_{j=1}^{h_r}\left(1-\left(1-\left(1-\left(1-\gamma_{ij}/s\right)^p\right)^{\omega_j}\right)\left(1-\left(\prod_{\substack{g=1 \\ g\neq j}}^{h_r}\left(1-\left(1-\gamma_{ig}/s\right)^q\right)^{\omega_g}\right)^{\frac{1}{\sum_{\substack{g=1 \\ g\neq j}}^{h_r}\omega_g}}\right)\right)\right)^{\frac{1}{\sum_{j=1}^{h_r}\omega_j}}. \end{aligned} \tag{23}$$

where $1 \leq i \leq m$, $p, q \geq 0$, and $p + q > 0$; $|F_2|$ denotes the number of attributes in F_2, h_r indicates the number of attributes in cluster P_r, and $\sum_{r=1}^{t} h_r = n - |F_2|$.

Step 4. Calculate the score value $C(y_i)$ and the accuracy value $A(y_i)$ of the synthesized evaluation value y_i in the light of Definitions 2 and 3, where $1 \leq i \leq m$.

Step 5. Compare the obtained score values $C(y_1)$, $C(y_2)$, ..., and $C(y_m)$ based on Definition 4. The larger the value of $C(y_i)$, the more front the order of alternative X_i, where $1 \leq i \leq m$. If the value of $C(y_i)$ is the same, then compare the obtained accuracy values $A(y_1)$, $A(y_2)$, ..., and $A(y_m)$ to determine the ranking orders of alternatives.

Step 6. Ends.

5. A Practical Application on Selecting Green Suppliers

In this section, we use a realistic example to illustrate the effectiveness and advantage of the proposed MAGDM method by the proposed *LNGWPBM* operator.

Example 2. *The example is about the selection of green suppliers. A car manufacturer wants to choose parts, and there are four alternative green suppliers expressed by* $\{X_1, X_2, X_3, X_4\}$*, which can be seen as evaluation objects. The car manufacturer establishes seven criteria to assess the four green suppliers and the measured evaluation criteria* $G = \{G_1, G_2, G_3, G_4, G_5, G_6, G_7\}$ *are shown as follows: price (* G_1*), green degree (* G_2 *), quality (* G_3*), service level (* G_4*), environment for development (* G_5*), response time (* G_6*), and innovation ability (* G_7*). Their weight vector is* $\omega = (0.1, 0.2, 0.2, 0.1, 0.1, 0.2, 0.1)^T$*. Since the green degree shows the influence degree of the green suppliers on the environment and resources, the criterion* G_2 *has nothing to do with the other criteria. Besides, according to the interrelationship patterns, we are able to divide the other evaluation criteria into three partition structures:* $P_1 = \{G_1, G_3\}$*,* $P_2 = \{G_4, G_6\}$*, and* $P_3 = \{G_5, G_7\}$*. The car manufacturer assembled a panel of three related principals to conduct field explorations and surveys in depth, so that the optimal green supplier can be selected. We use* D_k $(k = 1, 2, 3)$ *to denote each related principal and their weight vector* $\pi = (0.4, 0.3, 0.3)^T$*. On the basis of their investigation, professional knowledge and experience, every related principal* D_k $(k = 1, 2, 3)$ *needs to assess each green supplier* X_i $(i = 1, 2, 3, 4)$ *under each evaluation criterion* G_j $(j = 1, 2, \cdots, 7)$ *by using scores or linguistic information directly. Suppose the linguistic term set* $L = \{l_0, l_1, l_2, l_3, l_4, l_5, l_6, l_7, l_8\}$*, which expresses, from left to right: extremely low, very low, low, slightly low, medium, slightly high, high, very high, and extremely high, respectively. The corresponding relationships between score and LV are detailed in Table 1 [7]. Therefore, we can unify evaluation information with LNNs to depict the fuzziness and uncertainty of evaluation criteria. Finally, these related principals' evaluation information constructs the three following decision matrices* $Z^k = [z_{ij}^k]_{m\times n}$ $(k = 1, 2, 3)$ *described in Tables 2–4, where* z_{ij}^k *can be depicted as* $\left(l_{\alpha_{ij}^k}, l_{\beta_{ij}^k}, l_{\gamma_{ij}^k}\right)$.

Table 1. The corresponding relationships between score and linguistic values (LV).

Score	0~19	20~29	30~39	40~49	50~59	60~69	70~79	80~89	90~100
Evaluation	extremely low	very low	low	slightly low	medium	slightly high	high	very high	very high
Linguistic value	l_0	l_1	l_2	l_3	l_4	l_5	l_6	l_7	l_8

Table 2. Evaluation matrix Z^1 given by the related principal D_1.

	G_1	G_2	G_3	G_4	G_5	G_6	G_7
X_1	(l_4, l_4, l_3)	(l_3, l_5, l_1)	(l_6, l_3, l_4)	(l_7, l_1, l_2)	(l_4, l_1, l_3)	(l_2, l_1, l_3)	(l_4, l_4, l_3)
X_2	(l_4, l_3, l_2)	(l_5, l_4, l_2)	(l_5, l_3, l_2)	(l_6, l_3, l_1)	(l_5, l_4, l_3)	(l_2, l_7, l_2)	(l_2, l_4, l_1)
X_3	(l_5, l_1, l_2)	(l_3, l_1, l_1)	(l_7, l_1, l_2)	(l_7, l_1, l_2)	(l_4, l_6, l_1)	(l_4, l_3, l_3)	(l_1, l_4, l_1)
X_4	(l_3, l_4, l_3)	(l_6, l_3, l_3)	(l_6, l_4, l_2)	(l_5, l_1, l_1)	(l_5, l_2, l_2)	(l_3, l_1, l_2)	(l_4, l_6, l_3)

Table 3. Evaluation matrix Z^2 given by the related principal D_2.

	G_1	G_2	G_3	G_4	G_5	G_6	G_7
X_1	(l_3, l_5, l_2)	(l_3, l_1, l_4)	(l_5, l_2, l_3)	(l_6, l_2, l_1)	(l_5, l_1, l_3)	(l_3, l_1, l_2)	(l_3, l_2, l_3)
X_2	(l_5, l_1, l_2)	(l_4, l_4, l_3)	(l_6, l_1, l_1)	(l_7, l_2, l_2)	(l_7, l_4, l_4)	(l_3, l_3, l_4)	(l_3, l_1, l_1)
X_3	(l_4, l_3, l_1)	(l_3, l_5, l_1)	(l_7, l_4, l_3)	(l_5, l_3, l_1)	(l_6, l_1, l_2)	(l_4, l_1, l_2)	(l_2, l_3, l_3)
X_4	(l_3, l_4, l_3)	(l_5, l_2, l_2)	(l_7, l_2, l_4)	(l_7, l_3, l_4)	(l_4, l_2, l_1)	(l_2, l_3, l_4)	(l_4, l_4, l_3)

Table 4. Evaluation matrix Z^3 given by the related principal D_3.

	G_1	G_2	G_3	G_4	G_5	G_6	G_7
X_1	(l_4, l_1, l_2)	(l_4, l_2, l_3)	(l_6, l_3, l_2)	(l_6, l_1, l_4)	(l_6, l_3, l_1)	(l_3, l_4, l_5)	(l_4, l_1, l_2)
X_2	(l_5, l_3, l_4)	(l_5, l_4, l_3)	(l_5, l_1, l_2)	(l_5, l_3, l_5)	(l_5, l_3, l_3)	(l_2, l_1, l_2)	(l_3, l_2, l_1)
X_3	(l_3, l_1, l_2)	(l_3, l_1, l_1)	(l_7, l_1, l_3)	(l_5, l_2, l_2)	(l_4, l_1, l_1)	(l_3, l_2, l_3)	(l_2, l_2, l_1)
X_4	(l_4, l_1, l_4)	(l_4, l_2, l_3)	(l_5, l_3, l_5)	(l_6, l_1, l_5)	(l_7, l_2, l_4)	(l_3, l_1, l_2)	(l_3, l_2, l_1)

5.1. The Evaluation Procedures

[Step 1] Normalize the LNNs in the evaluation matrix. Since the price (G_1) and the response time (G_6) belong to the cost attributes, we need to transform the corresponding LNNs of the attributes G_1 and G_6 into the benefit attributes values according to Formula (21) in the evaluation matrices Z^k $(k = 1, 2, 3)$. The normalized matrices are $Y^k = \left[y_{ij}^k\right]_{4\times 7}$ $(k = 1, 2, 3)$, which are displayed in Tables 5–7.

Table 5. The normalized matrix Y^1.

	G_1	G_2	G_3	G_4	G_5	G_6	G_7
X_1	(l_4, l_4, l_5)	(l_3, l_5, l_1)	(l_6, l_3, l_4)	(l_7, l_1, l_2)	(l_4, l_1, l_3)	(l_6, l_7, l_5)	(l_4, l_4, l_3)
X_2	(l_4, l_5, l_6)	(l_5, l_4, l_2)	(l_5, l_3, l_2)	(l_6, l_3, l_1)	(l_5, l_4, l_3)	(l_6, l_1, l_6)	(l_2, l_4, l_1)
X_3	(l_3, l_7, l_6)	(l_3, l_1, l_1)	(l_7, l_1, l_2)	(l_7, l_1, l_2)	(l_4, l_6, l_1)	(l_4, l_5, l_5)	(l_1, l_4, l_1)
X_4	(l_5, l_4, l_5)	(l_6, l_3, l_3)	(l_6, l_4, l_2)	(l_5, l_1, l_1)	(l_5, l_2, l_2)	(l_5, l_7, l_6)	(l_4, l_6, l_3)

Table 6. The normalized matrix Y^2.

	G_1	G_2	G_3	G_4	G_5	G_6	G_7
X_1	(l_5, l_3, l_6)	(l_3, l_1, l_4)	(l_5, l_2, l_3)	(l_6, l_2, l_1)	(l_5, l_1, l_3)	(l_5, l_7, l_6)	(l_3, l_2, l_3)
X_2	(l_3, l_7, l_6)	(l_4, l_4, l_3)	(l_6, l_1, l_1)	(l_7, l_2, l_2)	(l_7, l_4, l_4)	(l_5, l_5, l_4)	(l_3, l_1, l_1)
X_3	(l_4, l_5, l_7)	(l_3, l_5, l_1)	(l_7, l_4, l_3)	(l_5, l_3, l_1)	(l_6, l_1, l_2)	(l_4, l_7, l_6)	(l_2, l_3, l_3)
X_4	(l_5, l_4, l_5)	(l_5, l_2, l_2)	(l_7, l_2, l_4)	(l_7, l_3, l_4)	(l_4, l_2, l_1)	(l_6, l_5, l_4)	(l_4, l_4, l_3)

Table 7. The normalized matrix Y^3.

	G_1	G_2	G_3	G_4	G_5	G_6	G_7
X_1	(l_4, l_7, l_6)	(l_4, l_2, l_3)	(l_6, l_3, l_2)	(l_6, l_1, l_4)	(l_6, l_3, l_1)	(l_5, l_4, l_3)	(l_4, l_1, l_2)
X_2	(l_3, l_5, l_4)	(l_5, l_4, l_3)	(l_5, l_1, l_2)	(l_5, l_3, l_5)	(l_5, l_3, l_3)	(l_6, l_7, l_6)	(l_3, l_2, l_1)
X_3	(l_5, l_7, l_6)	(l_3, l_1, l_1)	(l_7, l_1, l_3)	(l_5, l_2, l_2)	(l_4, l_1, l_1)	(l_5, l_6, l_5)	(l_2, l_2, l_1)
X_4	(l_4, l_7, l_4)	(l_4, l_2, l_3)	(l_5, l_3, l_5)	(l_6, l_1, l_5)	(l_7, l_2, l_4)	(l_5, l_7, l_6)	(l_3, l_2, l_1)

[Step 2] Calculate the collective decision information by the *LNNWBM* operator in Formula (22). In order to reduce the complexity of computing, we fix $p = q = 1$.

As an example, we can calculate the collective decision value y_{11}, and the below is its calculative process:

$$y_{11} = LNNWBM^{p,q}(y_{11}^1, y_{11}^2, y_{11}^3) =$$

$$= \left(l_{s\left(1-\prod\limits_{k=1}^{3}\left(1-\left(1-\left(1-\left(\delta_{11}^k/s\right)\right)^{\pi_k}\right)\left(1-\left(\prod\limits_{h=1, h\neq k}^{3}\left(1-\left(\delta_{11}^h/s\right)\right)^{\pi_h}\right)^{\frac{1}{1-\pi_k}}\right)\right)\right)^{\frac{1}{2}}}, l_{s-s\left(1-\prod\limits_{k=1}^{3}\left(1-\left(1-\left(\eta_{11}^k/s\right)^{\pi_k}\right)\left(1-\left(\prod\limits_{h=1, h\neq k}^{3}\left(\eta_{11}^h/s\right)^{\pi_h}\right)^{\frac{1}{1-\pi_k}}\right)\right)\right)^{\frac{1}{2}}}, l_{s-s\left(1-\prod\limits_{k=1}^{3}\left(1-\left(1-\left(\sigma_{11}^k/s\right)^{\pi_k}\right)\left(1-\left(\prod\limits_{h=1, h\neq k}^{3}\left(\sigma_{11}^h/s\right)^{\pi_h}\right)^{\frac{1}{1-\pi_k}}\right)\right)\right)^{\frac{1}{2}}} \right)$$

$$= (l_{4.5685}, l_{4.3818}, l_{5.5541})$$

The other collective decision values y_{ij} are shown in the following:

$$y_{21} = (l_{3.5344}, l_{5.5586}, l_{5.2871});\ y_{31} = (l_{4.1699}, l_{6.4201}, l_{6.2805});\ y_{41} = (l_{4.9669}, l_{4.7841}, l_{4.5176});$$

$$y_{12} = (l_{3.4824}, l_{2.2406}, l_{2.1420});\ y_{22} = (l_{4.9669}, l_{3.7793}, l_{2.3128});$$
$$y_{32} = (l_{3.1424}, l_{1.5104}, l_{0.7488});\ y_{42} = (l_{5.3832}, l_{2.0407}, l_{2.3634});$$

$$y_{13} = (l_{6.0202}, l_{2.3634}, l_{2.7300});\ y_{23} = (l_{5.6366}, l_{1.2948}, l_{1.3362});$$
$$y_{33} = (l_{7.2512}, l_{1.3489}, l_{2.3128});\ y_{43} = (l_{6.4202}, l_{2.7300}, l_{3.2926});$$

$$y_{14} = (l_{6.7107}, l_{0.9833}, l_{1.7986});\ y_{24} = (l_{6.4202}, l_{2.3634}, l_{1.9954});$$
$$y_{34} = (l_{6.1444}, l_{1.5443}, l_{1.3362});\ y_{44} = (l_{6.3515}, l_{1.1760}, l_{2.8122});$$

$$y_{15} = (l_{5.2700}, l_{1.1760}, l_{1.9394});\ y_{25} = (l_{6.0606}, l_{3.4315}, l_{3.0331});$$
$$y_{35} = (l_{4.9485}, l_{2.0139}, l_{0.9833});\ y_{45} = (l_{5.7982}, l_{1.6889}, l_{1.7986});$$

$$y_{16} = (l_{5.6872}, l_{6.1687}, l_{4.5018});\ y_{26} = (l_{6.0202}, l_{4.0018}, l_{5.2871});$$
$$y_{36} = (l_{4.5685}, l_{5.9320}, l_{5.2005});\ y_{46} = (l_{5.6366}, l_{6.4201}, l_{5.2871});$$

$$y_{17} = (l_{3.8863}, l_{1.9465}, l_{2.3634});\ y_{27} = (l_{2.7306}, l_{1.9465}, l_{0.7488});$$
$$y_{37} = (l_{1.6456}, l_{2.7300}, l_{1.1760});\ y_{47} = (l_{3.8863}, l_{3.8560}, l_{1.9394}).$$

[Step 3] According to Formula (23), we can get the comprehensive value y_i of each alternative X_i $(i = 1, 2, 3, 4)$ (suppose $p = q = 1$); the results are shown below:

$$y_1 = (l_{5.5760}, l_{0.0007}, l_{0.0016});\ y_2 = (l_{5.4447}, l_{0.0019}, l_{0.0009});$$
$$y_3 = (l_{5.0616}, l_{0.0036}, l_{0.0004});\ y_4 = (l_{5.8684}, l_{0.0046}, l_{0.0017}).$$

[Step 4] According to Formula (2), we can obtain the score values $C(y_1)$, $C(y_2)$, $C(y_3)$, and $C(y_4)$ of the comprehensive values y_1, y_2, y_3, and y_4, respectively, which are displayed as follows:

$$C(y_1) = 0.8989; C(y_2) = 0.8934; C(y_3) = 0.8774; C(y_4) = 0.9109.$$

[Step 5] Since $C(y_4) > C(y_1) > C(y_2) > C(y_3)$, which is based on Definition 4, we can see that the ranking order of the alternatives X_1, X_2, X_3 and X_4 is: $X_4 \succ X_1 \succ X_2 \succ X_3$, where the most suitable alternative is X_4.

According to the upper computation of the proposed method, we can find that the most suitable green supplier is X_4, the second is X_1, and the worst is X_2 or X_3. So, we recommend that the car manufacturer choose green supplier X_4.

5.2. Exploration of the Parameters' Influence

In the above steps, we fix parameters p and q with 1, but we can easily find that the parameters p and q play an important role in the procedures of the proposed method, based on the *LNGWPBM* operator. When we change the values of parameters p and q, the integration results are usually different, so that the ranking order may be changed accordingly. Table 8 shows the ranking orders of the green suppliers when we assign the parameters p and q to different values. Then, we further explore the influence of parameters p and q on the ranking order.

Table 8. Ranking orders of the green suppliers under different values of the parameters p and q.

Parameters p and q	Score Value y_i $(i = 1, 2, 3, 4)$	Ranking Orders
$p = 1, q = 1$	$C(y_1) = 0.8989; C(y_2) = 0.8934;$ $C(y_3) = 0.8774; C(y_4) = 0.9109.$	$X_4 \succ X_1 \succ X_2 \succ X_3$
$p = 1, q = 0.01$	$C(y_1) = 0.8775; C(y_2) = 0.8796;$ $C(y_3) = 0.8742; C(y_4) = 0.8921.$	$X_4 \succ X_2 \succ X_1 \succ X_3$
$p = 0.01, q = 1$	$C(y_1) = 0.9977; C(y_2) = 0.9969;$ $C(y_3) = 0.9951; C(y_4) = 0.9984.$	$X_4 \succ X_1 \succ X_2 \succ X_3$
$p = 1, q = 2$	$C(y_1) = 0.9019; C(y_2) = 0.8954;$ $C(y_3) = 0.8765; C(y_4) = 0.9136.$	$X_4 \succ X_1 \succ X_2 \succ X_3$
$p = 1, q = 5$	$C(y_1) = 0.9009; C(y_2) = 0.8993;$ $C(y_3) = 0.8840; C(y_4) = 0.9140.$	$X_4 \succ X_1 \succ X_2 \succ X_3$
$p = 1, q = 10$	$C(y_1) = 0.9028; C(y_2) = 0.9073;$ $C(y_3) = 0.9007; C(y_4) = 0.9200.$	$X_4 \succ X_2 \succ X_1 \succ X_3$
$p = 2, q = 1$	$C(y_1) = 0.8835; C(y_2) = 0.8800;$ $C(y_3) = 0.8694; C(y_4) = 0.8938.$	$X_4 \succ X_1 \succ X_2 \succ X_3$
$p = 5, q = 1$	$C(y_1) = 0.8792; C(y_2) = 0.8790;$ $C(y_3) = 0.8828; C(y_4) = 0.8872.$	$X_4 \succ X_3 \succ X_1 \succ X_2$
$p = 10, q = 1$	$C(y_1) = 0.8914; C(y_2) = 0.8923;$ $C(y_3) = 0.9061; C(y_4) = 0.8999.$	$X_3 \succ X_4 \succ X_2 \succ X_1$
$p = 2, q = 2$	$C(y_1) = 0.8846; C(y_2) = 0.8788;$ $C(y_3) = 0.8634; C(y_4) = 0.8938.$	$X_4 \succ X_1 \succ X_2 \succ X_3$
$p = 5, q = 5$	$C(y_1) = 0.8734; C(y_2) = 0.8683;$ $C(y_3) = 0.8583; C(y_4) = 0.8780.$	$X_4 \succ X_1 \succ X_2 \succ X_3$
$p = 9, q = 9$	$C(y_1) = 0.8788; C(y_2) = 0.8781;$ $C(y_3) = 0.8659; C(y_4) = 0.8820.$	$X_4 \succ X_1 \succ X_2 \succ X_3$
$p = 10, q = 10$	$C(y_1) = 0.8806; C(y_2) = 0.8809;$ $C(y_3) = 0.8677; C(y_4) = 0.8835.$	$X_4 \succ X_2 \succ X_1 \succ X_3$

From Table 8, it is easy to find that the bigger the value of parameter p or q is, the more chaotic the ranking order. Let's explain with an example. When $p = 1$ $q = 10$, the ranking order is $X_4 \succ X_2 \succ X_1 \succ X_3$; when $p = 5$ $q = 1$, the ranking order is $X_4 \succ X_3 \succ X_1 \succ X_2$; however, when $p = 10$ $q = 1$, the ranking order is $X_3 \succ X_4 \succ X_2 \succ X_1$. So, it's hard to get the regularity of arrangements under this situation. However, when the parameters p and q are equal and less than 10, the ranking orders are relatively stable, and the best green supplier is X_4, and the worst is X_3.

Generally, the bigger the values of parameters p and q, the more complex the calculation becomes, and the more the interrelations between the attributes are emphasized. DMs usually choose the right parameters p and q according to their preferences. However, there is a special case, i.e., $q = 0$, and the proposed method cannot reflect inner connections between attributes, which is similar to another case such as $|F_2| = $ n. Hence, this is not in conformity with this example, and we only allow q to be close to 0 infinitely when discussing. When $p = 1$ $q = 0.01$, the ranking order is $X_4 \succ X_2 \succ X_1 \succ X_3$, and the best green supplier is still X_4. Therefore, in real decision making, we generally recommend that the parameter values be 1 from a practical point of view, which is not only intuitionistic and simple, but is also able to consider the inner connections between attributes.

5.3. Comparison with Other Existing Methods

In this subsection, in order to illustrate the validity and advantage of the proposed MAGDM method related to the *LNGWPBM* operator, we plan to compare it with Fang and Ye's MAGDM method [29], which is related to the *LNNWAA* operator, and Liang et al.'s MAGDM method [7], which is about improving classical TOPSIS with LNNs based on Example 2; their ranking results are displayed in Table 9.

Table 9. A comparison of the ranking results of the alternatives for different multiple attribute group decision-making (MAGDM) methods for Example 2. *LNGWPBM*: linguistic neutrosophic generalized weighted partitioned Bonferroni mean operator.

Methods	Score Values	Ranking Orders
Fang and Ye's MAGDM method [29] by the *LNNWAA* operator	$C_1 = 0.6403$, $C_2 = 0.6508$, $C_3 = 0.6748$, $C_4 = 0.6300$.	$X_3 \succ X_2 \succ X_1 \succ X_4$
Liang et al.'s MAGDM method [7] by improving classical TOPSIS	No	$X_3 \succ X_2 \succ X_1 \succ X_4$
Our proposed MAGDM method by the *LNGWPBM* operator (when $p = q = 1$)	$C_1 = 0.8989$, $C_2 = 0.8934$, $C_3 = 0.8774$, $C_4 = 0.9109$.	$X_4 \succ X_1 \succ X_2 \succ X_3$
Our proposed MAGDM method by the *LNGWPBM* operator (when $p = 1, q = 0$ and $\mid F_2 \mid = 7$)	$C_1 = 0.6403$, $C_2 = 0.6508$, $C_3 = 0.6748$, $C_4 = 0.6300$.	$X_3 \succ X_2 \succ X_1 \succ X_4$

Note: C_i is the abbreviation of the score value $C(y_i)$ of the collective decision information y_i $(i = 1, 2, 3, 4)$, respectively.

(1) Since Fang and Ye's method [29] by the *LNNWAA* operator can only address the MAGDM problems where the attributes are not associated with each other, in order to complete the comparison between it and our proposed MAGDM method by the *LNGWPBM* operator, we suppose that the seven attributes are independent of each other in Example 2, i.e., $\mid F_2 \mid =7$; then, we compare their ranking results. In terms of Table 9, we can find that the ranking order of Fang and Ye's method [29] by the *LNNWAA* operator is consistent with the one of our proposed MAGDM method by the *LNGWPBM* operator (when $p = 1$, $q = 0$ and $\mid F_2 \mid =7$), which is $X_3 \succ X_2 \succ X_1 \succ X_4$. However, the ranking order of Fang and Ye's method [13] by the *LNNWAA* operator has a great difference from the one of our proposed MAGDM method by the *LNGWPBM* operator (when $p = q = 1$); even the best alternatives are not the same. In the following, we explain the reason for the ranking results.

Fang and Ye's method [29] by the *LNNWAA* operator cannot capture inner connections between attributes. In this practical application about the selection of green suppliers, if our assumption is that the attributes have nothing to do with any other attribute, then, i.e., $\mid F_2 \mid = 7$. Besides, we take $p = 1$, $q = 0$ to make the *LNNWBM* operator become the *LNNWAA* operator in integrating the evaluation information given by DMs, which is consistent with step 1 in Fang and Ye's method [29]. Then, the ranking result of Fang and Ye's method [29] by the *LNNWAA* operator should be consistent with the one of our proposed MAGDM method by the *LNGWPBM* operator (when $p = 1$, $q = 0$ and $\mid F_2 \mid = 7$). By using the two methods to deal with Example 2, respectively, we find that the ranking result of Fang and Ye's method [29] by the *LNNWAA* operator is equal to the one of our proposed MAGDM method by the *LNGWPBM* operator (when $p = 1$, $q = 0$ and $\mid F_2 \mid = 7$), which is $X_3 \succ X_2 \succ X_1 \succ X_4$. Therefore, this can explain that our proposed MAGDM method is tried and true. However, the ranking result of Fang and Ye's method [29] by the *LNNWAA* operator has a great difference from the one of our proposed MAGDM method by the *LNGWPBM* operator (when $p = q = 1$); even the best alternatives are not the same. In Example 2, we can find that inner connections exist between the attributes and a special condition where the criterion G_2 has nothing to do with other criteria, i.e., $\mid F_2 \mid = 1$; this can be solved by the *LNGWPBM* operator well, but the *LNNWAA* operator does not have the same ability. It is easy to compute in our proposed MAGDM method; we assume $p = q = 1$, and then the ranking result by the *LNGWPBM* operator is $X_4 \succ X_1 \succ X_2 \succ X_3$, which is very different from Fang and Ye's method [29] by the *LNNWAA* operator. In addition, DMs can choose the right value of the parameters p and q according to the actual decision-making situation and their personal preferences, so our proposed MAGDM method is universal and elastic. Meanwhile, Fang and Ye's method [29] can only solve the MAGDM problems with independent attributes, and is not suitable for this kind of question, such as in Example 2. Therefore, our proposed MAGDM method by the *LNGWPBM* operator is more workable and elastic than Fang and Ye's method [29] by the *LNNWAA* operator.

(2) From Table 9, we find that the ranking result of Liang et al.'s MAGDM method [7] by improving classical TOPSIS is the same as the one of our proposed MAGDM method by the *LNGWPBM* operator (when $p = 1$, $q = 0$ and $|F_2|$=7), which is $X_3 \succ X_2 \succ X_1 \succ X_4$; however, it is inconsistent with the one of our proposed MAGDM method by the *LNGWPBM* operator (when $p = q = 1$). Then, we elaborate what leads to the ranking results.

Liang et al.'s MAGDM method [7] uses the *LNNWAA* operator to integrate the evaluation information given by DMs, and then adopts the extended TOPSIS model to rank the alternatives. To compare our proposed MAGDM method with Liang et al.'s MAGDM method [7], we also take $p = 1$, $q = 0$, and $|F_2| = 7$ similar to in the previous analysis; so, the ranking result of Liang et al.'s MAGDM method [7] should be consistent with the one of our proposed MAGDM method. It is important to note that when using Liang et al.'s MAGDM method [7] to solve Example 2, we use the weights of the attributes given in Example 2. By calculating separately, the ranking result of Liang et al.'s MAGDM method [7] by improving classical TOPSIS is the same as the one of our proposed MAGDM method by the *LNGWPBM* operator (when $p = 1$, $q = 0$ and $|F_2| = 7$), which is $X_3 \succ X_2 \succ X_1 \succ X_4$. This proves the validity of our proposed MAGDM method again. However, Liang et al.'s MAGDM method [7] cannot integrate evaluation information, and does not reflect inner connections between attributes, while our proposed MAGDM method by the *LNGWPBM* operator can easily achieve these two points. Furthermore, in the extended TOPSIS model used by Liang et al.'s MAGDM method [7], the correlation coefficient cannot guarantee that the best solution should have the closest distance from the positive ideal solution and the farthest distance from the negative ideal solution, simultaneously [49]. At the same time, Liang et al.'s MAGDM method [7] by improving the classical TOPSIS model neglects DMs' utilities or preferences, whereas our proposed MAGDM method can draw attention to the influence of DMs' utilities or preferences on the final results, and select the appropriate parameters p and q. When $p = q = 1$, the ranking result of our proposed MAGDM method is $X_4 \succ X_1 \succ X_2 \succ X_3$, which is even the opposite result of Liang et al.'s MAGDM method [7]. Therefore, our proposed MAGDM method is more appropriate and effective than Liang et al.'s MAGDM method [7] in solving the problem, such as in Example 2.

(3) To further interpret the effectiveness of our proposed MAGDM method by the *LNGWPBM* operator, we use our proposed MAGDM method to solve the illustrative examples in [29] and [7], and compare our proposed MAGDM method by the *LNGWPBM* operator with Fang and Ye's MAGDM method [29] by the *LNNWAA* operator and Liang et al.'s MAGDM method [7] by improving the classical TOPSIS model. Of course, because the attributes are independent of each other in these two illustrative examples, we still fix with $p = 1$, $q = 0$ and $|F_2| = n$, where n denotes the numbers of the attributes. By applying our proposed MAGDM method to these two illustrative examples, we can find that the ranking result of our proposed MAGDM method is consistent with that of Fang and Ye's MAGDM method [29] and Liang et al.'s MAGDM method [7], respectively, which are detailed in Tables 10 and 11. This further illustrate the effectiveness of our proposed MAGDM method by the *LNGWPBM* operator.

Table 10. A ranking comparison of the alternatives for different MAGDM methods for example described by Fang and Ye in [29].

Methods	Score Values	Ranking Order
Fang and Ye's MAGDM method [29] by the *LNNWAA* operator	$C_1 = 0.7528$, $C_2 = 0.7777$, $C_3 = 0.7613$, $C_4 = 0.8060$.	$X_4 \succ X_2 \succ X_3 \succ X_1$
Our proposed MAGDM method by the *LNGWPBM* operator (when $p = 1, q = 0$ and $\mid F_2 \mid = 3$)	$C_1 = 0.7528$, $C_2 = 0.7777$, $C_3 = 0.7613$, $C_4 = 0.8060$.	$X_4 \succ X_2 \succ X_3 \succ X_1$

Note: C_i is abbreviation of score value $C(y_i)$ of the collective decision information y_i $(i = 1, 2, 3, 4)$, respectively.

Table 11. A ranking comparison of the alternatives for different MAGDM methods for example described by Liang et al. in [7].

Methods	Score Values	Ranking Order
Liang et al.'s MAGDM method [7] by improving classical TOPSIS	No	$X_4 \succ X_2 \succ X_3 \succ X_1$
Our proposed MAGDM method by the *LNGWPBM* operator (when $p = 1, q = 0$ and $\mid F_2 \mid = 5$)	$C_1 = 0.4941$, $C_2 = 0.7901$, $C_3 = 0.6495$, $C_4 = 0.7925$.	$X_4 \succ X_2 \succ X_3 \succ X_1$

Note: C_i is abbreviation of score value $C(y_i)$ of the collective decision information y_i $(i = 1, 2, 3, 4)$, respectively.

In the following, we compare the desirable properties of our proposed MAGDM method with the ones of Fang and Ye's MAGDM method [29] and Liang et al.'s MAGDM method [7] to go even further in the advantages of our proposed MAGDM method. Table 12 describes the final comparison results.

Table 12. A comparison of the properties for different MAGDM methods. DM: decision makers.

Properties \ Methods	Fang and Ye's MAGDM Method [29] by the *LNNWAA* Operator	Liang et al.'s MAGDM Method [7] by Improving Classical TOPSIS	Our proposed MAGDM Method by the *LNGWPBM* Operator
Integrate evaluation information	Yes	No	Yes
Reflect DMs' preferences	No	No	Yes
Consider inner relations between attributes in the same cluster	No	No	Yes
Consider the clusters of the input arguments	No	No	Yes

From Table 12, the following conclusions are drawn:

(1) Our proposed MAGDM method and Fang and Ye's MAGDM method [29] can integrate evaluation information, while Liang et al.'s MAGDM method [7] cannot do this and only rank the alternatives by comparing the relative closeness of the positive ideal alternative and the negative ideal alternative.

(2) Although Fang and Ye's MAGDM method [29] can integrate evaluation information, it ignores DMs' preferences, and does not capture the inherent relation pattern between attributes. Besides, Liang et al.'s MAGDM method [7] also cannot reflect DMs' preferences and the inherent relation patterns between attributes.

(3) Our proposed MAGDM method contains regulatory factors that are determined by DMs' preferences, and considers the clusters of the input arguments and the inner relations between the attributes in the same cluster. So, our proposed MAGDM method can effectively address the problems with the heterogeneous relationship among attributes. However, the other two methods do not have these advantages, which show that the application scopes of the two methods are relatively narrow.

In summary, the contrastive analysis further illustrates the validity and merit of our proposed MAGDM method, compared with Fang and Ye's MAGDM method [29] and Liang et al.'s MAGDM method [7].

6. Conclusions

The GPBM operator can model the average of the respective satisfaction of the independent and dependent inputs, and is an extended form of the *PBM* operator, the arithmetic mean operator, and the BM operator. Its merit is to capture the heterogeneous relationship among attributes where all of the attributes are sorted into two groups: F_1 and F_2. In F_1, the elements are divided into several clusters, and the members have inherent connections in the same cluster, but independence in different clusters; in F_2, the elements do not belong to any cluster of the correlated input arguments in F_1. Besides, LNNs can depict the qualitative information more appropriately than the SNNs, and are also an extension of the LIFNs. However, now, based on LNNs, we yet have not seen any studies addressing the MAGDM problems with the heterogeneous relationships among attributes. Therefore, in order to fill this gap, we have expanded the GPBM operator to adapt the linguistic neutrosophic environment, and have proposed the *LNGPBM* operator in this paper. At the same time, its desired properties and special cases have been discussed. Moreover, aiming at the condition where different attributes have different weights in practical applications, we also have introduced its weighted version, namely the *LNGWPBM* operator, including discussing its desired properties and special cases. Then, based on the developed *LNGWPBM* operator, we have developed a novel MAGDM method with LNNs to solve the MAGDM problems with the heterogeneous relationship among attributes. By comparing with Fang and Ye's MAGDM method [29] and Liang et al.'s MAGDM method [7], we find that the developed MAGDM method is more valid and general for solving the MAGDM problems with co-dependent attributes. This is because the developed MAGDM method can intuitively and realistically depict qualitative information and reflect the heterogeneous relationship among attributes. In further research, our developed operators can be improved by considering the unknown weights, objective data, or other forms of information, such as unbalanced linguistic information [50]. Besides, we can apply our developed operators to the other practices such as medical diagnosis, clustering analysis, pattern recognition, discordance analysis, and so on.

Author Contributions: P.L. proposed the *LNGPBM* and *LNGWPBM* operators and investigated their properties, and Y.W. provided the calculation and comparative analysis of examples. We wrote the paper together.

Acknowledgments: This work is supported by the National Natural Science Foundation of China (Nos. 71471172 and 71271124), the Special Funds of Taishan Scholars Project of Shandong Province (No. ts201511045) and the Humanities and Social Sciences Research Project of Ministry of Education of China (17YJA630065 and 17YJC630077).

Conflicts of Interest: The authors declare no conflict of interest.

References

1. Boran, F.E.; Genç, S.; Kurt, M.; Akay, D. A multi-criteria intuitionistic fuzzy group decision making for supplier selection with TOPSIS method. *Expert Syst. Appl.* **2009**, *36*, 11363–11368. [CrossRef]
2. Liu, P.; Wang, P. Some interval-valued intuitionistic fuzzy Schweizer–Sklar power aggregation operators and their application to supplier selection. *Int. J. Syst. Sci.* **2018**. [CrossRef]
3. Sanayei, A.; Mousavi, S.F.; Yazdankhah, A. Group decision making process for supplier selection with VIKOR under fuzzy environment. *Expert Syst. Appl.* **2010**, *37*, 24–30. [CrossRef]
4. Banerjee, D.; Dutta, B.; Guha, D.; Goh, M. Generalized partition Bonferroni mean for multi-attribute group decision making problems based on interval data. *IEEE Trans. Syst. Man Cybern. Syst.* **2017**, 1–18, in press.
5. Herrera, F.; Herrera-Viedma, E. A model of consensus in group decision making under linguistic assessments. *Fuzzy Sets Syst.* **1996**, *78*, 73–87. [CrossRef]
6. Herrera, F.; Herrera-Viedma, E. Linguistic decision analysis: Steps for solving decision problems under linguistic information. *Fuzzy Sets Syst.* **2000**, *115*, 67–82. [CrossRef]

7. Liang, W.; Zhao, G.; Wu, H. Evaluating investment risks of metallic mines using an extended TOPSIS method with linguistic neutrosophic numbers. *Symmetry* **2017**, *9*, 149. [CrossRef]
8. Liao, X.; Li, Y.; Lu, B. A model for selecting an ERP system based on linguistic information processing. *Inf. Syst.* **2007**, *32*, 1005–1017. [CrossRef]
9. Wang, T.; Liu, J.; Li, J.; Niu, C. An integrating OWA–TOPSIS framework in intuitionistic fuzzy settings for multiple attribute decision making. *Comput. Ind. Eng.* **2016**, *98*, 185–194. [CrossRef]
10. Yager, R.R. Multicriteria decision making with ordinal/linguistic intuitionistic fuzzy sets for mobile apps. *IEEE Trans. Fuzzy Syst.* **2016**, *24*, 590–599. [CrossRef]
11. Zhang, Z.; Chu, X. Fuzzy group decision making for multi-format and multi-granularity linguistic judgments in quality function deployment. *Expert Syst. Appl.* **2009**, *36*, 9150–9158. [CrossRef]
12. Zadeh, L.A. The concept of a linguistic variable and its application to approximate reasoning Part I. *Inf. Sci.* **1975**, *8*, 199–249. [CrossRef]
13. Bordogna, G.; Fedrizzi, M.; Pasi, G. A linguistic modelling of consensus in group decision making based on OWA operator. *IEEE Trans. Syst. Man Cybern. Syst.* **1997**, *27*, 126–133. [CrossRef]
14. Pei, Z.; Shi, P. Fuzzy risk analysis based on linguistic aggregation operators. *Int. J. Innov. Comput. Inf. Control* **2011**, *7*, 7105–7118.
15. Xu, Z. A method based on linguistic aggregation operators for group decision making with linguistic preference relations. *Inf. Sci.* **2004**, *166*, 19–30. [CrossRef]
16. Atanassov, K.T. Intuitionistic fuzzy sets. *Fuzzy Sets Syst.* **1986**, *20*, 87–96. [CrossRef]
17. Zadeh, L.A. Fuzzy sets. *Inf. Control* **1965**, *8*, 338–356. [CrossRef]
18. Montajabiha, M. An extended PROMETHE II multi-criteria group decision making technique based on intuitionistic fuzzy logic for sustainable energy planning. *Group Decis. Negot.* **2016**, *25*, 221–244. [CrossRef]
19. Peng, J.; Yeh, W.; Lai, T.; Hsu, C. The incorporation of the Taguchi and the VIKOR methods to optimize multi-response problems in intuitionistic fuzzy environments. *J. Chin. Inst. Eng.* **2015**, *38*, 897–907. [CrossRef]
20. Qin, Q.; Liang, F.; Li, L. A TODIM-based multi-criteria group decision making with triangular intuitionistic fuzzy numbers. *Appl. Soft Comput.* **2017**, *55*, 93–107. [CrossRef]
21. Xu, Z. Intuitionistic fuzzy aggregation operators. *IEEE Trans. Fuzzy Syst.* **2007**, *15*, 1179–1187.
22. Xu, Z.; Yager, R.R. Some geometric aggregation operators based on intuitionistic fuzzy sets. *Int. J. Gen. Syst.* **2006**, *35*, 417–433. [CrossRef]
23. Xu, Z.; Yager, R.R. Intuitionistic fuzzy Bonferroni means. *IEEE Trans. Syst. Man Cybern. Part B Cybern.* **2011**, *41*, 568–578.
24. Chen, Z.C.; Liu, P.H.; Pei, Z. An approach to multiple attribute group decision making based on linguistic intuitionistic fuzzy numbers. *Int. J. Comput. Intell. Syst.* **2015**, *8*, 747–760. [CrossRef]
25. Liu, P.; Liu, J.; Merigó, J.M. Partitioned Heronian means based on linguistic intuitionistic fuzzy numbers for dealing with multi-attribute group decision making. *Appl. Soft Comput.* **2018**, *62*, 395–422. [CrossRef]
26. Liu, P.; Wang, P. Some improved linguistic intuitionistic fuzzy aggregation operators and their applications to multiple-attribute decision making. *Int. J. Inf. Technol. Decis. Mak.* **2017**, *16*, 817–850. [CrossRef]
27. Peng, H.; Wang, J.; Cheng, P. A linguistic intuitionistic multi-criteria decision-making method based on the Frank Heronian mean operator and its application in evaluating coal mine safety. *Int. J. Mach. Learn. Cybern.* **2017**. [CrossRef]
28. Zhang, H.; Peng, H.; Wang, J.; Wang, J.Q. An extended outranking approach for multi-criteria decision-making problems with linguistic intuitionistic fuzzy numbers. *Appl. Soft Comput.* **2017**, *59*, 462–474. [CrossRef]
29. Fang, Z.B.; Ye, J. Multiple attribute group decision-making method based on linguistic neutrosophic numbers. *Symmetry* **2017**, *9*, 111. [CrossRef]
30. Smarandache, F. *A Unifying Field in Logics. Neutrosophy: Neutrosophic Probability, Set and Logic*; American Research Press: Rehoboth, NM, USA, 1999.
31. Wang, H.; Smarandache, F.; Zhang, Y.; Sunderraman, R. Single valued neutrosophic sets. In Proceedings of the 10th International Conference on Fuzzy Theory and Technology, Salt Lake City, UT, USA, 21–26 July 2005.
32. Wang, H.; Smarandache, F.; Zhang, Y.Q.; Sunderraman, R. *Interval Neutrosophic Sets and Logic: Theory and Applications in Computing*; Hexis: Phoenix, AZ, USA, 2005.
33. Ye, J. A multicriteria decision-making method using aggregation operators for simplified neutrosophic sets. *J. Intell. Fuzzy Syst.* **2014**, *26*, 2459–2466.

34. Baušys, R.; Juodagalvienė, B. Garage location selection for residential house by WASPAS-SVNS method. *J. Civ. Eng. Manag.* **2017**, *23*, 421–429. [CrossRef]
35. Baušys, R.; Zavadskas, E.K. Multicriteria decision making approach by vikor under interval neutrosophic set environment. *Econ. Comput. Econ. Cybern. Stud. Res.* **2015**, *49*, 33–48.
36. Baušys, R.; Zavadskas, E.K.; Kaklauskas, A. Application of neutrosophic set to multi-criteria decision making by copras. *Econ. Comput. Econ. Cybern. Stud. Res.* **2015**, *49*, 91–105.
37. Kazimieras Zavadskas, E.; Baušys, R.; Lazauskas, M. Sustainable assessment of alternative sites for the construction of a waste incineration plant by applying WASPAS method with single-valued neutrosophic set. *Sustainability* **2015**, *7*, 15923–15936. [CrossRef]
38. Ma, H.; Zhu, H.; Hu, Z.; Li, K.; Tang, W. Time-aware trustworthiness ranking prediction for cloud services using interval neutrosophic set and ELECTRE. *Knowl.-Based Syst.* **2017**, *138*, 27–45. [CrossRef]
39. Peng, X.; Liu, C. Algorithms for neutrosophic soft decision making based on EDAS, new similarity measure and level soft set. *J. Intell. Fuzzy Syst.* **2017**, *32*, 955–968. [CrossRef]
40. Zavadskas, E.K.; Bausys, R.; Juodagalviene, B.; Garnyte-Sapranavicienec, I. Model for residential house element and material selection by neutrosophic MULTIMOORA method. *Eng. Appl. Artif. Intell.* **2017**, *64*, 315–324. [CrossRef]
41. Zavadskas, E.K.; Bausys, R.; Kaklauskas, A.; Ubarte, I.; Kuzminske, A.; Gudiene, N. Sustainable market valuation of buildings by the single-valued neutrosophic MAMVA method. *Appl. Soft Comput.* **2017**, *57*, 74–87. [CrossRef]
42. Liu, P.; Zhang, X. Research on the supplier selection of a supply chain based on entropy weight and improved ELECTRE-III method. *Int. J. Prod. Res.* **2011**, *49*, 637–646. [CrossRef]
43. Jiang, Y.; Liang, X.; Liang, H. An I-TODIM method for multi-attribute decision making with interval numbers. *Soft Comput.* **2017**, *21*, 5489–5506. [CrossRef]
44. Bonferroni, C. Sulle medie multiple di potenze. *Bollettino dell' Unione Matematica Italiana* **1950**, *5*, 267–270.
45. Maclaurin, C. A second letter to Martin Folkes, Esq.; concerning the roots of equations, with the demonstartion of other rules in algebra. *Philos. Trans. R. Soc. Lond. Ser. A* **1729**, *36*, 59–96.
46. Hara, T.; Uchiyama, M.; Takahasi, S.-E. A refinement of various mean inequalities. *J. Inequal. Appl.* **1998**, *2*, 387–395. [CrossRef]
47. Wang, J.Q.; Yang, Y.; Li, L. Multi-criteria decision-making method based on single-valued neutrosophic linguistic Maclaurin symmetric mean operators. *Neural Comput. Appl.* **2016**. [CrossRef]
48. Dutta, B.; Guha, D. Partitioned Bonferroni mean based on linguistic 2-tuple for dealing with multi-attribute group decision making. *Appl. Soft Comput.* **2015**, *37*, 166–179. [CrossRef]
49. Hadi-Vencheh, A.; Mirjaberi, M. Fuzzy inferior ratio method for multiple attribute decision making problems. *Inf. Sci.* **2014**, *277*, 263–272. [CrossRef]
50. Cabrerizo, F.J.; Al-Hmouz, R.; Morfeq, A.; Balamash, A.S.; Martinez, M.A.; Herrera-Viedma, E. Soft consensus measures in group decision making using unbalanced fuzzy linguistic information. *Soft Comput.* **2017**, *21*, 3037–3050. [CrossRef]