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MADM Using m-Generalized q-Neutrosophic Sets

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Abstract: Although the single valued neutrosophic sets (SVNSs) are effective tool to express uncertain information and are superior to the fuzzy sets, intuitionistic fuzzy sets, Pythagorean fuzzy sets and *q*-rung orthopair fuzzy sets, there is not yet reported an operation which can provide desirable generality and flexibility under single valued neutrosophic environment, although many operations have been developed earlier to meet above such eventualities. So, the primary aim of this paper is to propose the concept of m-generalized q-neutrosophic sets (*mGqNSs*) as a further generalization of fuzzy sets, intuitionistic fuzzy sets, Pythagorean fuzzy sets and *q*-rung orthopair fuzzy sets, single valued neutrosophic sets, *n*-hyperspherical neutrosophic sets and single valued spherical neutrosophic sets. Under the m-generalized *q*-neutrosophic environment, we develop some new operational laws and study their properties. Using these operations, we define *m*-generalized *q*-neutrosophic weighted aggregation operators. The distinguished features of these proposed weighted aggregation operators are studied in detail. Furthermore, based on these proposed operators, a MADM (multi-attribute decision making) approach is developed. Finally, an illustrative example is provided to show the feasibility and effectiveness of the proposed approach.

Keywords: Single valued neutrosophic set, *m*-generalized *q*-neutrosophic set, *m*-generalized *q*-neutrosophic weighted averaging aggregation operator (*mGqNWAA*), *m*-generalized *q*-neutrosophic weighted geometric aggregation operator (*mGqNWGA*), score value, decision making.

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1. Introduction

Multi-attribute decision making (MADM) is basically a process of selecting an optimal alternative from a set of chosen ones. In our daily life, we come across various types of multi-attribute decision making problems. Therefore, all of us need to learn the techniques to make decisions. The area of decision making problems has attracted the interest of many researchers. Many authors have worked in this field by utilizing various approaches. All the traditional decision making processes involve crisp data set but in many real life problems, data may not be in crisp form always. Fuzzy set theory is one such extremely useful tool that helps us to deal with non-crisp data. In 1965, Lotfi A. Zadeh **[1]** first published the famous research paper on fuzzy sets that originated due to mainly the inclusion of vague human assessments in computing problems and it can deal with uncertainty, vagueness, partially trueness, impreciseness, Sharpless boundaries etc. Basically, the theory of fuzzy set is founded on the concept of relative graded membership which deals with the partial belongings of an element in a set in order to process inexact information. Later on, fuzzy sets have been generalized to intuitionistic fuzzy sets **[2]** by adding a non-membership function by Atanassov in 1986 in order to deal with problems that possess incomplete information. In the context of fuzzy sets or intuitionistic fuzzy sets, it is known that the membership (or non-membership) value of an element in a set admits a unique value in the closed interval [0,1]. However, the application range of intuitionistic fuzzy set is narrow because it has the constraint that sum of membership degree and non-membership degree of an element is not greater than one. But, in complex decision-making problems, decision makers/experts may choose the preferences in such a way that the above condition gets violated. For instance, if an expert gives his preference with membership degree 0.8 and non-membership degree 0.7, then clearly their sum is 1.5, which is greater than 1. Therefore, this situation can't be not properly handled by the intuitionistic fuzzy sets. To solve this problem, Yager **[3, 4]** introduced the nonstandard fuzzy set named as Pythagorean fuzzy sets with membership degree ζ and non-membership degree ϑ with the condition $\zeta^2 + \vartheta^2 \leq 1$. Obviously, the Pythagorean fuzzy sets accommodate more uncertainties than the intuitionistic fuzzy sets. Yager **[5]** defined *q*-rung orthopair fuzzy sets (*q-ROFSs*) by enlarging the scope of Pythagorean fuzzy sets. The *q*-rung orthopair fuzzy sets allows the result of the *q*th power of the membership grade plus the *q*th power of the non-membership grade to be limited in interval [0,1]. If q=1, the *q*-rung orthopair fuzzy set transforms into the intuitionistic fuzzy set; if q=2, the *q*-rung orthopair fuzzy set transforms into the Pythagorean fuzzy set, which means that the *q*-rung orthopair fuzzy sets are extensions of intuitionistic fuzzy sets and Pythagorean fuzzy sets.

In 1999, Smarandache **[6]** introduced the notion neutrsophic set as a generalization of the classical set, fuzzy set, intuitionistic fuzzy set, Pythagorean fuzzy set and *q*-rung orthopair fuzzy set. The characterization of this neutrosophic set is explicitly done by truth-membership function, indeterminacy membership function and falsity membership function. The concept of single valued neutrosophic set was developed by Wang et al. **[7]** as an extension of fuzzy sets, Pythagorean fuzzy sets, *q*-rung orthopair fuzzy sets, intuitionistic fuzzy sets, single valued spherical neutrosophic sets **[8]**, *n*-hyperspherical neutrosophic sets **[8]**. The possible applications of neutrosophic sets and single valued neutrosophic sets on image segmentation have been studied in Gou and Cheng **[9]**, Gou and Sensur **[10]**. Also, we find their probable infliction on clustering analysis in Karaaslan **[11]** and on medical diagnosis problems in Ansari et al. **[12]** respectively. Furthermore, the subject of the neutrosophic set theory has been practiced in Wang et al. **[13]**, Gou et al. **[14]**, Ye **[15]**, Sun et al. **[16]**, Ye **[17-19]** and Abdel Basset et al. **[20, 21]**. Some recent studies on this area can be found in **[22-37]**.

The growing capacity of decision complexity induces the real-life decision-making problems that indulge both generality and flexibility of the operations used. Some of the basic operations of single valued spherical neutrosophic sets fail to generalize the basic operations of fuzzy sets, intuitionistic fuzzy sets, Pythagorean fuzzy sets and *q*-rung orthopair fuzzy sets. Getting inspired and provoked with this fact, in this paper, we have tried to propose a new concept called "*m*-generalized *q*-neutrosophic sets (*mGqNSs*)" and develop some aggregation operators in *m*-generalized *q*-neutrosophic environment to deal with *MADM* problems. The aims in this article are pursued below:

(1) To propose the concept of *m*-generalized *q*-neutrosophic sets (*mGqNSs*) as a further generalization of fuzzy sets, Pythagorean fuzzy sets, *q*-rung orthopair fuzzy sets, intuitionistic fuzzy sets, single valued neutrosophic sets, *n*-hyperspherical neutrosophic sets and single valued spherical neutrosophic sets.

(2) To define few operations between the *m*-generalized *q*-neutrosophic numbers.

(3) To develop the weighted aggregation operators such as *m*-generalized *q*-neutrosophic weighted averaging aggregation operator (*mGqNWAA*) and *m*-generalized *q*-neutrosophic weighted geometric aggregation operator (*mGqNWGA*) and study their properties.

(4) To propose a multi-attribute decision making method based on the m-generalized q-neutrosophic weighted aggregation operators.

To do so, the rest of the article is arranged as follows:

In section 2, we review some basic concepts. In Section 3, we first define m-generalized q-neutrosophic sets (*mGqNSs*) and *m*-generalized *q*-neutrosophic numbers (*mGqNNs*) and then propose few operations between the mGqNNs. Furthermore, we introduce the score of a *mGqNN* to ranking the *mGqNNs*. In section 4, we propose two types of m-generalized q-neutrosophic weighted aggregation operators to aggregate the m-generalized q-neutrosophic information. In section 5, based on the *m*-generalized *q*-neutrosophic weighted aggregation operators and score of *mGqNNs*, we develop a multi attribute decision making approach, in which the evaluation values of alternatives on the attribute are represented in terms of *mGqNNs* and the alternatives are ranked according to the values of the score of *mGqNNs* to select the best (most desirable) one. Also, we present a practical example to demonstrate the application and effectiveness of the proposed method. In final section, we present the conclusion of the study.

2. Preliminaries:

In this section, first we recall some basic notions that are relevant to our study.

2.1 Definition: [7] A single-valued neutrosophic set ς on the universe set U is given by

$$\varsigma = \{< x, \xi(x), \vartheta(x), \eta(x) >: x \in U\}$$

where the functions $\xi, \vartheta, \eta : U \to [0,1]$ satisfy the condition $0 \le \xi(x) + \vartheta(x) + \eta(x) \le 3$ for every $x \in U$. The functions $\xi(x), \vartheta(x), \eta(x)$ define the degree of truth-membership, indeterminacy-membership and falsity-membership, respectively of $x \in U$.

2.2 Definition: [7] Suppose ς and ς' be two single-valued neutrosophic sets on U and are given by

$\varsigma = \{< x, \xi(x), \vartheta(x), \eta(x) >: x \in U\}$ and $\varsigma' = \{< x, \xi'(x), \vartheta'(x), \eta'(x) >: x \in U\}$. Then

(i) $\varsigma \subseteq \varsigma'$ if and only if $\xi(x) \le \xi'(x), \vartheta(x) \ge \vartheta'(x), \eta(x) \ge \eta'(x)\ \forall x \in U.$

(ii) $\varsigma^c = \{< x, \eta(x), 1 - \vartheta(x), \xi(x) >: x \in U\}$

(iii) $\varsigma \cup \varsigma' = \{< x, \max(\xi(x), \xi'(x)), \min(\vartheta(x), \vartheta'(x)),\ \min(\eta(x), \eta'(x)) >: x \in U\}.$

(iv) $\varsigma \cap \varsigma' = \{< x, \min(\xi(x), \xi'(x)), \max(\vartheta(x), \vartheta'(x)),\ \max(\eta(x), \eta'(x)) >: x \in U\}.$

3. *m*-GENERALIZED *q*-NEUTROSOPHIC SETS:

In this section first we define a *m*-generalized *q*-neutrosophic set as a further generalization of fuzzy set, Pythagorean fuzzy set, *q*-rung orthopair fuzzy set, intuitionistic fuzzy set, single valued neutrosophic set, single valued *n*-hyperspherical neutrosophic set and single valued spherical neutrosophic set. Then we present few operations between the *m*-generalized *q*-neutrosophic numbers.

3.1 Definition: Suppose *U* is a universe set and $x \in U$. A *m*-generalized *q*-neutrosophic set (*mGqNs*) in *U* is described as:

$$\psi = \{< x, \xi(x), \vartheta(x), \eta(x) >: x \in U\}$$

where $\xi, \vartheta, \eta : U \to [0, r]$ $(0 < r \le 1)$ are functions such that $0 \le \xi(x), \vartheta(x), \eta(x) \le 1$ and

$$0 \le (\xi(x))^{\frac{qm}{3}} + (\vartheta(x))^{\frac{qm}{3}} + (\eta(x))^{\frac{qm}{3}} \le \frac{3}{m} \quad (m, q \ge 1) .$$

Here $\xi(x), \vartheta(x), \eta(x)$ represent *m*-generalized truth membership, *m*-generalized indeterminacy membership and *m*-generalized falsity membership respectively of $x \in U$. The triplet $\psi = < \xi, \vartheta, \eta >$ is termed as *m*-generalized *q*-neutrosophic number (*mGqNN* for short).

In particular,

(i) when *m*=*r*=1 and *q*=3, ψ reduces to a single valued neutrosophic set **[7]**.

(ii) when *m*=3, *r*=*q*=1 and $\eta(x) = 0\ \forall x \in U$, ψ reduces to an intuitionistic fuzzy set **[2]**.

(iii) when *m*=3, *r*=*q*=1 and $\eta(x) = \vartheta(x) = 0\ \forall x \in U$, ψ reduces to a fuzzy set **[1]**.

(iv) when *m*=3, *r*= 1 and $\eta(x) = 0\ \forall x \in U$, ψ reduces to a q-Rung orthopair fuzzy set **[5]**.

(v) when *m*=3, *r*= 1, *q*=2 and $\eta(x) = 0\ \forall x \in U$, ψ reduces to a Pythagorean fuzzy set **[3, 4]**.

(vi) For $r = \sqrt[n]{3}$, *m*=1 and *q*=3*n* $(n \ge 1)$, ψ reduces to a single valued n-hyperspherical neutrosophic set **[8]**.

(vii) For $r = \sqrt{3}$, *m*=1 and *q*=6, ψ reduces to a single valued spherical neutrosophic set **[8]**.

Next we define few operations between *m*-generalized *q*-neutrosophic numbers.

3.2 Definition: Suppose $\psi_1 = < \xi_1, \vartheta_1, \eta_1 >$ and $\psi_2 = < \xi_2, \vartheta_2, \eta_2 >$ be two *m*-generalized *q*-neutrosophic numbers defined on *U* and λ be any real number >0. We define

(i) $\psi_1 \oplus \psi_2 = \left\langle \left(\frac{3}{m} - \left(\frac{3}{m} - \xi_1^{\frac{qm}{3}} \right) \left(\frac{3}{m} - \xi_2^{\frac{qm}{3}} \right) \right)^{\frac{3}{qm}}, \vartheta_1 \vartheta_2, \eta_1 \eta_2 \right\rangle$

(ii) $\psi_1 \otimes \psi_2 = \left\langle \xi_1 \xi_2, \left(\frac{3}{m} - \left(\frac{3}{m} - \vartheta_1^{\frac{qm}{3}} \right) \left(\frac{3}{m} - \vartheta_2^{\frac{qm}{3}} \right) \right)^{\frac{3}{qm}}, \left(\frac{3}{m} - \left(\frac{3}{m} - \eta_1^{\frac{qm}{3}} \right) \left(\frac{3}{m} - \eta_2^{\frac{qm}{3}} \right) \right)^{\frac{3}{qm}} \right\rangle$

(iii) $\lambda * \psi_1 = \left\langle \left(\frac{3}{m} - \left(\frac{3}{m} - \xi_1^{\frac{qm}{3}} \right)^{\lambda} \right)^{\frac{3}{qm}}, \vartheta_1^{\lambda}, \eta_1^{\lambda} \right\rangle$

(iv) $\lambda \circ \psi_1 = \left\langle \xi_1^{\lambda}, \left(\frac{3}{m} - \left(\frac{3}{m} - \vartheta_1^{\frac{qm}{3}} \right)^{\lambda} \right)^{\frac{3}{qm}}, \left(\frac{3}{m} - \left(\frac{3}{m} - \eta_1^{\frac{qm}{3}} \right)^{\lambda} \right)^{\frac{3}{qm}} \right\rangle$

3.3 Theorem: Suppose $\psi_1 = <\xi_1, \vartheta_1, \eta_1>$ and $\psi_2 = <\xi_2, \vartheta_2, \eta_2>$ be two *m*-generalized *q*-neutrosophic numbers defined on *U* and $\lambda, \lambda_1, \lambda_2$ be any three real numbers >0. Then

(i) $\psi_1 \oplus \psi_2 = \psi_2 \oplus \psi_1$

(ii) $\psi_1 \otimes \psi_2 = \psi_2 \otimes \psi_1$

(iii) $\lambda * (\psi_1 \oplus \psi_2) = (\lambda * \psi_1) \oplus (\lambda * \psi_2)$

(iv) $\lambda \circ (\psi_1 \otimes \psi_2) = (\lambda \circ \psi_1) \otimes (\lambda \circ \psi_2)$

(v) $(\lambda_1 + \lambda_2) * \psi_1 = (\lambda_1 * \psi_1) \oplus (\lambda_2 * \psi_1)$

(vi) $(\lambda_1 + \lambda_2) \circ \psi_1 = (\lambda_1 \circ \psi_1) \otimes (\lambda_2 \circ \psi_1)$

Proof: (i), (ii) are straight forward.

(iii) We have, $\psi_1 \oplus \psi_2 = \left\langle \left(\frac{3}{m} - \left(\frac{3}{m} - \xi_1^{\frac{qm}{3}} \right) \left(\frac{3}{m} - \xi_2^{\frac{qm}{3}} \right) \right)^{\frac{3}{qm}}, \vartheta_1 \vartheta_2, \eta_1 \eta_2 \right\rangle$.

$\therefore \lambda * (\psi_1 \oplus \psi_2)$

$$= \left\langle \left(\frac{3}{m} - \left(\frac{3}{m} - \left(\frac{3}{m} - \left(\frac{3}{m} - \xi_1^{\frac{qm}{3}} \right) \left(\frac{3}{m} - \xi_2^{\frac{qm}{3}} \right) \right) \right)^{\lambda} \right)^{\frac{3}{qm}}, (\vartheta_1 \vartheta_2)^{\lambda}, (\eta_1 \eta_2)^{\lambda} \right\rangle$$

$$= \left\langle \left(\frac{3}{m} - \left(\frac{3}{m} - \xi_1^{\frac{qm}{3}} \right)^{\lambda} \left(\frac{3}{m} - \xi_2^{\frac{qm}{3}} \right)^{\lambda} \right)^{\frac{3}{qm}}, \vartheta_1^{\lambda} \vartheta_2^{\lambda}, \eta_1^{\lambda} \eta_2^{\lambda} \right\rangle$$

On the other hand, we have,

$(\lambda * \psi_1) \oplus (\lambda * \psi_2)$

$$= \left\langle \left(\frac{3}{m} - \left(\frac{3}{m} - \xi_1^{\frac{qm}{3}} \right)^{\lambda} \right)^{\frac{3}{qm}}, \vartheta_1^{\lambda}, \eta_1^{\lambda} \right\rangle \oplus \left\langle \left(\frac{3}{m} - \left(\frac{3}{m} - \xi_2^{\frac{qm}{3}} \right)^{\lambda} \right)^{\frac{3}{qm}}, \vartheta_2^{\lambda}, \eta_2^{\lambda} \right\rangle$$

$$= \left\langle \left(\frac{3}{m} - \left(\frac{3}{m} - \left(\frac{3}{m} - \left(\frac{3}{m} - \xi_1^{\frac{qm}{3}} \right)^{\lambda} \right) \right) \left(\frac{3}{m} - \left(\frac{3}{m} - \left(\frac{3}{m} - \xi_2^{\frac{qm}{3}} \right)^{\lambda} \right) \right) \right)^{\frac{3}{qm}}, \vartheta_1^{\lambda} \vartheta_2^{\lambda}, \eta_1^{\lambda} \eta_2^{\lambda} \right\rangle$$

$$= \left\langle \left(\frac{3}{m} - \left(\frac{3}{m} - \xi_1^{\frac{qm}{3}} \right)^{\lambda} \left(\frac{3}{m} - \xi_2^{\frac{qm}{3}} \right)^{\lambda} \right)^{\frac{3}{qm}}, \vartheta_1^{\lambda} \vartheta_2^{\lambda}, \eta_1^{\lambda} \eta_2^{\lambda} \right\rangle$$

Thus, we get, $\lambda * (\psi_1 \oplus \psi_2) = (\lambda * \psi_1) \oplus (\lambda * \psi_2)$.

(iv) Similar to (iii)

(v) We have,

$$(\lambda_1 + \lambda_2) * \psi_1 = \left\langle \left(\frac{3}{m} - \left(\frac{3}{m} - \xi_1^{\frac{qm}{3}} \right)^{\lambda_1 + \lambda_2} \right)^{\frac{3}{qm}}, \vartheta_1^{\lambda_1 + \lambda_2}, \eta_1^{\lambda_1 + \lambda_2} \right\rangle$$

On the other hand,

$$(\lambda_1*\psi_1)\oplus(\lambda_2*\psi_1)$$

$$=\left\langle\left(\frac{3}{m}-\left(\frac{3}{m}-\xi_1^{\frac{qm}{3}}\right)^{\lambda_1}\right)^{\frac{3}{qm}},\vartheta_1^{\lambda_1},\eta_1^{\lambda_1}\right\rangle\oplus\left\langle\left(\frac{3}{m}-\left(\frac{3}{m}-\xi_1^{\frac{qm}{3}}\right)^{\lambda_2}\right)^{\frac{3}{qm}},\vartheta_1^{\lambda_2},\eta_1^{\lambda_2}\right\rangle$$

$$=\left\langle\left(\frac{3}{m}-\left(\frac{3}{m}-\left(\frac{3}{m}-\left(\frac{3}{m}-\xi_1^{\frac{qm}{3}}\right)^{\lambda_1}\right)\right)\left(\frac{3}{m}-\left(\frac{3}{m}-\left(\frac{3}{m}-\xi_1^{\frac{qm}{3}}\right)^{\lambda_2}\right)\right)\right)^{\frac{3}{qm}},\vartheta_1^{\lambda_1}\vartheta_1^{\lambda_2},\eta_1^{\lambda_1}\eta_1^{\lambda_2}\right\rangle$$

$$=\left\langle\left(\frac{3}{m}-\left(\frac{3}{m}-\xi_1^{\frac{qm}{3}}\right)^{\lambda_1+\lambda_2}\right)^{\frac{3}{qm}},\vartheta_1^{\lambda_1+\lambda_2},\eta_1^{\lambda_1+\lambda_2}\right\rangle$$

Thus we get, $(\lambda_1+\lambda_2)*\psi_1=(\lambda_1*\psi_1)\oplus(\lambda_2*\psi_1)$.

(vi) Similar to (v).

3.4 Definition: The score of the *mGqNN* $\psi=<\xi,\vartheta,\eta>$ is defined as: $S(\psi)=\dfrac{2+\xi-\vartheta-\eta}{3}$.

The ranking method for ranking the *mGqNNs* is given below:

If $\psi=<\xi,\vartheta,\eta>$ and $\psi'=<\xi',\vartheta',\eta'>$ be two *mGqNNs*, then

(I) if $S(\psi)>S(\psi')$, then $\psi>\psi'$

(II) if $S(\psi)=S(\psi')$, then $\psi=\psi'$

4. *m*-GENERALIZED *q*-NEUTROSOPHIC WEIGHTED AGGREGATION OPERATORS:

In this section first we define *m*-generalized *q*-neutrosophic weighted averaging aggregation operator (*mGqNWAA*) and *m*-generalized *q*-neutrosophic weighted geometric aggregation operator (*mGqNWGA*) and study their basic properties.

4.1 Definition: Suppose $\psi_k=<\xi_k,\vartheta_k,\eta_k>(k=1,2,3,.......,n)$ be a collection of *mGqNNs* defined on the universe set *U*. Then a *m*-generalized *q*-neutrosophic weighted averaging aggregation operator (*mGqNWAA* for short) is given as $mGqNWAA:\Theta^n\to\Theta$ and is defined as:

$$mGqNWAA(\psi_1,\psi_2,\psi_3,........,\psi_n)=(w_1*\psi_1)\oplus(w_2*\psi_2)\oplus(w_3*\psi_3)\oplus.........\oplus(w_n*\psi_n)$$

where Θ is the collection of all *mGqNNs* defined on the universe set *U*, $w=(w_1,w_2,w_3,.......,w_n)^T$ is the weight vector of $(\psi_1,\psi_2,\psi_3,........,\psi_n)$ such that $w_k\geq 0(k=1,2,3,....,n)$ and $\sum_{k=1}^{n}w_k=1$.

On the basis of the operational rules of the *mGqNNs,* we can get the aggregation result as described as Theorem 4.2.

4.2 Theorem: Suppose $\psi_k=<\xi_k,\vartheta_k,\eta_k>(k=1,2,3,.......,n)$ be a collection of *mGqNNs* defined on the universe set *U* and $w=(w_1,w_2,w_3,.......,w_n)^T$ is the weight vector of $(\psi_1,\psi_2,\psi_3,........,\psi_n)$ such that

$w_k \geq 0 (k=1,2,3,....,n)$ and $\sum_{k=1}^{n} w_k = 1$. Then $mGqNWAA(\psi_1, \psi_2, \psi_3, ..., \psi_n)$ is also a *mGqNN*.

Moreover, we have,

$$mGqNWAA(\psi_1, \psi_2, \psi_3, ..., \psi_n) = \left\langle \left(\frac{3}{m} - \prod_{k=1}^{n} \left(\frac{3}{m} - \xi_k^{\frac{qm}{3}} \right)^{w_k} \right)^{\frac{3}{qm}}, \prod_{k=1}^{n} \vartheta_k^{w_k}, \prod_{k=1}^{n} \eta_k^{w_k} \right\rangle.$$

Proof:

The first part of the theorem can be proved easily. To show the rest part, let us use the method of mathematical induction on *n*.

<u>Step-1:</u> For *n*=1, the proof is straight forward. So first take *n*=2.

Then,

$$mGqNWAA(\psi_1, \psi_2)$$

$$= (w_1 * \psi_1) \oplus (w_2 * \psi_2)$$

$$= \left\langle \left(\frac{3}{m} - \left(\frac{3}{m} - \xi_1^{\frac{qm}{3}} \right)^{w_1} \right)^{\frac{3}{qm}}, \vartheta_1^{w_1}, \eta_1^{w_1} \right\rangle \oplus \left\langle \left(\frac{3}{m} - \left(\frac{3}{m} - \xi_2^{\frac{qm}{3}} \right)^{\lambda_2} \right)^{\frac{3}{qm}}, \vartheta_1^{w_2}, \eta_1^{w_2} \right\rangle$$

$$= \left\langle \left(\frac{3}{m} - \left(\frac{3}{m} - \left(\frac{3}{m} - \left(\frac{3}{m} - \xi_1^{\frac{qm}{3}} \right)^{w_1} \right) \right) \left(\frac{3}{m} - \left(\frac{3}{m} - \left(\frac{3}{m} - \xi_2^{\frac{qm}{3}} \right)^{w_2} \right) \right) \right)^{\frac{3}{qm}}, \vartheta_1^{w_1} \vartheta_1^{w_2}, \eta_1^{w_1} \eta_1^{w_2} \right\rangle$$

$$= \left\langle \left(\frac{3}{m} - \left(\frac{3}{m} - \xi_1^{\frac{qm}{3}} \right)^{w_1} \left(\frac{3}{m} - \xi_2^{\frac{qm}{3}} \right)^{w_2} \right)^{\frac{3}{qm}}, \vartheta_1^{w_1} \vartheta_1^{w_2}, \eta_1^{w_1} \eta_1^{w_2} \right\rangle$$

$$= \left\langle \left(\frac{3}{m} - \prod_{k=1}^{2} \left(\frac{3}{m} - \xi_k^{\frac{qm}{3}} \right)^{w_k} \right)^{\frac{3}{qm}}, \prod_{k=1}^{2} \vartheta_k^{w_k}, \prod_{k=1}^{2} \eta_k^{w_k} \right\rangle.$$

Thus the result is true for *n*=2.

<u>Step-2:</u> Suppose that the result is true for *n*=*p* i.e;

$$mGqNWAA(\psi_1, \psi_2, \psi_3, ..., \psi_p) = \left\langle \left(\frac{3}{m} - \prod_{k=1}^{p} \left(\frac{3}{m} - \xi_k^{\frac{qm}{3}} \right)^{w_k} \right)^{\frac{3}{qm}}, \prod_{k=1}^{p} \vartheta_k^{w_k}, \prod_{k=1}^{p} \eta_k^{w_k} \right\rangle$$

<u>Step-3:</u> Take *n*=*p*+1.Then we have,

$$mGqNWAA(\psi_1,\psi_2,\psi_3,........,\psi_{p+1})$$

$$=((w_1*\psi_1)\oplus(w_2*\psi_2)\oplus(w_3*\psi_3)\oplus.........\oplus(w_p*\psi_p))\oplus(w_{p+1}*\psi_{p+1})$$

$$=\left\langle\left(\frac{3}{m}-\prod_{k=1}^{p}\left(\frac{3}{m}-\xi_k^{\frac{qm}{3}}\right)^{w_k}\right)^{\frac{3}{qm}},\prod_{k=1}^{p}\vartheta_k^{w_k},\prod_{k=1}^{p}\eta_k^{w_k}\right\rangle$$

$$\oplus\left\langle\left(\frac{3}{m}-\left(\frac{3}{m}-\xi_{p+1}^{\frac{qm}{3}}\right)^{w_{p+1}}\right)^{\frac{3}{qm}},\vartheta_{p+1}^{w_{p+1}},\eta_{p+1}^{w_{p+1}}\right\rangle$$

$$=\left\langle\left(\frac{3}{m}-\left(\frac{3}{m}-\left(\frac{3}{m}-\prod_{k=1}^{p}\left(\frac{3}{m}-\xi_k^{\frac{qm}{3}}\right)^{w_k}\right)\right)\left(\frac{3}{m}-\left(\frac{3}{m}-\left(\frac{3}{m}-\xi_{p+1}^{\frac{qm}{3}}\right)^{w_{p+1}}\right)\right)\right)^{\frac{3}{qm}},\right.$$

$$\left.\vartheta_{p+1}^{w_{p+1}}\prod_{k=1}^{p}\vartheta_k^{w_k},\eta_{p+1}^{w_{p+1}}\prod_{k=1}^{p}\eta_k^{w_k}\right\rangle$$

$$=\left\langle\left(\frac{3}{m}-\prod_{k=1}^{p}\left(\frac{3}{m}-\xi_k^{\frac{qm}{3}}\right)^{w_k}\times\left(\frac{3}{m}-\xi_{p+1}^{\frac{qm}{3}}\right)^{w_{p+1}}\right)^{\frac{3}{qm}},\vartheta_{p+1}^{w_{p+1}}\prod_{k=1}^{p}\vartheta_k^{w_k},\eta_{p+1}^{w_{p+1}}\prod_{k=1}^{p}\eta_k^{w_k}\right\rangle$$

$$=\left\langle\left(\frac{3}{m}-\prod_{k=1}^{p+1}\left(\frac{3}{m}-\xi_k^{\frac{qm}{3}}\right)^{w_k}\right)^{\frac{3}{qm}},\prod_{k=1}^{p+1}\vartheta_k^{w_k},\prod_{k=1}^{p+1}\eta_k^{w_k}\right\rangle$$

Thus the result is true for *n*=*p*+1 also. Hence, by the method of induction, the result is true for all *n*.

Let us explore some more results related to $mGqNWAA$ operator in the form of theorems 4.3-4.6.

4.3 Theorem: Suppose $\psi_k=<\xi_k,\vartheta_k,\eta_k>(k=1,2,3,.......,n)$ be a collection of m-Gq-NNs defined on the universe set *U* and $w=(w_1,w_2,w_3,.......,w_n)^T$ is the weight vector of $(\psi_1,\psi_2,\psi_3,........,\psi_n)$ such that $w_k\geq 0(k=1,2,3,....,n)$ and $\sum_{k=1}^{n}w_k=1$. Then for $\psi_0=<\xi_0,\vartheta_0,\eta_0>\in\Theta$ (where Θ is the collection of all *mGqNNs* defined on the universe set *U*), we have $mGqNWAA(\psi_0\oplus\psi_1,\psi_0\oplus\psi_2,\psi_0\oplus\psi_3,........,\psi_0\oplus\psi_n)=\psi_0\oplus mGqNWAA(\psi_1,\psi_2,\psi_3,........,\psi_n)$.

Proof:

$$\psi_0 \oplus \psi_k = \left\langle \left(\frac{3}{m} - \left(\frac{3}{m} - \xi_0^{\frac{qm}{3}}\right)\left(\frac{3}{m} - \xi_k^{\frac{qm}{3}}\right)\right)^{\frac{3}{qm}}, \vartheta_0\vartheta_k, \eta_0\eta_k \right\rangle \ (k = 1,2,3,........,n)$$

$$\therefore mGqNWAA(\psi_0 \oplus \psi_1, \psi_0 \oplus \psi_2, \psi_0 \oplus \psi_3, ..., \psi_0 \oplus \psi_n)$$

$$= \left\langle \left(\frac{3}{m} - \prod_{k=1}^{n} \left(\frac{3}{m} - \left(\frac{3}{m} - \left(\frac{3}{m} - \xi_0^{\frac{qm}{3}}\right)\left(\frac{3}{m} - \xi_k^{\frac{qm}{3}}\right)\right)\right)^{w_k} \right)^{\frac{3}{qm}}, \prod_{k=1}^{n} (\vartheta_0\vartheta_k)^{w_k}, \prod_{k=1}^{n} (\eta_0\eta_k)^{w_k} \right\rangle$$

$$= \left\langle \left(\frac{3}{m} - \left(\frac{3}{m} - \xi_0^{\frac{qm}{3}}\right)^{\sum_{k=1}^{n} w_k} \prod_{k=1}^{n} \left(\frac{3}{m} - \xi_k^{\frac{qm}{3}}\right)^{w_k} \right)^{\frac{3}{qm}}, \vartheta_0^{\sum_{k=1}^{n} w_k} \prod_{k=1}^{n} \vartheta_k^{w_k}, \eta_0^{\sum_{k=1}^{n} w_k} \prod_{k=1}^{n} \eta_k^{w_k} \right\rangle$$

$$= \left\langle \left(\frac{3}{m} - \left(\frac{3}{m} - \xi_0^{\frac{qm}{3}}\right) \prod_{k=1}^{n} \left(\frac{3}{m} - \xi_k^{\frac{qm}{3}}\right)^{w_k} \right)^{\frac{3}{qm}}, \vartheta_0 \prod_{k=1}^{n} \vartheta_k^{w_k}, \eta_0 \prod_{k=1}^{n} \eta_k^{w_k} \right\rangle$$

On the other hand, $\psi_0 \oplus mGqNWAA(\psi_1, \psi_2, \psi_3, ..., \psi_n)$

$$= < \xi_0, \vartheta_0, \eta_0 > \oplus \left\langle \left(\frac{3}{m} - \prod_{k=1}^{n} \left(\frac{3}{m} - \xi_1^{\frac{qm}{3}}\right)^{w_k} \right)^{\frac{3}{qm}}, \prod_{k=1}^{n} \vartheta_k^{w_k}, \prod_{k=1}^{n} \eta_k^{w_k} \right\rangle$$

$$= \left\langle \left(\frac{3}{m} - \left(\frac{3}{m} - \xi_0^{\frac{qm}{3}}\right) \prod_{k=1}^{n} \left(\frac{3}{m} - \left(\frac{3}{m} - \prod_{k=1}^{n} \left(\frac{3}{m} - \xi_1^{\frac{qm}{3}}\right)^{w_k} \right)\right)\right)^{\frac{3}{qm}}, \vartheta_0 \prod_{k=1}^{n} \vartheta_k^{w_k}, \eta_0 \prod_{k=1}^{n} \eta_k^{w_k} \right\rangle$$

$$= \left\langle \left(\frac{3}{m} - \left(\frac{3}{m} - \xi_0^{\frac{qm}{3}}\right) \prod_{k=1}^{n} \left(\frac{3}{m} - \xi_k^{\frac{qm}{3}}\right)^{w_k} \right)^{\frac{3}{qm}}, \vartheta_0 \prod_{k=1}^{n} \vartheta_k^{w_k}, \eta_0 \prod_{k=1}^{n} \eta_k^{w_k} \right\rangle$$

Hence,

$$mGqNWAA(\psi_0 \oplus \psi_1, \psi_0 \oplus \psi_2, \psi_0 \oplus \psi_3, ..., \psi_0 \oplus \psi_n) = \psi_0 \oplus mGqNWAA(\psi_1, \psi_2, \psi_3, ..., \psi_n)$$

.

4.4 Theorem: *(Idempotency)* Suppose $\psi_k = < \xi_k, \vartheta_k, \eta_k > (k = 1,2,3,.......,n)$ be a collection of m-Gq-NNs defined on the universe set U and $w = (w_1, w_2, w_3, ..., w_n)^T$ is the weight vector of

$(\psi_1,\psi_2,\psi_3,........,\psi_n)$ such that $w_k \geq 0\,(k=1,2,3,....,n)$ and $\sum_{k=1}^{n} w_k = 1$. If $\psi_0 =<\xi_0,\vartheta_0,\eta_0>\in\Theta$ (where Θ is the collection of all *mGqNNs* defined on the universe set *U*) such that $\psi_k=\psi_0\ \forall\, k=1,2,3,.......,n$, then we have $mGqNWAA(\psi_1,\psi_2,\psi_3,........,\psi_n)=\psi_0$.

Proof: We have, $mGqNWAA(\psi_1,\psi_2,\psi_3,........,\psi_n)$

$$=\left\langle \left(\frac{3}{m}-\prod_{k=1}^{n}\left(\frac{3}{m}-\xi_k^{\frac{qm}{3}}\right)^{w_k}\right)^{\frac{3}{qm}},\prod_{k=1}^{n}\vartheta_k^{\,w_k},\prod_{k=1}^{n}\eta_k^{\,w_k}\right\rangle$$

$$=\left\langle \left(\frac{3}{m}-\prod_{k=1}^{n}\left(\frac{3}{m}-\xi_0^{\frac{qm}{3}}\right)^{w_k}\right)^{\frac{3}{qm}},\prod_{k=1}^{n}\vartheta_0^{\,w_k},\prod_{k=1}^{n}\eta_0^{\,w_k}\right\rangle$$

$$=\left\langle \left(\frac{3}{m}-\left(\frac{3}{m}-\xi_0^{\frac{qm}{3}}\right)^{\sum_{k=1}^{n} w_k}\right)^{\frac{3}{qm}},\vartheta_0^{\sum_{k=1}^{n} w_k},\prod_{k=1}^{n}\eta_0^{\sum_{k=1}^{n} w_k}\right\rangle$$

$$=\left\langle \left(\frac{3}{m}-\left(\frac{3}{m}-\xi_0^{\frac{qm}{3}}\right)\right)^{\frac{3}{qm}},\vartheta_0,\eta_0\right\rangle =<\xi_0,\vartheta_0,\eta_0>=\psi_0$$

4.5 Theorem: *(Monotonocity)* Suppose $\psi_k=<\xi_k,\vartheta_k,\eta_k>$ and $\psi_k'=<\xi_k',\vartheta_k',\eta_k'>$ $(k=1,2,3,.......,n)$ be two collections of *mGqNNs* defined on the universe set *U* and $w=(w_1,w_2,w_3,.......,w_n)^T$ is the weight vector of $(\psi_1,\psi_2,\psi_3,........,\psi_n)$ as well as $(\psi_1',\psi_2',\psi_3',........,\psi_n')$ such that $w_k \geq 0\,(k=1,2,3,....,n)$ and $\sum_{k=1}^{n} w_k=1$. If $\xi_k\geq\xi_k',\vartheta_k\leq\vartheta_k',\eta_k\leq\eta_k'>$ $(k=1,2,3,.......,n)$, then $mGqNWAA(\psi_1,\psi_2,\psi_3,........,\psi_n)\geq mGqNWAA(\psi_1',\psi_2',\psi_3',........,\psi_n')$.

Proof: Since $\xi_k\geq\xi_k',\vartheta_k\leq\vartheta_k',\eta_k\leq\eta_k'>\text{for all}\,k$,

so $\prod_{k=1}^{n}\left(\frac{3}{m}-\xi_k^{\frac{qm}{3}}\right)^{w_k}\leq\prod_{k=1}^{n}\left(\frac{3}{m}-\xi_k'^{\frac{qm}{3}}\right)^{w_k},\prod_{k=1}^{n}\vartheta_k^{\,w_k}\leq\prod_{k=1}^{n}\vartheta_k'^{\,w_k},\prod_{k=1}^{n}\eta_k^{\,w_k}\leq\prod_{k=1}^{n}\eta_k'^{\,w_k}$

$$\Rightarrow \left(\frac{3}{m}-\prod_{k=1}^{n}\left(\frac{3}{m}-\xi_k^{\frac{qm}{3}}\right)^{w_k}\right)^{\frac{3}{qm}} \geq \left(\frac{3}{m}-\prod_{k=1}^{n}\left(\frac{3}{m}-\xi_k'^{\frac{qm}{3}}\right)^{w_k}\right)^{\frac{3}{qm}}, \prod_{k=1}^{n}\vartheta_k^{w_k} \leq \prod_{k=1}^{n}\vartheta_k'^{w_k},$$

$$\prod_{k=1}^{n}\eta_k^{w_k} \leq \prod_{k=1}^{n}\eta_k'^{w_k}$$

$$\Rightarrow \left(\frac{3}{m}-\prod_{k=1}^{n}\left(\frac{3}{m}-\xi_k^{\frac{qm}{3}}\right)^{w_k}\right)^{\frac{3}{qm}} - \left(\frac{3}{m}-\prod_{k=1}^{n}\left(\frac{3}{m}-\xi_k'^{\frac{qm}{3}}\right)^{w_k}\right)^{\frac{3}{qm}} - \left(\prod_{k=1}^{n}\vartheta_k^{w_k} - \prod_{k=1}^{n}\vartheta_k'^{w_k}\right)$$

$$-\left(\prod_{k=1}^{n}\eta_k^{w_k} - \prod_{k=1}^{n}\eta_k'^{w_k}\right) \geq 0$$

$$\Rightarrow \frac{2+\left(\frac{3}{m}-\prod_{k=1}^{n}\left(\frac{3}{m}-\xi_k^{\frac{qm}{3}}\right)^{w_k}\right)^{\frac{3}{qm}} - \prod_{k=1}^{n}\vartheta_k^{w_k} - \prod_{k=1}^{n}\eta_k^{w_k}}{3}$$

$$-\frac{2+\left(\frac{3}{m}-\prod_{k=1}^{n}\left(\frac{3}{m}-\xi_k'^{\frac{qm}{3}}\right)^{w_k}\right)^{\frac{3}{qm}} - \prod_{k=1}^{n}\vartheta_k'^{w_k} - \prod_{k=1}^{n}\eta_k'^{w_k}}{3} \geq 0$$

$$\Rightarrow S(mGqNWAA(\psi_1,\psi_2,\psi_3,........,\psi_n)) \geq S(mGqNWAA(\psi_1',\psi_2',\psi_3',........,\psi_n'))$$

Hence by definition of score value and ranking method, we have, $mGqNWAA(\psi_1,\psi_2,\psi_3,........,\psi_n) \geq mGqNWAA(\psi_1',\psi_2',\psi_3',........,\psi_n')$.

4.6 Theorem: *(Boundedness)* Suppose $\psi_k = <\xi_k, \vartheta_k, \eta_k> (k = 1,2,3,.......,n)$ be a collection of *mGqNNs* defined on the universe set *U* and $w = (w_1, w_2, w_3, ..., w_n)^T$ is the weight vector of $(\psi_1,\psi_2,\psi_3,........,\psi_n)$ such that $w_k \geq 0 (k = 1,2,3,....,n)$ and $\sum_{k=1}^{n} w_k = 1$. Let us define two *mGqNNs* by: $\psi^- = <\min_{1\leq k\leq n}\xi_k, \max_{1\leq k\leq n}\vartheta_k, \max_{1\leq k\leq n}\eta_k>, \psi^+ = <\max_{1\leq k\leq n}\xi_k, \min_{1\leq k\leq n}\vartheta_k, \min_{1\leq k\leq n}\eta_k>$

Then, $\psi^- \leq mGqNWAA(\psi_1,\psi_2,\psi_3,........,\psi_n) \leq \psi^+$.

Proof: Suppose $mGqNWAA(\psi_1,\psi_2,\psi_3,........,\psi_n) = <\xi, \vartheta, \eta>$.

Then we have, $\frac{3}{m} - \max_{1\leq k\leq n}\xi_k^{\frac{qm}{3}} \leq \frac{3}{m} - \xi_k^{\frac{qm}{3}} \leq \frac{3}{m} - \min_{1\leq k\leq n}\xi_k^{\frac{qm}{3}}$

$$\Rightarrow \left(\frac{3}{m} - \max_{1\leq k\leq n}\xi_k^{\frac{qm}{3}}\right)^{w_k} \leq \left(\frac{3}{m} - \xi_k^{\frac{qm}{3}}\right)^{w_k} \leq \left(\frac{3}{m} - \min_{1\leq k\leq n}\xi_k^{\frac{qm}{3}}\right)^{w_k}$$

$$\Rightarrow\left(\frac{3}{m}-\prod_{k=1}^{n}\left(\frac{3}{m}-\min_{1\le k\le n}\xi_k^{\frac{qm}{3}}\right)^{w_k}\right)^{\frac{3}{qm}}\le\left(\frac{3}{m}-\prod_{k=1}^{n}\left(\frac{3}{m}-\xi_k^{\frac{qm}{3}}\right)^{w_k}\right)^{\frac{3}{qm}}\le\left(\frac{3}{m}-\prod_{k=1}^{n}\left(\frac{3}{m}-\max_{1\le k\le n}\xi_k^{\frac{qm}{3}}\right)^{w_k}\right)^{\frac{3}{qm}}$$

$$\Rightarrow\left(\frac{3}{m}-\left(\frac{3}{m}-\min_{1\le k\le n}\xi_k^{\frac{qm}{3}}\right)^{\sum_{k=1}^{n}w_k}\right)^{\frac{3}{qm}}\le\xi\le\left(\frac{3}{m}-\left(\frac{3}{m}-\max_{1\le k\le n}\xi_k^{\frac{qm}{3}}\right)^{\sum_{k=1}^{n}w_k}\right)^{\frac{3}{qm}}$$

$$\Rightarrow\min_{1\le k\le n}\xi_k\le\xi\le\max_{1\le k\le n}\xi_k$$

Again, $\prod_{k=1}^{n}(\min_{1\le k\le n}\vartheta_k)^{w_k}\le\prod_{k=1}^{n}\vartheta_k^{\,w_k}\le\prod_{k=1}^{n}(\max_{1\le k\le n}\vartheta_k)^{w_k}$

$$\Rightarrow(\min_{1\le k\le n}\vartheta_k)^{\sum_{k=1}^{n}w_k}\le\prod_{k=1}^{n}\vartheta_k^{\,w_k}\le(\max_{1\le k\le n}\vartheta_k)^{\sum_{k=1}^{n}w_k}\Rightarrow\min_{1\le k\le n}\vartheta_k\le\vartheta\le\max_{1\le k\le n}\vartheta_k$$

Similarly, we can get, $\min_{1\le k\le n}\eta_k\le\eta\le\max_{1\le k\le n}\eta_k$.

Hence $\min_{1\le k\le n}\xi_k-\max_{1\le k\le n}\vartheta_k-\max_{1\le k\le n}\eta_k\le\xi-\vartheta-\eta\le\max_{1\le k\le n}\xi_k-\min_{1\le k\le n}\vartheta_k-\min_{1\le k\le n}\eta_k$

$$\Rightarrow\frac{2+\min_{1\le k\le n}\xi_k-\max_{1\le k\le n}\vartheta_k-\max_{1\le k\le n}\eta_k}{3}\le\frac{2+\xi-\vartheta-\eta}{3}\le\frac{2+\max_{1\le k\le n}\xi_k-\min_{1\le k\le n}\vartheta_k-\min_{1\le k\le n}\eta_k}{3}$$

$$\Rightarrow S(\psi^-)\le S(mGqNWAA(\psi_1,\psi_2,\psi_3,........,\psi_n))\le S(\psi^+)$$

Therefore by definition of score value and ranking method, we have, $\psi^-\le mGqNWAA(\psi_1,\psi_2,\psi_3,........,\psi_n)\le\psi^+$.

4.7 Definition: Suppose $\psi_k=<\xi_k,\vartheta_k,\eta_k>(k=1,2,3,.......,n)$ be a collection of *mGqNNs* defined on the universe set *U*. Then a *m*-generalized *q*-neutrosophic weighted geometric aggregation operator (*mGqNWGA* for short) is given as $mGqNWGA:\Theta^n\to\Theta$ and is defined as:

$$mGqNWGA(\psi_1,\psi_2,\psi_3,........,\psi_n)=(w_1\circ\psi_1)\otimes(w_2\circ\psi_2)\otimes(w_3\circ\psi_3)\otimes.........\otimes(w_n\circ\psi_n)$$

where Θ is the collection of all *mGqNNs* defined on the universe set *U*, $w=(w_1,w_2,w_3,.......,w_n)^T$ is the weight vector of $(\psi_1,\psi_2,\psi_3,........,\psi_n)$ such that $w_k\ge 0\,(k=1,2,3,....,n)$ and $\sum_{k=1}^{n}w_k=1$.

On the basis of the operational rules of the *mGqNNs*, we can get the aggregation result as described as Theorem 4.2.

4.8 Theorem: Suppose $\psi_k=<\xi_k,\vartheta_k,\eta_k>(k=1,2,3,.......,n)$ be a collection of *mGqNNs* defined on the universe set *U* and $w=(w_1,w_2,w_3,.......,w_n)^T$ is the weight vector of $(\psi_1,\psi_2,\psi_3,........,\psi_n)$ such that $w_k\ge 0\,(k=1,2,3,....,n)$ and $\sum_{k=1}^{n}w_k=1$. Then $mGqNWGA(\psi_1,\psi_2,\psi_3,........,\psi_n)$ is also a *mGqNN*. Moreover, we have, $mGqNWGA(\psi_1,\psi_2,\psi_3,........,\psi_n)$

$$= \left\langle \prod_{k=1}^{n} \xi_k^{w_k}, \left(\frac{3}{m} - \prod_{k=1}^{n} \left(\frac{3}{m} - \vartheta_k^{\frac{qm}{3}} \right)^{w_k} \right)^{\frac{3}{qm}}, \left(\frac{3}{m} - \prod_{k=1}^{n} \left(\frac{3}{m} - \eta_k^{\frac{qm}{3}} \right)^{w_k} \right)^{\frac{3}{qm}} \right\rangle.$$

Proof: Similar to theorem 4.2.

4.9 Theorem: Suppose $\psi_k =< \xi_k, \vartheta_k, \eta_k > (k=1,2,3,.......,n)$ be a collection of *mGqNNs* defined on the universe set *U* and $w = (w_1, w_2, w_3, ..., w_n)^T$ is the weight vector of $(\psi_1, \psi_2, \psi_3, ..., \psi_n)$ such that $w_k \geq 0 (k=1,2,3,....,n)$ and $\sum_{k=1}^{n} w_k = 1$. Then for $\psi_0 =< \xi_0, \vartheta_0, \eta_0 > \in \Theta$ (where Θ is the collection of all *mGqNNs* defined on the universe set *U*), we have

$$mGqNWGA(\psi_0 \otimes \psi_1, \psi_0 \otimes \psi_2, \psi_0 \otimes \psi_3, ..., \psi_0 \otimes \psi_n) = \psi_0 \otimes mGqNWGA(\psi_1, \psi_2, \psi_3, ..., \psi_n).$$

Proof: Similar to theorem 4.3.

4.10 Theorem: *(Idempotency)* Suppose $\psi_k =< \xi_k, \vartheta_k, \eta_k > (k=1,2,3,.......,n)$ be a collection of *mGqNNs* defined on the universe set *U* and $w = (w_1, w_2, w_3, ..., w_n)^T$ is the weight vector of $(\psi_1, \psi_2, \psi_3, ..., \psi_n)$ such that $w_k \geq 0 (k=1,2,3,....,n)$ and $\sum_{k=1}^{n} w_k = 1$. If $\psi_0 =< \xi_0, \vartheta_0, \eta_0 > \in \Theta$ (where Θ is the collection of all *mGqNNs* defined on the universe set *U*) such that $\psi_k = \psi_0 \, \forall k = 1,2,3,.......,n$, then we have $mGqNWGA(\psi_1, \psi_2, \psi_3, ..., \psi_n) = \psi_0$.

Proof: Similar to theorem 4.4.

4.11 Theorem: *(Monotonocity)* Suppose $\psi_k =< \xi_k, \vartheta_k, \eta_k >$ and $\psi'_k =< \xi'_k, \vartheta'_k, \eta'_k >$ $(k=1,2,3,.......,n)$ be two collections of *mGqNNs* defined on the universe set *U* and $w = (w_1, w_2, w_3, ..., w_n)^T$ is the weight vector of $(\psi_1, \psi_2, \psi_3, ..., \psi_n)$ as well as $(\psi'_1, \psi'_2, \psi'_3, ..., \psi'_n)$ such that $w_k \geq 0 (k=1,2,3,....,n)$ and $\sum_{k=1}^{n} w_k = 1$. If $\xi_k \geq \xi'_k, \vartheta_k \leq \vartheta'_k, \eta_k \leq \eta'_k > (k=1,2,3,.......,n)$, then $mGqNWGA(\psi_1, \psi_2, \psi_3, ..., \psi_n) \geq mGqNWGA(\psi'_1, \psi'_2, \psi'_3, ..., \psi'_n)$.

Proof: Similar to theorem 4.5.

4.12 Theorem: *(Boundedness)* Suppose $\psi_k =< \xi_k, \vartheta_k, \eta_k > (k=1,2,3,.......,n)$ be a collection of *mGqNNs* defined on the universe set *U* and $w = (w_1, w_2, w_3, ..., w_n)^T$ is the weight vector of $(\psi_1, \psi_2, \psi_3, ..., \psi_n)$ such that $w_k \geq 0 (k=1,2,3,....,n)$ and $\sum_{k=1}^{n} w_k = 1$. Let us define two *mGqNNs* by: $\psi^- =< \min_{1 \leq k \leq n} \xi_k, \max_{1 \leq k \leq n} \vartheta_k, \max_{1 \leq k \leq n} \eta_k >, \psi^+ =< \max_{1 \leq k \leq n} \xi_k, \min_{1 \leq k \leq n} \vartheta_k, \min_{1 \leq k \leq n} \eta_k >$

Then $\psi^- \leq mGqNWGA(\psi_1, \psi_2, \psi_3, ..., \psi_n) \leq \psi^+$.

Proof: Similar to theorem 4.6.

5. MULTI ATTRIBUTE DECISION MAKING:

Consider a multi-attribute decision making problem which consists of *m* different alternatives $A_1, A_2, ...,$ A_l which are evaluated under the set of *n* different attributes $C_1, C_2, ..., C_n$. Assume that an expert evaluates the given alternatives $A_i\ (i=1,2,...,l)$ under the attribute $C_j\ (j=1,2,...,n)$ and the evaluation result is presented by the form of *m*-generalized *q*-neutrosophic numbers $\zeta_{ij}=\left\langle \xi_{ij},\vartheta_{ij},\delta_{ij}\right\rangle$ such that $0\le \xi_{ij},\vartheta_{ij},\delta_{ij}\le 1$ and $0\le \left(\xi_{ij}\right)^{\frac{qm}{3}}+\left(\vartheta_{ij}\right)^{\frac{qm}{3}}+\left(\delta_{ij}\right)^{\frac{qm}{3}}\le \frac{3}{m}$ where $i=1,2,...,l;\ j=1,2,...,n$. Further assume that $w_j\ (j=1,2,...,n)$ is the weight of the attribute such C_j such that $w_j>0\,(j=1,2,...,n)$ and $\sum_{j=1}^{n} w_j=1$. Then to determine the most desirable alternative (s), the proposed operators are utilized to develop a multi-attribute decision making with *m*-generalized *q*-neutrosophic information, which involves the following steps:

Step-1 Arrange the rating values of the expert in the form of decision matrix $\tilde{D}=\left(\zeta_{ij}\right)_{l\times n}=\left(\left\langle \xi_{ij},\vartheta_{ij},\delta_{ij}\right\rangle\right)_{l\times n}$.

Step-2: Construct aggregated *m*-generalized *q*-neutrosophic decision matrix. In order to do that, the proposed operators can be utilized as follows:

Let $\tilde{R}=\left(\tilde{R}_i\right)_{l\times 1}$ be the aggregated *m*-generalized *q*-neutrosophic decision matrix, where

$$\tilde{R}_i=mGqNWAA\left(\zeta_{i1},\zeta_{i2},.....,\zeta_{in}\right)=\left(w_1*\zeta_{i1}\right)\oplus\left(w_2*\zeta_{i2}\right)\oplus..........\oplus\left(w_n*\zeta_{in}\right)$$

OR

$$\tilde{R}_i=mGqNWGA\left(\zeta_{i1},\zeta_{i2},.....,\zeta_{in}\right)=\left(w_1\circ\zeta_{i1}\right)\otimes\left(w_2\circ\zeta_{i2}\right)\otimes..........\otimes\left(w_n\circ\zeta_{in}\right)$$

Step-3: Calculate the score values $S\left(\tilde{R}_i\right)\ (i=1,2,...,l)$ of *m*-generalized *q*-neutrosophic numbers $\tilde{R}_i\,(i=1,2,...,m)$.

Step-4: Rank all the alternatives $A_i\ (i=1,2,...,l)$ and hence select the most desirable alternative(s).

- **CASE STUDY**:

We consider a multi attribute decision making problem adapted from **[15, 17, 18, 19]** to demonstrate the application of the proposed decision making method.

"Suppose there is an investment company that wants to invest a sum of money in the best option available. There is a panel with four possible alternatives in which to invest the money: (i) A_1 is a car company, (ii) A_2 is a food company, (iii) A_3 is a computer company and (iv) A_4 is an arms company. The investment company must take a decision according to the following attributes:

(1) C_1 is the risk,

(2) C_2 is the growth and

(3) C_3 is the environmental impact.

The attribute weight vector is given as: $w=(0.35, 0.25, 0.40)^T$. The four alternatives $A_i\ (i=1,2,3,4)$ are to be evaluated using the *m*-generalized *q*-neutrosophic information by some decision makers or experts under the attributes $C_j\ (j=1,2,3)$**"**.

Step-1: The rating values of the expert(s) are given in the form of the following decision matrix $\tilde{D}$:

	c_1	c_2	c_3
A_1	<0.3, 0.1, 0.4>	<0.5, 0.3, 0.4>	<0.3, 0.2, 0.6>
A_2	<0.8, 0.2, 0.3>	<0.7, 0.1, 0.3>	<0.7, 0.2 0.2>
A_3	<0.5, 0.4, 0.3>	<0.6, 0.3, 0.4>	<0.5, 0.1, 0.3>
A_4	<0.6, 0.1, 0.2>	<0.7, 0.1, 0.2>	<0.3, 0.2, 0.3>

Step-2: Using the operator $mGqNWAA$, we construct the aggregated *m*-generalized *q*-neutrosophic decision matrix $\tilde{R}$ given below (taking *m*=3 and *q*=3):

A_1	<0.374405104, 0.173657007, 0.470431609>
A_2	<0.741650663, 0.168179283, 0.2550849>
A_3	<0.529784239, 0.213796854, 0.322237098>
A_4	<0.56691263, 0.131950791, 0.235215805>

Step-3: The score values of the alternatives are calculated as:

S(A_1)=0.5767, S(A_2)=0.7727, S(A_3)=0.6645, S(A_4)=0.7332

Step-4: The ranking order of the alternatives are: $A_2 > A_4 > A_3 > A_1$ which coincides with the ranking order determined by Jun Ye **[15, 17, 18, 19]** and hence the most desirable alternative is A_2.

Now if we want to utilize the $mGqNWGA$ operator instead of $mGqNWAA$ operator, then the steps for solving the multi attribute decision making problem are as follows:

Step-1: The rating values of the expert(s) are given in the form of the following decision matrix $\tilde{D}$:

	c_1	c_2	c_3
A_1	<0.3, 0.1, 0.4>	<0.5, 0.3, 0.4>	<0.3, 0.2, 0.6>
A_2	<0.8, 0.2, 0.3>	<0.7, 0.1, 0.3>	<0.7, 0.2 0.2>
A_3	<0.5, 0.4, 0.3>	<0.6, 0.3, 0.4>	<0.5, 0.1, 0.3>
A_4	<0.6, 0.1, 0.2>	<0.7, 0.1, 0.2>	<0.3, 0.2, 0.3>

Step-2: Using the operator $mGqNWGA$, we construct the aggregated *m*-generalized *q*-neutrosophic decision matrix $\tilde{R}$ given below (taking *m*=3 and *q*=3):

A_1	<0.34086581, 0.2179417, 0.504033104>
A_2	<0.73349173, 0.184246871, 0.268903022>
A_3	<0.52331757, 0.310497774, 0.331365356>
A_4	<0.472580665, 0.156129905, 0.250101937>

Step-3: The score values of the alternatives are calculated as:

S(A_1)=0.5396, S(A_2)=0.7601, S(A_3)=0.6271, S(A_4)=0.6887

Step-4: The ranking order of the alternatives are: $A_2 > A_4 > A_3 > A_1$ which also coincides with the ranking order determined by Jun Ye **[15, 17, 18, 19]** and hence the most desirable alternative is still A_2.

6. CONCLUSIONS:

In this paper, the notion of *m*-generalized *q*- neutrosophic sets is proposed and the basic properties of *m*-generalized *q*- neutrosophic numbers (*mGqNNs* for short) are presented. Also, various types of operations between the *mGqNNs* are discussed. Then, two types of *m*-generalized *q*- neutrosophic weighted aggregation operators are proposed to aggregate the *m*-generalized *q*- neutrosophic information. Furthermore, score of *a mGqNN* is proposed to ranking the *mGqNNs*. Utilizing the *m*-generalized *q*- neutrosophic weighted aggregation operators and score of *a mGqNN*, a multi attribute decision making method is developed, in which the evaluation values of alternatives on the attribute are represented in terms of *mGqNNs* and the alternatives are ranked according to the values of the score of *mGqNNs* to select the most desirable one. Finally, a practical example for investment decision making is presented to demonstrate the application and effectiveness of the proposed method. The advantage of the proposed method is that it is more suitable for solving multi attribute decision making problems because *m*-generalized *q*-neutrosophic sets (*mGqNS*s) are extensions of fuzzy sets, Pythagorean fuzzy sets, *q*-rung orthopair fuzzy sets, intuitionistic fuzzy sets, single valued neutrosophic sets, *n*-hyperspherical neutrosophic sets and single valued spherical neutrosophic sets.

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