m-polar Neutrosophic Topology and its Application to Medical Diagnosis

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Abstract

In the present study we aim to introduce novel concepts of m-polar neutrosophic set (MPNS) and m-polar neutrosophic topology. For this aim, we first investigate several characterizations of the notion of m-polar neutrosophic set and discuss its fundamental properties. We establish some operations on m-polar neutrosophic set. We propose score functions for the comparison of m-polar neutrosophic numbers (MPNNs). Then we introduce m-polar neutrosophic topology and define interior, closure, exterior and frontier for m-polar neutrosophic sets (MPNSs) with illustrative examples. We discuss some results which holds for classical set theory but do not hold for m-polar neutrosophic set theory. We introduce cosine similarity measure and set theoretic similarity measures for MPNSs. Furthermore, we present two algorithms for multi-criteria decision-making (MCDM) in medical diagnosis by using m-polar neutrosophic set (MPNS) and m-polar neutrosophic topology.

Keywords: m-polar neutrsphic set, score functions for MPNNs, m-polar neutrsphic topological space, similarity measures for MPNSs, multi-criteria decision-making for medical diagnosis.

1 Introduction and background

A number of useful mathematical tools such as fuzzy sets, m-polar fuzzy sets, neutrosophic sets and soft sets have been developed to deal with uncertainties. These theories have been found to be particularly useful in decision making under uncertainty. Multi-criteria decision-making (MCDM) is a process that explicitly evaluates best alternative(s) among the feasible options. In primitive times, decisions were framed without handling the uncertainties in the data, which may lead to inadequate results toward the real-life operating situations. If we amass the data and deduce the result without handling uncertainties, then given results will be undecided, indefinite or equivocal. Since all these facilitate the uncertainties to a great extent, they cannot withstand situations where the decision maker has to consider the falsity corresponding to the truth value ranging over an interval. MCDM is an integral part in modern management, business, medical diagnosis and many other real wold problems. Essentially, rational or sound decision is necessary for a decision

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maker. Every decision maker takes hundreds of decisions subconsciously or consciously making it as the key component of his performance. Medical diagnosis with MCDM provides solutions for the doctors to determine symptoms of disease and kind of illness.

Zadeh introduced fuzzy set [60] as a significant mathematical model to characterize and assembling of the objects whose boundary is obscure. A fuzzy set $\mathfrak{F}$ on the set of objects $\mathcal{Q}$ is a mathematical mapping $\sigma : \mathcal{Q} \to [0, 1]$. After Zadeh, many extensions of fuzzy sets have been presented and investigated such as, intuitionistic fuzzy set (IFS) [4], single valued neutrosophic set (SVNS) [38, 39], picture fuzzy set [11], bipolar fuzzy sets [62]-[64], m-polar fuzzy set (MPFS) [8], interval valued fuzzy set (IVFS) [61] and Pythagorean fuzzy set (PFS) [53]-[55]. A fuzzy neutrosophic set $\mathfrak{N}$ is defined by $\mathfrak{N} = \{\langle\varsigma, \mathfrak{A}(\varsigma), \mathfrak{S}(\varsigma), \mathfrak{Y}(\varsigma)\rangle, \varsigma \in \mathcal{Q}\}$, where $\mathfrak{A}, \mathfrak{S}, \mathfrak{Y} : \mathcal{Q} \to]^{-}0, 1^{+}[$ and $^{-}0 \leq \mathfrak{A}(\varsigma) + \mathfrak{S}(\varsigma) + \mathfrak{Y}(\varsigma) \leq 3^{+}$ The neutrosophic set yields the value from real standard or non-standard subsets of $]^{-}0, 1^{+}[$. It is difficult to utilize these values in daily life science and technology problems. Consequently, the neutrosophic set which takes the value from the subset of $[0, 1]$ is to be regarded here. An abstraction of bipolar fuzzy set was inaugurated by Chen [8] named as MPFS. An MPFS $\mathfrak{C}$ on a non-empty universal set $\mathcal{Q}$ is a mathematical function $\mathfrak{C} : \mathcal{Q} \to [0, 1]^m$, symbolized by $\mathfrak{C} = \{\langle\varsigma, P_i o\Lambda(\varsigma)\rangle : \varsigma \in \mathcal{Q}; i = 1, 2, 3, ..., m\}$ where and $P_i : [0, 1]^m \to [0, 1]$ is the i-th projection mathematical function ($i \in m$). $\mathfrak{C}_\phi(\varsigma) = (0, 0, ..., 0)$ is the smallest value in $[0, 1]^m$ and $\mathfrak{C}_{\widetilde{X}}(\varsigma) = (1, 1, ..., 1)$ is the greatest value in $[0, 1]^m$.

Multi-criteria decision making is used in solving problems that contain complex and multiple criteria. In MCDM, we have to identify the problem by determining the possible alternatives, evaluate each alternative based upon the criteria given by the decision maker or group of decision makers and lastly select the best alternative. MCDM is a very efficient tool in handling complex problems. In the problems, it is useful to find the best alternative. MCDM allows us to focus on what is easy to use, consistent and reliable. MCDM problems are applied in many disciplines, including software engineering, medical sciences, information systems, social sciences and economics. MCDM problems under fuzzy environment were first introduced by Bellman and Zadeh in (1970) [7].

In the last few decades, many mathematicians worked on similarity measures and correlation coefficients. These measures have different formulae according to the different sets and give better solution to decision-making problems. It has numerous applications in the field of pattern recognition, medical diagnosis, artificial intelligence, social sciences, business and multi-attribute decision-making problems.

Akram *et al.* [1, 2],[26] presented certain applications of m-polar fuzzy set and neutrosophic incidence fuzzy graphs in decision making problems. Ali *et al.* [3] presented various properties of soft sets and rough sets with fuzzy soft set. Garg [12] introduced new generalized Pythagorean fuzzy information aggregation using Einstein operations and its application to decision making. Garg [13] introduced generalized intuitionistic fuzzy interactive geometric interaction operators using Einstein t-norm and t-conorm and their application to decision making. Kaur and Garg [17] introduced cubic intuitionistic fuzzy aggregation operators. Kumar and Garg [18] introduced TOPSIS method based on the connection number of set pair analysis under interval-valued intuitionistic fuzzy set environment. Karaaslan [19] introduced neutrosophic Soft Set with applications in Decision Making. Xu *et al.* [47, 48, 49, 50] introduced some results on hesitant fuzzy set theory and weighted averaging operators, geometric operators and induced generalized operators based on intuitionistic fuzzy set (IFS). Jose and Kuriaskose [16] investigated aggregation operators with the corresponding score function for MCDM in the context of IFNs. Mahmood *et al.* [23] established generalized aggregation operators for CHFNs and use it into MCDM. After fuzzy topology many researchers have been introduced topologies and their properties on different hybrid structures of fuzzy set theory. In 1968, Chang [10] interpreted fuzzy topology on fuzzy set. Pao-Ming and Ying-Ming [24, 25] introduced the structure of neighborhood of fuzzy-point. They provided the concept of fuzzy quasi-coincident and Q-neighborhood. They also discussed important properties of fuzzy topology by using fuzzy Q-neighborhood. Shabir and Naz [40] established soft topological spaces. Riaz and Hashmi [28, 29, 30, 31, 32] investigated certain applications of FPFS-sets, FPFS-topology and FPFS-compact spaces. They developed fixed point theorems of FNS-mapping with its decision-making. Riaz *et al.* [33, 34] introduced soft rough topology with multi-attribute group decision making problems (MAGDM). Riaz and Tahrim [35, 36, 37] established the idea of bipolar fuzzy soft topology and cubic bipolar fuzzy ordered weighted geometric aggregation operators and their application using internal and external cubic bipolar fuzzy data. They presented various illustrations and decision-making applications of these concepts by using different algorithms.

Qurashi and Shabir [27] presented generalized approximations of $(\in, \in \vee q)$-fuzzy ideals in quantales. Feng *et al.* [14, 15] introduced properties of soft sets combined with fuzzy soft set and MADM models in the environment of generalized IF soft set and fuzzy soft set. Boran *et al.* [5] use TOPSIS decision-making method for the supplier selection in the context of IFS. Liu *et al.* [20] worked on hesitant IF linguistic operators and presented its MAGDM problem. Wei *et al.* [44] established hesitant triangular fuzzy operators in MADGDM problems. Wei *et al.* [45, 46] worked on similarity measures on picture fuzzy set and correlation coefficient to interval-valued intuitionistic fuzzy set with application in decision-making. Ye [56, 57, 58, 59] introduced prioritized aggregation operators in the context of IVHFS and worked on its MAGDM. He also established MCDM methods for interval neutrosophic sets and correlation coefficient under single-value neutrosophic environment. He established cosine similarity measures for intuitionistic fuzzy sets with application in decision-making problems. Zhang *et al.* [65] introduced aggregation operators with MCDM by using interval valued FNS (IVFNS). An extended TOPSIS method for decision-making was developed by Chi and Lui [9] on IVFNS. Zhao [66] *et al.* worked on generalized aggregation operators in the context of IFS. Zhang *et al.* [67, 68] introduced fuzzy soft β-covering based fuzzy rough sets, covering-based generalized IF rough sets and novel classes of fuzzy soft with corresponding decision-making applications. Li and Cheng [21] established new similarity measures of IFSs and its applications to pattern recognition. Lin *et al.* [22] worked on hesitant fuzzy linguistic information and its application to models of selecting an ERP system. Bhattachayra [6] worked on measure on divergence of two multinomial populations. Salton and McGill [41] introduced modern information retrieval. Singh [42] introduced correlation coefficients of picture fuzzy sets. Son [43] inaugurated a novel distributed picture fuzzy clustering method on picture fuzzy sets. Xu and Chen [51, 52] established correlation, distance and similarity measures on intuitionistic fuzzy sets.

In this era, experts believe that the world is moving towards multi-polarity. Therefore it comes as no surprise that multi-polarity in data and information plays a vital role in various fields of science and technology. In neurobiology, multipolar neurons in brain gather a great deal of information from other neurons. In information technology, multipolar technology can be exploited to operate large scale systems. The motivation of this extended and hybrid work is given step by step in the whole manuscript. We show that other hybrid structures of fuzzy sets become special cases of MPNS under some suitable conditions. We discuss about the validity, flexibility, simplicity and superiority of our proposed model and algorithms. This model is most generalized form and use to collect data at a large scale and applicable in medical, engineering, artificial intelligence, agriculture and other daily life problems. In future, this work can be gone easily for other approaches and different types of hybrid structures.

The layout of this paper is systematized as follows. Section 2, implies a novel idea of m-polar neutrosophic set (MPNS). We establish some of its operations, score function and improved score function. In section 3, we use MPNS to establish m-polar neutrosophic topological space. We define various topological structures such as interior, closure, exterior and frontier for MPNSs with the help of illustrations. We establish various results with their counter examples, which holds for classical set theory, but do not hold for m-polar neutrosophic set theory. We introduce cosine and set theoretic similarity measures for MPNSs. In section 4, we establish a method for the solution of MCDM problem based on medical diagnosis using MPNTS and MPNSs. We proposed two algorithms with linguistic information based on m-polar neutrosophic data using MPNTS and similarity measures. It is interesting to note that both algorithms yields the same result. Furthermore, we present advantages, simplicity, flexibility and validity of the proposed algorithms. We give a brief discussion and comparative analysis of our proposed approach with some existing methodologies. Finally, the conclusion of this research is summarized in section 5.

2 m-polar Neutrosophic Set (MPNS)

Chen et al. [8] have proposed the concept of m-polar fuzzy set (MPFS) in 2014, which have the capability to deal with the data having vagueness and uncertainty under multi-criteria, multi-source, multi-sensor and multipolar information. The membership grades of m-polar fuzzy sets range over the interval $[0,1]^m$, which represent m attributes of the object, but it cannot deal with the falsity and indeterminacy part of the object. Neutrosophic set (NS) deals with truth, falsity and indeterminacy for one criteria of the attribute, but cannot deal with the multi-criteria, multi-source, multi-process information fusion of the attribute. To overcome this problem, we introduce a new model of m-polar neutrosophic set (MPNS) by combining the concepts of m-polar fuzzy set (MPFS) and neutrosophic set (NS). MPNS has the ability to deal with the m attributes and to deal with the truth, falsity and indeterminacy grades for each attribute. In fact, m-polar neutrosophic set is the extension of bipolar neutrosophic set. We establish various properties and operations on m-polar neutrosophic set. We propose score functions for the comparison of m-polar neutrosophic numbers (MPNNs). In the whole manuscript, we use $\mathcal{Q}$ as a fixed sample space and Δ as an indexing set.

Definition 2.1. An object $\mathcal{M}_{\mathfrak{N}}$ on the reference set $\mathcal{Q}$ is called m-polar neutrosophic set (MPNS), if it can be expressed as

$$\mathcal{M}_{\mathfrak{N}} = \{(\varsigma, \langle \mathfrak{A}_\alpha(\varsigma), \mathfrak{S}_\alpha(\varsigma), \mathfrak{Y}_\alpha(\varsigma)\rangle) : \varsigma \in \mathcal{Q}, \alpha = 1,2,3,...,m\}$$

where $\mathfrak{A}_\alpha, \mathfrak{S}_\alpha, \mathfrak{Y}_\alpha : \mathcal{Q} \to [0,1]$ and $0 \leq \mathfrak{A}_\alpha(\varsigma) + \mathfrak{S}_\alpha(\varsigma) + \mathfrak{Y}_\alpha(\varsigma) \leq 3; \quad \alpha = 1,2,3,...,m$. This condition shows that all the three grades $\mathfrak{A}_\alpha, \mathfrak{S}_\alpha$ and $\mathfrak{Y}_\alpha; \quad (\alpha = 1,2,3,...,m)$ are independent and represents the truth, indeterminacy and falsity of the considered object or attribute for multiple criteria respectively. Simply an m-polar neutrosophic number (MPNN) can be represented as $\Im = (\langle \mathfrak{A}_\alpha, \mathfrak{S}_\alpha, \mathfrak{Y}_\alpha \rangle)$, where $0 \leq \mathfrak{A}_\alpha + \mathfrak{S}_\alpha + \mathfrak{Y}_\alpha \leq 3; \quad \alpha = 1,2,3,...,m$. In tabular form the MPNS can be represented as Table 1.

Table 1: m-polar neutrosophic set

$\mathfrak{M}_{\mathfrak{N}}$	$MPNS$					
ς_1	$(\langle\mathfrak{A}_1(\varsigma_1),\mathfrak{S}_1(\varsigma_1),\mathfrak{Y}_1(\varsigma_1)\rangle,\langle\mathfrak{A}_2(\varsigma_1),\mathfrak{S}_2(\varsigma_1),\mathfrak{Y}_2(\varsigma_1)\rangle,...,\langle\mathfrak{A}_m(\varsigma_1),\mathfrak{S}_m(\varsigma_1),\mathfrak{Y}_m(\varsigma_1)\rangle)$					
ς_2	$(\langle\mathfrak{A}_1(\varsigma_2),\mathfrak{S}_1(\varsigma_2),\mathfrak{Y}_1(\varsigma_2)\rangle,\langle\mathfrak{A}_2(\varsigma_2),\mathfrak{S}_2(\varsigma_2),\mathfrak{Y}_2(\varsigma_2)\rangle,...,\langle\mathfrak{A}_m(\varsigma_2),\mathfrak{S}_m(\varsigma_2),\mathfrak{Y}_m(\varsigma_2)\rangle)$					
...	...	...	...	...	...	...
$\varsigma_{\mathfrak{N}}$	$(\langle\mathfrak{A}_1(\varsigma_{\mathfrak{N}}),\mathfrak{S}_1(\varsigma_{\mathfrak{N}}),\mathfrak{Y}_1(\varsigma_{\mathfrak{N}})\rangle,\langle\mathfrak{A}_2(\varsigma_{\mathfrak{N}}),\mathfrak{S}_2(\varsigma_{\mathfrak{N}}),\mathfrak{Y}_2(\varsigma_{\mathfrak{N}})\rangle,...,\langle\mathfrak{A}_m(\varsigma_{\mathfrak{N}}),\mathfrak{S}_m(\varsigma_{\mathfrak{N}}),\mathfrak{Y}_m(\varsigma_{\mathfrak{N}})\rangle)$					

Example 2.2. Let $\mathcal{Q} = \{\varsigma_1, \varsigma_2, \varsigma_3\}$ be the collection of some well-known mobile phones. Then 4-polar neutrosophic set over $\mathcal{Q}$ can be written as
$\mathcal{M}_{\mathfrak{N}} = \{(\varsigma_1, \langle 0.512, 0.231, 0.321\rangle, \langle 0.653, 0.223, 0.116\rangle, \langle 0.875, 0.114, 0.243\rangle, \langle 0.961, 0.115, 0.431\rangle),$
$(\varsigma_2, \langle 0.657, 0.114, 0.226\rangle, \langle 0.765, 0.224, 0.245\rangle, \langle 0.875, 0.465, 0.213\rangle, \langle 0.961, 0.141, 0.212\rangle),$
$(\varsigma_3, \langle 0.876, 0.221, 0.321\rangle, \langle 0.657, 0.115, 0.116\rangle, \langle 0.987, 0.114, 0.322\rangle, \langle 0.675, 0.221, 0.423\rangle)\}.$
In this set multi-polarity of each attribute ς shows its characteristic or qualities according to the considered criteria such as "affordable", "long lasting battery", "extra storage" and "good camera quality". For each ς and each of its criteria, we have neutrosophic values to represent the truth, indeterminacy and falsity of corresponding attribute according to the considered criteria under the influence of expert's opinion. In the set $\mathcal{M}_{\mathfrak{N}}$ for ς_1 the first triplet $\langle 0.512, 0.231, 0.321\rangle$ shows that the mobile phone ς_1 has 51.2% truth value, 23.1% indeterminacy and 32.1% falsity value for the criteria "affordable". Similarly we can see the next values for all attributes and corresponding criteria.

There is a relationship between MPNS and other hybrid structures of fuzzy set. This relationship can be elaborated in the given flow chart diagram of Figure 1, where $\alpha = 1, 2, 3, .., m$.

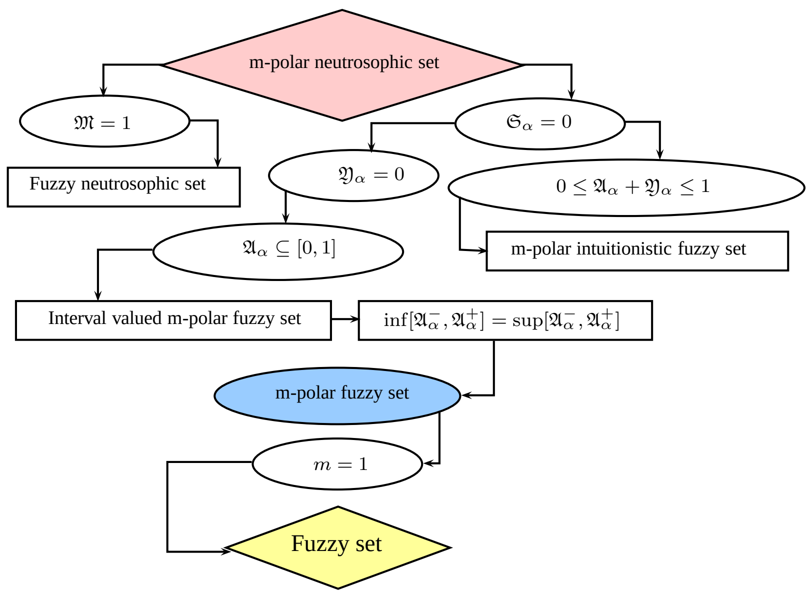

Figure 1: Relationship between MPNS and other hybrid fuzzy sets

Definition 2.3. An MPNS $\mathcal{M}_{\mathfrak{N}}$ is said to be an empty MPNS, if $\mathfrak{A}_\alpha(\varsigma) = 0, \mathfrak{S}_\alpha(\varsigma) = 1$ and $\mathfrak{Y}_\alpha(\varsigma) = 1, \forall\alpha =$

$1, 2, 3, ..., m$ and it can be scripted as

$$^{0}\mathcal{M}_{\mathfrak{N}} = \{\varsigma, (\langle 0,1,1\rangle, \langle 0,1,1\rangle, ..., \langle 0,1,1\rangle) : \varsigma \in \mathcal{Q}\}$$

and for absolute MPNS we have $\mathfrak{A}_\alpha(\varsigma) = 1, \mathfrak{S}_\alpha(\varsigma) = 0$ and $\mathfrak{Y}_\alpha(\varsigma) = 0, \forall \alpha = 1, 2, 3, ..., m$ and it can be written as

$$^{1}\mathcal{M}_{\mathfrak{N}} = \{\varsigma, (\langle 1,0,0\rangle, \langle 1,0,0\rangle, ..., \langle 1,0,0\rangle) : \varsigma \in \mathcal{Q}\}$$

The assembling of all MPNSs over $^{1}\mathcal{M}_{\mathfrak{N}}$ is represented as $\mathfrak{mpn}(^{1}\mathcal{M}_{\mathfrak{N}})$.

We define some operations for MPNSs.

Definition 2.4. Let $\mathcal{M}_{\mathfrak{N}}, \mathcal{M}_{\mathfrak{N}_\wp} \in \mathfrak{mpn}(^{1}\mathcal{M}_{\mathfrak{N}})$, where $\mathcal{M}_{\mathfrak{N}} = \{(\varsigma, \langle \mathfrak{A}_\alpha(\varsigma), \mathfrak{S}_\alpha(\varsigma), \mathfrak{Y}_\alpha(\varsigma)\rangle) : \varsigma \in \mathcal{Q}, \alpha = 1, 2, 3, ..., m\}$ and $\mathcal{M}_{\mathfrak{N}_\wp} = \{(\varsigma, \langle {}^{\wp}\mathfrak{A}_\alpha(\varsigma), {}^{\wp}\mathfrak{S}_\alpha(\varsigma), {}^{\wp}\mathfrak{Y}_\alpha(\varsigma)\rangle) : \varsigma \in \mathcal{Q}, \wp \in \Delta, \alpha = 1, 2, 3, ..., m\}$, then:
(i): $\mathcal{M}^c_{\mathfrak{N}} = \{(\varsigma, \langle \mathfrak{Y}_\alpha(\varsigma), 1 - \mathfrak{S}_\alpha(\varsigma), \mathfrak{A}_\alpha(\varsigma)\rangle) : \varsigma \in \mathcal{Q}, \alpha = 1, 2, 3, ..., m\}$
(ii): $\mathcal{M}_{\mathfrak{N}_1} = \mathcal{M}_{\mathfrak{N}_2} \Leftrightarrow \langle {}^{1}\mathfrak{A}_\alpha(\varsigma), {}^{1}\mathfrak{S}_\alpha(\varsigma), {}^{1}\mathfrak{Y}_\alpha(\varsigma)\rangle = \langle {}^{2}\mathfrak{A}_\alpha(\varsigma), {}^{2}\mathfrak{S}_\alpha(\varsigma), {}^{2}\mathfrak{Y}_\alpha(\varsigma)\rangle; \varsigma \in \mathcal{Q},\ \alpha = 1, 2, 3, ..., m$
(iii): $\mathcal{M}_{\mathfrak{N}_1} \subseteq \mathcal{M}_{\mathfrak{N}_2} \Leftrightarrow {}^{1}\mathfrak{A}_\alpha(\varsigma) \leq {}^{2}\mathfrak{A}_\alpha(\varsigma), {}^{1}\mathfrak{S}_\alpha(\varsigma) \geq {}^{2}\mathfrak{S}_\alpha(\varsigma), {}^{1}\mathfrak{Y}_\alpha(\varsigma) \geq {}^{2}\mathfrak{Y}_\alpha(\varsigma); \varsigma \in \mathcal{Q},\ \alpha = 1, 2, 3, ..., m$
(iv): $\bigcup_\wp \mathcal{M}_{\mathfrak{N}_\wp} = \{(\varsigma, \langle \sup_\wp {}^{\wp}\mathfrak{A}_\alpha(\varsigma), \inf_\wp {}^{\wp}\mathfrak{S}_\alpha(\varsigma), \inf_\wp {}^{\wp}\mathfrak{Y}_\alpha(\varsigma)\rangle); \varsigma \in \mathcal{Q}, \wp \in \Delta, \alpha = 1, 2, 3, ..., m\}$
(v): $\bigcap_\wp \mathcal{M}_{\mathfrak{N}_\wp} = \{(\varsigma, \langle \inf_\wp {}^{\wp}\mathfrak{A}_\alpha(\varsigma), \sup_\wp {}^{\wp}\mathfrak{S}_\alpha(\varsigma), \sup_\wp {}^{\wp}\mathfrak{Y}_\alpha(\varsigma)\rangle); \varsigma \in \mathcal{Q}, \wp \in \Delta, \alpha = 1, 2, 3, ..., m\}$

Example 2.5. Consider two 4-polar neutrosophic sets $\mathcal{M}_{\mathfrak{N}_1}$ and $\mathcal{M}_{\mathfrak{N}_2}$ given in tabular form as

Table 2: 4-polar neutrosophic sets

$\mathcal{Q}$	$4PNSs$
$\mathcal{M}_{\mathfrak{N}_1}$	$(\langle 0.611, 0.111, 0.251\rangle, \langle 0.821, 0.631, 0.111\rangle, \langle 0.721, 0.381, 0.591\rangle, \langle 0.211, 0.321, 0.411\rangle)$
$\mathcal{M}_{\mathfrak{N}_2}$	$(\langle 0.321, 0.621, 0.511\rangle, \langle 0.831, 0.111, 0.921\rangle, \langle 0.521, 0.431, 0.391\rangle, \langle 0.181, 0.931, 0.821\rangle)$

Now we calculate complement, union and intersection by using Definition 2.4 and results can be seen in tabular form as

Table 3: 4-polar neutrosophic sets

$\mathcal{Q}$	$4PNSs$
$\mathcal{M}^c_{\mathfrak{N}}$	$(\langle 0.251, 0.889, 0.611\rangle, \langle 0.111, 0.369, 0.821\rangle, \langle 0.591, 0.619, 0.721\rangle, \langle 0.411, 0.679, 0.211\rangle)$
$\mathcal{M}_{\mathfrak{N}_1} \cup \mathcal{M}_{\mathfrak{N}_2}$	$(\langle 0.611, 0.111, 0.251\rangle, \langle 0.831, 0.111, 0.111\rangle, \langle 0.721, 0.381, 0.391\rangle, \langle 0.211, 0.321, 0.411\rangle)$
$\mathcal{M}_{\mathfrak{N}_1} \cap \mathcal{M}_{\mathfrak{N}_2}$	$(\langle 0.321, 0.621, 0.511\rangle, \langle 0.821, 0.631, 0.921\rangle, \langle 0.521, 0.431, 0.591\rangle, \langle 0.181, 0.931, 0.821\rangle)$

Definition 2.6. If we want to do mathematical modeling with m-polar fuzzy neutrosophic numbers (MPNNs) to the decision-making problems or any application to multi-attribute decision-making, then it is necessary to rank these numbers. For this we have to define some score functions corresponding to MPNN, $\Im = (\langle \mathfrak{A}_\alpha, \mathfrak{S}_\alpha, \mathfrak{Y}_\alpha\rangle; \alpha = 1, 2, 3, ..., m)$ given as:

$$\pounds_1(\Im) = \frac{1}{2m}\left(m + \sum_{\alpha=1}^{m}(\mathfrak{A}_\alpha - 2\mathfrak{S}_\alpha - \mathfrak{Y}_\alpha)\right); \quad \pounds_1(\Im) \in [0,1]$$

$$\pounds_2(\Im) = \frac{1}{m}\sum_{\alpha=1}^{m}(\mathfrak{A}_\alpha - 2\mathfrak{S}_\alpha - \mathfrak{Y}_\alpha); \quad \pounds_2(\Im) \in [-1, 1]$$

By using above score functions there must be a possibility when score of two MPNNs will be same. For this purpose we define an improved score function for ranking of MPNNs scripted as

$$\pounds_3(\Im) = \frac{1}{2m}\left(m + \sum_{\alpha=1}^{m}\left((\mathfrak{A}_\alpha - 2\mathfrak{S}_\alpha - \mathfrak{Y}_\alpha)(2 - \mathfrak{A}_\alpha - \mathfrak{Y}_\alpha)\right)\right); \quad \pounds_3(\Im) \in [-1, 1]$$

In some cases when $\mathfrak{A}_\alpha + \mathfrak{Y}_\alpha = 1; \ \forall\ \alpha = 1, 2, ..., m$ then $\pounds_3(\Im)$ reduces to $\pounds_1(\Im)$.

Definition 2.7. Let $\Im_1$ and $\Im_2$ be two MPNNs, then by using score function we can define an order relation between these MPNNs given as:
(a): If $\pounds_1(\Im_1) \succ \pounds_1(\Im_2)$ then $\Im_1 \succ \Im_2$.
(b): If $\pounds_1(\Im_1) = \pounds_1(\Im_2)$ then
(1): If $\pounds_2(\Im_1) \succ \pounds_2(\Im_2)$ then $\Im_1 \succ \Im_2$.
(2): If $\pounds_2(\Im_1) = \pounds_2(\Im_2)$ then
(i): If $\pounds_3(\Im_1) \succ \pounds_3(\Im_2)$ then $\Im_1 \succ \Im_2$.
(ii): If $\pounds_3(\Im_1) \prec \pounds_3(\Im_2)$ then $\Im_1 \prec \Im_2$.
(iii): If $\pounds_3(\Im_1) = \pounds_3(\Im_2)$ then $\Im_1 \sim \Im_2$.

Example 2.8. Consider two 2-polar neutrosophic numbers $\Im_1$ and $\Im_2$ given in tabular form as

Table 4: 2-polar neutrosophic numbers

$\mathcal{Q}$	$2PNNs$
$\Im_1$	$(\langle 0.5, 0.3, 0.4\rangle, \langle 0.5, 0.1, 0.8\rangle)$
$\Im_2$	$(\langle 0.2, 0.3, 0.1\rangle, \langle 0.2, 0.1, 0.5\rangle)$

Then by using Definition 2.6 $\pounds_1(\Im_1) = \frac{1}{2(2)}[2 + 0.5 - 2(0.3) - 0.4 + 0.5 - 2(0.1) - 0.8] = 0.25$. Similarly, $\pounds_1(\Im_2) = 0.25$. This shows that $\pounds_1$ fails to give the ranking between both 2PNNs. Now we will use second score function $\pounds_2$. By using Definition 2.6 we obtain the score values $\pounds_2(\Im_1) = -0.5 = \pounds_2(\Im_2)$. This shows that $\pounds_2$ also fails to evaluate the ranking. Now we will use improved score function for the ranking of 2PNNs. After calculations we get $\pounds_3(\Im_1) = 0.275$ and $\pounds_3(\Im_2) = 0.125$. Hence $\pounds_3(\Im_1) \succ \pounds_3(\Im_2)$, so $\Im_1 \succ \Im_2$.

Remark. • For null MPNN ${}^0\Im$ we have $\pounds_3({}^0\Im) = -1$.
• For absolute MPNN ${}^1\Im$ we have $\pounds_3({}^1\Im) = 1$.

Proposition 2.9. *Let $\mathcal{M}_\mathfrak{N} \in \mathfrak{mpn}({}^1\mathcal{M}_\mathfrak{N})$ and ${}^0\mathcal{M}_\mathfrak{N}$ and ${}^1\mathcal{M}_\mathfrak{N}$ are null and absolute MPNSs, respectively. Then the following axioms holds:*
(i): $\mathcal{M}_\mathfrak{N} \subseteq \mathcal{M}_\mathfrak{N} \cup \mathcal{M}_\mathfrak{N}$,
(ii): $\mathcal{M}_\mathfrak{N} \cap \mathcal{M}_\mathfrak{N} \subseteq \mathcal{M}_\mathfrak{N}$,
(iii): $\mathcal{M}_\mathfrak{N} \cup {}^0\mathcal{M}_\mathfrak{N} = \mathcal{M}_\mathfrak{N}$,
(iv): $\mathcal{M}_\mathfrak{N} \cap {}^0\mathcal{M}_\mathfrak{N} = {}^0\mathcal{M}_\mathfrak{N}$,
(v): $\mathcal{M}_\mathfrak{N} \cup {}^1\mathcal{M}_\mathfrak{N} = {}^1\mathcal{M}_\mathfrak{N}$,
(vi): $\mathcal{M}_\mathfrak{N} \cap {}^1\mathcal{M}_\mathfrak{N} = \mathcal{M}_\mathfrak{N}$

Proof. The proof is obvious and can be easily done by using Definition 2.4. □

Proposition 2.10. *Let $\mathcal{M}_{\mathfrak{N}_1}, \mathcal{M}_{\mathfrak{N}_1}, \mathcal{M}_{\mathfrak{N}_3} \in \mathfrak{mpn}(^1\mathcal{M}_{\mathfrak{N}})$, then the following results holds:*
(i): $\mathcal{M}_{\mathfrak{N}_1} \cup \mathcal{M}_{\mathfrak{N}_2} = \mathcal{M}_{\mathfrak{N}_2} \cup \mathcal{M}_{\mathfrak{N}_1}$,
(ii): $\mathcal{M}_{\mathfrak{N}_1} \cap \mathcal{M}_{\mathfrak{N}_2} = \mathcal{M}_{\mathfrak{N}_2} \cap \mathcal{M}_{\mathfrak{N}_1}$,
(iii): $\mathcal{M}_{\mathfrak{N}_1} \cup (\mathcal{M}_{\mathfrak{N}_2} \cup \mathcal{M}_{\mathfrak{N}_3}) = (\mathcal{M}_{\mathfrak{N}_1} \cup \mathcal{M}_{\mathfrak{N}_2}) \cup \mathcal{M}_{\mathfrak{N}_3}$,
(iv): $\mathcal{M}_{\mathfrak{N}_1} \cap (\mathcal{M}_{\mathfrak{N}_2} \cap \mathcal{M}_{\mathfrak{N}_3}) = (\mathcal{M}_{\mathfrak{N}_1} \cap \mathcal{M}_{\mathfrak{N}_2}) \cap \mathcal{M}_{\mathfrak{N}_3}$,
(v): $(\mathcal{M}_{\mathfrak{N}_1} \cup \mathcal{M}_{\mathfrak{N}_2})^c = \mathcal{M}^c_{\mathfrak{N}_1} \cap \mathcal{M}^c_{\mathfrak{N}_2}$,
(vi): $(\mathcal{M}_{\mathfrak{N}_1} \cap \mathcal{M}_{\mathfrak{N}_2})^c = \mathcal{M}^c_{\mathfrak{N}_1} \cup \mathcal{M}^c_{\mathfrak{N}_2}$

Proof. The proof is obvious and can be easily done by using Definition 2.4. □

3 m-polar Neutrosophic Topology

In this section, we introduce the m-polar neutrosophic topology on m-polar neutrosophic set and discuss interior, closure, exterior and frontier of MPNSs with the help of illustrations. We introduce various results which holds for classical set theory but do not hold for MPN data. We present cosine similarity measure and set theoretic similarity measures to find the similarity between MPNSs.

Definition 3.1. Let $\mathcal{Q}$ be the non-empty reference set and $\mathfrak{mpn}(^1\mathcal{M}_{\mathfrak{N}})$ be the collection of all MPNSs over $\mathcal{Q}$. Then $\mathcal{T}_{\mathcal{M}_{\mathfrak{N}}}$ is the collection of MPN-subsets of $\mathfrak{mpn}(^1\mathcal{M}_{\mathfrak{N}})$ is called m-polar neutrosophic topological space (MPNTS) if it satisfies the following properties:
(i): $^0\mathcal{M}_{\mathfrak{N}}, {}^1\mathcal{M}_{\mathfrak{N}} \in \mathcal{T}_{\mathcal{M}_{\mathfrak{N}}}$.
(ii): If $(\mathcal{M}_{\mathfrak{N}})_\wp \in \mathcal{T}_{\mathcal{M}_{\mathfrak{N}}}$, $\forall \wp \in \Delta$, then $\bigcup_{\wp \in \Delta} (\mathcal{M}_{\mathfrak{N}})_\wp \in \mathcal{T}_{\mathcal{M}_{\mathfrak{N}}}$.
(iii): If $\mathcal{M}_{\mathfrak{N}_1}, \mathcal{M}_{\mathfrak{N}_2} \in \mathcal{T}_{\mathcal{M}_{\mathfrak{N}}}$, then $\mathcal{M}_{\mathfrak{N}_1} \cap \mathcal{M}_{\mathfrak{N}_2} \in \mathcal{T}_{\mathcal{M}_{\mathfrak{N}}}$.
Then the pair $(\mathcal{Q}, \mathcal{T}_{\mathcal{M}_{\mathfrak{N}}})$ is called MPNTS. The members of $\mathcal{T}_{\mathcal{M}_{\mathfrak{N}}}$ are called open MPNSs and their complements are called closed MPNSs.

Theorem 3.2. *Let $(\mathcal{Q}, \mathcal{T}_{\mathcal{M}_{\mathfrak{N}}})$ be an MPNTS. Then the following conditions are satisfied:*
(i): *$^0\mathcal{M}_{\mathfrak{N}}$ and $^1\mathcal{M}_{\mathfrak{N}}$ are open MPNSs.*
(ii): *Union of any number of open MPNSs is open.*
(iii): *Intersection of finite number of closed MPNSs is closed.*

Proof. The proof is obvious. □

Example 3.3. Let $\mathcal{Q} = \{\varsigma_1, \varsigma_2, \varsigma_3, \varsigma_4\}$ be an assembling of books. Then $\mathfrak{mpn}(^1\mathcal{M}_{\mathfrak{N}})$ be the collection of all MPNSs over $\mathcal{Q}$. We consider two 3-polar neutrosophic subsets of $\mathfrak{mpn}(^1\mathcal{M}_{\mathfrak{N}})$ given as
$\mathcal{M}_{\mathfrak{N}_1} = \{(\varsigma_1, \langle 0.871, 0.451, 0.412\rangle, \langle 0.317, 0.412, 0.321\rangle, \langle 0.187, 0.213, 0.118\rangle),$
$(\varsigma_2, \langle 0.547, 0.158, 0.413\rangle, \langle 0.518, 0.152, 0.118\rangle, \langle 0.618, 0.418, 0.321\rangle),$
$(\varsigma_3, \langle 0.618, 0.341, 0.231\rangle, \langle 0.815, 0.118, 0.527\rangle, \langle 0.511, 0.431, 0.215\rangle),$
$(\varsigma_4, \langle 0.518, 0.391, 0.812\rangle, \langle 0.815, 0.321, 0.415\rangle, \langle 0.911, 0.321, 0.512\rangle)\}$ and
$\mathcal{M}_{\mathfrak{N}_2} = \{(\varsigma_1, \langle 0.611, 0.512, 0.611\rangle, \langle 0.218, 0.531, 0.415\rangle, \langle 0.035, 0.311, 0.211\rangle),$

$(\varsigma_2, \langle 0.212, 0.218, 0.513\rangle, \langle 0.435, 0.218, 0.315\rangle, \langle 0.519, 0.511, 0.438\rangle)$,
$(\varsigma_3, \langle 0.418, 0.432, 0.321\rangle, \langle 0.639, 0.218, 0.357\rangle, \langle 0.211, 0.531, 0.316\rangle)$,
$(\varsigma_4, \langle 0.219, 0.491, 0.815\rangle, \langle 0.716, 0.421, 0.518\rangle, \langle 0.712, 0.421, 0.618\rangle)\}$.
Then clearly the collection $\mathcal{T}_{\mathcal{M}_{\mathfrak{N}}} = \{{}^0\mathcal{M}_{\mathfrak{N}}, {}^1\mathcal{M}_{\mathfrak{N}}, \mathcal{M}_{\mathfrak{N}_1}, \mathcal{M}_{\mathfrak{N}_2}\}$ is 3-polar neutrosophic topological space.

Definition 3.4. Let $(\mathcal{Q}, \mathcal{T}_{\mathcal{M}_{\mathfrak{N}}})$ and $(\mathcal{Q}, \mathcal{T}'_{\mathcal{M}_{\mathfrak{N}}})$ be two MPNTSs over $\mathcal{Q}$. If $\mathcal{T}_{\mathcal{M}_{\mathfrak{N}}} \subseteq \mathcal{T}'_{\mathcal{M}_{\mathfrak{N}}}$, then $\mathcal{T}_{\mathcal{M}_{\mathfrak{N}}}$ is courser or weaker than $\mathcal{T}'_{\mathcal{M}_{\mathfrak{N}}}$ and $\mathcal{T}'_{\mathcal{M}_{\mathfrak{N}}}$ is stronger and finer than $\mathcal{T}_{\mathcal{M}_{\mathfrak{N}}}$. Two MPNTSs are said to be comparable if $\mathcal{T}_{\mathcal{M}_{\mathfrak{N}}} \subseteq \mathcal{T}'_{\mathcal{M}_{\mathfrak{N}}}$ or $\mathcal{T}'_{\mathcal{M}_{\mathfrak{N}}} \subseteq \mathcal{T}_{\mathcal{M}_{\mathfrak{N}}}$.

Theorem 3.5. *Let $(\mathcal{Q}, \mathcal{T}_{\mathcal{M}_{\mathfrak{N}}})$ be an MPNTS. Then the following conditions are satisfied:*
(i): *${}^0\mathcal{M}_{\mathfrak{N}}$ and ${}^1\mathcal{M}_{\mathfrak{N}}$ are closed MPNSs.*
(ii): *Intersection of any number of closed MPNSs is closed.*
(iii): *Union of finite number od closed MPNSs is closed.*

Proof. **(i):** $({}^1\mathcal{M}_{\mathfrak{N}})^c = {}^0\mathcal{M}_{\mathfrak{N}}$ and $({}^0\mathcal{M}_{\mathfrak{N}})^c = {}^1\mathcal{M}_{\mathfrak{N}}$ are both open and closed MPNSs.
(ii): If $\{\mathcal{M}_{\mathfrak{N}_\alpha} : \mathcal{M}^c_{\mathfrak{N}_\alpha} \in \mathcal{T}_{\mathcal{M}_{\mathfrak{N}}}, \alpha \in \Delta\}$ is an assembling of closed MPNSs then $(\bigcap_{\alpha\in\Delta} \mathcal{M}_{\mathfrak{N}_\alpha})^c = \bigcup_{\alpha\in\Delta} \mathcal{M}^c_{\mathfrak{N}_\alpha}$ is open. This shows that $\bigcap_{\alpha\in\Delta} \mathcal{M}_{\mathfrak{N}_\alpha}$ is closed MPNS.
(iii): Since $\mathcal{M}_{\mathfrak{N}_\beta}$ is closed for $\beta = 1, 2, ..., z$, then $(\bigcup_{\beta=1}^{z} \mathcal{M}_{\mathfrak{N}_\beta})^c = \bigcap_{\beta=1}^{z} \mathcal{M}^c_{\mathfrak{N}_\beta}$ is open MPNS. Thus $\bigcup_{\beta=1}^{z} \mathcal{M}_{\mathfrak{N}_\beta}$ is closed MPNS. □

Definition 3.6. Let $(\mathcal{Q}, \mathcal{T}_{\mathcal{M}_{\mathfrak{N}}})$ be MPNTS and $\mathcal{M}_{\mathfrak{N}} \in \mathfrak{mpn}({}^1\mathcal{M}_{\mathfrak{N}})$, then interior of $\mathcal{M}_{\mathfrak{N}}$ is denoted as $\mathcal{M}^o_{\mathfrak{N}}$ and defined as the union of all open MPN subsets contained in $\mathcal{M}_{\mathfrak{N}}$. It is the greatest open MPNS contained in $\mathcal{M}_{\mathfrak{N}}$.

Example 3.7. We consider the 3-polar neutrosophic topological space constructed in Example 3.3 and let $\mathcal{M}_{\mathfrak{N}_3} \in \mathfrak{mpn}({}^1\mathcal{M}_{\mathfrak{N}})$ given as
$\mathcal{M}_{\mathfrak{N}_3} = \{(\varsigma_1, \langle 0.713, 0.412, 0.311\rangle, \langle 0.318, 0.418, 0.311\rangle, \langle 0.451, 0.211, 0.218\rangle)$,
$(\varsigma_2, \langle 0.312, 0.117, 0.418\rangle, \langle 0.513, 0.212, 0.218\rangle, \langle 0.613, 0.411, 0.438\rangle)$,
$(\varsigma_3, \langle 0.518, 0.321, 0.311\rangle, \langle 0.718, 0.118, 0.257\rangle, \langle 0.317, 0.461, 0.217\rangle)$,
$(\varsigma_1, \langle 0.319, 0.219, 0.615\rangle, \langle 0.719, 0.321, 0.418\rangle, \langle 0.811, 0.321, 0.417\rangle)\}$.
Then $\mathcal{M}^o_{\mathfrak{N}_3} = {}^o\mathcal{M}_{\mathfrak{N}} \cup \mathcal{M}_{\mathfrak{N}_2} = \mathcal{M}_{\mathfrak{N}_2}$ is open MPNS.

Theorem 3.8. *Let $(\mathcal{Q}, \mathcal{T}_{\mathcal{M}_{\mathfrak{N}}})$ be MPNTS and $\mathcal{M}_{\mathfrak{N}} \in \mathfrak{mpn}({}^1\mathcal{M}_{\mathfrak{N}})$. $\mathcal{M}_{\mathfrak{N}}$ is open MPNS iff $\mathcal{M}^o_{\mathfrak{N}} = \mathcal{M}_{\mathfrak{N}}$.*

Proof. If $\mathcal{M}_{\mathfrak{N}}$ is open MPNS then greatest open MPNS contained in $\mathcal{M}_{\mathfrak{N}}$ is itself $\mathcal{M}_{\mathfrak{N}}$. Thus $\mathcal{M}^o_{\mathfrak{N}} = \mathcal{M}_{\mathfrak{N}}$. Conversely, if $\mathcal{M}^o_{\mathfrak{N}} = \mathcal{M}_{\mathfrak{N}}$ then $\mathcal{M}^o_{\mathfrak{N}}$ is open MPNS. This implies that $\mathcal{M}_{\mathfrak{N}}$ is open MPNS. □

Theorem 3.9. *Let $(\mathcal{Q}, \mathcal{T}_{\mathcal{M}_{\mathfrak{N}}})$ be MPNTS and $\mathcal{M}_{\mathfrak{N}_1}, \mathcal{M}_{\mathfrak{N}_2} \in \mathfrak{mpn}({}^1\mathcal{M}_{\mathfrak{N}})$, then*
(i): $(\mathcal{M}^o_{\mathfrak{N}_1})^o = \mathcal{M}^o_{\mathfrak{N}_1}$,
(ii): $\mathcal{M}_{\mathfrak{N}_1} \subseteq \mathcal{M}_{\mathfrak{N}_2} \Rightarrow \mathcal{M}^o_{\mathfrak{N}_1} \subseteq \mathcal{M}^o_{\mathfrak{N}_2}$,
(iii): $(\mathcal{M}_{\mathfrak{N}_1} \cap \mathcal{M}_{\mathfrak{N}_2})^o = \mathcal{M}^o_{\mathfrak{N}_1} \cap \mathcal{M}^o_{\mathfrak{N}_2}$,
(iv): $(\mathcal{M}_{\mathfrak{N}_1} \cup \mathcal{M}_{\mathfrak{N}_2})^o \supseteq \mathcal{M}^o_{\mathfrak{N}_1} \cup \mathcal{M}^o_{\mathfrak{N}_2}$.

Proof. The proof is obvious. □

Definition 3.10. Let $(\mathcal{Q}, \mathcal{T}_{\mathcal{M}_{\mathfrak{N}}})$ be MPNTS and $\mathcal{M}_{\mathfrak{N}} \in \mathfrak{mpn}({}^1\mathcal{M}_{\mathfrak{N}})$, then closure of $\mathcal{M}_{\mathfrak{N}}$ is denoted as $\overline{\mathcal{M}}_{\mathfrak{N}}$ and defined as the intersection of all closed MPN supersets of $\mathcal{M}_{\mathfrak{N}}$. It is the smallest closed MPN superset of $\mathcal{M}_{\mathfrak{N}}$.

Example 3.11. We consider the 3-polar neutrosophic topological space constructed in Example 3.3, then closed MPNSs are given as,
${}^o\mathcal{M}^c_{\mathfrak{N}} = {}^1\mathcal{M}_{\mathfrak{N}}$, ${}^1\mathcal{M}^c_{\mathfrak{N}} = {}^o\mathcal{M}_{\mathfrak{N}}$,
$\mathcal{M}^c_{\mathfrak{N}_1} = \{(\varsigma_1, \langle 0.412, 0.549, 0.871\rangle, \langle 0.321, 0.588, 0.317\rangle, \langle 0.118, 0.787, 0.187\rangle),$
$(\varsigma_2, \langle 0.413, 0.842, 0.547\rangle, \langle 0.118, 0.848, 0.518\rangle, \langle 0.321, 0.582, 0.618\rangle),$
$(\varsigma_3, \langle 0.231, 0.659, 0.618\rangle, \langle 0.257, 0.882, 0.815\rangle, \langle 0.215, 0.569, 0.511\rangle),$
$(\varsigma_4, \langle 0.812, 0.609, 0.518\rangle, \langle 0.415, 0.679, 0.815\rangle, \langle 0.512, 0.679, 0.911\rangle)\}$ and
$\mathcal{M}^c_{\mathfrak{N}_2} = \{(\varsigma_1, \langle 0.611, 0.488, 0.611\rangle, \langle 0.415, 0.487, 0.218\rangle, \langle 0.211, 0.689, 0.035\rangle),$
$(\varsigma_2, \langle 0.513, 0.782, 0.212\rangle, \langle 0.315, 0.782, 0.435\rangle, \langle 0.438, 0.489, 0.519\rangle),$
$(\varsigma_3, \langle 0.321, 0.568, 0.418\rangle, \langle 0.357, 0.782, 0.639\rangle, \langle 0.316, 0.469, 0.211\rangle),$
$(\varsigma_4, \langle 0.815, 0.509, 0.219\rangle, \langle 0.518, 0.579, 0.716\rangle, \langle 0.618, 0.579, 0.712\rangle)\}$.
Let $\mathcal{M}_{\mathfrak{N}_4} \in \mathfrak{mpn}({}^1\mathcal{M}_{\mathfrak{N}})$ given as
$\mathcal{M}_{\mathfrak{N}_4} = \{(\varsigma_1, \langle 0.319, 0.615, 0.888\rangle, \langle 0.217, 0.618, 0.411\rangle, \langle 0.115, 0.817, 0.345\rangle),$
$(\varsigma_2, \langle 0.312, 0.888, 0.617\rangle, \langle 0.113, 0.878, 0.678\rangle, \langle 0.231, 0.598, 0.765\rangle),$
$(\varsigma_3, \langle 0.112, 0.767, 0.887\rangle, \langle 0.213, 0.889, 0.889\rangle, \langle 0.114, 0.667, 0.665\rangle),$
$(\varsigma_4, \langle 0.319, 0.768, 0.615\rangle, \langle 0.321, 0.778, 0.898\rangle, \langle 0.435, 0.767, 0.987\rangle)\}$.
Then $\overline{\mathcal{M}}_{\mathfrak{N}_4} = {}^1\mathcal{M}_{\mathfrak{N}} \cap \mathcal{M}^c_{\mathfrak{N}_1} \cap \mathcal{M}^c_{\mathfrak{N}_2} = \mathcal{M}^c_{\mathfrak{N}_1}$ is closed MPNS.

Theorem 3.12. *Let $(\mathcal{Q}, \mathcal{T}_{\mathcal{M}_{\mathfrak{N}}})$ be MPNTS and $\mathcal{M}_{\mathfrak{N}} \in \mathfrak{mpn}({}^1\mathcal{M}_{\mathfrak{N}})$. $\mathcal{M}_{\mathfrak{N}}$ is closed MPNS iff $\overline{\mathcal{M}}_{\mathfrak{N}} = \mathcal{M}_{\mathfrak{N}}$.*

Proof. The proof is obvious. □

Definition 3.13. Let $\mathcal{M}_{\mathfrak{N}}$ be an MPN-subset of $(\mathcal{Q}, \mathcal{T}_{\mathcal{M}_{\mathfrak{N}}})$, then its frontier or boundary can be represented as $F_r(\mathcal{M}_{\mathfrak{N}})$ and defined as $F_r(\mathcal{M}_{\mathfrak{N}}) = \overline{\mathcal{M}_{\mathfrak{N}}} \cap \overline{\mathcal{M}^c_{\mathfrak{N}}}$.

Definition 3.14. Let $\mathcal{M}_{\mathfrak{N}}$ be an MPN-subset of $(\mathcal{Q}, \mathcal{T}_{\mathcal{M}_{\mathfrak{N}}})$, then its exterior can be represented as $Ext(\mathcal{M}_{\mathfrak{N}})$ and defined as $Ext(\mathcal{M}_{\mathfrak{N}}) = (\overline{\mathcal{M}_{\mathfrak{N}}})^c = (\mathcal{M}^c_{\mathfrak{N}})^o$.

Example 3.15. We consider the MPNTS constructed in Example 3.3 and consider the MPNSs $\mathcal{M}_{\mathfrak{N}_3}$ and $\mathcal{M}_{\mathfrak{N}_4}$ given in Examples 3.7 and 3.11. Then by using previous definitions we can write that
$\mathcal{M}^o_{\mathfrak{N}_3} = \mathcal{M}_{\mathfrak{N}_2}$, $\overline{\mathcal{M}_{\mathfrak{N}_3}} = {}^1\mathcal{M}_{\mathfrak{N}}$,
$F_r(\mathcal{M}_{\mathfrak{N}_3}) = {}^1\mathcal{M}_{\mathfrak{N}}$, $Ext(\mathcal{M}_{\mathfrak{N}_3}) = {}^0\mathcal{M}_{\mathfrak{N}}$,
$\mathcal{M}^o_{\mathfrak{N}_4} = {}^0\mathcal{M}_{\mathfrak{N}}$, $\overline{\mathcal{M}_{\mathfrak{N}_4}} = \mathcal{M}^c_{\mathfrak{N}_1}$,
$F_r(\mathcal{M}_{\mathfrak{N}_4}) = \mathcal{M}^c_{\mathfrak{N}_1}$, $Ext(\mathcal{M}_{\mathfrak{N}_4}) = \mathcal{M}_{\mathfrak{N}_1}$.

Now we present some results which does not hold in MPNTS but hold in crisp set theory due to the law of contradiction and law of excluded middle.

Remark. **(i):** In MPNTS the members of discrete topology are infinite due to the infinite subsets of an arbitrary MPNS.
(ii): In MPNTS law of contradiction $\mathcal{M}_{\mathfrak{N}} \cap \mathcal{M}^c_{\mathfrak{N}} = {}^0\mathcal{M}_{\mathfrak{N}}$ and law of excluded middle $\mathcal{M}_{\mathfrak{N}} \cup \mathcal{M}^c_{\mathfrak{N}} = {}^1\mathcal{M}_{\mathfrak{N}}$ do

not hold in general. In Example 3.15, we can observe that $\mathcal{M}_{\mathfrak{N}_3} \cap \mathcal{M}^c_{\mathfrak{N}_3} \neq {}^0\mathcal{M}_{\mathfrak{N}}$ and $\mathcal{M}_{\mathfrak{N}_3} \cup \mathcal{M}^c_{\mathfrak{N}_3} \neq {}^1\mathcal{M}_{\mathfrak{N}}$.
(iii): In m-polar neutrosophic set theory an assembling $\mathcal{T}_{\mathcal{M}_{\mathfrak{N}}} = \{{}^0\mathcal{M}_{\mathfrak{N}}, {}^1\mathcal{M}_{\mathfrak{N}}, \mathcal{M}_{\mathfrak{N}}, \mathcal{M}^c_{\mathfrak{N}}\}$ is not an MPNTS in general. But this result hold in classical set theory. This result can be easily seen by using Example 3.15.

Theorem 3.16. *Let* $\mathcal{M}_{\mathfrak{N}} \in \mathfrak{mpn}({}^1\mathcal{M}_{\mathfrak{N}})$, *then*
(1): $(\mathcal{M}^o_{\mathfrak{N}})^c = \overline{(\mathcal{M}^c_{\mathfrak{N}})}$,
(2): $(\overline{\mathcal{M}_{\mathfrak{N}}})^c = (\mathcal{M}^c_{\mathfrak{N}})^o$,
(3): $Ext(\mathcal{M}^c_{\mathfrak{N}}) = \mathcal{M}^o_{\mathfrak{N}}$,
(4): $Ext(\mathcal{M}_{\mathfrak{N}}) = (\mathcal{M}^c_{\mathfrak{N}})^o$,
(5): $Ext(\mathcal{M}_{\mathfrak{N}}) \cup F_r(\mathcal{M}_{\mathfrak{N}}) \cup \mathcal{M}^o_{\mathfrak{N}} \neq {}^1\mathcal{M}_{\mathfrak{N}}$,
(6): $F_r(\mathcal{M}_{\mathfrak{N}}) = F_r(\mathcal{M}^c_{\mathfrak{N}})$,
(7): $\mathcal{M}^o_{\mathfrak{N}} \cap F_r(\mathcal{M}_{\mathfrak{N}}) \neq {}^0\mathcal{M}_{\mathfrak{N}}$.

Proof. **(1):** and **(2):** are obvious.
(3): $Ext(\mathcal{M}^c_{\mathfrak{N}}) = (\overline{\mathcal{M}^c_{\mathfrak{N}}})^c$
$\Rightarrow Ext(\mathcal{M}^c_{\mathfrak{N}}) = [(\mathcal{M}^c_{\mathfrak{N}})^c]^o$
$\Rightarrow Ext(\mathcal{M}^c_{\mathfrak{N}}) = \mathcal{M}^o_{\mathfrak{N}}$.
(4): $Ext(\mathcal{M}_{\mathfrak{N}}) = (\overline{\mathcal{M}_{\mathfrak{N}}})^c$
$\Rightarrow Ext(\mathcal{M}_{\mathfrak{N}}) = (\mathcal{M}^c_{\mathfrak{N}})^o$.
(5): $Ext(\mathcal{M}_{\mathfrak{N}}) \cup F_r(\mathcal{M}_{\mathfrak{N}}) \cup \mathcal{M}^o_{\mathfrak{N}} \neq {}^1\mathcal{M}_{\mathfrak{N}}$. By Example 3.15, we can see that $\mathcal{M}_{\mathfrak{N}_1} \cup \mathcal{M}^c_{\mathfrak{N}_1} \cup {}^0\mathcal{M}_{\mathfrak{N}} \neq {}^1\mathcal{M}_{\mathfrak{N}}$.
(6): $F_r(\mathcal{M}^c_{\mathfrak{N}}) = \overline{(\mathcal{M}^c_{\mathfrak{N}})} \cap \overline{[(\mathcal{M}^c_{\mathfrak{N}})]^c}$
$\Rightarrow F_r(\mathcal{M}^c_{\mathfrak{N}}) = \overline{(\mathcal{M}^c_{\mathfrak{N}})} \cap \overline{(\mathcal{M}_{\mathfrak{N}})} = F_r(\mathcal{M}_{\mathfrak{N}}$.
(7):$\mathcal{M}^o_{\mathfrak{N}} \cap F_r(\mathcal{M}_{\mathfrak{N}}) \neq {}^0\mathcal{M}_{\mathfrak{N}}$. Example 3.15 shows that $\mathcal{M}_{\mathfrak{N}_2} \cap {}^1\mathcal{M}_{\mathfrak{N}} \neq {}^0\mathcal{M}_{\mathfrak{N}}$. □

3.1 Similarity measures

In this part, we present two different formulae for similarity measures to find the similarity between MPNSs. This concept will help us in the section of multi–criteria decision-making.

Definition 3.17. (**Cosine similarity measure for MPNSs**)
We define the cosine similarity measure for m-polar neutrosophic sets based on Bhattacharyas distance [6, 41, 59]. Suppose that $\mathcal{M}_{\mathfrak{N}_1}, \mathcal{M}_{\mathfrak{N}_2} \in \mathfrak{mpn}({}^1\mathcal{M}_{\mathfrak{N}})$, over $\mathcal{Q} = \{\varsigma_1, \varsigma_2, ..., \varsigma_l\}$. A cosine similarity measure between $\mathcal{M}_{\mathfrak{N}_1}$ $\mathcal{M}_{\mathfrak{N}_2}$ is given as

$$\mathfrak{C}^1_{MPNS}(\mathcal{M}_{\mathfrak{N}_1}, \mathcal{M}_{\mathfrak{N}_2}) = \frac{1}{ml}\sum_{\eta=1}^{l}\sum_{\alpha=1}^{m}\frac{{}^1\mathfrak{A}_\alpha(\varsigma_\eta)^2\mathfrak{A}_\alpha(\varsigma_\eta)+{}^1\mathfrak{S}_\alpha(\varsigma_\eta)^2\mathfrak{S}_\alpha(\varsigma_\eta)+{}^1\mathfrak{Y}_\alpha(\varsigma_\eta)^2\mathfrak{Y}_\alpha(\varsigma_\eta)}{\sqrt{({}^1\mathfrak{A}_\alpha(\varsigma_\eta))^2+({}^1\mathfrak{S}_\alpha(\varsigma_\eta))^2+({}^1\mathfrak{Y}_\alpha(\varsigma_\eta))^2}\sqrt{({}^2\mathfrak{A}_\alpha(\varsigma_\eta))^2+({}^2\mathfrak{S}_\alpha(\varsigma_\eta))^2+({}^2\mathfrak{Y}_\alpha(\varsigma_\eta))^2}}.$$

$\mathfrak{C}^1_{MPNS}$ satisfies the following properties,
(1): $0 \leq \mathfrak{C}^1_{MPNS} \leq 1$,
(2): $\mathfrak{C}^1_{MPNS}(\mathcal{M}_{\mathfrak{N}_1}, \mathcal{M}_{\mathfrak{N}_2}) = \mathfrak{C}^1_{MPNS}(\mathcal{M}_{\mathfrak{N}_2}, \mathcal{M}_{\mathfrak{N}_1})$,
(3): $\mathfrak{C}^1_{MPNS}(\mathcal{M}_{\mathfrak{N}_1}, \mathcal{M}_{\mathfrak{N}_2}) = 1$ if $\mathcal{M}_{\mathfrak{N}_1} = \mathcal{M}_{\mathfrak{N}_2}$,
(4): If $\mathcal{M}_{\mathfrak{N}_1} \subseteq \mathcal{M}_{\mathfrak{N}_2} \subseteq \mathcal{M}_{\mathfrak{N}_3}$ then $\mathfrak{C}^1_{MPNS}(\mathcal{M}_{\mathfrak{N}_1}, \mathcal{M}_{\mathfrak{N}_3}) \leq \mathfrak{C}^1_{MPNS}(\mathcal{M}_{\mathfrak{N}_1}, \mathcal{M}_{\mathfrak{N}_2})$ and
$\mathfrak{C}^1_{MPNS}(\mathcal{M}_{\mathfrak{N}_1}, \mathcal{M}_{\mathfrak{N}_3}) \leq \mathfrak{C}^1_{MPNS}(\mathcal{M}_{\mathfrak{N}_2}, \mathcal{M}_{\mathfrak{N}_3})$. The proof of these properties can be easily done by using the above definition.

Definition 3.18. (**Set theoretic similarity measure of MPNSs**)
We define the set theoretic similarity measure for m-polar neutrosophic sets based on set theoretic viewpoint [52]. Suppose that $\mathcal{M}_{\mathfrak{N}_1}, \mathcal{M}_{\mathfrak{N}_2} \in \mathfrak{mpn}(^1\mathcal{M}_{\mathfrak{N}})$, over $\mathcal{Q} = \{\varsigma_1, \varsigma_2, ..., \varsigma_l\}$. A set theoretic similarity measure between $\mathcal{M}_{\mathfrak{N}_1}$ $\mathcal{M}_{\mathfrak{N}_2}$ is given as
$$\mathfrak{C}^2_{MPNS}(\mathcal{M}_{\mathfrak{N}_1}, \mathcal{M}_{\mathfrak{N}_2}) = \frac{1}{ml}\sum_{\eta=1}^{l}\sum_{\alpha=1}^{m}\frac{{}^1\mathfrak{A}_\alpha(\varsigma_\eta)^2\mathfrak{A}_\alpha(\varsigma_\eta)+{}^1\mathfrak{S}_\alpha(\varsigma_\eta)^2\mathfrak{S}_\alpha(\varsigma_\eta)+{}^1\mathfrak{Y}_\alpha(\varsigma_\eta)^2\mathfrak{Y}_\alpha(\varsigma_\eta)}{\max[({}^1\mathfrak{A}_\alpha(\varsigma_\eta))^2+({}^1\mathfrak{S}_\alpha(\varsigma_\eta))^2+({}^1\mathfrak{Y}_\alpha(\varsigma_\eta))^2,({}^2\mathfrak{A}_\alpha(\varsigma_\eta))^2+({}^2\mathfrak{S}_\alpha(\varsigma_\eta))^2+({}^2\mathfrak{Y}_\alpha(\varsigma_\eta))^2]}.$$

$\mathfrak{C}^2_{MPNS}$ satisfies the following properties,
(1): $0 \leq \mathfrak{C}^2_{MPNS} \leq 1$,
(2): $\mathfrak{C}^2_{MPNS}(\mathcal{M}_{\mathfrak{N}_1}, \mathcal{M}_{\mathfrak{N}_2}) = \mathfrak{C}^2_{MPNS}(\mathcal{M}_{\mathfrak{N}_2}, \mathcal{M}_{\mathfrak{N}_1})$,
(3): $\mathfrak{C}^2_{MPNS}(\mathcal{M}_{\mathfrak{N}_1}, \mathcal{M}_{\mathfrak{N}_2}) = 1$ if $\mathcal{M}_{\mathfrak{N}_1} = \mathcal{M}_{\mathfrak{N}_2}$,
(4): If $\mathcal{M}_{\mathfrak{N}_1} \subseteq \mathcal{M}_{\mathfrak{N}_2} \subseteq \mathcal{M}_{\mathfrak{N}_3}$ then $\mathfrak{C}^2_{MPNS}(\mathcal{M}_{\mathfrak{N}_1}, \mathcal{M}_{\mathfrak{N}_3}) \leq \mathfrak{C}^2_{MPNS}(\mathcal{M}_{\mathfrak{N}_1}, \mathcal{M}_{\mathfrak{N}_2})$ and
$\mathfrak{C}^2_{MPNS}(\mathcal{M}_{\mathfrak{N}_1}, \mathcal{M}_{\mathfrak{N}_3}) \leq \mathfrak{C}^2_{MPNS}(\mathcal{M}_{\mathfrak{N}_2}, \mathcal{M}_{\mathfrak{N}_3})$. The proof of these properties can be easily done by using the above definition.

4 Multi-Criteria Decision Making for Medical Diagnosis

Multi-criteria decision making (MCDM) is a process to find an optimal alternative that has the highest degree of satisfaction from a set of feasible alternatives characterized with multiple criteria, and these kinds of MCDM problems arise in many real-world situations. In this section, we discuss an application of medical diagnosis with the help of m-polar fuzzy neutrosophic data. We present two novel algorithms to multi-criteria decision-making (MCDM) with linguistic information based on the MPNTS and MPFNSs for medical diagnosis to determine kind of illness under the experts opinion.
Proposed Technique:
In this part of our manuscript, we establish two different techniques based on MPNTS and on similarity measures to investigate the disease with m-polar neutrosophic information. The flow chart diagram of proposed algorithms can be seen in Figure 2.

Algorithm 1 (Algorithm for m-polar neutrosophic topological space)

Input:

Step 1: Input the set $\mathfrak{P}$ for a patient according to his doctor, corresponding to the "m" number of symptoms appearing to the patient. All the input data leads to those "p" diseases which will be possible outcome according to the appearing symptoms in the form of m-polar neutrosophic set.

Step 2: Input the sets $\Im_\xi; \xi = 1, 2, ..., z$, for "p" diseases $\eth_\delta; \delta = 1, 2, ..., p$, according to "z" number of experts, corresponding to the "m" number of symptoms in the form of m-polar neutrosophic sets (MPNSs).

Calculations:

Step 3: Construct m-polar neutrosophic topological space (MPNTS) $\mathcal{T}_{\mathcal{M}_\mathfrak{N}}$ using MPNSs $\Im_\xi; \xi = 1, 2, ..., z$ given by "z" number of experts.

Step 4: Find interior $\mathfrak{P}^o$ of $\mathfrak{P}$ by using Definition 3.6 under the constructed $\mathcal{T}_{\mathcal{M}_\mathfrak{N}}$. $\mathfrak{P}^o$ shows the actual condition of the patient according to the "z" number of experts and give better decision to diagnosis.

Step 5: Calculate scores of each disease corresponding to "m" number of symptoms by using Definition 2.6.

Output:

Step 6: We rank the alternative (disease) on the basis of score values according to the Definition 2.7.

Step 7: Alternative (disease) with the higher score has the maximum rank according to the given numerical example. This implies that patient is suffering from that disease.

Algorithm 2 (Algorithm for m-polar neutrosophic sets using similarity measures)

Input:

Step 1: Input the set $\mathfrak{P}$ for a patient according to his doctor, corresponding to the "m" number of symptoms appearing to the patient. All the input data leads to those "p" diseases which will be possible outcome according to the appearing symptoms in the form of m-polar neutrosophic set.

Step 2: Input the sets $\Im_\xi; \xi = 1, 2, ..., z$, for "p" diseases $\eth_\delta; \delta = 1, 2, ..., p$, according to "z" number of experts, corresponding to the "m" number of symptoms in the form of m-polar neutrosophic sets (MPNSs).

Calculations:

Step 3: calculate cosine similarity measure using Definition 3.17 between $\Im_\xi; \xi = 1, 2, ..., z$ and $\mathfrak{P}$.

Step 3': calculate set theoretic similarity measure using Definition 3.18 between $\Im_\xi; \xi = 1, 2, ..., z$ and $\mathfrak{P}$.

Step 4: Choose the MPNS from $\Im_\xi; \xi = 1, 2, ..., z$ having highest cosine similarity measure with $\mathfrak{P}$. That $\Im_\xi$ gives the best decision for diagnosis of patient.

Step 4': Choose the MPNS from $\Im_\xi; \xi = 1, 2, ..., z$ having highest set theoretic similarity measure with $\mathfrak{P}$. That $\Im_\xi$ gives the best decision for diagnosis of patient.

Step 5: Calculate scores of each disease $\eth_\delta$ of selected $\Im_\xi$ after finding cosine and set theoretic similarity measures corresponding to "m" number of symptoms by using Definition 2.6. From this method we get two different results (rankings) according to two different similarity measures.

Output:

Step 6: We rank the alternative (disease) on the basis of score values according to the Definition 2.7.

Step 7: Alternative (disease) with the higher score has the maximum rank according to the given numerical example. This implies that patient is suffering from that disease.

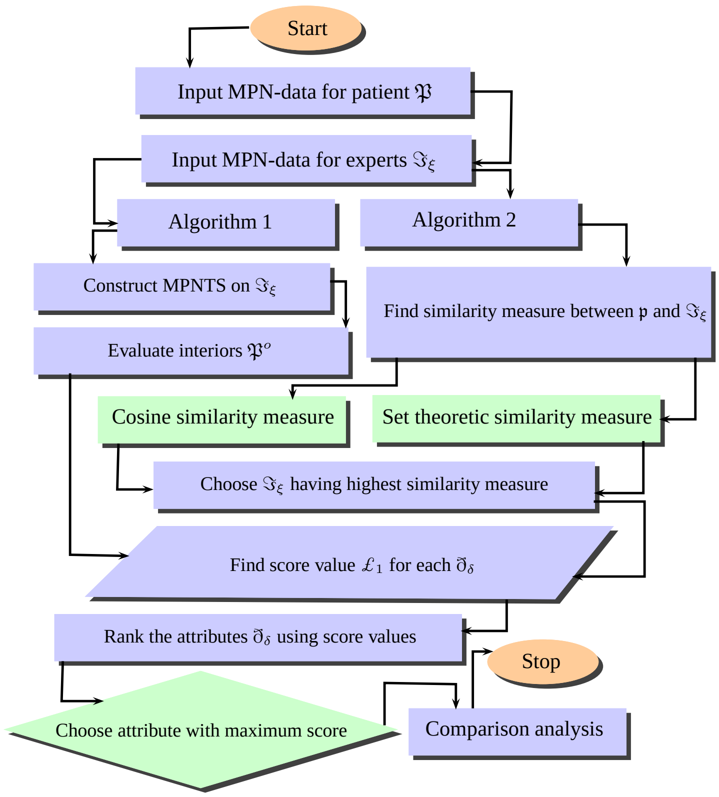

Figure 2: Flowchart diagram of proposed algorithms

4.1 Numerical Example

Suppose that a patient is facing some health issues and the symptoms are temperature, headache, fatigue, loss of appetite, stomach pain, inadequate immune system, muscle and joint pain. According to the doctor's opinion all these symptoms lead to the following diseases Tuberculosis, Hepatitis C and Typhoid fever. Let us consider the set $\mathcal{Q} = \{\eth_1, \eth_2, \eth_3\}$ of the alternatives consisting of three diseases and the set $\mathfrak{Z} = \{\mathcal{J}_1, \mathcal{J}_2, \mathcal{J}_3, \mathcal{J}_4\}$ of symptoms, where

$$\eth_1 = \text{Tuberculosis}$$
$$\eth_2 = \text{Hepatitis C}$$
$$\eth_3 = \text{Typhoid fever}$$
$$\mathcal{J}_1 = \text{Fever}$$
$$\mathcal{J}_2 = \text{Poor immune system}$$
$$\mathcal{J}_3 = \text{Muscle and joint pain, fatigue}$$
$$\mathcal{J}_4 = \text{Unintentional weight loss, loss of appetite}$$

Table 5: 4-polar neutrosophic data of patient $\mathfrak{P}$

$\mathfrak{P}$	4-polar neutrosophic sets
$\eth_1$	$(\langle 0.635, 0.115, 0.114\rangle, \langle 0.813, 0.239, 0.115\rangle, \langle 0.513, 0.431, 0.513\rangle\ \langle 0.911, 0.119, 0.238\rangle)$
$\eth_2$	$(\langle 0.739, 0.119, 0.115\rangle, \langle 0.923, 0.111, 0.108\rangle, \langle 0.889, 0.108, 0.117\rangle, \langle 0.835, 0.113, 0.218\rangle)$
$\eth_3$	$(\langle 0.919, 0.113, 0.122\rangle, \langle 0.818, 0.112, 0.211\rangle, \langle 0.611, 0.513, 0.618\rangle, \langle 0.713, 0.218, 0.319\rangle)$

Table 6: 4-polar neutrosophic data for expert $\Im_1$

$\Im_1$	4-polar neutrosophic sets
$\eth_1$	$(\langle 0.511, 0.311, 0.213\rangle, \langle 0.631, 0.431, 0.211\rangle, \langle 0.328, 0.611, 0.782\rangle\ \langle 0.713, 0.348, 0.411\rangle)$
$\eth_2$	$(\langle 0.638, 0.324, 0.237\rangle, \langle 0.816, 0.118, 0.119\rangle, \langle 0.717, 0.115, 0.218\rangle, \langle 0.719, 0.222, 0.249\rangle)$
$\eth_3$	$(\langle 0.889, 0.212, 0.213\rangle, \langle 0.699, 0.189, 0.232\rangle, \langle 0.413, 0.718, 0.818\rangle, \langle 0.518, 0.421, 0.518\rangle)$

Table 7: 4-polar neutrosophic data for expert $\Im_2$

$\Im_2$	4-polar neutrosophic sets
$\eth_1$	$(\langle 0.611, 0.213, 0.118\rangle, \langle 0.711, 0.321, 0.118\rangle, \langle 0.412, 0.511, 0.611\rangle\ \langle 0.813, 0.211, 0.341\rangle)$
$\eth_2$	$(\langle 0.718, 0.211, 0.117\rangle, \langle 0.916, 0.113, 0.112\rangle, \langle 0.817, 0.113, 0.211\rangle, \langle 0.815, 0.211, 0.234\rangle)$
$\eth_3$	$(\langle 0.918, 0.116, 0.132\rangle, \langle 0.713, 0.116, 0.213\rangle, \langle 0.511, 0.611, 0.713\rangle, \langle 0.613, 0.321, 0.416\rangle)$

Table 8: 4-polar neutrosophic data for expert $\Im_3$

$\Im_3$	4-polar neutrosophic sets
$\eth_1$	$(\langle 0.711, 0.118, 0.108\rangle, \langle 0.811, 0.213, 0.108\rangle, \langle 0.512, 0.421, 0.521\rangle\ \langle 0.815, 0.118, 0.213\rangle)$
$\eth_2$	$(\langle 0.723, 0.119, 0.111\rangle, \langle 0.928, 0.112, 0.110\rangle, \langle 0.888, 0.111, 0.119\rangle, \langle 0.889, 0.181, 0.201\rangle)$
$\eth_3$	$(\langle 0.929, 0.115, 0.128\rangle, \langle 0.813, 0.112, 0.211\rangle, \langle 0.611, 0.511, 0.613\rangle, \langle 0.718, 0.213, 0.325\rangle)$

We input the data of patient according to his doctor in the form of 4-polar neutrosophic set for each disease corresponding to every symptom. In this data the numeric values corresponding to each symptom shows that how many chances he have to be suffered from the considered disease. In Table 5 for disease $\eth_1$ =Tuberculosis, the first triplet $\langle 0.635, 0.115, 0.114\rangle$ shows that according to his "$\mathcal{J}_1$ =fever" patient has 63, 5% truth chances, 11.5% indeterminacy and 11.4% falsity chances to have tuberculosis. Similarly, we can observe all values of patient according to his symptoms for all diseases.
We consider that we have "z=3" highly qualified experts, then according to these experts the data of each disease corresponding to each symptom is given in tabular form of 4-polar neutrosophic sets as Table 6, 7 and 8. Each $\Im_\xi; \xi = 1, 2, 3$ represent the data of each disease corresponding to each symptom according to 3 experts. This means that for expert $\Im_1$ and disease $\eth_1$ =tuberculosis the first triplet $\langle 0.511, 0.311, 0.213\rangle$ shows that according to symptom "$\mathcal{J}_1$ =fever" there are 63, 5% truth chances, 11.5% indeterminacy and 11.4% falsity chances to have tuberculosis. On the same pattern, we can observe all values of diseases according to the corresponding symptoms for each expert.
Algorithm 1:
Now we construct 4-polar neutrosophic topological space $\mathcal{T}_{\mathcal{M}_\mathfrak{N}}$ on $\Im_\xi; \xi = 1, 2, 3$ given as
$\mathcal{T}_{\mathcal{M}_\mathfrak{N}} = \{\Im_1, \Im_2, \Im_3, {}^0\mathcal{M}_\mathfrak{N}, {}^1\mathcal{M}_\mathfrak{N}\}$. We find the interior $\mathfrak{P}^o$ of $\mathfrak{P}$ by using Definition 3.6 under the 4PNTS $\mathcal{T}_{\mathcal{M}_\mathfrak{N}}$. Thus $\mathfrak{P}^o = {}^0\mathcal{M}_\mathfrak{N} \cup \Im_1 \cup \Im_2 = \Im_2$. Now we use Definition 2.6 on $\Im_2$ to find scores of all the diseases

$\eth_\delta, \delta = 1, 2, 3$.
$\pounds_1(\Im_{2\eth_1}) = \frac{1}{2\times 4}(4 + (0.611 - 2(0.213) - 0.118) + (0.711 - 2(0.321) - 0.118) +$
$(0.412 - 2(0.511) - 0.611) + (0.813 - 2(0.211) - 0.341)) = 0.3558$.
Similarly we can find $\pounds_1(\Im_{2\eth_2}) = 0.662$ and $\pounds_1(\Im_{2\eth_3}) = 0.3691$. By Definition 2.7 we can write that $\eth_2 \succ \eth_3 \succ \eth_1$. Hence patient is suffering from Hepatitis C.

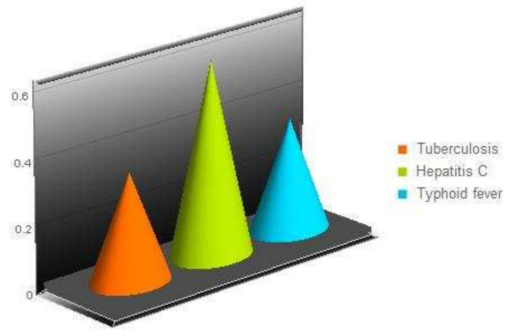

Figure 3: Ranking of attributes under MPNTS

Algorithm 2:
Now by using Tables 5, 6, 7 and 8, we find cosine similarity measures between $(\Im_1, \mathfrak{P}), (\Im_2, \mathfrak{P})$ and $(\Im_3, \mathfrak{P})$ by using Definition 3.17 given as;
$\mathfrak{C}^1_{MPNS}(\Im_2, \mathfrak{P}) = \frac{1}{3\times 4}\Big(\frac{(0.611)(0.635)+(0.213)(0.115)+(0.118)(0.114)}{\sqrt{(0.611)^2+(0.213)^2+(0.118)^2}\sqrt{(0.635)^2+(0.115)^2+(0.114)^2}} +$
$\frac{(0.711)(0.813)+(0.321)(0.329)+(0.118)(0.115)}{\sqrt{(0.711)^2+(0.321)^2+(0.118)^2}\sqrt{(0.813)^2+(0.329)^2+(0.115)^2}} + ... + \frac{(0.613)(0.713)+(0.321)(0.218)+(0.416)(0.319)}{\sqrt{(0.613)^2+(0.321)^2+(0.416)^2}\sqrt{(0.713)^2+(0.218)^2+(0.319)^2}}\Big)$.
$\mathfrak{C}^1_{MPNS}(\Im_2, \mathfrak{P}) = \frac{11.89053}{12} = 0.990878$. Similarly we can find similarity between other MPNSs given as;
$\mathfrak{C}^1_{MPNS}(\Im_1, \mathfrak{P}) = \frac{11.50807}{12} = 0.95900$, $\mathfrak{C}^1_{MPNS}(\Im_3, \mathfrak{P}) = \frac{11.996}{12} = 0.99966$. This shows that $\mathfrak{C}^1_{MPNS}(\Im_3, \mathfrak{P}) \succ \mathfrak{C}^1_{MPNS}(\Im_2, \mathfrak{P}) \succ \mathfrak{C}^1_{MPNS}(\Im_1, \mathfrak{P})$. From this ranking it is clear to see that opinion of expert $\Im_3$ is most related and similar to the condition of patient $\mathfrak{P}$. So, we select $\Im_3$ and calculate score values of all diseases $\eth_\delta; \delta = 1, 2, 3$ by using Definition 2.6. This implies that
$\pounds_1(\Im_{3\eth_1}) = 0.5198$, $\pounds_1(\Im_{3\eth_2}) = 0.7301$, $\pounds_1(\Im_{3\eth_3}) = 0.4977$. By Definition 2.7 we can write that $\eth_2 \succ \eth_1 \succ \eth_3$. Hence patient is suffering from Hepatitis C.
Now we use set theoretic similarity measure $\mathfrak{C}^2_{MPNS}$ to find similarity between $(\Im_1, \mathfrak{P}), (\Im_2, \mathfrak{P})$ and $(\Im_3, \mathfrak{P})$ by using Definition 3.18 given as;
$\mathfrak{C}^2_{MPNS}(\Im_2, \mathfrak{P}) = \frac{1}{3\times 4}\Big(\frac{(0.611)(0.635)+(0.213)(0.115)+(0.118)(0.114)}{\max((0.611)^2+(0.213)^2+(0.118)^2,(0.635)^2+(0.115)^2+(0.114)^2)} +$
$\frac{(0.711)(0.813)+(0.321)(0.329)+(0.118)(0.115)}{\max((0.711)^2+(0.321)^2+(0.118)^2,(0.813)^2+(0.329)^2+(0.115)^2)} + ... + \frac{(0.613)(0.713)+(0.321)(0.218)+(0.416)(0.319)}{\max((0.613)^2+(0.321)^2+(0.416)^2,(0.713)^2+(0.218)^2+(0.319)^2)}\Big)$.
$\mathfrak{C}^2_{MPNS}(\Im_2, \mathfrak{P}) = \frac{10.44972}{12} = 0.87081$. Similarly we can find similarity between other MPNSs given as;
$\mathfrak{C}^2_{MPNS}(\Im_1, \mathfrak{P}) = \frac{10.51971}{12} = 0.87664$, $\mathfrak{C}^2_{MPNS}(\Im_3, \mathfrak{P}) = \frac{11.2283}{12} = 0.9355$. This shows that $\mathfrak{C}^2_{MPNS}(\Im_3, \mathfrak{P}) \succ \mathfrak{C}^2_{MPNS}(\Im_1, \mathfrak{P}) \succ \mathfrak{C}^2_{MPNS}(\Im_2, \mathfrak{P})$. From this ranking it is clear to see that opinion of expert $\Im_3$ is most related and similar to the condition of patient $\mathfrak{P}$. So, we select $\Im_3$ and calculate score values of all diseases $\eth_\delta; \delta = 1, 2, 3$ by using Definition 2.6. This implies that
$\pounds_1(\Im_{3\eth_1}) = 0.5198$, $\pounds_1(\Im_{3\eth_2}) = 0.7301$, $\pounds_1(\Im_{3\eth_3}) = 0.4977$. By Definition 2.7 we can write that $\eth_2 \succ \eth_1 \succ \eth_3$. Hence patient is suffering from Hepatitis C.

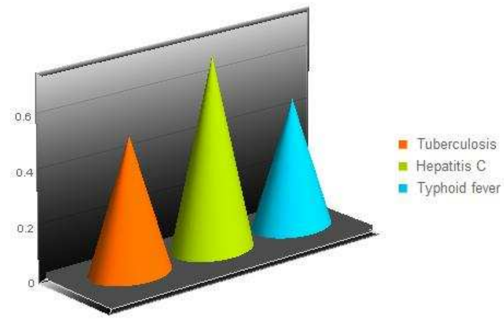

Figure 4: Ranking of attributes under similarity measures

4.2 Discussion and Comparison Analysis

In this section, we discuss advantages validity, simplicity, flexibility and superiority of our proposed approach and algorithms. We also give a brief comparison analysis of proposed method with existing approaches.

Advantages of Proposed Approach:

Now we discuss some advantages of the proposed technique based on MPNSs.

(i) Validity of the method:

The suggested method is valid and suitable for all types of input data. we present two algorithms in this manuscript one for MPNTS and other for similarity measures. We introduced two similarity measures between MPNSs. It is interesting to note that both algorithms and both formulas of similarity gives the same result (see Table 9). In this work, both algorithms have their own importance and can be used according to the requirement of decision-maker. Both algorithms are valid and give best decision in multi-criteria decision-making (MCDM) problems.

Table 9: Score values for diseases under both algorithms

Algorithm	Method	$\eth_1$	$\eth_2$	$\eth_3$	Ranking of alternatives
Algorithm1	m-polar neutrosophic topological space	0.3558	0.622	0.3691	$\eth_2 \succ \eth_3 \succ \eth_1$
Algorithm2	cosine similarity on m-polar neutrosophic sets	0.5198	0.7301	0.4977	$\eth_2 \succ \eth_1 \succ \eth_3$
Algorithm2	set theooretic similarity on m-polar neutrosophic sets	0.5198	0.7301	0.4977	$\eth_2 \succ \eth_1 \succ \eth_3$

(ii) Simplicity and flexibility dealing with different criteria:

In MCDM problems we experience different types of criteria and input data according to the given situations. The proposed algorithms are simple and easy to understand which can be applied easily on whatever type of alternatives and measures. Both algorithms are flexible and easily altered according to the different situations, inputs and outputs. There is a slightly difference between the ranking of the proposed approaches because different formulae have different ordering strategies so they can afford the slightly different effect according to their deliberations.

(iii) Superiority of proposed method:

From all above discussion, we observe that our proposed models of m-polar neutrosophic set and m-polar neutrosophic topological space are superior to previous approaches including fuzzy neutrosophic set, m-polar intuitionistic fuzzy set, interval valued m-polar fuzzy set, m-polar fuzzy set. Moreover, many hybrid structures of fuzzy set become the special cases of m-polar neutrosophic set with the addition of some suitable conditions (see Figure 1). So our proposed approach is valid, flexible, simple and superior to other hybrid structures of fuzzy set.

Comparison Analysis:

(1) In our proposed method, we define m-polar neutrosophic topological space and two algorithms based on MPN input data. The impressive point of this model is that we can use it for mathematical modeling at a large scale or "m" numbers of degrees with its truth, falsity and indeterminacy part. These m-degrees basically show the corresponding properties or any set criteria about the alternatives. As in given numerical example, we use $m = 4$ to analyze the data for four symptoms appearing to the patient. The vale of "m" can be taken as large as possible which is not possible for other approaches. Moreover, many hybrid structures of fuzzy set become the special cases of m-polar neutrosophic set with the addition of some suitable conditions (see Figure 1).

Table 10: Comparison with some existing approaches

Methods	Similarity measures on sets	Ranking of alternatives
Wei [45]	picture fuzzy set	$\eth_2 \succ \eth_1 \succ \eth_3$
Xu an Chen [51, 52]	intuitionistic fuzzy set and correlation measures	$\eth_2 \succ \eth_1 \succ \eth_3$
Ye [57]	correlation coefficient of neutrosophic set	$\eth_2 \succ \eth_1 \succ \eth_3$
Ye [59]	intuitionistic fuzzy set	$\eth_2 \succ \eth_3 \succ \eth_1$
Li and Cheng [21]	intuitionistic fuzzy set	$\eth_2 \succ \eth_3 \succ \eth_1$
Lin [22]	hesitant fuzzy linguistic information	$\eth_2 \succ \eth_1 \succ \eth_3$
Wei [46]	interval-valued intuitionistic fuzzy set	$\eth_2 \succ \eth_3 \succ \eth_1$
Proposed algorithm1	m-polar neutrosophic topological space	$\eth_2 \succ \eth_3 \succ \eth_1$
Proposed algorithm2	cosine similarity on m-polar neutrosophic sets	$\eth_2 \succ \eth_1 \succ \eth_3$
Proposed algorithm2	set theooretic similarity on m-polar neutrosophic sets	$\eth_2 \succ \eth_1 \succ \eth_3$

(2) Table 10 as given above listing the results of the comparison in the final ranking of top 3 alternatives (diseases). As it could be observed in the comparison Table 10, the best selection made by the proposed methods is comparable with the already established methods which is expressive in itself and approves the reliability and validity of the proposed method. Now the question turns out that why we need to specify a novel algorithm based on this novel structure? There are many arguments which show that proposed operator is most suitable than other existing methods. As we know that intuitionistic fuzzy set, picture fuzzy set, fuzzy set, hesitant fuzzy set, neutrosophic set and other existing hybrid structures of fuzzy sets have some limitations and not able to present full information about the situation. But our proposed model of m-polar neutrosophic set is most suitable for MCDM methods and deals with multi-criteria having truth, indeterminacy and falsity values. Due to addition of netrosophic nature in multi-polarity these three grades goes independent to each other and give a lot of information about the multiple-criteria for the attributes.

(3) The similarity measures for other existing hybrid structures of fuzzy set becomes special cases of similarity measures of m-polar neutrosophic set. So this model is most generalized and can easily deal with the problems involving intuitionistic, neutrosophy, hesitant, picture and fuzziness. The constructed topological space on MPNS become superior than existing topological spaces and easily deals with the problems in MCDM methods.

5 Conclusion

Decision analysis has been intensively studied by numerous scholars and researchers. The techniques developed for this task mainly depend on the type of decision problem under consideration. Most of its relating issues are associated with uncertain, imprecise and multipolar information, which cannot be tackled properly through fuzzy set. To overcome this particular deficiency of fuzzy set, Chen et al. [8] have proposed the concept of m-polar fuzzy set (MPFS) in 2014, which has the capability to deal with the data having vagueness and uncertainty under multipolar information. Neutrosophic set deal with the MCDM having truth, falsity and indeterminacy grades for the corresponding attributes. In this manuscript, we have established the idea of m-polar neutrosophic set (MPNS) by combining two independent concepts of MPFS and FNS. We have established the notion of m-polar neutrosophic topology and its different structures such as, interior, closure, exterior and frontier in the context of MPNS with the help of illustrations. We have introduced various results which holds for classical set theory but do not hold for MPN data. We have presented cosine and set theoretic similarity measures to find the similarity between MPNSs. The score function and improved score function have manifested for the comparison of MPNNs. Two novel algorithms for multi-criteria decision-making (MCDM) with linguistic information have developed, which based on the MPNTS and similarity measures for medical diagnosis to determine symptoms of disease, kind of illness of the patient. Furthermore, we have presentesd advantages, simplicity, flexibility and validity of the proposed algorithms. We have discussed and compare our results with some existing methodologies. m-polar fuzzy neutrosophic set is an important mathematical model to deal with uncertainties in MCDM of the real world problems. We shall extend the proposed work to solve other MCDM of real world problems by using TOPSIS, AHP, VIKOR, ELECTRE family and PROMETHEE family.

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