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Minimal solution of fuzzy neutrosophic soft matrix

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Abstract. The aim of this article is to study the concept of unique solvability of max-min fuzzy neutrosophic soft matrix equation and strong regularity of fuzzy neutrosophic soft matrices over Fuzzy Neutrosophic Soft Algebra (FNSA). A Fuzzy Neutrosophic Soft Matrix (FNSM) is said to have Strong Linear Independent (SLI) column (or, in the case of fuzzy neutrosophic soft square matrices, to be strongly regular) if for some fuzzy neutrosophic soft vector b the system $A \otimes x = b$ has a unique solution. A necessary and sufficient condition for a linear system of equation over a FNSA to have a unique solution is formulated and the equivalent condition for FNSM to have SLI column and Strong Regular (SR) are presented . Moreover Trapezoidal algorithm for testing these properties are reviewed.

Keywords: Fuzzy neutrosophic soft set (FNSS), fuzzy neutrosophic soft matrices (FNSMs), linear system of equation, unique solvability, strong regularity algorithms.

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1. Introduction

Neutrosophic was proposed by Prof. Florentin Smarandache [9] in 1995. Neutrosophic is a new branch of Philosophy that studies the origin, nature, and scope of neutralities, as well as its interactions with different ideational spectra. This theory considers every notion or idea $< A >$ together with its opposite or negation $< Anti - A >$ and the spectrum of "neutralities" $< Neut - A >$ (i.e.notions or ideas located between the two extremes,

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supporting neither $< A >$ nor $< Anti - A >$). The $< neut - A >$ and $< Anit - A >$ ideas together are referred to as $< Non - A >$. Neutrosophy is the base of Neutrosophic logic, Neutrosophic set, Neutrosophic Probability and Statistics used in Engineering application (especially for software and information fusion), Medicine, Military, Cybernetics, and Physics. Neutrosophic Logic (NL) is a general framework for unification of many existing logics, such as fuzzy logic (especially intuitionistic fuzzy logic) and paraconsistent logic. The main idea of NL is to characterize each logical statement in a 3D Neutrosophic Space, where each dimension of the space represents the truth(T), the falsehood (F) and the indeterminancy (I) of the statement under consideration, where T,I,F are standard or non-standard real subsets of $]-0, 1+[$ without necessarily connection between them.

One of the reasons for giving Nobel Prize for the achievement done in Physics for the year 2011 is "for the discovery of the accelerating expansion of the universe through observations of distant supernovae". But according to neutrosophy , "the accelerating expansion of the universe" is debatable.

Supposing that "the expansion of the universe" is an idea $< A >$, its opposite or negation $< Anti - A >$ should be "the contraction of the universe", and the spectrum of "neutralities" $< Neut - A >$ should be "the stable or indeterminate state of the universe" (i.e. the state located between the two extremes, supporting neither expansion nor contraction). In fact, the area nearby a black hole is in the state of contraction, because the mass of black hole (or similar black hole) is immense, and it produces a very strong gravitational field, so that all matters and radiations (including the electromagnetic wave or light) will be unable to escape if they enter a critical range around the black hole. The viewpoint of "the accelerating expansion of the universe" unexpectedly turns a blind eye to the fact that partial universe (such as the area nearby a black hole) is in the state of contraction. As for "the stable or indeterminate state of the universe" it should be located at the transition area between expansion area and contraction area. Again, running the same program to the state of "the expansion of the universe", supposing that "the accelerating expansion of the universe" is an idea $< A >$, its opposite or negation $< Anti - A >$ should be "the decelerating expansion of the universe", and the spectrum of "neutralities" $< Neut - A >$ should be "the uniform expansion of the universe".

Similarly, running the same program to the state of "the contraction of the universe", it can be divided into three cases, "the accelerating contraction of the universe", "the decelerating contraction of the universe" and "the uniform contraction of the universe". To sum up, there exist seven states in the universe namely accelerating expansion, decelerating expansion, uniform expansion, accelerating contraction, decelerating contraction, uniform contraction, and stable state. In addition, according to neutrosophy, another kind of seven states are as follows: long-term expansion, short-term expansion, medium-term expansion, long-term contraction, short-term contraction, medium-term contraction, and stable state. It should be noted that, the stable state can also be divided into three cases, such us, "long-term stable state" "short-term stable state", and "medium-term stable state." Thus there exist nine states in the universe. Considering all possible situations, besides these seven or nine states, due to the limitations of human knowledge, there may also exist some unknown states. From this example we can conclude that, all of the absolute, solitary and one-sided viewpoints, are completely wrong. But with the help of neutrosophy, many of these mistakes can be avoived.

2. Literature review

The complexity of problems in Economics, Engineering, Enviromental science and Social science which cannot be solved by the well known methods of Classical Mathematics poses a great difficulty in today's practical world (as various types of uncertainites are presented in these problems). To handle this type of situations many tools have been suggested. Some of them are Probability theory, Fuzzy set theory, Rough set theory etc. The Fuzzy Mathematics, since the seminal paper [23] by Zadeh first appeared, the number of researchers who are devoted to investigating both the theoretical and practical application of fuzzy sets has increased daily. This traditional fuzzy set may sometimes be very difficult to assign the membership value for fuzzy sets. In current scenario Intuitionistic Fuzzy Set (IFS) initiated [2] by Atanassov is appropriate for such a situation. The IFS can only handle the incomplete information considering both the truth membership (simple membership) and falsity-membership (or non -membership) value. It does not handle the indeterminate and inconsistent information which exist in belief system.

In 1999 Molodtsov [14] initiated the novel concept of soft set theory which was a completely new approach for modeling uncertainty. In [15] Maji et al., initiated the concept of fuzzy soft sets with some properties regarding fuzzy soft union, intersection, complement of fuzzy soft set. Moreover in [16] Maji et al., extended soft set to intuitionistic fuzzy soft set and neutrosophic soft sets. One of the important theories of Methamatics which has vast application in Science and Engineering is the theory of matrices. More over the last three decades have seen a lot of effort given to the study of "simple system of linear equation in the form

$$A \otimes x = b \tag{1}$$

where A is a matrix "(fuzzy matrix, intuitionistic fuzzy matrix), b and x are vectors (fuzzy vector, intuitionistic fuzzy vector) of suitable dimensions. While considering fuzzy concept Sanchez [19], Higashi and Klir [11], Guo et.al, [10], and Li-jian-Xin [12], used max-min fuzzy algebra to solve (1). And it has been proved that the maximum solution is unique and minimum solution need not be unique. Further Li-Jian-Xin [12] constructed a necessary and sufficient condition for the existence of unique minimal solution of (1), when A is a square matrix. The solution set of (1) is denoted by

$$S(A,b) = \{x \in \mathcal{R}_n | A \otimes x = b\}$$

where S(A,b) represents the set of all solutions for $A \otimes x = b$. Solution method for max-min system (1) was derived by several authors [4, 10–12, 19, 22]. This paper is a special case of an ordered algebraic structure $(A, \leqslant)$, where A is a FNSM. Also every entry in A is linearly ordered. For any two entries $a, b \in A$,

$$a \oplus b = max\{a, b\},$$
$$a \otimes b = min\{a, b\}.$$

Let $\mathcal{N}$ denote the set of all FNSM such that $\mathcal{N} = (A, \leqslant, \oplus, \otimes)$. In what follows we shall denote by $\mathcal{N}_{mn}$, the set of all $m \times n$ FNSM over $\mathcal{N}$, by $\mathcal{N}_n$ the set of column n-vectors over $\mathcal{N}$. The detailed nature of $\oplus$ and $\otimes$ is provided in preliminary section. M will normally denote the set $\{1, 2, ..., m\}$. A_i stands for the i^{th} row of a FNSM A and P_n for the set of all permutations of n elements. $A \in \mathcal{N}_{mn}$ is to have (SLI) columns if

$|S(A,b)| = 1$ for some $b \in \mathcal{N}_m$. In the case $m = n$, A is called Strongly Rgular (SR). The terms, SLI and SR were introduced originally in [7] as the starting point for a theory of rank and dimension in max algebra. Rajarajeswari and Dhanalakshimi have [17] recently introduced Intuitionistic Fuzzy Soft Matrices (IFSMs) that has been an effective tool in the application of Medical diagnosis. Arockiarani and Sumathi have [18] recently investigated some new operation on FNSMs. Uma et al [21] introduced FNSMs of Type-I and Type-II. Let us restrict our further discussion in this section to FNSM equation of the form

$$A \otimes x = b \quad \text{with} \quad b = [\langle b_i^T, b_i^I, b_i^F\rangle | i \in M], \quad x = [\langle x_j^T, x_j^I, x_j^F\rangle | j \in 1,2,...,n]$$

where $A \in FNSM$.

This paper is an attempt to find unique minimal solution of (1) using minimal covering technique, by considering A as FNSM, x and b as Fuzzy neutrosophic soft vector. Aim of this paper is

(I) To find an alternative description of the solution set of system (1) and of its standand solution.
(II) To find the unique solvability of (1) in case of a rectagular matrix A.
(III) To use the notion of SLI and SR in FNSM (Fuzzy neutrosophic soft algebra) and
(IV) To derive a necessary and sufficient condition for strong regularity of FNSM in FNSA.

3. Preliminary

In this section some basic definition of NS, FNSS, FNSM, and FNSMs of type-I.

Definition 3.1 [20] A neutrosophic set A on the universe of discourse X is defined as

$$A = \Big\{\langle x, T_A(x), I_A(x), F_A(x)\rangle | \quad x \in X\Big\},$$

where $T, I, F : X \to \;]^-0, 1^+[$ and

$$^-0 \leqslant T_A(x) + I_A(x) + F_A(x) \leqslant 3^+. \qquad (2)$$

From philosophical point of view the neutrosophic set takes the value from real standard or non-standard subsets of $]^-0, 1^+[$. But in real life application especially in scientific and engineering problems it is difficult to use neutrosophic set with value from real standard or non-standard subset of $]^-0, 1^+[$. Hence we consider the neutrosophic set which takes the value from the subset of $[0,1]$. Therefore we can rewrite equation (2) as

$$0 \leqslant T_A(x) + I_A(x) + F_A(x) \leqslant 3.$$

In short an element $\widetilde{a}$ in the neutrosophic set A, can be written as $\widetilde{a} = \langle a^T, a^I, a^F\rangle$, where a^T denotes degree of truth, a^I denotes degree of indeterminacy and a^F denotes degree of falsity such that $0 \leqslant a^T + a^I + a^F \leqslant 3$.

***Example* 3.2** Assume that the universe of discourse $X = \{x_1,\ x_2,\ x_3\}$ where $x_1,\ x_2$ and x_3 characterize the quality, reliability, and the price of the objects respcetively. It may be further assumed that the values of $\{x_1,\ x_2,\ x_3\}$ are in [0,1] and they are obtained from some investigations of some experts. The experts may impose their opinion in three components viz; the degree of goodness, the degree of indeterminacy and the degree of poorness to explain the characteristics of the objects. Suppose A is a Neutrosophic Set (NS) of X, such that

$$A = \Big\{\langle x_1,\ 0.4,\ 0.5,\ 0.3\rangle, \langle x_2,\ 0.7,\ 0.2,\ 0.4\rangle, \langle x_3, 0.8,\ 0.3,\ 0.4\rangle\Big\}$$

where for x_1 the degree of goodness of quality is 0.4, degree of indeterminacy of quality is 0.5 and degree of falsity of quality is 0.3 etc,.

Definition 3.3 [14] Let U be an initial universe set and E be a set of parameters. Let $P(U)$ denotes the power set of U. Consider a nonempty set A, $A \subset E$. A pair (F, A) is called a soft set over U, where F is a mapping given by $F : A \to P(U)$.

Definition 3.4 [20] Let U be the initial universe set and E be a set of parameter. Consider a non-empty set $A, A \subset E$. Let $P(U)$ denote the set of all fuzzy neutrosophic sets of U. The collection (F, A) is termed to be the fuzzy neutrosophic soft set (FNSS) over U, where F is a mapping given by $F : A \to\ P(U)$. Here after we simply consider A as FNSS over U instead of (F, A).

Definition 3.5 [1] Let $U = \{c_1,\ c_2,\ ...c_m\}$ be the universal set and E be the set of parameters given by $E = \{e_1,\ e_2,\ ...e_m\}$. Let $A \subset E$. A pair (F, A) be a $FNSS$ over U. Then the subset of $U \times E$ is defined by

$$R_A = \big\{(u, e);\ e \in A,\ u \in F_A(e)\big\}$$

which is called a relation form of $(F_A,\ E)$. The membership function, indeterminacy membership function and non membership function are written by

$$\begin{aligned} T_{R_A} &: U \times E \to [0, 1], \\ I_{R_A} &: U \times E \to [0, 1] \\ F_{R_A} &: U \times E \to [0, 1] \end{aligned}$$

where $T_{R_A}(u, e) \in [0, 1]$, $I_{R_A}(u, e) \in [0, 1]$ and $F_{R_A}(u, e) \in [0, 1]$ are the membership value, indeterminacy value and non membership value respectively of $u \in U$ for each $e \in E$. If $[(T_{ij},\ I_{ij},\ F_{ij})] = [T_{ij}(u_i, e_j),\ I_{ij}(u_i, e_j)\ , F_{ij}(u_i, e_j)]$, we define a matrix

$$[\langle T_{ij}\ , I_{ij},\ F_{ij}\rangle]_{m\times n} = \begin{bmatrix} \langle T_{11},\ I_{11},\ F_{11}\rangle & \cdots & \langle T_{1n},\ I_{1n},\ F_{1n}\rangle \\ \langle T_{21},\ I_{21},\ F_{21}\rangle & \cdots & \langle T_{2n},\ I_{2n},\ F_{2n}\rangle \\ \vdots & \vdots & \vdots \\ \langle T_{m1},\ I_{m1},\ F_{m1}\rangle & \cdots & \langle T_{mn},\ I_{mn},\ F_{mn}\rangle \end{bmatrix},$$

which is called an $m \times n$ $FNSM$ of the FNSS (F_A, E) over U.

FNSMs of Type-I

Definition 3.6 [21] Let $A = (\langle a^T_{ij},\ a^I_{ij},\ a^F_{ij}\rangle),\ B = \langle(b^T_{ij},\ b^I_{ij},\ b^F_{ij}\rangle) \in \mathcal{N}_{m\times n}$. The component wise addition and component wise multiplication is defined as

$$A \oplus B = \big(\sup\{a_{ij}^T,\ b_{ij}^T\},\quad \sup\{a_{ij}^I,\ b_{ij}^I\},\quad \inf\{a_{ij}^F,\ b_{ij}^F\}\big)$$
$$A \otimes B = \big(\inf\{a_{ij}^T,\ b_{ij}^T\},\quad \inf\{a_{ij}^I,\ b_{ij}^I\},\quad \sup\{a_{ij}^F,\ b_{ij}^F\}\big)$$

Definition 3.7 Let $A \in \mathcal{N}_{m\times n},\ B \in \mathcal{N}_{n\times p}$, the composition of A and B is defined as

$$A \circ B = \left(\sum_{k=1}^{n}(a_{ik}^T \wedge\ b_{kj}^T),\quad \sum_{k=1}^{n}(a_{ik}^I \wedge\ b_{kj}^I),\quad \prod_{k=1}^{n}(a_{ik}^F \vee\ b_{kj}^F)\right)$$

equivalently we can write the same as

$$A \circ B = \left(\bigvee_{k=1}^{n}(a_{ik}^T \wedge b_{kj}^T),\quad \bigvee_{k=1}^{n}(a_{ik}^I \wedge b_{kj}^I),\quad \bigwedge_{k=1}^{n}(a_{ik}^F \vee b_{kj}^F)\right).$$

The product $A \circ B$ is defined if and only if the number of columns of A is same as the number of rows of B. A and B are said to be conformable for multiplication. We shall use AB instead of $A \circ B$, where

$$\sum(a_{ik}^T \wedge\ b_{kj}^T)$$

means max-min operation and

$$\prod_{k=1}^{n}(a_{ik}^F \vee\ b_{kj}^F)$$

means min-max operation.

4. Main results

In this section, to find an alternative description of the solution set of system (1) and its standard solution.

Recall that we are studying system of the form in (1) with $A \in \mathcal{N}_{mn}$ and $b \in \mathcal{N}_m$ given. In what follows we shall always suppose that $\langle b_i^T, b_i^I, b_i^F\rangle > \langle 0,0,1\rangle$ for all $i \in M$ in system (1). To justify this assumption, we show how to get rid of zeros on the right-hand sides. Denoting by

$$M_0 = \Big\{i \in M \mid \langle b_i^T, b_i^I, b_i^F\rangle = \langle 0,0,1\rangle\Big\}.$$

Then any solution x of (1) has $\langle x_j^T, x_j^I, x_j^F\rangle = \langle 0,0,1\rangle$ for all $j \in N_0$, where

$$N_0 = \Big\{j \in N | \langle a_{ij}^T, a_{ij}^I, a_{ij}^F\rangle > \langle 0,0,1\rangle\Big\}$$

for some $i \in M_0$. Therefore it is possible to omit the equations with indices from M_0 and the columns of A with indices from N_0 and the solutions of the original and reduced systems correspond to each other by setting $\langle x_j^T, x_j^I, x_j^F\rangle = \langle 0,0,1\rangle$ for $j \in N_0$.

Now we extend the notation introduced in [4]. For each $j \in N$, let

$$M_j(A,b) = \Big\{ i \in M \mid \langle a_{ij}^T, a_{ij}^I, a_{ij}^F \rangle > \langle b_i^T, b_i^I, b_i^F \rangle \Big\},$$
$$N'(A,b) = \Big\{ j \in N \mid M_j(A,b) \neq \phi \Big\}, \tag{3}$$

where $N = \{1, 2, ..., n\}$

$$N''(A,b) = \Big\{ j \in N \mid M_j(A,b) = \phi \Big\} = N - N'(A,b),$$
$$\widetilde{M}_j(A,b) = \Big\{ i \in M \mid \langle a_{ij}^T, a_{ij}^I, a_{ij}^F \rangle = \langle b_i^T, b_i^I, b_i^F \rangle \Big\}.$$

For $j \in N'(A,b)$ then define

$$\langle \bar{x}_j^T, \bar{x}_j^I, \bar{x}_j^F \rangle = \min \Big\{ \langle b_i^T, b_i^I, b_i^F \rangle \mid i \in M_j(A,b) \Big\},$$
$$I_j(A,b) = \Big\{ i \in M_j(A,b) \mid \langle b_i^T, b_i^I, b_i^F \rangle = \langle \bar{x}_j^T, \bar{x}_j^I, \bar{x}_j^F \rangle \Big\},$$
$$K_j(A,b) = \Big\{ i \in \widetilde{M}_j(A,b) \mid \langle a_{ij}^T, a_{ij}^I, a_{ij}^F \rangle = \langle b_i^T, b_i^I, b_i^F \rangle \leqslant \langle \bar{x}_j^T, \bar{x}_j^I, \bar{x}_j^F \rangle \Big\}, \tag{4}$$
$$L_j(A,b) = I_j(A,b) \cup K_j(A,b).$$

and for $j \in N^{''}(A,b)$, let

$$\langle \bar{x}_j^T, \bar{x}_j^I, \bar{x}_j^F \rangle = \langle 1, 1, 0 \rangle, \quad L_j(A,b) = \widetilde{M}_j(A,b). \tag{5}$$

This notation will be fixed in the sequel and the specification (A,b) will be dropped if the corresponding matrix A and vector b are understood from the context. The significance of vector $\bar{x}$ is expressed in the following assertions.

Lemma 4.1 Let $A \in \mathcal{N}_{mn}$ and $b \in \mathcal{N}_m$ be given. Then

(a) If $A \otimes x = b$ for some $x \in \mathcal{N}_n$, then $x \leqslant \bar{x}$ and
(b) $A \otimes \bar{x} \leqslant b$.

Proof. If $A \otimes x = b$, then the inequality $\langle x_j^T, x_j^I, x_j^F \rangle \leqslant \langle b_i^T, b_i^I, b_i^F \rangle$ must be fulfilled for all $j \in N^{'}$ and $i \in M_j$, hence $x \leqslant \bar{x}$. The inequality $A \otimes \bar{x} \leqslant b$ can be verified by a detailed checking of all the possible cases. ■

Vector $\bar{x}$ is in a sense a standard solution of (1), since we have.

Theorem 4.2 Let $A \in \mathcal{N}_{mn}$ and $b \in \mathcal{N}_m$ be given. Then $S(A,b) \neq \phi$ if and only if $\bar{x} \in S(A,b)$.

Proof. The 'if' part is trivial. For the converse implication suppose that there exists, $x \in S(A,b)$ and $\bar{x}$ is defined by (3)-(5). The inequality $A \otimes \bar{x} \leqslant b$ has already been stated in Lemma 4.1; on the other hand, since x is a solution of (1) and $x \leqslant \bar{x}$, we have

$$A \otimes \bar{x} \geqslant A \otimes x = b.$$

Thus there is a straightforward procedure for testing the solvability of a given system: find the vector $\bar{x}$ according to (3)-(5) and check whether it is a solution of (1). If the answer is 'yes', (1) is obviously solvable, in the negative case the solvability of (1) has been disproved too. ■

***Example* 4.3** We illustrate the previous assertions by the following example:

$$A = \begin{pmatrix} \langle 0.3 \ \ 0.2 \ \ 0.4\rangle & \langle 0.8 \ \ 0.7 \ \ 0.1\rangle & \langle 0.1 \ \ 0.2 \ \ 0.3\rangle & \langle 0.4 \ \ 0.3 \ \ 0.3\rangle \\ \langle 0.7 \ \ 0.5 \ \ 0.4\rangle & \langle 0.3 \ \ 0.2 \ \ 0.7\rangle & \langle 0.1 \ \ 0.2 \ \ 0.7\rangle & \langle 0.9 \ \ 0.8 \ \ 0.4\rangle \\ \langle 0.2 \ \ 0.1 \ \ 0.4\rangle & \langle 0.3 \ \ 0.2 \ \ 0.4\rangle & \langle 0.2 \ \ 0.1 \ \ 0.4\rangle & \langle 0.2 \ \ 0.1 \ \ 0.4\rangle \\ \langle 0.4 \ \ 0.3 \ \ 0.5\rangle & \langle 0.5 \ \ 0.3 \ \ 0.4\rangle & \langle 0.3 \ \ 0.2 \ \ 0.4\rangle & \langle 0.7 \ \ 0.5 \ \ 0.3\rangle \end{pmatrix}$$

$$b = \begin{pmatrix} \langle 0.5 \ \ 0.4 \ \ 0.3\rangle \\ \langle 0.7 \ \ 0.5 \ \ 0.4\rangle \\ \langle 0.3 \ \ 0.2 \ \ 0.4\rangle \\ \langle 0.6 \ \ 0.3 \ \ 0.4\rangle \end{pmatrix}.$$

We have

$$M_1(A,b) = \{\phi\}, \quad M_2(A,b) = \{1\}, \quad M_3(A,b) = \{\phi\}, \quad M_4(A,b) = \{2,4\},$$

hence

$$N'(A,b) = \{2,4\} \quad \text{and} \quad N''(A,b) = \{1,3\}.$$

Therefore we obtain

$$\begin{aligned} \bar{x}_1 &= \langle 1 \ \ 1 \ \ 0\rangle, \\ \bar{x}_2 &= \min\{b_1\} = \langle 0.5 \ \ 0.4 \ \ 0.3\rangle, \\ \bar{x}_3 &= \langle 1 \ \ 1 \ \ 0\rangle \\ \bar{x}_4 &= \min\{b_2, b_4\} = \big\{\langle 0.7 \ \ 0.5 \ \ 0.4\rangle, \ \ \langle 0.6 \ \ 0.3 \ \ 0.4\rangle\big\} = \langle 0.6 \ \ 0.3 \ \ 0.4\rangle. \end{aligned}$$

Therefore a simple computation shows that

$$\bar{x} = (\langle 1 \ \ 1 \ \ 0\rangle \ \ \langle 0.5 \ \ 0.4 \ \ 0.3\rangle \ \ \ \langle 1 \ \ 1 \ \ 0\rangle \ \ \ \langle 0.6 \ \ 0.3 \ \ 0.4\rangle)^t$$

is a solution of system(1). Another necessary and sufficient condition for the solvability of (1) can be formulated in terms of sets $L_j(A,b)$ is provided in the following lemma.

Lemma 4.4 Let $A \in \mathcal{N}_{mn}$, $b \in \mathcal{N}_m$ be given and let M_j, $\widetilde{M_j}$, I_j, K_j, L_j and $\langle \bar{x}_j^T, \bar{x}^I, \bar{x}^F\rangle$ be as defined in (3)-(5). Then

$$\langle a_{ij}^T, a_{ij}^I, a_{ij}^F\rangle \otimes \langle \bar{x}_j^T, \bar{x}^I, \bar{x}^F\rangle \geqslant \langle b_i^T, b_i^I, b_i^F\rangle$$

if and only if $i \in L_j$.

Proof. Suppose first that $\langle a_{ij}^T, a_{ij}^I, a_{ij}^F\rangle \otimes \langle \bar{x}_j^T, \bar{x}_j^I, \bar{x}_j^F\rangle \geqslant \langle b_i^T, b_i^I, b_i^F\rangle$. This means that both

$$\langle a_{ij}^T, a_{ij}^I, a_{ij}^F\rangle \geqslant \langle b_i^T, b_i^I, b_i^F\rangle,$$
$$\langle \bar{x}_j^T, \bar{x}_j^I, \bar{x}_j^F\rangle \geqslant \langle b_i^T, b_i^I, b_i^F\rangle.$$

Now we distinguish two cases:

(i) If $\langle a_{ij}^T, a_{ij}^I, a_{ij}^F\rangle > \langle b_i^T, b_i^I, b_i^F\rangle$, then $i \in M_j$ and

$$\langle \bar{x}_j^T, \bar{x}_j^I, \bar{x}_j^F\rangle = \min\left\{b_k | k \in M_j\right\} \leqslant \langle b_i^T, b_i^I, b_i^F\rangle,$$

hence $\langle \bar{x}_j^T, \bar{x}_j^I, \bar{x}_j^F\rangle = \langle b_i^T, b_i^I, b_i^F\rangle$ and $i \in I_j$.

(ii) If $\langle a_{ij}^T, a_{ij}^I, a_{ij}^F\rangle = \langle b_i^T, b_i^I, b_i^F\rangle$, then $i \in \widetilde{M}_j$.

Now if $M_j = \phi$, then $L_j = \widetilde{M}_j$ and $i \in L_j$. On the other hand, if $M_j \neq \phi$, then the conditions

$$\langle \bar{x}_j^T, \bar{x}_j^I, \bar{x}_j^F\rangle \geqslant \langle b_i^T, b_i^I, b_i^F\rangle, \text{ and } \langle a_{ij}^T, a_{ij}^I, a_{ij}^F\rangle = \langle b_i^T, b_i^I, b_i^F\rangle$$

mean $i \in K_j$, thus in both cases, $i \in L_j$. In the proof of the converse implication suppose that $i \in L_j$. Then either $j \in N''$ or $j \in N'$. In the former case $L_j = \widetilde{M}_j$ and

$$\langle \bar{x}_j^T, \bar{x}_j^T, \bar{x}_j^T\rangle = \langle 1, 1, 0\rangle,$$
$$\langle a_{ij}^T, a_{ij}^I, a_{ij}^F\rangle = \langle b_i^T, b_i^I, b_i^F\rangle.$$

From this we have

$$\langle a_{ij}^T, a_{ij}^I, a_{ij}^F\rangle \otimes \langle \bar{x}_j^T, \bar{x}_j^I, \bar{x}_j^F\rangle = \langle b_i^T, b_i^I, b_i^F\rangle.$$

In the second case $M_j \neq \phi$ and either

(i) $i \in I_j$, hence $\langle a_{ij}^T, a_{ij}^I, a_{ij}^F\rangle > \langle b_i^T, b_i^I, b_i^F\rangle, \langle b_i^T, b_i^I, b_i^F\rangle = \langle \bar{x}_j^T, \bar{x}_j^I, \bar{x}_j^F\rangle$ or

(ii) $i \in k_j$, hence $\langle a_{ij}^T, a_{ij}^I, a_{ij}^F\rangle = \langle b_i^T, b_i^I, b_i^F\rangle \leqslant \langle \bar{x}_j^T, \bar{x}_j^I, \bar{x}_j^F\rangle$

In both case

$$\langle a_{ij}^T, a_{ij}^I, a_{ij}^F\rangle \otimes \langle \bar{x}_j^T, \bar{x}_j^I, \bar{x}_j^F\rangle \geqslant \langle b_i^T, b_i^I, b_i^F\rangle.$$

■

Theorem 4.5 Let $A \in \mathcal{N}_{mn},\ b \in \mathcal{N}_m$ be given. Then $S(A, b) \neq \phi$ iff $\bigcup\limits_{j \in N} L_j(A, b) = M$.

Proof. If $S(A, b) \neq \phi$, then

$$\langle \bar{x}^T, \bar{x}^I, \bar{x}^F\rangle = \left(\langle \bar{x}_1^T, \bar{x}_1^I, \bar{x}_1^F\rangle, \cdots, \langle \bar{x}_n^T, \bar{x}_n^I, \bar{x}_n^F\rangle\right)^t$$

defined by (4) and (5) is a solution of (1). If we suppose that there exists

$$i \in M - \bigcup_{j \in N} L_j(A, b),$$

then according to Lemma 4.4,

$$\langle a_{ij}^T, a_{ij}^I, a_{ij}^F\rangle \otimes \langle \bar{x}_j^T, \bar{x}_j^I, \bar{x}_j^F\rangle < \langle b_i^T, b_i^I, b_i^F\rangle$$

for all $j \in N$, a controdiction. For the if part recall that $A \otimes \bar{x} \leqslant b$ and suppose that

$$\bigcup_{k \in N} L_k(A, b) = M.$$

This means that for each $i \in M$ there exist $j \in N$ such that $i \in L_j$. Again Lemma 4.4 gives that $\langle a_{ij}^T, a_{ij}^I, a_{ij}^F\rangle \otimes \langle \bar{x}_j^T, \bar{x}_j^I, \bar{x}_j^F\rangle \geqslant \langle b_i^T, b_i^I, b_i^F\rangle$ for these indices, hence $(A \otimes \bar{x})_i \geqslant b_i$ and $\bar{x} \in S(A, b)$. ■

***Example* 4.6** We compute further the following in Example 4.3

$$\begin{aligned}
&\widetilde{M}_1 = \{2\}, \widetilde{M}_2 = \{3\}, \widetilde{M}_3 = \{\phi\}, \widetilde{M}_4 = \{\phi\}\\
&I_1 = \{\phi\}, I_2 = \{1\}, I_3 = \{\phi\}, I_4 = \{4\}\\
&K_1 = \{2\}, k_2 = \{3\}, k_3 = \{\phi\}, k_4 = \{\phi\}\\
&L_j(A, b) = I_j(A, b) \cup K_j(A, b)\\
&L_1 = \{2\}, L_2 = \{1, 3\}, L_3 = \{\phi\}, L_4 = \{4\}.
\end{aligned}$$

and check that for this system we really have $\bigcup_{j \in N} L_j = M$.

5. Unique solvability

In this previous section, we studied system (1), if solvable, always has a maximum solution, namely $\bar{x}$. In this section, the following theorem states a sufficient condition for (1) to have a minimum solution.

Theorem 5.1 Let $A \in \mathcal{N}_{mn}$, $b \in \mathcal{N}_m$ be given such that $\{L_1, ..., L_n\}$ is a minimal covering of M. Then $A \otimes x = b$ has a minimum solution.

Proof. If $\{L_1, ..., L_n\}$ is a minimal covering of M, then Theorem 4.5 implies that $S(A, b) \neq \phi$ and we proceed to find the minimum element of $S(A, b)$. Hence minimality of the covering means

$$(\forall j \in N)(\exists i \in M) i \in L_j - \bigcup_{k \neq j} L_k. \tag{6}$$

In other words, using Lemma 4.4.

$$\begin{aligned}
&(\forall j \in N)(\exists i \in M)[\langle a_{ij}^T, a_{ij}^I, a_{ij}^F\rangle \otimes \langle \bar{x}_j^T, \bar{x}_j^I, \bar{x}_j^F\rangle \geqslant \langle b_i^T, b_i^I, b_i^F\rangle\\
&(\forall k \neq j)(\langle a_{ik}^T, a_{ik}^I, a_{ik}^F\rangle \otimes \langle \bar{x}_k^T, \bar{x}_k^I, \bar{x}_k^F\rangle < \langle b_i^T, b_i^I, b_i^F\rangle)]
\end{aligned} \tag{7}$$

The inequality

$$\langle a_{ik}^T, a_{ik}^I, a_{ik}^F\rangle \otimes \langle \bar{x}_k^T, \bar{x}_k^I, \bar{x}_k^F\rangle < \langle b_i^T, b_i^I, b_i^F\rangle$$

yields that the same holds for any solution x. Thus if x is a solution of (1), we have

$$\forall j \in N \;\; \exists i \in M \;\; \langle a_{ij}^T, a_{ij}^I, a_{ij}^F \rangle \otimes \langle \bar{x}_j^T, \bar{x}_j^I, \bar{x}_j^F \rangle \geqslant \langle b_i^T, b_i^I, b_i^F \rangle,$$
$$\forall k \neq j \;\; \langle a_{ik}^T, a_{ik}^I, a_{ik}^F \rangle \otimes \langle x_k^T, x_k^I, x_k^F \rangle < \langle b_i^T, b_i^I, b_i^F \rangle$$

Denote for each $j \in N$ by $N_j(A, b)$ the set of all indices i fulfilling (6). Thus we have $N_j(A, b) \neq \phi$ for each $j \in N$ and for any solution x

$$\forall j \in N \;\; \forall i \in N_j(A, b) \;\; \langle a_{ij}^T, a_{ij}^I, a_{ij}^F \rangle \otimes \langle x_j^T, x_j^I, x_j^F \rangle \geqslant \langle b_i^T, b_i^I, b_i^F \rangle,$$

or

$$\forall j \in N \;\; \langle x_j^T, x_j^I, x_j^F \rangle \geqslant \max \Big\{ \langle b_i^T, b_i^I, b_i^F \rangle \mid i \in N_j(A, b) \Big\}$$

Now it is easy to see that $\underline{x}$ is defined by

$$\underline{x}_j = \max \Big\{ \langle b_i^T, b_i^I, b_i^F \rangle \mid i \in N_j(A, b) \Big\} \tag{8}$$

is a solution, moreover, the minimum solution of (1). However, the minimality of the covering is not a necessary condition, as we can see in the following example. ■

***Example* 5.2** By example 4.3 $\{L_1, L_2, L_3, L_4\}$ is not a minimal covering because $L_3 = \phi$. In spite of this

$$N_1 = \{2\}, N_2 = \{1, 3\}, N_3 = \{\phi\}, N_4 = \{4\}.$$

Therefore

$$\underline{x}_j(A, b) = \max \big\{ b_i | i \in N_j(A, b) \big\}$$

and

$$\underline{x}_1 = \max\{b_2\} \geqslant \{0.7 \;\; 0.5 \;\; 0.4\}$$
$$\underline{x}_2 = \max\{b_1\} \geqslant \{0.5 \;\; 0.4 \;\; 0.3\}$$
$$\underline{x}_4 = \max\{b_4\} \geqslant \{0.6 \;\; 0.3 \;\; 0.4\}.$$

The sets N_1, N_2, N_4 cover M, hence $x_3 = \langle 0, \; 0, \; 1 \rangle$ and the minimum solution is

$$\underline{x} = \big(\langle 0.7 \;\; 0.5 \;\; 0.4 \rangle, \langle 0.5 \;\; 0.4 \;\; 0.3 \rangle, \langle 0 \;\; 0 \;\; 1 \rangle, \langle 0.6 \;\; 0.3 \;\; 0.4 \rangle \big)^t.$$

***Example* 5.3** Let

$$A = \begin{pmatrix} \langle 0.2 \;\; 0.3 \;\; 0.4\rangle & \langle 0.1 \;\; 0.1 \;\; 0.4\rangle & \langle 0.1 \;\; 0.1 \;\; 0.4\rangle \\ \langle 0.1 \;\; 0.1 \;\; 0.6\rangle & \langle 0.3 \;\; 0.2 \;\; 0.5\rangle & \langle 0.1 \;\; 0.1 \;\; 0.6\rangle \\ \langle 0.1 \;\; 0.1 \;\; 0.6\rangle & \langle 0.1 \;\; 0.1 \;\; 0.6\rangle & \langle 0.3 \;\; 0.2 \;\; 0.5\rangle \\ \langle 0.4 \;\; 0.5 \;\; 0.4\rangle & \langle 0.4 \;\; 0.5 \;\; 0.4\rangle & \langle 0.4 \;\; 0.5 \;\; 0.4\rangle \end{pmatrix},$$

$$b = \begin{pmatrix} \langle 0.2 \;\; 0.3 \;\; 0.4\rangle \\ \langle 0.3 \;\; 0.2 \;\; 0.5\rangle \\ \langle 0.3 \;\; 0.2 \;\; 0.5\rangle \\ \langle 0.3 \;\; 0.2 \;\; 0.5\rangle \end{pmatrix}.$$

We compute successively

$$\begin{aligned} &M_1 = \{4\}, M_2 = \{4\},\ M_3 = \{4\}, \\ &\bar{x}_1 = \bar{x}_2 = \bar{x}_3 = \{0.3 \;\; 0.2 \;\; 0.5\}, \\ &I_1 = I_2 = I_3 = \{4\}, \\ &K_1 = \{1\},\ K_2 = \{2\},\ K_3 = \{3\}, \\ &L_1 = \{1, 4\}, L_2 = \{2, 4\},\ L_3 = \{3, 4\}, \end{aligned}$$

hence $\{L_1, L_2, L_3\}$ is a covering of M, the system is solvable and the maximum solution is

$$\bar{x} = \big(\langle 0.3 \;\; 0.2 \;\; 0.5\rangle, \langle 0.3 \;\; 0.2 \;\; 0.5\rangle, \langle 0.3 \;\; 0.2 \;\; 0.5\rangle\big)^t.$$

Further, $\{L_1, L_2, L_3\}$ is a minimal covering and

$$N_1 = \{1\},\ N_2 = \{2\},\ N_3 = \{3\}.$$

Therefore the minimum solution is

$$\underline{x} = \big(\langle 0.2 \;\; 0.3 \;\; 0.4\rangle, \langle 0.3 \;\; 0.2 \;\; 0.5\rangle, \langle 0.3 \;\; 0.2 \;\; 0.5\rangle\big)^t$$

and all the solutions are of the form

$$x = \big(\langle \alpha^T \;\; \alpha^I \;\; \alpha^F\rangle, \langle 0.3 \;\; 0.2 \;\; 0.5\rangle, \langle 0.3 \;\; 0.2 \;\; 0.5\rangle\big)^t,$$

where $\langle 0.2 \;\; 0.3 \;\; 0.4\rangle \leqslant \langle \alpha^T \;\; \alpha^I \;\; \alpha^F\rangle \leqslant \langle 0.3 \;\; 0.2 \;\; 0.5\rangle$).

***Example* 5.4** If

$$A = \begin{pmatrix} \langle 0.7 \;\; 0.6 \;\; 0.5\rangle & \langle 0.8 \;\; 0.7 \;\; 0.4\rangle & \langle 0.7 \;\; 0.6 \;\; 0.5\rangle \\ \langle 0.3 \;\; 0.2 \;\; 0.7\rangle & \langle 0.5 \;\; 0.4 \;\; 0.6\rangle & \langle 0.5 \;\; 0.4 \;\; 0.6\rangle \end{pmatrix}$$

$$b = \begin{pmatrix} \langle 0.7 \;\; 0.6 \;\; 0.5\rangle \\ \langle 0.5 \;\; 0.4 \;\; 0.6\rangle \end{pmatrix},$$

then

$$M_1 = \{\phi\},\ M_2 = \{1\},\ M_3 = \{\phi\},$$
$$\widetilde{M_1} = \{1\},\ \widetilde{M_2} = \{2\},\ \widetilde{M_3} = \{1, 2\}.$$

Therefore

$$L_1 = \{1\},\ L_2 = \{1, 2\},\ L_3 = \{1, 2\}$$

and

$$\bar{x}_j = \big(\langle 1 \ \ 1 \ \ 0\rangle, \langle 0.7 \ \ 0.6 \ \ 0.5\rangle, \langle 1 \ \ 1 \ \ 0\rangle\big)^t.$$

Moreover, it can be easily seen that

$$x^{'} = \big(\langle 0 \ \ 0 \ \ 1\rangle, \langle 0.7 \ \ 0.6 \ \ 0.5\rangle, \langle 0 \ \ 0 \ \ 1\rangle\big)^t$$
$$x^{''} = \big(\langle 0 \ \ 0 \ \ 1\rangle, \langle 0 \ \ 0 \ \ 1\rangle, \langle 0.7 \ \ 0.6 \ \ 0.5\rangle\big)^t$$

are both minimal, but not minimum solution. The minimality of the covering is necessary for the uniqueness of the solutions.

Theorem 5.5 Let $A \in \mathcal{N}_{mn},\ b \in \mathcal{N}_m,\ b > \langle 0, 0, 1\rangle$ be given such that $\{L_1, ..., L_n\}$ is a covering of M, but not a minimal one. Then $|S(A, b)| > 1$.

Proof. Suppose that $\{L_1, ..., L_n\}$ is a covering of M, but not a minimal one. Then $\bar{x}$ is a solution of (1) and let $\{L_1, ..., L_{k-1}, L_{k+1}, ..., L_n\}$ also be a covering of M. Take

$$x' = \Big(\langle \bar{x}_1^T, \bar{x}_1^I, \bar{x}_1^F\rangle, ..., \langle \bar{x}_{k-1}^T, \bar{x}_{k-1}^I, \bar{x}_{k-1}^F\rangle, \langle \alpha^T, \alpha^I, \alpha^F\rangle, \langle \bar{x}_{k+1}^T, \bar{x}_{k+1}^I, \bar{x}_{k+1}^F\rangle, ..., \langle \bar{x}_n^T, \bar{x}_n^I, \bar{x}_n^F\rangle\Big)^t$$

for an arbitrary $\langle \alpha^T, \alpha^I, \alpha^F\rangle < \langle \bar{x}_k^T, \bar{x}_k^I, \bar{x}_k^F\rangle$ (recall that $\bar{x}_k$ is equal either to $\langle 1, 1, 0\rangle$or to some $\langle b_i^T, b_i^I, b_i^F\rangle$ and those are all by assumption positive). Clearly,

$$A \otimes x^{'} \leqslant A \otimes \bar{x} = b$$

and for the proof of the opposite inequality suppose that for each $i \in M$ there exists $j \neq k$ such that $i \in L_j$. For these j's we have $\bar{x}_j = x_j^{'}$. Lemma 4.4 now gives

$$\langle a_{ij}^T, a_{ij}^I, a_{ij}^F\rangle \otimes \langle \bar{x}_j^T, \bar{x}_j^I, \bar{x}_j^F\rangle \geqslant \langle b_i^T, b_i^I, b_i^F\rangle,$$

hence $(A \otimes x^{'})_i \geqslant \langle a_{ij}^T, a_{ij}^I, a_{ij}^F\rangle \otimes \langle \bar{x}_j^T, \bar{x}_j^I, \bar{x}_j^F\rangle \geqslant \langle b_i^T, b_i^I, b_i^F\rangle$. This means that $\bar{x}$ is not a unique solution. ■

6. The case of square matrices

The assertions of the following lemma are well known from combinatorics.

Lemma 6.1 Let $H_1, ..., H_k$ be arbitrary finite sets and $H = \bigcup\limits_{j=1}^{k} H_j, |H| = l$.

(i) If $\{H_1, ..., H_k\}$ is a minimal covering of H, then $k \leqslant l$.
(ii) For $k = l$, $\{H_1, ..., H_k\}$ is a minimal covering of H if and only if $H_1, ..., H_k$ are one element pairwise disjoint sets.

Remark 1 *Hence in the case $m = n$, if $\{L_1, ..., L_n\}$ is a minimal covering, then $L_1, ..., L_n$ are one-element pairwise disjoint sets. Therefore, there exists a permutation $\Pi \in P_n$ such that $L_{\pi(i)} = \{i\}$. We shall use this notation in the following theorem.*

Theorem 6.2 Let a fuzzy neutrosophic soft square matrix $A \in \mathcal{N}_{mn}$ and $b \in \mathcal{N}_n$ be given. Then $|S(A, b)| = 1$ if and only if $\{L_1, ..., L_n\}$ is a minimal covering of the form $L_{\pi(i)} = \{i\}$ for a permutation $\pi \in P_n$ and for all i with $\langle a^T_{i\pi(i)}, a^I_{i\pi(i)}, a^F_{i\pi(i)}\rangle = \langle b^T_i, b^I_i, b^F_i\rangle$ it holds $\langle b^T_i, b^I_i, b^F_i\rangle = \langle 1, 1, 0\rangle$.

Proof. If (1) has a unique solution, then $\{L_1, ..., L_n\}$ is a minimal covering by Theorem 5.5 and according to the previous remark 6.2 there exists a permutation π such that $L_{\pi(i)} = \{i\}$. Let us fix i and for the sake of brevity denote $\pi(i)$ by j. Now we need to show that if $\langle a^T_{ij}, a^I_{ij}, a^F_{ij}\rangle = \langle b^T_i, b^I_i, b^F_i\rangle$, then $\langle b^T_i, b^I_i, b^F_i\rangle = \langle 1, 1, 0\rangle$. Realize first, that the inequality $M_j \neq \phi$ would imply $I_j \neq \phi$ and $I_j \subseteq L_j$, but since $\langle a^T_{ij}, a^I_{ij}, a^F_{ij}\rangle = \langle b^T_i, b^I_i, b^F_i\rangle, i \notin I_j$ and L_j is a singleton, we have a contradiction. Thus $M_j = \phi$ and $\langle \bar{x}^T_j, \bar{x}^I_j, \bar{x}^F_j\rangle = \langle 1, 1, 0\rangle$. The fact $L_j = \{i\}$ means, according to (5), that $\langle a^T_{kj}, a^I_{kj}, a^F_{kj}\rangle < \langle b^T_k, b^I_k, b^F_k\rangle$ for all $k \neq i$. If we set $x' = (\langle \bar{x}^T_1, \bar{x}^1_k, \bar{x}^F_1\rangle, ..., \langle \bar{x}^T_j, \bar{x}^I_j, \bar{x}^F_j\rangle, \langle b^T_i, b^I_i, b^F_i\rangle, \langle \bar{x}^T_{j+1}, \bar{x}^I_{j+1}, \bar{x}^F_{j+1}\rangle, ..., \langle \bar{x}^T_n, \bar{x}^I_n, \bar{x}^F_n\rangle)^t$ then x' is also a solution of (1) because all the rows other than the i^{th} are covered by some L_s disjoint from $\{j\}$ and for i^{th} equation we have $(A \otimes x')_i \geqslant \langle a^T_{ij}, a^I_{ij}, a^F_{ij}\rangle \otimes \langle b^T_i, b^I_i, b^F_i\rangle = \langle b^T_i, b^I_i, b^F_i\rangle$ –if $\langle b^T_i, b^I_i, b^F_i\rangle < \langle 1, 1, 0\rangle$: this is a different solution and we have a contradiction with the uniqueness of the solution.
For the converse implication suppose that $\{L_1, ..., L_n\}$ is a minimal covering. Then (1) has the maximum solution $\bar{x}$ and the minimum solution $\underline{x}$. We want to show $\bar{x} = \underline{x}$. Recall that $L_j = \{i\}, N_j \neq \phi$ for all j and distinguish two cases:

(i) If $M_j \neq \phi$, then $L_j = \underline{N_j} = I_j$, hence $\langle \bar{x}^T_j, \bar{x}^I_j, \bar{x}^F_j\rangle = \langle \underline{x}^T_j, \underline{x}^I_j, \underline{x}^F_j\rangle = \langle b^T_i, b^I_i, b^F_i\rangle$.
(ii) If $M_j = \phi$, then $L_j = \widetilde{M_j} = \{i\}$.

This means that $\langle a^T_{ij}, a^I_{ij}, a^F_{ij}\rangle = \langle b^T_i, b^I_i, b^F_i\rangle, \langle b^T_i, b^I_i, b^F_i\rangle = \langle 1, 1, 0\rangle$ and the second assumption of the theorem says that $\langle b^T_i, b^I_i, b^F_i\rangle = \langle 1, 1, 0\rangle$, which gives $\langle \underline{x}^T_j, \underline{x}^I_j, \underline{x}^F_j\rangle = \langle 1, 1, 0\rangle$ too. This completes the proof. ■

Theorem 6.3 Let $A \in \mathcal{N}_{n,n}$ and $b \in \mathcal{N}_n$ be given. Then $|S(A, b)| = 1$ if and only if there exists a permutation $\pi \in P_n$ such that

$$\langle a^T_{i\pi(i)}, a^I_{i\pi(i)}, a^F_{i\pi(i)}\rangle \geqslant \langle b^T_i, b^I_i, b^F_i\rangle > \sum_{j \in N, j \neq i} \langle a^T_{i\pi(i)}, a^I_{i\pi(i)}, a^F_{i\pi(i)}\rangle \wedge \langle b^T_j, b^I_j, b^F_j\rangle, \tag{9}$$

and the inequality is strict for each i with $\langle b^T_i, b^I_i, b^F_i\rangle < \langle 1, 1, 0\rangle$.

Proof. It suffices to prove the equivalence of this condition to the condition of Theorem 6.2. For simplicity suppose that $\pi = id$, that the covering in Theorem 6.2 is of the form $L_i = \{i\}$ and that $\langle b^T_1, b^I_1, b^F_1\rangle \leqslant \langle b^T_2, b^I_2, b^F_2\rangle \leqslant ... \leqslant \langle b^T_n, b^I_n, b^F_n\rangle$. First assume that (9) is fulfilled, i.e. we have

$$\langle a^T_{ii}, a^I_{ii}, a^F_{ii}\rangle \geqslant \langle b^T_i, b^I_i, b^F_i\rangle > \sum_{j \neq i} \langle a^T_{ij}, a^I_{ij}, a^F_{ij}\rangle \wedge \langle b^T_j, b^I_j, b^F_j\rangle, \tag{10}$$

with strict inequality whenever $\langle b_i^T, b_i^I, b_i^F\rangle < \langle 1,1,0\rangle$. Now, it is easy to see that $i \notin L_j$ for $j > i$, since the inequality $\langle b_i^T, b_i^I, b_i^F\rangle > \sum_{j>i} \langle a_{ij}^T, a_{ij}^I, a_{ij}^F\rangle \wedge \langle b_j^T, b_j^I, b_j^F\rangle$ with

$$\langle b_i^T, b_i^I, b_i^F\rangle \leqslant \langle b_j^T, b_j^I, b_j^F\rangle,$$

for $i < j$ gives $\langle b_i^T, b_i^I, b_i^F\rangle > \sum_{j>i} \langle a_{ij}^T, a_{ij}^I, a_{ij}^F\rangle$, thus

$$\langle b_i^T, b_i^I, b_i^F\rangle > \langle a_{ij}^T, a_{ij}^I, a_{ij}^F\rangle, \qquad \forall j > i. \tag{11}$$

Further we show $i \in L_i$. Distinguish two cases:
(a)$\langle a_{ii}^T, a_{ii}^I, a_{ii}^F\rangle > \langle b_i^T, b_i^I, b_i^F\rangle$.This means that
$\langle \bar{x}_i^T, \bar{x}_i^I, \bar{x}_i^F\rangle = min\{\langle b_k^T, b_k^I, b_k^F\rangle; \langle a_{ki}^T, a_{ki}^I, a_{ki}^F\rangle > \langle b_k^T, b_k^I, b_k^F\rangle\}$, which together with the ordering of right-hand sides and (11) implies $\langle \bar{x}_i^T, \bar{x}_i^I, \bar{x}_i^F\rangle = \langle b_i^T, b_i^I, b_i^F\rangle$, thus $i \in L_i$.
(b) $\langle a_{ii}^T, a_{ii}^I, a_{ii}^F\rangle = \langle b_i^T, b_i^I, b_i^F\rangle$. This implies $\langle b_i^T, b_i^I, b_i^F\rangle = \langle 1,1,0\rangle$, and
$\langle b_k^T, b_k^I, b_k^F\rangle = \langle 1,1,0\rangle$ for all $k > i$.
Now, in the light of (10) and (11), we have $M_i = \phi$, thus $L_i = \widetilde{M_i}$ which gives $i \in L_i$.
Now it remains only to show that $i \notin L_j$ for $j < i$. Again distinguish two cases:
(a)$\langle a_{jj}^T, a_{jj}^I, a_{jj}^F\rangle > \langle b_j^T, b_j^I, b_j^F\rangle$. Thus $\langle \bar{x}_j^T, \bar{x}_j^I, \bar{x}_j^F\rangle \leqslant \langle b_j^T, b_j^I, b_j^F\rangle$ and if
$\langle b_j^T, b_j^I, b_j^F\rangle < \langle b_i^T, b_i^I, b_i^F\rangle$, we are ready because $\langle a_{ij}^T, a_{ij}^I, a_{ij}^F\rangle \otimes \langle \bar{x}_j^T, \bar{x}_j^I, \bar{x}_j^F\rangle < \langle b_i^T, b_i^I, b_i^F\rangle$ and Lemma 4.4 gives $i \notin L_j$. If $\langle b_j^T, b_j^I, b_j^F\rangle = \langle b_i^T, b_i^I, b_i^F\rangle$,
(10) gives $\langle b_i^T, b_i^I, b_i^F\rangle > \langle a_{ij}^T, a_{ij}^I, a_{ij}^F\rangle \otimes \langle b_j^T, b_j^I, b_j^F\rangle$, hence $\langle b_i^T, b_i^I, b_i^F\rangle > \langle a_{ij}^T, a_{ij}^I, a_{ij}^F\rangle$ and $i \notin L_j$.
(b)$\langle a_{jj}^T, a_{jj}^I, a_{jj}^F\rangle = \langle b_j^T, b_j^I, b_j^F\rangle$ means $\langle b_j^T, b_j^I, b_j^F\rangle = \langle 1,1,0\rangle$,
hence from $\langle a_{ij}^T, a_{ij}^I, a_{ij}^F\rangle \otimes \langle b_j^T, b_j^I, b_j^F\rangle < \langle b_i^T, b_i^I, b_i^F\rangle$ we have $\langle b_i^T, b_i^I, b_i^F\rangle > \langle a_{ij}^T, a_{ij}^I, a_{ij}^F\rangle$ and $i \notin L_j$.
For the converse implication assume the condition of Theorem 6.2. This implies directly that $\langle a_{ii}^T, a_{ii}^I, a_{ii}^F\rangle \geqslant \langle b_i^T, b_i^I, b_i^F\rangle$ with strict inequality for $\langle b_i^T, b_i^I, b_i^F\rangle < \langle 1,1,0\rangle$. For the second inequality in (10) let us fix i and take $j \neq i$. Distinguish two cases:
(a)$M_j = \phi$. In this cases $\langle a_{kj}^T, a_{kj}^I, a_{kj}^F\rangle \leqslant \langle b_k^T, b_k^I, b_k^F\rangle$ for all k, moreover we know $\langle a_{ij}^T, a_{ij}^I, a_{ij}^F\rangle < \langle b_i^T, b_i^I, b_i^F\rangle$ because $\langle a_{ij}^T, a_{ij}^I, a_{ij}^F\rangle = \langle b_i^T, b_i^I, b_i^F\rangle$ would give $i \in L_j$, a contradiction with the condition of Theorem 6.2.
(b)If $M_j \neq \phi$, then $\langle \bar{x}_j^T, \bar{x}_j^I, \bar{x}_j^F\rangle = \langle b_j^T, b_j^I, b_j^F\rangle$, but Lemma 4.4 together with $i \notin L_j$ gives $\langle a_{ij}^T, a_{ij}^I, a_{ij}^F\rangle \otimes \langle \bar{x}_j^T, \bar{x}_j^I, \bar{x}_j^F\rangle = \langle a_{ij}^T, a_{ij}^I, a_{ij}^F\rangle \otimes \langle b_j^T, b_j^I, b_j^F\rangle < \langle b_i^T, b_i^I, b_i^F\rangle$. Hence for a fixed i we have $\langle b_i^T, b_i^I, b_i^F\rangle > \langle a_{ij}^T, a_{ij}^I, a_{ij}^F\rangle \otimes \langle b_j^T, b_j^I, b_j^F\rangle$ for all $j \neq i$, thus
$\langle b_i^T, b_i^I, b_i^F\rangle > \sum_{j>i} \langle a_{ij}^T, a_{ij}^I, a_{ij}^F\rangle \wedge \langle b_j^T, b_j^I, b_j^F\rangle$.
We extend the above theorem to rectangular matrix. ■

Lemma 6.4 Let $A \in \mathcal{N}_{mn}, b \in \mathcal{N}_m$ be such that $|S(A,b)| = 1$ and (1) is in a normal form. Then

$$\langle a_{ii}^T, a_{ii}^I, a_{ii}^F\rangle \geqslant \langle b_i^T, b_i^I, b_i^F\rangle > \sum_{j \in N-\{i\}} \langle a_{ij}^T, a_{ij}^I, a_{ij}^F\rangle \wedge \langle b_j^T, b_j^I, b_j^F\rangle. \tag{12}$$

holds for all $i \in N$, with strict inequality whenever $\langle b_i^T, b_i^I, b_i^F\rangle < \langle 1,1,0\rangle$.

7. Strong regularity

In this section, first we show that the strong linear independence (SLI) of columns of a fuzzy neutrosophic soft rectangular matrix can be reduced to the strong regularity of its fuzzy neutrosophic soft square submatrix. Then to derive a necessary and sufficient condition for strong regularity of FNSM in FNSA.

Definition 7.1 (Strong Linear Independent): We say that a Fuzzy neutrosophic soft matrices A has (SLI) column if for some b the system $A \otimes x = b$ is uniquely solvable.

Definition 7.2 (Strongly regular): An (n, n) fuzzy neutrosophic soft matrices

$$A = \langle a_{ij}^T, a_{ij}^I, a_{ij}^F \rangle,$$

is said to be FNSMs of strongly regular (1) if it has FNSMs of SLI column (2) to have a strong permanent if the equality $per(A) = \prod_{i=1}^{n} a_{i,\pi(i)}$ hold for unique $\pi \in P_n$.

Theorem 7.3 *$A \in \mathcal{N}_{mn}$ has SLI columns if and only if A contains a fuzzy neutrosophic soft strongly regular submatrix of order n.*

Proof. Suppose $A \in \mathcal{N}_{m,n}$ contains a fuzzy neutrosophic soft strongly regular submatrix C of order n, and without loss of generality let C consist of the first n row of A. Denote by y and $c = (\langle c_1^T, c_1^I, c_1^F \rangle, ..., \langle c_n^T, c_n^I, c_n^F \rangle)^t$ a vector satisfying $S(C, c) = \{y\}$. Put $\langle b_i^T, b_i^I, b_i^F \rangle = \langle c_i^T, c_i^I, c_i^F \rangle$ for $i = 1, 2, ..., n$ and $\langle b_i^T, b_i^I, b_i^F \rangle = A_i \otimes y$ for $i = n = 1, ..., m$. Then evidently $S(A, b) = \{y\}$. Now suppose that $|S(A, b)| = 1$
for some $b = (\langle b_1^T, b_1^I, b_1^F \rangle, ..., \langle b_m^T, b_m^I, b_m^F \rangle)^t \in \mathcal{N}_m$. Without loss of generality let the system be in a normal form. We show that the fuzzy neutrosophic soft submatrix A' of A consisting of its first n rows is strongly regular. According to Theorem 6.3 it is sufficient to find $\langle d_1^T, d_1^I, d_1^F \rangle, ..., \langle d_m^T, d_m^I, d_F^T \rangle \in \mathcal{N}_n$ satisfying

$$\langle a_{ii}^T, a_{ii}^I, a_{ii}^F \rangle \geqslant \langle d_i^T, d_i^I, d_i^F \rangle > \sum_{j \in N - \{i\}} \langle a_{ij}^T, a_{ij}^I, a_{ij}^F \rangle \wedge \langle d_j^T, d_j^I, d_j^F \rangle, \tag{13}$$

for all $i \in N$ and ensuring strict inequality for $\langle d_i^T, d_i^I, d_i^F \rangle < \langle 1, 1, 0 \rangle$, because then $|S(A', d)| = 1$. We can take arbitrary $\langle d_1^T, d_1^I, d_1^F \rangle, ..., \langle d_n^T, d_n^I, d_n^F \rangle$ fulfilling the following conditions:

$\langle b_1^T, b_1^I, b_1^F \rangle > \langle d_1^T, d_1^I, d_1^T \rangle > \sum_{j=2}^{n} \langle a_{ij}^T, a_{ij}^I, a_{ij}^F \rangle$, and

$\langle b_i^T, b_i^I, b_i^F \rangle > \langle d_i^T, d_i^I, d_i^F \rangle > \langle d_{i-1}^T, d_{i-1}^I, d_{i-1}^F \rangle \oplus \sum_{j=i+1}^{n} \langle a_{ij}^T, a_{ij}^I, a_{ij}^F \rangle$

for i=2,...,n. Clearly, $\langle d_1^T, d_1^I, d_1^F \rangle, ..., \langle d_n^T, d_n^I, d_n^F \rangle$ are well defined if the inequality $\langle b_i^T, b_i^I, b_i^F \rangle > \sum_{j>i}^{n} \langle a_{ij}^T, a_{ij}^I, a_{ij}^F \rangle$

for all $i \in \{1, 2, ..., n\}$ is fulfilled; but for a fuzzy neutrosophic soft matrix in a normal form this follows from the proof of Theorem 6.3. Now it remains to verify (13) for each $i \in \{1, 2, ..., n\}$.The first inequality follows from $\langle a_{ii}^T, a_{ii}^I, a_{ii}^F \rangle \geqslant \langle b_i^T, b_i^I, b_i^F \rangle$ (since $i \in L_i$), and one can easily see that

$\langle d_i^T, d_i^I, d_i^F \rangle > \sum_{j=1}^{i-1} \langle d_j^T, d_j^I, d_j^F \rangle \oplus \sum_{j=i+1}^{n} \langle a_{ij}^T, a_{ij}^I, a_{ij}^F \rangle \geqslant \sum_{j=1}^{i-1} (\langle a_{ij}^T, a_{ij}^I, a_{ij}^F \rangle \wedge \langle d_j^T, d_j^I, d_j^F \rangle) \oplus$

$$\sum_{j=i+1}^{n}(\langle a_{ij}^T, a_{ij}^I, a_{ij}^F\rangle \wedge \langle d_j^T, d_j^I, d_j^F\rangle) = \sum_{j\in N-\{i\}}\langle a_{ij}^T, a_{ij}^I, a_{ij}^F\rangle \wedge \langle d_j^T, d_j^I, d_j^F\rangle. \qquad \blacksquare$$

Definition 7.4 For a given fuzzy neutrosophic soft square matrix $A = (\langle a_{ij}^T, a_{ij}^I, a_{ij}^F\rangle) \in \mathcal{N}_{nn}$ define
$A_k = \sum_{i=1}^{k}\sum_{j=i+1}^{n}\langle a_{ij}^T, a_{ij}^I, a_{ij}^F\rangle$ and say that A is trapezoidal if $\langle a_{kk}^T, a_{kk}^I, a_{kk}^F\rangle > A_k$ holds for all $k = 1, 2, ..., n$.

Theorem 7.5 A FNSM A over FNSA is strongly regular if and only if A is equivalent to a Fuzzy Neutrosophic soft trapezoidal matrix.

Proof. For the 'if' part, let A itself be Fuzzy Neutrosophic soft trapezoidal. Hence we have. $\langle a_{ii}^T, a_{ii}^I, a_{ii}^F\rangle > \sum_{k=1}^{i}\sum_{i=k+1}^{n}\langle a_{kj}^T, a_{kj}^I, a_{kj}^F\rangle = A_i$. Now it is sufficient to take $\langle b_i^T, b_i^I, b_i^F\rangle \in (A_i, \langle a_{ii}^T, a_{ii}^I, a_{ii}^F\rangle)$ for each i and such that $\langle b_1^T, b_1^I, b_1^F\rangle < \langle b_2^T, b_2^I, d_2^F\rangle < ... < \langle b_n^T, b_n^I, d_n^F\rangle$ Due to the trapezoidal property and the density of FNSM these intervals are always nonempty and since $A_1 \leqslant A_2 \leqslant ... \leqslant A_n$, the desired increasing $\langle b^T, b^I, b^F\rangle$ exists. We show that the inequality (10) for A, b is fulfilled, even strictly. For $i = 1$ we have
$\langle a_{11}^T, a_{11}^I, a_{11}^F\rangle > \langle b_1^T, b_1^I, b_1^F\rangle > \sum_{j>2}\langle a_{ij}^T, a_{ij}^I, a_{ij}^F\rangle \geqslant \sum_{j>2}\langle a_{ij}^T, a_{ij}^I, a_{ij}^F\rangle \wedge \langle b_j^T, b_j^I, b_j^F\rangle$.
For $i > 1$,
$\langle a_{ii}^T, a_{ii}^I, a_{ii}^T\rangle > \langle b_i^T, b_i^I, b_i^F\rangle > A_i \geqslant \sum_{j>i}\langle a_{ij}^T, a_{ij}^I, a_{ij}^F\rangle \geqslant \sum_{j>i}\langle a_{ij}^T, a_{ij}^I, a_{ij}^F\rangle \wedge \langle b_j^T, b_j^I, b_j^F\rangle$ and for $j < i$ we get $\langle b_i^T, b_i^I, b_i^F\rangle > \sum_{j<i}\langle b_j^T, b_I^I, b_j^F\rangle \geqslant \sum_{j<i}^{\oplus}\langle a_{ij}^T, a_{ij}^I, a_{ij}^F\rangle \wedge \langle b_j^T, b_j^I, b_j^F\rangle$ and (10) is thus proved. For the 'only if' part take a vector b such that $|S(A, b)| = 1$ and suppose that (1) is in a normal form. Thus (10) is fulfilled with
$\pi = id : \langle a_{ii}^T, a_{ii}^I, a_{ii}^F\rangle \geqslant \langle b_i^T, b_i^I, b_i^F\rangle > \sum_{j=1}\langle a_{ij}^T, a_{ij}^I, a_{ij}^F\rangle \wedge \langle b_j^T, b_j^I, b_j^F\rangle$ and we proceed to show the trapezoidal property. For $i = 1$
$\langle b_1^T, b_1^I, b_1^F\rangle > \sum_{j>1}\langle a_{ij}^T, a_{ij}^I, a_{ij}^F\rangle \wedge \langle b_j^T, b_j^I, b_j^F\rangle$
and since $\langle b_1^T, b_1^I, b_1^F\rangle \leqslant b_j^T, b_j^I, b_j^F$ we have
$\langle b_1^T, b_1^I, b_1^F\rangle > \sum_{j\neq 1}\langle a_{ij}^T, a_{ij}^I, a_{ij}^F\rangle = A_1$.
Now suppose that for all $k < i$ it is $b_k > A_k$. Further we get
$\langle b_i^T, b_i^I, b_i^F\rangle > \sum_{j\neq 1}\langle a_{ij}^T, a_{ij}^I, a_{ij}^F\rangle \wedge \langle b_j^T, b_j^I, b_j^F\rangle \geqslant \sum_{j>1}\langle a_{ij}^T, a_{ij}^I, a_{ij}^F\rangle \wedge \langle b_j^T, b_j^I, b_j^F\rangle$
and since $\langle b_i^T, b_i^I, b_i^F\rangle \leqslant \langle b_j^T, b_j^I, b_j^F\rangle$ for $i < j$, it is $\langle b_i^T, b_i^I, b_i^F\rangle > \sum_{j>1}\langle a_{ij}^T, a_{ij}^I, a_{ij}^F\rangle$.
Moreover, $\langle b_i^T, b_i^I, b_i^F\rangle \geqslant \sum_{j<1}\langle b_j^T, b_j^I, b_j^F\rangle > \sum_{j<1} A_j$ which together with the previous inequality gives $\langle b_i^T, b_i^I, b_i^F\rangle > A_i$.
Thus we have proved $\langle a_{ii}^T, a_{ii}^I, a_{ii}^F\rangle \geqslant \langle b_i^T, b_i^I, b_i^F\rangle > A_i$ for each i, which means that A is trapezoidal. Theorem 7.3 and Theorem 7.5 imply that $A \in FNSM_{mn}$ has SLI columns over the algebra if and only if A contains a square submatrix of order n equivalent to a trapezoidal one. $\blacksquare$

8. The Trapezoidal Algorithm

In the following section is trapezoidal algorithm of fuzzy neutrosophic soft matrix.
Trapezoidal Algorithm:

Step 1 Set $k := 0$, $d := 0$ and $A_1 := A$. (variable k counts the number of diagonal entries defined so far, d stores the maximum of obtained over-daigonal entries, A_1 denotes the reduced fuzzy neutrosophic soft matrix.)
Step 2 Choose an arbitrary row of A_1, say row i, such that
 (a) row i has a unique maximum entry in A_1, say $\langle a_{ij}^T, a_{ij}^I, a_{ij}^F \rangle$
 (b) $\langle a_{ij}^T, a_{ij}^I, a_{ij}^F \rangle > d$
 (c) the second greatest entry of A_1 in row i, say a_{is}, is minimum of all the second greatest entries of A_1 in rows fulfilling (a) and (b).
Step 3 If such a row does not exist, then SLI:= false and STOP.
Step 4 Otherwise set $k := k+1$ and $d := d \otimes a_{is}$, shift row i and column j to the k^{th} position in FNSM A and further consider as A_1 only the reduced fuzzy neutrosophic soft submatrix is obtained by deleting the first k rows and columns of A.
Step 5 If k=n then SLI:= true and STOP. Otherwise go to **Step 2**.

We illustrate the trapezoidal algorithm by its application to the following FNSM A.

***Example* 8.1** We illustrate the previous assertions by the following example:

$$A = \begin{pmatrix} \langle 0.1 \; 0.2 \; 0.3\rangle & \langle 0.2 \; 0.3 \; 0.3\rangle & \langle 0.3 \; 0.4 \; 0.2\rangle \\ \langle 0.2 \; 0.3 \; 0.3\rangle & \langle 0.1 \; 0.2 \; 0.3\rangle & \langle 0.2 \; 0.3 \; 0.4\rangle \\ \langle 0 \; 0 \; 1\rangle & \langle 0.1 \; 0.2 \; 0.3\rangle & \langle 0.1 \; 0.2 \; 0.3\rangle \\ \langle 0 \; 0 \; 1\rangle & \langle 0.2 \; 0.3 \; 0.3\rangle & \langle 0.4 \; 0.5 \; 0.3\rangle \end{pmatrix}.$$

Le us write the corresponding row maximum and second greatest entries to the right of each row X means that the maximum in the corresponding row is not unique. Hence we get

$$A = \begin{pmatrix} \langle 0.1 \; 0.2 \; 0.3\rangle & \langle 0.2 \; 0.3 \; 0.4\rangle & \boxed{0.3 \; 0.4 \; 0.2} \\ \langle 0.2 \; 0.3 \; 0.3\rangle & \langle 0.1 \; 0.2 \; 0.3\rangle & \langle 0.1 \; 0.2 \; 0.3\rangle \\ \langle 0 \; 0 \; 1\rangle & \langle 0.1 \; 0.2 \; 0.3\rangle & \langle 0.1 \; 0.2 \; 0.3\rangle \\ \langle 0 \; 0 \; 1\rangle & \langle 0.2 \; 0.3 \; 0.3\rangle & \langle 0.4 \; 0.5 \; 0.3\rangle \end{pmatrix} \begin{matrix} \langle 0.3 \; 0.4 \; 0.2\rangle \; \langle 0.2 \; 0.3 \; 0.3\rangle \\ X \\ X \\ \langle 0.4 \; 0.5 \; 0.3\rangle \; \langle 0.2 \; 0.3 \; 0.3\rangle \end{matrix}$$

$$A = \left(\begin{array}{c|cc} \langle 0.3 \; 0.4 \; 0.2\rangle & \langle 0.1 \; 0.2 \; 0.3\rangle & \langle 0.2 \; 0.3 \; 0.3\rangle \\ \hline \langle 0.2 \; 0.3 \; 0.3\rangle & \langle 0.2 \; 0.3 \; 0.3\rangle & \langle 0.1 \; 0.2 \; 0.3\rangle \\ \langle 0.1 \; 0.2 \; 0.3\rangle & \langle 0 \; 0 \; 1\rangle & \langle 0.1 \; 0.2 \; 0.3\rangle \\ \langle 0.4 \; 0.5 \; 0.3\rangle & \langle 0 \; 0 \; 1\rangle & \langle 0.2 \; 0.3 \; 0.3\rangle \end{array}\right) \begin{matrix} \langle 0.2 \; 0.3 \; 0.3\rangle \; \langle 0.1 \; 0.2 \; 0.3\rangle \\ \langle 0.1 \; 0.2 \; 0.3\rangle \; \langle 1 \; 0 \; 0\rangle \\ \langle 0.2 \; 0.3 \; 0.3\rangle \; \langle 0 \; 0 \; 1\rangle \end{matrix}$$

Since none of the remaining rows contains an entry greater than the current over diagonal maximum $d = \langle 0.2 \;\; 0.3 \;\; 0.3\rangle$, the algorithm stops and the columns of A are not SLI.

***Example* 8.2** Now we check the strong linear independence of columns of the FNSMs

$$A=\begin{pmatrix} \langle 0.5\ \ 0.4\ \ 0.2\rangle & \langle 0.5\ \ 0.4\ \ 0.2\rangle & \langle 0.1\ \ 0.2\ \ 0.3\rangle & \langle 0.4\ \ 0.3\ \ 0.2\rangle \\ \langle 0.2\ \ 0.3\ \ 0.4\rangle & \langle 0.6\ \ 0.7\ \ 0.1\rangle & \langle 0.2\ \ 0.3\ \ 0.4\rangle & \langle 0.7\ \ 0.8\ \ 0.1\rangle \\ \langle 0.3\ \ 0.4\ \ 0.2\rangle & \langle 0.5\ \ 0.4\ \ 0.2\rangle & \langle 0.1\ \ 0.2\ \ 0.7\rangle & \langle 0.2\ \ 0.3\ \ 0.4\rangle \\ \langle 0.1\ \ 0.2\ \ 0.7\rangle & \langle 0.4\ \ 0.3\ \ 0.2\rangle & \langle 0.4\ \ 0.3\ \ 0.2\rangle & \langle 0.3\ \ 0.4\ \ 0.5\rangle \\ \langle 0.2\ \ 0.3\ \ 0.4\rangle & \langle 0.1\ \ 0.2\ \ 0.7\rangle & \langle 0.3\ \ 0.4\ \ 0.5\rangle & \langle 0.6\ \ 0.7\ \ 0.1\rangle \\ \langle 0.6\ \ 0.7\ \ 0.1\rangle & \langle 0.1\ \ 0.2\ \ 0.7\rangle & \langle 0.5\ \ 0.4\ \ 0.2\rangle & \langle 0\ \ \ 0\ \ \ 1\rangle \end{pmatrix}$$

with the same formalism as in the previous example

$$A=\begin{pmatrix} \langle 0.5,0.4,0.2\rangle & \langle 0.5,0.4,0.2\rangle & \langle 0.1,0.2,0.3\rangle & \langle 0.4,0.3,0.2\rangle \\ \langle 0.2,0.3,0.4\rangle & \langle 0.6,0.7,0.1\rangle & \langle 0.2,0.3,0.4\rangle & \langle 0.7,0.8,0.1\rangle \\ \langle 0.3,0.4,0.5\rangle & \boxed{0.5,\ 0.4,\ 0.2} & \langle 0.1,0.2,0.7\rangle & \langle 0.2,0.3,0.4\rangle \\ \langle 0.1,0.2,0.3\rangle & \langle 0.4,0.3,0.2\rangle & \langle 0.4,0.3,0.2\rangle & \langle 0.3,0.4,0.5\rangle \\ \langle 0.2,0.3,0.4\rangle & \langle 0.1,0.2,0.7\rangle & \langle 0.3,0.4,0.5\rangle & \langle 0.6,0.7,0.1\rangle \\ \langle 0.6,0.7,0.1\rangle & \langle 0.1,0.2,0.7\rangle & \langle 0.5,0.4,0.2\rangle & \langle 0,\ \ 0,\ \ 0\rangle \end{pmatrix} \begin{matrix} X \\ \langle 0.7,0.8,0.1\rangle\ \langle 0.6,0.7,0.1\rangle \\ \langle 0.5,0.4,0.2\rangle\ \langle 0.3,0.4,0.5\rangle \\ X \\ \langle 0.6,0.7,0.1\rangle\ \langle 0.3,0.4,0.5\rangle \\ \langle 0.6,0.7,0.1\rangle\ \langle 0.3,0.4,0.5\rangle \\ \langle 0.6,0.7,0.1\rangle\ \langle 0.5,0.4,0.2\rangle \end{matrix}$$

The first candidate for the main diagonal is in the box, $d=\langle 0.3\ \ 0.4\ \ 0.5\rangle$, the third row and the second column come to

$$A=\begin{pmatrix} \langle 0.5,0.4,0.2\rangle & \langle 0.1,0.2,0.7\rangle & \langle 0.2,0.3,0.4\rangle & \langle 0.3,0.4,0.5\rangle \\ \langle 0.6,0.7,0.1\rangle & \langle 0.2,0.3,0.4\rangle & \boxed{0.7\ ,0.8\ ,0.1} & \langle 0.2,0.3,0.4\rangle \\ \langle 0.5,0.4,0.2\rangle & \langle 0.1,0.2,0.7\rangle & \langle 0.4,0.3,0.2\rangle & \langle 0.5,0.4,0.2\rangle \\ \langle 0.4,0.3,0.2\rangle & \langle 0.4,0.5,0.2\rangle & \langle 0.3,0.4,0.5\rangle & \langle 0.1,0.2,0.7\rangle \\ \langle 0.1,0.2,0.7\rangle & \langle 0.3,0.4,0.5\rangle & \langle 0.6,0.7,0.1\rangle & \langle 0.2,0.3,0.4\rangle \\ \langle 0.1,0.2,0.7\rangle & \langle 0.5,0.4,0.2\rangle & \langle 0\ ,\ \ 0,\ \ 1\rangle & \langle 0.6,0.7,0.1\rangle \end{pmatrix} \begin{matrix} \langle 0.7,0.8,0.1\rangle\ \langle 0.2,0.3,0.4\rangle \\ \langle 0.5,0.4,0.2\rangle\ \langle 0.4,0.3,0.2\rangle \\ \langle 0.4,0.5,0.7\rangle\ \langle 0.3,0.4,0.5\rangle \\ \langle 0.6,0.7,0.1\rangle\ \langle 0.3,0.4,0.5\rangle \\ \langle 0.6,0.7,0.1\rangle\ \langle 0.5,0.4,0.2\rangle \end{matrix}$$

Now the condition (a) and (b) of **Step 2** are fulfilled by the rows $2,3,4$ and 6 the second greatest entry from them is achieved in rows 2, so the third column is switched to the second position $d=d\oplus\langle 0.2\ \ 0.3\ \ 0.4\rangle=\langle 0.3\ \ 0.4\ \ 0.4\rangle$

$$A=\begin{pmatrix} \langle 0.5,0.4,0.2\rangle & \langle 0.2,0.3,0.4\rangle & \langle 0.1,0.2,0.7\rangle & \langle 0.3,0.4,0.5\rangle \\ \langle 0.6,0.7,0.1\rangle & \langle 0.7,0.8,0.1\rangle & \langle 0.2,0.3,0.4\rangle & \langle 0.2,0.3,0.4\rangle \\ \langle 0.5,0.4,0.2\rangle & \langle 0.4,0.3,0.2\rangle & \langle 0.1,0.2,0.3\rangle & \boxed{0.5,\ 0.4,\ 0.2} \\ \langle 0.4,0.5,0.2\rangle & \langle 0.3,0.4,0.5\rangle & \langle 0.4,0.5,0.7\rangle & \langle 0.1,0.2,0.3\rangle \\ \langle 0.1,0.2,0.3\rangle & \langle 0.6,0.7,0.1\rangle & \langle 0.3,0.4,0.5\rangle & \langle 0.2,0.3,0.4\rangle \\ \langle 0.1,0.2,0.7\rangle & \langle 0\ ,\ \ 0,\ \ 1\rangle & \langle 0.5,0.4,0.2\rangle & \langle 0.6,0.7,0.1\rangle \end{pmatrix} \begin{matrix} \langle 0.5,0.4,0.2\rangle\ \langle 0.1,0.2,0.3\rangle \\ \langle 0.4,0.3,0.2\rangle\ \langle 0.1,0.2,0.3\rangle \\ \langle 0.3,0.4,0.5\rangle\ \langle 0.2,0.3,0.4\rangle \\ \langle 0.6,0.7,0.1\rangle\ \langle 0.5,0.4,0.5\rangle \end{matrix}$$

with the chosen entry in a box and after one more step we obtain the described trapezoided submatrix.

$$T=\begin{pmatrix} \langle 0.5\ \ 0.4\ \ 0.2\rangle & \langle 0.2\ \ 0.3\ \ 0.4\rangle & \langle 0.3\ \ 0.4\ \ 0.5\rangle & \langle 0.1\ \ 0.2\ \ 0.7\rangle \\ \langle 0.6\ \ 0.7\ \ 0.1\rangle & \langle 0.7\ \ 0.8\ \ 0.1\rangle & \langle 0.2\ \ 0.3\ \ 0.4\rangle & \langle 0.2\ \ 0.3\ \ 0.4\rangle \\ \langle 0.5\ \ 0.4\ \ 0.2\rangle & \langle 0.4\ \ 0.3\ \ 0.2\rangle & \langle 0.5\ \ 0.4\ \ 0.2\rangle & \langle 0.1\ \ 0.2\ \ 0.3\rangle \\ \langle 0.4\ \ 0.3\ \ 0.2\rangle & \langle 0.3\ \ 0.4\ \ 0.2\rangle & \langle 0.1\ \ 0.2\ \ 0.3\rangle & \langle 0.4\ \ 0.3\ \ 0.2\rangle \end{pmatrix}$$

9. Conclusion

In this work, the authors obtain a minimal solution of FNSM and study their unique solvability, strong linearly independent and strong regularity of FNSM. We formulate a necessary and sufficent condition for a linear system of equations over a FNSA to have a unique solution and prove the equivalence of strong regularity and trapezoidal property.

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