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Multi-Attribute Decision Making Based on Probabilistic Neutrosophic Hesitant Fuzzy Choquet Aggregation Operators

Songtao Shao ^{1,†} , Xiaohong Zhang ^{2,3,*}  and Quan Zhao ^{1,†}

¹ College of Information Engineering, Shanghai Maritime University, Shanghai 201306, China; 201740310005@stu.shmtu.edu.cn (S.S.); quanzhao@shmtu.edu.cn (Q.Z.)

² Department of Mathematics, School of Arts and Sciences, Shaanxi University of Science & Technology, Xi'an 710021, China

³ Department of Mathematics, College of Arts and Sciences, Shanghai Maritime University, Shanghai 201306, China

* Correspondence: zhangxiaohong@sust.edu.cn or zhangxh@shmtu.edu.cn

† Current address: 1550, Haigang Main Road, Shanghai 201306, China

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Abstract: Take the third-party logistics providers (3PLs) as an example, according to the characteristics of correlation between attributes in multi-attribute decision-making, two Choquet aggregation operators adopting probabilistic neutrosophic hesitation fuzzy elements (PNHFES) are proposed to cope with the situations of correlation among criterions. This measure not only provides support for the correlation phenomenon between internal attributes, but also fully concerns the incidental uncertainty of the external space. Our goal is to make it easier for decision makers to cope with this uncertainty, thus we establish the notion of probabilistic neutrosophic hesitant fuzzy Choquet averaging (geometric) (PNHFCA, PNHFCAOG) operator. Based on this foundation, a method for aggregating decision makers' information is proposed, and then the optimal decision scheme is obtained. Finally, an example of selecting optimal 3PL is given to demonstrate the objectivity of the above-mentioned standpoint.

Keywords: probabilistic neutrosophic hesitant fuzzy set (PNHFS); decision-making; Choquet integral

1. Introduction

In the process of enterprise development, business leaders often encounter various multi-criteria decision-making (MADM) situations. In order for a company to survive in today's increasingly competitive real life, decision makers (DMs) must decide on the best solution when encountering MADM issues [1,2]. Therefore, how to effectively make optimal decisions in the MADM problems has become an emergency problem that global enterprises urgently need to solve. Establishing and perfecting research methods to suitable for MADM situations has attracted more and more attention from DMs [3]. The key to MADM is to choose the appropriate decision theory and computer software to aggregate the information of DMs and make the best decision in the decision process according to relevance of the information. In order to choose the optimal solution, different MADM schemes have been established to increase the development and competitiveness of enterprises. Since the most common feature in decision information is the ambiguity of the information. Therefore, some related researches based on fuzzy information have been proposed [4–12].

In reality, there is a correlation among attributes in the process of MADM. In addition, some researchers have begun to pay attention to this issue. For example, Brito et al. [13] proposed a new-type multi-criteria model based on the Choquet integral and epistemic mapping technique

for evaluating eco-friendly cities. Krishnan et al. [14] introduced a new λ^0 -measure authentication approach that expresses the mutual relation between the attributes. To more effectively highlight the correlation between attributes, Beg et al. [15] introduced the diminishing Choquet hesitant 2-tuple average (DCH2TA) operator and established a MADM approach.

Compared with reality, the current research method does not consider the fuzzy information with possibility in the MADM problem. Thus, when there is a situation with possibility information, the existing methods will be invalid. To deal with this type of problem, there are two aspects that need to be solved: one is the representation of the PNHF information and the probability information of external environment. The other is the establishment of the MADM model considering the interrelationship among the attributes.

According to the analysis of the common fuzzy conditions in practice, fuzzy sets [16] have been used in many situations. Diversity based on fuzzy information, the fuzzy set theory has been continuously improved. For example, intuitionistic FS [17], hesitant FS [18], and dual hesitant FS [19]. When dealing with fuzzy data, DMs will often encounter the following three kinds of independent fuzzy information: truth fuzzy information, indeterminacy fuzzy information and falsity fuzzy information. The definition of neutrosophic set (NS) was constructed [20] to express the feature. For application to engineering projects, different types of generalized NSs are constructed such as single-valued NS [21], interval NS [22], and neutrosophic hesitant fuzzy (NHF) set [23]. These extended NS theories have been applied to medical diagnosis and other engineering fields [24–28]. Fuzzy set and neutrosophic set are also extended to the field of algebra [29–33]. In order to express three independent hesitant messages, NHFS was proposed and applied to MADM [34–36]. However, as can be seen from these references, those theories can only express information about cognitive uncertainty. Thus, in this article, we use PNHFS [3,37] to express the occasional uncertainty of information and the uncertainty of cognition. Under the MADM environment, due to the different cognitive situations of DMs and their own hesitation, the final evaluation value may not be unique. Depending on the complexity of the external environment, the probability of taking values also affects the evaluation values. Thus, probability plays a key role in interpreting the evaluation value, which avoid the loss of evaluation values, simultaneously. Not only subjective information of NHFS, but also objective probability information of each membership was clearly described.

The MADM problem under attribute correlation is one of the main research questions in this paper. As an important part of fuzzy mathematics, fuzzy integral can help DMs to better deal with MADM problems through modeling methods when attributes are associated with each other. The classic weight information is to satisfy the additivity condition, and the advantage of fuzzy integral is that it is not limited by the additivity condition, that is, the sum of the fuzzy integral may be greater than 1. When the Choquet integral (CI) [38] satisfies the additivity, the Choquet integral is converted into classic weight information. Therefore, the application range of Choquet integral is extensive, and it is more suitable for coping with MADM problems with uncertain information. For example, Khan et al. [39] proposed the (generalized) Pythagorean hesitant fuzzy Choquet averaging (geometric) operators under the MADM environment. Based on the hierarchical and interacting standards, Corrente et al. [40] Choquet integral preference model that can be robust ranking of universities evaluation. Labreuche et al. [41] developed two new Choquet models. Yager [42] used the standard Shapley value as an approximation of Choquet integra. Liu et al. [43] effectively investigate the MADM situations that the interrelationship between attributes, globally. Absolutely, above aggregation operators can not integrate PNHF information. Thus, we construct a new integration method by CI. Thus, we extend these operators to the PNHFSs and propose PNHF Choquet averaging (geometric) operators and establish a process to handle MADM situations.

Based on the above analysis: In Section 2, fundamental concepts are reviewed. In Section 3, the operators are included, the comparison approaches are described, and some basic properties are studied. In Section 4, an approach to MADM based on the PNHFCOA (PNHF COG) operator is proposed. In Section 5, an illustrative situation is given to confirm the proposed method. In Section 6, our results are analyzed with the results of other methods. Finally, conclusions and future work are summarized.

2. Preliminaries

Some basic definitions can be referred to [3,23,37,38,44].

Definition 1. A NHF set (NHFS) in a reference domain X set is depicted by:

$$N = \{ \langle x, \tilde{t}(x), \tilde{i}(x), \tilde{f}(x) \rangle | x \in X \}$$

in which $\tilde{t}(x)$ indicates that a set includes some truth-membership hesitant degrees of x , $\tilde{i}(x)$ indicates that a set includes some indeterminacy-membership hesitant degrees, and $\tilde{f}(x)$ indicates that a set includes some falsity-membership hesitant degrees. The following conditions are held: $\delta \in \tilde{t}(x), \gamma \in \tilde{i}(x), \eta \in \tilde{f}(x), \delta, \gamma, \eta \in [0, 1], \max\{\delta\} + \max\{\gamma\} + \max\{\eta\} \in [0, 3]$.

Definition 2. A PNHFS in a reference domain X is depicted by:

$$N = \{ \langle x, T(x)|P^T(x), I(x)|P^I(x), F(x)|P^F(x) \rangle | x \in X \}. \tag{1}$$

$T(x)|P^T(x), I(x)|P^I(x)$ and $F(x)|P^F(x)$ describes three components of N , $T(x), I(x)$ and $F(x)$ is depicted the three types hesitant degrees of x . $P^T(x), P^I(x)$ and $P^F(x)$ describe the corresponding probability of $T(x), I(x)$ and $F(x)$, The conditions that need to be held:

$$\alpha \in T(x), \beta \in I(x), \gamma \in F(x), \alpha, \beta, \gamma \in [0, 1], \max\{\alpha\} + \max\{\beta\} + \max\{\gamma\} \in [0, 3];$$

$$P_a^T \in P^T, P_b^I \in P^I, P_c^F \in P^F, P_a^T, P_b^I, P_c^F \in [0, 1]; \sum_{a=1}^{\#\tilde{t}} P_a^T \leq 1, \sum_{b=1}^{\#\tilde{i}} P_b^I \leq 1, \sum_{c=1}^{\#\tilde{f}} P_c^F \leq 1.$$

$\#\tilde{t}, \#\tilde{i}$ and $\#\tilde{f}$ describe the cardinal number of $T(x)|P^T(x), I(x)|P^I(x), F(x)|P^F(x)$, respectively.

Generally, $N = \{T|P^T, I|P^I, F|P^F\}$ is described a an PNHF number (PNHFE) of $N = \langle T(x)|P^T(x), I(x)|P^I(x), F(x)|P^F(x) \rangle$.

Definition 3. A normalized PNHFE N satisfies the following conditions:

$$\tilde{N} = \langle T(x)|\tilde{P}^T(x), I(x)|\tilde{P}^I(x), F(x)|\tilde{P}^F(x) \rangle, \tag{2}$$

where $\tilde{P}_a^T = \frac{P_a^T}{\sum P_a^T}, \tilde{P}_b^I = \frac{P_b^I}{\sum P_b^I}, \tilde{P}_c^F = \frac{P_c^F}{\sum P_c^F}$.

Definition 4. Supposing that $N_1 = \{T_1|P^{T_1}, I_1|P^{I_1}, F_1|P^{F_1}\}, N_2 = \{T_2|P^{T_2}, I_2|P^{I_2}, F_2|P^{F_2}\}$ are two PNHFEs, some basic algorithms are defined by:

- (1) $(N_1)^c = \bigcup_{\alpha_1 \in T_1, \beta_1 \in I_1, \gamma_1 \in F_1} \{ \gamma_1 | P_1^{F_1}, 1 - \beta_1 | P_1^{I_1}, \alpha_1 | P_1^{T_1} \},$
- (2) $(N_1)^\lambda = \bigcup_{\alpha_1 \in T_1, \beta_1 \in I_1, \gamma_1 \in F_1} \{ \{ (\alpha_1)^\lambda | P_1^{T_1} \}, \{ 1 - (1 - \beta_1)^\lambda | P_1^{I_1} \}, \{ 1 - (1 - \gamma_1)^\lambda | P_1^{F_1} \} \},$
- (3) $\lambda(N_1) = \bigcup_{\alpha_1 \in T_1, \beta_1 \in I_1, \gamma_1 \in F_1} \{ \{ 1 - (1 - \lambda_1)^\lambda | P_1^{T_1} \}, \{ (\beta_1)^\lambda | P_1^{I_1} \}, \{ (\gamma_1)^\lambda | P_1^{F_1} \} \},$
- (4) $N_1 \oplus N_2 = \bigcup_{\substack{\alpha_1 \in T_1, \beta_1 \in I_1, \gamma_1 \in F_1, \\ \eta_2 \in T_2, \pi_2 \in I_2, \mu_2 \in F_2}} \{ \{ \alpha_1 + \eta_2 - \alpha_2 \eta_2 | P_1^{T_1} P_2^{T_2} \}, \{ \beta_1 \pi_2 | P_1^{I_1} P_2^{I_2} \}, \{ \gamma_1 \mu_2 | P_1^{F_1} P_2^{F_2} \} \},$
- (5) $N_1 \otimes N_2 = \bigcup_{\substack{\alpha_1 \in T_1, \beta_1 \in I_1, \gamma_1 \in F_1, \\ \eta_2 \in T_2, \pi_2 \in I_2, \mu_2 \in F_2}} \{ \{ \alpha_1 \eta_2 | P_1^{T_1} P_2^{T_2} \}, \{ \beta_1 + \pi_2 - \beta_1 \pi_2 | P_1^{I_1} P_2^{I_2} \}, \{ \gamma_1 + \mu_2 - \gamma_1 \mu_2 | P_1^{F_1} P_2^{F_2} \} \},$

where $P_1^{T_1}; P_1^{I_1}$ and $P_1^{F_1}$ are hesitant probabilities of $\alpha_1 \in T_1, \beta_1 \in I_1$ and $\gamma_1 \in F_1$, respectively. $P_2^{T_2}; P_2^{I_2}$ and $P_2^{F_2}$ are corresponding hesitant probabilities of $\eta_2 \in T_2, \pi_2 \in I_2$ and $\mu_2 \in F_2$.

Definition 5. $P(Y)$ depicts the power set of a domain Y . Fuzzy measure $\mu: P(Y) \rightarrow [0, 1]$ satisfies conditions:

- (1) $\mu(\emptyset) = 0, \mu(Y) = 1$;
- (2) $A \subseteq B$, then $\mu(A) \leq \mu(B), \forall A, B \subseteq P(Y)$;

Fuzzy measure μ satisfies property, $\forall Y_1, Y_2 \in P(Y), A \cap B = \emptyset$

$$\mu(Y_1 \cup Y_2) = \mu(Y_1) + \mu(Y_2) + \lambda\mu(Y_1)\mu(Y_2), \lambda \in (-1, \infty).$$

Then, μ is described a λ -fuzzy measure.

Theorem 1. A λ -fuzzy measure μ in a discourse $Y = \{y_1, y_2, \dots, y_n\}$ satisfies the following formula:

$$\mu(X) = \begin{cases} \frac{1}{\lambda}(\prod_{i=1}^n [1 + \lambda\mu(y_i)] - 1) & \text{if } \lambda \neq 0, \\ \sum_{i=1}^n \mu(y_i) & \text{if } \lambda = 0. \end{cases} \tag{3}$$

Since $\mu(Y) = 1$, parameter λ can be determined by

$$\lambda = \prod_{i=1}^n (1 + \lambda\mu(x_i)) - 1.$$

Definition 6. f is a real function on $Y = \{y_1, y_2, \dots, y_n\}$. The Choquet integral about fuzzy measure μ is depicted by:

$$\int f d\mu = \sum_{a=1}^n f(y_{\pi(a)})[\mu(A_{(a)}) - \mu(A_{(a+1)})]$$

in which $\{x_{\pi(1)}, x_{\pi(2)}, \dots, x_{\pi(n)}\}$ is a new rank of $Y, f(y_{\pi(1)}) \leq f(y_{\pi(2)}) \leq \dots \leq f(y_{\pi(n)})$, $A_{(a)} = \{y_{\pi(a)}, y_{\pi(a+1)}, \dots, x_{\pi(n)}\}$ and $A_{(n+1)} = \emptyset$.

3. PNHFSS and Aggregation Operators

The PNHFSS and PNHFSS operators are proposed in this section. Some basic properties are verified.

3.1. The Comparison Method of PNHFSS

When we describe decision information with PNHFSS theory, a comparison method of PNHFSS is necessary. Thus, a approach of ranking PNHFSS is established.

Definition 7. Supposing that $N = \{T|P^T, I|P^I, F|P^F\}$ is an PNHFSS, then the score function of the PNHFSS is expressed by the following formula:

$$S(N) = \sum_{a=1}^{\#T} \alpha_a P_a^T + \sum_{b=1}^{\#I} (1 - \beta_b) P_b^I - \sum_{c=1}^{\#F} \gamma_c P_c^F. \tag{4}$$

Definition 8. Supposing that $N = \{T|P^T, I|P^I, F|P^F\}$ is an PNHFSS, then we can find the deviation function $D(N)$:

$$D(N) = \sum_{a=1}^{\#T} (\alpha_a - S(N))^2 \cdot P_a^T + \sum_{b=1}^{\#I} (1 - \beta_b - S(N))^2 \cdot P_b^I + \sum_{c=1}^{\#F} (\gamma_c - S(N))^2 \cdot P_c^F. \tag{5}$$

The distance from the score valued in the PNHFSS N is described by the deviation function. Thus, the deviation value is called a consistency indicator of the PNHFSS N . The higher value of $D(N)$, the lower consistency of N . Based on Definitions 7 and 8, a method for ranking two PNHFSS is developed.

Definition 9. For PNHFEs N_1 and N_2 , the sort of N_1 and N_2 is determined by:

- (1) If $S(N_1) > S(N_2)$, it indicates that PNHFE N_1 is superior to N_2 ;
- (2) If $S(N_1) = S(N_2), D(N_1) > D(N_2)$, it indicates that PNHFE N_1 is inferior to N_2 ;
- (3) If $S(N_1) = S(N_2), D(N_1) = D(N_2)$, it indicates that PNHFE N_1 is equal to N_2 .

3.2. The PNHFCOA Operator and PNHFCOG Operator

This section constructed the PNHFCOA operator and PNHFCOG operator under the probabilistic neutrosophic environment, and some basic properties are investigated. In this section, μ describes a fuzzy measure on a domain $X, k = 1, 2, \dots, n$.

Definition 10. Suppose that $N_k = \{T_k|P^{T_k}, I_k|P^{I_k}, F_k|P^{F_k}\}$ describes an PNHFE in a reference set X . Then, the PNHFCOA operator is expressed by the following formula:

$$\begin{aligned}
 \text{PNHFCOA}(N_1, N_2, \dots, N_n) &= \bigoplus_{\pi(k)=1}^n \mu_{\pi(k)} N_{\pi(k)} \\
 &= \bigcup_{\alpha_{\pi(k)} \in T_{\pi(k)}, \beta_{\pi(k)} \in I_{\pi(k)}, \gamma_{\pi(k)} \in F_{\pi(k)}} \left\{ \left\{ 1 - \prod_{\pi(k)=1}^n (1 - \alpha_{\pi(k)})^{\mu_{\pi(k)}} \mid \prod_{\pi(k)=1}^n P_{\pi(k)}^{T_{\pi(k)}} \right\}, \right. \\
 &\quad \left. \left\{ \prod_{\pi(k)=1}^n (\beta_{\pi(k)})^{\mu_{\pi(k)}} \mid \prod_{\pi(k)=1}^n P_{\pi(k)}^{I_{\pi(k)}} \right\}, \left\{ \prod_{\pi(k)=1}^n (\gamma_{\pi(k)})^{\mu_{\pi(k)}} \mid \prod_{\pi(k)=1}^n P_{\pi(k)}^{F_{\pi(k)}} \right\} \right\}, \tag{6}
 \end{aligned}$$

where $P_{\pi(k)}^{T_{\pi(k)}}, P_{\pi(k)}^{I_{\pi(k)}}$ and $P_{\pi(k)}^{F_{\pi(k)}}$ are corresponding probability data of $\alpha_{\pi(k)}, \beta_{\pi(k)}$ and $\gamma_{\pi(k)}$. $\mu_{\pi(k)} = \mu(F_{\pi(k)} - F_{\pi(i+1)}), F_{\pi(k)} = \{x_{\pi(1)}, x_{\pi(2)}, \dots, x_{\pi(k)}\}$ and $F_{\pi(0)} = 0$. $\{n_{\pi(k)}\}$ is a sequence such that $n_{\pi(1)} \leq n_{\pi(2)} \leq \dots \geq n_{\pi(m)}$.

Theorem 2. Supposing that N_k describes an PNHFE, PNHFCOA operator $\text{PNHFCOA}(N_1, N_2, \dots, N_n)$ is still an PNHFE.

Proof. The mathematical induction can be utilized.

- (1) When $n = 1$, we have the following equation by Definition 10:

$$\text{PNHFCOA}(N_1) = \mu_{\pi(1)} \oplus N_{\pi(1)} = N_1. \tag{7}$$

Obviously, $\text{PNHFCOA}\{N_1\}$ is an PNHFE.

- (2) When $n = 2$, we have

$$\begin{aligned}
 \text{PNHFCOA}(N_1, N_2) &= (\mu_{\pi(1)} N_{\pi(1)}) \oplus (\mu_{\pi(2)} N_{\pi(2)}) \\
 &= \bigcup_{\alpha_{\pi(1)} \in T_{\pi(1)}, \beta_{\pi(1)} \in I_{\pi(1)}, \gamma_{\pi(1)} \in F_{\pi(1)}} \left\{ 1 - (1 - \alpha_{\pi(1)})^{\mu_{\pi(1)}} \mid P_{\pi(1)}^{T_{\pi(1)}}, \beta_{\pi(1)}^{\mu_{\pi(1)}} \mid P_{\pi(1)}^{I_{\pi(1)}}, \gamma_{\pi(1)}^{\mu_{\pi(1)}} \mid P_{\pi(1)}^{F_{\pi(1)}} \right\} \\
 &\oplus \bigcup_{\alpha_{\pi(2)} \in T_{\pi(2)}, \beta_{\pi(2)} \in I_{\pi(2)}, \gamma_{\pi(2)} \in F_{\pi(2)}} \left\{ 1 - (1 - \alpha_{\pi(2)})^{\mu_{\pi(2)}} \mid P_{\pi(2)}^{T_{\pi(2)}}, \beta_{\pi(2)}^{\mu_{\pi(2)}} \mid P_{\pi(2)}^{I_{\pi(2)}}, \gamma_{\pi(2)}^{\mu_{\pi(2)}} \mid P_{\pi(2)}^{F_{\pi(2)}} \right\} \\
 &= \bigcup_{\alpha_{\pi(k)} \in T_{\pi(k)}, \beta_{\pi(k)} \in I_{\pi(k)}, \gamma_{\pi(k)} \in F_{\pi(k)}} \left\{ 1 - \prod_{\pi(k)=1}^2 (1 - \alpha_{\pi(k)})^{\mu_{\pi(k)}} \mid \prod_{\pi(k)=1}^2 P_{\pi(k)}^{T_{\pi(k)}}, \right. \\
 &\quad \left. \prod_{\pi(k)=1}^2 \beta_{\pi(k)}^{\mu_{\pi(k)}} \mid \prod_{\pi(k)=1}^2 P_{\pi(k)}^{I_{\pi(k)}}, \prod_{\pi(k)=1}^2 \gamma_{\pi(k)}^{\mu_{\pi(k)}} \mid \prod_{\pi(k)=1}^2 P_{\pi(k)}^{F_{\pi(k)}} \right\}.
 \end{aligned}$$

Thus, we know $\text{PNHFCOA}\{N_1, N_2\}$ is an PNHFE.

- (3) When $n = k$, Equation (9) is true, and we have

$$\begin{aligned}
 PNHFCOA(N_1, N_2, \dots, N_k) &= \bigoplus_{\pi(k)=1}^k \mu_{\pi(k)} N_{\pi(k)} \\
 &= \bigcup_{\alpha_{\pi(k)} \in T_{\pi(k)}, \beta_{\pi(k)} \in I_{\pi(k)}, \gamma_{\pi(k)} \in F_{\pi(k)}} \left\{ \left\{ 1 - \prod_{\pi(k)=1}^k (1 - \alpha_{\pi(k)})^{\mu_{\pi(k)}} \mid \prod_{\pi(k)=1}^k P_{\pi(k)}^{T_{\pi(k)}} \right\}, \right. \\
 &\quad \left. \left\{ \prod_{\pi(k)=1}^k (\beta_{\pi(k)})^{\mu_{\pi(k)}} \mid \prod_{\pi(k)=1}^k P_{\pi(k)}^{I_{\pi(k)}} \right\}, \left\{ \prod_{\pi(k)=1}^k (\gamma_{\pi(k)})^{\mu_{\pi(k)}} \mid \prod_{\pi(k)=1}^k P_{\pi(k)}^{F_{\pi(k)}} \right\} \right\}.
 \end{aligned}$$

Thus, the next formula is obtained, $n = k + 1$,

$$\begin{aligned}
 PNHFCOA(N_1, N_2, \dots, N_k, N_{k+1}) &= (\bigoplus_{\pi(k)=1}^k \mu_{\pi(k)} N_{\pi(k)}) \oplus (\mu_{\pi(k+1)} N_{\pi(k+1)}) \\
 &= \bigcup_{\alpha_{\pi(k)} \in T_{\pi(k)}, \beta_{\pi(k)} \in I_{\pi(k)}, \gamma_{\pi(k)} \in F_{\pi(k)}} \left\{ \left\{ 1 - \prod_{\pi(k)=1}^k (1 - \alpha_{\pi(k)})^{\mu_{\pi(k)}} \mid \prod_{\pi(k)=1}^k P_{\pi(k)}^{T_{\pi(k)}} \right\}, \right. \\
 &\quad \left. \left\{ \prod_{\pi(k)=1}^k (\beta_{\pi(k)})^{\mu_{\pi(k)}} \mid \prod_{\pi(k)=1}^k P_{\pi(k)}^{I_{\pi(k)}} \right\}, \left\{ \prod_{\pi(k)=1}^k (\gamma_{\pi(k)})^{\mu_{\pi(k)}} \mid \prod_{\pi(k)=1}^k P_{\pi(k)}^{F_{\pi(k)}} \right\} \right\} \\
 \oplus &\bigcup_{\alpha_{\pi(k+1)} \in T_{\pi(k+1)}, \beta_{\pi(k+1)} \in I_{\pi(k+1)}, \gamma_{\pi(k+1)} \in F_{\pi(k+1)}} \left\{ 1 - (1 - \alpha_{\pi(k+1)})^{\mu_{\pi(k+1)}} \mid P_{\pi(k+1)}^{T_{\pi(k+1)}}, \right. \\
 &\quad \left. \beta_{\pi(k+1)}^{\mu_{\pi(k+1)}} \mid P_{\pi(k+1)}^{I_{\pi(k+1)}}, \gamma_{\pi(k+1)}^{\mu_{\pi(k+1)}} \mid P_{\pi(k+1)}^{F_{\pi(k+1)}} \right\} \\
 &= \bigcup_{\alpha_{\pi(k)} \in T_{\pi(k)}, \beta_{\pi(k)} \in I_{\pi(k)}, \gamma_{\pi(k)} \in F_{\pi(k)}} \left\{ \left\{ 1 - \prod_{\pi(k)=1}^{k+1} (1 - \alpha_{\pi(k)})^{\mu_{\pi(k)}} \mid \prod_{\pi(k)=1}^{k+1} P_{\pi(k)}^{T_{\pi(k)}} \right\}, \right. \\
 &\quad \left. \left\{ \prod_{\pi(k)=1}^{k+1} (\beta_{\pi(k)})^{\mu_{\pi(k)}} \mid \prod_{\pi(k)=1}^{k+1} P_{\pi(k)}^{I_{\pi(k)}} \right\}, \left\{ \prod_{\pi(k)=1}^{k+1} (\gamma_{\pi(k)})^{\mu_{\pi(k)}} \mid \prod_{\pi(k)=1}^{k+1} P_{\pi(k)}^{F_{\pi(k)}} \right\} \right\}.
 \end{aligned}$$

Thus, for any n , the conclusion is right.

□

Next, when the fuzzy measure satisfies different conditions, different types of PNHFCOA can be obtained.

- (1) Assume $\mu(F) = 1$, then

$$PNHFCOA(N_1, N_2, \dots, N_n) = \max\{N_1, N_2, \dots, N_n\}.$$

- (2) Assume $\mu(F) = 0$, then

$$PNHFCOA(N_1, N_2, \dots, N_n) = \min\{N_1, N_2, \dots, N_n\}.$$

- (3) Assume the condition $\mu(x_{\pi(k)}) = \mu(F_{\pi(k)} - F_{\pi(k-1)})$ is independent, the PNHFCOA operator is described an PNHFWA operator,

$$\begin{aligned}
 PNHFWA(N_1, N_2, \dots, N_n) &= \bigoplus_{k=1}^n \mu(x_k) N_k \\
 &= \bigcup_{\alpha_k \in T_k, \beta_k \in I_k, \gamma_k \in F_k} \left\{ \left\{ 1 - \prod_{k=1}^n (1 - \alpha_k)^{\mu(x_k)} \mid \prod_{k=1}^n P_k^{T_k} \right\}, \left\{ \prod_{k=1}^n (\beta_k)^{\mu(x_k)} \mid \prod_{k=1}^n P_k^{I_k} \right\}, \left\{ \prod_{k=1}^n (\gamma_k)^{\mu(x_k)} \mid \prod_{k=1}^n P_k^{F_k} \right\} \right\}.
 \end{aligned}$$

- (4) Assume the condition $\mu(x_{\pi(k)}) = \frac{1}{n}$, the PNHFCOA operator and PNHFWA operator reduce to the PNHFA operator,

$$\begin{aligned}
 \text{PNHFWA}(N_1, N_2, \dots, N_n) &= \bigoplus_{k=1}^n \mu(x_k) N_k \\
 &= \bigcup_{\alpha_k \in T_k, \beta_k \in I_k, \gamma_k \in F_k} \left\{ \left[1 - \prod_{k=1}^n (1 - \alpha_k)^{\frac{1}{n}} \right] \prod_{k=1}^n P_k^{T_k}, \left[\prod_{k=1}^n (\beta_k)^{\frac{1}{n}} \right] \prod_{k=1}^n P_k^{I_k}, \left[\prod_{k=1}^n (\gamma_k)^{\frac{1}{n}} \right] \prod_{k=1}^n P_k^{F_k} \right\}.
 \end{aligned}$$

Theorem 3. (Monotonicity) Suppose $N_k = \{\{\alpha_k | P_k^{T_k}\}, \{\beta_k | P_k^{I_k}\}, \{\gamma_k | P_k^{F_k}\}\}$ and $\tilde{N}_k = \{\{\tilde{\alpha}_k | P_k^{\tilde{T}_k}\}, \{\tilde{\beta}_k | P_k^{\tilde{I}_k}\}, \{\tilde{\gamma}_k | P_k^{\tilde{F}_k}\}\}$ indicate two PNHFEs. The factor $\pi(k)$ satisfies $N_{\pi(1)} \geq N_{\pi(2)} \geq \dots \geq N_{\pi(n)}$ and $\tilde{N}_{\pi(1)} \geq \tilde{N}_{\pi(2)} \geq \dots \geq \tilde{N}_{\pi(n)}$. For any $N_{\pi(k)}, \tilde{N}_{\pi(k)}$, there are $\alpha_{\pi(k)} \leq \tilde{\alpha}_{\pi(k)}, \beta_{\pi(k)} \geq \tilde{\beta}_{\pi(k)}, \gamma_{\pi(k)} \geq \tilde{\gamma}_{\pi(k)}$ and $P_{\pi(k)}^{T_{\pi(k)}} = P_{\pi(k)}^{\tilde{T}_{\pi(k)}}, P_{\pi(k)}^{I_{\pi(k)}} = P_{\pi(k)}^{\tilde{I}_{\pi(k)}}, P_{\pi(k)}^{F_{\pi(k)}} = P_{\pi(k)}^{\tilde{F}_{\pi(k)}}$. Then,

$$\text{PNHFCOA}\{N_1, N_2, \dots, N_n\} \leq \text{PNHFCOA}\{\tilde{N}_1, \tilde{N}_2, \dots, \tilde{N}_n\}.$$

Proof. By Definition 7, $N_{\pi(k)} \leq \tilde{N}_{\pi(k)}$. By Definition 10, the following inequality is obtained:

$$\begin{aligned}
 &(1 - \prod(1 - \alpha_{\pi(k)})^{\mu_{\pi(k)}}) \prod P_{\pi(k)}^{T_{\pi(k)}} + (1 - \prod(\beta_{\pi(k)})^{\mu_{\pi(k)}}) \prod P_{\pi(k)}^{I_{\pi(k)}} - \prod(\gamma_{\pi(k)})^{\mu_{\pi(k)}} \prod P_j^{F_{\pi(k)}} \leq \\
 &(1 - \prod(1 - \tilde{\alpha}_{\pi(k)})^{\mu_{\pi(k)}}) \prod P_{\pi(k)}^{\tilde{T}_{\pi(k)}} + (1 - \prod(\tilde{\beta}_{\pi(k)})^{\mu_{\pi(k)}}) \prod P_{\pi(k)}^{\tilde{I}_{\pi(k)}} - \prod(\tilde{\gamma}_{\pi(k)})^{\mu_{\pi(k)}} \prod P_j^{F_{\pi(k)}}.
 \end{aligned}$$

Then, by Definitions 7 and 9, the result is proved

$$\text{PNHFCOA}(N_1, N_2, \dots, N_n) \leq \text{PNHFCOA}(\tilde{N}_1, \tilde{N}_2, \dots, \tilde{N}_n).$$

□

Theorem 4. (Boundedness) Suppose $N_k = \{\{\alpha_k | P_k^{T_k}\}, \{\beta_k | P_k^{I_k}\}, \{\gamma_k | P_k^{F_k}\}\}$ indicate an PNHFE,

$$\begin{aligned}
 N^- &= \{\{\min\{\alpha_k\} | \min\{P_k^{T_k}\}\}, \{\max\{\beta_k\} | \max\{P_k^{I_k}\}\}, \{\max\{\gamma_k\} | \max\{P_k^{F_k}\}\}\}, \\
 N^+ &= \{\{\max\{\alpha_k\} | \max\{P_k^{T_k}\}\}, \{\min\{\beta_k\} | \min\{P_k^{I_k}\}\}, \{\min\{\gamma_k\} | \min\{P_k^{F_k}\}\}\}.
 \end{aligned}$$

Then,

$$\text{PNHFCOA}(N^-) \leq \text{PNHFCOA}(N_1, N_2, \dots, N_n) \leq \text{PNHFCOA}(N^+).$$

Proof. $\forall N_k$, we know:

$$\begin{aligned}
 \min\{\alpha_k\} &\leq \alpha_{\pi(k)} \leq \max\{\alpha_k\}, \min\{\beta_k\} \leq \beta_{\pi(k)} \leq \max\{\beta_k\}, \min\{\gamma_k\} \leq \gamma_{\pi(k)} \leq \max\{\gamma_k\}; \\
 \min\{P_k^{T_k}\} &\leq P_{\pi(k)}^{T_{\pi(k)}} \leq \max\{P_k^{T_k}\}, \min\{P_k^{I_k}\} \leq P_{\pi(k)}^{I_{\pi(k)}} \leq \max\{P_k^{I_k}\}, \min\{P_k^{F_k}\} \leq P_{\pi(k)}^{F_{\pi(k)}} \leq \max\{P_k^{F_k}\}.
 \end{aligned}$$

Thus,

$$\begin{aligned}
 1 - \prod(1 - \alpha_{\pi(k)})^{(\mu(F_{\pi(k)}) - \mu(F_{\pi(k-1)}))} &\geq 1 - \prod(1 - \min\{\alpha_i\})^{(\mu(F_{\pi(k)}) - \mu(F_{\pi(k-1)}))} \\
 &= 1 - (1 - \min\{\alpha_k\})^{\sum(\mu(F_{\pi(k)}) - \mu(F_{\pi(k-1)}))} \\
 &= \min\{\alpha_k\} = \min\{\alpha_{\pi(k)}\}, \\
 \prod(\beta_{\pi(k)})^{(\mu(F_{\pi(k)}) - \mu(F_{\pi(k-1)}))} &\leq \prod(\max\{\beta_k\})^{(\mu(F_{\pi(k)}) - \mu(F_{\pi(k-1)}))} \\
 &= (\max\{\beta_k\})^{\sum(\mu(F_{\pi(k)}) - \mu(F_{\pi(k-1)}))} \\
 &= \max\{\beta_k\} = \max\{\beta_{\pi(k)}\},
 \end{aligned}$$

$$\begin{aligned} \prod (\gamma_{\pi(k)})^{(\mu(F_{\pi(k)}) - \mu(F_{\pi(k-1)}))} &\leq \prod (\max\{\gamma_k\})^{(\mu(F_{\pi(k)}) - \mu(F_{\pi(k-1)}))} \\ &= (\max\{\gamma_k\})^{\sum(\mu(F_{\pi(k)}) - \mu(F_{\pi(k-1)}))} \\ &= \max\{\gamma_k\} = \max\{\gamma_{\pi(k)}\}. \end{aligned}$$

For the probabilities, it is easy to get

$$\begin{aligned} \prod \min\{P_k^{T_k}\} &= \min\{P_{\pi(k)}^{T_{\pi(k)}}\} \leq \prod P_{\pi(k)}^{T_{\pi(k)}}, \\ \prod P_{\pi(k)}^{I_{\pi(k)}} &\leq \prod \max\{P_{\pi(k)}^{I_{\pi(k)}}\} = \prod \max\{P_i^{I_i}\}, \\ \prod P_{\pi(k)}^{F_{\pi(k)}} &\leq \prod \max\{P_{\pi(k)}^{F_{\pi(k)}}\} = \prod \max\{P_i^{F_i}\}. \end{aligned}$$

Therefore, we have:

$$PNHFCOA(N^-) = \bigcup \{ \{ \min\{\alpha_{\pi(k)}\} | \prod \min\{P_k^{T_k}\}, \{ \max\{\beta_{\pi(k)}\} | \prod \max\{P_{\pi(k)}^{I_{\pi(k)}}\}, \{ \max\{\gamma_{\pi(k)}\} | \prod \max\{P_{\pi(k)}^{F_{\pi(k)}}\} \} \}.$$

By Definitions 7 and 9,

$$PNHFCOA(N^-) \leq PNHFCOA(N_1, N_2, \dots, N_n).$$

Similarly, we can get

$$PNHFCOA(N_1, N_2, \dots, N_n) \leq PNHFCOA(N^+).$$

□

Theorem 5. (Idempotency) Supposing $N_i = \{ \{ \alpha | P_1 \}, \{ \beta | P_2 \}, \{ \gamma | P_3 \} \}$ ($i = 1, 2, \dots, n$) is a normalized PNHF, μ is a fuzzy measure on X ; then,

$$PNHFCOA(N_1, N_2, \dots, N_X) = \{ \{ \alpha | P_1 \}, \{ \beta | P_2 \}, \{ \gamma | P_3 \} \}. \tag{8}$$

Proof. When $P_1, P_2, P_3 \in \{1\}$, we have the following equation:

$$\prod P_j = P_j \quad (j = 1, 2, 3).$$

Based on Definition 10, it is expressed by the following formula:

$$\begin{aligned} &PNHFCOA(N_1, N_2, \dots, N_n) \\ &= \{ \{ 1 - \prod (1 - \alpha)^{\mu_{\pi(k)}} | \prod P_1 \}, \{ \prod (\beta)^{\mu_{\pi(k)}} | \prod P_2 \}, \{ \prod (\gamma)^{\mu_{\pi(k)}} | \prod P_3 \} \} \\ &= \{ \{ 1 - (1 - \alpha)^{\sum(\mu(F_{\pi(k)}) - \mu(F_{\pi(k-1)}))} | P_1 \}, \{ (\beta)^{\sum(\mu(F_{\pi(k)}) - \mu(F_{\pi(k-1)}))} | P_2 \}, \{ (\gamma)^{\sum(\mu(F_{\pi(k)}) - \mu(F_{\pi(k-1)}))} | P_3 \} \} \\ &= \{ \{ \alpha | P_1 \}, \{ \beta | P_2 \}, \{ \gamma | P_3 \} \}. \end{aligned}$$

□

Theorem 6. (Commutativity) Suppose $A = \{N_1, N_2, \dots, N_n\}$ and $B = \{N_{\lambda(1)}, N_{\lambda(2)}, \dots, N_{\lambda(n)}\}$ are two finite sets. If the position of the element in $\{N_{\pi(1)}, N_{\pi(2)}, \dots, N_{\pi(n)}\}$ is changed arbitrarily to get $\{N_1, N_2, \dots, N_n\}$. Then,

$$PNHFCOA(N_1, N_2, \dots, N_n) = PNHFCOA\{N_{\lambda(1)}, N_{\lambda(2)}, \dots, N_{\lambda(n)}\}.$$

Proof. Based on Definition 10, the result is easy to get. □

Definition 11. Suppose $N_k = \{T_k|P^{T_k}, I_k|P^{I_k}, F_k|P^{F_k}\}$ indicates an PNHFPE. Then, the PNHFCEG operator is described by the following formula:

$$\begin{aligned}
 \text{PNHFCEG}(N_1, N_2, \dots, N_n) &= \otimes_{\pi(k)=1}^n \mu_{\pi(k)} N_{\pi(k)} \\
 &= \bigcup_{\alpha_{\pi(k)} \in T_{\pi(k)}, \beta_{\pi(k)} \in I_{\pi(k)}, \gamma_{\pi(k)} \in F_{\pi(k)}} \left\{ \prod_{\pi(k)=1}^n (\alpha_{\pi(k)})^{\mu_{\pi(k)}} \mid \prod_{\pi(k)=1}^n P_{\pi(k)}^{T_{\pi(k)}} \right\}, \\
 &\left\{ 1 - \prod_{\pi(k)=1}^n (1 - \beta_{\pi(k)})^{\mu_{\pi(k)}} \mid \prod_{\pi(k)=1}^n P_{\pi(k)}^{I_{\pi(k)}} \right\}, \left\{ 1 - \prod_{\pi(k)=1}^n (1 - \gamma_{\pi(k)})^{\mu_{\pi(k)}} \mid \prod_{\pi(k)=1}^n P_{\pi(k)}^{F_{\pi(k)}} \right\},
 \end{aligned} \tag{9}$$

where $P_{\pi(k)}^{T_{\pi(k)}}$, $P_{\pi(k)}^{I_{\pi(k)}}$ and $P_{\pi(k)}^{F_{\pi(k)}}$ are the corresponding probabilities of $\alpha_{\pi(k)}$, $\beta_{\pi(k)}$ and $\gamma_{\pi(k)}$. $\mu_{\pi(k)} = \mu(F_{\pi(k)} - F_{\pi(k-1)})$, $F_{\pi(k)} = \{x_{\pi(1)}, x_{\pi(2)}, \dots, x_{\pi(k)}\}$ and $F_{\pi(0)} = 0$. The factor $n_{\pi(k)}$ hold $n_{\pi(1)} \geq n_{\pi(2)} \geq \dots \geq n_{\pi(n)}$.

Theorem 7. Suppose that N_k indicates an PNHFPE, $\text{PNHFCEG}(N_1, N_2, \dots, N_n)$ is still an PNHFPE.

Similarly, the fuzzy measure satisfies different conditions, different types of PNHFCEA can be obtained.

- (1) Assume $\mu(F) = 1$, then

$$\text{PNHFCEG}(N_1, N_2, \dots, N_n) = \max\{N_1, N_2, \dots, N_n\}.$$

- (2) Assume $\mu(F) = 0$, then

$$\text{PNHFCEG}(N_1, N_2, \dots, N_n) = \min\{N_1, N_2, \dots, N_n\}.$$

- (3) Assume the prerequisite $\mu(x_{\pi(k)}) = \mu(F_{\pi(k)} - F_{\pi(i-1)})$ is independent, the PNHFCEG operator indicates an PNHFCEG operator:

$$\begin{aligned}
 \text{PNHFCEG}(N_1, N_2, \dots, N_n) &= \otimes_{k=1}^n \mu(x_k) N_k \\
 &= \bigcup_{\alpha_k \in T_k, \beta_k \in I_k, \gamma_k \in F_k} \left\{ \prod_{k=1}^n (\alpha_k)^{\mu(x_k)} \mid \prod_{k=1}^n P_k^{T_k} \right\}, \left\{ 1 - \prod_{k=1}^n (1 - \beta_k)^{\mu(x_k)} \mid \prod_{k=1}^n P_k^{I_k} \right\}, \left\{ 1 - \prod_{k=1}^n (1 - \gamma_k)^{\mu(x_k)} \mid \prod_{k=1}^n P_k^{F_k} \right\}.
 \end{aligned}$$

- (4) Assume the precondition $\mu(x_{\pi(k)}) = \frac{1}{n}$, the PNHFCEG operator and PNHFCEG operator reduce to the PNHFCEG operator:

$$\begin{aligned}
 \text{PNHFCEG}(N_1, N_2, \dots, N_n) &= \otimes_{k=1}^n \mu(x_k) N_k \\
 &= \bigcup_{\alpha_k \in T_k, \beta_k \in I_k, \gamma_k \in F_k} \left\{ \prod_{k=1}^n (\alpha_k)^{\frac{1}{n}} \mid \prod_{k=1}^n P_k^{T_k} \right\}, \left\{ 1 - \prod_{k=1}^n (1 - \beta_k)^{\frac{1}{n}} \mid \prod_{k=1}^n P_k^{I_k} \right\}, \left\{ 1 - \prod_{k=1}^n (1 - \gamma_k)^{\frac{1}{n}} \mid \prod_{k=1}^n P_k^{F_k} \right\}.
 \end{aligned}$$

Theorem 8. By analyzing the PNHFCEA operator, we can obtain the following theorems, obviously.

- (1) (Monotonicity) Assume $N_k = \{\{\alpha_k|P_k^{T_k}\}, \{\beta_k|P_k^{I_k}\}, \{\gamma_k|P_k^{F_k}\}\}$ and $\tilde{N}_k = \{\{\tilde{\alpha}_k|P_k^{T_k}\}, \{\tilde{\beta}_k|P_k^{I_k}\}, \{\tilde{\gamma}_k|P_k^{F_k}\}\}$ indicate two PNHFPEs. The factor $\pi(k)$ satisfies condition $N_{\pi(1)} \geq N_{\pi(2)} \geq \dots \geq N_{\pi(n)}$ and $\tilde{N}_{\pi(1)} \geq \tilde{N}_{\pi(2)} \geq \dots \geq \tilde{N}_{\pi(n)}$. With $\forall N_{\pi(k)}$ and $\forall \tilde{N}_{\pi(k)}$, there are $\alpha_{\pi(k)} \leq \tilde{\alpha}_{\pi(k)}$, $\beta_{\pi(k)} \geq \tilde{\beta}_{\pi(k)}$, $\gamma_{\pi(k)} \geq \tilde{\gamma}_{\pi(k)}$ and $P_{\pi(k)}^{T_{\pi(k)}} = P_{\pi(k)}^{\tilde{T}_{\pi(k)}}$, $P_{\pi(k)}^{I_{\pi(k)}} = P_{\pi(k)}^{\tilde{I}_{\pi(k)}}$, $P_{\pi(k)}^{F_{\pi(k)}} = P_{\pi(k)}^{\tilde{F}_{\pi(k)}}$. Then,

$$\text{PNHFCEG}\{N_1, N_2, \dots, N_n\} \leq \text{PNHFCEG}\{\tilde{N}_1, \tilde{N}_2, \dots, \tilde{N}_n\}.$$

- (2) (Boundedness) Assume $N_k = \{\{\alpha_k|P_k^{T_k}\}, \{\beta_k|P_k^{I_k}\}, \{\gamma_k|P_k^{F_k}\}\}$ indicates an PNHFPE,

$$N^- = \{ \{ \min\{\alpha_k\} | \min\{P_k^{T_k}\} \}, \{ \max\{\beta_k\} | \max\{P_k^{I_k}\} \}, \{ \max\{\gamma_k\} | \max\{P_k^{F_k}\} \} \},$$

$$N^+ = \{ \{ \max\{\alpha_k\} | \max\{P_k^{T_k}\} \}, \{ \min\{\beta_k\} | \min\{P_k^{I_k}\} \}, \{ \min\{\gamma_k\} | \min\{P_k^{F_k}\} \} \}.$$

Then,

$$PNHFCOG(N^-) \leq PNHFCA(N_1, N_2, \dots, N_n) \leq PNHFCOG(N^+).$$

(3) (Idempotency) Assume $N_k = \{ \{\alpha | P_1\}, \{\beta | P_2\}, \{\gamma | P_3\} \}$ is a normalized PNHF, then

$$PNHFCOG(N_1, N_2, \dots, N_n) = \{ \{\alpha | P_1\}, \{\beta | P_2\}, \{\gamma | P_3\} \}. \tag{10}$$

(4) (Commutativity) Assume $A = \{N_1, N_2, \dots, N_n\}$ and $B = \{N_{\lambda(1)}, N_{\lambda(2)}, \dots, N_{\lambda(n)}\}$ are two finite sets. If the position of the element in $\{N_{\pi(1)}, N_{\pi(2)}, \dots, N_{\pi(n)}\}$ is changed arbitrarily to get $\{N_1, N_2, \dots, N_n\}$, then:

$$PNHFCOG(N_1, N_2, \dots, N_n) = PNHFCOG\{N_{\lambda(1)}, N_{\lambda(2)}, \dots, N_{\lambda(n)}\}.$$

Lemma 1. By reference [37], if $x_k \geq 0, w_k \geq 0, \sum_{k=1}^n w_k = 1$, we have

$$\prod_{k=1}^n (x_k)^{w_k} \leq \sum_{k=1}^n w_k x_k.$$

Theorem 9. Suppose $N_k = \{ \{\alpha_k | P_k^{T_k}\}, \{\beta_k | P_k^{I_k}\}, \{\gamma_k | P_k^{F_k}\} \}$ indicates an PNHF, we have

$$PNHFCOG(N_1, N_2, \dots, N_n) \leq PNHFCA(N_1, N_2, \dots, N_n). \tag{11}$$

Proof. Based on Lemma 1, Definitions 10 and 11, the following formula is obtained:

$$\prod (\alpha_{\pi(k)})^{\mu_{\pi(k)}} \leq \sum (\mu_{\pi(k)}) \alpha_{\pi(k)} = 1 - \sum \mu_{\pi(k)} (1 - \alpha_{\pi(k)}) \leq 1 - \prod (1 - \alpha_{\pi(k)})^{\mu_{\pi(k)}}.$$

Obviously,

$$\prod (\alpha_{\pi(k)})^{\mu_{\pi(k)}} \prod P_{\pi(k)}^{T_{\pi(k)}} \leq (1 - \prod (1 - \alpha_{\pi(k)})^{\mu_{\pi(k)}}) P_{\pi(k)}^{T_{\pi(k)}}.$$

Similarly, we know

$$\prod (\beta_{\pi(k)})^{\mu_{\pi(k)}} \prod P_{\pi(k)}^{T_{\pi(k)}} \leq (1 - \prod (1 - \beta_{\pi(k)})^{\mu_{\pi(k)}}) P_{\pi(k)}^{T_{\pi(k)}},$$

$$\prod (\gamma_{\pi(k)})^{\mu_{\pi(k)}} \prod P_{\pi(k)}^{T_{\pi(k)}} \leq (1 - \prod (1 - \gamma_{\pi(k)})^{\mu_{\pi(k)}}) P_{\pi(k)}^{T_{\pi(k)}}.$$

Thus, based on Equation (4) and Definition 9,

$$PNHFCOG(N_1, N_2, \dots, N_n) \leq PNHFCA(N_1, N_2, \dots, N_n).$$

□

4. A MADM method in PNHF Environment

For a MADM problem under the PNHF environment, assume Z_1, Z_2, \dots, Z_n indicates all the alternatives, D_1, D_2, \dots, D_k indicates all the attributes. The evaluated information of Z_i with D_j is indicated by PNHF $N_{ij} = \{ \{T_{ij} | P_{ij}^{T_{ij}}\}, \{I_{ij} | P_{ij}^{I_{ij}}\}, \{F_{ij} | P_{ij}^{F_{ij}}\} \}$.

Based on these necessary prerequisites, we elicit specific steps.

- Step 1. Construct a PNHF decision matrix (PNHFD) $E = (N_{ij})_{m \times k}$.

Rank all PNHFES from smallest to largest according to Definition 9. Then, the reorder decision matrix can be obtained such that $N_{i\pi(1)} \leq N_{i\pi(2)} \leq \dots \leq N_{i\pi(n)}$, $\{\pi(1), \pi(2), \dots, \pi(n)\}$ indicates a new arrangement.

- Step 2. Calculate μ_j of criterion D_j .

In order to consider more interrelationships among criteria, by taking the g_λ fuzzy measure, the measure μ of each criterion could be determined.

- Step 3. Based on the goal, select a PNHFCA (PNHFCA) operator to aggregate all PNHFES Z_i ($i = 1, 2, \dots, n$).

When we need to consider the group's major points, the PNHFCA operator should be utilized. However, the individual major points could be emphasized based on the PNHFCA operator. Thus, different types of operators can be used based on the different demands.

- Step 4. Reorder the alternatives

By the value of $S(N_i)$, $D(N_i)$ and Definition 9, all the Z_i are ranked in decreasing order, DM picks an optimal option Z_i .

5. The Program of the Proposed Approach

Choosing the suitable 3PL plays a key role in business development, like improving efficiency and reducing costs, improving market share and service quality. ABC Machinery Manufacturing Company Limited as an automotive manufacturing company. The decision maker needs to select an optimal third part logistics supplier. There are four possible logistics suppliers that are denoted as Z_1, Z_2, Z_3, Z_4 . The decision maker selects the following four attributes to access these alternatives: D_1 : equipment system; D_2 : effectiveness; D_3 : safety; D_4 : Correlation. The PNHFDM E is obtained, depicted in Table 1. Suppose that fuzzy measures of D_j are $\mu(D_1) = 0.3$, $\mu(D_2) = 0.3$, $\mu(D_3) = 0.3$, $\mu(D_4) = 0.2$, respectively.

- Step 1. Calculate the score values of all Z_i . The results are depicted in Table 2.

Thus, we can get reordered PNHFDM E' . Because of space constraints, the decision matrix E' is omitted.

- Step 2. Since the information of fuzzy measure is $\mu(C_1) = 0.3$, $\mu(C_2) = 0.3$, $\mu(C_3) = 0.3$, $\mu(C_4) = 0.2$, respectively. By Equation (3), we get $\lambda = -0.2317$. Thus, taking Z_1 as an example, we can get

$$\mu_{\pi(1)} = 0.2477, \mu_{\pi(2)} = 0.1732, \mu_{\pi(3)} = 0.2791, \mu_{\pi(4)} = 0.3.$$

- Step 3. Utilizing the PNHFCA operator, by Equation (9), we can get

$$S(Z_1) = 0.6466, S(Z_2) = 0.6436, S(Z_3) = 0.5822, S(Z_4) = 0.6950.$$

- Step 4. Rank the PNHFES by Definition 9,

$$Z_4 > Z_1 > Z_2 > Z_3.$$

The 3PL Company Z_1 is an optimal option.

Next, suppose that the PNHFCA operator is utilized to solve this problem. Similarly, the score value of alternative A_i is obtained:

$$S(Z_1) = 0.6181, S(Z_2) = 0.6167, S(Z_3) = 0.5639, S(Z_4) = 0.6686.$$

Thus, the final ranking of alternatives is determined, as follows:

$$Z_4 > Z_1 > Z_2 > Z_3.$$

The 3PL Company A_1 is an optimal option.

Table 1. A PNHFDM Information E .

C_1	
A_1	$\{\{0.5 0.3, 0.57 0.22, 0.58 0.27, 0.64 0.21\}, \{0.43 0.25, 0.48 0.2, 0.49 0.30, 0.55 0.25\}, \{0.41 0.27, 0.47 0.23, 0.52 0.23, 0.46 0.27\}\}$
A_2	$\{\{0.44 0.27, 0.49 0.24, 0.48 0.26, 0.52 0.23\}, \{0.46 0.47, 0.53 0.53\}, \{0.29 0.18, 0.33 0.14, 0.36 0.20, 0.41 0.16, 0.41 0.18, 0.47 0.14\}\}$
A_3	$\{\{0.41 0.30, 0.48 0.22, 0.47 0.27, 0.53 0.21\}, \{0.46 0.23, 0.49 0.26, 0.49 0.24, 0.53 0.27\}, \{0.39 0.24, 0.41 0.25, 0.48 0.26, 0.45 0.25\}\}$
A_4	$\{\{0.47 0.25, 0.51 0.24, 0.50 0.26, 0.53 0.25\}, \{0.34 0.33, 0.43 0.35, 0.5 0.32\}, \{0.42 0.28, 0.45 0.21, 0.53 0.29, 0.56 0.22\}\}$
C_2	
A_1	$\{\{0.40 0.26, 0.51 0.25, 0.49 0.25, 0.58 0.24\}, \{0.56 0.27, 0.59 0.24, 0.60 0.26, 0.63 0.23\}, \{0.39 0.23, 0.43 0.29, 0.42 0.21, 0.47 0.27\}\}$
A_2	$\{\{0.51 0.53, 0.54 0.47\}, \{0.49 0.25, 0.52 0.22, 0.57 0 : 28, 0.60 0.25\}, \{0.43 0.18, 0.46 0.18, 0.48 0.17, 0.50 0.16, 0.53 0.16, 0.55 0.15\}\}$
A_3	$\{\{0.54 0.26, 0.60 0.25, 0.63 0.25, 0.68 0.24\}, \{0.50 0.48, 0.56 0.52\}, \{0.43 0.26, 0.46 0.24, 0.46 0.26T, 0.50 0.24\}\}$
A_4	$\{\{0.61 0.54, 0.67 0.46\}, \{0.43 0.27, 0.50 0.26, 0.46 0.24, 0.53 0.23\}, \{0.42 0.23, 0.50 0.24T, 0.45 0.26, 0.53 0.27\}\}$
C_3	
A_1	$\{\{0.56 0.24, 0.62 0.24, 0.59 0.26, 0.64 0.26\}, \{0.33 0.25, 0.36 0.24, 0.37 0.26, 0.41 0.25\}, \{0.36 0.33, 0.42 0.36, 0.45 0.31\}\}$
A_2	$\{\{0.65 0.24T, 0.69 0.27, 0.67 0.23, 0.71 0.26\}, \{0.43 0.31, 0.52 0.23, 0.46 0.27, 0.55 0.19\}, \{0.43 0.26, 0.46 0.25, 0.50 0.25, 0.53 0.24\}\}$
A_3	$\{\{0.51 0.26, 0.54 0.26, 0.57 0.24, 0.60 0.24\}, \{0.43 0.26, 0.46 0.24, 0.48 0.26, 0.52 0.24\}, \{0.49 0.25, 0.54 0.26, 0.57 0.24, 0.62 0.25\}\}$
A_4	$\{\{0.57 0.24, 0.66 0.28, 0.66 0.22, 0.73 0.26\}, \{0.43 0.54, 0.49 0.46\}, \{0.47 0.16, 0.53 0.17, 0.56 0.16, 0.50 0.17, 0.57 0.18, 0.59 0.17\}\}$
C_4	
A_1	$\{\{0.48 0.47, 0.57 0.53\}, \{0.40 0.51, 0.47 0.49\}, \{0.47 0.16, 0.50 0.15, 0.53 0.15, 0.49 0.19, 0.54 0.18, 0.56 0.17\}\}$
A_2	$\{\{0.51 0.27, 0.62 0.26, 0.54 0.24, 0.64 0.23\}, \{0.40 0.25, 0.46 0.28, 0.46 0.22, 0.53 0.25\}, \{0.39 0.33, 0.42 0.37, 0.45 0.30\}\}$
A_3	$\{\{0.48 0.28, 0.58 0.23, 0.51 0.26, 0.61 0 : 23\}, \{0.42 0.25, 0.45 0.24, 0.47 0.26, 0.50 0.25\}, \{0.42 0.27, 0.50 0.26, 0.45 0.24, 0.53 0.23\}\}$
A_4	$\{\{0.66 0.27, 0.73 0.24, 0.71 0 : 26, 0.77 0.23\}, \{0.43 0.38, 0.49 0.33, 0.54 0.29\}, \{0.36 0.27, 0.41 0.24, 0.39 0.26T, 0.45 0.23\}\}$

Table 2. The score values of PNHF N_{ij} .

	D_1	D_2	D_3	D_4
Z_1	0.6185	0.4700	0.8259	0.5782
Z_2	0.6081	0.4885	0.7204	0.6941
Z_3	0.5395	0.6181	0.5273	0.6072
Z_4	0.5907	0.6825	0.6562	0.8329

6. Comparison with Other Approaches

Based on the same problem background, the comparison results are described.

Wu's method emphasizes the individual (group's) major points, which utilize the MADM problems with a simplified neutrosophic environment.

Peng put forward the TOPSIS-based QUALIFIEX method and the cross-entropy measurement to manage MADM situations with probability multi-valued neutrosophic elements. Then, the effectiveness of this method is demonstrated by an illustrative example.

However, in many actual problems, attributes are not independent. But by comparing the above methods, the association between attributes is not considered. However, attribute correlation is considered in our proposed method. The decision results are more reasonable and effective. The final results by different approaches are indicated in Table 3.

Table 3. Choice of optimal result.

Method	Sort of Results	Optimal Alternative	Worst Alternative
<i>TOPSIS-based QUALIFLEX method [3]</i>	$Z_4 > Z_2 > Z_1 > Z_3$	Z_4	Z_3
<i>SNNPWA operator [45]</i>	$Z_3 > Z_1 > Z_2 > Z_4$	Z_3	Z_4
<i>SNNPWG operator [45]</i>	$Z_3 > Z_2 > Z_1 > Z_4$	Z_2	Z_1
<i>PNHFCA operator</i>	$Z_4 > Z_1 > Z_2 > Z_3$	Z_4	Z_3
<i>PNHFCA operator</i>	$Z_4 > Z_1 > Z_2 > Z_3$	Z_4	Z_3

Through the above analysis, the DMs' evaluation information is represented by PNHFes. PNHFes are more flexible in describing the hesitant MADM information and reporting the probabilistic values of all hesitant values. The Choquet integral and aggregation operators are integrated in our method. Next, the alternatives' information is aggregated and ranked. In this model, the interrelationship between attributes are involved by Choquet integral. The MADM problems are effectively resolved by utilizing interdependencies or interactions between attributes. The results are closer to the actual situations.

7. Conclusions

Firstly, our goal is to aggregate the PNHF information by the notion of PNHFS. By applying the Choquet integral, the weight information is extended, more information about the correlation between attributes is mined. The advantage of PNHFS is that it can explain the preferences of DMS without information loss. By investigating, we found both of the PNHFCA and PNHFCA operators satisfying the following characteristics: monotonicity, boundedness, idempotency and commutativity. Those operators can reduce to some common aggregation operators. Moreover, those aggregation operators were used to an PNHF background, where fuzzy measure of attributes are recognized. All alternatives are reordered and choose an optimal option. Next, we present an illustrative situation to explain the objectivity of our method. The result received by the PNHFCA and PNHFCA operators are effective and flexible. The results are more suitable for the actual situations, because more information can be considered based on our method. Thus, when some uncertain problems or inconsistent and indeterminate information needs to be resolved, our proposed approach shows great advantages. In regard to the next jobs, more types of aggregation operators are investigated and applied in other practical situations, like medical diagnoses, group decision-making, risk evaluations, and fractal-wavelet modeling [46–50].

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