Int. J. Mach. Learn. & Cyber.
DOI 10.1007/s13042-016-0552-9

ORIGINAL ARTICLE

Multi-criteria decision-making based on generalized prioritized aggregation operators under simplified neutrosophic uncertain linguistic environment

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Received: 10 October 2015 / Accepted: 23 May 2016

Abstract A simplified neutrosophic uncertain linguistic set that integrates quantitative and qualitative evaluation can serve as an extension of both an uncertain linguistic variable and a simplified neutrosophic set. It can describe the real preferences of decision-makers and reflect their uncertainty, incompleteness and inconsistency. This paper focuses on multi-criteria decision-making (MCDM) problems in which the criteria occupy different priority levels and the criteria values take the form of simplified neutrosophic uncertain linguistic elements. Having reviewed the relevant literatures, this paper develops some generalized simplified neutrosophic uncertain linguistic prioritized weighted aggregation operators and applies them to solve MCDM problems. Finally, an illustrative example is given, and two cases of comparison analysis are conducted with other representative methods to demonstrate the effectiveness and feasibility of the developed approach.

Keywords Multi-criteria decision-making · Simplified neutrosophic uncertain linguistic element · Linguistic scale function · Generalized prioritized aggregation operator

1 Introduction

Since Zadeh [1] proposed fuzzy sets (FSs) in 1965, they have come to be regarded as a powerful tool with applications across various fields [2], such as fuzzy classification [3, 4], learning rules discovery [5], intrusion detection [6], and modeling fuzzy-in fuzzy-out systems [7]. In order to deal with the uncertainty of non-membership degree, Atanassov [8] introduced intuitionistic fuzzy sets (IFSs) as an extension of Zadeh's FSs. IFSs have been widely applied in solving multi-criteria decision-making (MCDM) problems [9–11].

Although FSs and IFSs have been developed and generalized, they cannot deal with every sort of fuzziness in real problems. In some cases, linguistic variables are effective in coping with complex or ill-defined situations [12]. In recent years, linguistic variables have been studied in depth, leading to the development of numerous MCDM methods associated with other theories. These include intuitionistic linguistic sets (ILSs), which combine IFSs and linguistic variables [13]; gray linguistic sets, which integrate gray sets and linguistic variables [14]; and hesitant fuzzy linguistic sets (HFLSs) and linguistic hesitant fuzzy sets (LHFSs) [15], both of which are based on linguistic term sets and hesitant fuzzy sets (HFSs). HFLSs and LHFSs are used to express decision-makers' hesitance, utilizing linguistic scale functions to model linguistic information. Hesitant fuzzy linguistic term sets, which describe decision-makers' preferences using several linguistic terms, are more suitable than traditional fuzzy linguistic sets in expressions [16]. Another method based on the 2-tuple linguistic information model [17, 18], can effectively avoid the information distortion that has hitherto occurred in linguistic information processing [19]. In some cases, decision-makers cannot completely express information by selecting linguistic labels, but they can describe opinions with interval linguistic labels. They are called uncertain linguistic variables. Xu [20, 21] developed some methods for solving MCDM problems with uncertain linguistic variables. Similarly, strategies like using intuitionistic uncertain linguistic variables [22] and hesitant

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Published online: 04 June 2016

fuzzy uncertain linguistic elements (HFULEs) [23] have been proposed to solve MCDM problems associated with some aggregation operators. Broadly speaking, linguistic variables and uncertain linguistic variables can express uncertain information but not incomplete or inconsistent information. For example, when a paper is sent to a reviewer, he or she may state that the paper is perhaps higher than "good" but lower than "very good". Furthermore, he or she may estimate the possibility that a statement is true at 60 %, the possibility that it is false at 50 %, and the degree to which he or she is unsure at 20 %. This issue cannot be addressed effectively with FSs or IFSs. Therefore, new theories are required.

Smarandache [24] proposed neutrosophic sets (NSs). Since it is difficult to apply NSs in real scientific and engineering situations, single-valued neutrosophic sets [25], interval neutrosophic sets (INSs) [26, 27], neutrosophic refined sets [28, 29], neutrosophic soft sets [30–34], and neutrosophic graphs [35–39] were introduced. Subsequently, studies of the various aspects of NSs have concentrated on defining operations and aggregation operators [40–42], correlation coefficients [43–46], entropy measures [47–50], similarity measures [51], and subsethood measures [52] to cope with MCDM problems. In addition, an outranking approach was developed to deal with MCDM problems with simplified neutrosophic sets (SNSs) [53] and INSs [54]. A neutrosophic normal cloud model was constructed and applied to tackle single-value neutrosophic MCDM problems [55]. To overcome the problems involved in using linguistic variables and uncertain linguistic variables associated with IFSs and HFSs, single valued neutrosophic linguistic sets (SVNLSs) [56], interval neutrosophic linguistic sets (INLSs) [57, 58], interval neutrosophic uncertain linguistic sets (INULSs) [59, 60], and single valued neutrosophic trapezoid linguistic sets [61] were introduced. However, in some cases, the operations [56–59] may be irrational. This paper will discuss these limitations and define new operations for simplified neutrosophic uncertain linguistic elements (SNULEs). As for the aforementioned example, it can be expressed as $\langle [h_5, h_6], (0.6, 0.2, 0.5) \rangle$ using SNULEs. SNULEs have enabled great progress in describing qualitative information, and to some extent they can be considered an innovative construct.

In general, aggregation operators are significant tools for addressing information fusion in MCDM problems. Aggregation operators for SVNLSs, INLSs, and INULSs include the strict assumption that the aggregated arguments all hold the same priority level. However, this assumption is not always valid in real-world applications. Consider a situation where we are selecting a car on the basis of safety and cost. We generally would not allow a benefit in terms of cost to compensate for a loss in safety. Consequently, it is significant to describe this practical issue in precise mathematical terms. Yager [62] first addressed this issue by developing the prioritized aggregation (PA) operator, which holds many advantages over other operators. For example, the PA operator does not require weight vectors, and it only needs to be informed of the priorities among the criteria. A drawback of Yager's research [62] is that it can only deal with precise situations where criteria values and weights are in the real number domain. This paper addresses the simplified neutrosophic uncertain linguistic information aggregation method, which includes priority relationships between aggregated arguments. In addition, some generalized PA operators for aggregating SNULEs that are a special case of simplified neutrosophic uncertain linguistic sets (SNULSs) are developed.

The rest of the paper is organized as follows. Section 2 reviews uncertain linguistic term sets as well as the concepts of NSs and SNSs. Section 3 provides operations for SNULEs and proposes a method for comparing SNULEs based on the linguistic scale function. Section 4 extends the traditional PA operator to the simplified neutrosophic uncertain linguistic environment. A generalized simplified neutrosophic uncertain linguistic prioritized weighted aggregation (GSNULPWA) operator is developed and some properties and special cases are discussed. In addition, a MCDM approach based on the GSNULPWA operator is introduced. Section 5 gives an illustrative example based on the proposed approach and analyzes the influence on ranking results of linguistic scale function f^* and parameter λ in GSNULPWA operators. In addition, two cases of comparison analysis between the proposed approach and the existing method are conducted. Section 6 gives some summary remarks.

2 Preliminaries

This section introduces some basic concepts and definitions related to SNULSs, including uncertain linguistic term sets, linguistic scale functions, NSs, and SNSs. These concepts will be utilized latter in analysis.

2.1 Uncertain linguistic term set

Suppose that $H = \{h_0, h_1, h_2, \ldots, h_{2t}\}$ is a finite and totally ordered discrete term set, where t is a nonnegative real number. It is required that h_i and h_j must satisfy the following characteristics [63, 64].

1. The set is ordered: $h_i < h_j$ if and only if $i < j$;
2. Negation operator: $\mathrm{neg}(h_i) = h_{(2t-i)}$.

To preserve all the given information, the discrete linguistic label $H = \{h_0, h_1, h_2, \ldots, h_{2t}\}$ is extended to a continuous label $H = \{h_i | 0 \le i \le L\}$, in which $h_i < h_j$ if and only if $i < j$, and L $(L > 2t)$ is a sufficiently large positive integer. If $h_i \in H$, then h_i is called the original linguistic term; otherwise h_i is called the virtual linguistic term. In general, the decision-maker uses the original linguistic terms to evaluate criteria and alternatives, and the virtual linguistic terms can only appear in calculation [20].

Let $\hat{h} = [h_\alpha, h_\beta]$, where, $h_\alpha, h_\beta \in H$, h_α and h_β are the lower and upper bounds, respectively. Then $\hat{h}$ is called an uncertain linguistic variable. Let $\hat{H}$ be the set of all uncertain linguistic variables [20].

Consider any two uncertain linguistic variables $\hat{h}_1 = [h_{\alpha_1}, h_{\beta_1}]$, $\hat{h}_2 = [h_{\alpha_2}, h_{\beta_2}]$, $\hat{h}_1, \hat{h}_2 \in \hat{H}$ and $\lambda \in [0,1]$, then the operational laws are defined as follows [20, 21]:

$$(1)\ \hat{h}_1 \oplus \hat{h}_2 = [h_{\alpha_1}, h_{\beta_1}] \oplus [h_{\alpha_2}, h_{\beta_2}] = [h_{\alpha_1} \oplus h_{\alpha_2}, h_{\beta_1} \oplus h_{\beta_2}] = [h_{\alpha_1+\alpha_2}, h_{\beta_1+\beta_2}];$$

$$(2)\ \hat{h}_1 \otimes \hat{h}_2 = [h_{\alpha_1}, h_{\beta_1}] \otimes [h_{\alpha_2}, h_{\beta_2}] = [h_{\alpha_1} \otimes h_{\alpha_2}, h_{\beta_1} \otimes h_{\beta_2}] = [h_{\alpha_1\alpha_2}, h_{\beta_1\beta_2}];$$

$$(3)\ \lambda\hat{h}_1 = \lambda[h_{\alpha_1}, h_{\beta_1}] = [\lambda h_{\alpha_1}, \lambda h_{\beta_1}] = [h_{\lambda\alpha_1}, h_{\lambda\beta_1}];$$

$$(4)\ (\hat{h}_1)^\lambda = \left([h_{\alpha_1}, h_{\beta_1}]\right)^\lambda = \left[(h_{\alpha_1})^\lambda, (h_{\beta_1})^\lambda\right] = \left[h_{(\alpha_1)^\lambda}, h_{(\beta_1)^\lambda}\right].$$

2.2 Linguistic scale functions

To use data more efficiently and to express semantics more flexibly, linguistic scale functions assign different semantic values to linguistic terms under different situations [15]. These functions are preferable in practice because they are flexible and can produce more deterministic results according to different semantics. For linguistic term h_i in linguistic set H, where $H = \{h_i | i = 0, 1, 2, \ldots, 2t\}$, the relationship between the element h_i and its subscript i is strictly monotonically increasing [65]. Linguistic scale functions are defined below.

Definition 1 [15]. If $\theta_i \in [0,1]$ is a numeric value, then the linguistic scale function f that conducts the mapping from h_i to θ_i $(i = 0, 1, 2, \ldots, 2t)$ is defined as follows:

$$f : h_i \to \theta_i \quad (i = 0, 1, 2, \ldots, 2t),$$

where $0 \le \theta_0 < \theta_1 < \theta_2 < \cdots < \theta_{2t}$.

Clearly, function f is strictly monotonically increasing with respect to subscript i. The symbol θ_i $(i = 0, 1, 2, \ldots, 2t)$ reflects the preferences of the decision-makers when using the linguistic term $h_i \in H$ $(i = 0, 1, 2, \ldots, 2t)$. Therefore, the function or value in fact denotes the semantics of the linguistic terms.

$$(1)\ f_1(h_i) = \theta_i = \frac{i}{2t} \quad (i = 0, 1, 2, \ldots, 2t).$$

The evaluation scale of the linguistic information given above is divided on average.

$$(2)\ f_2(h_i) = \theta_i = \begin{cases} \dfrac{\alpha^t - \alpha^{t-i}}{2\alpha^t - 2} & (i = 0, 1, 2, \ldots, t) \\ \dfrac{\alpha^t + \alpha^{i-t} - 2}{2\alpha^t - 2} & (i = t+1, t+2, \ldots, 2t) \end{cases}.$$

The value of α can be determined using a subjective approach. Let A and B be two indicators. Assume that A is far more significant than B and the importance ratio is m. Then $\alpha^k = m$ (k represents the scale level), and $\alpha = \sqrt[k]{m}$. At present, the vast majority of researchers believe that $m = 9$ is the upper limit of the importance ratio. If the scale level is 7, then $\alpha = \sqrt[7]{9} \approx 1.37$ [66]. Extending the middle of the given linguistic term set to both ends also increases, the absolute deviation between adjacent linguistic subscripts.

$$(3)\ f_3(h_i) = \theta_i = \begin{cases} \dfrac{t^\beta - (t-i)^\beta}{2t^\beta} & (i = 0, 1, 2, \ldots, t) \\ \dfrac{t^\gamma + (i-t)^\gamma}{2t^\gamma} & (i = t+1, t+2, \ldots, 2t) \end{cases}.$$

The values $\beta, \gamma \in [0,1]$ denote the curvatures of the subjective value functions for gains and losses, respectively [67]. Kahneman and Tversky [67] experimentally determined that $\beta = \gamma = 0.88$, which is consistent with empirical data. As the middle of the given linguistic term set extends to both ends, the absolute deviation between adjacent linguistic subscripts decreases.

To preserve all the given information and facilitate calculations, the above function can be expanded to $f^* : \bar{H} \to R^+0(R^+ = \{r | r \ge 0, r \in R\})$, which satisfies $f^*(h_i) = \theta_i$, and is a strictly monotonically increasing and continuous function. Therefore, the mapping from $\bar{H}$ to R^+ is one-to-one because of its monotonicity, and the inverse function of f^* exists and is denoted by f^{*-1}.

2.3 NSs and SNSs

Definition 2 [24]. Let X be a space of points (objects) with a generic element in X, denoted by x. A NS A in X is characterized by a truth-membership function $t_A(x)$, an indeterminacy-membership function $i_A(x)$ and a falsity-membership function $f_A(x)$. $t_A(x)$, $i_A(x)$ and $f_A(x)$ are real standard or nonstandard subsets of $]0^-, 1^+[$, that is, $t_A(x) : X \to]0^-, 1^+[$, $i_A(x) : X \to]0^-, 1^+[$, and $f_A(x) : X \to]$

$0^-, 1^+[$. There is no restriction on the sum of $t_A(x)$, $i_A(x)$ and $f_A(x)$, so $0^- \leq \sup t_A(x) + \sup i_A(x) + \sup f_A(x) \leq 3^+$.

Since it is hard to apply NSs to practical problems, Ye [56] reduced NSs of nonstandard interval numbers into a kind of SNSs of standard interval numbers.

Definition 3 [68]. Let X be a space of points (objects) with a generic element in X, denoted by x. A NS A in X is characterized by $t_A(x)$, $i_A(x)$ and $f_A(x)$, which are single subintervals/subsets in the real standard $[0,1]$, that is, $t_A(x): X \to [0,1]$, $i_A(x): X \to [0,1]$, and $f_A(x): X \to [0,1]$. And the sum of $t_A(x)$, $i_A(x)$ and $f_A(x)$ satisfies the condition $0 \leq t_A(x) + i_A(x) + f_A(x) \leq 3$. Then a simplification of A is denoted by $A = \{(x, t_A(x), i_A(x), f_A(x)) | x \in X\}$, which is a SNS (a subclass of NS).

For a SNS $\{(x, t_A(x), i_A(x), f_A(x)) | x \in X\}$, the ordered triple components $(t_A(x), i_A(x), f_A(x))$, are described as a simplified neutrosophic number (SNN), and each SNN can be expressed as $A = (t_A, i_A, f_A)$, where $t_A \in [0,1]$, $i_A \in [0,1]$, $f_A \in [0,1]$, and $0 \leq t_A + i_A + f_A \leq 3$.

3 SNULSs and their operations

This section introduces the advantages and applications of SNULEs. Then, operations and comparison rules for SNULEs are presented.

Definition 4 Let X be a space of points (objects) with a generic element in X, denoted by x and $\hat{H}$ be a set of uncertain linguistic variables. Then a SNULS A in X is characterized as

$$A = \left\{ \left\langle x, [h^L_{\theta(x)}, h^U_{\theta(x)}], (t(x), i(x), f(x)) \right\rangle | x \in X \right\},$$

where $[h^L_{\theta(x)}, h^U_{\theta(x)}] \in \hat{H}$, $t(x) \in [0,1]$, $i(x) \in [0,1]$ and $f(x) \in [0,1]$, with the condition $0 \leq t(x) + i(x) + f(x) \leq 3$ for any $x \in X$. And $t(x)$, $i(x)$ and $f(x)$ represent, respectively, the degrees of truth-membership, indeterminacy-membership and falsity-membership of the element x in X to the uncertain linguistic variable $[h^L_{\theta(x)}, h^U_{\theta(x)}]$.

For convenience, $a = \left\langle [h^L_{\theta_a}, h^U_{\theta_a}], (t_a, i_a, f_a) \right\rangle$ is called a SNULE and A is the set of all SNULEs. Furthermore, a degenerates to an uncertain linguistic variable if $t_a = 1$, $i_a = 0$ and $f_a = 0$.

A SNULE is an extension of an uncertain linguistic element and a SNN. Compared to uncertain linguistic variables, SNULEs can embody an evaluation object that attaches to an uncertain linguistic variable, and they can depict uncertainty and fuzziness more accurately than uncertain linguistic variables can. When compared to SNSs, SNULEs integrate uncertain linguistic variables and SNNs, in addition to assigning truth-membership, indeterminacy-membership and falsity-membership functions to a specific linguistic assessment value. This establishes the functions relative to uncertain linguistic variables rather than to a fuzzy concept. Thus, SNULEs are effective tools in solving problems defined by qualitative expressions that involve incomplete and inconsistent information.

Operations of SNULEs represent one of the essential themes. It is not appropriate to extend the operations of single valued neutrosophic linguistic numbers (SVNLNs) [56], interval neutrosophic linguistic numbers (INLNs) [57] or interval neutrosophic uncertain linguistic variables (INULVs) [59] to SNULEs because of irrational aspects in the operations of SVNLNs, INLNs, and INULVs, and the limitations are not eliminated. In the following section, discussion regarding limitations in the operations of INLNs [57] are conducted as an example, and modified operations are defined.

3.1 Operations of SNULEs

Definition 5 [57]. Let $a_1 = \langle h_{\theta_1}, ([t^L_1, t^U_1], [i^L_1, i^U_1], [f^L_1, f^U_1],) \rangle$ and $a_2 = \langle h_{\theta_2}, ([t^L_2, t^U_2], [i^L_2, i^U_2], [f^L_2, f^U_2],) \rangle$ be any two INLNs and $\lambda \geq 0$. Then the following operations of INLNs can be defined.

$$(1)\, a_1 \oplus a_2 = \langle h_{\theta_1+\theta_2}, ([t^L_1 + t^L_2 - t^L_1 t^L_2, t^U_1 + t^U_2 - t^U_1 t^U_2], [i^L_1 i^L_2, i^U_1 i^U_2], [f^L_1 f^L_2, f^U_1 f^U_2]) \rangle;$$

$$(2)\, a_1 \otimes a_2 = \langle h_{\theta_1 \theta_2}, ([t^L_1 t^L_2, t^U_1 t^U_2], [i^L_1 + i^L_2 - i^L_1 i^L_2, i^U_1 + i^U_2 - i^U_1 i^U_2], [f^L_1 + f^L_2 - f^L_1 f^L_2, f^U_1 + f^U_2 - f^U_1 f^U_2]) \rangle;$$

$$(3)\, \lambda a_1 = \left\langle h_{\lambda\theta_1}, \left(\left[1 - (1 - t^L_1)^\lambda, 1 - (1 - t^U_1)^\lambda\right], \left[(i^L_1)^\lambda, (i^U_1)^\lambda\right], \left[(f^L_1)^\lambda, (f^U_1)^\lambda\right] \right) \right\rangle;$$

$$(4)\, a_1^\lambda = \left\langle h_{\theta_1^\lambda}, \left(\left[(t^L_1)^\lambda, (t^U_1)^\lambda\right], \left[1-(1-i^L_1)^\lambda, 1-(1-i^U_1)^\lambda\right], \left[1-(1-f^L_1)^\lambda, 1-(1-f^U_1)^\lambda\right] \right) \right\rangle.$$

However, the operations presented in Definition 5 have some obvious limitations:

1. All operations are carried out directly on the basis of the subscripts of linguistic terms, which cannot reveal

critical differences in final results under various semantic situations.

2. The two parts of INLNs are processed separately in the additive operation, ignoring the correlation between them. In some situations, this might be irrational, as is shown in the following example.

Example 1 In an example of the performance evaluation of a house on two equally important parameters, let c_1 stand for the parameter "beautiful" and c_2 stand for the parameter "wooden". Suppose that $a_1 = \langle h_5, ([1,1],[0,0],[0,0])\rangle$ represents the statement that all decision-makers think the house is good, and $a_2 = \langle h_5, ([0,0],[1,1],[1,1])\rangle$ represents the statement that none of the decision-makers think the house is good. Then the comprehensive performance evaluation can be calculated using Definition 5, $a_{12} = 0.5a_1 \oplus 0.5a_2 = \langle h_5, ([1,1],[0,0],[0,0])\rangle$.

Obviously, the above result is contradictory and unreasonable. Since $([1,1],[0,0],[0,0])$ is the maximum of INSs and $([0,0],[1,1],[1,1])$ is the minimum of INSs, $a_1 = \langle h_5, ([1,1],[0,0],[0,0])\rangle$ is superior to $a_2 = \langle h_5, ([0,0],[1,1],[1,1])\rangle$. It stands to reason that the comprehensive evaluation should be between a_1 and a_2. However, according to the result, $a_{12} = a_1$, which apparently goes against logical thinking.

The evaluation information in this example can be expressed in the form of SNULEs, in which $b_1 = \langle [h_5, h_5], (1,0,0)\rangle$ and $b_2 = \langle [h_5, h_5], (0,1,1)\rangle$. Then a comprehensive performance evaluation can be calculated by extending the operations in Definition 5, $b_{12} = 0.5b_1 \oplus 0.5b_2 = \langle [h_5, h_5], (1,0,0)\rangle$. However, similar limitations remain in the operations of SNULEs. Moreover, similar problems also exist in the operations of SVNLNs [56] and INULVs [59].

Therefore, it would be more reasonable if the two parts of INLNs and SNULEs were simultaneously taken into account, which requires new operations.

In order to overcome the limitations described above, we introduce a new definition of operations of SNULEs, inspired by the work of Tian et al. [69].

Definition 6 Let $a_1 = \left\langle [h^L_{\theta_1}, h^U_{\theta_1}], (t_1, i_1, f_1)\right\rangle$ and $a_2 = \left\langle [h^L_{\theta_2},\ h^U_{\theta_2}], (t_2, i_2, f_2)\right\rangle$ be any two SNULEs, f^* be a linguistic scale function and $\lambda \geq 0$. Then the following operations of SNULEs can be defined:

(1) $$a_1 \oplus a_2 = \left\langle \left[f^{*-1}\left(f^*(h^L_{\theta_1}) + f^*(h^L_{\theta_2})\right), f^{*-1}\left(f^*(h^U_{\theta_1}) + f^*(h^U_{\theta_2})\right)\right], \left(\frac{\left(f^*(h^L_{\theta_1}) + f^*(h^U_{\theta_1})\right)t_1 + \left(f^*(h^L_{\theta_2}) + f^*(h^U_{\theta_2})\right)t_2}{\left(f^*(h^L_{\theta_1}) + f^*(h^U_{\theta_1})\right) + \left(f^*(h^L_{\theta_2}) + f^*(h^U_{\theta_2})\right)}, \frac{\left(f^*(h^L_{\theta_1}) + f^*(h^U_{\theta_1})\right)i_1 + \left(f^*(h^L_{\theta_2}) + f^*(h^U_{\theta_2})\right)i_2}{\left(f^*(h^L_{\theta_1}) + f^*(h^U_{\theta_1})\right) + \left(f^*(h^L_{\theta_2}) + f^*(h^U_{\theta_2})\right)}, \frac{\left(f^*(h^L_{\theta_1}) + f^*(h^U_{\theta_1})\right)f_1 + \left(f^*(h^L_{\theta_2}) + f^*(h^U_{\theta_2})\right)f_2}{\left(f^*(h^L_{\theta_1}) + f^*(h^U_{\theta_1})\right) + \left(f^*(h^L_{\theta_2}) + f^*(h^U_{\theta_2})\right)}\right)\right\rangle;$$

(2) $$a_1 \otimes a_2 = \left\langle \left[f^{*-1}\left(f^*(h^L_{\theta_1})f^*(h^L_{\theta_2})\right), f^{*-1}\left(f^*(h^U_{\theta_1})f^*(h^U_{\theta_2})\right)\right], (t_1t_2, i_1 + i_2 - i_1i_2, f_1 + f_2 - f_1f_2)\right\rangle;$$

(3) $$\lambda a_1 = \left\langle \left[f^{*-1}\left(\lambda f^*(h^L_{\theta_1})\right), f^{*-1}\left(\lambda f^*(h^U_{\theta_1})\right)\right], (t_1, i_1, f_1)\right\rangle;$$

(4) $$a_1^\lambda = \left\langle \left[f^{*-1}\left(\left(f^*(h^L_{\theta_1})\right)^\lambda\right), f^{*-1}\left(\left(f^*(h^U_{\theta_1})\right)^\lambda\right)\right], \left(t_1^\lambda, 1 - (1 - i_1)^\lambda, 1 - (1 - f_1)^\lambda\right)\right\rangle;$$

(5) $$\mathrm{neg}(a_1) = \left\langle \left[f^{*-1}\left(f^*(h_{2t}) - f^*(h^U_{\theta_1})\right), f^{*-1}(f^*(h_{2t}) - f^*(h^L_{\theta_1}))\right], (f_1, 1 - i_1, t_1)\right\rangle.$$

According to Definition 1, f^* is a mapping from the linguistic term h_i to the numeric value θ_i, and f^{*-1} is a mapping from θ_i to h_i. Therefore, the first parts of (1–5) are uncertain linguistic variables consisting of linguistic terms, and the second parts of (1–5) are SNNs. In other words, the results obtained by Definition 6 are also SNULEs.

The operations defined above occur on the basis of the linguistic scale function, which can present different results when a different linguistic function f^* is employed. Therefore, decision-makers can flexibly select f^* depending on their own personal preferences and the actual semantic situations.

In practical applications, $a_1 \oplus a_2$, $a_1 \otimes a_2$, λa_1 and a_1^λ necessarily appear in defining basic operations, but their results have no practical meaning. Only in the aggregation process do $a_1 \oplus a_2$ combined with λa_1 or $a_1 \otimes a_2$ combined with a_1^λ make sense.

Example 2 Assume $H = \{h_0, h_1, h_2, \ldots, h_6\}$, $a_1 = \langle [h_1, h_2],\ (0.6, 0.4, 0.2)\rangle$, $a_2 = \langle [h_2, h_4], (0.5, 0.6, 0.2)\rangle$ and λ=2. Then the following results can be calculated.

If α=1.4 and $f_2^*(x) = \begin{cases} \dfrac{\alpha^t - \alpha^{t-x}}{2\alpha^t - 2} & (0 \leq x \leq t) \\ \dfrac{\alpha^t - \alpha^{x-t} - 2}{2\alpha^t - 2} & (t < x \leq 2t) \end{cases}$,

then

(1) $a_1 \oplus a_2 = \langle [h_{3.9658}, h_6], (0.5379, 0.5242, 0.2) \rangle$;

(2) $a_1 \otimes a_2 = \langle [h_{0.3466}, h_{1.0646}], (0.3, 0.76, 0.36) \rangle$;

(3) $a_1^2 = \langle [h_{0.1973}, h_{0.6216}], (0.36, 0.64, 0.36) \rangle$;

(4) $2a_1 = \langle [h_{2.5182}, h_{4.9756}], (0.6, 0.4, 0.2) \rangle$;

(5) $\mathrm{neg}(a_1) = \langle [h_4, h_5], (0.2, 0.6, 0.6) \rangle$.

As for the issue discussed in Example 1, the comprehensive performance evaluation result can be amended using Definition 6, such that $a'_{12} = 0.5a_1 \oplus 0.5a_2 = \langle h_5, ([0.5, 0.5], [0.5, 0.5], [0.5, 0.5]) \rangle$. This shows that a'_{12} is inferior to a_1 and superior to a_2, or, $a_1 > a'_{12} > a_2$; this can normally describe the comprehensive evaluation information, and it is preferable in practice. It also can be modified such that $b'_{12} = 0.5b_1 \oplus 0.5b_2 = \langle [h_5, h_5], (0.5, 0.5, 0.5) \rangle$.

All of the results provided above are also SNULEs. In terms of the corresponding operations of SNULEs, we offer the following theorem:

Theorem 1 Let $a_1 = \left\langle [h^L_{\theta_1}, h^U_{\theta_1}], (t_1, i_1, f_1) \right\rangle$, $a_2 = \left\langle [h^L_{\theta_2}, h^U_{\theta_2}], (t_2, i_2, f_2) \right\rangle$ and $a_3 = \left\langle [h^L_{\theta_3}, h^U_{\theta_3}], (t_3, i_3, f_3) \right\rangle$ be three arbitrary SNULEs and f^* be a linguistic scale function. Then the following properties are true.

(1) $a_1 \oplus a_2 = a_2 \oplus a_1$;

(2) $(a_1 \oplus a_2) \oplus a_3 = a_1 \oplus (a_2 \oplus a_3)$;

(3) $a_1 \otimes a_2 = a_2 \otimes a_1$;

(4) $(a_1 \otimes a_2) \otimes a_3 = a_1 \otimes (a_2 \otimes a_3)$;

(5) $\lambda a_1 \oplus \lambda a_2 = \lambda(a_1 \oplus a_2), \quad \lambda \geq 0$;

(6) $\lambda_1 a_1 \oplus \lambda_2 a_1 = (\lambda_1 + \lambda_2) a_1, \quad \lambda_1, \lambda_2 \geq 0$;

(7) $(a_1 \otimes a_2)^\lambda = a_1^\lambda \otimes a_2^\lambda, \quad \lambda \geq 0$;

(8) $a_1^{\lambda_1} \otimes a_1^{\lambda_2} = a_1^{\lambda_1 + \lambda_2}, \quad \lambda_1, \lambda_2 \geq 0$.

3.2 Comparison method for SNULEs

Based on the score function and accuracy function of ILSs, we can determine the score function, accuracy function and certainty function of a SNULE, which are significant indexes for ranking alternatives in decision-making problems.

Definition 7 Let $a_1 = \left\langle [h^L_{\theta_1}, h^U_{\theta_1}], (t_1, i_1, f_1) \right\rangle$ be a SNULE, where $\hat{h}_{\theta_1} = [h^L_{\theta_1}, h^U_{\theta_1}]$, and let f^* be a linguistic scale function. An aggregation expression $E(\hat{h}_{\theta_1})$ can be expressed as

$$E(\hat{h}_{\theta_1}) = f_\rho\left([h^L_{\theta_1}, h^U_{\theta_1}]\right) = \int_0^1 (d\rho(y)/dy)\left(f^*(h^U_{\theta_1}) - y\left(f^*(h^U_{\theta_1}) - f^*(h^L_{\theta_1})\right)\right) dy,$$

in which f_ρ is the continuous order weighted averaging operator and the function ρ is denoted by a basic unit-interval monotonic function developed by Yager [70]. If $(\delta \geq 0)$, then $\rho(y) = y^\delta$. Furthermore, $E(\hat{h}_{\theta_1})$ is an increasing function with respect to $h^L_{\theta_1}$ and $h^U_{\theta_1}$, and $E(\hat{h}_{\theta_1})$ satisfies $0 \leq E(\hat{h}_{\theta_1}) \leq 1$. Thus, the score function, accuracy function, and certainty function can be represented as follows:

(1) $S(a_1) = \dfrac{1}{3} E(\hat{h}_{\theta_1})(t_1 + 2 - i_1 - f_1)$;

(2) $A(a_1) = E(\hat{h}_{\theta_1})(t_1 - f_1)$;

(3) $C(a_1) = E(\hat{h}_{\theta_1}) t_1$.

For SNULE a_1, if the truth-membership t_1 with respect to the uncertain linguistic variable $\hat{h}_{\theta_1}$ is bigger and the determinacy-membership i_1 and the falsity-membership f_1 corresponding to $\hat{h}_{\theta_1}$ are smaller, then a_1 is greater and the reliability of $\hat{h}_{\theta_1}$ is higher. For the accuracy function $A(a_1)$, if the difference between t_1 and f_1 with respect to $\hat{h}_{\theta_1}$ is bigger, then the statement is more affirmative, meaning that a_1 has higher accuracy. As for the certainty function $C(a_1)$, the certainty of a_1 positively depends on the value of t_1. The bigger $S(a_1)$, $A(a_1)$ and $C(a_1)$ are, the greater the corresponding a_1 is.

Example 3 Use the data of Example 2, and let $\rho(y) = y^2$. Then the following results can be calculated.

$S(a_1) = 0.1855,\ A(a_1) = 0.1113,\ C(a_1) = 0.1670$;

$S(a_2) = 0.2617,\ A(a_2) = 0.1385,\ C(a_2) = 0.2309$.

On the basis of Definition 7, the method to compare SNULEs can be defined as follows:

Definition 8 Let $a_1 = \left\langle [h^L_{\theta_1}, h^U_{\theta_1}], (t_1, i_1, f_1) \right\rangle$ and $a_2 = \left\langle [h^L_{\theta_2}, h^U_{\theta_2}], (t_2, i_2, f_2) \right\rangle$ be two SNULEs.

(1) If $S(a_1) > S(a_2)$, then $a_1 > a_2$;
(2) If $S(a_1) = S(a_2)$ and $A(a_1) > A(a_2)$, then $a_1 > a_2$;
(3) If $S(a_1) = S(a_2)$, $A(a_1) = A(a_2)$ and $C(a_1) > C(a_2)$, then $a_1 > a_2$;

(4) If $S(a_1) = S(a_2)$, $A(a_1) = A(a_2)$ and $C(a_1) = C(a_2)$, then $a_1 = a_2$.

Example 4 Assume $H = \{h_0, h_1, h_2, \ldots, h_6\}$ and $\rho(y) = y^2$. Then the following results can be calculated.

If α=1.4 and $f_2^*(x) = \begin{cases} \dfrac{\alpha^t - \alpha^{t-x}}{2\alpha^t - 2} & (0 \le x \le t) \\ \dfrac{\alpha^t + \alpha^{x-t} - 2}{2\alpha^t - 2} & (t < x \le 2t) \end{cases}$, then

1. For two SNULEs $a_1 = \langle [h_3, h_5], (0.8, 0.4, 0.3) \rangle$ and $a_2 = \langle [h_3, h_4], (0.9, 0.6, 0.6) \rangle$, according to Definition 8, $S(a_1) = 0.4142 > S(a_2) = 0.19735$. Therefore, $a_1 > a_2$.
2. For two SNULEs $a_1 = \langle [h_3, h_5], (0.8, 0.4, 0.3) \rangle$ and $a_2 = \langle [h_3, h_5], (0.9, 0.3, 0.5) \rangle$, according to Definition 8, $S(a_1) = S(a_2) = 0.4142$ and $A(a_1) = 0.0986 > A(a_2) = 0.0789$. Therefore, $a_1 > a_2$.
3. For two SNULEs $a_1 = \langle [h_3, h_5], (0.8, 0.4, 0.3) \rangle$ and $a_2 = \langle [h_3, h_5], (0.7, 0.4, 0.2) \rangle$, according to Definition 8, $S(a_1) = S(a_2) = 0.4142$, $A(a_1) = A(a_2) = 0.0986$, and $C(a_1) = 0.4734 > C(a_2) = 0.4142$. Therefore, $a_1 > a_2$.

4 GSNULPWA operator and its application in MCDM problems

This section develops a GSNULPWA operator and analyzes some of its desirable properties. Subsequently, some special cases with respect to the parameter λ are discussed. Finally, a MCDM approach is proposed, based on the GSNULPWA operator.

4.1 PA operator

Definition 9 [62]. Let $G = \{G_1, G_2, \ldots, G_n\}$ be a collection of criteria that ensures prioritization between the criteria expressed as the linear ordering $G_1 \succ G_2 \succ G_3 \succ \cdots \succ G_n$, which indicates that criterion G_j has a higher priority than G_k, if $j < k$. The value $G_j(x)$ is the performance of any alternative x under criterion G_j, satisfying $G_j(x) \in [0, 1]$. If

$$PA(G_i(x)) = \sum\nolimits_{j=1}^{n} w_j G_j(x),$$

where $w_j = \frac{T_j}{\sum_{i=1}^{n} T_i}$, $T_1 = 1$ and $T_j = \prod_{k=1}^{j-1} G_k(x)$ $(j = 2, 3, \ldots, n)$, then $PA(G_i(x))$ is called the PA operator.

4.2 GSNULPWA operator

This subsection investigates the prioritized weighted average operator under a simplified neutrosophic uncertain linguistic environment. The definition of GSNULPWA operator and its relevant theorems are provided below.

Definition 10 Let $a_j = \left\langle [h^L_{\theta_j}, h^U_{\theta_j}], (t_j, i_j, f_j) \right\rangle$ $(j = 1, 2, \ldots, n)$ be a collection of SNULEs, and let $GSNULPWA$: $\Omega^n \to \Omega$. If

$$\begin{aligned} &GSNULPWA(a_1, a_2, \ldots, a_n) \\ &= \left(\frac{T_1}{\sum_{i=1}^{n} T_i} a_1^{\lambda} \oplus \frac{T_2}{\sum_{i=1}^{n} T_i} a_2^{\lambda} \oplus \cdots \oplus \frac{T_n}{\sum_{i=1}^{n} T_i} a_n^{\lambda} \right)^{\frac{1}{\lambda}} \\ &= \left(\mathop{\oplus}_{j=1}^{n} \frac{a_j^{\lambda} T_j}{\sum_{i=1}^{n} T_i} \right)^{\frac{1}{\lambda}}, \end{aligned}$$

then the function $GSNULPWA$ is called a GSNULPWA operator, where $\lambda > 0$, $T_1 = 1$, and $T_j = \prod_{k=1}^{j-1} S(a_k)$ $(j = 2, 3, \ldots, n)$. Furthermore, $S(a_k)$ is the score of a_k, and satisfies $S(a_k) \in [0, 1]$.

Based on the operational laws of SNULEs described in Sect. 3, Theorem 2 can be proven as follows:

Theorem 2 Let $a_j = \left\langle [h^L_{\theta_j}, h^U_{\theta_j}], (t_j, i_j, f_j) \right\rangle$ $(j = 1, 2, \ldots, n)$ be a collection of SNULEs. Let $\lambda > 0$, $T_1 = 1$, and $T_j = \prod_{k=1}^{j-1} S(a_k)$ $(j = 2, 3, \ldots, n)$, and let $S(a_k)$ be the score of a_k. The aggregation value, obtained by using the GSNULPWA operator is also a SNULE, and

$$\begin{aligned} &GSNULPWA(a_1, a_2, \ldots, a_n) = \left(\frac{a_1^{\lambda} T_1}{\sum_{i=1}^{n} T_i} \oplus \frac{a_2^{\lambda} T_2}{\sum_{i=1}^{n} T_i} \oplus \cdots \oplus \frac{a_n^{\lambda} T_n}{\sum_{i=1}^{n} T_i} \right)^{\frac{1}{\lambda}} \\ &= \left\langle \left[f^{*-1}\left(\left(\sum_{j=1}^{n} \frac{\left(f^*\left(h^L_{\theta_j}\right)\right)^{\lambda} T_j}{\sum_{i=1}^{n} T_i} \right)^{\frac{1}{\lambda}} \right), f^{*-1}\left(\left(\sum_{j=1}^{n} \frac{\left(f^*\left(h^U_{\theta_j}\right)\right)^{\lambda} T_j}{\sum_{i=1}^{n} T_i} \right)^{\frac{1}{\lambda}} \right) \right], \right. \\ &\times \left(\left(\frac{\sum_{j=1}^{n} \left(\left(f^*\left(h^L_{\theta_j}\right)\right)^{\lambda} + \left(f^*\left(h^U_{\theta_j}\right)\right)^{\lambda} \right) T_j t_j^{\lambda}}{\sum_{j=1}^{n} \left(\left(f^*\left(h^L_{\theta_j}\right)\right)^{\lambda} + \left(f^*\left(h^U_{\theta_j}\right)\right)^{\lambda} \right) T_j} \right)^{\frac{1}{\lambda}}, \right. \\ &1 - \left(1 - \frac{\sum_{j=1}^{n} \left(\left(f^*\left(h^L_{\theta_j}\right)\right)^{\lambda} + \left(f^*\left(h^U_{\theta_j}\right)\right)^{\lambda} \right) T_j \left(1 - (1 - i_j)^{\lambda}\right)}{\sum_{j=1}^{n} \left(\left(f^*\left(h^L_{\theta_j}\right)\right)^{\lambda} + \left(f^*\left(h^U_{\theta_j}\right)\right)^{\lambda} \right) T_j} \right)^{\frac{1}{\lambda}}, \\ &\left. \left. 1 - \left(1 - \frac{\sum_{j=1}^{n} \left(\left(f^*\left(h^L_{\theta_j}\right)\right)^{\lambda} + \left(f^*\left(h^U_{\theta_j}\right)\right)^{\lambda} \right) T_j \left(1 - (1 - f_j)^{\lambda}\right)}{\sum_{j=1}^{n} \left(\left(f^*\left(h^L_{\theta_j}\right)\right)^{\lambda} + \left(f^*\left(h^U_{\theta_j}\right)\right)^{\lambda} \right) T_j} \right)^{\frac{1}{\lambda}} \right) \right\rangle, \end{aligned} \tag{1}$$

where $\lambda > 0$, $T_1 = 1$, $T_j = \prod_{k=1}^{j-1} S(a_k)$ $(j = 2, 3, \ldots, n)$, and $S(a_k)$ is the score of a_k.

The following verifies Eq. (1) using the mathematical induction on n.

Proof. First, the following equation needs to be proved.

$$\mathop{\oplus}_{j=1}^{n}\frac{a_j^{\lambda}T_j}{\sum_{i=1}^{n}T_i}=\left\langle\left[f^{*-1}\left(\sum_{j=1}^{n}\frac{\left(f^{*}\left(h_{\theta_j}^{L}\right)\right)^{\lambda}T_j}{\sum_{i=1}^{n}T_i}\right),f^{*-1}\left(\sum_{j=1}^{n}\frac{\left(f^{*}\left(h_{\theta_j}^{U}\right)\right)^{\lambda}T_j}{\sum_{i=1}^{n}T_i}\right)\right],\left(\frac{\sum_{j=1}^{n}\left(\left(f^{*}\left(h_{\theta_j}^{L}\right)\right)^{\lambda}+\left(f^{*}\left(h_{\theta_j}^{U}\right)\right)^{\lambda}\right)T_jt_j^{\lambda}}{\sum_{j=1}^{n}\left(\left(f^{*}\left(h_{\theta_j}^{L}\right)\right)^{\lambda}+\left(f^{*}\left(h_{\theta_j}^{U}\right)\right)^{\lambda}\right)T_j},\frac{\sum_{j=1}^{n}\left(\left(f^{*}\left(h_{\theta_j}^{L}\right)\right)^{\lambda}+\left(f^{*}\left(h_{\theta_j}^{U}\right)\right)^{\lambda}\right)T_j\left(1-(1-i_j)^{\lambda}\right)}{\sum_{j=1}^{n}\left(\left(f^{*}\left(h_{\theta_j}^{L}\right)\right)^{\lambda}+\left(f^{*}\left(h_{\theta_j}^{U}\right)\right)^{\lambda}\right)T_j},\frac{\sum_{j=1}^{n}\left(\left(f^{*}\left(h_{\theta_j}^{L}\right)\right)^{\lambda}+\left(f^{*}\left(h_{\theta_j}^{U}\right)\right)^{\lambda}\right)T_j\left(1-(1-f_j)^{\lambda}\right)}{\sum_{j=1}^{n}\left(\left(f^{*}\left(h_{\theta_j}^{L}\right)\right)^{\lambda}+\left(f^{*}\left(h_{\theta_j}^{U}\right)\right)^{\lambda}\right)T_j}\right)\right\rangle \tag{2}$$

1. When $n=2$, the following equation can be calculated.

$$\begin{aligned}
&\mathop{\oplus}_{j=1}^{n}\frac{a_j^{\lambda}T_j}{\sum_{i=1}^{n}T_i}=\frac{a_1^{\lambda}T_1}{\sum_{i=1}^{n}T_i}\oplus\frac{a_2^{\lambda}T_2}{\sum_{i=1}^{n}T_i}\\
&=\left\langle\left[f^{*-1}\left(\frac{\left(f^{*}\left(h_{\theta_1}^{L}\right)\right)^{\lambda}T_1}{\sum_{i=1}^{n}T_i}\right),f^{*-1}\left(\frac{\left(f^{*}\left(h_{\theta_1}^{U}\right)\right)^{\lambda}T_1}{\sum_{i=1}^{n}T_i}\right)\right],\left(t_1^{\lambda},\left(1-(1-i_1)^{\lambda}\right),\left(1-(1-f_1)^{\lambda}\right)\right)\right\rangle\oplus\\
&\left\langle\left[f^{*-1}\left(\frac{\left(f^{*}\left(h_{\theta_2}^{L}\right)\right)^{\lambda}T_2}{\sum_{i=1}^{n}T_i}\right),f^{*-1}\left(\frac{\left(f^{*}\left(h_{\theta_2}^{U}\right)\right)^{\lambda}T_2}{\sum_{i=1}^{n}T_i}\right)\right],\left(t_2^{\lambda},\left(1-(1-i_2)^{\lambda}\right),\left(1-(1-f_2)^{\lambda}\right)\right)\right\rangle\\
&=\left\langle\left[f^{*-1}\left(\frac{\left(f^{*}\left(h_{\theta_1}^{L}\right)\right)^{\lambda}T_1}{\sum_{i=1}^{n}T_i}+\frac{\left(f^{*}\left(h_{\theta_2}^{L}\right)\right)^{\lambda}T_2}{\sum_{i=1}^{n}T_i}\right),f^{*-1}\left(\frac{\left(f^{*}\left(h_{\theta_1}^{U}\right)\right)^{\lambda}T_1}{\sum_{i=1}^{n}T_i}+\frac{\left(f^{*}\left(h_{\theta_2}^{U}\right)\right)^{\lambda}T_2}{\sum_{i=1}^{n}T_i}\right)\right],\right.\\
&\left(\frac{\left(\left(f^{*}\left(h_{\theta_1}^{L}\right)\right)^{\lambda}+\left(f^{*}\left(h_{\theta_1}^{U}\right)\right)^{\lambda}\right)T_1t_1^{\lambda}+\left(\left(f^{*}\left(h_{\theta_2}^{L}\right)\right)^{\lambda}+\left(f^{*}\left(h_{\theta_2}^{U}\right)\right)^{\lambda}\right)T_2t_2^{\lambda}}{\left(\left(f^{*}\left(h_{\theta_1}^{L}\right)\right)^{\lambda}+\left(f^{*}\left(h_{\theta_1}^{U}\right)\right)^{\lambda}\right)T_1+\left(\left(f^{*}\left(h_{\theta_2}^{L}\right)\right)^{\lambda}+\left(f^{*}\left(h_{\theta_2}^{U}\right)\right)^{\lambda}\right)T_2},\right.\\
&\frac{\left(\left(f^{*}\left(h_{\theta_1}^{L}\right)\right)^{\lambda}+\left(f^{*}\left(h_{\theta_1}^{U}\right)\right)^{\lambda}\right)T_1\left(1-(1-i_1)^{\lambda}\right)+\left(\left(f^{*}\left(h_{\theta_2}^{L}\right)\right)^{\lambda}+\left(f^{*}\left(h_{\theta_2}^{U}\right)\right)^{\lambda}\right)T_2\left(1-(1-i_2)^{\lambda}\right)}{\left(\left(f^{*}\left(h_{\theta_1}^{L}\right)\right)^{\lambda}+\left(f^{*}\left(h_{\theta_1}^{U}\right)\right)^{\lambda}\right)T_1+\left(\left(f^{*}\left(h_{\theta_2}^{L}\right)\right)^{\lambda}+\left(f^{*}\left(h_{\theta_2}^{U}\right)\right)^{\lambda}\right)T_2},\\
&\left.\left.\frac{\left(\left(f^{*}\left(h_{\theta_1}^{L}\right)\right)^{\lambda}+\left(f^{*}\left(h_{\theta_1}^{U}\right)\right)^{\lambda}\right)T_1\left(1-(1-f_1)^{\lambda}\right)+\left(\left(f^{*}\left(h_{\theta_2}^{L}\right)\right)^{\lambda}+\left(f^{*}\left(h_{\theta_2}^{U}\right)\right)^{\lambda}\right)T_2\left(1-(1-f_2)^{\lambda}\right)}{\left(\left(f^{*}\left(h_{\theta_1}^{L}\right)\right)^{\lambda}+\left(f^{*}\left(h_{\theta_1}^{U}\right)\right)^{\lambda}\right)T_1+\left(\left(f^{*}\left(h_{\theta_2}^{L}\right)\right)^{\lambda}+\left(f^{*}\left(h_{\theta_2}^{U}\right)\right)^{\lambda}\right)T_2}\right)\right\rangle.
\end{aligned}$$

That is, when $n = 2$, Eq. (2) is true.

2. Suppose that when $n = k$, Eq. (2) is true. That is,

$$\overset{k}{\underset{j=1}{\oplus}} \frac{a_j^{\lambda} T_j}{\sum_{i=1}^{n} T_i} = \left\langle \left[f^{*-1}\left(\sum_{j=1}^{k} \frac{\left(f^*\left(h_{\theta_j}^L\right)\right)^{\lambda} T_j}{\sum_{i=1}^{n} T_i} \right), f^{*-1}\left(\sum_{j=1}^{k} \frac{\left(f^*\left(h_{\theta_j}^U\right)\right)^{\lambda} T_j}{\sum_{i=1}^{n} T_i} \right) \right], \left(\frac{\sum_{j=1}^{k}\left(\left(f^*\left(h_{\theta_j}^L\right)\right)^{\lambda} + \left(f^*\left(h_{\theta_j}^U\right)\right)^{\lambda}\right) T_j t_j^{\lambda}}{\sum_{j=1}^{k}\left(\left(f^*\left(h_{\theta_j}^L\right)\right)^{\lambda} + \left(f^*\left(h_{\theta_j}^U\right)\right)^{\lambda}\right) T_j}, \frac{\sum_{j=1}^{k}\left(\left(f^*\left(h_{\theta_j}^L\right)\right)^{\lambda} + \left(f^*\left(h_{\theta_j}^U\right)\right)^{\lambda}\right) T_j \left(1-(1-i_j)^{\lambda}\right)}{\sum_{j=1}^{k}\left(\left(f^*\left(h_{\theta_j}^L\right)\right)^{\lambda} + \left(f^*\left(h_{\theta_j}^U\right)\right)^{\lambda}\right) T_j}, \frac{\sum_{j=1}^{k}\left(\left(f^*\left(h_{\theta_j}^L\right)\right)^{\lambda} + \left(f^*\left(h_{\theta_j}^U\right)\right)^{\lambda}\right) T_j \left(1-(1-f_j)^{\lambda}\right)}{\sum_{j=1}^{k}\left(\left(f^*\left(h_{\theta_j}^L\right)\right)^{\lambda} + \left(f^*\left(h_{\theta_j}^U\right)\right)^{\lambda}\right) T_j} \right) \right\rangle.$$

Then, when $n = k + 1$, the following result can be calculated.

$$\overset{k+1}{\underset{j=1}{\oplus}} \frac{a_j^{\lambda} T_j}{\sum_{i=1}^{n} T_i} = \overset{k}{\underset{j=1}{\oplus}} \frac{a_j^{\lambda} T_j}{\sum_{i=1}^{n} T_i} \oplus \frac{a_{k+1}^{\lambda} T_{k+1}}{\sum_{i=1}^{n} T_i}$$

$$= \left\langle \left[f^{*-1}\left(\sum_{j=1}^{k} \frac{\left(f^*\left(h_{\theta_j}^L\right)\right)^{\lambda} T_j}{\sum_{i=1}^{n} T_i} \right), f^{*-1}\left(\sum_{j=1}^{k} \frac{\left(f^*\left(h_{\theta_j}^U\right)\right)^{\lambda} T_j}{\sum_{i=1}^{n} T_i} \right) \right], \left(\frac{\sum_{j=1}^{k}\left(\left(f^*\left(h_{\theta_j}^L\right)\right)^{\lambda} + \left(f^*\left(h_{\theta_j}^U\right)\right)^{\lambda}\right) T_j t_j^{\lambda}}{\sum_{j=1}^{k}\left(\left(f^*\left(h_{\theta_j}^L\right)\right)^{\lambda} + \left(f^*\left(h_{\theta_j}^U\right)\right)^{\lambda}\right) T_j}, \frac{\sum_{j=1}^{k}\left(\left(f^*\left(h_{\theta_j}^L\right)\right)^{\lambda} + \left(f^*\left(h_{\theta_j}^U\right)\right)^{\lambda}\right) T_j \left(1-(1-i_j)^{\lambda}\right)}{\sum_{j=1}^{k}\left(\left(f^*\left(h_{\theta_j}^L\right)\right)^{\lambda} + \left(f^*\left(h_{\theta_j}^U\right)\right)^{\lambda}\right) T_j}, \frac{\sum_{j=1}^{k}\left(\left(f^*\left(h_{\theta_j}^L\right)\right)^{\lambda} + \left(f^*\left(h_{\theta_j}^U\right)\right)^{\lambda}\right) T_j \left(1-(1-f_j)^{\lambda}\right)}{\sum_{j=1}^{k}\left(\left(f^*\left(h_{\theta_j}^L\right)\right)^{\lambda} + \left(f^*\left(h_{\theta_j}^U\right)\right)^{\lambda}\right) T_j} \right) \right\rangle \oplus$$

$$\left\langle \left[f^{*-1}\left(\frac{\left(f^*\left(h_{\theta_{k+1}}^L\right)\right)^{\lambda} T_{k+1}}{\sum_{i=1}^{n} T_i} \right), f^{*-1}\left(\frac{\left(f^*\left(h_{\theta_{k+1}}^U\right)\right)^{\lambda} T_{k+1}}{\sum_{i=1}^{n} T_i} \right) \right], \left(t_{k+1}^{\lambda}, \left(1-(1-i_{k+1})^{\lambda}\right), \left(1-(1-f_{k+1})^{\lambda}\right) \right) \right\rangle$$

$$= \left\langle \left[f^{*-1}\left(\sum_{j=1}^{k+1} \frac{\left(f^*\left(h_{\theta_j}^L\right)\right)^{\lambda} T_j}{\sum_{i=1}^{n} T_i} \right), f^{*-1}\left(\sum_{j=1}^{k+1} \frac{\left(f^*\left(h_{\theta_j}^U\right)\right)^{\lambda} T_j}{\sum_{i=1}^{n} T_i} \right) \right], \left(\frac{\sum_{j=1}^{k+1}\left(\left(f^*\left(h_{\theta_j}^L\right)\right)^{\lambda} + \left(f^*\left(h_{\theta_j}^U\right)\right)^{\lambda}\right) T_j t_j^{\lambda}}{\sum_{j=1}^{k+1}\left(\left(f^*\left(h_{\theta_j}^L\right)\right)^{\lambda} + \left(f^*\left(h_{\theta_j}^U\right)\right)^{\lambda}\right) T_j}, \frac{\sum_{j=1}^{k+1}\left(\left(f^*\left(h_{\theta_j}^L\right)\right)^{\lambda} + \left(f^*\left(h_{\theta_j}^U\right)\right)^{\lambda}\right) T_j \left(1-(1-i_j)^{\lambda}\right)}{\sum_{j=1}^{k+1}\left(\left(f^*\left(h_{\theta_j}^L\right)\right)^{\lambda} + \left(f^*\left(h_{\theta_j}^U\right)\right)^{\lambda}\right) T_j}, \frac{\sum_{j=1}^{k+1}\left(\left(f^*\left(h_{\theta_j}^L\right)\right)^{\lambda} + \left(f^*\left(h_{\theta_j}^U\right)\right)^{\lambda}\right) T_j \left(1-(1-f_j)^{\lambda}\right)}{\sum_{j=1}^{k+1}\left(\left(f^*\left(h_{\theta_j}^L\right)\right)^{\lambda} + \left(f^*\left(h_{\theta_j}^U\right)\right)^{\lambda}\right) T_j} \right) \right\rangle.$$

That is, when $n = k + 1$, Eq. (2) is true.

3. Therefore, Eq. (2) holds for all n.

Then, through Eq. (2), Eq. (1) can be proved right.

Theorem 3 (*Idempotency*). Let $a_j = \left\langle [h^L_{\theta_j}, h^U_{\theta_j}], (t_j, i_j, f_j) \right\rangle$ $(j = 1, 2, \ldots, n)$ be a collection of SNULEs. Let $\lambda > 0$, $T_1 = 1$, and $T_j = \prod_{k=1}^{j-1} S(a_k)$ $(j = 2, 3, \ldots, n)$, and let $S(a_k)$ be the score of a_k. If all a_j $(j = 1, 2, \ldots, n)$ are equal, that is, $a_j = a$ for all j, then $GSNULPWA(a_1, a_2, \ldots, a_n) = a$.

Proof. Since $a_j = a$ for all j, according to Eq. (6) in Theorem 1, the following equation can be obtained.

$$
\begin{aligned}
&GSNULPWA(a_1, a_2, \ldots, a_n) \\
&= \left(\frac{a_1^\lambda T_1}{\sum_{i=1}^n T_i} \oplus \frac{a_2^\lambda T_2}{\sum_{i=1}^n T_i} \oplus \cdots \oplus \frac{a_n^\lambda T_n}{\sum_{i=1}^n T_i} \right)^{\frac{1}{\lambda}} \\
&= \left(a^\lambda \left(\frac{T_1}{\sum_{i=1}^n T_i} + \frac{T_2}{\sum_{i=1}^n T_i} + \cdots + \frac{T_n}{\sum_{i=1}^n T_i} \right) \right)^{\frac{1}{\lambda}} = a.
\end{aligned}
$$

Theorem 4 (*Boundedness*). Let $a_j = \left\langle [h^L_{\theta_j}, h^U_{\theta_j}], (t_j, i_j, f_j) \right\rangle$ $(j = 1, 2, \ldots, n)$ be a collection of SNULEs. Let $\lambda > 0$, $T_1 = 1$, and $T_j = \prod_{k=1}^{j-1} S(a_k)$ $(j = 2, 3, \ldots, n)$, and let $S(a_k)$ be the score of a_k. Further, let

$$a^- = \left\langle \left[\min_j \left\{ h^L_{\theta_j} \right\}, \min_j \left\{ h^U_{\theta_j} \right\} \right], \left(\min_j \{t_j\}, \max_j \{i_j\}, \max_j \{f_j\} \right) \right\rangle,$$

$$a^+ = \left\langle \left[\max_j \left\{ h^L_{\theta_j} \right\}, \max_j \left\{ h^U_{\theta_j} \right\} \right], \left(\max_j \{t_j\}, \min_j \{i_j\}, \min_j \{f_j\} \right) \right\rangle.$$

Then, $S(a^-) \le S(GSNULPWA(a_1, a_2, \ldots, a_n)) \le S(a^+)$.

Proof. Let $GSNULPWA(a_1, a_2, \ldots, a_n) = a = \left\langle [h^L_{\theta_a}, h^U_{\theta_a}], (t_a, i_a, f_a) \right\rangle$, $\hat{h}_a = [h^L_{\theta_a}, h^U_{\theta_a}]$, $\hat{h}_{a^-} = \left[\min_j \left\{ h^L_{\theta_j} \right\}, \min_j \left\{ h^U_{\theta_j} \right\} \right]$ and $\hat{h}_{a^+} = \left[\max_j \left\{ h^L_{\theta_j} \right\}, \max_j \left\{ h^U_{\theta_j} \right\} \right]$ then

1. For the linguistic term part
   Since $\min_j \left\{ h^L_{\theta_j} \right\} \le h^L_{\theta_j} \le \max_j \left\{ h^L_{\theta_j} \right\}$ and $\min_j \left\{ h^U_{\theta_j} \right\} \le h^U_{\theta_j} \le \max_j \left\{ h^U_{\theta_j} \right\}$ for all j, based on Theorem 3, the following equations hold.

$$\min_j \left\{ h^L_{\theta_j} \right\} = f^{*-1} \left(\left(\sum_{k=1}^n \frac{T_k}{\sum_{i=1}^n T_i} \left(f^* \left(\min_j \left\{ h^L_{\theta_j} \right\} \right) \right)^\lambda \right)^{\frac{1}{\lambda}} \right),$$

$$\max_j \left\{ h^L_{\theta_j} \right\} = f^{*-1} \left(\left(\sum_{k=1}^n \frac{T_k}{\sum_{i=1}^n T_i} \left(f^* \left(\max_j \left\{ h^L_{\theta_j} \right\} \right) \right)^\lambda \right)^{\frac{1}{\lambda}} \right),$$

$$\min_j \left\{ h^U_{\theta_j} \right\} = f^{*-1} \left(\left(\sum_{k=1}^n \frac{T_k}{\sum_{i=1}^n T_i} \left(f^* \left(\min_j \left\{ h^U_{\theta_j} \right\} \right) \right)^\lambda \right)^{\frac{1}{\lambda}} \right)$$

and

$$\max_j \left\{ h^U_{\theta_j} \right\} = f^{*-1} \left(\left(\sum_{k=1}^n \frac{T_k}{\sum_{i=1}^n T_i} \left(f^* \left(\max_j \left\{ h^U_{\theta_j} \right\} \right) \right)^\lambda \right)^{\frac{1}{\lambda}} \right).$$

In addition, since both f^* and f^{*-1} are strictly monotonically increasing and continuous functions, then

$$
\begin{aligned}
&f^{*-1} \left(\left(\sum_{k=1}^n \frac{T_k}{\sum_{i=1}^n T_i} \left(f^* \left(\min_j \left\{ h^L_{\theta_j} \right\} \right) \right)^\lambda \right)^{\frac{1}{\lambda}} \right) \\
&\le f^{*-1} \left(\left(\sum_{k=1}^n \frac{T_k}{\sum_{i=1}^n T_i} \left(f^* \left(h^L_{\theta_j} \right) \right)^\lambda \right)^{\frac{1}{\lambda}} \right) \\
&\le f^{*-1} \left(\left(\sum_{k=1}^n \frac{T_k}{\sum_{i=1}^n T_i} \left(f^* \left(\max_j \left\{ h^L_{\theta_j} \right\} \right) \right)^\lambda \right)^{\frac{1}{\lambda}} \right),
\end{aligned}
$$

$$
\begin{aligned}
&f^{*-1} \left(\left(\sum_{k=1}^n \frac{T_k}{\sum_{i=1}^n T_i} \left(f^* \left(\min_j \left\{ h^U_{\theta_j} \right\} \right) \right)^\lambda \right)^{\frac{1}{\lambda}} \right) \\
&\le f^{*-1} \left(\left(\sum_{k=1}^n \frac{T_k}{\sum_{i=1}^n T_i} \left(f^* \left(h^U_{\theta_j} \right) \right)^\lambda \right)^{\frac{1}{\lambda}} \right) \\
&\le f^{*-1} \left(\left(\sum_{k=1}^n \frac{T_k}{\sum_{i=1}^n T_i} \left(f^* \left(\max_j \left\{ h^U_{\theta_j} \right\} \right) \right)^\lambda \right)^{\frac{1}{\lambda}} \right).
\end{aligned}
$$

That is, $\min_j \left\{ h^L_{\theta_j} \right\} \le h^L_{\theta_a} \le \max_j \left\{ h^L_{\theta_j} \right\}$ and $\min_j \left\{ h^U_{\theta_j} \right\} \le h^U_{\theta_a} \le \max_j \left\{ h^U_{\theta_j} \right\}$.
Therefore, based on the description in Definition 7, $E(\hat{h}_{a-}) \le E(\hat{h}_a) \le E(\hat{h}_{a^+})$.

2. Similarly, for the truth-membership part, indeterminacy-membership part and falsity-membership part, the following inequalities hold.

$$
\begin{aligned}
&\left(\frac{\sum_{j=1}^n \left(\left(f^* \left(h^L_{\theta_j} \right) \right)^\lambda + \left(f^* \left(h^U_{\theta_j} \right) \right)^\lambda \right) T_j \left(\min_j \{t_j\} \right)^\lambda}{\sum_{j=1}^n \left(\left(f^* \left(h^L_{\theta_j} \right) \right)^\lambda + \left(f^* \left(h^U_{\theta_j} \right) \right)^\lambda \right) T_j} \right)^{\frac{1}{\lambda}} \\
&\le \left(\frac{\sum_{j=1}^n \left(\left(f^* \left(h^L_{\theta_j} \right) \right)^\lambda + \left(f^* \left(h^U_{\theta_j} \right) \right)^\lambda \right) T_j t_j^\lambda}{\sum_{j=1}^n \left(\left(f^* \left(h^L_{\theta_j} \right) \right)^\lambda + \left(f^* \left(h^U_{\theta_j} \right) \right)^\lambda \right) T_j} \right)^{\frac{1}{\lambda}} \\
&\le \left(\frac{\sum_{j=1}^n \left(\left(f^* \left(h^L_{\theta_j} \right) \right)^\lambda + \left(f^* \left(h^U_{\theta_j} \right) \right)^\lambda \right) T_j \left(\max_j \{t_j\} \right)^\lambda}{\sum_{j=1}^n \left(\left(f^* \left(h^L_{\theta_j} \right) \right)^\lambda + \left(f^* \left(h^U_{\theta_j} \right) \right)^\lambda \right) T_j} \right)^{\frac{1}{\lambda}};
\end{aligned}
$$

$$1-\left(1-\frac{\sum_{j=1}^{n}\left(\left(f^{*}\left(h_{\theta_j}^{L}\right)\right)^{\lambda}+\left(f^{*}\left(h_{\theta_j}^{U}\right)\right)^{\lambda}\right)T_j\left(\max_j\{i_j\}\right)^{\lambda}}{\sum_{j=1}^{n}\left(\left(f^{*}\left(h_{\theta_j}^{L}\right)\right)^{\lambda}+\left(f^{*}\left(h_{\theta_j}^{U}\right)\right)^{\lambda}\right)T_j}\right)^{\frac{1}{\lambda}}$$

$$\geq 1-\left(1-\frac{\sum_{j=1}^{n}\left(\left(f^{*}\left(h_{\theta_j}^{L}\right)\right)^{\lambda}+\left(f^{*}\left(h_{\theta_j}^{U}\right)\right)^{\lambda}\right)T_j i_j^{\lambda}}{\sum_{j=1}^{n}\left(\left(f^{*}\left(h_{\theta_j}^{L}\right)\right)^{\lambda}+\left(f^{*}\left(h_{\theta_j}^{U}\right)\right)^{\lambda}\right)T_j}\right)^{\frac{1}{\lambda}}$$

$$\geq 1-\left(1-\frac{\sum_{j=1}^{n}\left(\left(f^{*}\left(h_{\theta_j}^{L}\right)\right)^{\lambda}+\left(f^{*}\left(h_{\theta_j}^{U}\right)\right)^{\lambda}\right)T_j\left(\min_j\{i_j\}\right)^{\lambda}}{\sum_{j=1}^{n}\left(\left(f^{*}\left(h_{\theta_j}^{L}\right)\right)^{\lambda}+\left(f^{*}\left(h_{\theta_j}^{U}\right)\right)^{\lambda}\right)T_j}\right)^{\frac{1}{\lambda}};$$

$$\begin{aligned}S(a^{-})&=\frac{1}{3}E(\hat{h}_{a-})\left(\min_j\{t_j\}+2-\max_j\{i_j\}-\max_j\{f_j\}\right)\\&\leq\frac{1}{3}E(\hat{h}_a)(t_a+2-i_a-f_a)=S(GSNULPWA(a_1,a_2,\ldots,a_n))\\&\leq\frac{1}{3}E(\hat{h}_{a^+})\left(\max_j\{t_j\}+2-\min_j\{i_j\}-\min_j\{f_j\}\right)=S(a^{+}).\end{aligned}$$

Therefore, $S(a^{-})\leq S(GSNULPWA(a_1,a_2,\ldots,a_n))\leq S(a^{+})$.

The following discusses some special cases of the GSNULPWA operator.

1. If $\lambda=1$, then the GSNULPWA operator degenerates into the SNULPWA operator.

$$GSNULPWA(a_1,a_2,\ldots,a_n)=\mathop{\oplus}_{j=1}^{n}\frac{T_j}{\sum_{i=1}^{n}T_i}a_j$$
$$=\left\langle\left[f^{*-1}\left(\sum_{j=1}^{n}\frac{f^{*}\left(h_{\theta_j}^{L}\right)T_j}{\sum_{i=1}^{n}T_i}\right),f^{*-1}\left(\sum_{j=1}^{n}\frac{f^{*}\left(h_{\theta_j}^{U}\right)T_j}{\sum_{i=1}^{n}T_i}\right)\right],\right.$$
$$\left.\left(\frac{\sum_{j=1}^{n}\left(f^{*}\left(h_{\theta_j}^{L}\right)+f^{*}\left(h_{\theta_j}^{U}\right)\right)T_jt_j}{\sum_{j=1}^{n}\left(f^{*}\left(h_{\theta_j}^{L}\right)+f^{*}\left(h_{\theta_j}^{U}\right)\right)T_j},\frac{\sum_{j=1}^{n}\left(f^{*}\left(h_{\theta_j}^{L}\right)+f^{*}\left(h_{\theta_j}^{U}\right)\right)T_ji_j}{\sum_{j=1}^{n}\left(f^{*}\left(h_{\theta_j}^{L}\right)+f^{*}\left(h_{\theta_j}^{U}\right)\right)T_j},\frac{\sum_{j=1}^{n}\left(f^{*}\left(h_{\theta_j}^{L}\right)+f^{*}\left(h_{\theta_j}^{U}\right)\right)T_jf_j}{\sum_{j=1}^{n}\left(f^{*}\left(h_{\theta_j}^{L}\right)+f^{*}\left(h_{\theta_j}^{U}\right)\right)T_j}\right)\right\rangle.$$

$$1-\left(1-\frac{\sum_{j=1}^{n}\left(\left(f^{*}\left(h_{\theta_j}^{L}\right)\right)^{\lambda}+\left(f^{*}\left(h_{\theta_j}^{U}\right)\right)^{\lambda}\right)T_j\left(\max_j\{f_j\}\right)^{\lambda}}{\sum_{j=1}^{n}\left(\left(f^{*}\left(h_{\theta_j}^{L}\right)\right)^{\lambda}+\left(f^{*}\left(h_{\theta_j}^{U}\right)\right)^{\lambda}\right)T_j}\right)^{\frac{1}{\lambda}}$$

$$\geq 1-\left(1-\frac{\sum_{j=1}^{n}\left(\left(f^{*}\left(h_{\theta_j}^{L}\right)\right)^{\lambda}+\left(f^{*}\left(h_{\theta_j}^{U}\right)\right)^{\lambda}\right)T_jf_j^{\lambda}}{\sum_{j=1}^{n}\left(\left(f^{*}\left(h_{\theta_j}^{L}\right)\right)^{\lambda}+\left(f^{*}\left(h_{\theta_j}^{U}\right)\right)^{\lambda}\right)T_j}\right)^{\frac{1}{\lambda}}$$

$$\geq 1-\left(1-\frac{\sum_{j=1}^{n}\left(\left(f^{*}\left(h_{\theta_j}^{L}\right)\right)^{\lambda}+\left(f^{*}\left(h_{\theta_j}^{U}\right)\right)^{\lambda}\right)T_j\left(\min_j\{f_j\}\right)^{\lambda}}{\sum_{j=1}^{n}\left(\left(f^{*}\left(h_{\theta_j}^{L}\right)\right)^{\lambda}+\left(f^{*}\left(h_{\theta_j}^{U}\right)\right)^{\lambda}\right)T_j}\right)^{\frac{1}{\lambda}}.$$

That is, $\min_j\{t_j\}\leq t_a\leq\max_j\{t_j\}$, $\max_j\{i_j\}\geq i_a\geq\min_j\{i_j\}$ and $\max_j\{f_j\}\geq f_a\geq\min_j\{f_j\}$.

Thus, $(\min_j\{t_j\}+2-\max_j\{i_j\}-\max_j\{f_j\})\leq(t_a+2-i_a-f_a)\leq(\max_j\{t_j\}+2-\min_j\{i_j\}-\min_j\{f_j\})$, and

2. If $\lambda\to 0$, then the GSNULPWA operator degenerates into the SNULPWG operator.

$$\lim_{\lambda\to 0}GSNULPWA(a_1,a_2,\ldots,a_n)=\mathop{\otimes}_{j=1}^{n}a_j^{\left(T_j/\sum_{i=1}^{n}T_i\right)}$$
$$=\left\langle\left[f^{*-1}\left(\prod_{j=1}^{n}\left(f^{*}(h_{\theta_j}^{L})\right)^{\left(T_j/\sum_{i=1}^{n}T_i\right)}\right),\right.\right.$$
$$\left.f^{*-1}\left(\prod_{j=1}^{n}\left(f^{*}(h_{\theta_j}^{U})\right)^{\left(T_j/\sum_{i=1}^{n}T_i\right)}\right)\right],$$
$$\left(\prod_{j=1}^{n}t_j^{\left(T_j/\sum_{i=1}^{n}T_i\right)},1-\prod_{j=1}^{n}(1-i_j)^{\left(T_j/\sum_{i=1}^{n}T_i\right)},\right.$$
$$\left.\left.1-\prod_{j=1}^{n}(1-f_j)^{\left(T_j/\sum_{i=1}^{n}T_i\right)}\right)\right\rangle.$$

3. If $\lambda\to\infty$, then the GSNULPWA operator degenerates into the following form.

$$\lim_{\lambda\to\infty} GSNULPWA(a_1,a_2,\ldots,a_n) = \left\langle \left[\max_j\left\{h^L_{\theta_j}\right\}, \max_j\left\{h^U_{\theta_j}\right\}\right], \left(\max_j\{t_j\}, \min_j\{i_j\}, \min_j\{f_j\}\right)\right\rangle.$$

4.3 MCDM approach based on GSNULPWA operator

This subsection applies the GSNULPWA operator to solve MCDM problems with simplified neutrosophic uncertain linguistic information.

For MCDM problems with simplified neutrosophic uncertain linguistic information, assume that there is a set of criteria $C=\{c_1,c_2,\ldots,c_n\}$, and the prioritization relationship that exists among them is $c_1 \succ c_2 \succ \cdots \succ c_n$. This indicates that criterion c_j has a higher priority than c_k if $j<k$. Under these criteria, there exists a set of the alternatives $A=\{a_1,a_2,\ldots,a_m\}$ and the criteria values of alternatives are expressed as SNULEs $r_{ij}=\left\langle [h^L_{\theta_{ij}}, h^U_{\theta_{ij}}], (t_{ij}, i_{ij}, f_{ij})\right\rangle$ $(i=1,2,\ldots,m;\ j=1,2,\ldots,n)$. Suppose that $R=(r_{ij})_{m\times n}$ is the decision matrix, in which $r_{ij}=\left\langle [h^L_{\theta_{ij}}, h^U_{\theta_{ij}}], (t_{ij}, i_{ij}, f_{ij})\right\rangle$ takes the form of a SNULE for a_i with respect to c_j. Subsequently, a ranking of alternatives is required.

In general, the following are the main procedures of the MCDM approach described above.

Step 1. Normalize the decision matrix.

In general, there are two types of criteria, called maximizing criteria and minimizing criteria. In order to ensure uniform criterion types, the minimizing criteria need to be transformed into maximizing criteria using the negation operator in Definition 6.

For convenience, the normalized criterion values of a_i $(i=1,2,\ldots,m)$ with respect to c_j $(j=1,2,\ldots,n)$ are also expressed as $\left\langle [h^L_{\theta_{ij}}, h^U_{\theta_{ij}}], (t_{ij}, i_{ij}, f_{ij})\right\rangle$.

Step 2. Calculate the comprehensive evaluation values for each alternative.

Use Eq. (1) to calculate the comprehensive evaluation values, denoted by r_i $(i=1,2,\ldots,m)$ for each alternative a_i.

Step 3. Calculate the score values, accuracy values and certainty values of r_i $(i=1,2,\ldots,m)$.

Use the equations in Definition 7 to calculate the score values, accuracy values and certainty values, denoted by $S(r_i)$, $A(r_i)$ and $C(r_i)$ of r_i $(i=1,2,\ldots,m)$, respectively.

Step 4. Rank all alternatives and select the best one(s).

Use the comparison method described in Definition 8 to rank all the alternatives and select the best one(s) according to $S(r_i)$, $A(r_i)$ and $C(r_i)$ $(i=1,2,\ldots,m)$.

5 Illustrative example

This section employs an investment appraisal project to apply the proposed decision-making approach, and to demonstrate its validity and effectiveness.

5.1 Background

The following case is adapted from [69].

ABC Nonferrous Metals Co. Ltd. is a large state-owned company whose main business is producing and selling nonferrous metals. It is also the largest manufacturer of multi-species nonferrous metals in China, with the exception of aluminum. To expand its main business, the company is engaged in overseas investment, and a department consisting of executive managers and several experts in the field has been established specifically to make decisions on global mineral investments.

Recently, the overseas investment department decided to select a pool of alternatives from several foreign countries based on preliminary surveys. After thorough investigation, five countries (alternatives) are taken into consideration, that is, $\{a_1,a_2,\ldots,a_5\}$. Many factors affect the investment environment, and four factors are prioritized based on the experience of the department personnel. These include c_1, resources (including the suitability of the minerals and their exploration); c_2, politics and policy (including corruption and political risks); c_3, economy (including development vitality and stability); and c_4, infrastructure (including railway and highway facilities).

The decision-makers, including experts and executive managers, gather to determine the decision information. The linguistic term set $H=\{h_0,h_1,h_2,\ldots,h_6\}=\{very\ poor,\ poor,\ slightly\ poor,\ fair,.\ slightly\ good,\ good,\ very\ good\}$ is employed here, and the evaluation information is given in the form of SNULEs. Following a heated discussion, they come to a consensus on the final evaluations which are expressed by SNULEs in Table 1.

5.2 An illustration of the proposed approach

The following section presents the main procedures for obtaining the optimal ranking of alternatives. Assume that the prioritization relationship for the criteria is $c_1 \succ c_2 \succ c_3 \succ c_4$. Let $\lambda=1, f^*(h_i)=\frac{i}{2t}$ and $\rho(y)=y^2$.

Step 1. Normalize the decision matrix.

Because all of the criteria are maximizing criteria, the performance values of alternatives a_i $(i=1,2,3,4,5)$ do not need to be normalized.

Step 2. Calculate the comprehensive evaluation values for each alternative.

Table 1 Simplified neutrosophic uncertain linguistic decision information

	c_1	c_2	c_3	c_4
a_1	$\langle [h_3, h_6], (0.6, 0.6, 0.1) \rangle$	$\langle [h_4, h_5], (0.7, 0.4, 0.3) \rangle$	$\langle [h_3, h_6], (0.8, 0.5, 0.1) \rangle$	$\langle [h_1, h_4], (0.8, 0.3, 0.1) \rangle$
a_2	$\langle [h_1, h_4], (0.7, 0.5, 0.1) \rangle$	$\langle [h_3, h_6], (0.6, 0.4, 0.2) \rangle$	$\langle [h_2, h_5], (0.6, 0.2, 0.4) \rangle$	$\langle [h_3, h_6], (0.7, 0.4, 0.3) \rangle$
a_3	$\langle [h_2, h_5], (0.5, 0.1, 0.2) \rangle$	$\langle [h_3, h_6], (0.6, 0.5, 0.3) \rangle$	$\langle [h_5, h_6], (0.7, 0.6, 0.1) \rangle$	$\langle [h_1, h_4], (0.5, 0.5, 0.2) \rangle$
a_4	$\langle [h_1, h_4], (0.4, 0.5, 0.3) \rangle$	$\langle [h_2, h_5], (0.5, 0.3, 0.4) \rangle$	$\langle [h_3, h_6], (0.6, 0.8, 0.2) \rangle$	$\langle [h_4, h_5], (0.9, 0.3, 0.1) \rangle$
a_5	$\langle [h_4, h_5], (0.6, 0.4, 0.4) \rangle$	$\langle [h_4, h_5], (0.8, 0.3, 0.1) \rangle$	$\langle [h_2, h_5], (0.7, 0.5, 0.1) \rangle$	$\langle [h_3, h_6], (0.6, 0.5, 0.2) \rangle$

Table 2 Ranking results with different λ using f^*

λ	f_1^*	f_2^*	f_3^*
$\lambda \to 0$	$a_5 \succ a_1 \succ a_3 \succ a_2 \succ a_4$	$a_5 \succ a_1 \succ a_3 \succ a_2 \succ a_4$	$a_5 \succ a_1 \succ a_3 \succ a_2 \succ a_4$
$\lambda = 1$	$a_5 \succ a_1 \succ a_3 \succ a_2 \succ a_4$	$a_5 \succ a_1 \succ a_3 \succ a_2 \succ a_4$	$a_5 \succ a_1 \succ a_3 \succ a_2 \succ a_4$
$\lambda = 2$	$a_5 \succ a_1 \succ a_3 \succ a_2 \succ a_4$	$a_5 \succ a_1 \succ a_3 \succ a_2 \succ a_4$	$a_5 \succ a_1 \succ a_3 \succ a_2 \succ a_4$
$\lambda = 3$	$a_5 \succ a_1 \succ a_3 \succ a_2 \succ a_4$	$a_5 \succ a_1 \succ a_3 \succ a_2 \succ a_4$	$a_5 \succ a_1 \succ a_3 \succ a_2 \succ a_4$
$\lambda = 4$	$a_5 \succ a_1 \succ a_3 \succ a_2 \succ a_4$	$a_5 \succ a_1 \succ a_3 \succ a_2 \succ a_4$	$a_5 \succ a_1 \succ a_3 \succ a_2 \succ a_4$
$\lambda = 5$	$a_5 \succ a_3 \succ a_1 \succ a_2 \succ a_4$	$a_5 \succ a_3 \succ a_1 \succ a_2 \succ a_4$	$a_5 \succ a_3 \succ a_1 \succ a_2 \succ a_4$
$\lambda = 6$	$a_5 \succ a_3 \succ a_1 \succ a_2 \succ a_4$	$a_3 \succ a_5 \succ a_1 \succ a_2 \succ a_4$	$a_5 \succ a_3 \succ a_1 \succ a_2 \succ a_4$
$\lambda = 7$	$a_3 \succ a_5 \succ a_1 \succ a_2 \succ a_4$	$a_3 \succ a_5 \succ a_1 \succ a_2 \succ a_4$	$a_5 \succ a_3 \succ a_1 \succ a_2 \succ a_4$
$\lambda = 8$	$a_3 \succ a_5 \succ a_1 \succ a_2 \succ a_4$	$a_3 \succ a_5 \succ a_1 \succ a_2 \succ a_4$	$a_3 \succ a_5 \succ a_1 \succ a_2 \succ a_4$
$\lambda = 9$	$a_3 \succ a_5 \succ a_1 \succ a_2 \succ a_4$	$a_3 \succ a_5 \succ a_1 \succ a_2 \succ a_4$	$a_3 \succ a_5 \succ a_1 \succ a_2 \succ a_4$
$\lambda = 10$	$a_3 \succ a_5 \succ a_1 \succ a_2 \succ a_4$	$a_3 \succ a_5 \succ a_1 \succ a_2 \succ a_4$	$a_3 \succ a_5 \succ a_1 \succ a_2 \succ a_4$

Use Eq. (1) to calculate the comprehensive evaluation values, denoted by r_i $(i = 1, 2, 3, 4, 5)$ for each alternative a_i.

$r_1 = \langle [h_{3.1295}, h_{5.6400}], (0.6559, 0.5278, 0.1502) \rangle$,
$r_2 = \langle [h_{1.4662}, h_{4.662}], (0.6653, 0.4436, 0.1602) \rangle$,

$r_3 = \langle [h_{2.4498}, h_{5.2665}], (0.5529, 0.2863, 0.2137) \rangle$, $r_4 = \langle [h_{1.2731}, h_{4.2427}], (0.4444, 0.4798, 0.3061) \rangle$ and $r_5 = \langle [h_{3.6679}, h_{5.0495}], (0.6619, 0.3912, 0.2800) \rangle$.

Step 3. Calculate the score values, accuracy values and certainty values of r_i $(i = 1, 2, 3, 4, 5)$.

Use the equations in Definition 7 to calculate the score values, accuracy values and certainty values, denoted by $S(r_i)$, $A(r_i)$ and $C(r_i)$ of r_i $(i = 1, 2, 3, 4, 5)$, respectively.

$S(r_1) = 0.4358$, $S(r_2) = 0.2824$, $S(r_3) = 0.3865$, $S(r_4) = 0.2085$, $S(r_5) = 0.4566$; $A(r_1) = 0.3343$, $A(r_2) = 0.2076$, $A(r_3) = 0.1916$, $A(r_4) = 0.0521$, $A(r_5) = 0.2628$; $C(r_1) = 0.4336$, $C(r_2) = 0.2735$, $C(r_3) = 0.3122$, $C(r_4) = 0.1676$, $C(r_5) = 0.4554$.

Step 4. Rank the alternatives and select the best one(s).

Use the comparison method described in Definition 8 to rank all the alternatives and select the best one(s) according to $S(r_i)$, $A(r_i)$ and $C(r_i)$\; $(i = 1, 2, 3, 4, 5)$.

$a_5 \succ a_1 \succ a_3 \succ a_2 \succ a_4$ and a_5 is the best one.

In order to illustrate the influence of the linguistic scale function f^* and parameter λ on the decision-making result in this example, different f^* and values of λ can be taken into consideration. The results are shown in Table 2. (Let $\alpha = \sqrt[7]{9} \approx 1.37$ and $\beta = \gamma = 0.88$)

The results in Table 2, show that the ranking of alternatives may change as the linguistic scale function f^* or parameter λ in the GSNULPWA operator changes. If $\lambda \leq 5$, the best alternative is a_5; if $\lambda \geq 8$, the best alternative is a_3. The worst alternative is always a_4. Moreover, the rankings of alternatives may differ slightly when f^* changes. Thus, decision-makers can appropriately select f^* in accordance with their interests and the actual semantic situations.

5.3 Comparison analysis and discussion

To further illustrate the advantages of the proposed approach based on GSNULPWA operator under a simplified neutrosophic uncertain linguistic environment, a comparative study was conducted using other methods. The comparison analysis includes two cases. One applies the method outlined in Ye [57], which is compared to the proposed approach using INLNs. In the other, the method introduced in Li et al. [23] is compared to an approach using HFULEs.

Case 1 The proposed approach is compared with the methods using INLNs.

For comparison, the transformation from SNULEs to INLNs is accomplished by substituting each uncertain linguistic variable with the mean value of its upper and

Table 3 Ranking results according to the method in Ye [57]

Methods	Scores	Ranking results
INLWAA	$S(a_1) = 2.6814$, $S(a_2) = 1.7288$, $S(a_3) = 2.4339$, $S(a_4) = 1.2582$, $S(a_5) = 2.8355$	$a_5 \succ a_1 \succ a_3 \succ a_2 \succ a_4$
INLWGA	$S(a_1) = 2.5738$, $S(a_2) = 1.6221$, $S(a_3) = 2.2351$, $S(a_4) = 1.1950$, $S(a_5) = 2.6927$	$a_5 \succ a_1 \succ a_3 \succ a_2 \succ a_4$

Table 4 Scores of alternatives according to the methods in Li et al. [23]

Alternatives	a_1	a_2	a_3	a_4	a_5
Scores	$[h_{1.9982}, h_{3.7073}]$	$[h_{0.9022}, h_{3.0418}]$	$[h_{1.5787}, h_{3.5968}]$	$[h_{0.6449}, h_{2.3016}]$	$[h_{2.3587}, h_{3.3284}]$

Table 5 Complementary matrix

	a_1	a_2	a_3	a_4	a_5
a_1	0.5	0.7288	0.5711	0.9099	0.5034
a_2	0.2712	0.5	0.3519	0.6314	0.2197
a_3	0.4289	0.6481	0.5	0.8033	0.4144
a_4	0.0901	0.3686	0.1967	0.5	0
a_5	0.4966	0.7803	0.5856	1	0.5

lower bounds, and then converting each SNN to an INN with equal upper and lower bounds. For example, the SNULE $\langle [h_3, h_6], (0.7, 0.4, 0.3) \rangle$ becomes the INLN $\langle [h_{4.5}], ([0.7, 0.7], [0.4, 0.4], [0.3, 0.3]) \rangle$.

Ye [57] used the interval neutrosophic linguistic weighted arithmetic average (INLWAA) operator and interval neutrosophic linguistic weighted geometric average (INLWGA) operator with known criteria weights to obtain comprehensive values for the alternatives, and according to score function values. The aggregation operators are then used in a prioritized situation and the criteria weights are generated by the PA operator ($w_{ij} = \frac{T_{ij}}{\sum_{j=1}^{n} T_{ij}}$, $T_1 = 1$, and $T_{ij} = \prod_{k=1}^{j-1} S(a_{ik})$, where $S(a_{ik})$ [57] is the score function value of INLN a_{ik}). The calculated scores of all alternatives are shown in Table 3.

It is shown that the rankings of the two methods are $a_5 \succ a_1 \succ a_3 \succ a_2 \succ a_4$, and the optimal alternative is a_5. Obviously, these results are the same as those obtained by the proposed approach. This demonstrates both the validity and the advantages of the proposed approach, as it can capture different results in different semantic situations or with different parameters λ while the methods in [57] ignore differing semantics. In addition, uncertain linguistic variables are more flexible than linguistic terms in expressing qualitative information. Furthermore, SNSs can state evaluation information more succinctly than INSs. In other words, by incorporating SNULEs, the proposed approach is much more convenient and practical.

Case 2 The proposed approach is compared to a method using HFULEs.

To begin, the SNULE $\left\langle [h^L_{\theta_a}, h^U_{\theta_a}], (t_a, i_a, f_a) \right\rangle$ is transformed into HFULE $\left\langle [h^L_{\theta_a}, h^U_{\theta_a}], \{\Gamma_a\} \right\rangle$. Then, Γ_a can be replaced by the score value of the SNN (t_a, i_a, f_a), and $\Gamma_a = \frac{1}{3}(t_a + 2 - i_a - f_a)$, as described in Definition 8. For example, $(0.8, 0.6, 0.4)$ can be replaced by the hesitant number $\{0.6\}$ because $\Gamma_a = \frac{1}{3}(0.8 + 2 - 0.6 - 0.4) = 0.6$. Therefore SNULE $\langle [h_3, h_5], (0.8, 0.6, 0.4) \rangle$ can be replaced by HFULE $\langle [h_3, h_5], \{0.6\} \rangle$. Assume the criteria weights are generated by the PA operator ($w_{ij} = \frac{T_{ij}}{\sum_{j=1}^{n} T_{ij}}$, $T_1 = 1$, and $T_{ij} = \prod_{k=1}^{j-1} I(a_{ik})$, where $I(a_{ik})$ [23] is the expected value of HFULE a_{ik}).

Li et al. [23] utilized the hesitant fuzzy uncertain linguistic weighted geometric (HFULWG) operator to aggregate hesitant fuzzy uncertain linguistic information with known criteria weights in order to derive the comprehensive overall HFULEs of alternatives. Subsequently, the overall scores of the HFULEs were calculated. Then, a likelihood method was employed to rank the scores. Applying their method produces the scores $S(a_i)$ for the overall HFULEs and the complementary matrix P as shown in Tables 4 and 5.

The alternatives can be ordered by summing all the elements in each line of matrix P with the summing values $p_i = \sum_{j=1}^{n} p_{ij}$. The results are $p_1 = 3.2132$, $p_2 = 1.9742$, $p_3 = 2.7947$, $p_4 = 1.1554$ and $p_5 = 3.3625$.

Since $p_5 > p_1 > p_3 > p_2 > p_4$, the ordering is $a_5 \succ a_1 \succ a_3 \succ a_2 \succ a_4$, and the most desirable alternative is a_5. The rankings obtained by the method in [23] are consistent with some results derived through the proposed approach. Though the ranking results are the same in some situations, the approach developed in this paper is more reliable. The addition operation in SNULEs is more reasonable and reliable because the final simplified neutrosophic number is closely combined with each element of the original SNULE, effectively eliminating information loss. According to Ref. [71], the operations in [23] have some limitations and may produce information distortion. In addition, the proposed approach's use of SNSs allows it to flexibly express incomplete and inconsistent information that is widespread in scientific and engineering situations, which hesitant fuzzy sets cannot deal with effectively.

To summarize the above analysis, the proposed approach for solving MCDM problems with SNULEs has the following advantages.

First, SNULEs that integrate uncertain linguistic elements and SNSs can express evaluation information more flexibly. Although the representation of SNULEs appears complex, it can closely depict uncertain linguistic information while retaining the completeness of the original data and considering the perspectives of decision-makers, all of which can help ensure the accuracy of final outcomes. Furthermore, in applying, SNULEs is allowed, decision-makers can trade off between the characteristics of SNULEs and the interrelated computational costs. Considering that a practical decision-making problem encompasses a huge amount of information, the difficulty of implementing the required computation can be largely overcome with the assistance of powerful computer software.

Second, the operations of SNULEs discussed in this paper are defined on the basis of linguistic scale functions, which can yield different results when a different linguistic scale function f^* is used. Thus, decision-makers can flexibly select values for f^* depending on their preferences and the actual semantic environment.

Third, the proposed GSNULPWA operator can deal with MCDM problems in the simplified neutrosophic uncertain linguistic environment where criteria occupy different priority levels. Moreover, the criteria weights calculated by the PA operator according to the priority levels, are more objective and reasonable than a set of known ones.

6 Conclusions

This paper proposes a new class of fuzzy sets named SNULEs that reflect uncertain, imprecise, incomplete, and inconsistent information in order to address decision-making situations that involve qualitative information rather than numerical information. Based on related research achievements in the literature, this paper also extends the PA operator to the simplified neutrosophic uncertain linguistic environment. Thus, a MCDM approach based on the GSNULPWA operator is developed. Finally, we offer an illustrative example and conduct two cases of comparison analysis with representative methods.

The main advantages of this study are that the proposed approach based on the GSNULPWA operator can accommodate situations where the input arguments consist of SNULEs. In addition, the proposed operator can take into account different priority levels among the criteria. Furthermore, the results can change according to different values of the linguistic scale function f^* or parameter λ. This allows decision-makers to select the most appropriate linguistic scale function and input parameter for their interests and actual semantic situations. In other words, the proposed approach is more feasible and practical for real-world applications because the operator can both accommodate a simplified neutrosophic uncertain linguistic environment, and can consider priority levels among the criteria.

Acknowledgments The authors would like to thank the editors and anonymous reviewers for their helpful comments and suggestions that improved the paper. This work was supported by the National Natural Science Foundation of China (Nos. 71571193 and 71431006).

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