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Research Article

Multiple-Attribute Decision-Making Method Based on Normalized Geometric Aggregation Operators of Single-Valued Neutrosophic Hesitant Fuzzy Information

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As a generalization of both single-valued neutrosophic element and hesitant fuzzy element, single-valued neutrosophic hesitant fuzzy element (SVNHFE) is an efficient tool for describing uncertain and imprecise information. Thus, it is of great significance to deal with single-valued neutrosophic hesitant fuzzy information for many practical problems. In this paper, we study the aggregation of SVNHFEs based on some normalized operations from geometric viewpoint. Firstly, two normalized operations are defined for processing SVNHFEs. Then, a series of normalized aggregation operators which fulfill some basic conditions of a valid aggregation operator are proposed. Additionally, a decision-making method is developed for resolving multiattribute decision-making problems based on the proposed operators. Finally, a numerical example is provided to illustrate the feasibility and effectiveness of the method.

1. Introduction

Being different from the fuzzy set which assigns one value from [0, 1] for the membership degree of an element, the neutrosophic set [1, 2] is composed of three independent functions, i.e., truth-membership function, indeterminacy-membership function, and falsity-membership function. Neutrosophic set can describe the indeterminacy of information data independently which conforms to human beings' recognition mode better actually. Therefore, many scholars focused their attention to promote its development. Wang et al. [3] presented the single-valued neutrosophic set (SVNS) in which all the three membership degrees belong to unit interval $[0, 1]$ which brings about convenience to adopt neutrosophic theory in many real-life situations. Combining the single-valued neutrosophic set with the rough set, Yang et al. [4] introduced the single-valued neutrosophic rough set. Furthermore, Bao et al. [5] studied the characterization of the single-valued neutrosophic rough set from logic point of view. Besides, Bao et al. [6] put forward the single-valued neutrosophic refined rough set model. By means of the single-valued refined neutrosophic set, Vasantha et al. [7] did some meaningful research on imaginative play of children. In addition, the single-valued neutrosophic set contributes a lot to decision-making problems due to its flexibility and practicability. In particular, Ye [8] introduced cross-entropy in single-valued neutrosophic environment for solving decision-making problems. Liu and Wang [9] developed the decision-making method under the single-valued neutrosophic framework by using normalized weighted Bonferroni mean operator. Subsequently, Ye [10] also explored the single-valued neutrosophic decision-making method based on the correlation coefficient. Biswas et al. [11] studied the single-valued neutrosophic TOPSIS method for multiattribute group decision-making. Yang et al. [12] analyzed triangular single-valued neutrosophic data envelopment and applied it to hospital performance measurement.

In an era of information explosion, people find it difficult to determine the specific membership degree of an element to a set due to various reasons. To solve this problem, Torra [13] proposed the hesitant fuzzy set (HFS) in which the membership degree of an element to a set can be some different values rather than a single one. Furthermore, Xia and Xu [14] characterized the hesitant fuzzy set through a mathematical symbol and defined some basic operations on it. Since presented, the hesitant fuzzy set has contributed a lot to decision-making problems by combining with aggregation operators. Firstly, Xia and Xu [14] put forward a number of hesitant fuzzy aggregation operators from arithmetic and geometric viewpoint, respectively. In addition, Xia et al. [15] also came up with some aggregation operators for hesitant fuzzy information based on quasi-arithmetic means. Meanwhile, Wei [16] developed hesitant fuzzy prioritized operators and applied them to resolve multiple attribute decision-making problems. Zhu et al. [17] put forward hesitant fuzzy geometric Bonferroni means which take full advantage of geometric as well as the Bonferroni aggregation operator. Later, Zhang [18] introduced power aggregation in hesitant fuzzy framework and proposed the hesitant fuzzy power aggregation operator. Wang et al. [19] gave some aggregation operators under dual hesitant fuzzy set environment and explored their application to multiple attribute decision-making.

As described above, both SVNS and HFS have contributed a lot to decision-making problems. Nevertheless, there is only one truth-membership hesitant function in the hesitant fuzzy set which cannot describe indeterminacy-membership degree and falsity-membership degree effectively. On the contrary, an SVNS cannot describe the three membership degrees with different values, which maybe usual in real life due to hesitancy of decision makers. Therefore, Ye [20] first introduced the single-valued neutrosophic hesitant fuzzy set and developed a series of aggregation operators of single-valued neutrosophic hesitant fuzzy elements. Then, Şahin and Liu [21] explored correlation coefficient of the single-valued neutrosophic hesitant fuzzy set as well as its applications to decision-making. Additionally, Liu and Zhang [22] investigated neutrosophic hesitant fuzzy elements aggregation by the aid of Heronian mean aggregation operators. Liu and Luo [23] presented ordered weighted arithmetic and hybrid weighted arithmetic operator under single-valued neutrosophic hesitant fuzzy environment. However, Mishra and Kumar [24] identified the problem that the aggregation operators proposed in [23] do not satisfy monotonicity actually. Wang and Bao [25] also pointed out that the aggregation operators in [23] do not fulfill idempotency either. In fact, all the existing aggregation operators concerned with SVNHFEs do not satisfy the basic properties of a valid aggregation operator such as idempotency and monotonicity. Hence, it is necessary to give some novel aggregation operators to improve earlier results. In this paper, we focus on defining some normalized operations for SVNHFEs and developing a series of normalized geometric single-valued neutrosophic hesitant fuzzy geometric aggregation operators to provide theoretical foundation for decision-making problems.

To achieve the above goal, we design the rest of paper as follows. In Section 2, some basic concepts about the hesitant fuzzy set, single-valued neutrosophic hesitant fuzzy set, and several existing single-valued neutrosophic hesitant fuzzy aggregation operators are provided. In Section 3, we put forward a number of normalized single-valued neutrosophic hesitant fuzzy aggregation operators and explore some basic properties. In Section 4, a method is developed for solving multiattribute decision-making problems. Additionally, a numerical example demonstrates specific process of the method. Finally, we draw a conclusion in Section 5.

2. Preliminaries

In this section, we mainly recall some basic notions and operations of the hesitant fuzzy set and single-valued neutrosophic hesitant fuzzy set which are necessary for understanding the article.

Definition 1 (see [13]). Let U be a fixed set, and a hesitant fuzzy set A on U is defined in terms of function h_A that returns a set of several values in $[0,1]$ when applied to U.

For convenience and directness, Xia and Xu [14] characterized the hesitant fuzzy set as $A = \{\langle x, h_A(x)\rangle | x \in U\}$, where $h_A(x)$ is a point subset of unit interval $[0,1]$, representing the possible membership degrees of the element $x \in U$ to A. For any $x \in U$, $h_A(x)$ is termed as a hesitant fuzzy element, and the set of all hesitant fuzzy elements is denoted by H.

Definition 2 (see [14]). For a hesitant fuzzy element (HFE) h, $s(h) = (\sum_{\gamma\in h}\gamma/\delta(h))$ is termed as the score of h, where $\delta(h)$ is the number of elements in h. For any two HFEs, h_1 and h_2, if $s(h_1) > s(h_2)$, then $h_1 > h_2$; if $s(h_1) = s(h_2)$, then $h_1 = h_2$.

Let HFEs $h_1 = \{0.5, 0.4\}$ and $h_2 = \{\{0.1, 0.8\}$, and it is obvious that $s(h_1) = s(h_2) = 0.45$, which implies $h_1 = h_2$. However, from the data itself we can find that h_1 is much more stable than h_2; thus, it is not reasonable enough to judge the order between HFEs only by the score of element. In the following, we introduce a novel comparison rule to improve Definition 2. First, we need to introduce a creative method to extend an HFE to a fixed length. For convenience, given two HFEs, h_1 and h_2, and $\delta(h_1) = n_1 < n_2 = \delta(h_2)$, Xia and Xu [14] suggested h_1 should be extended by adding the minimum value in it until it reaches the same length with h_2. Zhang [18] pointed out the selection of the appended value depends primarily on the decision makers' risk preferences. Optimists would append the maximum value, while pessimists would append the minimum value. However, both of the methods cannot guarantee the steady of data. In fact, the best choice to extend an HFE is the closest number to the given data, and it is merited to add the score of the HFE repeatedly until it reaches the fixed length. In the present paper, we adopt this method to extend an HFE to a fixed length if without other explanation.

Example 1. Let $h_1 = \{0.2, 0.5\}$ and $h_2 = \{0.1, 0.3, 0.7, 0.8\}$; in order to extend h_1 to reach the same length with h_2, we need to calculate $s(h_1) = 0.35$; then, h_1 should be extended as $h_1' = \{0.2, 0.5, 0.35, 0.35\}$.

Definition 3. For an HFE h, we define $s(h) = (\sum_{\gamma\in h}\gamma/\delta(h))$ as the score of h, $a(h) = \max(\{\gamma|\gamma \in h\}) - \min(\{\gamma|\gamma \in h\})$ as the amplitude of h, and $v(h) = (\sum_{\gamma\in h}(\gamma - s(h))^2/\delta(h))$ as the variance of h.

Definition 4. For any two HFEs, h_1 and h_2, the order relation is defined as follows:

(1) If $s(h_1) < s(h_2)$, then h_1 is smaller than h_2, denoted by $h_1 \prec h_2$

(2) If $s(h_1) = s(h_2)$ and $a(h_1) > a(h_2)$, then h_1 is smaller than h_2, denoted by $h_1 \prec h_2$

(3) If $s(h_1) = s(h_2), a(h_1) = a(h_2)$ and $v(h_1) > v(h_2)$, then h_1 is smaller than h_2, denoted by $h_1 \prec h_2$

(4) If $s(h_1) = s(h_2), a(h_1) = a(h_2)$ and $v(h_1) = v(h_2)$, then h_1 is equivalent to h_2, denoted by $h_1 \sim h_2$

(5) If $\{\gamma_1|\gamma_1 \in h_1\} = \{\gamma_2|\gamma_2 \in h_2\}$, then h_1 is equal to h_2, denoted by $h_1 = h_2$

(6) Suppose $h_i = \{\gamma_{ij}|j = 1, 2, \ldots, n_i\}\ (i = 1, 2)$ with $n_1 \leq n_2$, if $\gamma_{1\sigma(j)} < \gamma_{2\sigma(j)}\ (j = 1, 2, \ldots, n_2)$, then h_1 is strictly smaller than h_2, denoted by $h_1 \prec_s h_2$, where $\gamma_{i\sigma(j)}$ is the j th largest element of h_i, and it should be pointed that there are $n_2 - n_1$ elements $s(h_1)$ inserted in h_1 to ensure the lengths of h_1 and h_2 are the same in the process of comparison

Example 2. Let $h_1 = \{0.1, 0.5, 0.6\}, h_2 = \{0.3, 0.5\}, h_3 = \{0.1, 0.5, 0.6\}, h_4 = \{0.2, 0.3, 0.4\}$, and $h_5 = \{0.2, 0.3, 0.7\}$; then, we can obtain that

$$\begin{aligned} s(h_1) &= 0.4000,\\ s(h_2) &= 0.4000,\\ s(h_3) &= 0.4000,\\ s(h_4) &= 0.3000,\\ s(h_5) &= 0.4000,\\ a(h_1) &= 0.5000,\\ a(h_2) &= 0.2000,\\ a(h_3) &= 0.5000,\\ a(h_5) &= 0.5000,\\ v(h_1) &= 0.0467,\\ v(h_3) &= 0.0467,\\ v(h_5) &= 0.0467, \end{aligned} \tag{1}$$

which indicates $h_4 \prec h_1 = h_3 \sim h_5 \prec h_2$ and $h_4 \prec_s h_2$.

For any three hesitant fuzzy elements, h, h_1, and h_2, Torra [13] and Xia and Xu [14] gave the operations between them as follows:

(i) $h^c = \cup_{\gamma\in h}\{1 - \gamma\}$

(ii) $h_1 \cup h_2 = \cup_{\gamma_1\in h_1,\gamma_2\in h_2}\max\{\gamma_1, \gamma_2\}$

(iii) $h_1 \cap h_2 = \cup_{\gamma_1\in h_1,\gamma_2\in h_2}\min\{\gamma_1, \gamma_2\}$

(iv) $h^\lambda = \cup_{\gamma\in h}\{\gamma^\lambda\}, \lambda > 0$

(v) $\lambda h = \cup_{\gamma\in h}\{1 - (1 - \gamma)^\lambda\}, \lambda > 0$

(vi) $h_1 \oplus h_2 = \cup_{\gamma_1\in h_1,\gamma_2\in h_2}\{\gamma_1 + \gamma_2 - \gamma_1\gamma_2\}$

(vii) $h_1 \otimes h_2 = \cup_{\gamma_1\in h_1,\gamma_2\in h_2}\{\gamma_1\gamma_2\}$

When defining some new operation rules, people always expect they are convenient to implement and satisfy some basic properties, such as distributive law and associative law. Whereas, in the aforementioned definition, we can find out that some desirable properties do not hold. For instance, let an HFE $h = \{0.2, 0.3\}$; then,

$$\begin{aligned} h\oplus h &= \{0.2 + 0.2 - 0.2 \times 0.2, 0.2 + 0.3 - 0.2 \times 0.3, 0.3 + 0.2\\ &\quad - 0.3 \times 0.2, 0.3 + 0.3 - 0.3 \times 0.3\}\\ &= \{0.36, 0.44, 0.44, 0.51\}, \end{aligned} \tag{2}$$

whereas $2h = \{1 - 0.8^2, 1 - 0.7^2\} = \{0.36, 0.51\}$, which means that $h\oplus h \neq 2h$. In addition, $h \otimes h = \{0.2 \times 0.2, 0.2 \times 0.3, 0.3 \times 0.2, 0.3 \times 0.3\} = \{0.04, 0.06, 0.06, 0.09\}$ and $h^2 = \{0.2^2, 0.3^2\} = \{0.04, 0.09\}$, and it is obvious that $h \otimes h \neq h^2$.

In what follows, we give some new normalized operations which turn out to satisfy a number of basic desirable properties.

Definition 5. Given HFEs $h_1 = \cup_{i=1}^{n_1}\{\xi_i\}$ and $h_2 = \cup_{i=1}^{n_2}\{\eta_i\}$ with $n_1 \leq n_2$, normalized sum $\oplus_N$ and normalized product $\otimes_N$ are defined as follows:

(1) $h_1 \oplus_N h_2 = \cup_{i=1}^{n_2}\{\xi_{\sigma(i)} + \eta_{\sigma(i)} - \xi_{\sigma(i)}\eta_{\sigma(i)}\}$

(2) $h_1 \otimes_N h_2 = \cup_{i=1}^{n_2}\{\xi_{\sigma(i)}\eta_{\sigma(i)}\}$

where $\xi_{\sigma(i)}$ is the ith largest element of h_1, $\eta_{\sigma(i)}$ is the ith largest element of h_2, and there are $n_2 - n_1$ elements $s(h_1)$ inserted in h_1 such that the lengths of two HHEs are the same.

Proposition 1. *Let h, h_1, and h_2 be three HFEs and $\lambda, \lambda_1, \lambda_2 > 0$; then, the following operation rules hold:*

(1) $h_1 \oplus_N h_2 = h_2 \oplus_N h_1, h_1 \otimes_N h_2 = h_2 \otimes_N h_1$

(2) $(h \oplus_N h_1) \oplus_N h_2 = h \oplus_N (h_1 \oplus_N h_2), (h \otimes_N h_1) \otimes_N h_2 = h \otimes_N (h_1 \otimes_N h_2)$

(3) $\lambda(h_1 \oplus_N h_2) = \lambda h_1 \oplus_N \lambda h_2$

(4) $(\lambda_1 + \lambda_2)h = \lambda_1 h \oplus_N \lambda_2 h$

(5) $(h_1 \otimes_N h_2)^\lambda = h_1^\lambda \otimes_N h_2^\lambda$

(6) $h^{\lambda_1+\lambda_2} = h^{\lambda_1} \otimes_N h^{\lambda_2}$

Proof. (1) and

(2) can be quickly proved by Definition 5. Next, we detail the rest. Suppose $h_1 = \cup_{i=1}^{n_1}\{\xi_i\}, h_2 = \cup_{i=1}^{n_2}\{\eta_i\}, n_1 \leq n_2, h = \cup_{i=1}^{n}\{\gamma_i\}$; then, we have

(3) $h_1 \oplus_N h_2 = \cup_{i=1}^{n_2}\{\xi_{\sigma(i)} + \eta_{\sigma(i)} - \xi_{\sigma(i)}\eta_{\sigma(i)}\}$,

$$\begin{aligned} \lambda(h_1 \oplus_N h_2) &= \bigcup_{i=1}^{n_2}\left\{1 - \left(1 - \xi_{\sigma(i)} - \eta_{\sigma(i)} + \xi_{\sigma(i)}\eta_{\sigma(i)}\right)^\lambda\right\}\\ &= \bigcup_{i=1}^{n_2}\left\{1 - \left(1 - \xi_{\sigma(i)}\right)^\lambda\left(1 - \eta_{\sigma(i)}\right)^\lambda\right\}. \end{aligned} \tag{3}$$

On the contrary, $\lambda h_1 = \cup_{i=1}^{n_1}\{1-(1-\xi_i)^\lambda\}$, $\lambda h_2 = \cup_{i=1}^{n_2}\{1-(1-\eta_i)^\lambda\}$,

$$\begin{aligned}\lambda h_1 \oplus_N \lambda h_2 &= \bigcup_{i=1}^{n_2}\left\{1-\left(1-\xi_{\sigma(i)}\right)^\lambda + 1-\left(1-\eta_{\sigma(i)}\right)^\lambda \right. \\ &\quad \left. -\left(1-\left(1-\xi_{\sigma(i)}\right)^\lambda\right)\left(1-\left(1-\eta_{\sigma(i)}\right)^\lambda\right)\right\} \\ &= \bigcup_{i=1}^{n_2}\left\{1-\left(1-\xi_{\sigma(i)}\right)^\lambda\left(1-\eta_{\sigma(i)}\right)^\lambda\right\}.\end{aligned} \tag{4}$$

Therefore, (3) is proved.

(4) $(\lambda_1+\lambda_2)h = \cup_{i=1}^{n}\{1-(1-\gamma_i)^{\lambda_1+\lambda_2}\}$, $\lambda_1 h = \cup_{i=1}^{n}\{1-(1-\gamma_i)^{\lambda_1}\}$, $\lambda_2 h = \cup_{i=1}^{n}\{1-(1-\gamma_i)^{\lambda_2}\}$,

$$\begin{aligned}\lambda_1 h \oplus_N \lambda_2 h &= \bigcup_{i=1}^{n}\left\{1-\left(1-\gamma_{\sigma(i)}\right)^{\lambda_1} + 1-\left(1-\gamma_{\sigma(i)}\right)^{\lambda_2}\right. \\ &\quad \left. -\left(1-\left(1-\gamma_{\sigma(i)}\right)^{\lambda_1}\right)\left(1-\left(1-\gamma_{\sigma(i)}\right)^{\lambda_2}\right)\right\} \\ &= \bigcup_{i=1}^{n}\left\{1-\left(1-\gamma_{\sigma(i)}\right)^{\lambda_1}\left(1-\gamma_{\sigma(i)}\right)^{\lambda_2}\right\} \\ &= \bigcup_{i=1}^{n}\left\{1-(1-\gamma_i)^{\lambda_1+\lambda_2}\right\}.\end{aligned} \tag{5}$$

(5) $h_1 \otimes_N h_2 = \cup_{i=1}^{n_2}\{\xi_{\sigma(i)}\eta_{\sigma(i)}\}$, $(h_1 \otimes_N h_2)^\lambda = \cup_{i=1}^{n_2}\{(\xi_{\sigma(i)}\eta_{\sigma(i)})^\lambda\} = \cup_{i=1}^{n_2}\{\xi_{\sigma(i)}^\lambda \eta_{\sigma(i)}^\lambda\}$,

$$h_1^\lambda \otimes_N h_2^\lambda = \bigcup_{i=1}^{n_1}\left\{\xi_i^\lambda\right\} \otimes_N \bigcup_{i=1}^{n_2}\left\{\eta_i^\lambda\right\} = \bigcup_{i=1}^{n_2}\left\{\xi_{\sigma(i)}^\lambda \eta_{\sigma(i)}^\lambda\right\}. \tag{6}$$

(6) $h^{\lambda_1+\lambda_2} = \cup_{i=1}^{n}\{\gamma_i^{\lambda_1+\lambda_2}\}$, and $h^{\lambda_1} = \cup_{i=1}^{n}\{\gamma_i^{\lambda_1}\}$, $h^{\lambda_2} = \cup_{i=1}^{n}\{\gamma_i^{\lambda_2}\}$; then,

$$h^{\lambda_1} \otimes_N h^{\lambda_2} = \bigcup_{i=1}^{n}\left\{\gamma_{\sigma(i)}^{\lambda_1}\gamma_{\sigma(i)}^{\lambda_2}\right\} = \bigcup_{i=1}^{n}\left\{\gamma_i^{\lambda_1+\lambda_2}\right\}. \tag{7}$$

□

Definition 6 (see [20]). Let X be a fixed set; then, a single-valued neutrosophic hesitant fuzzy set (SVNHFS) N on X is defined as follows:

$$N = \left\{\langle x, (\tilde{t}(x), \tilde{i}(x), \tilde{f}(x))\rangle | x \in X\right\}, \tag{8}$$

in which $\tilde{t}(x)$, $\tilde{i}(x)$, and $\tilde{f}(x)$ are three point subsets of $[0,1]$, denoting the possible truth hesitant membership degree, indeterminacy hesitant membership degree, and falsity hesitant membership degree of the element x to N, respectively, with the condition $0 \le \gamma, \delta, \eta \le 1$ and $0 \le \gamma^+ + \delta^+ + \eta^+ \le 3$, where $\gamma \in \tilde{t}(x)$, $\delta \in \tilde{i}(x)$, $\eta \in \tilde{f}(x)$, $\gamma^+ = \max(\tilde{t}(x))$, $\delta^+ = \max(\tilde{i}(x))$, $\eta^+ = \max(\tilde{f}(x))$. For each $x \in X$, the triplet $n(x) = (\tilde{t}(x), \tilde{i}(x), \tilde{f}(x))$ is termed as a single-valued neutrosophic hesitant fuzzy element (SVNHFE), which can be denoted by the simplified symbol $n = (\tilde{t}, \tilde{i}, \tilde{f})$, and the set of all SVNHFEs is represented by Ω.

It should be pointed out that the single-valued neutrosophic hesitant fuzzy is the same with the hesitant neutrosophic set essentially in literature [26]. In order to compare SVNHFEs, we give the following concept.

Definition 7. For an SVNHFE $n = (\tilde{t}, \tilde{i}, \tilde{f})$, we define $s(n) = (1/3)(2 + s(\tilde{t}) - s(\tilde{i}) - s(\tilde{f}))$ as the score of n, $a(n) = (1/3)(a(\tilde{t}) + a(\tilde{i}) + a(\tilde{f}))$ as the amplitude of n, and $v(n) = (1/3)(v(\tilde{t}) + v(\tilde{i}) + v(\tilde{f}))$ as the variance of n.

Definition 8. For any two SVNHFEs $n_1 = (\tilde{t}_1, \tilde{i}_1, \tilde{f}_1)$ and $n_2 = (\tilde{t}_2, \tilde{i}_2, \tilde{f}_2)$, the order relation is defined as follows:

(1) If $s(n_1) < s(n_2)$, then n_1 is smaller than n_2, denoted by $n_1 \prec n_2$

(2) If $s(n_1) = s(n_2)$ and $a(n_1) > a(n_2)$, then n_1 is smaller than n_2, denoted by $n_1 \prec n_2$

(3) If $s(n_1) = s(n_2)$, $a(n_1) = a(n_2)$ and $v(n_1) > v(n_2)$, then n_1 is smaller than n_2, denoted by $n_1 \prec n_2$

(4) If $s(n_1) = s(n_2)$, $a(n_1) = a(n_2)$ and $v(n_1) = v(n_2)$, then n_1 is equivalent to n_2, denoted by $n_1 \sim n_2$

(5) If $\tilde{t}_1 = \tilde{t}_2$, $\tilde{i}_1 = \tilde{i}_2$ and $\tilde{f}_1 = \tilde{f}_2$ then n_1 is equal to n_2, denoted by $n_1 = n_2$

(6) If $\tilde{t}_1 \prec_s \tilde{t}_2$, $\tilde{i}_2 \prec_s \tilde{i}_1$ and $\tilde{f}_2 \prec_s \tilde{f}_1$, then n_1 is strictly smaller than n_2, denoted by $n_1 \prec_s n_2$

Example 3. Suppose SVNHFEs $n_1 = (\{0.1, 0.4, 0.5\}, \{0.1, 0.3, 0.5\}, \{0.2, 0.3, 0.4\})$, $n_2 = (\{0.1, 0.5\}, \{0.2, 0.3, 0.7\}, \{0.3, 0.5, 0.6\})$, $n_3 = (\{0.1, 0.4, 0.5\}, \{0.2, 0.3, 0.4\}, \{0.2, 0.3, 0.4\})$, and $n_4 = (\{0.3, 0.5\}, \{0.1, 0.3, 0.7\}, \{0.2, 0.3, 0.4\})$; then, we can calculate that $s(n_1) = 0.5778$, $s(n_2) = 0.4778$, $s(n_3) = 0.5778$, and $s(n_4) = 0.5778$ which indicates $n_2 \prec n_3$, $n_2 \prec n_4$, and $n_2 \prec n_1$. In addition, $a(n_3) = 0.2667 < 0.3333 = a(n_1) = a(n_4)$ means $n_1 \prec n_3$ and $n_4 \prec n_3$. Furthermore, $v(n_1) = 0.02963$ and $v(n_4) = 0.0263$ imply that $n_1 \prec n_4$. Therefore, $n_2 \prec n_1 \prec n_4 \prec n_3$.

For any two single-valued neutrosophic hesitant fuzzy elements $n_1 = (\tilde{t}_1, \tilde{i}_1, \tilde{f}_1)$ and $n_2 = (\tilde{t}_2, \tilde{i}_2, \tilde{f}_2)$, some operations between them are given as follows [20]:

(i) $n_1 \oplus n_2 = (\tilde{t}_1 \oplus \tilde{t}_2, \tilde{i}_1 \otimes \tilde{i}_2, \tilde{f}_1 \otimes \tilde{f}_2) = \cup_{\substack{\gamma_1 \in \tilde{t}_1, \delta_1 \in \tilde{i}_1, \eta_1 \in \tilde{f}_1 \\ \gamma_2 \in \tilde{t}_2, \delta_2 \in \tilde{i}_2, \eta_2 \in \tilde{f}_2}} (\{\gamma_1 + \gamma_2 - \gamma_1\gamma_2\}, \{\delta_1\delta_2\}, \{\eta_1\eta_2\})$

(ii) $n_1 \otimes n_2 = (\tilde{t}_1 \otimes \tilde{t}_2, \tilde{i}_1 \oplus \tilde{i}_2, \tilde{f}_1 \oplus \tilde{f}_2) = \cup_{\gamma_1 \in \tilde{t}_1, \delta_1 \in \tilde{i}_1, \eta_1 \in \tilde{f}_1 \gamma_2 \in \tilde{t}_2, \delta_2 \in \tilde{i}_2, \eta_2 \in \tilde{f}_2} (\{\gamma_1\gamma_2\}, \{\delta_1 + \delta_2 - \delta_1\delta_2\}, \{\eta_1 + \eta_2 - \eta_1\eta_2\})$

(iii) $\lambda n_1 = (\lambda\tilde{t}_1, \tilde{i}_1^\lambda, \tilde{f}_i^\lambda) = \cup_{\gamma_1 \in \tilde{t}_1, \delta_1 \in \tilde{i}_1, \eta_1 \in \tilde{f}_1} (\{1-(1-\gamma_1)^\lambda\}, \{\delta_1^\lambda\}, \{\eta_i^\lambda\}), \lambda > 0$

(iv) $n_1^\lambda = (t_1^\lambda, \lambda\tilde{i}_1, \lambda\tilde{f}_1) = \cup_{\gamma_1 \in \tilde{t}_1, \delta_1 \in \tilde{i}_1, \eta_1 \in \tilde{f}_1} (\{\gamma_1^\lambda\}, \{1-(1-\delta_1)^\lambda\}, \{1-(1-\eta_1)^\lambda\}), \lambda > 0$

For the aforementioned operations, we can find out that some desirable properties do not hold. For instance, let an SVNHFE be $n = (\{0.2, 0.3\}, \{0.2\}, \{0.3\})$; then,

$$\begin{aligned} n\oplus n &= (\{0.2+0.2-0.2\times 0.2, 0.2+0.3-0.2\times 0.3, \\ &\quad 0.3+0.2-0.3\times 0.2, 0.3+0.3-0.3\times 0.3\}, \\ &\quad \{0.2\times 0.2\}, \{0.3\times 0.3\}) \\ &= (\{0.36, 0.44, 0.44, 0.51\}, \{0.04\}, \{0.09\}). \end{aligned} \tag{9}$$

On the contrary, $2n = (\{1-0.8^2, 1-0.7^2\}, \{0.2^2\}, \{03^2\}) = (\{0.36, 0.51\}, \{0.04\}, \{0.09\})$, which means that $n\oplus n \neq 2n$. In addition,

$$\begin{aligned} n\otimes n &= (\{0.2\times 0.2, 0.2\times 0.3, 0.3\times 0.2, 0.3\times 0.3\}, \\ &\quad \{0.2+0.2-0.2\times 0.2\}, \{0.3+0.3-0.3\times 0.3\}) \\ &= (\{0.04, 0.06, 0.06, 0.09\}, \{0.36\}, \{0.51\}), \\ n^2 &= \left(\left\{0.2^2, 0.3^2\right\}, \left\{1-0.8^2\right\}, \left\{1-0.7^2\right\}\right) \\ &= (\{0.04, 0.09\}, \{0.36\}, \{0.51\}), \end{aligned} \tag{10}$$

and it is obvious that $n\otimes n \neq n^2$.

In what follows, we introduce two normalized single-valued neutrosophic hesitant fuzzy operations which obviously satisfy a number of basic operational rules.

Definition 9. Given SVNHFEs $n_1 = (\tilde{t}_1, \tilde{i}_1, \tilde{f}_1) = (\{\gamma_{1i}|i = 1, 2, \ldots, \delta(\tilde{t}_1)\}, \{\delta_{1i}|i = 1, 2, \ldots, \delta(\tilde{i}_1)\}, \{\eta_{1i}|i = 1, 2, \ldots, \delta(\tilde{f}_1)\})$ and $n_2 = (\tilde{t}_2, \tilde{i}_2, \tilde{f}_2) = (\{\gamma_{2i}|i = 1, 2, \ldots, \delta(\tilde{t}_2)\}, \{\delta_{2i}|i = 1, 2, \ldots, \delta(\tilde{i}_2)\}, \{\eta_{2i}|i = 1, 2, \ldots, \delta(\tilde{f}_2)\})$ with $l = \max\{\delta(\tilde{t}_1), \delta(\tilde{t}_2)\}$, $p = \max\{\delta(\tilde{i}_1), \delta(\tilde{i}_2)\}$, $q = \max\left\{\delta(\tilde{f}_1), \delta(\tilde{f}_2)\right\}$, then normalized sum $\oplus_N$ and normalized product $\otimes_N$ are defined as follows:

(1) $n_1\oplus_N n_2 = (\tilde{t}_1\oplus_N\tilde{t}_2, \tilde{i}_1\otimes_N\tilde{i}_2, \tilde{f}_1\otimes_N\tilde{f}_2) = (\cup_{i=1}^{l}\{\gamma_{1\sigma(i)} + \gamma_{2\sigma(i)} - \gamma_{1\sigma(i)}\quad \gamma_{2\sigma(i)}\}, \cup_{i=1}^{p}\{\delta_{1\sigma(i)}\quad \delta_{2\sigma(i)}\}, \cup_{i=}\ 1^{q}\{\eta_{1}\ \sigma(i)\eta_{2\sigma(i)}\})$

(2) $n_1\otimes_N n_2 = (\tilde{t}_1\otimes_N\tilde{t}_2, \quad \tilde{i}_1\oplus_N\tilde{i}_2, \tilde{f}_1\oplus_N\tilde{f}_2) = (\cup_{i=1}^{l}\left\{\gamma_{1\sigma(i)}\gamma_{2\sigma(i)}\right\}, \quad \cup_{i=1}^{p}\left\{\delta_{1\sigma(i)} + \delta_{2\sigma(i)} - \quad \delta_{1\sigma(i)}\delta_{2\sigma(i)}\right\}, \cup_{i=1}^{q}\left\{\eta_{1\sigma(i)} + \eta_{2\sigma(i)} - \eta_{1\sigma(i)}\eta_{2\sigma(i)}\right\})$

$\gamma_{j\sigma(i)}$ is the ith largest element of $\tilde{t}_j$ and there are $l-\delta(\tilde{t}_j)$ element $s(\tilde{t}_j)$ inserted in $\tilde{t}_j$. Similarly, $\delta_{j\sigma(i)}$ is the ith largest element of $\tilde{i}_j$ and there are $p-\delta(\tilde{i}_j)$ elements $s(\tilde{i}_j)$ inserted in $\tilde{i}_j$ and $\eta_{j\sigma(i)}$ is the i th largest elements of $\tilde{f}_j$ and there are $q-\delta(\tilde{f}_j)$ elements $s(\tilde{f}_j)$ inserted in $\tilde{f}_j$.

Example 4. Given two SVNHFEs $n_1 = (\{0.1, 0.4, 0.5\}, \{0.3\}, \{0.2, 0.3, 0.4\})$ and $n_2 = (\{0.1, 0.5\}, \{0.2, 0.3, 0.7\}, \{0.3, 0.5, 0.6\})$, then

$$\begin{aligned} n_1\oplus_N n_2 &= (\{0.5+0.5-0.5\times 0.5, 0.3+0.4-0.3\times 0.4, 0.1+0.1-0.1\times 0.1\}, \{0.3\times 0.7, 0.3\times 0.3, 0.3\times 0.2\}, \\ &\quad \cdot \{0.4\times 0.6, 0.3\times 0.5, 0.2\times 0.3\}) \\ &= (\{0.75, 0.58, 0.19\}, \{0.21, 0.09, 0.06\}, \{0.24, 0.15, 0.06\}), \\ n_1\otimes_N n_2 &= (\{0.5\times 0.5, 0.3\times 0.4, 0.1\times 0.1\}, \{0.3+0.2-0.3\times 0.2, 0.3+0.3-0.3\times 0.3, 0.3+0.7-0.3\times 0.7\}, \\ &\quad \cdot \{0.2+0.3-0.2\times 0.3, 0.3+0.5-0.3\times 0.5, 0.4+0.6-0.4\times 0.6\}) \\ &= (\{0.01, 0.12, 0.15\}, \{0.44, 0.51, 0.79\}, \{0.44, 0.65, 0.76\}). \end{aligned} \tag{11}$$

Proposition 2. *Let n, n_1, and n_2 be three SVNHFEs and $\lambda, \lambda_1, \lambda_2 > 0$; then, the following properties hold:*

(1) $n_1\oplus_N n_2 = n_2\oplus_N n_1, n_1\otimes_N n_2 = n_2\otimes_N n_1$

(2) $(n\oplus_N n_1)\oplus_N n_2 = n\oplus_N(n_1\oplus_N n_2), (n\otimes_N n_1)\otimes_N n_2 = n\otimes_N(n_1\otimes_N n_2)$

(3) $\lambda(n_1\oplus_N n_2) = \lambda n_1\oplus_N\lambda n_2$

(4) $(\lambda_1+\lambda_2)n = \lambda_1 n\oplus_N\lambda_2 n$

(5) $(n_1\otimes_N n_2)^{\lambda} = n_1^{\lambda}\otimes_N n_2^{\lambda}$

(6) $n^{\lambda_1+\lambda_2} = n^{\lambda_1}\otimes_N n^{\lambda_2}$

Proof. Suppose $n_1 = (\tilde{t}_1, \tilde{i}_1, \tilde{f}_1), n_2 = (\tilde{t}_2, \tilde{i}_2, \tilde{f}_2)$, and $n = (\tilde{t}, \tilde{i}, \tilde{f})$; then, we have

(1) $n_1\oplus_N n_2 = (\tilde{t}_1\oplus_N\tilde{t}_2, \tilde{i}_1\otimes_N\tilde{i}_2, \tilde{f}_1\otimes_N\tilde{f}_2) = (\tilde{t}_2\oplus_N\tilde{t}_1, \tilde{i}_2\otimes_N\tilde{i}_1, \tilde{f}_2\otimes_N\tilde{f}_1) = n_2\oplus_N n_1$, $n_1\otimes_N n_2 = (\tilde{t}_1\otimes_N\tilde{t}_2, \tilde{i}_1\oplus_N\tilde{i}_2, \tilde{f}_1\oplus_N\tilde{f}_2) = (\tilde{t}_2\otimes_N\tilde{t}_1, \tilde{i}_2\oplus_N\tilde{i}_1, \tilde{f}_2\oplus_N\tilde{f}_1) = n_2\otimes_N n_1$

(2) $(n\oplus_N n_1)\oplus_N n_2 = ((\tilde{t}\oplus_N\tilde{t}_1)\oplus_N\tilde{t}_2, (\tilde{i}\otimes_N\tilde{i}_1)\otimes_N\tilde{i}_2, (\tilde{f}\otimes_N\tilde{f}_1)\otimes_N\tilde{f}_2) = (\tilde{t}\oplus_N(\tilde{t}_1\oplus_N\tilde{t}_2), \tilde{i}\otimes_N(\tilde{i}_1\otimes_N\tilde{i}_2), \tilde{f}\otimes_N(\tilde{f}_1\otimes_N\tilde{f}_2)) = n\oplus_N(n_1\oplus_N n_2)$, $(n\otimes_N n_1)\otimes_N n_2 = ((\tilde{t}\otimes_N\tilde{t}_1)\otimes_N\tilde{t}_2, (\tilde{i}\oplus_N\tilde{i}_1)\oplus_N\tilde{i}_2, (\tilde{f}\oplus_N\tilde{f}_1)\oplus_N\tilde{f}_2) = (\tilde{t}\otimes_N(\tilde{t}_1\otimes_N\tilde{t}_2), \tilde{i}\oplus_N(\tilde{i}_1\oplus_N\tilde{i}_2), \tilde{f}\oplus_N(\tilde{f}_1\oplus_N\tilde{f}_2)) = n\otimes_N(n_1\otimes_N n_2)$

(3) $\lambda(n_1\oplus_N n_2) = \lambda(\tilde{t}_1\oplus_N\tilde{t}_2, \tilde{i}_1\otimes_N\tilde{i}_2, \tilde{f}_1\otimes_N\tilde{f}_2) = (\lambda(\tilde{t}_1\oplus_N\tilde{t}_2), (\tilde{i}_1\otimes_N\tilde{i}_2)^{\lambda}, (\tilde{f}_1\otimes_N\tilde{f}_2)^{\lambda}) = (\lambda\tilde{t}_1\oplus_N\lambda\tilde{t}_2, \tilde{i}_1^{\lambda}\otimes_N\tilde{i}_2^{\lambda}, \tilde{f}_1^{\lambda}\otimes_N\tilde{f}_2^{\lambda}) = (\lambda\tilde{t}_1, \tilde{i}_1^{\lambda}, \tilde{f}_1^{\lambda})\oplus_N(\lambda\tilde{t}_2, \tilde{i}_2^{\lambda}, \tilde{f}_2^{\lambda}) = \lambda n_1\oplus_N\lambda n_2$

(4) $(\lambda_1+\lambda_2)n = ((\lambda_1+\lambda_2)\tilde{t}, \tilde{i}^{\lambda_1+\lambda_2}, \tilde{f}^{\lambda_1+\lambda_2}) = (\lambda_1\tilde{t}\oplus_N\lambda_2\tilde{t}, \tilde{i}^{\lambda_1}\otimes_N\tilde{i}^{\lambda_2}, \tilde{f}^{\lambda_1}\otimes_N\tilde{f}^{\lambda_2}) = \lambda_1 n\oplus_N\lambda_2 n$

(5) $(n_1\otimes_N n_2)^{\lambda} = ((\tilde{t}_1\otimes_N\tilde{t}_2)^{\lambda}, \lambda(\tilde{i}_1\oplus_N\tilde{i}_2), \lambda(\tilde{f}_1\oplus_N\tilde{f}_2))$

(6) $n^{\lambda_1+\lambda_2} = (\tilde{t}^{\lambda_1+\lambda_2}, (\lambda_1+\lambda_2)\tilde{i}, (\lambda_1+\lambda_2)\tilde{f}) = (\tilde{t}^{\lambda_1}\otimes_N\tilde{t}^{\lambda_2}, \lambda_1\tilde{i}\oplus_N\lambda_2\tilde{i}, \lambda_1\tilde{f}\oplus_N\lambda_2\tilde{f}) = n^{\lambda_1}\oplus_N n^{\lambda_2}$ □

Definition 10 (see [14]). For a collection of HFEs h_j $(j = 1, 2, \ldots, n)$, some hesitant fuzzy aggregation operators are defined as follows:

(1) The hesitant fuzzy weighted geometric operator HFWG:

$$\mathrm{HFWG}(h_1, h_2, \ldots, h_n) = \otimes_{j=1}^{n} (h_j)^{w_j} = \bigcup_{\gamma_j \in h_j} \left\{ \prod_{j=1}^{n} \gamma_j^{w_j} \right\}, \quad (12)$$

where $w = (w_1, w_2, \ldots, w_n)^T$ is the weight vector of $(h_1, h_2, \ldots, h_n)$ with $w_j \in [0,1]\,(j = 1, 2, \ldots, n)$ and $\sum_{j=1}^{n} w_j = 1$.

(2) The hesitant fuzzy ordered weighted geometric operator HFOWG:

$$\mathrm{HFOWG}(h_1, h_2, \ldots, h_n) = \otimes_{j=1}^{n} (h_{\sigma(j)})^{\omega_j} = \bigcup_{\gamma_{\sigma(j)} \in h_{\sigma(j)}} \left\{ \prod_{j=1}^{n} (\gamma_{\sigma(j)})^{\omega_j} \right\}, \quad (13)$$

where $h_{\sigma(j)}$ is the jth largest element of $h_i\,(i = 1, 2, \ldots, n)$ and $\omega = (\omega_1, \omega_2, \ldots, \omega_n)^T$ is the aggregation-associated vector such that $\omega_j \in [0,1]\,(j = 1, 2, \ldots, n)$ and $\sum_{j=1}^{n} \omega_j = 1$.

(3) The hesitant fuzzy hybrid geometric operator HFHG:

$$\mathrm{HFHG}(h_1, h_2, \ldots, h_n) = \otimes_{j=1}^{n} (\dot{h}_{\sigma(j)})^{\omega_j} = \bigcup_{\dot{\gamma}_{\sigma(j)} \in \dot{h}_{\sigma(j)}} \left\{ \prod_{j=1}^{n} (\dot{\gamma}_{\sigma(j)})^{\omega_j} \right\}, \quad (14)$$

where $\dot{h}_{\sigma(j)}$ is the jth largest element of $\dot{h}_i = h_i^{nw_i}\,(i = 1, 2, \ldots, n)$, $w = (w_1, w_2, \ldots, w_n)^T$ is the weight vector of $(h_1, h_2, \ldots, h_n)$ with $w_j \in [0,1]$ $(j = 1, 2, \ldots, n)$ and $\sum_{j=1}^{n} w_j = 1$, and $\omega = (\omega_1, \omega_2, \ldots, \omega_n)^T$ is the aggregation-associated vector such that $\omega_j \in [0,1]\,(j = 1, 2, \ldots, n)$ and $\sum_{j=1}^{n} \omega_j = 1$.

3. Normalized Single-Valued Neutrosophic Hesitant Fuzzy Geometric Aggregation Operators

In this part, we propose some normalized aggregation operators based on the normalized product operation.

Definition 11. For a collection of SVNHFEs $n_j = (\tilde{t}_j, \tilde{i}_j, \tilde{f}_j) \in \Omega\,(j = 1, 2, \ldots, k)$, a normalized single-valued neutrosophic hesitant fuzzy geometric mean aggregation operator NSVNHFG:$\Omega^k \longrightarrow \Omega$ is defined as

$$\mathrm{NSVNHFG}(n_1, n_2, \ldots, n_k) = \left(\otimes_{j=1N}^{k} n_j\right)^{(1/k)} = \left(\left(\otimes_{j=1N}^{k} \tilde{t}_j\right)^{(1/k)}, \frac{1}{k} \oplus_{j=1N}^{k} \tilde{i}_j, \frac{1}{k} \oplus_{j=1N}^{k} \tilde{f}_j\right). \quad (15)$$

Definition 12. For a collection of SVNHFEs $n_j = (\tilde{t}_j, \tilde{i}_j, \tilde{f}_j) \in \Omega\,(j = 1, 2, \ldots, k)$ and $w = (w_1, w_2, \ldots, w_k)^T$ which is the weight vector of $(n_1, n_2, \ldots, n_k)$ with $w_j \in [0,1]$, $\sum_{j=i}^{k} w_j = 1$, a normalized single-valued neutrosophic hesitant fuzzy weighted geometric aggregation operator NSVNHFWG: $\Omega^k \longrightarrow \Omega$ is a mapping such that

$$\mathrm{NSVNHFWG}(n_1, n_2, \ldots, n_k) = \otimes_{j=1N}^{k} n_j^{w_j} = \left(\otimes_{j=1N}^{k} \tilde{t}_j^{w_j}, \oplus_{j=1N}^{k} w_j \tilde{i}_j, \oplus_{j=1N}^{k} w_j \tilde{f}_j\right). \quad (16)$$

It can be observed that if $w = ((1/k), (1/k), \ldots, (1/k))^T$, then the operator NSVNHFWG reduces to the operator NSVNHFG.

Theorem 1. *Let $n_j = (\tilde{t}_j, \tilde{i}_j, \tilde{f}_j) = (\{\gamma_{ji} \mid i = 1, 2, \ldots, l_j\}, \{\delta_{ji} \mid i = 1, 2, \ldots, p_j\}, \{\eta_{ji} \mid i = 1, 2, \ldots, q_j\})\,(j = 1, 2, \ldots, k)$ be a collection of SVNHFEs; if the weight vector is $w = (w_1, w_2, \ldots, w_n)^T$, then the aggregated result by operator NSVNHFWG can be expressed as*

$$\mathrm{NSVNHFWG}(n_1, n_2, \ldots, n_k) = \left(\bigcup_{i=1}^{l} \left\{ \prod_{j=1}^{k} \gamma_{j\sigma(i)}^{w_j} \right\}, \bigcup_{i=1}^{p} \left\{ 1 - \prod_{j=1}^{k} (1 - \delta_{j\sigma(i)})^{w_j} \right\}, \bigcup_{i=1}^{q} \left\{ 1 - \prod_{j=1}^{k} (1 - \eta_{j\sigma(i)})^{w_j} \right\} \right), \quad (17)$$

where $\gamma_{j\sigma(i)}, \delta_{j\sigma(i)}$, *and* $\eta_{j\sigma(i)}$ *are the ith largest element of* $\widetilde{t}_j, \widetilde{i}_j$, *and* $\widetilde{f}_j$, *respectively,* $l = \max_{j=1,\ldots,k}(l_j)$, $p = \max_{j=1,\ldots,k}(p_j)$, *and* $q = \max_{j=1,\ldots,k}(q_j)$.

Proof. We prove the result by mathematical induction on k.

First, we demonstrate (17) holds for $k = 2$. Since

$$\begin{aligned} n_1^{w_1} &= \left(\widetilde{t}_1^{w_1}, w_1\widetilde{i}_1, w_1\widetilde{f}_1\right) = \left(\bigcup_{i=1}^{l_1}\{\gamma_{1i}^{w_1}\}, \bigcup_{i=1}^{p_1}\{1-(1-\delta_{1i})^{w_1}\}, \bigcup_{i=1}^{q_1}\{1-(1-\eta_{1i})^{w_1}\}\right), \\ n_2^{w_2} &= \left(\widetilde{t}_2^{w_2}, w_2\widetilde{i}_2, w_2\widetilde{f}_2\right) = \left(\bigcup_{i=1}^{l_2}\{\gamma_{2i}^{w_2}\}, \bigcup_{i=1}^{p_2}\{1-(1-\delta_{2i})^{w_2}\}, \bigcup_{i=1}^{q_2}\{1-(1-\eta_{2i})^{w_2}\}\right), \end{aligned} \tag{18}$$

then

$$\begin{aligned} \text{NSVNHFWG}(n_1, n_2) &= n_1^{w_1} \otimes_N n_2^{w_2} \\ &= \left(\bigcup_{i=1}^{\max(l_1,l_2)}\{\gamma_{1\sigma(i)}^{w_1}\gamma_{2\sigma(i)}^{w_2}\}, \bigcup_{i=1}^{\max(p_1,p_2)}\{1-(1-\delta_{1\sigma(i)})^{w_1} + 1-(1-\delta_{2\sigma(i)})^{w_2} - (1-(1-\delta_{1\sigma(i)})^{w_1})(1-(1-\delta_{2\sigma(i)})^{w_2})\},\right. \\ &\quad \left.\bigcup_{i=1}^{\max(q_1,q_2)}\{1-(1-\eta_{1\sigma(i)})^{w_1} + 1-(1-\eta_{2\sigma(i)})^{w_2} - (1-(1-\eta_{1\sigma(i)})^{w_1})(1-(1-\eta_{2\sigma(i)})^{w_2})\}\right) \\ &= \left(\bigcup_{i=1}^{\max(l_1,l_2)}\left\{\prod_{j=1}^{2}\gamma_{j\sigma(i)}^{w_j}\right\}, \bigcup_{i=1}^{\max(p_1,p_2)}\{1-(1-\delta_{1\sigma(i)})^{w_1}(1-\delta_{2\sigma(i)})^{w_2}\}, \bigcup_{i=1}^{\max(q_1,q_2)}\{1-(1-\eta_{1\sigma(i)})^{w_1}(1-\eta_{2\sigma(i)})^{w_2}\}\right) \\ &= \left(\bigcup_{i=1}^{\max(l_1,l_2)}\left\{\prod_{j=1}^{2}\gamma_{j\sigma(i)}^{w_j}\right\}, \bigcup_{i=1}^{\max(p_1,p_2)}\left\{1-\prod_{j=1}^{2}(1-\delta_{j\sigma(i)})^{w_j}\right\}, \bigcup_{i=1}^{\max(q_1,q_2)}\left\{1-\prod_{j=1}^{2}(1-\eta_{j\sigma(i)})^{w_j}\right\}\right). \end{aligned} \tag{19}$$

If (17) holds for $k = m$, that is,

$$\text{NSVNHFWG}(n_1, n_2, \ldots, n_m) = \left(\bigcup_{i=1}^{l\prime}\left\{\prod_{j=1}^{m}\gamma_{j\sigma(i)}^{w_j}\right\}, \bigcup_{i=1}^{p\prime}\left\{1-\prod_{j=1}^{m}(1-\delta_{j\sigma(i)})^{w_j}\right\}, \bigcup_{i=1}^{q\prime}\left\{1-\prod_{j=1}^{m}(1-\eta_{j\sigma(i)})^{w_j}\right\}\right), \tag{20}$$

where $l\prime = \max_{i=1,\ldots,m}(l_i)$, $p\prime = \max_{i=1,\ldots,m}(p_i)$, and $q\prime = \max_{i=1,\ldots,m}(q_i)$, then when $k = m+1$, let $l = \max(l\prime, l_{m+1})$, $p = \max(p\prime, p_{m+1})$, and $q = \max(q\prime, q_{m+1})$, by Proposition 2, we have

$$\begin{aligned}
\text{NSVNHFWG}(n_1,n_2,\ldots,n_m,n_{m+1}) &= \left(\bigcup_{i=1}^{l\prime}\left\{\prod_{j=1}^{m}\gamma_{j\sigma(i)}^{w_j}\right\},\bigcup_{i=1}^{p\prime}\left\{1-\prod_{j=1}^{m}\left(1-\delta_{j\sigma(i)}\right)^{w_j}\right\},\bigcup_{i=1}^{q\prime}\left\{1-\prod_{j=1}^{m}\left(1-\eta_{j\sigma(i)}\right)^{w_j}\right\}\right)\\
&\otimes_N\left(\widetilde{t}_{m+1}^{w_{m+1}}, w_{m+1}\widetilde{i}_{m+1}, w_{m+1}\widetilde{f}_{m+1}\right)\\
&= \left(\bigcup_{i=1}^{l\prime}\left\{\prod_{j=1}^{m}\gamma_{j\sigma(i)}^{w_j}\right\},\bigcup_{i=1}^{p\prime}\left\{1-\prod_{j=1}^{m}\left(1-\delta_{j\sigma(i)}\right)^{w_j}\right\},\bigcup_{i=1}^{q\prime}\left\{1-\prod_{j=1}^{m}\left(1-\eta_{j\sigma(i)}\right)^{w_j}\right\}\right)\otimes_N\left(\bigcup_{i=1}^{l_{m+1}}\left\{\gamma_{(m+1)i}^{w_{m+1}}\right\},\right.\\
&\left.\bigcup_{i=1}^{p_{m+1}}\left\{1-\left(1-\delta_{(m+1)i}\right)^{w_{m+1}}\right\},\bigcup_{i=1}^{q_{m+1}}\left\{1-\left(1-\eta_{(m+1)i}\right)^{w_{m+1}}\right\}\right)\\
&= \left(\bigcup_{i=1}^{l}\left\{\prod_{j=1}^{m+1}\gamma_{j\sigma(i)}^{w_j}\right\},\bigcup_{i=1}^{p}\left\{1-\prod_{j=1}^{m}\left(1-\delta_{j\sigma(i)}\right)^{w_j}+1-\left(1-\delta_{(m+1)\sigma(i)}\right)^{w_{m+1}}\right.\right.\\
&\left.-\left(1-\prod_{j=1}^{m}\left(1-\delta_{j\sigma(i)}\right)^{w_j}\right)\left(1-\left(1-\delta_{(m+1)\sigma(i)}\right)^{w_{m+1}}\right)\right\},\\
&\bigcup_{i=1}^{q}\left\{1-\prod_{j=1}^{m}\left(1-\eta_{j\sigma(i)}\right)^{w_j}+1-\left(1-\eta_{(m+1)\sigma(i)}\right)^{w_{m+1}}-\left(1-\prod_{j=1}^{m}\left(1-\eta_{j\sigma(i)}\right)^{w_j}\right)\right.\\
&\left.\left.\left(1-\left(1-\eta_{(m+1)\sigma(i)}\right)^{w_{m+1}}\right)\right\}\right),\\
&= \left(\bigcup_{i=1}^{l}\left\{\prod_{j=1}^{m+1}\gamma_{j\sigma(i)}^{w_j}\right\},\bigcup_{i=1}^{p}\left\{1-\prod_{j=1}^{m+1}\left(1-\delta_{j\sigma(i)}\right)^{w_j}\right\},\bigcup_{i=1}^{q}\left\{1-\prod_{j=1}^{m+1}\left(1-\eta_{j\sigma(i)}\right)^{w_j}\right\}\right).
\end{aligned} \tag{21}$$

That is to say, (17) holds for $k=m+1$.

Therefore, (17) holds for all $k\in\mathbb{N}$, which completes the proof. □

Theorem 2. *Let $n_j\,(j=1,2,\ldots,k)$ be a collection of SVNHFEs; then, for the proposed aggregation operator NSVNHFWG, the following properties always hold:*

(1) Idempotency: if all $n_j\,(j=1,2,\ldots,k)$ are equal, i.e., $n_j=n\,(j=1,2,\ldots,k)$; then,

$$\text{NSVNHFWG}(n_1,n_2,\ldots,n_k)=n. \tag{22}$$

(2) Boundary: if there is a pair of SVNHFEs n_c and n_d such that $n_c\prec_s n_j\,(j=1,2,\ldots,k,\, j\neq c)$ and $n_j\prec_s n_d\,(j=1,2,\ldots,k,\, j\neq d)$, then

$$n_c\prec_s \text{NSVNHFWG}(n_1,n_2,\ldots,n_k)\prec_s n_d. \tag{23}$$

(3) Monotonicity: if there are a collection of SVNHFEs $n_j^*\,(j=1,2,\ldots,k)$ such that $n_j\prec_s n_j^*$, then

$$\text{NSVNHFWG}(n_1,n_2,\ldots,n_k)\prec_s \text{NSVNHFWG}\cdot(n_1^*,n_2^*,\ldots,n_k^*). \tag{24}$$

Proof.

(1) It is not difficult to achieve the above results from Definition 9, herein we omit it.

(2) Suppose $n_c=(\widetilde{t}_c,\widetilde{i}_c,\widetilde{f}_c)$ and $n_d=(\widetilde{t}_d,\widetilde{i}_d,\widetilde{f}_d)$. Since $n_c\prec_s n_j=(\widetilde{t}_j,\widetilde{i}_j,\widetilde{f}_j)\prec_s n_d$ for any $j\neq c,d$, we have $\widetilde{t}_c\prec_s\widetilde{t}_j\prec_s\widetilde{t}_d$, $\widetilde{i}_d\prec_s\widetilde{i}_j\prec_s\widetilde{i}_c$, and $\widetilde{f}_d\prec_s\widetilde{f}_j\prec_s\widetilde{f}_c$; furthermore, $\widetilde{t}_c\prec_s\otimes_{j=1N}^{k}\widetilde{t}_j^{w_j}\prec_s\widetilde{t}_d$, $\widetilde{i}_d\prec_s\otimes_{j=1N}^{k}\widetilde{i}_j^{w_j}\prec_s\widetilde{i}_c$, and $\widetilde{f}_d\prec_s\otimes_{j=1}N^{k}\widetilde{f}_j^{w_j}\prec_s\widetilde{f}_c$, where w_j is the weight of n_j.
Hence, $n_c\prec_s(\otimes_{j=1N}^{k}\widetilde{t}_j^{w_j},\oplus_{j=1N}^{k}w_j\widetilde{i}_j,\oplus_{j=1N}^{k}w_j\widetilde{f}_j)\prec_s n_d$.
On the contrary, $\text{NSVNHFWG}(n_1,n_2,\ldots,n_k)=(\otimes_{j=1N}^{k}\widetilde{t}_j^{w_j},\oplus_{j=}1N^{k}w_j\widetilde{i}_j,\oplus_{j=1N}^{k}w_j\widetilde{f}_j)$.
Therefore, $n_c\prec_s\text{NSVNHFWG}(n_1,n_2,\ldots,n_k)\prec_s n_d$.

(3) Suppose $n_j=(\widetilde{t}_j,\widetilde{i}_j,\widetilde{f}_j)$ and $n_j^*=(\widetilde{t}_j^*,\widetilde{i}_j^*,\widetilde{f}_j^*)$ $(j=1,2,\ldots,k)$. $n_j\prec_s n_j^*$ implies $\widetilde{t}_j\prec_s\widetilde{t}_j^*$, $\widetilde{i}_j^*\prec_s\widetilde{i}_j$, and $\widetilde{f}_j^*\prec_s\widetilde{f}_j$; hence, $(\otimes_{j=1N}^{k}\widetilde{t}_j^{w_j},\oplus_{j=1N}^{k}w_j\widetilde{i}_j,\oplus_{j=1N}^{k}w_j\widetilde{f}_j)\prec_s(\otimes_{j=1N}^{k}(\widetilde{t}_j^*)^{w_j},\oplus_{j=1N}^{k}w_j\widetilde{i}_j^*,\oplus_{j=1N}^{k}w_j\widetilde{f}_j^*)$. Therefore, $\text{NSVNHFWG}(n_1,n_2,\ldots,n_k)\prec_s\text{NSVNHFWG}(n_1^*,n_2^*,\ldots,n_k^*)$. □

Definition 13. Let $n_j=(\widetilde{t}_j,\widetilde{i}_j,\widetilde{f}_j)\in\Omega\,(j=1,2,\ldots,k)$ be a collection of SVNHFEs, and a normalized single-valued neutrosophic hesitant fuzzy ordered weighted geometric operator NSVNHFOWG is defined as

$$\text{NSVNHFOWG}(n_1,n_2,\ldots,n_k)=\otimes_{j=1N}^{k}n_{\sigma(j)}^{\omega_j}=\left(\otimes_{j=1N}^{k}\widetilde{t}_{\sigma(j)}^{\omega_j},\oplus_{j=1N}^{k}\omega_j\widetilde{i}_{\sigma(j)},\oplus_{j=1N}^{k}\omega_j\widetilde{f}_{\sigma(j)}\right), \tag{25}$$

where $n_{\sigma(j)} = (\tilde{t}_{\sigma(j)}, \tilde{i}_{\sigma(j)}, \tilde{f}_{\sigma(j)})$ is the jth largest element of $n_j\,(j = 1, 2, \ldots, k)$ and $\omega = (\omega_1, \omega_2, \ldots, \omega_k)^T$ is the aggregation-associated vector such that $\omega_j \in [0, 1]\,(j = 1, 2, \ldots, k)$ and $\sum_{j=1}^{k} \omega_j = 1$.

Especially, if $\omega = ((1/k), (1/k), \ldots, (1/k))^T$, then by the commutativity of $\otimes_N$, the operator NSVNHFOWG is also reduced to operator NSVNHFG. Similar to Theorem 1, we can give the following result.

Theorem 3. *Let $n_j = (\tilde{t}_j, \tilde{i}_j, \tilde{f}_j) = (\{\gamma_{ji} | i = 1, 2, \ldots, l_j\}, \{\delta_{ji} | i = 1, 2, \ldots, p_j\}, \{\eta_{ji} | i = 1, 2, \ldots, q_j\})\,(j = 1, 2, \ldots, k)$ be a collection of SVNHFEs; if the aggregation-associated vector is $\omega = (\omega_1, \omega_2, \ldots, \omega_n)^T$ and $l = \max_{j=1,\ldots,k}(l_j)$, $p = \max_{j=1,\ldots,k}(p_j)$, and $q = \max_{j=1,\ldots,k}(q_j)$, then the aggregated result by operator NSVNHFOWG can be expressed as*

$$\mathrm{NSVNHFOWG}(n_1, n_2, \ldots, n_k) = \left(\bigcup_{i=1}^{l} \left\{ \prod_{j=1}^{k} \gamma_{\sigma(j)\sigma(i)}^{\omega_j} \right\}, \bigcup_{i=1}^{p} \left\{ 1 - \prod_{j=1}^{k} \left(1 - \delta_{\sigma(j)\sigma(i)}\right)^{\omega_j} \right\}, \bigcup_{i=1}^{q} \left\{ 1 - \prod_{j=1}^{k} \left(1 - \eta_{\sigma(j)\sigma(i)}\right)^{\omega_j} \right\} \right), \tag{26}$$

where $\gamma_{\sigma(j)\sigma(i)}$, $\delta_{\sigma(j)\sigma(i)}$, and $\eta_{\sigma(j)\sigma(i)}$ are the ith largest element of $\tilde{t}_{\sigma(j)}, \tilde{i}_{\sigma(j)}$, and $\tilde{f}_{\sigma(j)}$, respectively, and $n_{\sigma(j)} = (\tilde{t}_{\sigma(j)}, \tilde{i}_{\sigma(j)}, \tilde{f}_{\sigma(j)})\,(j = 1, 2, \ldots, k)$ is a permutation of $n_j = (\tilde{t}_j, \tilde{i}_j, \tilde{f}_j)\,(j = 1, 2, \ldots, k)$ such that $n_{\sigma(j-1)} \succ n_{\sigma(j)}$.

Proof. It can be proved similar to Theorem 1, herein we omit it. □

Example 5. For three SVNHFEs $n_1 = (\{0.1, 0.4\}, \{0.3\}, \{0.2, 0.6\})$, $n_2 = (\{0.6\}, \{0.5\}, \{0.3, 0.5\})$, and $n_3 = (\{0.2, 0.3\}, \{0.6\}, \{0.7\})$ with aggregation-associated vector $\omega = (0.3, 0.2, 0.5)^T$, then $s(n_1) = 0.5167$, $s(n_2) = 0.5667$, and $s(n_3) = 0.3167$, which means $n_2 \succ n_1 \succ n_3$, i.e., $n_{\sigma(1)} = n_2$, $n_{\sigma(2)} = n_1$, and $n_{\sigma(3)} = n_3$; thus,

$$\begin{aligned}
&\mathrm{NSVNHFOWG}(n_1, n_2, n_3) \\
&= \left(\bigcup_{i=1}^{2} \left\{ \prod_{j=1}^{3} \gamma_{\sigma(j)\sigma(i)}^{\omega_j} \right\}, \left\{ 1 - \prod_{j=1}^{3} \left(1 - \delta_{\sigma(j)\sigma(1)}\right)^{\omega_j} \right\}, \bigcup_{i=1}^{2} \left\{ 1 - \prod_{j=1}^{3} \left(1 - \eta_{\sigma(j)\sigma(i)}\right)^{\omega_j} \right\} \right) \\
&= \left(\left\{ 0.6^{0.3} 0.4^{0.2} 0.3^{0.5}, 0.6^{0.3} 0.1^{0.2} 0.2^{0.5} \right\}, \left\{ 1 - (1 - 0.5)^{0.3} (1 - 0.3)^{0.2} (1 - 0.6)^{0.5} \right\}, \right. \\
&\left. \left\{ 1 - (1 - 0.5)^{0.3} (1 - 0.6)^{0.2} (1 - 0.7)^{0.5}, 1 - (1 - 0.3)^{0.3} (1 - 0.2)^{0.2} (1 - 0.7)^{0.5} \right\} \right) \\
&= (\{0.3912, 0.2421\}, \{0.5217\}, \{0.6296, 0.5293\}).
\end{aligned} \tag{27}$$

Theorem 4. *Let $n_j\,(j = 1, 2, \ldots, k)$ be a collection of SVNHFEs; then, for the aggregation operator NSVNHFOWG with aggregation-associated weight vector $\omega = (\omega_1, \omega_2, \ldots, \omega_n)^T$, and the following properties always hold.*

(1) Idempotency: if all $n_j\,(j = 1, 2, \ldots, k)$ are equal, i.e., $n_j = n\,(j = 1, 2, \ldots, k)$, then

$$\mathrm{NSVNHFOWG}(n_1, n_2, \ldots, n_k) = n. \tag{28}$$

(2) Boundary: if there are a pair of SVNHFEs n_c and n_d such that $n_c \prec_s n_j\,(j = 1, 2, \ldots, k, j \neq c)$ and $n_j \prec_s n_d\,(j = 1, 2, \ldots, k, j \neq d)$, then

$$n_c \prec_s \mathrm{NSVNHFOWG}(n_1, n_2, \ldots, n_k) \prec_s n_d. \tag{29}$$

(3) Monotonicity: if there are a collection of SVNHFEs $n_j^*\,(j = 1, 2, \ldots, k)$ such that $n_j \prec_s n_j^*$, then

$$\mathrm{NSVNHFOWG}(n_1, n_2, \ldots, n_k) \prec_s \mathrm{NSVNHFOWG}(n_1^*, n_2^*, \ldots, n_k^*). \tag{30}$$

Proof. It can be proved similar to Theorem 2, herein we omit it.

In what follows, we develop a sort of hybrid aggregation operator which weights the given arguments as well as their ordered positions simultaneously. □

Definition 14. Let $n_j \in \Omega\,(j = 1, 2, \ldots, k)$ be a collection of SVNHFEs; then, a normalized single-valued neutrosophic hesitant fuzzy hybrid weighted geometric operator NSVNHFHWG which has an aggregation-associated vector $\omega = (\omega_1, \omega_2, \ldots, \omega_k)^T$ with $\omega_j \in [0, 1]$, $\sum_{j=1}^{k} \omega_j = 1$ is defined as

$$\text{NSVNHFHWG}(n_1, n_2, \ldots, n_k) = \otimes_{j=1N}^{k} \dot{n}_{\sigma(j)}^{\omega_j} = \left(\otimes_{j=1N}^{k} \dot{\tilde{t}}_{\sigma(j)}^{\omega_j}, \oplus_{j=1N}^{k} \omega_j \dot{\tilde{i}}_{\sigma(j)}, \oplus_{j=1N}^{k} \omega_j \dot{\tilde{f}}_{\sigma(j)} \right), \tag{31}$$

where $\dot{n}_{\sigma(j)}$ is the jth largest element of $\dot{n}_j = n_j^{kw_j}$ $(j = 1, 2, \ldots, k)$, $w = (w_1, w_2, \ldots, w_k)^T$ is the weight vector of $n_j (j = 1, 2, \ldots, k)$ such that $w_j \in [0, 1]$, $\sum_{j=1}^{k} w_j = 1$, and k is the balancing coefficient.

Especially, if $\omega = (\omega_1, \omega_2, \ldots, \omega_k)^T = ((1/k), (1/k), \ldots, (1/k))^T$, then the aggregation operator NSVNHFHWG can be reduced to operator NSVNHFWG. On the contrary, if $w = (w_1, w_2, \ldots, w_k)^T = ((1/k), (1/k), \ldots, (1/k))^T$, then the aggregation operator NSVNHFHWG can be reduced to operator NSVNHFOWG. Furthermore, if $(w_1, w_2, \ldots, w_k)^T = (\omega_1, \omega_2, \ldots, \omega_k)^T = ((1/k), (1/k), \ldots, (1/k))^T$, then the aggregation operator NSVNHFHWG can be reduced to operator NSVNHFG. Put it another way, the hybrid operator NSVNHFHG is a generalization of aggregation operators NSVNHFG, NSVNHFWG, and NSVNHFOWG.

Theorem 5. *Let* $n_j = (\tilde{t}_j, \tilde{i}_j, \tilde{f}_j) = (\{\gamma_{ji} | i = 1, 2, \ldots, l_j\}, \{\delta_{ji} | i = 1, 2, \ldots, p_j\}, \{\eta_{ji} | i = 1, 2, \ldots, q_j\}) (j = 1, 2, \ldots, k)$ *be a collection of SVNHFEs with weight vector* $w = (w_1, w_2, \ldots, w_n)^T$ *and* $l = \max_{j=1,\ldots,k} (l_j)$, $p = \max_{j=1,\ldots,k} (p_j)$, *and* $q = \max_{j=1,\ldots,k} (q_j)$; *then, the aggregated result by using operator NSVNHFHWG with aggregation-associated vector* $\omega = (\omega_1, \omega_2, \ldots, \omega_n)^T$ *can be expressed as*

$$\text{NSVNHFHWG}(n_1, n_2, \ldots, n_k) = \left(\bigcup_{i=1}^{l} \left\{ \prod_{j=1}^{k} \dot{\gamma}_{\sigma(j)\sigma(i)}^{\omega_j} \right\}, \bigcup_{i=1}^{p} \left\{ 1 - \prod_{j=1}^{k} \left(1 - \dot{\delta}_{\sigma(j)\sigma(i)}\right)^{\omega_j} \right\}, \bigcup_{i=1}^{q} \left\{ 1 - \prod_{j=1}^{k} \left(1 - \dot{\eta}_{\sigma(j)\sigma(i)}\right)^{\omega_j} \right\} \right), \tag{32}$$

where $\dot{\gamma}_{\sigma(j)\sigma(i)}$, $\dot{\delta}_{\sigma(j)\sigma(i)}$, *and* $\dot{\eta}_{\sigma(j)\sigma(i)}$ *are the ith largest element of* $\tilde{t}_{\sigma(j)}$, $\tilde{i}_{\sigma(j)}$, *and* $\tilde{f}_{\sigma(j)}$, *respectively, and* $\dot{n}_{\sigma(j)} = (\tilde{t}_{\sigma(j)}, \tilde{i}_{\sigma(j)}, \tilde{f}_{\sigma(j)}) (j = 1, 2, \ldots, k)$ *is a permutation of* $\dot{n}_j = n_j^{kw_j} = (\tilde{t}_j^{kw_j}, kw_j \tilde{i}_j, kw_j \tilde{f}_j) (j = 1, 2, \ldots, k)$ *such that* $\dot{n}_{\sigma(j-1)} \succ \dot{n}_{\sigma(j)}$.

Proof. It can be proved similar to Theorem 1, herein we omit it. □

Example 6. Suppose SVNHFEs $n_1 = (\{0.1, 0.4\}, \{0.3\}, \{0.2, 0.6\})$, $n_2 = (\{0.6\}, \{0.5\}, \{0.3, 0.5\})$, and $n_3 = (\{0.2, 0.3\}, \{0.6\}, \{0.7\})$ with the weight vector $w = (0.1, 0.3, 0.6)^T$; then, their aggregated result by operator NSVNHFHG with aggregation-associated vector $\omega = (0.7, 0.2, 0.1)^T$ can be calculated through the following process:

(1) $\dot{n}_1 = n_1^{3\times 0.1} = (\{0.1, 0.4\}, \{0.3\}, \{0.2, 0.6\})^{0.3} = (\{0.5012, 0.7597\}, \{0.1015\}, \{0.0648, 0.2403\})$, $\dot{n}_2 = n_2^{3\times 0.3} = (\{0.6\}, \{0.5\}, \{0.3, 0.5\})^{0.9} = (\{0.6314\}, \{0.4641\}, \{0.2746, 0.4641\})$, $\dot{n}_3 = n_3^{3\times 0.6} = (\{0.2, 0.3\}, \{0.6\}, \{0.7\})^{1.8} = (\{0.0552, 0.1145\}, \{0.8078\}, \{0.8855\})$

(2) $s(\dot{n}_1) = 0.7921$, $s(\dot{n}_2) = 0.5993$, $s(\dot{n}_3) = 0.1305$

(3) $\dot{n}_{\sigma(1)} = \dot{n}_1, \dot{n}_{\sigma(2)} = \dot{n}_2, \dot{n}_{\sigma(3)} = \dot{n}_3$

(4) $\text{NSVNHFHWG}(n_1, n_2, n_3) = (\cup_{i=1}^{2} \{\prod_{j=1}^{3} \dot{\gamma}_{\sigma(j)\sigma(i)}^{\omega_j}\}, \{1 - \prod_{j=1}^{3} (1 - \dot{\delta}_{\sigma(j)\sigma(1)})^{\omega_j}\}, \cup_{i=1}^{2} \{1 - \prod_{j=1}^{3} (1 - \dot{\eta}_{\sigma(j)\sigma(i)})^{\omega_j}\}) = (\{0.7597^{0.7} 0.6314^{0.2} 0.1145^{0.1}, 0.5012^{0.7} 0.6314^{0.2} 0.0552^{0.1}\}, \{1 - (1 - 0.1015)^{0.7} (1 - 0.4641)^{0.2} (1 - 0.8078)^{0.1}\}, \{1 - (1 - 0.2403)^{0.7} (1 - 0.4641)^{0.2} (1 - 0.8855)^{0.1}, 1 - (1 - 0.0648)^{0.7} (1 - 0.2746)^{0.2} (1 - 0.8855)^{0.1}\}) = (\{0.6059, 0.4210\}, \{0.3055\}, \{0.4137, 0.2795\})$

Remark 1. From Theorems 2 and 4, we can conclude that the aggregation operator NSVNHFHWG also satisfy idempotency, boundary, and monotonicity.

4. Decision-Making Method Based on Normalized Single-Valued Neutrosophic Hesitant Fuzzy Geometric Aggregation Operators

4.1. Decision-Making Method. Assume there are m alternatives $A_i (i = 1, 2, \ldots, m)$ under consideration for a decision-making problem, and they are estimated in terms of attributes $C_j (j = 1, 2, \ldots, k)$ which possess weight vector $w = (w_1, w_2, \ldots, w_k)^T$ such that $w_j \in [0, 1] (j = 1, 2, \ldots, k)$ and $\sum_{j=1}^{k} w_j = 1$. Some decision makers provide their evaluation values for alternative A_i with respect to attribute C_j which is characterized by a single-valued neutrosophic hesitant fuzzy element $n_{ij} = (\tilde{t}_{ij}, \tilde{i}_{ij}, \tilde{f}_{ij})$, where $\tilde{t}_{ij}, \tilde{i}_{ij}$, and $\tilde{f}_{ij}$ represent the truth degree, uncertain degree, and falsity degree of alternative of A_i satisfying attribute C_j. It should be noticed that if two decision makers give the same evaluation value on one alternative, then the values cannot be merged in n_{ij} since it will affect the score of a hesitant fuzzy element actually. Next, take operator NSVNHFHWG with aggregation-associated vector $\omega = (\omega_1, \omega_2, \ldots, \omega_k)^T$ as an example, and we demonstrate the following decision-making process in detail:

Step 1. Collect evaluation values from all decision makers and construct SVNHFEs information matrix $\tilde{N} = (n_{ij})_{m\times k}$.

Step 2. For any $i = 1, 2, \ldots, m$, calculate $\dot{n}_{ij} = n_{ij}^{kw_j}$ $(j = 1, 2, \ldots, k)$.

Step 3. For any $i = 1, 2, \ldots, m$, rank $\dot{n}_{ij}\,(j = 1, 2, \ldots, k)$ according to the score, amplitude, or variance of $\dot{n}_{ij}\,(j = 1, 2, \ldots, k)$ and obtain $\dot{n}_{i\sigma(j)} = (\dot{\tilde{t}}_{i\sigma(j)}, \dot{\tilde{i}}_{i\sigma(j)}, \dot{\tilde{f}}_{i\sigma(j)}) = (\{\dot{\gamma}_{i\sigma(j)}\}, \{\dot{\delta}_{i\sigma(j)}\}, \{\dot{\eta}_{i\sigma(j)}\})$.

Step 4. For any $i = 1, 2, \ldots, m$, aggregate the collection $n_{ij}\,(j = 1, 2, \ldots, k)$ by aggregation operator NSVNHFHWG and denote the result as n_i, that is,

$$n_i = \mathrm{NSVNHFHWG}(n_{i1}, n_{i2}, \ldots, n_{ik}) = \left(\bigcup_{t=1}^{l} \left\{ \prod_{j=1}^{k} \dot{\gamma}_{i\sigma(j)\sigma(t)}^{\omega_j} \right\}, \bigcup_{t=1}^{p} \left\{ 1 - \prod_{j=1}^{k} \left(1 - \dot{\delta}_{i\sigma(j)\sigma(t)}\right)^{\omega_j} \right\}, \bigcup_{t=1}^{q} \left\{ 1 - \prod_{j=1}^{k} \left(1 - \dot{\eta}_{i\sigma(j)\sigma(t)}\right)^{\omega_j} \right\} \right), \tag{33}$$

where $l = \max(\dot{\gamma}_{i\sigma(j)})$, $p = \max(\dot{\delta}_{i\sigma(j)})$, $q = \max(\dot{\eta}_{i\sigma(j)})$, $\dot{\gamma}_{i\sigma(j)\sigma(t)}$ is the tth largest element of $\dot{\tilde{t}}_{i\sigma(j)}$, $\dot{\delta}_{i\sigma(j)\sigma(t)}$ is the tth largest element of $\dot{\tilde{i}}_{i\sigma(j)}$, and $\dot{\eta}_{i\sigma(j)\sigma(t)}$ is the tth largest element of $\dot{\tilde{f}}_{i\sigma(j)}$.

Step 5. Rank $n_i\,(i = 1, 2, \ldots, m)$ based on the score, amplitude, and variance of $n_i\,(i = 1, 2, \ldots, m)$.

Step 6. Rank the alternatives $A_i\,(i = 1, 2, \ldots, m)$ based on the order of $n_i\,(i = 1, 2, \ldots, m)$ and obtain optimal alternative.

4.2. Numerical Example and Analysis. An example from [27] is utilized to illustrate the applicability and validity of the proposed MADM method. An example from [27] is utilized to illustrate the applicability and validity of the proposed MADM method. Now, there are four alternatives $A_i\,(i = 1, 2, 3, 4)$ which were considered with respect to twelve attributes: C_1: functionality, C_2: reliability, C_3:usability, C_4: efficiency, C_5: maintainability, C_6: portability, C_7: acquisition, C_8: customization, C_9: training, C_{10}: operation, C_{11}: maintenance, and C_{12}: standards, which possess weight vector $w = (0.1, 0.12, 0.2, 0.05, 0.06, 0.04, 0.08, 0.05, 0.1, 0.08, 0.02, 0.1)^T$. Some decision makers estimate these alternatives and provide their evaluation information adequately that is listed in Table 1. Meanwhile, the weighted vector $\omega = (0.08, 0.12, 0.1, 0.06, 0.02, 0.06, 0.06, 0.04, 0.06, 0.1, 0.2, 0.1)^T$ of the operator NSVNHFHWG is given. We perform the following steps.

Step 1. Collect evaluation values from all decision makers and construct SVNHFEs information matrix $\tilde{N} = (n_{ij})_{4\times 12}$ (see Table 1).

Step 2. Utilize weight vector $w = (0.1, 0.12, 0.2, 0.05, 0.06, 0.04, 0.08, 0.05, 0.1, 0.08, 0.02, 0.1)^T$ to obtain $\dot{n}_{ij} = n_{ij}^{12w_j}$. Take $\dot{n}_{21}$ as an example:

$$\begin{aligned} \dot{n}_{21} &= (\{0.5\}, \{0.1\}, \{0.4\})^{12\times 0.1} \\ &= \left(\{0.5\}^{1.2}, 1.2\{0.1\}, 1.2\{0.4\}\right) \\ &= \left(\{0.5^{1.2}\}, \{1-(1-0.1)^{1.2}\}, \{1-(1-0.4)^{1.2}\}\right) \\ &= (\{0.4353\}, \{0.1188\}, \{0.4583\}). \end{aligned} \tag{34}$$

Further details are shown in Table 2.

Step 3. We utilize function to figure out the order relationship of $\dot{n}_{ij}\,(j = 1, 2, \ldots, 12)$. Take the alternative A_2 as an example: according to the score, amplitude, or variance of $\dot{n}_{2j}$, we obtain

$$\begin{aligned} s(\dot{n}_{21}) &= 0.6194, \\ s(\dot{n}_{22}) &= 0.6697, \\ s(\dot{n}_{23}) &= 0.6412, \\ s(\dot{n}_{24}) &= 0.5175, \\ s(\dot{n}_{25}) &= 0.5854, \\ s(\dot{n}_{26}) &= 0.5647, \\ s(\dot{n}_{27}) &= 0.5625, \\ s(\dot{n}_{28}) &= 0.6179, \\ s(\dot{n}_{29}) &= 0.7203, \\ s(\dot{n}_{210}) &= 0.5617, \\ s(\dot{n}_{211}) &= 0.4223, \\ s(\dot{n}_{212}) &= 0.7779. \end{aligned} \tag{35}$$

We rank the order as $\dot{n}_{212} \succ \dot{n}_{29} \succ \dot{n}_{22} \succ \dot{n}_{23} \succ \dot{n}_{21} \succ \dot{n}_{28} \succ \dot{n}_{25} \succ \dot{n}_{26} \succ \dot{n}_{27} \succ \dot{n}_{210} \succ \dot{n}_{24} \succ \dot{n}_{211}$. Similarly, we obtain

$$\begin{aligned} &\dot{n}_{13} \succ \dot{n}_{19} \succ \dot{n}_{112} \succ \dot{n}_{12} \succ \dot{n}_{110} \succ \dot{n}_{17} \succ \dot{n}_{11} \succ \dot{n}_{15} \succ \dot{n}_{14} \succ \dot{n}_{16} \succ \dot{n}_{18} \succ \dot{n}_{111}, \\ &\dot{n}_{32} \succ \dot{n}_{39} \succ \dot{n}_{312} \succ \dot{n}_{38} \succ \dot{n}_{33} \succ \dot{n}_{310} \succ \dot{n}_{31} \succ \dot{n}_{34} \succ \dot{n}_{37} \succ \dot{n}_{36} \succ \dot{n}_{35} \succ \dot{n}_{311}, \\ &\dot{n}_{42} \succ \dot{n}_{412} \succ \dot{n}_{43} \succ \dot{n}_{41} \succ \dot{n}_{49} \succ \dot{n}_{45} \succ \dot{n}_{48} \succ \dot{n}_{46} \succ \dot{n}_{410} \succ \dot{n}_{47} \succ \dot{n}_{44} \succ \dot{n}_{411}. \end{aligned} \tag{36}$$

Step 4. Utilize one aggregation operator to aggregate $n_{ij}\,(j = 1, 2, \ldots, 12)$ and obtain n_i, and we take operator NSVNHFHWG as an example:

Table 1: Single-valued neutrosophic hesitant fuzzy information.

	C_1	C_2	C_3	C_4
A_1	$(\{0.4\}, \{0.2\}, \{0.1, 0.3\})$	$(\{0.3, 0.4\}, \{0.2\}, \{0.1, 0.3\})$	$(\{0.6\}, \{0.3\}, \{0.2\})$	$(\{0.5\}, \{0.3\}, \{0.2\})$
A_2	$(\{0.5\}, \{0.1\}, \{0.4\})$	$(\{0.2, 0.4\}, \{0.2, 0.3\}, \{0.1\})$	$(\{0.3\}, \{0.2\}, \{0.4\})$	$(\{0.2\}, \{0.1\}, \{0.4\})$
A_3	$(\{\{0.3, 0.5\}, \{0.2\}, \{0.3\})$	$(\{0.6\}, \{0.3\}, \{0.2\})$	$(\{0.4\}, \{0.1\}, \{0.2\})$	$(\{0.5\}, \{0.2\}, \{0.3\})$
A_4	$(\{0.2\}, \{0.1\}, \{0.1\})$	$(\{0.6\}, \{0.1\}, \{0.3\})$	$(\{0.3, 0.5\}, \{0.2\}, \{0.3\})$	$(\{0.3\}, \{0.1\}, \{0.4\})$
	C_5	C_6	C_7	C_8
A_1	$(\{0.3\}, \{0.2\}, \{0.1, 0.2\})$	$(\{0.3, 0.4\}, \{0.1\}, \{0.3\})$	$(\{0.6\}, \{0.4\}, \{0.3\})$	$(\{0.4\}, \{0.2, 0.3\}, \{0.2\})$
A_2	$(\{0.2, 0.3\}, \{0.1\}, \{0.3\})$	$(\{0.5\}, \{0.2\}, \{0.3\})$	$(\{0.4\}, \{0.2\}, \{0.5\})$	$(\{0.3\}, \{0.1\}, \{0.2\})$
A_3	$(\{0.6\}, \{0.3\}, \{0.4\})$	$(\{0.2, 0.4\}, \{0.2\}, \{0.1\})$	$(\{0.3\}, \{0.1\}, \{0.4\})$	$(\{0.6\}, \{0.2\}, \{0.1\})$
A_4	$(\{0.6\}, \{0.2\}, \{0.4\})$	$(\{0.6\}, \{0.3\}, \{0.2\})$	$(\{0.3\}, \{0.2\}, \{0.4\})$	$(\{0.4\}, \{0.1\}, \{0.3\})$
	C_9	C_{10}	C_{11}	C_{12}
A_1	$(\{0.7\}, \{0.3\}, \{0.4\})$	$(\{0.6\}, \{0.4\}, \{0.2\})$	$(\{0.3\}, \{0.2\}, \{0.1\})$	$(\{0.5\}, \{0.3\}, \{0.2\})$
A_2	$(\{0.6\}, \{0.2\}, \{0.3\})$	$(\{0.2\}, \{0.3\}, \{0.2\})$	$(\{0.3\}, \{0.2\}, \{0.4\})$	$(\{0.6\}, \{0.2\}, \{0.1\})$
A_3	$(\{0.4\}, \{0.1\}, \{0.2\})$	$(\{0.5\}, \{0.2\}, \{0.3\})$	$(\{0.3\}, \{0.2\}, \{0.1\})$	$(\{0.4\}, \{0.1\}, \{0.2\})$
A_4	$(\{0.5\}, \{0.2\}, \{0.3, 0.4\})$	$(\{0.4\}, \{0.1, 0.3\}, \{0.5\})$	$(\{0.5\}, \{0.3\}, \{0.4\})$	$(\{0.4\}, \{0.1\}, \{0.2\})$

Table 2: Weighted single-valued neutrosophic hesitant fuzzy information.

	C_1	C_2
A_1	$(\{0.3330\}, \{0.2349\}, \{0.1188, 0.3482\})$	$(\{0.1766, 0.2673\}, \{0.2748\}, \{0.1408, 0.4017\})$
A_2	$(\{0.4353\}, \{0.1188\}, \{0.4583\})$	$(\{0.0985, 0.2673\}, \{0.2748, 0.4017\}, \{0.1408\})$
A_3	$(\{0.2358, 0.4353\}, \{0.2349\}, \{0.3482\})$	$(\{0.4792\}, \{0.4017\}, \{0.2748\})$
A_4	$(\{0.1450\}, \{0.1188\}, \{0.1188\})$	$(\{0.4792\}, \{0.1408\}, \{0.4017\})$
	C_3	C_4
A_1	$(\{0.2935\}, \{0.5752\}, \{0.4146\})$	$(\{0.6598\}, \{0.1927\}, \{0.1253\})$
A_2	$(\{0.0556\}, \{0.4146\}, \{0.7065\})$	$(\{0.3807\}, \{0.0613\}, \{0.2640\})$
A_3	$(\{0.1109\}, \{0.2234\}, \{0.4146\})$	$(\{0.6598\}, \{0.1253\}, \{0.1927\})$
A_4	$(\{0.0556, 0.1895\}, \{0.4146\}, \{0.5752\})$	$(\{0.4856\}, \{0.0613\}, \{0.2640\})$
	C_5	C_6
A_1	$(\{0.4203\}, \{0.1484\}, \{0.0731, 0.1484\})$	$(\{0.5611, 0.6442\}, \{0.0493\}, \{0.1574\})$
A_2	$(\{0.3139, 0.4203\}, \{0.0731\}, \{0.2265\})$	$(\{0.7170\}, \{0.1016\}, \{0.1574\})$
A_3	$(\{0.6923\}, \{0.2265\}, \{0.3077\})$	$(\{0.4618, 0.6442\}, \{0.1016\}, \{0.0493\})$
A_4	$(\{0.6923\}, \{0.1484\}, \{0.3077\})$	$(\{0.7826\}, \{0.1574\}, \{0.1016\})$
	C_7	C_8
A_1	$(\{0.6124\}, \{0.3876\}, \{0.2899\})$	$(\{0.5771\}, \{0.1253, 0.1927\}, \{0.1253\})$
A_2	$(\{0.4149\}, \{0.1928\}, \{0.4859\})$	$(\{0.4856\}, \{0.0613\}, \{0.1253\})$
A_3	$(\{0.3148\}, \{0.0962\}, \{0.3876\})$	$(\{0.7360\}, \{0.1253\}, \{0.0613\})$
A_4	$(\{0.3148\}, \{0.1928\}, \{0.3876\})$	$(\{0.5771\}, \{0.0613\}, \{0.1927\})$
	C_9	C_{10}
A_1	$(\{0.6518\}, \{0.3482\}, \{0.4583\})$	$(\{0.6124\}, \{0.3876\}, \{0.1928\})$
A_2	$(\{0.5417\}, \{0.2349\}, \{0.3482\})$	$(\{0.2133\}, \{0.2899\}, \{0.1928\})$
A_3	$(\{0.3330\}, \{0.1188\}, \{0.2349\})$	$(\{0.5141\}, \{0.1928\}, \{0.2899\})$
A_4	$(\{0.4353\}, \{0.2349\}, \{0.3482, 0.4583\})$	$(\{0.4149\}, \{0.0962, 0.2899\}, \{0.4859\})$
	C_{11}	C_{12}
A_1	$(\{0.7490\}, \{0.0521\}, \{0.0250\})$	$(\{0.4353\}, \{0.3482\}, \{0.2349\})$
A_2	$(\{0.7490\}, \{0.0521\}, \{0.1154\})$	$(\{0.5417\}, \{0.2349\}, \{0.1188\})$
A_3	$(\{0.7490\}, \{0.0521\}, \{0.0250\})$	$(\{0.3330\}, \{0.1188\}, \{0.2349\})$
A_4	$(\{0.8467\}, \{0.0820\}, \{0.1154\})$	$(\{0.3330\}, \{0.1188\}, \{0.2349\})$

$$
\begin{aligned}
n_2 &= \mathrm{NSVNHFHWG}\left(n_{21}, n_{22}, n_{23}, n_{24}, n_{25}, n_{26}, n_{27}, n_{28}, n_{29}, n_{210}, n_{211}, n_{212}\right) \\
&= \left(\bigcup_{t=1}^{2} \left\{ \prod_{j=1}^{12} \dot{\gamma}_{2\sigma(j)\sigma(t)}^{\omega_j} \right\}, \bigcup_{t=1}^{2} \left\{ 1 - \prod_{j=1}^{12} \left(1 - \dot{\delta}_{2\sigma(j)\sigma(t)}\right)^{\omega_j} \right\}, \bigcup_{t=1}^{2} \left\{ 1 - \prod_{j=1}^{12} \left(1 - \dot{\eta}_{2\sigma(j)\sigma(t)}\right)^{\omega_j} \right\} \right) \\
&= \left(\left\{ 0.5417^{0.08} 0.5417^{0.12} 0.2673^{0.1} 0.0556^{0.06} 0.4353^{0.02} 0.4856^{0.06} 0.4203^{0.06} 0.7170^{0.04} 0.4149^{0.06} 0.2133^{0.1} \right.\right. \\
&\quad 0.3807^{0.2} 0.7490^{0.1}, 0.5417^{0.08} 0.5417^{0.12} 0.0985^{0.1} 0.0556^{0.06} 0.4353^{0.02} 0.4856^{0.06} 0.4203^{0.06} 0.7170^{0.04} \\
&\quad \left. 0.4149^{0.06} 0.2133^{0.1} 0.3807^{0.2} 0.7490^{0.1} \right\}, \left\{ 1 - (1 - 0.1450)^{0.08} (1 - 0.1450)^{0.12} (1 - 0.1766)^{0.1} \right. \\
&\quad (1 - 0.0210)^{0.06} (1 - 0.1188)^{0.02} (1 - 0.2512)^{0.06} (1 - 0.1905)^{0.06} (1 - 0.4618)^{0.04} (1 - 0.2133)^{0.06} \\
&\quad (1 - 0.3148)^{0.1} (1 - 0.2512)^{0.2} (1 - 0.6796)^{0.1}, 1 - (1 - 0.1450)^{0.08} (1 - 0.1450)^{0.12} (1 - 0.0985)^{0.1} \\
&\quad (1 - 0.0210)^{0.06} (1 - 0.1188)^{0.02} (1 - 0.2512)^{0.06} (1 - 0.1905)^{0.06} (1 - 0.4618)^{0.04} (1 - 0.2133)^{0.06} \\
&\quad \left. (1 - 0.3148)^{0.1} (1 - 0.2512)^{0.2} (1 - 0.6796)^{0.1} \right\}, \left\{ 1 - (1 - 0.0631)^{0.08} (1 - 0.2358)^{0.12} (1 - 0.0363)^{0.1} \right. \\
&\quad (1 - 0.1109)^{0.06} (1 - 0.4583)^{0.02} (1 - 0.3807)^{0.06} (1 - 0.4203)^{0.06} (1 - 0.5611)^{0.04} (1 - 0.5141)^{0.06} \\
&\quad \left.\left. (1 - 0.2133)^{0.1} (1 - 0.5771)^{0.2} (1 - 0.8026)^{0.1} \right\} \right) \\
&= (\{0.3329, 0.3744\}, \{0.2769, 0.2834\}, \{0.4268\}).
\end{aligned} \tag{37}
$$

Similarly, we can get n_1, n_3, and n_4 as follows:

$$
\begin{aligned}
n_1 &= \left(\bigcup_{t=1}^{2} \left\{ \prod_{j=1}^{12} \dot{\gamma}_{1\sigma(j)\sigma(t)}^{\omega_j} \right\}, \bigcup_{t=1}^{2} \left\{ 1 - \prod_{j=1}^{12} \left(1 - \dot{\delta}_{1\sigma(j)\sigma(t)}\right)^{\omega_j} \right\}, \bigcup_{t=1}^{2} \left\{ 1 - \prod_{j=1}^{12} \left(1 - \dot{\eta}_{1\sigma(j)\sigma(t)}\right)^{\omega_j} \right\} \right) \\
&= (\{0.4975, 0.5171\}, \{0.3519, 0.3755\}, \{0.3326, 0.3550\}), \\
n_3 &= \left(\bigcup_{t=1}^{2} \left\{ \prod_{j=1}^{12} \dot{\gamma}_{1\sigma(j)\sigma(t)}^{\omega_j} \right\}, \left\{ 1 - \prod_{j=1}^{12} \left(1 - \dot{\delta}_{1\sigma(j)\sigma(1)}\right)^{\omega_j} \right\}, \left\{ 1 - \prod_{j=1}^{12} \left(1 - \dot{\eta}_{1\sigma(j)\sigma(1)}\right)^{\omega_j} \right\} \right) \\
&= (\{0.4682, 0.5022\}, \{0.3253\}, \{0.3492\}), \\
n_4 &= \left(\bigcup_{t=1}^{2} \left\{ \prod_{j=1}^{12} \dot{\gamma}_{1\sigma(j)\sigma(t)}^{\omega_j} \right\}, \bigcup_{t=1}^{2} \left\{ 1 - \prod_{j=1}^{12} \left(1 - \dot{\delta}_{1\sigma(j)\sigma(t)}\right)^{\omega_j} \right\}, \bigcup_{t=1}^{2} \left\{ 1 - \prod_{j=1}^{12} \left(1 - \dot{\eta}_{1\sigma(j)\sigma(t)}\right)^{\omega_j} \right\} \right) \\
&= (\{0.3655, 0.4131\}, \{0.2738, 0.2851\}, \{0.4451, 0.4466\}).
\end{aligned} \tag{38}
$$

Step 5. Based on the score function of SVNHFEs, we get $s(n_1) = 0.5999, s(n_2) = 0.5489, s(n_3) = 0.6036$, and $s(n_4) = 0.5547$.

Step 6. Since $s(n_3) > s(n_1) > s(n_4) > s(n_2)$, the ranking order of all the alternatives is $A_3 \succ A_1 \succ A_4 \succ A_2$ and the most desirable one is A_3.

Comparing to the decision result $A_1 \succ A_3 \succ A_4 \succ A_2$ in [23], we can observe that there exist a little difference from the result of the present paper $A_3 \succ A_1 \succ A_4 \succ A_2$. As Mishra and Kumar asserted in [24], it does not make any sense to apply the aggregation operator in [23] to solve decision-making problems since the aggregation operators do not fulfill monotonicity and idempotency. However, the aggregation operator proposed in the present paper definitely satisfies monotonicity and idempotency, and the normalized sum operation is completely available which means the decision-making result is convincing absolutely. Besides, the computation is less than the earlier method since we avoid crossover operation. Therefore, it is wise to apply the method to many other decision-making problems.

5. Conclusion

In this paper, we have defined two normalized single-valued neutrosophic hesitant fuzzy operations which are indeed meaningful for the processing of single-valued neutrosophic hesitant fuzzy elements, since it turned out that the operations satisfy some basic desirable operation rules such as associative law and distributive law. Moreover, a series of normalized geometric aggregation operators possessing all the basic properties of a valid aggregation operator such as indempotency, boundary, and monotonicity are proposed from the geometric point of view. Furthermore, a decision-making method based on the aggregation operators is developed to resolve multiattribute group decision-making problems, and its feasibility and validity have been illustrated with the help of a practical example. However, the information to be aggregated is mutually connected sometimes and the weight vector can be affected by other evaluation values. Thus, how to handle the relationship between single-valued neutrosophic hesitant fuzzy elements is a critical problem, and it is a research direction we will focus on in the future.

Data Availability

The data used to support the findings of this study are included within the article.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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