Neutrosophic BCC-ideals in BCC-algebras

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Abstract. The notions of a neutrosophic subalgebra and a neutrosohic ideal of a BCC-algebra are introduced and consider characterizations of a neutrosophic subalgebra and a neutrosophic ideal. We define the notion of a neutrosophic BCC-ideal of a BCC-algebra, and investigated some properties of it.

1. Introduction

Y. Kormori [8] introduced a notion of a BCC-algebras, and W. A. Dudek [4] redefined the notion of BCC-algebras by using a dual from of the ordinary definition of Y. Kormori. In [6], J. Hao introduced the notion of ideals in a BCC-algebra and studied some related properties. W. A. Dudek and X. Zhang [5] introdued a BCC-ideals in a BCC-algebra and described connections between such BCC-ideals and congruences. S. S. Ahn and S. H. Kwon [2] defined a topological BCC-algebra and investigated some properties of it.

Zadeh [10] introduced the degree of membership/truth (t) in 1965 and defined the fuzzy set. As a generalization of fuzzy sets, Atanassov [3] introduced the degree of nonmembership/falsehood (f) in 1986 and defined the intuitionistic fuzzy set. Smarandache introduced the degree of indeterminacy/neutrality (i) as independent component in 1995 (published in 1998) and defined the neutrosophic set on three components (t, i, f) = (truth, indeterminacy, falsehood). Jun et. al [7] introduced the notions of a neutrosophic $\mathscr{N}$-subalgebras and a (closed) neutrosophic $\mathscr{N}$-ideal in a BCK/BCI-algebras and investigated some related properties. subalgebras

In this paper, we introduce the notions of a neutrosophic subalgebra and a neutrosohic ideal of a BCC-algebra and consider characterizations of a neutrosophic subalgebra and a neutrosophic ideal. We define the notion of a neutrosophic BCC-ideal of a BCC-algebra, and investigate some properties of it.

2. Preliminaries

By a BCC*-algebra* [4] we mean an algebra $(X, *, 0)$ of type (2,0) satisfying the following conditions: for all $x, y, z \in X$,

(a1) $((x * y) * (z * y)) * (x * z) = 0$,

(a2) $0 * x = 0$,

(a3) $x * 0 = x$,

(a4) $x * y = 0$ and $y * x = 0$ imply $x = y$.

For brevity, we also call X a BCC*-algebra.* In X, we can define a partial order "$\leq$" by putting $x \leq y$ if and only if $x * y = 0$. Then $\leq$ is a partial order on X.

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A BCC-algebra X has the following properties: for any $x, y \in X$,

(b1) $x * x = 0$,

(b2) $(x * y) * x = 0$,

(b3) $x \leq y \Rightarrow x * z \leq y * z$ and $z * y \leq z * x$.

Any BCK-algebra is a BCC-algebra, but there are BCC-algebras which are not BCK-algebra [4]. Note that a BCC-algebra is a BCK-algebra if and only if it satisfies:

(b4) $(x * y) * z = (x * z) * y$, for all $x, y, z \in X$.

Let $(X, *, 0_X)$ and $(Y, *, 0_Y)$ be BCC-algebras. A mapping $\varphi : X \to Y$ is called a *homomorphism* if $\varphi(x *_X y) = \varphi(x) *_Y \varphi(y)$ for all $x, y \in X$. A non-empty subset S of a BCC-algebra X is called a *subalgebra* of X if $x * y \in S$ whenever $x, y \in S$. A non-empty subset I of a BCI-algebra X is called an *ideal* [6] of X if it satisfies:

(c1) $0 \in I$,

(c2) $x * y, y \in I \Rightarrow x \in I$ for all $x, y \in X$.

I is called an *BCC-ideal* [5] of X if it satisfies (c1) and

(c3) $(x * y) * z, y \in I \Rightarrow x * z \in I$, for all $x, y, z \in X$.

Theorem 2.1. [6] *In a BCC-algebra, an ideal is a subalgebra.*

Theorem 2.2. [5] *In a BCC-algebra, a BCC-ideal is an ideal.*

Corollary 2.3. [5] *Any BCC-ideal of a BCC-algebra is a subalgebra.*

Definition 2.4. Let X be a space of points (objects) with generic elements in X denoted by x. A simple valued neutrosophic set A in X is characterized by a truth-membership function $T_A(x)$, an indeterminacy-membership function $I_A(x)$, and a falsity-membership function $F_A(x)$. Then a simple valued neutrosophic set A can be denoted by

$$A := \{\langle x, T_A(x), I_A(x), F_A(x)\rangle | x \in X\},$$

where $T_A(x), I_A(x), F_A(x) \in [0, 1]$ for each point x in X. Therefore the sum of $T_A(x), I_A(x)$, and $F_A(x)$ satisfies the condition $0 \leq T_A(x) + I_A(x) + F_A(x) \leq 3$.

For convenience, "simple valued neutrosophic set" is abbreviated to "neutrosophic set" later.

Definition 2.5. Let A be a neutrosophic set in a B-algebra X and $\alpha, \beta, \gamma \in [0, 1]$ with $0 \leq \alpha + \beta + \gamma \leq 3$ and an (α, β, γ)-level set of X denoted by $A^{(\alpha,\beta,\gamma)}$ is defined as

$$A^{(\alpha,\beta,\gamma)} = \{x \in X | T_A(x) \leq \alpha, I_A(x) \geq \beta, F_A(x) \leq \gamma\}.$$

For any family $\{a_i | i \in \Lambda\}$, we define

$$\bigvee\{a_i | i \in \Lambda\} := \begin{cases} \max\{a_i | i \in \Lambda\} & \text{if } \Lambda \text{ is finite,} \\ \sup\{a_i | i \in \Lambda\} & \text{otherwise} \end{cases}$$

and

$$\bigwedge\{a_i|i\in\Lambda\} := \begin{cases}\min\{a_i|i\in\Lambda\} & \text{if } \Lambda \text{ is finite,}\\ \inf\{a_i|i\in\Lambda\} & \text{otherwise.}\end{cases}$$

3. Neutrosophic BCC-ideals

In what follows, let X be a BCC-algebra unless otherwise specified.

Definition 3.1. A neutrosophic set A in a BCC-algebra X is called a *neutrosophic subalgebra* of X if it satisfies:

(NSS) $T_A(x*y)\le\max\{T_A(x),T_A(y)\}, I_A(x*y)\ge\min\{I_A(x),I_A(y)\},$ and $F_A(x*y)\le\max\{F_A(x),F_A(y)\}$, for any $x,y\in X$.

Proposition 3.2. *Every neutrosophic subalgebra of a BCC-algebra X satisfies the following conditions:*

(3.1) $T_A(0)\le T_A(x), I_A(0)\ge I_A(x)$, *and* $F_A(0)\le F_A(x)$ *for any* $x\in X$.

Proof. Straightforward. □

Example 3.3. Let $X:=\{0,1,2,3\}$ be a BCC-algebra [6] with the following table:

$*$	0	1	2	3
0	0	0	0	0
1	1	0	0	1
2	2	1	0	1
3	3	3	3	0

Define a neutrosophic set A in X as follows:

$$T_A : X\to[0,1],\ x\mapsto\begin{cases}0.12 & \text{if } x\in\{0,1,2\}\\ 0.83 & \text{if } x=3,\end{cases}$$

$$I_A : X\to[0,1],\ x\mapsto\begin{cases}0.81 & \text{if } x\in\{0,1,2\}\\ 0.14 & \text{if } x=3,\end{cases}$$

and

$$F_A : X\to[0,1],\ x\mapsto\begin{cases}0.12 & \text{if } x\in\{0,1,2\}\\ 0.83 & \text{if } x=3.\end{cases}$$

It is easy to check that A is a neutrosophic subalgebra of X.

Theorem 3.4. *Let A be a neutrosophic set in a BCC-algebra X and let $\alpha,\beta,\gamma\in[0,1]$ with $0\le\alpha+\beta+\gamma\le 3$. Then A is a neutrosophic subalgebra of X if and only if all of (α,β,γ)-level set $A^{(\alpha,\beta,\gamma)}$ are subalgebras of X when $A^{(\alpha,\beta,\gamma)}\neq\emptyset$.*

Proof. Assume that A is a neutrosophic subalgebra of X. Let $\alpha,\beta,\gamma\in[0,1]$ be such that $0\le\alpha+\beta+\gamma\le 3$ and $A^{(\alpha,\beta,\gamma)}\neq\emptyset$. Let $x,y\in A^{(\alpha,\beta,\gamma)}$. Then $T_A(x)\le\alpha, T_A(y)\le\alpha, I_A(x)\ge\beta, I_A(y)\ge\beta$ and $F_A(x)\le\gamma, F_A(y)\le\gamma$. Using (NSS), we have $T_A(x*y)\le\max\{T_A(x),T_A(y)\}\le\alpha$, $I_A(x*y)\ge\min\{I_A(x),I_A(y)\}\ge\beta$, and $F_A(x*y)\le\max\{F_A(x),F_A(y)\}\le\gamma$. Hence $x*y\in A^{(\alpha,\beta,\gamma)}$. Therefore $A^{(\alpha,\beta,\gamma)}$ is a subalgebra of X.

Conversely, all of (α,β,γ)-level set $A^{(\alpha,\beta,\gamma)}$ are subalgebras of X when $A^{(\alpha,\beta,\gamma)}\neq\emptyset$. Assume that there exist $a_t,b_t,a_i,b_i\in X$ and $a_f,b_f\in X$ such that $T_A(a_t*b_t)>\max\{T_A(a_t),T_A(b_t)\}, I_A(a_i*b_i)<\min\{I_A(a_i),I_A(b_i)\}$

and $F_A(a_f * b_f) > \max\{F_A(a_f), F_A(b_f)\}$. Then $T_A(a_t * b_t) > \alpha_1 \geq \max\{T_A(a_t), T_A(b_t)\}, I_A(a_i * b_i) < \beta_1 \leq \min\{I_A(a_i), I_A(b_i)\}$ and $F_A(a_f * b_f) > \gamma_1 \geq \max\{F_A(a_f), F_A(b_f)\}$ for some $\alpha_1, \gamma_1 \in [0, 1)$ and $\beta_1 \in (0, 1]$. Hence $a_t, b_t, a_i, b_i \in A^{(\alpha_1,\beta_1,\gamma_1)}$, and $a_f, b_f \in A^{(\alpha_1,\beta_1,\gamma_1)}$. But $a_t * b_t, a_i * b_i \notin A^{(\alpha_1,\beta_1,\gamma_1)}$, and $a_f * b_f \notin A^{(\alpha_1,\beta_1,\gamma_1)}$, which is a contradiction. Hence $T_A(x * y) \leq \max\{T_A(x), T_A(y)\}, I_A(x * y) \geq \min\{I_A(x), I_A(y)\}$, and $F_A(x * y) \leq \max\{T_A(x), T_A(y)\}$, for any $x, y \in X$. Therefore A is a neutrosophic subalgebra of X. □

Since $[0, 1]$ is a completely distributive lattice with respect to the usual ordering, we have the following theorem.

Theorem 3.5. *If $\{A_i | i \in \mathbb{N}\}$ is a family of neutrosopic subalgebras of a BCC-algebra X, then $(\{A_i | i \in \mathbb{N}\}, \subseteq)$ forms a complete distributive lattice.*

Theorem 3.6. *Let A be a neutrosophic subalgebra of a BCC-algebra X. If there exists a sequence $\{a_n\}$ in X such that $\lim_{n\to\infty} T_A(a_n) = 0, \lim_{n\to\infty} I_A(a_n) = 1$, and $\lim_{n\to\infty} F_A(a_n) = 0$, then $T_A(0) = 0, I_A(0) = 1$, and $F_A(0) = 0$.*

Proof. By Proposition 3.2, we have $T_A(0) \leq T_A(x), I_A(0) \geq I_A(x)$, and $F_A(0) \leq F_A(x)$ for all $x \in X$. Hence we have $T_A(0) \leq T_A(a_n), I_A(0) \geq I_A(a_n)$, and $F_A(0) \leq F_A(a_n)$ for every positive integer n. Therefore $0 \leq T_A(0) \leq \lim_{n\to\infty} T_A(a_n) = 0, 1 = \lim_{n\to\infty} I_A(a_n) \leq I_A(0) \leq 1$, and $0 \leq F_A(0) \leq \lim_{n\to\infty} F_A(a_n) = 0$. Thus we have $T_A(0) = 0, I_A(0) = 1$, and $F_A(0) = 0$. □

Proposition 3.7. *If every neutrosophic subalgebra A of a BCC-algebra X satisfies the condition*

(3.2) $T_A(x * y) \leq T_A(y), I_A(x * y) \geq I_A(y), F_A(x * y) \leq F_A(y)$, *for any $x, y \in X$,*

then T_A, I_A, and F_A are constant functions.

Proof. It follows from (3.2) that $T_A(x) = T_A(x * 0) \leq T_A(0), I_A(x) = I_A(x * 0) \geq I_A(0)$, and $F_A(x) = F_A(x * 0) \leq F_A(0)$ for any $x \in X$. By Proposition 3.2, we have $T_A(x) = T_A(0), I_A(x) = I_A(0)$, and $F_A(x) = F_A(0)$ for any $x \in X$. Hence T_A, I_A, and F_A are constant functions. □

Theorem 3.8. *Every subalgebra of a BCC-algebra X can be represented as an (α, β, γ)-level set of a neutrosophic subalgebra A of X.*

Proof. Let S be a subalgebra of a BCC-algebra X and let A be a neutrosophic subalgebra of X. Define a neutrosophic set A in X as follows:

$$T_A : X \to [0, 1],\ x \mapsto \begin{cases} \alpha_1 & \text{if } x \in S \\ \alpha_2 & \text{otherwise,} \end{cases}$$

$$I_A : X \to [0, 1],\ x \mapsto \begin{cases} \beta_1 & \text{if } x \in S \\ \beta_2 & \text{otherwise,} \end{cases}$$

$$F_A : X \to [0, 1],\ x \mapsto \begin{cases} \gamma_1 & \text{if } x \in S \\ \gamma_2 & \text{otherwise,} \end{cases}$$

where $\alpha_1, \alpha_2, \gamma_1, \gamma_2 \in [0, 1)$ and $\beta_1, \beta_2 \in (0, 1]$ with $\alpha_1 < \alpha_2, \beta_1 > \beta_2, \gamma_1 < \gamma_2$, and $0 \leq \alpha_1 + \beta_1 + \gamma_1 \leq 3, 0 \leq \alpha_2 + \beta_2 + \gamma_2 \leq 3$. Obviously, $S = A^{(\alpha_1,\beta_1,\gamma_1)}$. We now prove that A is a neutrosophic subalgebra of X. Let $x, y \in X$. If $x, y \in S$, then $x * y \in S$ because S is a subalgebra of X. Hence $T_A(x) = T_A(y) = T_A(x * y) = \alpha_1$,

$I_A(x) = I_A(y) = I_A(x*y) = \beta_1$, $F_A(x) = F_A(y) = F_A(x*y) = \gamma_1$ and so $T_A(x*y) \leq \max\{T_A(x), T_A(y)\}$, $I_A(x*y) \geq \min\{I_A(x), I_A(y)\}$, $F_A(x*y) \leq \max\{F_A(x), F_A(y)\}$. If $x \in S$ and $y \notin S$, then $T_A(x) = \alpha_1, T_A(y) = \alpha_2$, $I_A(x) = \beta_1, I_A(y) = \beta_2$, $F_A(x) = \gamma_1, F_A(y) = \gamma_2$ and so $T_A(x*y) \leq \max\{T_A(x), T_A(y)\} = \alpha_2$, $I_A(x*y) \geq \min\{I_A(x), I_A(y)\} = \beta_2$, $F_A(x*y) \leq \max\{F_A(x), F_A(y)\} = \gamma_2$. Obviously, if $x \notin A$ and $y \notin A$, then $T_A(x*y) \leq \max\{T_A(x), T_A(y)\} = \alpha_2$, $I_A(x*y) \geq \min\{I_A(x), I_A(y)\} = \beta_2$, $F_A(x*y) \leq \max\{F_A(x), F_A(y)\} = \gamma_2$. Therefore A is a neutrosophic subalgebra of X. □

Definition 3.9. A neutrosophic set A in a BCC-algebra X is said to be *neutrosophic ideal* of X if it satisfies:

(NSI1) $T_A(0) \leq T_A(x), I_A(0) \geq I_A(x)$, and $F_A(0) \leq F_A(x)$ for any $x \in X$;

(NSI2) $T_A(x) \leq \max\{T_A(x*y), T_A(y)\}, I_A(x) \geq \min\{I_A(x*y), I_A(y)\}$, and $F_A(x) \leq \max\{F_A(x*y), F_A(y)\}$, for any $x, y \in X$.

Proposition 3.10. *Every neutrosophic ideal of a BCC-algebra X is a neutrosophic subalgebra of X.*

Proof. Let A be a neutrosophic ideal of X. Put $x := x*y$ and $y := x$ in (NSI2). Then we have $T_A(x*y) \leq \max\{T_A((x*y)*x), T_A(x)\}, I_A(x*y) \geq \min\{I_A((x*y)*x), I_A(x)\}$, and $F_A(x*y) \leq \max\{F_A((x*y)*x), F_A(x)\}$. It follows from (b2) and (NSI1) that $T_A(x*y) \leq \max\{T_A((x*y)*x), T_A(x)\} = \max\{T_A(0), T_A(x)\} \leq \max\{T_A(x), T_A(y)\}, I_A(x*y) \geq \min\{I_A((x*y)*x), I_A(x)\} = \max\{I_A(0), I_A(x)\} \geq \max\{I_A(x), I_A(y)\}$, and $F_A(x*y) \leq \max\{F_A((x*y)*x), F_A(x)\} = \max\{F_A(0), F_A(x)\} \leq \max\{F_A(x), F_A(y)\}$. Thus A is a neutrosophic subalgebra of X. □

Theorem 3.11. *Let A be a neutrosophic set in a BCC-algebra X and let $\alpha, \beta, \gamma \in [0,1]$ with $0 \leq \alpha+\beta+\gamma \leq 3$. Then A is a neutrosophic ideal of X if and only if all of (α, β, γ)-level set $A^{(\alpha,\beta,\gamma)}$ are ideals of X when $A^{(\alpha,\beta,\gamma)} \neq \emptyset$.*

Proof. Assume that A is a neutrosophic ideal of X. Let $\alpha, \beta, \gamma \in [0,1]$ be such that $0 \leq \alpha+\beta+\gamma \leq 3$ and $A^{(\alpha,\beta,\gamma)} \neq \emptyset$. Let $x, y \in X$ be such that $x*y, y \in A^{(\alpha,\beta,\gamma)}$. Then $T_A(x*y) \leq \alpha, T_A(y) \leq \alpha, I_A(x*y) \geq \beta, I_A(y) \geq \beta$, and $F_A(x*y) \leq \gamma, F_A(y) \leq \gamma$. By Definition 3.9, we have $T_A(0) \leq T_A(x) \leq \max\{T_A(x*y), T_A(y)\} \leq \alpha, I_A(0) \geq I_A(x) \geq \min\{I_A(x*y)), I_A(y)\} \geq \beta$, and $F_A(0) \leq F_A(x) \leq \max\{F_A(x*y), T_A(y)\} \leq \gamma$. Hence $0, x \in A^{(\alpha,\beta,\gamma)}$. Therefore $A^{(\alpha,\beta,\gamma)}$ is an ideal of X.

Conversely, suppose that there exist $a, b, c \in X$ such that $T_A(0) > T_A(a), I_A(0) < I_A(b)$, and $F_A(0) > F_A(c)$. Then there exist $a_t, c_t \in [0,1)$ and $b_t \in (0,1]$ such that $T_A(0) > a_t \geq T_A(a), I_A(0) < b_t \leq I_A(b)$ and $F_A(0) > c_t \geq F_A(c)$. Hence $0 \notin A^{(a_t,b_t,c_t)}$, which is a contradiction. Therefore $T_A(0) \leq T_A(x), I_A(0) \geq I_A(x)$ and $F_A(0) \leq F_A(x)$ for all $x \in X$. Assume that there exist $a_t, b_t, a_i, b_i, a_f, b_f \in X$ such that $T_A(a_t) > \max\{T_A(a_t*b_t), T_A(b_t)\}, I_A(a_i) < \min\{I_A(a_i*b_i), I_A(b_i)\}$, and $F_A(a_f) > \max\{T_A(a_f*b_f), T_A(b_f)\}$. Then there exist $s_t, s_f \in [0,1)$ and $s_i \in (0,1]$ such that $T_A(a_t) > s_t \geq \max\{T_A(a_t*b_t), T_A(b_t)\}, I_A(a_i) < s_i \leq \min\{I_A(a_i*b_i), I_A(b_i)\}$, and $F_A(a_f) > s_f \geq \max\{T_A(a_f*b_f), T_A(b_f)\}$. Hence $a_t*b_t, b_t, a_i*b_i, a_f*b_f \in A^{(s_t,s_i,s_f)}$, and $b_t, b_i, b_f \in A^{(s_t,s_i,s_f)}$. But $a_t, a_i \notin A^{(s_t,s_i,s_f)}$ and $a_f \notin A^{(s_t,s_i,s_f)}$. This is a contradiction. Therefore $T_A(x) \leq \max\{T_A(x*y), T_A(y)\}, I_A(x) \geq \min\{I_A(x*y)), I_A(y)\}$ and $F_A(x) \leq \max\{F_A(x*y), F_A(y)\}$, for any $x, y \in X$. Therefore A is a neutrosophic ideal of X □

Proposition 3.12. *Every neutrosophic ideal A of a BCC-algebra X satisfies the following properties:*

(i) $(\forall x, y \in X)(x \leq y \Rightarrow T_A(x) \leq T_A(y), I_A(x) \geq I_A(y), F_A(x) \leq F_A(y))$,

(ii) $(\forall x, y, z \in X)(x*y \leq z \Rightarrow T_A(x) \leq \max\{T_A(y), T_A(z)\}, I_A(x) \geq \min\{I_A(y), I_A(z)\}, F_A(x) \leq \max\{F_A(y), F_A(z)\})$.

Proof. (i) Let $x, y \in X$ be such that $x \leq y$. Then $x * y = 0$. Using (NSI2) and (NSI1), we have $T_A(x) \leq \max\{T_A(x*y), T_A(y)\} = \max\{T_A(0), T_A(y)\} = T_A(y), I_A(y) \geq \min\{I_A(x*y), I_A(y)\} = \min\{I_A(0), I_A(y)\} = I_A(y)$, and $F_A(x) \leq \max\{F_A(x*y), F_A(y)\} = \max\{F_A(0), F_A(y)\} = F_A(y)$.
(ii) Let $x, y, z \in X$ be such that $x * y \leq z$. By (NSI2) and (NSI1). we get $T_A(x*y) \leq \max\{T_A((x*y)*z), T_A(z)\} = \max\{T_A(0), T_A(z)\} = T_A(z), I_A(x*y) \geq \min\{I_A((x*y)*z), I_A(z)\} = \min\{I_A(0), I_A(z)\} = I_A(z)$, and $F_A(x*y) \leq \max\{F_A((x*y)*z), F_A(z)\} = \max\{F_A(0), F_A(z)\} = F_A(z)$. Hence $T_A(x) \leq \max\{T_A(x*y), T_A(y)\} \leq \max\{T_A(y), T_A(z)\}, I_A(x) \geq \min\{I_A(x*y), I_A(y)\} \geq \min\{I_A(y), I_A(z)\}$, and $F_A(x) \leq \max\{F_A(x*y), F_A(y)\} \leq \max\{F_A(y), F_A(z)\}$. □

The following corollary is easily proved by induction.

Corollary 3.13. *Every neutrosophic ideal A of a BCC-algebra X satisfies the following property:*

(3.3) $(\cdots(x*a_1)*\cdots)*a_n = 0 \Rightarrow T_A(x) \leq \bigvee_{k=1}^n T_A(a_k), I_A(x) \geq \bigwedge_{k=1}^n I_A(a_k), F_A(x) \leq \bigvee_{k=1}^n F_A(a_k)$, *for all* $x, a_1, \cdots, a_n \in X$.

Definition 3.14. Let A and B be neutrosophic sets of a set X. The *union* of A and B is defined to be a neutrosophic set

$$A \tilde{\cup} B := \{\langle x, T_{A\cup B}(x), I_{A\cup B}(x), F_{A\cup B}(x)\rangle | x \in X\},$$

where $T_{A\cup B}(x) = \min\{T_A(x), T_B(x)\}, I_{A\cup B}(x) = \max\{I_A(x), I_B(x)\}, F_{A\cup B}(x) = \min\{F_A(x), F_B(x)\}$, for all $x \in X$. The *intersection* of A and B is defined to be a neutrosophic set

$$A \tilde{\cap} B := \{\langle x, T_{A\cap B}(x), I_{A\cap B}(x), F_{A\cap B}(x)\rangle | x \in X\},$$

where $T_{A\cap B}(x) = \max\{T_A(x), T_B(x)\}, I_{A\cap B}(x) = \min\{I_A(x), I_B(x)\}, F_{A\cap B}(x) = \max\{F_A(x), F_B(x)\}$, for all $x \in X$.

Theorem 3.15. *The intersection of two neutrosophic ideals of a BCC-algebra X is a also a neutrosophic ideal of X.*

Proof. Let A and B be neutrosophic ideals of X. For any $x \in X$, we have $T_{A\cap B}(0) = \max\{T_A(0), T_B(0)\} \leq \max\{T_A(x), T_B(x)\} = T_{A\cap B}(x), I_{A\cap B}(0) = \min\{T_A(0), T_B(0)\} \geq \min\{I_A(x), I_B(x)\} = I_{A\cap B}(x)$, and $F_{A\cap B}(0) = \max\{F_A(0), F_B(0)\} \leq \max\{F_A(x), F_B(x)\} = F_{A\cap B}(x)$. Let $x, y \in X$. Then we have

$$\begin{aligned} T_{A\cap B}(x) &= \max\{T_A(x), T_B(x)\} \\ &\leq \max\{\max\{T_A(x*y), T_A(y)\}, \max\{T_B(x*y), T_B(y)\}\} \\ &= \max\{\max\{T_A(x*y), T_B(x*y)\}, \max\{T_A(y), T_B(y)\}\} \\ &= \max\{T_{A\cap B}(x*y), T_{A\cap B}(y)\}, \end{aligned}$$

$$\begin{aligned} I_{A\cap B}(x) &= \min\{I_A(x), I_B(x)\} \\ &\geq \min\{\min\{I_A(x*y), I_A(y)\}, \min\{I_B(x*y), I_B(y)\}\} \\ &= \min\{\min\{I_A(x*y), I_B(x*y)\}, \min\{I_A(y), I_B(y)\}\} \\ &= \min\{I_{A\cap B}(x*y), I_{A\cap B}(y)\}, \end{aligned}$$

and

$$\begin{aligned} F_{A\cap B}(x) &= \max\{F_A(x), F_B(x)\} \\ &\leq \max\{\max\{F_A(x*y), F_A(y)\}, \max\{F_B(x*y), F_B(y)\}\} \\ &= \max\{\max\{F_A(x*y), F_B(x*y)\}, \max\{F_A(y), F_B(y)\}\} \\ &= \max\{F_{A\cap B}(x*y), F_{A\cap B}(y)\}. \end{aligned}$$

Hence $A\tilde{\cap}B$ is a neutrosophic ideal of X. □

Corollary 3.16. *If $\{A_i | i \in \mathbb{N}\}$ is a family of neutrosophic ideals of a BCC-algebra X, then so is $\tilde{\cap}_{i\in\mathbb{N}} A_i$.*

The union of any set of neutrosophic ideals of a BCC-algebra X need not be a neutrosophic ideal of X.

Example 3.17. Let $X = \{0, 1, 2, 3, 4\}$ be a BCC-algebra [5] with the following table:

$*$	0	1	2	3	4
0	0	0	0	0	0
1	1	0	1	0	0
2	2	2	0	0	0
3	3	3	1	0	0
4	4	3	4	3	0

Define neutrosophic sets A and B of X as follows:

$$T_A : X \to [0,1],\ x \mapsto \begin{cases} 0.12, & \text{if } x \in \{0,1\} \\ 0.74 & \text{otherwise,} \end{cases}$$

$$I_A : X \to [0,1],\ x \mapsto \begin{cases} 0.63, & \text{if } x \in \{0,1\} \\ 0.11 & \text{otherwise,} \end{cases}$$

$$F_A : X \to [0,1],\ x \mapsto \begin{cases} 0.12, & \text{if } x \in \{0,1\} \\ 0.74 & \text{otherwise,} \end{cases}$$

$$T_B : X \to [0,1],\ x \mapsto \begin{cases} 0.13, & \text{if } x \in \{0,2\} \\ 0.63 & \text{otherwise,} \end{cases}$$

$$I_B : X \to [0,1],\ x \mapsto \begin{cases} 0.75, & \text{if } x \in \{0,2\} \\ 0.14 & \text{otherwise,} \end{cases}$$

and

$$F_B : X \to [0,1],\ x \mapsto \begin{cases} 0.13, & \text{if } x \in \{0,2\} \\ 0.63 & \text{otherwise.} \end{cases}$$

It is easy to check that A and B are neutrosophic ideals of X. But $A\tilde{\cup}B$ is not a neutrosophic ideal of X, since $T_{A\cup B}(3) = \min\{T_A(3), T_B(3)\} = 0.63 \not\leq \max\{T_{A\cup B}(3*2), T_{A\cup B}(2)\} = \max\{T_{A\cup B}(1), T_{A\cup B}(2)\} = \max\{\min\{T_A(1), T_B(1)\}, \min\{T_A(2), T_B(2)\}\} = \max\{0.12, 0.13\} = 0.13$.

Definition 3.18. A neutrosophic set A in a BCC-algebra X is said to be a *neutrosophic BCC-ideal* of X if it satisfies (NSI1) and

(NSI3) $T_A(x*z) \leq \max\{T_A((x*y)*z), T_A(y)\}, I_A(x*z) \geq \min\{I_A((x*y)*z), I_A(y)\}$, and $F_A(x*z) \leq \max\{F_A((x*y)*z), F_A(y)\}$, for any $x, y, z \in X$.

Lemma 3.19. *Every neutrosophic BCC-ideal of a BCC-algebra X is a neutrosophic ideal of X.*

Proof. Let A be a neutrosophic BCC-ideal of a BCC-algebra X. Put $z := 0$ in (NSI3). By (a3), we have $T_A(x*0) = T_A(x) \leq \max\{T_A((x*y)*0), T_A(y)\} = \max\{T_A(x*y), T_A(y)\}, I_A(x*0) = I_A(x) \geq \min\{I_A((x*y)*0), I_A(y)\} = \min\{I_A(x*y), I_A(y)\}$, and $F_A(x*0) = F_A(x) \leq \max\{F_A((x*y)*0), F_A(y)\} = \max\{F_A(x*y), F_A(y)\}$, for any $x, y \in X$. Hence A is a neutrosophic ideal of X. □

Corollary 3.20. *Every neutrosophic BCC-ideal of a BCC-algebra X is a neutrosophic subalgebra of X.*

The converse of Proposition 3.10 and Lemma 3.19 need not be true in general (see Example 3.21).

Example 3.21. Let $X = \{0, 1, 2, 3, 4\}$ be a BCC-algebra as in Example 3.17. Define a neutrosophic set A of X as follows:

$$T_A : X \to [0,1],\ x \mapsto \begin{cases} 0.13 & \text{if } x \in \{0,1,2,3\} \\ 0.83 & \text{if } x = 4, \end{cases}$$

$$I_A : X \to [0,1],\ x \mapsto \begin{cases} 0.82 & \text{if } x \in \{0,1,2,3\} \\ 0.11 & \text{if } x = 4, \end{cases}$$

and

$$F_A : X \to [0,1],\ x \mapsto \begin{cases} 0.13 & \text{if } x \in \{0,1,2,3\} \\ 0.83 & \text{if } x = 4, \end{cases}$$

It is easy to check that A is a neutrosophic subalgebra of X, but not a neutrosophic ideal of X, since $T_A(4) = 0.83 \not\leq \max\{T_A(4*3), T_A(3)\} = \max\{T_A(3), T_A(3)\} = 0.13$. Consider a neutrosophic set B of X which is given by

$$T_B : X \to [0,1],\ x \mapsto \begin{cases} 0.14 & \text{if } x \in \{0,1\}, \\ 0.84 & \text{if } x \in \{2,3,4\} \end{cases}$$

$$I_B : X \to [0,1],\ x \mapsto \begin{cases} 0.85 & \text{if } x \in \{0,1\} \\ 0.12 & \text{if } x \in \{2,3,4\}, \end{cases}$$

and

$$F_B : X \to [0,1],\ x \mapsto \begin{cases} 0.14 & \text{if } x \in \{0,1\} \\ 0.84 & \text{if } x \in \{2,3,4\}. \end{cases}$$

It is easy to show that B is a neutrosophic ideal of X, but not a neutrosophic BCC-ideal of X, since $T_B(4*3) = T_B(3) = 0.84 \not\leq \max\{T_B((4*1)*3), T_B(1)\} = \max\{T_B(0), T_B(1)\} = 0.14$.

Example 3.22. Let $X = \{0, 1, 2, 3, 4, 5\}$ be a BCC-algebra [5] with the following table:

$*$	0	1	2	3	4	5
0	0	0	0	0	0	0
1	1	0	0	0	0	1
2	2	2	0	0	1	1
3	3	2	1	0	1	1
4	4	4	4	4	0	1
5	5	5	5	5	5	0

Define a neutrosophic set A of X as follows:

$$T_A : X \to [0,1],\ x \mapsto \begin{cases} 0.43 & \text{if } x \in \{0,1,2,3,4\} \\ 0.55 & \text{if } x = 5, \end{cases}$$

$$I_A : X \to [0,1],\ x \mapsto \begin{cases} 0.54 & \text{if } x \in \{0,1,2,3\,4\} \\ 0.42 & \text{if } x = 5, \end{cases}$$

and

$$F_A : X \to [0,1],\ x \mapsto \begin{cases} 0.43 & \text{if } x \in \{0,1,2,3,4\} \\ 0.55 & \text{if } x = 5. \end{cases}$$

It is easy to check that A is a neutrosophic BCC-ideal of X.

Theorem 3.23. *Let A be a neutrosophic set in a BCC-algebra X and let $\alpha, \beta, \gamma \in [0, 1]$ with $0 \le \alpha + \beta + \gamma \le 3$. Then A is a neutrosophic BCC-ideal of X if and only if all of (α, β, γ)-level set $A^{(\alpha,\beta,\gamma)}$ are BCC-ideals of X when $A^{(\alpha,\beta,\gamma)} \ne \emptyset$.*

Proof. Similar to Theorem 3.11. □

Proposition 3.24. *Let A be a neutrosophic BCC-ideal of a BCC-algebra X. Then $X_T := \{x \in X | T_A(x) = T_A(0)\}, X_T := \{x \in X | I_A(x) = I_A(0)\}$, and $X_F := \{x \in X | F_A(x) = F_A(0)\}$ are BCC-ideals of X.*

Proof. Clearly, $0 \in X_T$. Let $(x * y) * z, y \in X_T$. Then $T_A((x * y) * z) = T_A(0)$ and $T_A(y) = T_A(0)$. It follows from (NSI3) that $T_A(x * z) \le \max\{T_A((x * y) * z), T_A(y)\} = T_A(0)$. By (NSI1), we get $T_A(x * z) = T_A(0)$. Hence $x * z \in X_T$. Therefore X_T is a BCC-ideal of X. By a similar way, X_I and X_F are BCC-ideals of X. □

References

[1] S. S. Ahn, Applications of soft sets to BCC-ideals in BCC-algebras, J. Comput. Anal. Appl. (to appear).

[2] S. S. Ahn and S. H. Kwon, Toplogical properties in BCC-algerbras, Commun. Korean Math. Soc. **23(2)** (2008), 169-178.

[3] K. Atanassov, *Intuitionistic fuzzy sets,* Fuzzy sets and Systems 20 (1986), 87–96.

[4] W. A. Dudek, *On constructions of BCC-algebras,* Selected Papers on BCK- and BCI-algebras **1** (1992), 93-96.

[5] W. A. Dudek and X. Zhang, *On ideals and congruences in BCC-algeras,* Czecho Math. J. **48** (1998), 21-29.

[6] J. Hao, *Ideal Theory of BCC-algebras,* Sci. Math. Japo. **3** (1998), 373-381.

[7] Y. B. Jun, F. Smarandache and H. Bordbar, *Neutrosophic $\mathcal{N}$-structures applied to BCK/BCI-algebras,* Information, (to submit).

[8] Y. Kormori, *The class of BCC-algebras is not a varity,* Math. Japo. **29** (1984), 391-394.

[9] F. Smarandache, Neutrosophy, Neutrosophic Probablity, Sets, and Logic, Amer. Res. Press, Rehoboth, USA, 1998.

[10] L. A. Zadeh, Fuzzy sets, Inform. Control **8** (1965), 338-353.