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Neutrosophic Closed Set and Neutrosophic Continuous Functions

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Abstract
In this paper, we introduce and study the concept of "neutrosophic closed set "and "neutrosophic continuous function". Possible application to GIS topology rules are touched upon.

Keywords: Neutrosophic Closed Set, Neutrosophic Set; Neutrosophic Topology; Neutrosophic Continuous Function.

1 INTRODUCTION

The idea of "neutrosophic set" was first given by Smarandache [11, 12]. Neutrosophic operations have been investigated by Salama at el. [1-10]. Neutrosophy has laid the foundation for a whole family of new mathematical theories, generalizing both their crisp and fuzzy counterparts [9, 13]. Here we shall present the neutrosophic crisp version of these concepts. In this paper, we introduce and study the concept of "neutrosophic closed set "and "neutrosophic continuous function".

2 TERMINOLOGIES

We recollect some relevant basic preliminaries, and in particular the work of Smarandache in [11, 12], and Salama at el. [1-10].

2.1 Definition [5]
A neutrosophic topology (NT for short) an a non empty set X is a family τ of neutrosophic subsets in X satisfying the following axioms

(NT_1) $O_N, 1_N \in \tau$,
(NT_2) $G_1 \cap G_2 \in \tau$ for any $G_1, G_2 \in \tau$,
(NT_3) $\cup G_i \in \tau \quad \forall \{G_i : i \in J\} \subseteq \tau$

In this case the pair (X, τ) is called a neutrosophic topological space (NTS for short) and any neutrosophic set in τ is known as neuterosophic open set (NOS for short) in X . The elements of τ are called open neutrosophic sets, A neutrosophic set F is closed if and only if it C (F) is neutrosophic open.

2.1 Definition [5]
The complement of (C (A) for short) of is called a neutrosophic closed set (for short) in A . NOSA NCS X**.**

3 Neutrosophic Closed Set .

3.1 Definition
Let (X, τ) be a neutrosophic topological space. A neutrosophic set A in (X, τ) is said to be neutrosophic closed (in shortly N-closed).
If Ncl (A) $\subseteq$ G whenever A $\subseteq$ G and G is neutrosophic open; the complement of neutrosophic closed set is Neutrosophic open.

3.1 Proposition
If A and B are neutrosophic closed sets then A$\cup$B is Neutrosophic closed set.

3.1 Remark
The intersection of two neutrosophic closed (N-closed for short) sets need not be neutrosophic closed set.

3.1 Example
Let X = {a, b, c} and

A = <(0.5,0.5,0.5) , (0.4,0.5,0.5) , (0.4,0.5,0.5)>
B = <(0.3,0.4,0.4) , (0.7,0.5,0.5) , (0.3,0.4,0.4)>
Then T = { 0_N ,1_N , A, B} is a neutrosophic topology on X. Define the two neutrosophic sets A_1 and A_2 as follows,
A_1 = <(0.5,0.5,0.5),(0.6,0.5,0.5),(0.6,0.5,0.5)>
A_2 = <(0.7,0.6,0.6)(0.3,0.5,0.5),(0.7,0.6,0.6)>
A_1 and A_2 are neutrosophic closed set but $A_1 \cap A_2$ is not a neutrosophic closed set.

3.2 Proposition

Let (X,τ) be a neutrosophic topological space. If B is neutrosophic closed set and B ⊆ A ⊆ Ncl (B), then A is N-closed.

3.4 Proposition

In a neutrosophic topological space (X,T), T=ℑ (the family of all neutrosophic closed sets) iff every neutrosophic subset of (X,T) is a neutrosophic closed set.

Proof.

suppose that every neutrosophic set A of (X,T) is N-closed. Let A∈T, since A ⊆ A and A is N-closed, Ncl (A) ⊆ A. But A ⊆ Ncl (A). Hence, Ncl (A) =A. thus, A ∈ ℑ. Therefore, T ⊆ ℑ. If B ∈ ℑ then 1-B ∈T ⊆ ℑ. and hence B∈T, That is, ℑ ⊆ T. Therefore T=ℑ conversely, suppose that A be a neutrosophic set in (X,T). Let B be a neutrosophic open set in (X,T). such that A ⊆ B. By hypothesis, B is neutrosophic N-closed. By definition of neutrosophic closure, Ncl (A) ⊆ B. Therefore A is N-closed.

3.5 Proposition

Let (X,T) be a neutrosophic topological space. A neutrosophic set A is neutrosophic open iff B ⊆ NInt (A), whenever B is neutrosophic closed and B ⊆ A.

Proof

Let A a neutrosophic open set and B be a N-closed, such that B ⊆ A. Now, B ⊆ A⇒ 1–A ⇒1–B and 1–A is a neutrosophic closed set ⇒ Ncl (1–A) ⊆ 1–B. That is, B=1–(1–B) ⊆ 1–Ncl (1–A). But 1–Ncl (1–A) = Nint (A). Thus, B ⊆ Nint (A). Conversely, suppose that A be a neutrosophic set, such that B ⊆ Nint (A) whenever B is neutrosophic closed and B ⊆ A. Let 1–A ⊆ B ⇒ 1–B ⊆ A. Hence by assumption 1–B ⊆ Nint (A). that is, 1–Nint (A) ⊆ B. But 1–Nint (A) =Ncl (1–A). Hence Ncl(1–A) ⊆ B. That is 1–A is neutrosophic closed set. Therefore, A is neutrosophic open set

3.6 Proposition

If Nint (A) ⊆ B ⊆ A and if A is neutrosophic open set then B is also neutrosophic open set.

4 Neutrosophic Continuous Functions

4.1 Definition

i) If $B = \langle \mu_B, \sigma_B, \nu_B \rangle$ is a NS in Y, then the preimage of B under f, denoted by $f^{-1}(B)$, is a NS in X defined by $f^{-1}(B) = \left\langle f^{-1}(\mu_B), f^{-1}(\sigma_B), f^{-1}(\nu_\nu) \right\rangle$.

ii) If $A = \langle \mu_A, \sigma_A, \nu_A \rangle$ is a NS in X, then the image of A under f, denoted by $f(A)$, is the a NS in Y defined by $f(A) = \left\langle f(\mu_A), f(\sigma_A), f(\nu_A)^c) \right\rangle$.

Here we introduce the properties of images and preimages some of which we shall frequently use in the following sections .

4.1 Corollary

Let A, $\{A_i : i \in J\}$, be NSs in X, and B, $\{B_j : j \in K\}$ NS in Y, and $f : X \to Y$ a function. Then

(a) $A_1 \subseteq A_2 \Leftrightarrow f(A_1) \subseteq f(A_2)$,
$B_1 \subseteq B_2 \Leftrightarrow f^{-1}(B_1) \subseteq f^{-1}(B_2)$,

(b) $A \subseteq f^{-1}(f(A))$ and if f is injective, then $A = f^{-1}(f(A))$.

(c) $f^{-1}(f(B)) \subseteq B$ and if f is surjective, then $f^{-1}(f(B)) = B$, .

(d) $f^{-1}(\cup B_i) = \cup f^{-1}(B_i)$, $f^{-1}(\cap B_i) = \cap f^{-1}(B_i)$,

(e) $f(\cup A_i) = \cup f(A_i)$; $f(\cap A_i) \subseteq \cap f(A_i)$; and if f is injective, then $f(\cap A_i) = \cap f(A_i)$;

(f) $f^{-1}(!_N) = 1_N$ $f^{-1}(0_N) = 0_N$.

(g) $f(0_N) = 0_N$, $f(1_N) = 1_N$ if f is subjective.

Proof

Obvious.

4.2 Definition

Let (X, Γ_1) and (Y, Γ_2) be two NTSs, and let $f : X \to Y$ be a function. Then f is said to be continuous iff the preimage of each NCS in Γ_2 is a NS in Γ_1.

4.3 Definition

Let (X, Γ_1) and (Y, Γ_2) be two NTSs and let $f : X \to Y$ be a function. Then f is said to be open iff the image of each NS in Γ_1 is a NS in Γ_2 .

4.1 Example

Let (X, Γ_o) and (Y, ψ_o) be two NTSs

(a) If $f : X \to Y$ is continuous in the usual sense, then in this case, f is continuous in the sense of Definition 5.1 too. Here we consider the NTs on X and Y, respectively, as follows : $\Gamma_1 = \left\{ \left\langle \mu_G, 0, \mu_G^c \right\rangle : G \in \Gamma_o \right\}$ and

$\Gamma_2 = \left\{\left\langle \mu_H, 0, \mu_H^c \right\rangle : H \in \Psi_o\right\}$,

In this case we have, for each $\left\langle \mu_H, 0, \mu_H^c \right\rangle \in \Gamma_2$, $H \in \Psi_o$,

$f^{-1}\left\langle \mu_H, 0, \mu_H^c \right\rangle = \left\langle f^{-1}(\mu_H), f^{-1}(0), f^{-1}(\mu_H^c)\right\rangle = \left\langle f^{-1}\mu_H, f(0), (f(\mu)^c \right\rangle \in \Gamma_1$.

(b) If $f: X \to Y$ is neutrosophic open in the usual sense, then in this case, f is neutrosophic open in the sense of Definition 3.2.

Now we obtain some characterizations of neutrosophic continuity:

4.1 Proposition

Let $f:(X,\Gamma_1) \to (Y,\Gamma_2)$.

f is neutrosop continuous iff the preimage of each NS (neutrosophic closed set) in Γ_2 is a NS in Γ_2.

4.2 Proposition

The following are equivalent to each other:

(a) $f:(X,\Gamma_1) \to (Y,\Gamma_2)$ is neutrosophic continuous .

(b) $f^{-1}(NInt(B) \subseteq NInt(f^{-1}(B))$ for each CNS B in Y.

(c) $NCl(f^{-1}(B)) \subseteq f^{-1}(NCl(B))$ for each NCB in Y.

4.2 Example

Let (Y,Γ_2) be a NTS and $f: X \to Y$ be a function. In this case $\Gamma_1 = \left\{f^{-1}(H): H \in \Gamma_2\right\}$ is a NT on X. Indeed, it is the coarsest NT on X which makes the function $f: X \to Y$ continuous. One may call it the initial neutrosophic crisp topology with respect to f.

4.4 Definition

Let (X,T) and (Y,S) be two neutrosophic topological space, then

(a) A map f : (X,T) → (Y,S) is called N-continuous (in short N-continuous) if the inverse image of every closed set in (Y,S) is Neutrosophic closedin (X,T).

(b) A map f:(X,T)→ (Y,S) is called neutrosophic-gc irresolute if the inverse image of every Neutrosophic closedset in (Y,S) is Neutrosophic closedin (X,T). Equivalently if the inverse image of every Neutrosophic open set in (Y,S) is Neutrosophic open in (X,T).

(c) A map f:(X,T)→ (Y,S) is said to be strongly neutrosophic continuous if f^{-1}(A) is both neutrosophic open and neutrosophic closed in (X,T) for each neutrosophic set A in (Y,S).

(d) A map f : (X,T) → (Y,S) is said to be perfectly neutrosophic continuous if f^{-1} (A) is both neutrosophic open and neutrosophic closed in (X,T) for each neutrosophic open set A in (Y,S).

(e) A map f:(X,T)→(Y,S) is said to be strongly N-continuous if the inverse image of every Neutrosophic open set in (Y,S) is neutrosophic open in (X,T).

(F) A map f:(X,T)→(Y,S) is said to be perfectly N-continuous if the inverse image of every Neutrosophic open set in (Y,S) is both neutrosophic open and neutrosophic closed in (X,T).

4.3 Proposition

Let (X,T) and (Y,S) be any two neutrosophic topological spaces. Let f : (X,T) → (Y,S) be generalized neutrosophic continuous. Then for every neutrosophic set A in X, f(Ncl(A)) ⊆ Ncl(f(A)).

4.4 Proposition

Let (X,T) and (Y,S) be any two neutrosophic topological spaces. Let f : (X,T) → (Y,S) be generalized neutrosophic continuous. Then for every neutrosophic set A in Y, Ncl(f^{-1}(A)) ⊆ f^{-1}(Ncl(A)).

4.5 Proposition

Let (X,T) and (Y,S) be any two neutrosophic topological spaces. If A is a Neutrosophic closedset in (X,T) and if f : (X,T) → (Y,S) is neutrosophic continuous and neutrosophic-closed then f(A) is Neutrosophic closedin (Y,S).

Proof.

Let G be a neutrosophic-open in (Y,S). If f(A) ⊆ G, then A ⊆ f^{-1}(G) in (X,T). Since A is neutrosophic closedand f^{-1}(G) is neutrosophic open in (X,T), Ncl(A) ⊆ f^{-1}(G), (i.e) f(Ncl(A)⊆G. Now by assumption, f(Ncl(A)) is neutrosophic closed and Ncl(f(A)) ⊆ Ncl(f(Ncl(A))) = f(Ncl(A)) ⊆ G. Hence, f(A) is N-closed.

4.5 Proposition

Let (X,T) and (Y,S) be any two neutrosophic topological spaces, If f : (X,T) → (Y,S) is neutrosophic continuous then it is N-continuous.

The converse of proposition 4.5 need not be true. See Example 4.3.

4.3 Example

Let X ={a,b,c} and Y ={a,b,c}. Define neutrosophic sets A and B as follows A = $\langle(0.4,0.4,0.5),(0.2,0.4,0.3),(0.4,0.4,0.5)\rangle$

B = $\langle(0.4,0.5,0.6),(0.3,0.2,0.3),(0.4,0.5,0.6)\rangle$

Then the family T = {0_N, 1_N, A} is a neutrosophic topology on X and S = {0_N, 1_N, B} is a neutrosophic topology on Y. Thus (X,T) and (Y,S) are neutrosophic topological spaces. Define f : (X,T) → (Y,S) as f(a) = b , f(b) = a, f(c) = c. Clearly f is N-continuous. Now f is not neutrosophic continuous, since f^{-1}(B) ∉ T for B ∈ S.

4.4 Example

Let X = {a,b,c}. Define the neutrosophic sets A and B as follows.

A = $\langle(0.4,0.5,0.4),(0.5,0.5,0.5),(0.4,0.5,0.4)\rangle$

B = $\langle(0.7,0.6,0.5),(0.3,0.4,0.5),(0.3,0.4,0.5)\rangle$
and C = $\langle(0.5,0.5,0.5),(0.4,0.5,0.5),(0.5,0.5,0.5)\rangle$
T = $\{0_N, 1_N, A, B\}$
and S = $\{0_N, 1_N, C\}$ are neutrosophic topologies on X. Thus (X,T) and (X,S) are neutrosophic topological spaces. Define f: (X,T) $\rightarrow$ (X,S) as follows f(a) = b, f(b) = b, f(c) = c. Clearly f is N-continuous. Since
D = $\langle(0.6,0.6,0.7),(0.4,0.4,0.3),(0.6,0.6,0.7)\rangle$
is neutrosophic open in (X,S), f^{-1}(D) is not neutrosophic open in (X,T).

4.6 Proposition
Let (X,T) and (Y,S) be any two neutrosophic topological space. If f : (X,T) $\rightarrow$ (Y,S) is strongly N-continuous then f is neutrosophic continuous.
The converse of Proposition 3.19 is not true. See Example 3.3

4.5 Example
Let X ={a,b,c}. Define the neutrosophic sets A and B as follows.
A = $\langle(0.9,0.9,0.9),(0.1,0.1,0.1),(0.9,0.9,0.9)\rangle$
B = $\langle(0.9,0.9,0.9),(0.1,0.1,0),(0.9,0.1,0.8)\rangle$
and C = $\langle(0.9,0.9,0.9),(0.1,0,0.1),(0.9,0.9,0.9)\rangle$
T = $\{0_N, 1_N, A, B\}$ and S = $\{0_N, 1_N, C\}$ are neutrosophic topologies on X. Thus (X,T) and (X,S) are neutrosophic topological spaces. Also define f :(X,T)$\rightarrow$ (X,S) as follows f(a) = a, f(b) = c, f(c) = b. Clearly f is neutrosophic continuous. But f is not strongly N-continuous. Since
D = $\langle(0.9,0.9,0.99),(0.05,0,0.01),(0.9,0.9,0.99)\rangle$
Is an Neutrosophic open set in (X,S), f^{-1}(D) is not neutrosophic open in (X,T).

4.7 Proposition
Let (X,T) and (Y, S) be any two neutrosophic topological spaces. If f: (X,T) $\rightarrow$ (Y,S) is perfectly N-continuous then f is strongly N-continuous.
The converse of Proposition 4.7 is not true. See Example 4.6

4.6 Example
Let X = {a,b,c}. Define the neutrosophic sets A and B as follows.
A = $\langle(0.9,0.9,0.9),(0.1,0.1,0.1),(0.9,0.9,0.9)\rangle$
B = $\langle(0.99,0.99,0.99),(0.01,0,0),(0.99,0.99,0.99)\rangle$
And C = $\langle(0.9,0.9,0.9),(0.1,0.1,0.05),(0.9,0.9,0.9)\rangle$
T = $\{0_N, 1_N, A, B\}$ and S = $\{0_N, 1_N, C\}$ are neutrosophic topologies space on X. Thus (X,T) and (X,S) are neutrosophic topological spaces. Also define f : (X,T) $\rightarrow$ (X,S) as follows f(a) = a, f(b) = f(c) = b. Clearly f is strongly N-continuous. But f is not perfectly N continuous. Since D = $\langle(0.9,0.9,0.9),(0.1,0.1,0),(0.9,0.9,0.9)\rangle$
Is an Neutrosophic open set in (X,S), f^{-1}(D) is neutrosophic open and not neutrosophic closed in (X,T).

4.8 Proposition
Let (X,T) and (Y,S) be any neutrosophic topological spaces. If f: (X,T) $\rightarrow$ (Y,S) is strongly neutrosophic continuous then f is strongly N-continuous.
The converse of proposition 3.23 is not true. See Example 4.7

4.7 Example
Let X = {a,b,c} and Define the neutrosophic sets A and B as follows.
A = $\langle(0.9,0.9,0.9),(0.1,0.1,0.1),(0.9,0.9,0.9)\rangle$
B = $\langle(0.99,0.99,0.99),(0.01,0,0),(0.99,0.99,0.99)\rangle$
and C = $\langle(0.9,0.9,0.9),(0.1,0.1,0.05),(0.9,0.9,0.9)\rangle$
T = $\{0_N, 1_N, A, B\}$ and S = $\{0_N, 1_N, C\}$ are neutrosophic topologies on X. Thus (X,T) and (X,S) are neutrosophic topological spaces. Also define f : (X,T) $\rightarrow$ (X,S) as follows: f(a) = a, f(b) = f(c) = b. Clearly f is strongly N-continuous. But f is not strongly neutrosophic continuous. Since
D = $\langle(0.9,0.9,0.9),(0.1,0.1,0),(0.9,0.9,0.9)\rangle$
be a neutrosophic set in (X,S), f^{-1}(D) is neutrosophic open and not neutrosophic closed in (X,T).

4.9 Proposition
Let (X,T),(Y,S) and (Z,R) be any three neutrosophic topological spaces. Suppose f : (X,T) $\rightarrow$ (Y,S), g : (Y,S) $\rightarrow$ (Z,R) be maps. Assume f is neutrosophic gc-irresolute and g is N-continuous then g o f is N-continuous.

4.10 Proposition
Let (X,T), (Y,S) and (Z,R) be any three neutrosophic topological spaces. Let f : (X,T) $\rightarrow$ (Y,S), g : (Y,S) $\rightarrow$ (Z,R) be map, such that f is strongly N-continuous and g is N-continuous. Then the composition g o f is neutrosophic continuous.

4.5 Definition
A neutrosophic topological space (X,T) is said to be neutrosophic $T_{1/2}$ if every Neutrosophic closed set in (X,T) is neutrosophic closed in (X,T).

4.11 Proposition
Let (X,T),(Y,S) and (Z,R) be any neutrosophic topological spaces. Let f : (X,T) $\rightarrow$ (Y,S) and g : (Y,S) $\rightarrow$ (Z,R) be mapping and (Y,S) be neutrosophic $T_{1/2}$ if f and g are N-continuous then the composition g o f is N-continuous.
The proposition 4.11 is not valid if (Y,S) is not neutrosophic $T_{1/2}$.

4.8 Example
Let X = {a,b,c}. Define the neutrosophic sets A,B and C as follows.
A = $\langle(0.4,0.4,0.6),(0.4,0.4,0.3)\rangle$
B = $\langle(0.4,0.5,0.6),(0.3,0.4,0.3)\rangle$
and C = $\langle(0.4,0.6,0.5),(0.5,0.3,0.4)\rangle$

Then the family T = {0_N, 1_N, A}, S = {0_N, 1_N, B} and R = {0_N, 1_N, C} are neutrosophic topologies on X. Thus (X,T),(X,S) and (X,R) are neutrosophic topological spaces. Also define f : (X,T) $\rightarrow$ (X,S) as f(a) = b, f(b) = a, f(c) = c and g : (X,S) $\rightarrow$ (X,R) as g(a) = b, g(b) = c, g(c) = b. Clearly f and g are N-continuous function. But g o f is not N-continuous. For 1 − C is neutrosophic closed in (X,R). $f^{-1}(g^{-1}(1-C))$ is not N closed in (X,T). g o f is not N-continuous.

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