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Neutrosophic Cubic Einstein Geometric Aggregation Operators with Application to Multi-Criteria Decision Making Method

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Abstract: Neutrosophic cubic sets (NCs) are amore generalized version of neutrosophic sets(Ns) and interval neutrosophic sets (INs). Neutrosophic cubic setsare better placed to express consistent, indeterminate and inconsistent information, which provides a better platform to deal with incomplete, inconsistent and vague data. Aggregation operators play a key role in daily life, and in relation to science and engineering problems. In this paper we defined the algebraic and Einstein sum, multiplication and scalar multiplication, score and accuracy functions. Using these operations we defined geometric aggregation operators and Einstein geometric aggregation operators. First, we defined the algebraic and Einstein operators of addition, multiplication and scalar multiplication. We defined score and accuracy function to compare neutrosophic cubic values. Then we definedthe neutrosophic cubic weighted geometric operator (NCWG), neutrosophic cubic ordered weighted geometric operator (NCOWG), neutrosophic cubic Einstein weighted geometric operator (NCEWG), and neutrosophic cubic Einstein ordered weighted geometric operator (NCEOWG) over neutrosophic cubic sets. A multi-criteria decision making method is developed as an application to these operators. This method is then applied to a daily life problem.

Keywords: neutrosophic cubic weighted geometric operator (NCWG); neutrosophic cubic ordered weighted geometric operator (NCOWG); neutrosophic cubic Einstein weighted geometric operator (NCEWG); neutrosophic cubic Einstein ordered weighted geometric operator (NCEOWG)

1. Introduction

The theory of fuzzy sets was introduced by Zadeh [1].Soon after, it attracted experts of sciences and engineering due to its possibilistic behavior. The applicability of fuzzy sets extended it to interval valued fuzzy sets(IVFs) [2,3]. In 1986, K. Atnassov developed the theory of intuitionistic fuzzy sets [4], which were further extended to interval valued intuitionistic fuzzy sets in 1989 [5]. In 2012, Y.B. Jun generalized the idea of fuzzy sets and intuitionistic fuzzy sets to form cubic sets [6]. Smarandache presented his theory regarding the inconsistent and indeterminate behavior of data in 1999, and named it the neutrosophic set [7]. Neutrosophic sets consist of three components:Truth, indeterminate and falsehood, which provides a more general platform to deal with vague and insufficient data. In 2005, Wang et al. [8] presented the idea of interval valued neutrosophic sets. Interval valued neutrosophic sets provide a range to experts which makes them more comfortable with making the choice. Jun et al. defined the neutrosophic cubic set [9,10]. Neutrosophic cubic sets are a generalization of neutrosophic sets and interval neutrosophic sets. They enable us to choose both interval values and single value membership. This characteristic of neutrosophic cubic sets enables us to deal with uncertain and vague data more efficiently.

Decision making is one of the most important factors in scienceand day-to-day life as well. Aggregation operators are an imperative part of modern decision making. A lack of data or information makes it difficult for decision makers to take an appropriatedecision. This uncertain situation can be minimized using the vague nature neutrosophic cubic set and its extensions. Neutrosophic cubic set (NCs) are a more generalized version of neutrosophic sets (Ns) and interval neutrosophic sets (INs). Neutrosophic cubic setsare better placed to express consistent, indeterminate, and inconsistent information, which provides a better platform to deal with incomplete, inconsistent, and vague data. Aggregation operators have a key role in daily life, science and engineering problems. Zhan et al. [11] in their workapplications of neutrosophic cubic sets in multi-criteria decision making in 2017. Banerjee et al. [12] usedgrey rational analysis in their workGRA for multi attribute decision making in neutrosophic cubic set environment in 2017.Lu and Ye [13] definedcosine measure for neutrosophic cubic sets for multiple attribte decision making in 2017. Pramanik et al. [14] defined neutrosophic cubic MCGDM method based on similarity measurein 2017. Shi and Ye [15] defined Dombi aggregation operators of neutrosophic cubic set for multiple attribute deicision makingin 2018. Baolin et al. [16] applied Einstein aggregations onneutrosophic sets in a novel generalized simplified neutrosophic number Einstein aggregation operator 2018. Alot of work has been done and is being done by different researchers in decision making using neutrosophic cubic sets.

In this paper, we define algebraic and Einstein sum, multiplication and scalar multiplication, score and accuracy functions. Using these operations, we define geometric aggregation operators and Einstein geometric aggregation operators. First, we define algebraic and Einstein operators of addition, multiplication and scalar multiplication. We then define score and accuracy functions to compare neutrosophic cubic values. Following this, we propose a neutrosophic cubic ordered weighted geometric operator (NCOWG), neutrosophic cubic Einstein weighted geometric operator (NCEWG), and a neutrosophic cubic Einstein ordered weighted geometric operator (NCEOWG) over neutrosophic cubic sets. A multi-criteria decision making method is then developed as an application for these operators. This method is then applied to a daily life problem.

2. Preliminaries

This section consists of two parts: Notations, which consists of notations with their descriptions and some previous definitions; and results. We recommend the reader to see [1–3,6–9,16].

2.1. Notations

This section consists of some notations with their descriptions, as shown in Table 1.

Table 1. Some notations with their descriptions.

S. No	Notation	Description
1	U	Ground set
2	u	Element of ground set (U).
3	ψ	Fuzzy set
4	$\widetilde{\Psi} = [\Psi^L, \Psi^U]$	Interval valued fuzzy set which is an interval of [0,1]. The left extreme ψ^L is referred as lower fuzzy and right extreme ψ^U is referred as upper fuzzy function.
5	(T_N, I_N, F_N)	components of neutrosophic sets each one is fuzzy sets.
6	$\left(\widetilde{T}_N, \widetilde{I}_N, \widetilde{F}_N\right)$	The components of interval neutrosophic each one is an interval valued fuzzy set.
7	$\left(\widetilde{T}_N, \widetilde{I}_N, \widetilde{F}_N, T_N, I_N, F_N\right)$	The components of neutrosophic cubic set. Referred to 5 and 6.
8	Γ^*, Γ	t-conorm, t-norm
9	$\oplus, \otimes$	Algebraic sum, product
10	$\oplus_E, \otimes_E$	Einstein sum, product

2.2. Pre-Defined Definitions

This section consists of some predefined definitions and results.

Definition 1 [1]. *A mapping* $\psi: U \to [0,1]$ *is called a fuzzy set, and* $\psi(u)$ *is called a membership function, simply denoted by* ψ.

Definition 2 [2,3]. *A mapping* $\widetilde{\Psi}: U \to D[0,1]$, *where* $D[0,1]$ *is the interval value of* $[0,1]$, *called the interval valued fuzzy set(IVF). For all* $u \in U$ $\widetilde{\Psi}(u) = \left\{ [\psi^L(u), \psi^U(u)] \,|\, \psi^L(u), \psi^U(u) \in [0,1] \text{ and } \psi^L(u) \le \psi^U(u) \right\}$ *is membership degree of u in* $\widetilde{\Psi}$. *This is simply denoted by* $\widetilde{\Psi} = [\Psi^L, \Psi^U]$.

Definition 3 [6]. *A structure* $C = \left\{ \left(u, \widetilde{\Psi}(u), \Psi(u)\right) \,|\, u \in U \right\}$ *is a cubic set in U in which* $\widetilde{\Psi}(u)$ *is IVF in U, that is,* $\widetilde{\Psi} = [\Psi^L, \Psi^U]$ *and* Ψ *is a fuzzy set in U. This can be simply denoted by* $C = \left(\widetilde{\Psi}, \Psi\right)$. C^U *denotes the collection of cubic sets in U.*

Definition 4 [7]. *A structure* $N = \{(T_N(u), I_N(u), F_N(u)) \,|\, u \in U\}$ *is a neutrosophic set(Ns), where* $\{T_N(u), I_N(u), F_N(u) \in [0,1]\}$ *are called truth, indeterminacy and falsity functions, respectively.This can be simply denoted by* $N = (T_N, I_N, F_N)$.

Definition 5 [8]. *An interval neutrosophic set (INs) in U is a structure* $N = \left\{ \left(\widetilde{T}_N(u), \widetilde{I}_N(u), \widetilde{F}_N(u)\right) \,|\, u \in U \right\}$, *where* $\left\{ \widetilde{T}_N(u), \widetilde{I}_N(u), \widetilde{F}_N(u) \in D[0,1] \right\}$ *is calledtruth, indeterminacy an falsity functionin U, respectively. This can be simply denoted by* $N = \left(\widetilde{T}_N, \widetilde{I}_N, \widetilde{F}_N\right)$. *For convenience, we denote* $N = \left(\widetilde{T}_N, \widetilde{I}_N, \widetilde{F}_N\right)$ *by* $N = \left(\widetilde{T}_N = [T_N^L, T_N^U], \widetilde{I}_N = [I_N^L, I_N^U], \widetilde{F}_N = [F_N^L, F_N^U]\right)$.

Definition 6 [9]. *A structure* $N = \left\{ \left(u, \widetilde{T}_N(u), \widetilde{I}_N(u), \widetilde{F}_N(u), T_N(u), I_N(u), F_N(u)\right) \,|\, u \in U \right\}$ *is neutrosophic cubic set(NCs) in U, in which* $\left(\widetilde{T}_N = [T_N^L, T_N^U], \widetilde{I}_N = [I_N^L, I_N^U], \widetilde{F}_N = [F_N^L, F_N^U]\right)$ *is an interval neutrosophic set and* (T_N, I_N, F_N) *is a neutrosophic set in U. Simply denoted by* $N = \left(\widetilde{T}_N, \widetilde{I}_N, \widetilde{F}_N, T_N, I_N, F_N\right)$, $[0,0] \le \widetilde{T}_N + \widetilde{I}_N + \widetilde{F}_N \le [3,3]$ *and* $0 \le T_N + I_N + F_N \le 3$. N^U *denotes the collection of neutrosophic cubic sets in U. Simply denoted by* $N = \left(\widetilde{T}_N, \widetilde{I}_N, \widetilde{F}_N, T_N, I_N, F_N\right)$.

Definition 7 [16]. *The t-operators are basically union and intersection operators in the theory of fuzzy sets, which are denoted by t-conorm* (Γ^*) *and t-norm* (Γ), *respectively. The role of t-operators is very important in fuzzy theory and its applications.*

Definition 8 [16]. $\Gamma^* : [0,1] \times [0,1] \to [0,1]$ *is called t-conorm if it satisfies the following axioms:*

Axiom 1. $\Gamma^*(1, u) = 1$ *and* $\Gamma^*(0, u) = 0$;
Axiom 2. $\Gamma^*(u, v) = \Gamma^*(v, u)$ *for all a and b;*
Axiom 3. $\Gamma^*(u, \Gamma^*(v, w)) = \Gamma^*(\Gamma^*(u, v), w)$ *for all a, b and c;*
Axiom 4. *If* $u \le u'$ *and* $v \le v'$, *then* $\Gamma^*(u, v) \le \Gamma^*(u', v')$.

Most known t-conorms are as follows:

1. The default t-conorm: $\Gamma^*_{\max}(u, v) = \max(u, v)$.
2. The bounded t-conorm: $\Gamma^*_{bounded}(u, v) = \min(1, u + v)$.
3. The algebraic t-conorm: $\Gamma^*_{algebraic}(u, v) = u + v - uv$.

Definition 9 [16]. $\Gamma : [0,1] \times [0,1] \to [0,1]$ *is called t-norm if it satisfies the following axioms:*

Axiom 5. $\Gamma(1, u) = u$ *and* $\Gamma(0, u) = 0$;
Axiom 6. $\Gamma(u, v) = \Gamma(v, u)$ *for all a and b;*
Axiom 7. $\Gamma(u, \Gamma(v, w)) = \Gamma(\Gamma(u, v), w)$ *for all a, b and c;*
Axiom 8. *If* $u \le u'$ *and* $v \le v'$, *then* $\Gamma(u, v) \le \Gamma(u', v')$.

Most well known t-norms are as follows:

1. The default t-norm: $\Gamma_{\min}(u, v) = \min(u, v)$.
2. The bounded t-norm: $\Gamma_{bounded}(u, v) = \max(0, u + v - 1)$.
3. The algebraic t-norm: $\Gamma_{algebraic}(u, v) = uv$.

If $\Gamma^*(u, v)$, $\Gamma(u, v)$ are continuous and $\Gamma^*(u, u) > u$, $\Gamma(u, u) < u$, then Γ^* and Γ are said to be Archimedes t-conorm and t-norm, respectively. Any pair of dual t-conorm (Γ^*) and t-norm (Γ) is used. It is known that t-norms and t-conorms operators satisfy the condition of conjunction and disjunction operators, respectively. However, the algebraic operations, like algebraic sum and product, are not unique and may correspond to union and intersection. The t-conorms and t-norms families have a vast range, which corresponds to unions and intersections. Among these, the Einstein sum and Einstein product are good choices since they give the smooth approximation like algebraic sum and algebraic product, respectively. Einstein sum $\oplus_E$ and Einstein product $\otimes_E$ are examples of t-conorm and t-norm, respectively:

$$\Gamma^*_E(u, v) = \frac{u + v}{1 + uv}$$

$$\Gamma_E(u, v) = \frac{uv}{1 + (1 - u)(1 - v)}$$

Group decision making is an important aspect of decision making theory. We are often in situationsin which we have to deal with more then one expert, attribute and alternative. Motivated by such situations, a multi-attribute decision making method for more then one expert is proposed on neutrosophic cubic aggregation operators.This whole work consisted of six sections. In Section 3, we define some algebraicEinstein operations and score and accuracy functions, along with some important results and examples. On the basis of these definitions and results, we define geometric and Einstein geometric aggregation operators on neutrosophic cubic sets in Section 4. In Section 5, an algorithm is proposed based on neutrosophic cubic geometric and Einstein geometric aggregation operators to deal with multi-attribute decision making problems. In the final section, a numerical example from daily life is presented as an application of the work.

3. Operations on Neutrosophic Cubic Sets

In this section, we introduce some new operations on neutrosophic cubic sets which are further used in the article.

3.1. Algebraic Addition, Multiplication and Scalar Multiplication

We introduce the algebraic addition, multiplication, and scalar multiplication on neutrosophic cubic sets(NCs). An important result of exponential multlipliction is established on the basis of these defintions, which provides the basis to define neutrosophic cubic geometric aggregation operators.

Definition 10. *The sum of two neutrosophic cubic sets(NCs),* $A = \left(\widetilde{T}_A, \widetilde{I}_A, \widetilde{F}_A, T_A, I_A, F_A\right)$, *where* $\widetilde{T}_A = [T_A^L, T_A^U], \widetilde{I}_A = [I_A^L, I_A^U], \widetilde{F}_A = [F_A^L, F_A^U]$, *and* $B = \left(\widetilde{T}_B, \widetilde{I}_B, \widetilde{F}_B, T_B, I_B, F_B\right)$, *where* $\widetilde{T}_B = [T_B^L, T_B^U], \widetilde{I}_B = [I_B^L, I_B^U], \widetilde{F}_B = [F_B^L, F_B^U]$ *is defined as*

$$
A \oplus B = \begin{pmatrix} [T_A^L + T_B^L - T_A^L T_B^L, T_A^U + T_B^U - T_A^U T_B^U], \\ [I_A^L + I_B^L - I_A^L I_B^L, I_A^U + I_B^U - I_A^U I_B^U], \\ [F_A^L F_B^L, F_A^U F_B^U], \\ T_A T_B, I_A I_B, F_A + F_B - F_A F_B \end{pmatrix}
$$

Definition 11. *The product between two neutrosophic cubic sets (NCs),* $A = \left(\widetilde{T}_A, \widetilde{I}_A, \widetilde{F}_A, T_A, I_A, F_A\right)$, *where* $\widetilde{T}_A = [T_A^L, T_A^U], \widetilde{I}_A = [I_A^L, I_A^U], \widetilde{F}_A = [F_A^L, F_A^U]$ *and* $B = \left(\widetilde{T}_B, \widetilde{I}_B, \widetilde{F}_B, T_B, I_B, F_B\right)$, *where* $\widetilde{T}_B = [T_B^L, T_B^U], \widetilde{I}_B = [I_B^L, I_B^U], \widetilde{F}_B = [F_B^L, F_B^U]$ *is defined as*

$$
A \otimes B = \begin{pmatrix} [T_A^L T_B^L, T_A^U T_B^U], \\ [I_A^L I_B^L, I_A^U I_B^U], \\ [F_A^L + F_B^L - F_A^L F_B^L, F_A^U + F_B^U - F_A^U F_B^U], \\ T_A + T_B - T_A T_B, I_A + I_B - I_A I_B, F_A F_B \end{pmatrix}
$$

Definition 12. *The scalar multiplication on a neutrosophic cubic set (NCs),* $A = \left(\widetilde{T}_A, \widetilde{I}_A, \widetilde{F}_A, T_A, I_A, F_A\right)$, *where* $\widetilde{T}_A = [T_A^L, T_A^U], \widetilde{I}_A = [I_A^L, I_A^U], \widetilde{F}_A = [F_A^L, F_A^U]$, *and a Scalar k is defined as*

$$
kA = \begin{pmatrix} \left[1 - (1 - T_A^L)^k, 1 - (1 - T_A^U)^k\right], \\ \left[1 - (1 - I_A^L)^k, 1 - (1 - I_A^U)^k\right], \\ \left[(F_A^L)^k, (F_A^U)^k\right], \\ (T_A)^k, (I_A)^k, 1 - (1 - F_A)^k \end{pmatrix}
$$

The following result is established to deal with the exponential multiplication on neutrosophic cubic values. This result enables us to define geometric aggregation operators along some important results on neutrosophic cubic sets.

Theorem 1. *Let* $A = \left(\widetilde{T}_A, \widetilde{I}_A, \widetilde{F}_A, T_A, I_A, F_A\right)$, *where* $\widetilde{T}_A = [T_A^L, T_A^U], \widetilde{I}_A = [I_A^L, I_A^U], \widetilde{F}_A = [F_A^L, F_A^U]$, *be a neutrosophic cubic value, then the exponential operation can be defined by*

$$
A^k = \begin{pmatrix} \left[(T_A^L)^k, (T_A^U)^k\right], \\ \left[(I_A^L)^k, (I_A^U)^k\right], \\ \left[1 - (1 - F_A^L)^k, 1 - (1 - F_A^U)^k\right], \\ 1 - (1 - T_A)^k, 1 - (1 - I_A)^k, (F_A)^k \end{pmatrix}
$$

where $A^k = A \otimes A \otimes, \ldots \otimes A(k - times)$, *and* A^k *is a neutrosophic cubic value for every positive value of k.*

Proof. We prove the theorem by mathematical induction, as the $k = 1$, $A^1 = A$ result holds. We assume that for $k = m$ the result is true:

$$A^m = \begin{pmatrix} \left[(T_A^L)^m, (T_A^U)^m \right], \\ \left[(I_A^L)^m, (I_A^U)^m \right], \\ \left[1 - (1 - F_A^L)^m, 1 - (1 - F_A^U)^m \right], \\ 1 - (1 - T_A)^m, 1 - (1 - I_A)^m, (F_A)^m \end{pmatrix}$$

That is A^m is neutrosophic cubic value. We prove that for $k = m + 1$ is also neutrosophic cubic value.

Since

$$A^m \otimes A = \begin{pmatrix} \left[(T_A^L)^m, (T_A^U)^m \right], \\ \left[(I_A^L)^m, (I_A^U)^m \right], \\ \left[1 - (1 - F_A^L)^m, 1 - (1 - F_A^U)^m \right], \\ 1 - (1 - T_A)^m, 1 - (1 - I_A)^m, (F_A)^m \end{pmatrix} \otimes \begin{pmatrix} \left[(T_A^L), (T_A^U) \right], \\ \left[(I_A^L), (I_A^U) \right], \\ \left[F_A^L, F_A^U \right], \\ T_A, I_A, F_A \end{pmatrix}$$

$$= \begin{pmatrix} \left[(T_A^L)^{m+1}, (T_A^U)^{m+1} \right], \\ \left[(I_A^L)^{m+1}, (I_A^U)^{m+1} \right], \\ \left[1 - (1 - F_A^L)^m + F_A^L - \left(1 - (1 - F_A^L)^m \right) F_A^L, 1 - (1 - F_A^U)^m + F_A^U - \left(1 - (1 - F_A^U)^m \right) F_A^U \right], \\ 1 - (1 - T_A)^m + T_A - (1 - (1 - T_A)^m) T_A, 1 - (1 - I_A)^m + I_A - (1 - (1 - I_A)^m I_A), (F_A)^{m+1} \end{pmatrix}$$

$$= \begin{pmatrix} \left[(T_A^L)^{m+1}, (T_A^U)^{m+1} \right], \\ \left[(I_A^L)^{m+1}, (I_A^U)^{m+1} \right], \\ \left[1 - (1 - F_A^L)^m + F_A^L - F_A^L + (1 - F_A^L)^m F_A^L, 1 - (1 - F_A^U)^m + F_A^U - F_A^U + (1 - F_A^U)^m F_A^U \right], \\ 1 - (1 - T_A)^m + T_A - T_A + (1 - T_A)^m T_A, 1 - (1 - I_A)^m + I_A - I_A + (1 - I_A)^m I_A, (F_A)^{m+1} \end{pmatrix}$$

$$= \begin{pmatrix} \left[(T_A^L)^{m+1}, (T_A^U)^{m+1} \right], \\ \left[(I_A^L)^{m+1}, (I_A^U)^{m+1} \right], \\ \left[1 - (1 - F_A^L)^m + (1 - F_A^L)^m F_A^L, 1 - (1 - F_A^U)^m + (1 - F_A^U)^m F_A^U \right], \\ 1 - (1 - T_A)^m + (1 - T_A)^m T_A, 1 - (1 - I_A)^m + (1 - I_A)^m I_A, (F_A)^{m+1} \end{pmatrix}$$

$$= \begin{pmatrix} \left[(T_A^L)^{m+1}, (T_A^U)^{m+1} \right], \\ \left[(I_A^L)^{m+1}, (I_A^U)^{m+1} \right], \\ \left[1 - (1 - F_A^L)^m (1 - F_A^L), 1 - (1 - F_A^U)^m (1 - F_A^U) \right], \\ 1 - (1 - T_A)^m (1 - T_A), 1 - (1 - I_A)^m (1 - I_A), (F_A)^{m+1} \end{pmatrix}$$

$$= \begin{pmatrix} \left[(T_A^L)^{m+1}, (T_A^U)^{m+1} \right], \\ \left[(I_A^L)^{m+1}, (I_A^U)^{m+1} \right], \\ \left[1 - (1 - F_A^L)^{m+1}, 1 - (1 - F_A^U)^{m+1} \right], \\ 1 - (1 - T_A)^{m+1}, 1 - (1 - I_A)^{m+1}, (F_A)^{m+1} \end{pmatrix}$$

$$= A^{m+1}.$$

□

3.2. Einstein Addition, Multiplication and Scalar Multiplication

Taking into account the dual t-conorm (Γ^*) and t-norm (Γ), the Einstein operations of union, intersection, addition, multiplication and scalar multiplication are defined on the neutrosophic cubic sets. An important result of Einstein exponential multlipliction is established on the basis of these definitions, which provides the base with which to define neutrosophic cubic Einstein geometric aggregation operators.

Definition 13. *The Einstein union between two neutrosophic cubic sets (NCs),* $A = \left(\widetilde{T}_A, \widetilde{I}_A, \widetilde{F}_A, T_A, I_A, F_A \right)$ *where* $\widetilde{T}_A = [T_A^L, T_A^U], \widetilde{I}_A = [I_A^L, I_A^U], \widetilde{F}_A = [F_A^L, F_A^U],$ *and* $B = \left(\widetilde{T}_B, \widetilde{I}_B, \widetilde{F}_B, T_B, I_B, F_B \right)$ *where* $\widetilde{T}_B = [T_B^L, T_B^U], \widetilde{I}_B = [I_B^L, I_B^U], \widetilde{F}_B = [F_B^L, F_B^U]$ *is defined as*

$$A \vee B = \left(\Gamma\left\{ \widetilde{T}_A, \widetilde{T}_B \right\}, \Gamma\left\{ \widetilde{I}_A, \widetilde{I}_B \right\}, \Gamma^*\left\{ \widetilde{F}_A, \widetilde{F}_B \right\}, \Gamma^*\{T_A, T_B\}, \Gamma^*\{I_A, I_B\}, \Gamma\{F_A, F_B\} \right)$$

Definition 14. *The Einstein intersection between two neutrosophic cubic sets(NCS),* $A = \left(\widetilde{T}_A, \widetilde{I}_A, \widetilde{F}_A, T_A, I_A, F_A \right),$ *where* $\widetilde{T}_A = [T_A^L, T_A^U], \widetilde{I}_A = [I_A^L, I_A^U], \widetilde{F}_A = [F_A^L, F_A^U]$ *and* $B = \left(\widetilde{T}_B, \widetilde{I}_B, \widetilde{F}_B, T_B, I_B, F_B \right),$ *where* $\widetilde{T}_B = [T_B^L, T_B^U], \widetilde{I}_B = [I_B^L, I_B^U], \widetilde{F}_B = [F_B^L, F_B^U]$ *is defined as*

$$A \wedge B = \left(\Gamma^*\left\{ \widetilde{T}_A, \widetilde{T}_B \right\}, \Gamma^*\left\{ \widetilde{I}_A, \widetilde{I}_B \right\}, \Gamma\left\{ \widetilde{F}_A, \widetilde{F}_B \right\}, \Gamma\{T_A, T_B\}, \Gamma\{I_A, I_B\}, \Gamma^*\{F_A, F_B\} \right).$$

On the basis of Einstein union and intersection the Einstein sum and product is defined over neutrosophic cubic values.

Definition 15. *The Einstein sum between two neutrosophic cubic sets (NCS),* $A = \left(\widetilde{T}_A, \widetilde{I}_A, \widetilde{F}_A, T_A, I_A, F_A \right),$ *where* $\widetilde{T}_A = [T_A^L, T_A^U], \widetilde{I}_A = [I_A^L, I_A^U], \widetilde{F}_A = [F_A^L, F_A^U]$ *and* $B = \left(\widetilde{T}_B, \widetilde{I}_B, \widetilde{F}_B, T_B, I_B, F_B \right),$ *where* $\widetilde{T}_B = [T_B^L, T_B^U], \widetilde{I}_B = [I_B^L, I_B^U], \widetilde{F}_B = [F_B^L, F_B^U]$ *is defined as*

$$A \oplus_E B = \left(\begin{array}{c} \left[\dfrac{T_A^L + T_B^L}{1 + T_A^L T_B^L}, \dfrac{T_A^U + T_B^U}{1 + T_A^U T_B^U} \right], \\ \left[\dfrac{I_A^L + I_B^L}{1 + I_A^L I_B^L}, \dfrac{I_A^U + I_B^U}{1 + I_A^U I_B^U} \right], \\ \left[\dfrac{F_A^L F_B^L}{1 + (1 - F_A^L)(1 - F_B^L)}, \dfrac{F_A^U F_B^U}{1 + (1 - F_A^U)(1 - F_B^U)} \right] \\ \dfrac{T_A T_B}{1 + (1 - T_A)(1 - T_B)}, \dfrac{I_A I_B}{1 + (1 - I_A)(1 - I_B)}, \dfrac{F_A + F_B}{1 + F_A F_B} \end{array} \right)$$

Definition 16. *The Einstein product between two neutrosophic cubic sets (NCS),* $A = \left(\widetilde{T}_A, \widetilde{I}_A, \widetilde{F}_A, T_A, I_A, F_A \right),$ *where* $\widetilde{T}_A = [T_A^L, T_A^U], \widetilde{I}_A = [I_A^L, I_A^U], \widetilde{F}_A = [F_A^L, F_A^U]$ *and* $B = \left(\widetilde{T}_B, \widetilde{I}_B, \widetilde{F}_B, T_B, I_B, F_B \right),$ *where* $\widetilde{T}_B = [T_B^L, T_B^U], \widetilde{I}_B = [I_B^L, I_B^U], \widetilde{F}_B = [F_B^L, F_B^U]$ *is defined as*

$$A \otimes_E B = \left(\begin{array}{c} \left[\dfrac{T_A^L T_B^L}{1 + (1 - T_A^L)(1 - T_B^L)}, \dfrac{T_A^U T_B^U}{1 + (1 - T_A^U)(1 - T_B^U)} \right], \\ \left[\dfrac{I_A^L I_B^L}{1 + (1 - I_A^L)(1 - I_B^L)}, \dfrac{I_A^U I_B^U}{1 + (1 - I_A^U)(1 - I_B^U)} \right], \\ \left[\dfrac{F_A^L + F_B^L}{1 + F_A^L F_B^L}, \dfrac{F_A^U + F_B^U}{1 + F_A^U F_B^U} \right] \\ \dfrac{T_A + T_B}{1 + T_A T_B}, \dfrac{I_A + I_B}{1 + I_A I_B}, \dfrac{F_A F_B}{1 + (1 - F_A)(1 - F_B)} \end{array} \right)$$

Definition 17. *The scalar multiplication on a neutrosophic cubic set(NCS),* $A = \left(\widetilde{T}_A, \widetilde{I}_A, \widetilde{F}_A, T_A, I_A, F_A \right),$ *where* $\widetilde{T}_A = [T_A^L, T_A^U], \widetilde{I}_A = [I_A^L, I_A^U], \widetilde{F}_A = [F_A^L, F_A^U],$ *and scalar k is defined as*

$$
k_E A = \begin{pmatrix}
\left[\dfrac{(1+T_A^L)^k - (1-T_A^L)^k}{(1+T_A^L)^k + (1-T_A^L)^k}, \dfrac{(1+T_A^U)^k - (1-T_A^U)^k}{(1+T_A^U)^k + (1-T_A^U)^k} \right], \\[2ex]
\left[\dfrac{(1+I_A^L)^k - (1-I_A^L)^k}{(1+I_A^L)^k + (1-I_A^L)^k}, \dfrac{(1+I_A^U)^k - (1-I_A^U)^k}{(1+I_A^U)^k + (1-I_A^U)^k} \right], \\[2ex]
\left[\dfrac{2(F_A^L)^k}{(2-F_A^L)^k + (F_A^L)^k}, \dfrac{2(F_A^U)^k}{(2-F_A^U)^k + (F_A^U)^k} \right], \\[2ex]
\dfrac{2(T_A)^k}{(2-T_A)^k + (T_A)^k}, \dfrac{2(I_A)^k}{(2-I_A)^k + (I_A)^k}, \dfrac{(1+F_A)^k - (1-F_A)^k}{(1+F_A)^k + (1-F_A)^k}
\end{pmatrix}
$$

After defining the scalar multiplication over the neutrosophic cubic set, we established the following result, which deals with the Einstein exponential multiplication on neutrosophic cubic values. This result enabled us to define Einstein geometric aggregation operators along with some important results on neutrosophic cubic sets.

Theorem 2. *Let* $A = \left(\widetilde{T}_A, \widetilde{I}_A, \widetilde{F}_A, T_A, I_A, F_A \right)$, *where* $\widetilde{T}_A = [T_A^L, T_A^U]$, $\widetilde{I}_A = [I_A^L, I_A^U]$, $\widetilde{F}_A = [F_A^L, F_A^U]$, *be a neutrosophic cubic value, then the exponential operation defined by*

$$
A^{E^k} = \begin{pmatrix}
\left[\dfrac{2(T_A^L)^k}{2(-T_A^L)^k + (T_A^L)^k}, \dfrac{2(T_A^U)^k}{(2-T_A^U)^k + (T_A^U)^k} \right], \\[2ex]
\left[\dfrac{2(I_A^L)^k}{(2-I_A^L)^k + (I_A^L)^k}, \dfrac{2(I_A^U)^k}{(2-I_A^U)^k + (I_A^U)^k} \right], \\[2ex]
\left[\dfrac{(1+F_A^L)^k - (1-F_A^L)^k}{(1+F_A^L)^k + (1-F_A^L)^k}, \dfrac{(1+F_A^U)^k - (1-F_A^U)^k}{(1+F_A^U)^k + (1-F_A^U)^k} \right], \\[2ex]
\dfrac{(1+T_A)^k - (1-T_A)^k}{(1+T_A)^k + (1-T_A)^k}, \dfrac{(1+I_A)^k - (1-I_A)^k}{(1+I_A)^k + (1-I_A)^k}, \dfrac{2(F_A)^k}{(2-F_A)^k + (F_A)^k}
\end{pmatrix}
$$

where $A^{E^k} = A \otimes_E A \otimes_E \ldots \otimes_E A(k - times)$, *moreover* A^{E^k} *is a neutrosophic cubic value for every positive value of k.*

Proof. We prove the theorem by mathematical induction. For $k = 1$

$$
A^E = \begin{pmatrix}
\left[\dfrac{2(T_A^L)}{(2-T_A^L) + (T_A^L)}, \dfrac{2(T_A^U)}{(2-T_A^U) + (T_A^U)} \right], \\[2ex]
\left[\dfrac{2(I_A^L)}{(2-I_A^L) + (I_A^L)}, \dfrac{2(I_A^U)}{(2-I_A^U) + (I_A^U)} \right], \\[2ex]
\left[\dfrac{(1+F_A^L) - (1-F_A^L)}{(1+F_A^L) + (1-F_A^L)}, \dfrac{(1+F_A^U) - (1-F_A^U)}{(1+F_A^U) + (1-F_A^U)} \right], \\[2ex]
\dfrac{(1+T_A) - (1-T_A)}{(1+T_A) + (1-T_A)}, \dfrac{(1+I_A) - (1-I_A)}{(1+I_A) + (1-I_A)}, \dfrac{2(F_A)}{(2-F_A) + (F_A)}
\end{pmatrix}
$$

We observe that the components $T_A^L, T_A^U, I_A^L, I_A^U, F_A$ are of the form $\frac{2x}{(2-x)+x}$, and F_A^L, F_A^U, T_A, I_A are of the form $\frac{(1+y)-(1-y)}{(1+y)+(1-y)}$,

For all $x, y \in [0, 1]$, clearly $x = \frac{2x}{(2-x)+x}$ and $y = \frac{(1+y)-(1-y)}{(1+y)+(1-y)}$

Hence A^E is neutrosophic cubic value.

Assuming $k = m$ is a neutrosophic cubic value i.e.,

$$
A^{E^m} = \begin{pmatrix}
\left[\dfrac{2(T_A^L)^m}{(2-T_A^L)^m + (T_A^L)^m}, \dfrac{2(T_A^U)^m}{(2-T_A^U)^m + (T_A^U)^m} \right], \\[2ex]
\left[\dfrac{2(I_A^L)^m}{(2-I_A^L)^m + (I_A^L)^m}, \dfrac{2(I_A^U)^m}{(2-I_A^U)^m + (I_A^U)^m} \right], \\[2ex]
\left[\dfrac{(1+F_A^L)^m - (1-F_A^L)^m}{(1+F_A^L)^m + (1-F_A^L)^m}, \dfrac{(1+F_A^U)^m - (1-F_A^U)^m}{(1+F_A^U)^m + (1-F_A^U)^m} \right], \\[2ex]
\dfrac{(1+T_A)^m - (1-T_A)^m}{(1+T_A)^m + (1-T_A)^m}, \dfrac{(1+I_A)^m - (1-I_A)^m}{(1+I_A)^m + (1-I_A)^m}, \dfrac{2(F_A)^m}{(2-F_A)^m + (F_A)^m}
\end{pmatrix}
$$

is a neutrosophic cubic value. Then we prove $A^{E^{k+1}}$ is neutrosophic cubic value.

Consider,

$$A^{E^m} \otimes_E A^E = \left(\begin{array}{c} \left[\frac{2(T_A^L)^m}{(2-T_A^L)^m+(T_A^L)^m}, \frac{2(T_A^U)^m}{(2-T_A^U)^m+(T_A^U)^m} \right], \\ \left[\frac{2(I_A^L)^m}{(2-I_A^L)^m+(I_A^L)^m}, \frac{2(I_A^U)^m}{(2-I_A^U)^m+(I_A^U)^m} \right], \\ \left[\frac{(1+F_A^L)^m-(1-F_A^L)^m}{(1+F_A^L)^m+(1-F_A^L)^m}, \frac{(1+F_A^U)^m-(1-F_A^U)^m}{(1+F_A^U)^m+(1-F_A^U)^m} \right], \\ \frac{(1+T_A)^m-(1-T_A)^m}{(1+T_A)^m+(1-T_A)^m}, \frac{(1+I_A)^m-(1-I_A)^m}{(1+I_A)^m+(1-I_A)^m}, \frac{2(F_A)^m}{(2-F_A)^m+(F_A)^m} \end{array} \right) \otimes_E \left(\begin{array}{c} \left[\frac{2(T_A^L)}{(2-T_A^L)+(T_A^L)}, \frac{2(T_A^U)}{(2-T_A^U)+(T_A^U)} \right], \\ \left[\frac{2(I_A^L)^1}{(2-I_A^L)+(I_A^L)}, \frac{2(I_A^U)^1}{(2-I_A^U)+(I_A^U)} \right], \\ \left[\frac{(1+F_A^L)-(1-F_A^L)}{(1+F_A^L)+(1-F_A^L)}, \frac{(1+F_A^U)-(1-F_A^U)}{(1+F_A^U)+(1-F_A^U)} \right], \\ \frac{(1+T_A)-(1-T_A)}{(1+T_A)+(1-T_A)}, \frac{(1+I_A)-(1-I_A)}{(1+I_A)+(1-I_A)}, \frac{2(F_A)}{(2-F_A)+(F_A)} \end{array} \right)$$

$$= \left(\begin{array}{c} \left[\frac{\frac{4\left(T_A^L\right)^{m+1}}{\left((2-T_A^L)^m+\left(T_A^L\right)^m\right)\left((2-T_A^L)+T_A^L\right)}}{1+\left(1-\frac{2\left(T_A^L\right)^m}{(2-T_A^L)^m+\left(T_A^L\right)^m}\right)\left(1-\frac{2T_A^L}{(2-T_A^L)+T_A^L}\right)}, \frac{\frac{4\left(T_A^U\right)^{m+1}}{\left((2-T_A^U)^m+\left(T_A^U\right)^m\right)\left((2-T_A^U)+T_A^U\right)}}{1+\left(1-\frac{2\left(T_A^U\right)^m}{(2-T_A^U)^m+\left(T_A^U\right)^m}\right)\left(1-\frac{2T_A^U}{(2-T_A^U)+T_A^U}\right)} \right], \\[20pt] \left[\frac{\frac{4\left(I_A^L\right)^{m+1}}{\left((2-I_A^L)^m+\left(I_A^L\right)^m\right)\left((2-I_A^L)+I_A^L\right)}}{1+\left(1-\frac{2\left(I_A^L\right)^m}{(2-I_A^L)^m+\left(I_A^L\right)^m}\right)\left(1-\frac{2I_A^L}{(2-I_A^L)+I_A^L}\right)}, \frac{\frac{4\left(I_A^U\right)^{m+1}}{\left((2-I_A^U)^m+\left(I_A^U\right)^m\right)\left((2-I_A^U)+I_A^U\right)}}{1+\left(1-\frac{2\left(I_A^U\right)^m}{(2-I_A^U)^m+\left(I_A^U\right)^m}\right)\left(1-\frac{2I_A^U}{(2-I_A^U)+I_A^U}\right)} \right], \\[20pt] \left[\frac{\frac{(1+F_A^L)^m-(1-F_A^L)^m}{(1+F_A^L)^m+(1-F_A^L)^m}+\left(\frac{(1+F_A^L)-(1-F_A^L)}{(1+F_A^L)+(1-F_A^L)}\right)}{1+\left(\frac{(1+F_A^L)^m-(1-F_A^L)^m}{(1+F_A^L)^m+(1-F_A^L)^m}\right)\left(\frac{(1+F_A^L)-(1-F_A^L)}{(1+F_A^L)+(1-F_A^L)}\right)}, \frac{\frac{(1+F_A^U)^m-(1-F_A^U)^m}{(1+F_A^U)^m+(1-F_A^U)^m}+\left(\frac{(1+F_A^U)-(1-F_A^U)}{(1+F_A^U)+(1-F_A^U)}\right)}{1+\left(\frac{(1+F_A^U)^m-(1-F_A^U)^m}{(1+F_A^U)^m+(1-F_A^U)^m}\right)\left(\frac{(1+F_A^U)-(1-F_A^U)}{(1+F_A^U)+(1-F_A^U)}\right)} \right], \\[20pt] \frac{\frac{(1+T_A)^m-(1-T_A)^m}{(1+T_A)^m+(1-T_A)^m}+\left(\frac{(1+T_A)-(1-T_A)}{(1+T_A)+(1-T_A)}\right)}{1+\left(\frac{(1+T_A)^m-(1-T_A)^m}{(1+T_A)^m+(1-T_A)^m}\right)\left(\frac{(1+T_A)-(1-T_A)}{(1+T_A)+(1-T_A)}\right)}, \frac{\frac{(1+I_A)^m-(1-I_A)^m}{(1+I_A)^m+(1-I_A)^m}+\left(\frac{(1+I_A)-(1-I_A)}{(1+I_A)+(1-I_A)}\right)}{1+\left(\frac{(1+I_A)^m-(1-I_A)^m}{(1+I_A)^m+(1-I_A)^m}\right)\left(\frac{(1+I_A)-(1-I_A)}{(1+I_A)+(1-I_A)}\right)}, \\[20pt] \frac{\frac{4(F_A)^{m+1}}{\left((2-F_A)^m+(F_A)^m\right)\left((2-F_A)+F_A\right)}}{1+\left(1-\frac{2(F_A)^m}{(2-F_A)^m+(F_A)^m}\right)\left(1-\frac{2F_A}{(2-F_A)+F_A}\right)} \end{array} \right)$$

$$= \left(\begin{array}{c} \left[\frac{\frac{4\left(T_A^L\right)^{m+1}}{\left((2-T_A^L)^m+\left(T_A^L\right)^m\right)\left((2-T_A^L)+T_A^L\right)}}{1+\left(\frac{(2-T_A^L)^m+\left(T_A^L\right)^m-2\left(T_A^L\right)^m}{(2-T_A^L)^m+\left(T_A^L\right)^m}\right)\left(\frac{(2-T_A^L)+T_A^L-2T_A^L}{(2-T_A^L)+T_A^L}\right)}, \frac{\frac{4\left(T_A^U\right)^{m+1}}{\left((2-T_A^U)^m+\left(T_A^U\right)^m\right)\left((2-T_A^U)+T_A^U\right)}}{1+\left(\frac{(2-T_A^U)^m+\left(T_A^U\right)^m-2\left(T_A^U\right)^m}{(2-T_A^U)^m+\left(T_A^U\right)^m}\right)\left(\frac{(2-T_A^U)+T_A^U-2T_A^U}{(2-T_A^U)+T_A^U}\right)} \right], \\[20pt] \left[\frac{\frac{4\left(I_A^L\right)^{m+1}}{\left((2-I_A^L)^m+\left(I_A^L\right)^m\right)\left((2-I_A^L)+I_A^L\right)}}{1+\left(\frac{(2-I_A^L)^m+\left(I_A^L\right)^m-2\left(I_A^L\right)^m}{(2-I_A^L)^m+\left(I_A^L\right)^m}\right)\left(\frac{(2-I_A^L)+I_A^L-2I_A^L}{(2-I_A^L)+I_A^L}\right)}, \frac{\frac{4\left(I_A^U\right)^{m+1}}{\left((2-I_A^U)^m+\left(I_A^U\right)^m\right)\left((2-I_A^U)+I_A^U\right)}}{1+\left(\frac{(2-I_A^U)^m+\left(I_A^U\right)^m-2\left(I_A^U\right)^m}{(2-I_A^U)^m+\left(I_A^U\right)^m}\right)\left(\frac{(2-I_A^U)+I_A^U-2I_A^U}{(2-I_A^U)+I_A^U}\right)} \right], \\[20pt] \left[\frac{\frac{\left(\left(1+F_A^L\right)^m-\left(1-F_A^L\right)^m\right)\left(\left(1+F_A^L\right)+\left(1-F_A^L\right)\right)+\left(\left(1+F_A^L\right)^m+\left(1-F_A^L\right)^m\right)\left(\left(1+F_A^L\right)-\left(1-F_A^L\right)\right)}{\left(1+F_A^L\right)^m+(1-F_A^L)^m\right)\left((1+F_A^L)+(1-F_A^L)\right)}}{\frac{\left(\left(1+F_A^L\right)^m+\left(1-F_A^L\right)^m\right)\left(\left(1+F_A^L\right)+\left(1-F_A^L\right)\right)+\left(1+F_A^L\right)^{m+1}-\left(1+F_A^L\right)^m\left(1-F_A^L\right)-\left(1-F_A^L\right)^m\left(1+F_A^L\right)+\left(1-F_A^L\right)^{m+1}}{\left(1+F_A^L\right)^m+(1-F_A^L)^m\right)\left((1+F_A^L)+(1-F_A^L)\right)}}, \\[20pt] \frac{\frac{\left(\left(1+F_A^U\right)^m-\left(1-F_A^U\right)^m\right)\left(\left(1+F_A^U\right)+\left(1-F_A^U\right)\right)+\left(\left(1+F_A^U\right)^m+\left(1-F_A^U\right)^m\right)\left(\left(1+F_A^U\right)-\left(1-F_A^U\right)\right)}{\left(1+F_A^U\right)^m+(1-F_A^U)^m\right)\left((1+F_A^U)+(1-F_A^U)\right)}}{\frac{\left(\left(1+F_A^U\right)^m+\left(1-F_A^U\right)^m\right)\left(\left(1+F_A^U\right)+\left(1-F_A^U\right)\right)+\left(1+F_A^U\right)^{m+1}-\left(1+F_A^U\right)^m\left(1-F_A^U\right)-\left(1-F_A^U\right)^m\left(1+F_A^U\right)+\left(1-F_A^U\right)^{m+1}}{\left(1+F_A^U\right)^m+(1-F_A^U)^m\right)\left((1+F_A^U)+(1-F_A^U)\right)}} \right], \\[20pt] \frac{\frac{\left(\left(1+T_A\right)^m-\left(1-T_A\right)^m\right)\left(\left(1+T_A\right)+\left(1-T_A\right)\right)+\left(\left(1+T_A\right)^m+\left(1-T_A\right)^m\right)\left(\left(1+T_A\right)-\left(1-T_A\right)\right)}{\left(1+T_A\right)^m+(1-T_A)^m\right)\left((1+T_A)+(1-T_A)\right)}}{\frac{\left(\left(1+T_A\right)^m+\left(1-T_A\right)^m\right)\left(\left(1+T_A\right)+\left(1-T_A\right)\right)+\left(1+T_A\right)^{m+1}-\left(1+T_A\right)^m\left(1-T_A\right)-\left(1-T_A\right)^m\left(1+T_A\right)+\left(1-T_A\right)^{m+1}}{\left(1+T_A\right)^m+(1-T_A)^m\right)\left((1+T_A)+(1-T_A)\right)}}, \\[20pt] \frac{\frac{\left(\left(1+I_A\right)^m-\left(1-I_A\right)^m\right)\left(\left(1+I_A\right)+\left(1-I_A\right)\right)+\left(\left(1+I_A\right)^m+\left(1-I_A\right)^m\right)\left(\left(1+I_A\right)-\left(1-I_A\right)\right)}{\left(1+I_A\right)^m+(1-I_A)^m\right)\left((1+I_A)+(1-I_A)\right)}}{\frac{\left(\left(1+I_A\right)^m+\left(1-I_A\right)^m\right)\left(\left(1+I_A\right)+\left(1-I_A\right)\right)+\left(1+I_A\right)^{m+1}-\left(1+I_A\right)^m\left(1-I_A\right)-\left(1-I_A\right)^m\left(1+I_A\right)+\left(1-I_A\right)^{m+1}}{\left(1+I_A\right)^m+(1-I_A)^m\right)\left((1+I_A)+(1-I_A)\right)}}, \\[20pt] \frac{\frac{4(F_A)^{m+1}}{\left((2-F_A)^m+(F_A)^m\right)\left((2-F_A)+F_A\right)}}{1+\left(\frac{(2-F_A)^m+(F_A)^m-2(F_A)^m}{(2-F_A)^m+(F_A)^m}\right)\left(\frac{(2-F_A)+F_A-2F_A}{(2-F_A)+F_A}\right)} \end{array} \right)$$

$$
\left(
\begin{bmatrix}
\dfrac{\dfrac{4\left(T_A^L\right)^{m+1}}{\left(\left(2-T_A^L\right)^m+\left(T_A^L\right)^m\right)\left(\left(2-T_A^L\right)+T_A^L\right)}}{\dfrac{\left(\left(\left(2-T_A^L\right)^m+\left(T_A^L\right)^m\right)\left(\left(2-T_A^L\right)+T_A^L\right)\right)+\left(\left(\left(2-T_A^L\right)^m-\left(T_A^L\right)^m\right)\left(\left(2-T_A^L\right)-T_A^L\right)\right)}{\left(\left(2-T_A^L\right)^m+\left(T_A^L\right)^m\right)\left(\left(2-T_A^L\right)+T_A^L\right)}}, \\[2em]
\dfrac{\dfrac{4\left(T_A^U\right)^{m+1}}{\left(\left(2-T_A^U\right)^m+\left(T_A^U\right)^m\right)\left(\left(2-T_A^U\right)+T_A^U\right)}}{\dfrac{\left(\left(\left(2-T_A^U\right)^m+\left(T_A^U\right)^m\right)\left(\left(2-T_A^U\right)+T_A^U\right)\right)+\left(\left(\left(2-T_A^U\right)^m-\left(T_A^U\right)^m\right)\left(\left(2-T_A^U\right)-T_A^U\right)\right)}{\left(\left(2-T_A^U\right)^m+\left(T_A^U\right)^m\right)\left(\left(2-T_A^U\right)+T_A^U\right)}}
\end{bmatrix},
\right.
$$

$$
\begin{bmatrix}
\dfrac{\dfrac{4\left(I_A^L\right)^{m+1}}{\left(\left(2-I_A^L\right)^m+\left(I_A^L\right)^m\right)\left(\left(2-I_A^L\right)+I_A^L\right)}}{\dfrac{\left(\left(\left(2-I_A^L\right)^m+\left(I_A^L\right)^m\right)\left(\left(2-I_A^L\right)+I_A^L\right)\right)+\left(\left(\left(2-I_A^L\right)^m-\left(I_A^L\right)^m\right)\left(\left(2-I_A^L\right)-I_A^L\right)\right)}{\left(\left(2-I_A^L\right)^m+\left(I_A^L\right)^m\right)\left(\left(2-I_A^L\right)+I_A^L\right)}}, \\[2em]
\dfrac{\dfrac{4\left(I_A^U\right)^{m+1}}{\left(\left(2-I_A^U\right)^m+\left(I_A^U\right)^m\right)\left(\left(2-I_A^U\right)+I_A^U\right)}}{\dfrac{\left(\left(\left(2-I_A^U\right)^m+\left(I_A^U\right)^m\right)\left(\left(2-I_A^U\right)+I_A^U\right)\right)+\left(\left(\left(2-I_A^U\right)^m-\left(I_A^U\right)^m\right)\left(\left(2-I_A^U\right)-I_A^U\right)\right)}{\left(\left(2-I_A^U\right)^m+\left(I_A^U\right)^m\right)\left(\left(2-I_A^U\right)+I_A^U\right)}}
\end{bmatrix},
$$

$$
= \quad
\begin{bmatrix}
\dfrac{\begin{array}{l}\left(1+F_A^L\right)^{m+1}+\left(1+F_A^L\right)^m\left(1-F_A^L\right)-\left(1-F_A^L\right)^m\left(1+F_A^L\right)-\left(1-F_A^L\right)^{m+1}+\\\left(1+F_A^L\right)^{m+1}-\left(1+F_A^L\right)^m\left(1-F_A^L\right)+\left(1-F_A^L\right)^m\left(1+F_A^L\right)-\left(1-F_A^L\right)^{m+1}\end{array}}{\begin{array}{l}\left(1+F_A^L\right)^{m+1}+\left(1+F_A^L\right)^m\left(1-F_A^L\right)-\left(1-F_A^L\right)^m\left(1+F_A^L\right)+\left(1-F_A^L\right)^{m+1}+\\\left(1+F_A^L\right)^{m+1}-\left(1+F_A^L\right)^m\left(1-F_A^L\right)+\left(1-F_A^L\right)^m\left(1+F_A^L\right)+\left(1-F_A^L\right)^{m+1}\end{array}}, \\[3em]
\dfrac{\begin{array}{l}\left(1+F_A^U\right)^{m+1}+\left(1+F_A^U\right)^m\left(1-F_A^U\right)-\left(1-F_A^U\right)^m\left(1+F_A^U\right)-\left(1-F_A^U\right)^{m+1}+\\\left(1+F_A^U\right)^{m+1}-\left(1+F_A^U\right)^m\left(1-F_A^U\right)+\left(1-F_A^U\right)^m\left(1+F_A^U\right)-\left(1-F_A^U\right)^{m+1}\end{array}}{\begin{array}{l}\left(1+F_A^U\right)^{m+1}+\left(1+F_A^U\right)^m\left(1-F_A^U\right)-\left(1-F_A^U\right)^m\left(1+F_A^U\right)+\left(1-F_A^U\right)^{m+1}+\\\left(1+F_A^U\right)^{m+1}-\left(1+F_A^U\right)^m\left(1-F_A^U\right)+\left(1-F_A^U\right)^m\left(1+F_A^U\right)+\left(1-F_A^U\right)^{m+1}\end{array}}
\end{bmatrix},
$$

$$
\dfrac{\begin{array}{l}\left(1+T_A\right)^{m+1}+\left(1+T_A\right)^m\left(1-T_A\right)-\left(1-T_A\right)^m\left(1+T_A\right)-\left(1-T_A\right)^{m+1}+\\\left(1+T_A\right)^{m+1}-\left(1+T_A\right)^m\left(1-T_A\right)+\left(1-T_A\right)^m\left(1+T_A\right)-\left(1-T_A\right)^{m+1}\end{array}}{\begin{array}{l}\left(1+T_A\right)^{m+1}+\left(1+T_A\right)^m\left(1-T_A\right)-\left(1-T_A\right)^m\left(1+T_A\right)+\left(1-T_A\right)^{m+1}+\\\left(1+T_A\right)^{m+1}-\left(1+T_A\right)^m\left(1-T_A\right)+\left(1-T_A\right)^m\left(1+T_A\right)+\left(1-T_A\right)^{m+1}\end{array}},
$$

$$
\dfrac{\begin{array}{l}\left(1+I_A\right)^{m+1}+\left(1+I_A\right)^m\left(1-I_A\right)-\left(1-I_A\right)^m\left(1+I_A\right)-\left(1-I_A\right)^{m+1}+\\\left(1+I_A\right)^{m+1}-\left(1+I_A\right)^m\left(1-I_A\right)+\left(1-I_A\right)^m\left(1+I_A\right)-\left(1-I_A\right)^{m+1}\end{array}}{\begin{array}{l}\left(1+I_A\right)^{m+1}+\left(1+I_A\right)^m\left(1-I_A\right)-\left(1-I_A\right)^m\left(1+I_A\right)+\left(1-I_A\right)^{m+1}+\\\left(1+I_A\right)^{m+1}-\left(1+I_A\right)^m\left(1-I_A\right)+\left(1-I_A\right)^m\left(1+I_A\right)+\left(1-I_A\right)^{m+1}\end{array}},
$$

$$
\left.
\dfrac{\dfrac{4\left(F_A\right)^{m+1}}{\left(\left(2-F_A\right)^m+\left(F_A\right)^m\right)\left(\left(2-F_A\right)+F_A\right)}}{\dfrac{\left(\left(\left(2-F_A\right)^m+\left(F_A\right)^m\right)\left(\left(2-F_A\right)+F_A\right)\right)+\left(\left(\left(2-F_A\right)^m-\left(F_A\right)^m\right)\left(\left(2-F_A\right)-F_A\right)\right)}{\left(\left(2-F_A\right)^m+\left(F_A\right)^m\right)\left(\left(2-F_A\right)+F_A\right)}}
\right)
$$

$$
= \quad
\left(
\begin{bmatrix}
\dfrac{4\left(T_A^L\right)^{m+1}}{\left(2-T_A^L\right)^{m+1}+T_A^L\left(2-T_A^L\right)^m+\left(T_A^L\right)^{m+1}+T_A^{Lm}\left(2-T_A^L\right)+\left(\left(2-T_A^L\right)^{m+1}-T_A^L\left(2-T_A^L\right)^m+\left(T_A^L\right)^{m+1}-\left(T_A^L\right)^m\left(2-T_A^L\right)\right)}, \\[2em]
\dfrac{4\left(T_A^U\right)^{m+1}}{\left(2-T_A^U\right)^{m+1}+T_A^U\left(2-T_A^U\right)^m+\left(T_A^U\right)^{m+1}+T_A^{Um}\left(2-T_A^U\right)+\left(\left(2-T_A^U\right)^{m+1}-T_A^U\left(2-T_A^U\right)^m+\left(T_A^U\right)^{m+1}-\left(T_A^U\right)^m\left(2-T_A^U\right)\right)}
\end{bmatrix},
\right.
$$

$$
\begin{bmatrix}
\dfrac{4\left(I_A^L\right)^{m+1}}{\left(2-I_A^L\right)^{m+1}+I_A^L\left(2-I_A^L\right)^m+\left(I_A^L\right)^{m+1}+I_A^{Lm}\left(2-I_A^L\right)+\left(\left(2-I_A^L\right)^{m+1}-I_A^L\left(2-I_A^L\right)^m+\left(I_A^L\right)^{m+1}-\left(I_A^L\right)^m\left(2-I_A^L\right)\right)}, \\[2em]
\dfrac{4\left(I_A^U\right)^{m+1}}{\left(2-I_A^U\right)^{m+1}+I_A^U\left(2-I_A^U\right)^m+\left(I_A^U\right)^{m+1}+I_A^{Um}\left(2-I_A^U\right)+\left(\left(2-I_A^U\right)^{m+1}-I_A^U\left(2-I_A^U\right)^m+\left(I_A^U\right)^{m+1}-\left(I_A^U\right)^m\left(2-I_A^U\right)\right)}
\end{bmatrix},
$$

$$
\begin{bmatrix}
\dfrac{2\left(\left(1+F_A^L\right)^{m+1}-\left(1-F_A^L\right)^{m+1}\right)}{2\left(\left(1+F_A^L\right)^{m+1}+\left(1-F_A^L\right)^{m+1}\right)}, \dfrac{2\left(\left(1+F_A^U\right)^{m+1}-\left(1-F_A^U\right)^{m+1}\right)}{2\left(\left(1+F_A^U\right)^{m+1}+\left(1-F_A^U\right)^{m+1}\right)}
\end{bmatrix},
$$

$$
\dfrac{2\left(\left(1+T_A\right)^{m+1}-\left(1-T_A\right)^{m+1}\right)}{2\left(\left(1+T_A\right)^{m+1}+\left(1-T_A\right)^{m+1}\right)},
$$

$$
\dfrac{2\left(\left(1+I_A\right)^{m+1}-\left(1-I_A\right)^{m+1}\right)}{2\left(\left(1+I_A\right)^{m+1}+\left(1-I_A\right)^{m+1}\right)},
$$

$$
\left.
\dfrac{4\left(F_A\right)^{m+1}}{\left(2-F_A\right)^{m+1}+F_A\left(2-F_A\right)^m+\left(F_A\right)^{m+1}+F_A^m\left(2-F_A\right)+\left(\left(2-F_A\right)^{m+1}-F_A\left(2-F_A\right)^m+\left(F_A\right)^{m+1}-\left(F_A\right)^m\left(2-F_A\right)\right)}
\right)
$$

$$= \left(\begin{array}{c} \left[\dfrac{4\left(T_A^L\right)^{m+1}}{2\left(\left(2-T_A^L\right)^{m+1}+\left(T_A^L\right)^{m+1}\right)}, \dfrac{4\left(T_A^U\right)^{m+1}}{2\left(\left(2-T_A^U\right)^{m+1}+\left(T_A^U\right)^{m+1}\right)} \right], \\[4mm] \left[\dfrac{4\left(I_A^L\right)^{m+1}}{2\left(\left(2-I_A^L\right)^{m+1}+\left(I_A^L\right)^{m+1}\right)}, \dfrac{4\left(I_A^U\right)^{m+1}}{2\left(\left(2-I_A^U\right)^{m+1}+\left(I_A^U\right)^{m+1}\right)} \right], \\[4mm] \left[\dfrac{\left(\left(1+F_A^L\right)^{m+1}-\left(1-F_A^L\right)^{m+1}\right)}{\left(\left(1+F_A^L\right)^{m+1}+\left(1-F_A^L\right)^{m+1}\right)}, \dfrac{\left(\left(1+F_A^U\right)^{m+1}-\left(1-F_A^U\right)^{m+1}\right)}{\left(\left(1+F_A^U\right)^{m+1}+\left(1-F_A^U\right)^{m+1}\right)} \right], \\[4mm] \dfrac{\left(\left(1+T_A\right)^{m+1}-\left(1-T_A\right)^{m+1}\right)}{\left(\left(1+T_A\right)^{m+1}+\left(1-T_A\right)^{m+1}\right)}, \\[4mm] \dfrac{\left(\left(1+I_A\right)^{m+1}-\left(1-I_A\right)^{m+1}\right)}{\left(\left(1+I_A\right)^{m+1}+\left(1-I_A\right)^{m+1}\right)}, \\[4mm] \dfrac{4\left(F_A\right)^{m+1}}{2\left(\left(2-F_A\right)^{m+1}+\left(F_A\right)^{m+1}\right)} \end{array} \right)$$

$$= \left(\begin{array}{c} \left[\dfrac{2\left(T_A^L\right)^{m+1}}{\left(\left(2-T_A^L\right)^{m+1}+\left(T_A^L\right)^{m+1}\right)}, \dfrac{2\left(T_A^U\right)^{m+1}}{\left(\left(2-T_A^U\right)^{m+1}+\left(T_A^U\right)^{m+1}\right)} \right], \\[4mm] \left[\dfrac{2\left(I_A^L\right)^{m+1}}{\left(\left(2-I_A^L\right)^{m+1}+\left(I_A^L\right)^{m+1}\right)}, \dfrac{2\left(I_A^U\right)^{m+1}}{\left(\left(2-I_A^U\right)^{m+1}+\left(I_A^U\right)^{m+1}\right)} \right], \\[4mm] \left[\dfrac{\left(\left(1+F_A^L\right)^{m+1}-\left(1-F_A^L\right)^{m+1}\right)}{\left(\left(1+F_A^L\right)^{m+1}+\left(1-F_A^L\right)^{m+1}\right)}, \dfrac{\left(\left(1+F_A^U\right)^{m+1}-\left(1-F_A^U\right)^{m+1}\right)}{\left(\left(1+F_A^U\right)^{m+1}+\left(1-F_A^U\right)^{m+1}\right)} \right], \\[4mm] \dfrac{\left(\left(1+T_A\right)^{m+1}-\left(1-T_A\right)^{m+1}\right)}{\left(\left(1+T_A\right)^{m+1}+\left(1-T_A\right)^{m+1}\right)}, \\[4mm] \dfrac{\left(\left(1+I_A\right)^{m+1}-\left(1-I_A\right)^{m+1}\right)}{\left(\left(1+I_A\right)^{m+1}+\left(1-I_A\right)^{m+1}\right)}, \\[4mm] \dfrac{2\left(F_A\right)^{m+1}}{\left(\left(2-F_A\right)^{m+1}+\left(F_A\right)^{m+1}\right)} \end{array} \right)$$

Which shows that $k = m + 1$ is a neutrosophic cubic value. □

3.3. Score and Accuracy Function of Neutrosophic Cubic Set

For the comparison of two neutrosophic values, the score and accuracy function are defined. The score function is used tocompare two neutrosophic cubic values; sometimes the score of two neutrosophic cubic values becomes equal, although they have different components of truth, indeterminancy and falsity functions. This situation can be overcome by the help of an accuracy function. The following definition, along with examples, providesa better view of understanding to the reader.

Definition 18. *Let* $N = \left(\widetilde{T}_N, \widetilde{I}_N, \widetilde{F}_N, T_N, I_N, F_N \right)$, *where* $\widetilde{T}_N = \left[T_N^L, T_N^U \right], \widetilde{I}_N = \left[I_N^L, I_N^U \right], \widetilde{F}_N = \left[F_N^L, F_N^U \right]$, *be a neutrosophic cubic value and we define the score function as*

$$S(N) = \left[T_N^L - F_N^L + T_N^U - F_N^U + T_N - F_N \right]$$

Sometimes the situation arises that the score of two neutrosophic cubic values are equal. In such a situation, a comparison is made on the basis of an accuracy function.

Definition 19. *Let* $N = \left(\widetilde{T}_N, \widetilde{I}_N, \widetilde{F}_N, T_N, I_N, F_N \right)$, *where* $\widetilde{T}_N = \left[T_N^L, T_N^U \right], \widetilde{I}_N = \left[I_N^L, I_N^U \right], \widetilde{F}_N = \left[F_N^L, F_N^U \right]$, *be a neutrosophic cubic value, the accuracy function is defined as*

$$H(u) = \frac{1}{9} \left\{ T_N^L + I_N^L + F_N^L + T_N^U + I_N^U + F_N^U + T_N + I_N + F_N \right\}$$

The following definition is accomplished for the comparison relation of the neutrosophic cubic values.

Definition 20. *Let N_1 and N_2 be two neutrosophic cubic values, where S_{N_1} and S_{N_2} are scores and H_{N_1} and H_{N_2} are accuracy functions of N_1 and N_2, respectively.*

1.　*If $S_{N_1} > S_{N_2} \Rightarrow N_1 > N_2$*
2.　*If $S_{N_1} = S_{N_2}$ and $H_{N_1} > H_{N_2} \Rightarrow N_1 > N_2$ $H_{N_1} = H_{N_2} \Rightarrow N_1 = N_2$*

Example 1.　　*Let $N_1 = ([0.5, 0.9][0.6, 0.9][0.1, 0.4], 0.3, 0.4, 0.4)$ and $N_2 = ([0.2, 0.8][0.5, 0.9][0.4, 0.8], 0.4, 0.45, 0.8)$ be two neutrosophic sets.*
　　Then

$$S_{N_1} = 0.8, and \, S_{N_2} = -0.6$$

$$S_{N_1} > S_{N_2} \Rightarrow N_1 > N_2$$

In the following example the score funtions are equal, so accuracy functions are used to compare neutrosophic cubic values.

Example 2.　　*Let $N_1 = ([0.4, 0.9][0.5, 0.8][0.1, 0.7], 0.4, 0.5, 0.8)$ and $N_2 = ([0.4, 0.6][0.5, 0.9][0.6, 0.7], 0.7, 0.5, 0.3)$ be two neutrosophic sets.*

$$S_{N_1} = 0.1, S_{N_2} = 0.1$$

$$S_{N_1} = S_{N_2} \Rightarrow N_1 = N_2$$

$$H_{N_1} = 0.566, \, H_{N_2} = 0.577$$

$$H_{N_1} < H_{N_2} \Rightarrow N_1 < N_2$$

4. Neutrosophic Cubic Geometric and Einstein Geometric Aggregation Operators

In this section, we introduce the concept of neutrosophic cubic geometric aggregation operators and neutrosophic cubic Einstein geometric aggregation operators.

This section consists of two sub-sections: In Section 4.1, the neutrosophic cubic geometric aggregation operators are defined on the basis of Section 3.1; and in Section 4.2, the neutrosophic cubic Einstein geometric aggregation operators are defined on the basis of Section 3.2.

4.1. Neutrosophic Cubic Weighted Geometric Aggregation Operator

We define neutrosophic cubic geometric aggregation operators using Section 3.1.

Definition 21. *We define the neutrosophic cubic weighted geometric operator(NCWG) as*

$$NCWG : R^m \rightarrow R \text{ defined by } NCWG_w(N_1, N_2, \ldots, N_m) = \overset{m}{\underset{j=1}{\otimes}} N_j^{w_j}$$

where the weight $W = (w_1, w_2, \ldots, w_m)^T$ of corresponding neutrosophic cubic values is such that each $w_j \in [0, 1]$ and $\sum\limits_{j=1}^{m} w_j = 1$.

In NCWG, the neutrosophic cubic values are first weighted then aggregated.

Definition 22. *We define the neutrosophic cubic ordered weighted geometric operator(NCOWG) as*

$$NCOWG : R^m \rightarrow R \text{ defined by } NCOWG_w(N_1, N_2, \ldots, N_m) = \overset{m}{\underset{j=1}{\otimes}} N_{(\gamma)_j}^{w_j}$$

where $N_{(\gamma)_j}$ are descending ordered neutrosophic cubic values, and the weight $W = (w_1, w_2, \ldots, w_m)^T$ of corresponding neutrosophic cubic values $N_j(j = 1, 2, 3, \ldots, m)$ is such that each $w_j \in [0, 1]$ and $\sum_{j=1}^{m} w_j = 1$.

In NCOWG, the neutrosophic cubic values are first arranged in decending order, weighted and then aggregated.

Theorem 3. *Let $N_j = \left(\widetilde{T}_{N_j}, \widetilde{I}_{N_j}, \widetilde{F}_{N_j}, T_{N_j}, I_{N_j}, F_{N_j} \right)$, where $\widetilde{T}_{N_j} = \left[T_{N_j}^L, T_{N_j}^U \right], \widetilde{I}_{N_j} = \left[I_{N_j}^L, I_{N_j}^U \right], \widetilde{F}_{N_j} = \left[F_{N_j}^L, F_{N_j}^U \right] (j = 1, 2, \ldots, n)$ are a collection of neutrosophic cubic values, then neutrosophic cubic weighted geometric(NCWG) operator of N_j is also a neutrosophic cubic value and*

$$NCWG(N_j) = \begin{pmatrix} \left[\prod_{j=1}^{m} \left(T_{N_j}^L \right)^{w_j}, \prod_{j=1}^{m} \left(T_{N_j}^U \right)^{w_j} \right], \\ \left[\prod_{j=1}^{m} \left(I_{N_j}^L \right)^{w_j}, \prod_{j=1}^{m} \left(I_{N_j}^U \right)^{w_j} \right] \\ \left[1 - \prod_{j=1}^{m} \left(1 - F_{N_j}^L \right)^{w_j}, 1 - \prod_{j=1}^{m} \left(1 - F_{N_j}^U \right)^{w_j} \right] \\ 1 - \prod_{j=1}^{m} \left(1 - \left(T_{N_j} \right) \right)^{w_j}, 1 - \prod_{j=1}^{m} \left(1 - \left(I_{N_j} \right) \right)^{w_j}, \prod_{j=1}^{m} \left(F_{N_j} \right)^{w_j} \end{pmatrix}$$

where the weight $W = (w_1, w_2, \ldots, w_m)^T$ of $N_j(j = 1, 2, 3, \ldots, m)$ such that $w_j \in [0, 1]$ and $\sum_{j=1}^{m} w_j = 1$.

Proof. By mathematical induction for $m = 2$, using

$$\overset{2}{\underset{j=1}{\otimes}} N_j^{w_j} = N_1^{w_1} \otimes N_2^{w_2}$$

$$= \begin{pmatrix} \left[(T_{N_j}^L)^{w_1}, (T_{N_j}^U)^{w_1} \right], \\ \left[(I_{N_j}^L)^{w_1}, (I_{N_j}^U)^{w_1} \right], \\ \left[1 - \left(1 - F_{N_j}^L \right)^{w_1}, 1 - \left(1 - F_{N_j}^U \right)^{w_1} \right], \\ 1 - \left(1 - \left(T_{N_j} \right) \right)^{w_1}, 1 - \left(1 - \left(I_{N_j} \right) \right)^{w_1}, \left(F_{N_j} \right)^{w_1} \end{pmatrix} \otimes \begin{pmatrix} \left[(T_{N_j}^L)^{w_2}, (T_{N_j}^U)^{w_2} \right], \\ \left[(I_{N_j}^L)^{w_2}, (I_{N_j}^U)^{w_2} \right], \\ \left[1 - \left(1 - F_{N_j}^L \right)^{w_2}, 1 - \left(1 - F_{N_j}^U \right)^{w_2} \right], \\ 1 - \left(1 - \left(T_{N_j} \right) \right)^{w_2}, 1 - \left(1 - \left(I_{N_j} \right) \right)^{w_2}, \left(F_{N_j} \right)^{w_2} \end{pmatrix}$$

$$= \begin{pmatrix} \left[\prod_{j=1}^{2} (T_{N_j}^L)^{w_j}, \prod_{j=1}^{2} (T_{N_j}^U)^{w_j} \right], \\ \left[\prod_{j=1}^{2} (I_{N_j}^L)^{w_j}, \prod_{j=1}^{2} (I_{N_j}^U)^{w_j} \right], \\ \left[1 - \prod_{j=1}^{2} \left(1 - F_{N_j}^L \right)^{w_j}, \prod_{j=1}^{2} \left(F_{N_j}^U \right)^{w_j} \right], \\ 1 - \prod_{j=1}^{2} \left(1 - T_{N_j} \right)^{w_j}, 1 - \prod_{j=1}^{2} \left(1 - I_{N_j} \right)^{w_j}, \prod_{j=1}^{2} \left(F_{N_j} \right)^{w_j} \end{pmatrix}$$

For $m = n$, we have

$$
\overset{n}{\underset{j=1}{\otimes}} N_j^{w_j} = \left(
\begin{array}{c}
\left[\prod_{j=1}^{n} (T_{N_j}^L)^{w_j}, \prod_{j=1}^{n} (T_{N_j}^U)^{w_j}\right], \\[2mm]
\left[\prod_{j=1}^{n} (I_{N_j}^L)^{w_j}, \prod_{j=1}^{n} (I_{N_j}^U)^{w_j}\right], \\[2mm]
\left[1 - \prod_{j=1}^{n} \left(1 - F_{N_j}^L\right)^{w_j}, \prod_{j=1}^{n} \left(F_{N_j}^U\right)^{w_j}\right], \\[2mm]
1 - \prod_{j=1}^{n} \left(1 - T_{N_j}\right)^{w_j}, 1 - \prod_{j=1}^{n} \left(1 - I_{N_j}\right)^{w_j}, \prod_{j=1}^{n} \left(F_{N_j}\right)^{w_j}
\end{array}
\right)
$$

We prove the result holds for $m = n + 1$,

$$
N_{n+1}^{w_{n+1}} = \left(
\begin{array}{c}
\left[(T_{N_{n+1}}^L)^{w_{n+1}}, (T_{N_{n+1}}^U)^{w_{n+1}}\right], \\[2mm]
\left[(I_{N_{j+1}}^L)^{w_{j+1}}, (I_{N_{j+1}}^U)^{w_{j+1}}\right], \\[2mm]
\left[1 - \left(1 - F_{N_{n+1}}^L\right)^{w_{n+1}}, 1 - \left(1 - F_{N_{n+1}}^U\right)^{w_{n+1}}\right], \\[2mm]
1 - \left(1 - T_{N_{n+1}}\right)^{w_{n+1}}, 1 - \left(1 - I_{N_{n+1}}\right)^{w_{n+1}}, (F_{N_{n+1}})^{w_{n+1}}
\end{array}
\right)
$$

$$
\overset{n}{\underset{j=1}{\otimes}} N_j^{w_j} \oplus N_{n+1}^{w_{n+1}}
$$

$$
= \left(
\begin{array}{c}
\left[\prod_{j=1}^{n} (T_{N_j}^L)^{w_j}, \prod_{j=1}^{n} (T_{N_j}^U)^{w_j}\right], \\[2mm]
\left[\prod_{j=1}^{n} (I_{N_j}^L)^{w_j}, \prod_{j=1}^{n} (I_{N_j}^U)^{w_j}\right], \\[2mm]
\left[1 - \prod_{j=1}^{n} \left(1 - F_{N_j}^L\right)^{w_j}, \prod_{j=1}^{n} \left(F_{N_j}^U\right)^{w_j}\right], \\[2mm]
1 - \prod_{j=1}^{n} \left(1 - T_{N_j}\right)^{w_j}, 1 - \prod_{j=1}^{n} \left(1 - I_{N_j}\right)^{w_j}, \prod_{j=1}^{n} \left(F_{N_j}\right)^{w_j}
\end{array}
\right) \oplus \left(
\begin{array}{c}
\left[(T_{N_{n+1}}^L)^{w_{n+1}}, (T_{N_{n+1}}^U)^{w_{n+1}}\right], \\[2mm]
\left[(I_{N_{j+1}}^L)^{w_{j+1}}, (I_{N_{j+1}}^U)^{w_{j+1}}\right], \\[2mm]
\left[1 - \left(1 - F_{N_{n+1}}^L\right)^{w_{n+1}}, 1 - \left(1 - F_{N_{n+1}}^U\right)^{w_{n+1}}\right], \\[2mm]
1 - \left(1 - T_{N_{n+1}}\right)^{w_{n+1}}, 1 - \left(1 - I_{N_{n+1}}\right)^{w_{n+1}}, (F_{N_{n+1}})^{w_{n+1}}
\end{array}
\right)
$$

$$
\overset{n+1}{\underset{j=1}{\otimes}} N_j^{w_j} = \left(
\begin{array}{c}
\left[\prod_{j=1}^{n} \left(T_{N_j}^L\right)^{w_j} \left(T_{N_{m+1}}^L\right)^{w_{m+1}}, \prod_{j=1}^{n} \left(T_{N_j}^U\right)^{w_j} \left(T_{N_{m+1}}^U\right)^{w_{m+1}}\right], \\[2mm]
\left[\prod_{j=1}^{n} \left(I_{N_j}^L\right)^{w_j} \left(I_{N_{m+1}}^L\right)^{w_{m+1}}, \prod_{j=1}^{n} \left(I_{N_j}^U\right)^{w_j} \left(I_{N_{m+1}}^U\right)^{w_{m+1}}\right], \\[2mm]
\left[\begin{array}{c}
1 - \prod_{j=1}^{n} (1 - F_{N_j}^L)^{w_j} + 1 - (1 - F_{N_{m+1}}^L)^{w_{m+1}} - \\[2mm]
\left(1 - \prod_{j=1}^{n} (1 - F_{N_j}^L)^{w_j}\right) \left(1 - (1 - F_{N_{m+1}}^L)^{w_{m+1}}\right), \\[2mm]
1 - \prod_{j=1}^{n} (1 - F_{N_j}^U)^{w_j} + 1 - (1 - F_{N_{m+1}}^U)^{w_{m+1}} - \\[2mm]
\left(1 - \prod_{j=1}^{n} (1 - F_{N_j}^U)^{w_j}\right) \left(1 - (1 - F_{N_{m+1}}^U)^{w_{m+1}}\right),
\end{array}\right], \\[2mm]
1 - \prod_{j=1}^{n} \left(1 - T_{N_j}\right)^{w_j} + 1 - \left(1 - T_{N_{m+1}}\right)^{w_{m+1}} - \\[2mm]
\left(1 - \prod_{j=1}^{n} \left(1 - T_{N_j}\right)^{w_j}\right) \left(1 - \left(1 - T_{N_{m+1}}\right)^{w_{m+1}}\right) \\[2mm]
1 - \prod_{j=1}^{n} \left(1 - I_{N_j}\right)^{w_j} + 1 - \left(1 - I_{N_{m+1}}\right)^{w_{m+1}} - \\[2mm]
\left(1 - \prod_{j=1}^{n} \left(1 - I_{N_j}\right)^{w_j}\right) \left(1 - \left(1 - I_{N_{m+1}}\right)^{w_{m+1}}\right) \\[2mm]
\prod_{j=1}^{n} \left(F_{N_j}\right)^{w_j} \left(F_{N_{m+1}}\right)^{w_{m+1}}
\end{array}
\right)
$$

$$
=
\begin{pmatrix}
\left[\prod\limits_{j=1}^{n+1} \left(T_{N_j}^L \right)^{w_j}, \prod\limits_{j=1}^{n+1} \left(T_{N_j}^U \right)^{w_j} \right] \\[2mm]
\left[\prod\limits_{j=1}^{n+1} \left(I_{N_j}^L \right)^{w_j}, \prod\limits_{j=1}^{n+1} \left(I_{N_j}^U \right)^{w_j} \right], \\[2mm]
\begin{bmatrix}
2 - \prod\limits_{j=1}^{n+1} \left(1 - F_{N_j}^L \right)^{w_j} - 1 + \prod\limits_{j=1}^{n} \left(1 - F_{N_j}^L \right)^{w_j} + \left(1 - F_{N_{m+1}}^L \right)^{w_{m+1}} \\[2mm]
- \left(\prod\limits_{j=1}^{n} \left(1 - F_{N_j}^L \right)^{w_j} \right) \left(1 - F_{N_{m+1}}^L \right)^{w_{m+1}}, \\[2mm]
2 - \prod\limits_{j=1}^{n+1} \left(1 - F_{N_j}^U \right)^{w_j} - 1 + \prod\limits_{j=1}^{n+1} \left(1 - F_{N_j}^U \right)^{w_j} + \left(1 - F_{N_{m+1}}^U \right)^{w_{m+1}} \\[2mm]
- \left(\prod\limits_{j=1}^{n} \left(1 - F_{N_j}^U \right)^{w_j} \right) \left(1 - F_{N_{m+1}}^U \right)^{w_{m+1}}
\end{bmatrix}, \\[2mm]
2 - \prod\limits_{j=1}^{n+1} \left(1 - T_{N_j} \right)^{w_j} - 1 + \prod\limits_{j=1}^{n} \left(1 - T_{N_j} \right)^{w_j} + \left(1 - T_{N_{m+1}} \right)^{w_{m+1}} \\[2mm]
- \left(\prod\limits_{j=1}^{n} \left(1 - T_{N_j} \right)^{w_j} \right) \left(1 - T_{N_{m+1}} \right)^{w_{m+1}}, \\[2mm]
2 - \prod\limits_{j=1}^{n+1} \left(1 - I_{N_j} \right)^{w_j} - 1 + \prod\limits_{j=1}^{n} \left(1 - I_{N_j} \right)^{w_j} + \left(1 - I_{N_{m+1}} \right)^{w_{m+1}} \\[2mm]
- \left(\prod\limits_{j=1}^{n} \left(1 - I_{N_j} \right)^{w_j} \right) \left(1 - I_{N_{m+1}} \right)^{w_{m+1}} \\[2mm]
, \prod\limits_{j=1}^{n+1} \left(F_{N_j} \right)^{w_j}
\end{pmatrix}
$$

$$
=
\begin{pmatrix}
\left[\prod\limits_{J=1}^{n+1} \left(T_{N_j}^L \right)^{w_j}, \prod\limits_{J=1}^{n+1} \left(T_{N_j}^U \right)^{w_j} \right] \\[2mm]
\left[\prod\limits_{J=1}^{n+1} \left(I_{N_j}^L \right)^{w_j}, \prod\limits_{J=1}^{n+1} \left(I_{N_j}^U \right)^{w_j} \right], \\[2mm]
\left[1 - \prod\limits_{J=1}^{n+1} \left(1 - F_{N_j}^L \right)^{w_j}, 1 - \prod\limits_{J=1}^{n+1} \left(1 - F_{N_j}^U \right)^{w_j} \right], \\[2mm]
, 1 - \prod\limits_{J=1}^{n+1} \left(1 - T_{N_j} \right)^{w_j}, 1 - \prod\limits_{J=1}^{n+1} \left(1 - I_{N_j} \right)^{w_j}, \\[2mm]
\prod\limits_{J=1}^{n+1} \left(1 - F_{N_j} \right)^{w_j}
\end{pmatrix}
$$

□

Theorem 4. *Let* $N_j = \left(\widetilde{T}_{N_j}, \widetilde{I}_{N_j}, \widetilde{F}_{N_j}, T_{N_j}, I_{N_j}, F_{N_j} \right)$, *where* $\widetilde{T}_{N_j} = \left[T_{N_j}^L, T_{N_j}^U \right], \widetilde{I}_{N_j} = \left[I_{N_j}^L, I_{N_j}^U \right], \widetilde{F}_{N_j} = \left[F_{N_j}^L, F_{N_j}^U \right]$, $(j = 1, 2, \ldots, m)$ *is a collection of neutrosophic cubic values The weight* $W = (w_1, w_2, \ldots, w_m)^T$ *of* $N_j (j = 1, 2, 3, \ldots, m)$, *be such that* $w_j \in [0, 1]$ *and* $\sum\limits_{j=1}^{m} w_j = 1$.

1. **Idempotency:** *If for all* $N_j = \left(\widetilde{T}_{N_j}, \widetilde{I}_{N_j}, \widetilde{F}_{N_j}, T_{N_j}, I_{N_j}, F_{N_j} \right)$, *where* $\widetilde{T}_{N_j} = \left[T_{N_j}^L, T_{N_j}^U \right]$, $\widetilde{I}_{N_j} = \left[I_{N_j}^L, I_{N_j}^U \right], \widetilde{F}_{N_j} = \left[F_{N_j}^L, F_{N_j}^U \right]$, $(j = 1, 2, \ldots, m)$ *are equal, that is,* $N_j = N$ *for all k, then* NCW $G_w(N_1, N_2, \ldots, N_m) = N$

2. **Monotonicity:** *Let* $B_j = \left(\widetilde{T}_{B_j}, \widetilde{I}_{B_j}, \widetilde{F}_{B_j}, T_{B_j}, I_{B_j}, F_{B_j} \right)$ *where* $\widetilde{T}_{B_j} = \left[T_{B_j}^L, T_{B_j}^U \right]$, $\widetilde{I}_{B_j} = \left[I_{B_j}^L, I_{B_j}^U \right], \widetilde{F}_{B_j} = \left[F_{B_j}^L, F_{B_j}^U \right]$ $(j = 1, 2, \ldots, m)$ *is the collection of neutrosophic cubic values. If* $S_{B_j}(u) \geq S_{N_j}(u)$ *and* $B_j(u) \geq N_j(u)$ *then* $NCWG_w(N_1, N_2, \ldots, N_m) \leq NCWG_w(B_1, B_2, \ldots, B_m)$.

3. **Boundary:** $N^- \leq NCWG_w\{(N_1)_T, (N_2)_T, \ldots, (N_m)_T\} \leq N^+$, where

$$N^- = \left\{ \min_j T_{N_j}^L, \min_j I_{N_j}^L, 1 - \max_j F_{N_j}^L, \min_j T_{N_j}, \min_j I_{N_j}, 1 - \max_j F_{N_j}^L, \min_j T_{N_j}, \min_j I_{N_j}, 1 - \max_j F_{N_j}^L \right\},$$

$$N^+ = \left\{ \max_j T_{N_j}^U, \max_j I_{N_j}^U, 1 - \min_j F_{N_j}^U, \max_j T_{N_j}, \max_j I_{N_j}, 1 - \min_j F_{N_j}, \max_j T_{N_j}, \max_j I_{N_j}, 1 - \min_j F_{N_j} \right\}$$

Proof.

1. **Idempotent**: Since $N_j = N$, so

$$NCWG(N_j) = \left(\begin{array}{c} \left[\prod_{j=1}^m (T_N^L)^{w_j}, \prod_{j=1}^m (T_N^U)^{w_j} \right], \\ \left[\prod_{j=1}^m (I_N^L)^{w_j}, \prod_{j=1}^m (I_N^U)^{w_j} \right], \\ \left[1 - \prod_{j=1}^m (1 - F_N^L)^{w_j}, 1 - \prod_{j=1}^m (1 - F_N^L)^{w_j} \right], \\ 1 - \prod_{j=1}^m (1 - T_N)^{w_j}, 1 - \prod_{j=1}^m (1 - I_N)^{w_j}, \prod_{j=1}^m (F_N)^{w_j} \end{array} \right)$$

$$= \left(\begin{array}{c} \left[(T_N^L)^{\sum_{j=1}^m w_j}, (T_N^U)^{\sum_{j=1}^m w_j} \right], \\ \left[(I_N^L)^{\sum_{j=1}^m w_j}, (I_N^U)^{\sum_{j=1}^m w_j} \right], \\ \left[1 - (1 - F_N^L)^{\sum_{j=1}^m w_j}, 1 - (1 - F_N^U)^{\sum_{j=1}^m w_j} \right], \\ 1 - (1 - T_N)^{\sum_{j=1}^m w_j}, 1 - (1 - I_N)^{\sum_{j=1}^m w_j}, (F_N)^{\sum_{j=1}^m w_j} \end{array} \right)$$

$$= \left(\widetilde{T}_N, \widetilde{I}_N, \widetilde{F}_N, T_N, I_N, F_N \right)$$

2. **Monotonicity:** Since NCOWG is strictly monotone function.
3. **Boundary:** Let $u = \min N^-$ and $y = \max N^+$, then by monotonicity we have $u \leq NCOWA(N_j) \leq y \Rightarrow N^- \leq NCOWG(N_j) \leq N^+$.

    $\square$

Theorem 5. *Let* $N_j = \left(\widetilde{T}_{N_j}, \widetilde{I}_{N_j}, \widetilde{F}_{N_j}, T_{N_j}, I_{N_j}, F_{N_j} \right)$, *where* $\widetilde{T}_{N_j} = \left[T_{N_j}^L, T_{N_j}^U \right], \widetilde{I}_{N_j} = \left[I_{N_j}^L, I_{N_j}^U \right], \widetilde{F}_{N_j} = \left[F_{N_j}^L, F_{N_j}^U \right]$, $(j = 1, 2, \ldots, n)$ *be the collection of neutrosophic cubic values and* $W = (w_1, w_2, \ldots, w_n)^T$ *is the weight of the NCOWG, with* $w_j \in [0,1]$ *and* $\sum_{j=1}^m w_j = 1$.

1. *If* $W = (1, 0, \ldots, 0)^T$, *then* $NCOWG(N_1, N_2, \ldots, N_n) = \max N_j$
2. *If* $W = (0, 0, \ldots, 1)^T$, *then* $NCOWG(N_1, N_2, \ldots, N_n) = \min N_j$
3. *If* $w_j = 1$, $w_l = 0$, *and* $j \neq l$, *then* $NCOWG(N_1, N_2, \ldots, N_n) = N_j$

where N_j *is the jth largest of* $(N_1, N_2, \ldots, N_n)$.

Proof. Since in NCOWG the neutrosophic values are ordered in descending order. $\square$

4.2. Neutrosophic Cubic Einstein Weighted Geometric Aggregation Operator

We define neutrosophic cubic Einstein geometric aggregation operators using Section 3.2.

Definition 23. *The neutrosophic cubic Einstein weighted geometric operator(NCEWA) is defined as*
$$NCEWG: R^m \rightarrow R, \text{ defined by } NCEWG_w(N_1, N_2, \dots, N_m) = \overset{m}{\underset{j=1}{\otimes}} \left(N_j^E \right)^{w_j}$$
where, $W = (w_1, w_2, \dots, w_m)^T$ *is the weight of* $N_j (j = 1, 2, 3, \dots, m)$*, such that* $w_j \in [0, 1]$ *and* $\sum\limits_{j=1}^{m} w_j = 1$*.*

That is, first all the neutrosophic values are weighted then aggregated using Einstein operations.

Definition 24. *Order neutrosophic cubic Einstein weighted geometric operator(NCEOWG) is defined as*
$$NCEOWG : R^m \rightarrow R \text{ by } NCEOWG_w(N_1, N_2, \dots, N_m) = \overset{m}{\underset{j=1}{\otimes}} \left(B_j^E \right)^{w_j}$$
where B_j *is the jth largest,* $W = (w_1, w_2, \dots, w_m)^T$ *is the weight of* $N_j (j = 1, 2, 3, \dots, m)$*, such that* $w_j \in [0, 1]$ *and* $\sum\limits_{j=1}^{m} w_j = 1$*.*

That is, first all the neutrosophic values are ordered and then weighted, after ordering weighted values are aggregated using Einstein operations.The fundamental concept of ordered weighted operators is to rearrange the neutrosophic cubic values in descending order.

Theorem 6. *Let* $N_j = \left(\widetilde{T}_{N_j}, \widetilde{I}_{N_j}, \widetilde{F}_{N_j}, T_{N_j}, I_{N_j}, F_{N_j} \right)$*, where* $\widetilde{T}_{N_j} = \left[T_{N_j}^L, T_{N_j}^U \right], \widetilde{I}_{N_j} = \left[I_{N_j}^L, I_{N_j}^U \right],$ $\widetilde{F}_{N_j} = \left[F_{N_j}^L, F_{N_j}^U \right], (j = 1, 2, \dots, m)$ *is a collection of neutrosophic cubic values, then their Einstein weighted geometric aggregated value by NCEWG operator is also a neutrosophic cubic value, and*

$$NCEWG(N_j) = \begin{pmatrix} \left[\dfrac{2\prod\limits_{j=1}^{m} \left(T_{N_j}^L \right)^{w_j}}{\prod\limits_{j=1}^{m} \left(2 - T_{N_j}^L \right)^{w_j} + \prod\limits_{j=1}^{m} \left(T_{N_j}^L \right)^{w_j}}, \dfrac{2\prod\limits_{j=1}^{m} \left(T_{N_j}^U \right)^{w_j}}{\prod\limits_{j=1}^{m} \left(2 - T_{N_j}^U \right)^{w_j} + \prod\limits_{j=1}^{m} \left(T_{N_j}^U \right)^{w_j}} \right], \\ \left[\dfrac{2\prod\limits_{j=1}^{m} \left(I_{N_j}^L \right)^{w_j}}{\prod\limits_{j=1}^{m} \left(2 - I_{N_j}^L \right)^{w_j} + \prod\limits_{j=1}^{m} \left(I_{N_j}^L \right)^{w_j}}, \dfrac{2\prod\limits_{j=1}^{m} \left(I_{N_j}^U \right)^{w_j}}{\prod\limits_{j=1}^{m} \left(2 - I_{N_j}^U \right)^{w_j} + \prod\limits_{j=1}^{m} \left(I_{N_j}^U \right)^{w_j}} \right], \\ \left[\dfrac{\prod\limits_{j=1}^{m} \left(1 + F_{N_j}^L \right)^{w_j} - \prod\limits_{j=1}^{m} \left(1 - F_{N_j}^L \right)^{w_j}}{\prod\limits_{j=1}^{m} \left(1 + F_{N_j}^L \right)^{w_j} + \prod\limits_{j=1}^{m} \left(1 - F_{N_j}^L \right)^{w_j}}, \dfrac{\prod\limits_{j=1}^{m} \left(1 + F_{N_j}^U \right)^{w_j} - \prod\limits_{j=1}^{m} \left(1 - F_{N_j}^U \right)^{w_j}}{\prod\limits_{j=1}^{m} \left(1 + F_{N_j}^U \right)^{w_j} + \prod\limits_{j=1}^{m} \left(1 - F_{N_j}^U \right)^{w_j}} \right], \\ \dfrac{\prod\limits_{j=1}^{m} \left(1 + T_{N_j} \right)^{w_j} - \prod\limits_{j=1}^{m} \left(1 - T_{N_j} \right)^{w_j}}{\prod\limits_{j=1}^{m} \left(1 + T_{N_j} \right)^{w_j} + \prod\limits_{j=1}^{m} \left(1 - T_{N_j} \right)^{w_j}}, \\ \dfrac{\prod\limits_{j=1}^{m} \left(1 + I_{N_j} \right)^{w_j} - \prod\limits_{j=1}^{m} \left(1 - I_{N_j} \right)^{w_j}}{\prod\limits_{j=1}^{m} \left(1 + I_{N_j} \right)^{w_j} + \prod\limits_{j=1}^{m} \left(1 - I_{N_j} \right)^{w_j}}, \\ \dfrac{2\prod\limits_{j=1}^{m} \left(F_{N_j} \right)^{w_j}}{\prod\limits_{j=1}^{m} \left(2 - F_{N_j} \right)^{w_j} + \prod\limits_{j=1}^{m} \left(F_{N_j} \right)^{w_j}} \end{pmatrix}$$

where $W = (w_1, w_2, \dots, w_m)^T$ *is the weight vector of* $N_j (j = 1, 2, 3, \dots, m)$*, such that* $w_j \in [0, 1]$ *and* $\sum\limits_{j=1}^{m} w_j = 1$*.*

Proof. We use mathematical induction to prove this result, for $m = 2$, using definition (Einstein sum and Einstein scalar multiplication).

$$
\left(N_1^E\right)^{w_1} = \left(\begin{array}{c}
\left[\dfrac{2\left(T_{N_1}^L\right)^{w_1}}{\left(2-T_{N_1}^L\right)^{w_1}+T_{N_1}^L}, \dfrac{2\left(T_{N_1}^U\right)^{w_1}}{\left(2-T_{N_1}^U\right)^{w_1}+T_{N_1}^U}\right], \\[2ex]
\left[\dfrac{2\left(I_{N_1}^L\right)^{w_1}}{\left(2-I_{N_1}^L\right)^{w_1}+I_{N_1}^L}, \dfrac{2\left(I_{N_1}^U\right)^{w_1}}{\left(2-I_{N_1}^U\right)^{w_1}+I_{N_1}^U}\right], \\[2ex]
\left[\dfrac{\left(1+F_{N_1}^L\right)^{w_1}-\left(1-F_{N_1}^L\right)^{w_1}}{\left(1+F_{N_1}^L\right)^{w_1}+\left(1-F_{N_1}^L\right)^{w_1}}, \dfrac{\left(1+F_{N_1}^U\right)^{w_1}-\left(1-F_{N_1}^U\right)^{w_1}}{\left(1+F_{N_1}^U\right)^{w_1}+\left(1-F_{N_1}^U\right)^{w_1}}\right], \\[2ex]
\dfrac{\left(1+T_{N_1}\right)^{w_1}-\left(1-T_{N_1}\right)^{w_1}}{\left(1+T_{N_1}\right)^{w_1}+\left(1-T_{N_1}\right)^{w_1}}, \\[2ex]
\dfrac{\left(1+I_{N_1}\right)^{w_1}-\left(1-I_{N_1}\right)^{w_1}}{\left(1+I_{N_1}\right)^{w_1}+\left(1-I_{N_1}\right)^{w_1}}, \\[2ex]
\dfrac{2\left(F_{N_1}\right)^{w_1}}{\left(2-F_{N_1}\right)^{w_1}+F_{N_1}}
\end{array}\right)
$$

$$
\left(N_2^E\right)^{w_2} = \left(\begin{array}{c}
\left[\dfrac{2\left(T_{N_2}^L\right)^{w_2}}{\left(2-T_{N_2}^L\right)^{w_2}+T_{N_2}^L}, \dfrac{2\left(T_{N_2}^U\right)^{w_2}}{\left(2-T_{N_2}^U\right)^{w_2}+T_{N_2}^U}\right], \\[2ex]
\left[\dfrac{2\left(I_{N_2}^L\right)^{w_2}}{\left(2-I_{N_2}^L\right)^{w_2}+I_{N_2}^L}, \dfrac{2\left(I_{N_2}^U\right)^{w_2}}{\left(2-I_{N_2}^U\right)^{w_2}+I_{N_2}^U}\right], \\[2ex]
\left[\dfrac{\left(1+F_{N_2}^L\right)^{w_2}-\left(1-F_{N_2}^L\right)^{w_2}}{\left(1+F_{N_2}^L\right)^{w_2}+\left(1-F_{N_2}^L\right)^{w_2}}, \dfrac{\left(1+F_{N_2}^U\right)^{w_2}-\left(1-F_{N_2}^U\right)^{w_2}}{\left(1+F_{N_2}^U\right)^{w_2}+\left(1-F_{N_2}^U\right)^{w_2}}\right], \\[2ex]
\dfrac{\left(1+T_{N_2}\right)^{w_2}-\left(1-T_{N_2}\right)^{w_2}}{\left(1+T_{N_2}\right)^{w_2}+\left(1-T_{N_2}\right)^{w_2}}, \dfrac{\left(1+I_{N_2}\right)^{w_2}-\left(1-I_{N_2}\right)^{w_2}}{\left(1+I_{N_2}\right)^{w_2}+\left(1-I_{N_2}\right)^{w_2}}, \\[2ex]
\dfrac{2\left(F_{N_2}\right)^{w_2}}{\left(2-F_{N_2}\right)^{w_2}+F_{N_2}}
\end{array}\right)
$$

$$
\overset{2}{\underset{j=1}{\otimes}}\left(N_j^E\right)^{w_j} = \left(\begin{array}{c}
\left[\dfrac{2\prod\limits_{j=1}^{2}\left(T_{N_j}^L\right)^{w_j}}{\prod\limits_{j=1}^{2}\left(2-T_{N_j}^L\right)^{w_j}+\prod\limits_{j=1}^{2}\left(T_{N_j}^L\right)^{w_j}}, \dfrac{2\prod\limits_{j=1}^{2}\left(T_{N_j}^U\right)^{w_j}}{\prod\limits_{j=1}^{2}\left(2-T_{N_j}^U\right)^{w_j}+\prod\limits_{j=1}^{2}\left(T_{N_j}^U\right)^{w_j}}\right], \\[3ex]
\left[\dfrac{2\prod\limits_{j=1}^{2}\left(I_{N_j}^L\right)^{w_j}}{\prod\limits_{j=1}^{2}\left(2-I_{N_j}^L\right)^{w_j}+\prod\limits_{j=1}^{2}\left(I_{N_j}^L\right)^{w_j}}, \dfrac{2\prod\limits_{j=1}^{2}\left(I_{N_j}^U\right)^{w_j}}{\prod\limits_{j=1}^{2}\left(2-I_{N_j}^U\right)^{w_j}+\prod\limits_{j=1}^{2}\left(I_{N_j}^U\right)^{w_j}}\right], \\[3ex]
\left[\dfrac{\prod\limits_{j=1}^{2}\left(1+F_{N_j}^L\right)^{w_j}-\prod\limits_{j=1}^{2}\left(1-F_{N_j}^L\right)^{w_j}}{\prod\limits_{j=1}^{2}\left(1+F_{N_j}^L\right)^{w_j}+\prod\limits_{j=1}^{2}\left(1-F_{N_j}^L\right)^{w_j}}, \dfrac{\prod\limits_{j=1}^{2}\left(1+F_{N_j}^U\right)^{w_j}-\prod\limits_{j=1}^{2}\left(1-F_{N_j}^U\right)^{w_j}}{\prod\limits_{j=1}^{2}\left(1+F_{N_j}^U\right)^{w_j}+\prod\limits_{j=1}^{2}\left(1-F_{N_j}^U\right)^{w_j}}\right], \\[3ex]
\dfrac{\prod\limits_{j=1}^{2}\left(1+T_{N_j}\right)^{w_j}-\prod\limits_{j=1}^{2}\left(1-T_{N_j}\right)^{w_j}}{\prod\limits_{j=1}^{2}\left(1+T_{N_j}\right)^{w_j}+\prod\limits_{j=1}^{2}\left(1-T_{N_j}\right)^{w_j}}, \dfrac{\prod\limits_{j=1}^{2}\left(1+I_{N_j}\right)^{w_j}-\prod\limits_{j=1}^{2}\left(1-I_{N_j}\right)^{w_j}}{\prod\limits_{j=1}^{2}\left(1+I_{N_j}\right)^{w_j}+\prod\limits_{j=1}^{2}\left(1-I_{N_j}\right)^{w_j}}, \\[3ex]
\dfrac{2\prod\limits_{j=1}^{2}\left(F_{N_j}\right)^{w_j}}{\prod\limits_{j=1}^{2}\left(2-F_{N_j}\right)^{w_j}+\prod\limits_{j=1}^{2}\left(F_{N_j}\right)^{w_j}}
\end{array}\right)
$$

for $m = n$

$$
\overset{n}{\underset{j=1}{\otimes}} \left(N_j^E\right)^{w_j} =
\left(
\begin{array}{c}
\left[
\dfrac{2\prod\limits_{j=1}^{n}\left(T_{N_j}^L\right)^{w_j}}{\prod\limits_{j=1}^{n}\left(2-T_{N_j}^L\right)^{w_j}+\prod\limits_{j=1}^{n}\left(T_{N_j}^L\right)^{w_j}},
\dfrac{2\prod\limits_{j=1}^{n}\left(T_{N_j}^U\right)^{w_j}}{\prod\limits_{j=1}^{n}\left(2-T_{N_j}^U\right)^{w_j}+\prod\limits_{j=1}^{n}\left(T_{N_j}^U\right)^{w_j}}
\right], \\[4ex]
\left[
\dfrac{2\prod\limits_{j=1}^{n}\left(I_{N_j}^L\right)^{w_j}}{\prod\limits_{j=1}^{n}\left(2-I_{N_j}^L\right)^{w_j}+\prod\limits_{j=1}^{n}\left(I_{N_j}^L\right)^{w_j}},
\dfrac{2\prod\limits_{j=1}^{n}\left(I_{N_j}^U\right)^{w_j}}{\prod\limits_{j=1}^{n}\left(2-I_{N_j}^U\right)^{w_j}+\prod\limits_{j=1}^{n}\left(I_{N_j}^U\right)^{w_j}}
\right], \\[4ex]
\left[
\dfrac{\prod\limits_{j=1}^{n}\left(1+F_{N_j}^L\right)^{w_j}-\prod\limits_{j=1}^{n}\left(1-F_{N_j}^L\right)^{w_j}}{\prod\limits_{j=1}^{n}\left(1+F_{N_j}^L\right)^{w_j}+\prod\limits_{j=1}^{n}\left(1-F_{N_j}^L\right)^{w_j}},
\dfrac{\prod\limits_{j=1}^{n}\left(1+F_{N_j}^U\right)^{w_j}-\prod\limits_{j=1}^{n}\left(1-F_{N_j}^U\right)^{w_j}}{\prod\limits_{j=1}^{n}\left(1+F_{N_j}^U\right)^{w_j}+\prod\limits_{j=1}^{n}\left(1-F_{N_j}^U\right)^{w_j}}
\right], \\[4ex]
\dfrac{\prod\limits_{j=1}^{n}\left(1+T_{N_j}\right)^{w_j}-\prod\limits_{j=1}^{n}\left(1-T_{N_j}\right)^{w_j}}{\prod\limits_{j=1}^{n}\left(1+T_{N_j}\right)^{w_j}+\prod\limits_{j=1}^{n}\left(1-T_{N_j}\right)^{w_j}},
\dfrac{\prod\limits_{j=1}^{n}\left(1+I_{N_j}\right)^{w_j}-\prod\limits_{j=1}^{n}\left(1-I_{N_j}\right)^{w_j}}{\prod\limits_{j=1}^{n}\left(1+I_{N_j}\right)^{w_j}+\prod\limits_{j=1}^{n}\left(1-I_{N_j}\right)^{w_j}}, \\[4ex]
\dfrac{2\prod\limits_{j=1}^{n}\left(F_{N_j}\right)^{w_j}}{\prod\limits_{j=1}^{n}\left(2-F_{N_j}\right)^{w_j}+\prod\limits_{j=1}^{n}\left(F_{N_j}\right)^{w_j}}
\end{array}
\right)
$$

We prove the result holds for $m = n+1$

$$
as\ \left(N_{n+1}^E\right)^{w_{n+1}} =
\left(
\begin{array}{c}
\left[
\dfrac{2\left(T_{N_{n+1}}^L\right)^{w_{n+1}}}{\left(2-T_{N_{n+1}}^L\right)^{w_{n+1}}+\left(T_{N_{n+1}}^L\right)^{w_{n+1}}},
\dfrac{2\left(T_{N_{n+1}}^U\right)^{w_{n+1}}}{\left(2-T_{N_{n+1}}^U\right)^{w_{n+1}}+\left(T_{N_{n+1}}^U\right)^{w_{n+1}}}
\right], \\[4ex]
\left[
\dfrac{2\left(I_{N_{n+1}}^L\right)^{w_{n+1}}}{\left(2-I_{N_{n+1}}^L\right)^{w_{n+1}}+\left(I_{N_{n+1}}^L\right)^{w_{n+1}}},
\dfrac{2\left(I_{N_{n+1}}^U\right)^{w_{n+1}}}{\left(2-I_{N_{n+1}}^U\right)^{w_{n+1}}+\left(I_{N_{n+1}}^U\right)^{w_{n+1}}}
\right], \\[4ex]
\left[
\dfrac{\left(1+F_{N_{n+1}}^L\right)^{w_{n+1}}-\left(1-F_{N_{n+1}}^L\right)^{w_{n+1}}}{\left(1+F_{N_{n+1}}^L\right)^{w_{n+1}}+\left(1-F_{N_{n+1}}^L\right)^{w_{n+1}}},
\dfrac{\left(1+F_{N_{n+1}}^U\right)^{w_{n+1}}-\left(1-F_{N_{n+1}}^U\right)^{w_{n+1}}}{\left(1+F_{N_{n+1}}^U\right)^{w_{n+1}}+\left(1-F_{N_{n+1}}^U\right)^{w_{n+1}}}
\right], \\[4ex]
\dfrac{\left(1+T_{N_{n+1}}\right)^{w_{n+1}}-\left(1-T_{N_{n+1}}\right)^{w_{n+1}}}{\left(1+T_{N_{n+1}}\right)^{w_{n+1}}+\left(1-T_{N_{n+1}}\right)^{w_{n+1}}},
\dfrac{\left(1+I_{N_{n+1}}\right)^{w_{n+1}}-\left(1-I_{N_{n+1}}\right)^{w_{n+1}}}{\left(1+I_{N_{n+1}}\right)^{w_{n+1}}+\left(1-I_{N_{n+1}}\right)^{w_{n+1}}}, \\[4ex]
\dfrac{2\left(F_{N_{n+1}}\right)^{w_{n+1}}}{\left(2-F_{N_{n+1}}\right)^{w_{n+1}}+\left(F_{N_{n+1}}\right)^{w_{n+1}}}
\end{array}
\right)
$$

so $\overset{n}{\underset{j=1}{\otimes}} \left(N_j^E\right)^{w_j} \otimes_E \left(N_{m+1}^E\right)^{w_{m+1}} =$

$$
\left(
\begin{array}{c}
\left[
\dfrac{2\prod\limits_{j=1}^{n}\left(T_{N_j}^L\right)^{w_j}}{\prod\limits_{j=1}^{n}\left(2-T_{N_j}^L\right)^{w_j}+\prod\limits_{j=1}^{n}\left(T_{N_j}^L\right)^{w_j}},
\dfrac{2\prod\limits_{j=1}^{n}\left(T_{N_j}^U\right)^{w_j}}{\prod\limits_{j=1}^{n}\left(2-T_{N_j}^U\right)^{w_j}+\prod\limits_{j=1}^{n}\left(T_{N_j}^U\right)^{w_j}}
\right], \\[3ex]
\left[
\dfrac{2\prod\limits_{j=1}^{n}\left(I_{N_j}^L\right)^{w_j}}{\prod\limits_{j=1}^{n}\left(2-I_{N_j}^L\right)^{w_j}+\prod\limits_{j=1}^{n}\left(I_{N_j}^L\right)^{w_j}},
\dfrac{2\prod\limits_{j=1}^{n}\left(I_{N_j}^U\right)^{w_j}}{\prod\limits_{j=1}^{n}\left(2-I_{N_j}^U\right)^{w_j}+\prod\limits_{j=1}^{n}\left(I_{N_j}^U\right)^{w_j}}
\right], \\[3ex]
\left[
\dfrac{\prod\limits_{j=1}^{n}\left(1+F_{N_j}^L\right)^{w_j}-\prod\limits_{j=1}^{n}\left(1-F_{N_j}^L\right)^{w_j}}{\prod\limits_{j=1}^{n}\left(1+F_{N_j}^L\right)^{w_j}+\prod\limits_{j=1}^{n}\left(1-F_{N_j}^L\right)^{w_j}},
\dfrac{\prod\limits_{j=1}^{n}\left(1+F_{N_j}^U\right)^{w_j}-\prod\limits_{j=1}^{n}\left(1-F_{N_j}^U\right)^{w_j}}{\prod\limits_{j=1}^{n}\left(1+F_{N_j}^U\right)^{w_j}+\prod\limits_{j=1}^{n}\left(1-F_{N_j}^U\right)^{w_j}}
\right], \\[3ex]
\dfrac{\prod\limits_{j=1}^{n}\left(1+T_{N_j}\right)^{w_j}-\prod\limits_{j=1}^{n}\left(1-T_{N_j}\right)^{w_j}}{\prod\limits_{j=1}^{n}\left(1+T_{N_j}\right)^{w_j}+\prod\limits_{j=1}^{n}\left(1-T_{N_j}\right)^{w_j}},
\dfrac{\prod\limits_{j=1}^{n}\left(1+I_{N_j}\right)^{w_j}-\prod\limits_{j=1}^{n}\left(1-I_{N_j}\right)^{w_j}}{\prod\limits_{j=1}^{n}\left(1+I_{N_j}\right)^{w_j}+\prod\limits_{j=1}^{n}\left(1-I_{N_j}\right)^{w_j}}, \\[3ex]
\dfrac{2\prod\limits_{j=1}^{n}\left(F_{N_j}\right)^{w_j}}{\prod\limits_{j=1}^{n}\left(2-F_{N_j}\right)^{w_j}+\prod\limits_{j=1}^{n}\left(F_{N_j}\right)^{w_j}}
\end{array}
\right)
\oplus_E
\left(
\begin{array}{c}
\left[
\dfrac{2\left(T_{N_{n+1}}^L\right)^{w_{n+1}}}{\left(2-T_{N_{n+1}}^L\right)^{w_{n+1}}+\left(T_{N_{n+1}}^L\right)^{w_{n+1}}},
\dfrac{2\left(T_{N_{n+1}}^U\right)^{w_{n+1}}}{\left(2-T_{N_{n+1}}^U\right)^{w_{n+1}}+\left(T_{N_{n+1}}^U\right)^{w_{n+1}}}
\right], \\[3ex]
\left[
\dfrac{2\left(I_{N_{n+1}}^L\right)^{w_{n+1}}}{\left(2-I_{N_{n+1}}^L\right)^{w_{n+1}}+\left(I_{N_{n+1}}^L\right)^{w_{n+1}}},
\dfrac{2\left(I_{N_{n+1}}^U\right)^{w_{n+1}}}{\left(2-I_{N_{n+1}}^U\right)^{w_{n+1}}+\left(I_{N_{n+1}}^U\right)^{w_{n+1}}}
\right], \\[3ex]
\left[
\dfrac{\left(1+F_{N_{n+1}}^L\right)^{w_{n+1}}-\left(1-F_{N_{n+1}}^L\right)^{w_{n+1}}}{\left(1+F_{N_{n+1}}^L\right)^{w_{n+1}}+\left(1-F_{N_{n+1}}^L\right)^{w_{n+1}}},
\dfrac{\left(1+F_{N_{n+1}}^U\right)^{w_{n+1}}-\left(1-F_{N_{n+1}}^U\right)^{w_{n+1}}}{\left(1+F_{N_{n+1}}^U\right)^{w_{n+1}}+\left(1-F_{N_{n+1}}^U\right)^{w_{n+1}}}
\right], \\[3ex]
\dfrac{\left(1+T_{N_{n+1}}\right)^{w_{n+1}}-\left(1-T_{N_{n+1}}\right)^{w_{n+1}}}{\left(1+T_{N_{n+1}}\right)^{w_{n+1}}+\left(1-T_{N_{n+1}}\right)^{w_{n+1}}}, \\[3ex]
\dfrac{\left(1+I_{N_{n+1}}\right)^{w_{n+1}}-\left(1-I_{N_{n+1}}\right)^{w_{n+1}}}{\left(1+I_{N_{n+1}}\right)^{w_{n+1}}+\left(1-I_{N_{n+1}}\right)^{w_{n+1}}}, \\[3ex]
\dfrac{2\left(F_{N_{n+1}}\right)^{w_{n+1}}}{\left(2-F_{N_{n+1}}\right)^{w_{n+1}}+\left(F_{N_{n+1}}\right)^{w_{n+1}}}
\end{array}
\right)
$$

$$\overset{n+1}{\underset{j=1}{\otimes}}\left(N_j^E\right)^{w_j} = \left(\begin{array}{c} \left[\dfrac{2\prod\limits_{j=1}^{n+1}\left(T_{N_j}^L\right)^{w_j}}{\prod\limits_{j=1}^{n+1}\left(2-T_{N_j}^L\right)^{w_j}+\prod\limits_{j=1}^{n+1}\left(T_{N_j}^L\right)^{w_j}}, \dfrac{2\prod\limits_{j=1}^{n+1}\left(T_{N_j}^U\right)^{w_j}}{\prod\limits_{j=1}^{n+1}\left(2-T_{N_j}^U\right)^{w_j}+\prod\limits_{j=1}^{n+1}\left(T_{N_j}^U\right)^{w_j}} \right], \\[4ex] \left[\dfrac{2\prod\limits_{j=1}^{n+1}\left(I_{N_j}^L\right)^{w_j}}{\prod\limits_{j=1}^{n+1}\left(2-I_{N_j}^L\right)^{w_j}+\prod\limits_{j=1}^{n+1}\left(I_{N_j}^L\right)^{w_j}}, \dfrac{2\prod\limits_{j=1}^{n+1}\left(I_{N_j}^U\right)^{w_j}}{\prod\limits_{j=1}^{n+1}\left(2-I_{N_j}^U\right)^{w_j}+\prod\limits_{j=1}^{n+1}\left(I_{N_j}^U\right)^{w_j}} \right], \\[4ex] \left[\dfrac{\prod\limits_{j=1}^{n+1}\left(1+F_{N_j}^L\right)^{w_j}-\prod\limits_{j=1}^{n+1}\left(1-F_{N_j}^L\right)^{w_j}}{\prod\limits_{j=1}^{n+1}\left(1+F_{N_j}^L\right)^{w_j}+\prod\limits_{j=1}^{n+1}\left(1-F_{N_j}^L\right)^{w_j}}, \dfrac{\prod\limits_{j=1}^{n+1}\left(1+F_{N_j}^U\right)^{w_j}-\prod\limits_{j=1}^{n+1}\left(1-F_{N_j}^U\right)^{w_j}}{\prod\limits_{j=1}^{n+1}\left(1+F_{N_j}^U\right)^{w_j}+\prod\limits_{j=1}^{n+1}\left(1-F_{N_j}^U\right)^{w_j}} \right], \\[4ex] \dfrac{\prod\limits_{j=1}^{n+1}\left(1+T_{N_j}\right)^{w_j}-\prod\limits_{j=1}^{n+1}\left(1-T_{N_j}\right)^{w_j}}{\prod\limits_{j=1}^{n+1}\left(1+T_{N_j}\right)^{w_j}+\prod\limits_{j=1}^{n+1}\left(1-T_{N_j}\right)^{w_j}}, \dfrac{\prod\limits_{j=1}^{n+1}\left(1+I_{N_j}\right)^{w_j}-\prod\limits_{j=1}^{n+1}\left(1-I_{N_j}\right)^{w_j}}{\prod\limits_{j=1}^{n+1}\left(1+I_{N_j}\right)^{w_j}+\prod\limits_{j=1}^{n+1}\left(1-I_{N_j}\right)^{w_j}}, \\[4ex] \dfrac{2\prod\limits_{j=1}^{n+1}\left(F_{N_j}\right)^{w_j}}{\prod\limits_{j=1}^{n+1}\left(2-F_{N_j}\right)^{w_j}+\prod\limits_{j=1}^{n+1}\left(F_{N_j}\right)^{w_j}} \end{array} \right)$$

so result holds for all values of *m*. $\square$

Theorem 7. *Let* $N_j = \left(\widetilde{T}_{N_j}, \widetilde{I}_{N_j}, \widetilde{F}_{N_j}, T_{N_j}, I_{N_j}, F_{N_j}\right)$, *where* $\widetilde{T}_{N_j} = \left[T_{N_j}^L, T_{N_j}^U\right]$, $\widetilde{I}_{N_j} = \left[I_{N_j}^L, I_{N_j}^U\right], \widetilde{F}_{N_j} = \left[F_{N_j}^L, F_{N_j}^U\right]$, $(j = 1,2,\ldots,m)$ *is a collection of neutrosophic cubic values and* $W = (w_1, w_2, \ldots, w_m)^T$ *is a weight vector of* $N_j (j = 1,2,3,\ldots,m)$, *with* $w_j \in [0,1]$ *and* $\sum\limits_{j=1}^{m} w_j = 1$.

1. **Idempotency:** *If for all* $N_j = \left(\widetilde{T}_{N_j}, \widetilde{I}_{N_j}, \widetilde{F}_{N_j}, T_{N_j}, I_{N_j}, F_{N_j}\right)$, *where* $\widetilde{T}_{N_j} = \left[T_{N_j}^L, T_{N_j}^U\right]$, $\widetilde{I}_{N_j} = \left[I_{N_j}^L, I_{N_j}^U\right], \widetilde{F}_{N_j} = \left[F_{N_j}^L, F_{N_j}^U\right]$, $(j = 1,2,\ldots,m)$ *are equal, that is,* $N_j = N$ *for all k, then* $NCEWG_w(N_1, N_2, \ldots, N_m) = N$

2. **Monotonicity:** *Let* $B_j = \left(\widetilde{T}_{B_j}, \widetilde{I}_{B_j}, \widetilde{F}_{B_j}, T_{B_j}, I_{B_j}, F_{B_j}\right)$, *where* $\widetilde{T}_{B_j} = \left[T_{B_j}^L, T_{B_j}^U\right]$, $\widetilde{I}_{B_j} = \left[I_{B_j}^L, I_{B_j}^U\right], \widetilde{F}_{B_j} = \left[F_{B_j}^L, F_{B_j}^U\right]$ $(j = 1,2,\ldots,m)$ *be the collection of cubic values. If* $S_B(u) \geq S_N(u)$ *and* $B_j(u) \geq N_j(u)$ *then* $NCW\,G_w(N_1, N_2, \ldots, N_m) \leq NCWG_w(B_1, B_2, \ldots, B_m)$

3. **Boundary:** $N^- \leq NCWG_w\{(N_1)_T, (N_2)_T, \ldots, (N_m)_T\} \leq N^+$, *where*

$$N^- = \left\{ \min_j T_{N_j}^L, \min_j I_{N_j}^L, 1 - \max_j F_{N_j}^L, \min_j T_{N_j}, \min_j I_{N_j}, 1 - \max_j F_{N_j}^L \right\},$$
$$N^+ = \left\{ \max_j T_{N_j}^U, \max_j I_{N_j}^U, 1 - \min_j F_{N_j}^U, \max_j T_{N_j}, \max_j I_{N_j}, 1 - \min_j F_{N_j} \right\}$$

Proof. Followed by Theorem 2. $\square$

Theorem 8. *Let* $N_j = \left(\widetilde{T}_{N_j}, \widetilde{I}_{N_j}, \widetilde{F}_{N_j}, T_{N_j}, I_{N_j}, F_{N_j}\right)$, *where* $\widetilde{T}_{N_j} = \left[T_{N_j}^L, T_{N_j}^U\right]$, $\widetilde{I}_{N_j} = \left[I_{N_j}^L, I_{N_j}^U\right], \widetilde{F}_{N_j} = \left[F_{N_j}^L, F_{N_j}^U\right]$, $(j = 1,2,\ldots,m)$ *be a collection of neutrosophic cubic values and* $W = (w_1, w_2, \ldots, w_m)^T$ *is a weight vector of the NCOWA, with* $w_j \in [0,1]$ *and* $\sum\limits_{j=1}^{m} w_j = 1$.

1. *If* $w = (1, 0, \ldots, 0)^T$, *then* $NCEOWG\,(N_1, N_2, \ldots, N_m) = max\,N_j$
2. *If* $w(0, 0, \ldots, 1)^T$, *then* $NCEOWG\,(N_1, N_2, \ldots, N_m) = min N_j$
3. *If* $w_j = 1$, $w_j = 0$, *and* $j \neq j$, *then* $NCEOWG\,(N_1, N_2, \ldots, N_m) = N_j$

where N_j *is the jth largest of* $(N_1, N_2, \ldots, N_m)$.

Proof. Followed by Theorem 3. □

5. An Application of Neutrosophic cubic Geometric and Einstein Geometric Aggregation Operator to Group Decision Making Problems

Group decision making is an important factor of decision making theory. We are often in a situation with more then one expert, attribute and alternative to deal with. Motivated by such situations, a multi-attribute decision making method for more then one expert is proposed in this section.

In this section, we develop an algorithm for group decision making problems using the geometric and Einstein geometric aggregations (NCWG and NCEWG) under the neutrosophic cubic environment.

Algorithm. *Let $F = \{F_1, F_2, \ldots, F_n\}$ be the set of n alternatives, $H = \{H_1, H_2, \ldots, H_m\}$ be the m attributes subject to their corresponding weight $W = \{w_1, w_2, \ldots, w_m\}$ such that $w_j \in [0,1]$ and $\sum_{j=1}^{m} w_j = 1$, and $D = \{D_1, D_2, \ldots D_r\}$ be the r decision makers with their corresponding weight $V = \{v_1, v_2, \ldots, v_r\}$. such that $v_j \in [0,1]$ and $\sum_{j=1}^{r} v_j = 1$ The method has the following steps:*

Step1. *First, we construct neutrosophic cubic decision matrices for each decision maker* $D^{(s)} = \left[N_{ij}^{(s)} \right]_{n \times m} (s = 1, 2, \ldots, r)$.

Step2. All decision matrices are aggregated to a single matrix consisting of m attributes, by NCWG and NCEWG corresponding to the weight assigned to the decision maker.

Step3. By using aggregation operators like NCWG and NCEWG, the decision matrix is aggregated by the weight assigned to the m attributes.

Step4. The n alternatives are ranked according to their scores and arranged in descending order to select the alternative with highest score.

6. Application

Mobile companies play a vital role in Pakistan's stock market. The performance of these companies affects resources of capital market and have become a common concern of shareholders, government authorities, creditors and other stakeholders. In this example, an investor company wants to invest his capital levy in listed companies. They acquire two types of experts: Attorney and market maker. The attorney is acquired to look at the legal matters and the market maker is aquired to provide his expertise in capital market matters. Data are collected on the basis of stock market analysis and growth in different areas. Let the listed mobile companies be (x_1) Zong, (x_2) Jazz, (x_3) Telenor and (x_4) Ufone, which have higher ratios of earnings than the others available in the market, from the three alternatives of (A_1) stock market trends, (A_2) policy directions and (A_3) the annual performance. The two experts evaluated the mobile companies $(x_j, j = 1, 2, 3, 4)$ with respect to the corresponding attributes $(A_i, i = 1, 2, 3)$, and proposed their decision making matrices consisting of neutrosophic cubic values in Equation (1) and Equation (2). The Equation (3) represents the single matrix as the aggregation of Equtiona1 and Equation (2) by NCWG or NCEWG. The Equation (4) is obtained by applying NCWG or NCEWG on attributes. The decision matrices are aggregated to a single decision matrix. At the end we rank the alternatives according to their score to get the desirable alternative(s).

Step 1. We construct the decision maker matrices in Equations (1) and (2).

Equation (1): Decision making matrix for the first expert(attorney) D_a is

$$
\begin{pmatrix}
 & A_1 & A_2 & A_3 \\
X_1 & \begin{pmatrix} [0.2,0.6],[0.4,0.6], \\ [0.5,0.8],0.7,0.4,0.3 \end{pmatrix} & \begin{pmatrix} [0.1,0.4],[0.5,0.8], \\ [0.4,0.8],0.6,0.7,0.5 \end{pmatrix} & \begin{pmatrix} [0.4,0.6],[0.2,0.7], \\ [0.5,0.9],0.4,0.5,0.3 \end{pmatrix} \\
X_2 & \begin{pmatrix} [0.3,0.5],[0.6,0.9], \\ [0.3,0.6],0.3,0.6,0.7 \end{pmatrix} & \begin{pmatrix} [0.5,0.9],[0.1,0.3], \\ [0.4,0.8],0.8,0.3,0.6 \end{pmatrix} & \begin{pmatrix} [0.2,0.7],[0.1,0.6], \\ [0.4,0.7],0.5,0.4,0.7 \end{pmatrix} \\
X_3 & \begin{pmatrix} [0.6,0.9],[0.2,0.7], \\ [0.4,0.9],0.5,0.5,0.6 \end{pmatrix} & \begin{pmatrix} [0.2,0.6],[0.7,0.3], \\ [0.3,0.8],0.4,0.6,0.5 \end{pmatrix} & \begin{pmatrix} [0.5,0.9],[0.7,0.9], \\ [0.1,0.5],0.5,0.6,0.4 \end{pmatrix} \\
X_4 & \begin{pmatrix} [0.4,0.8],[0.5,0.9], \\ [0.3,0.8],0.5,0.8,0.5 \end{pmatrix} & \begin{pmatrix} [0.2,0.7],[0.4,0.9], \\ [0.5,0.7],0.6,0.4,0.5 \end{pmatrix} & \begin{pmatrix} [0.3,0.5],[0.5,0.9], \\ [0.7,0.3],0.3,0.3,0.8 \end{pmatrix}
\end{pmatrix} \quad (1)
$$

Equation (2): Decision making matrix for the second expert(market maker) D_m is

$$
\begin{pmatrix}
 & A_1 & A_2 & A_3 \\
X_1 & \begin{pmatrix} [0.3,0.6],[0.2,0.6], \\ [0.2,0.6],0.8,0.7,0.2 \end{pmatrix} & \begin{pmatrix} [0.3,0.8],[0.4,0.8], \\ [0.3,0.8],0.6,0.7,0.4 \end{pmatrix} & \begin{pmatrix} [0.2,0.7],[0.2,0.6], \\ [0.3,0.8],0.5,0.3,0.5 \end{pmatrix} \\
X_2 & \begin{pmatrix} [0.2,0.5],[0.6,0.9], \\ [0.7,0.3],0.4,0.8,0.7 \end{pmatrix} & \begin{pmatrix} [0.4,0.9],[0.1,0.4], \\ [0.5,0.8],0.6,0.5,0.7 \end{pmatrix} & \begin{pmatrix} [0.4,0.9],[0.1,0.4], \\ [0.5,0.8],0.6,0.5,0.7 \end{pmatrix} \\
X_3 & \begin{pmatrix} [0.5,0.9],[0.2,0.6], \\ [0.3,0.8],0.7,0.7,0.8 \end{pmatrix} & \begin{pmatrix} [0.2,0.5],[0.2,0.7], \\ [0.5,0.8],0.6,0.7,0.2 \end{pmatrix} & \begin{pmatrix} [0.3,0.5],[0.3,0.9], \\ [0.2,0.5],0.6,0.5,0.4 \end{pmatrix} \\
X_4 & \begin{pmatrix} [0.3,0.5],[0.3,0.9], \\ [0.2,0.5],0.6,0.5,0.4 \end{pmatrix} & \begin{pmatrix} [0.4,0.7],[0.2,0.8], \\ [0.7,0.3],0.6,0.7,0.7 \end{pmatrix} & \begin{pmatrix} [0.2,0.6],[0.5,0.9], \\ [0.2,0.8],0.4,0.4,0.8 \end{pmatrix}
\end{pmatrix} \quad (2)
$$

Step2. Let $W = (0.4, 0.6)^T$, then the single matrix corresponding to weight W by use of NCWG operator is

Equation (3): The single decision matrix.

$$
\begin{pmatrix}
 & A_1 & A_2 & A_3 \\
X_1 & \begin{pmatrix} [0.2551,0.6000], \\ [0.2885,0.6732], \\ [0.3371,0.6968], \\ 0.7647,0.6041,0.2352 \end{pmatrix} & \begin{pmatrix} [0.1933,0.6062], \\ [0.4430,0.8001], \\ [0.3418,0.8680], \\ 0.6000,0.7000,0.4772 \end{pmatrix} & \begin{pmatrix} [0.2638,0.6581], \\ [0.1999,0.6381], \\ [0.3881,0.8484], \\ 0.4621,0.3881,0.2223 \end{pmatrix} \\
X_2 & \begin{pmatrix} [0.2352,0.5577], \\ [0.6000,0.9000], \\ [0.3000,0.6634], \\ 0.3618,0.7360,0.7000 \end{pmatrix} & \begin{pmatrix} [0.2352,0.5577], \\ [0.6000,0.9000], \\ [0.3000,0.6634], \\ 0.3618,0.7360,0.7000 \end{pmatrix} & \begin{pmatrix} [0.5253,0.8670], \\ [0.1515,0.6000], \\ [0.4621,0.8448], \\ 0.3371,0.4621,0.3301 \end{pmatrix} \\
X_3 & \begin{pmatrix} [0.5378,0.9000], \\ [0.3565,0.8385], \\ [0.3418,0.8484], \\ 0.6319,0.6319,0.7130 \end{pmatrix} & \begin{pmatrix} [0.2000,0.5378], \\ [0.2352,0.7000], \\ [0.4279,0.8000], \\ 0.5295,0.6634,0.2885 \end{pmatrix} & \begin{pmatrix} [0.3680,0.6325], \\ [0.4210,0.9000], \\ [0.1614,0.5000], \\ 0.5626,0.5426,0.4000 \end{pmatrix} \\
X_4 & \begin{pmatrix} [0.5101,0.8000], \\ [0.3465,0.6325], \\ [0.2416,0.7449], \\ 0.5000,0.6133,0.3807 \end{pmatrix} & \begin{pmatrix} [0.3031,0.7000], \\ [0.2639,0.8385], \\ [0.3881,0.7000], \\ 0.6000,0.6041,0.6118 \end{pmatrix} & \begin{pmatrix} [0.2352,0.5578], \\ [0.5000,0.9000], \\ [0.2416,0.7647], \\ 0.3618,0.3618,0.8000 \end{pmatrix}
\end{pmatrix} \quad (3)
$$

Step3. Let the weight of attributes are $W = \{0.35, 0.30, 0.35\}$, using NCWG operators on attributes A's we get Equation (4),

$$NCWG = \begin{pmatrix} X_1 & \begin{Bmatrix} [0.2375, 0.6195], \\ [0.2885, 0.7916], \\ [0.3567, 0.8146], \\ 0.6315, 0.5757, 0.2851 \end{Bmatrix} \\ X_2 & \begin{Bmatrix} [0.4426, 0.7657], \\ [0.2165, 0.5915], \\ [0.5382, 0.7804], \\ 0.4827, 0.5729, 0.5282 \end{Bmatrix} \\ X_3 & \begin{Bmatrix} [0.3500, 0.6616], \\ [0.3335, 0.8142], \\ [0.3131, 0.7498], \\ 0.5791, 0.6133, 0.4439 \end{Bmatrix} \\ X_4 & \begin{Bmatrix} [0.3327, 0.6774], \\ [0.3630, 0.7787], \\ [0.2888, 0.7396], \\ 0.4906, 0.5359, 0.5692 \end{Bmatrix} \end{pmatrix} \tag{4}$$

Step4. Using the score function we rank the alternatives as:
$S(X_1) = 0.0321$, $S(X_2) = 0.0548$, $S(X_3) = 0.0839$ and $S(X_4) = -0.0969$, $X_3 > X_2 > X_1 > X_4$
The most desirable alternative is X_3.

7. Conclusions

Dealing with real life problems, decision makers encounter incomplete and vague data. The characteristics of neutrosophic cubic sets enablesdecision makers to deal with such a situation. Consequently, for each situation we defined the algebraic and Einstein sum, product and scalar multiplication. It is often difficult to compare two or more neutrosophic cubic values. The score and accuracy functions are defined to compare the neutrosophic cubic values values. Using these operations we defined neutrosophic cubic geometric, neutrosophic cubic weighted geometric, neutrosophic cubic Einstein geometric, and neutrosophic cubic Einstein weighted geometric aggregation operators with some useful properties. In the next section, a multi-criteria decision making algorithm was constructed. In the last section, a daily life problem was solved usingmulti-criteria decision making method (MCDM). This paper is based on some basic definitions and aggregation operators, which can be further extended to new horizons, like neutrosophic cubic hybrid geometric and neutrosophic cubic Einstein hybrid geometric aggregation operators.

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