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# Neutrosophic $G$ -modules

**Kemale Veliyeva** $^{\star}$

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**Abstract.** *The concept of a neutrosophic set was introduced by Smarandache. This theory is a generalization of classical sets, fuzzy set theory, intuitionistic fuzzy set theory, etc. Some works have been done on neutrosophic sets by some researchers in many area of mathematics. In this paper, we introduce the notion of neutrosophic G -modules and established many results.*

**Keywords.** Neutrosophic set, neutrosophic module, fuzzy $G$ -modules, intuitionistic fuzzy $G$ -modules

**Mathematics Subject Classification (2010):** 03F55

## 1 Introduction.

The concept of a neutrosophic set was introduced by Smarandache [17]. This theory is a generalization of classical sets, fuzzy set theory [19], intuitionistic fuzzy set theory [1], etc. Some works have been done on neutrosophic sets by some researchers in many area of mathematics [3,6,11].

Algebraic structures play a vital role in Mathematics and numerous applications of these structures are seen in many disciplines such as computer sciences, information sciences, theoretical physics, control engineering and so on. This inspires researchers to study and carry out research in various concepts of abstract algebra in fuzzy setting. Biswas [4] applied the concept of intuitionistic fuzzy sets to the theory of groups and studied intuitionistic fuzzy subgroups of a group. Fuzzy submodules of a module $M$ over a ring $R$ were first introduced by Naevoita and Ralescu [5,8,12,13,20]. Since then different types of fuzzy submodules were investigated in the last two decades. Fuzzy soft modules and intuitionistic fuzzy soft modules was given and researched by C. Gunduz (Aras) and S.Bayramov [9,10]. Shery Fernadez introduced the notion of fuzzy $G$ -modules in [7].

P.K. Sharma and Tarandeep Kaur introduced the notion of intuitionistic fuzzy $G$ -modules on a $G$ -module $M$ over a field $K$ [14]. Also they defined and discussed the quotient intuitionistic fuzzy $G$ -modules and established a homomorphism of $G$ -module $M$ onto $M^*$ [15,16].

In this paper, we introduce the notion of neutrosophic $G$ -modules and established many results.

$^{\star}$ Corresponding author

K. Veliyeva
Baku State University, Baku, Azerbaijan
E-mail: kemale2607@mail.ru

## 2 Preliminares.

In this section, we will give some preliminary information for the present study.

**Definition 2.1** *[17] A neutrosophic set $A$ on the universe of discourse $X$ is defined as:*

$$A = \{\langle x, T_A(x), I_A(x), F_A(x)\rangle : x \in X\},$$

*where* $T, I, F : X \to ]^{-}0, 1^{+}[$ *and* $^{-}0 \leq T_A(x) + I_A(x) + F_A(x) \leq^{+} 3$.

**Definition 2.2** *[2] Let $M$ be a module over a ring $R$. An intuitionistic fuzzy set $A = (\mu_A, \lambda_A)$ in $M$ is called an intuitionistic fuzzy submodule of $M$ if*
*1)* $\mu_A(0) = 1$,
*2)* $\min\{\mu_A(x), \mu_A(y)\} \leq \mu_A(x - y)$ *for all* $x, y \in M$,
*3)* $\mu_A(x) \leq \mu_A(r, x)$ *for all* $x \in M$ *and* $r \in R$,
*4)* $\lambda_A(0) = 0$,
*5)* $\lambda_A(x - y) \leq max\{\lambda_A(x), \lambda_A(y)\}$ *for all* $x, y \in M$,
*6)*$\lambda_A(r, x) \leq \lambda_A(x)$ *for all* $x \in M$ *and* $r \in R$.
*Let $A = (\mu, \lambda)$ be intuitionistic fuzzy submodule of $M$. We denote this module by $(M, \mu, \lambda)$ . We say this module as intuitionistic fuzzy module.*

**Definition 2.3** *[18] Let $M$ be a left $R$ -module and let $A = (T, I, F)$ be a neutrosophic set over M. Then we say $(M, T, I, F)$ is a neutrosophic modul, if the following conditions are satisfied:*
*a)* $T(0) = I(0) = 1; F(0) = 0$
*b)* $T(x + y) \geq T(x) \wedge T(y); I(x + y) \geq I(x) \wedge I(y); F(x + y) \leq F(x) \vee F(y)$
*c)* $T(\lambda x) \geq T(x); I(\lambda x) \geq I(x); F(\lambda x) \leq F(x)$

**Definition 2.4** *[18] $f : (M_1, T_1, I_1, F_1) \to (M_2, T_2, I_2, F_2)$ is homomorphism of neutrosophic modules if and only if the condition $T_2(f(x)) \geq T(x)$, $I_2(f(x)) \geq I_1(x)$, and $F_2(f(x)) \leq F_1(x)$ are satisfied.*

**Definition 2.5** *[14] Let G be a group and $M$ be a vector space over a field $K$ . Then $M$ is called a $G$ -module if for every $g \in G$ and $m \in M, \exists$ a product (called the action of G on $M$) $gm \in M$ satisfies the following axioms*
*i)* $1_G \cdot m = m, \forall m \in M$ *($1_G$ being the identity of G)*
*ii)* $(g \cdot h) \cdot m = g \cdot (h \cdot m), \forall m \in M, g, h \in G$
*iii)* $g \cdot (k_1 m_1 + k_2 m_2) = k_1 (g \cdot m_1) + k_2 (g \cdot m_2), \forall k_1, k_2 \in K; m_1, m_2 \in M, g \in G$.

**Definition 2.6** *[14] Let G be a group and $M$ be a G-module over $K$ . Then a intuitionistic fuzzy G-module on $M$ is an intuitionistic fuzzy set $A = (\mu_A, \nu_A)$ of $M$ such that following conditions are satisfied*
*i)*$\mu_A(ax + by) \geq \mu_A(x) \wedge \mu_A(y)$ *and* $\nu_A(ax + by) \leq \nu_A(x) \wedge \nu_A(y)$, $\forall a, b \in K$ *and* $x, y \in M$
*ii)*$\mu_A(gm) \geq \mu_A(m)$ *and* $\nu_A(gm) \leq \nu_A(m), \forall g \in G, \quad m \in M$.

## 3 Neutrosophic $G$ -modules

**Definition 3.1** *Let $G$ be a group and $M$ be a $G$ -module over $K$ . Then neutrosophic $G$ -module on $M$ is a neutrosophic set $A = (T, I, F)$ of $M$ such that following conditions are satisfied*

$$(i) \quad \begin{array}{l} T_A(ax+by) \geq T_A(x) \wedge T_A(y) \\ I_A(ax+by) \geq I_A(x) \wedge I_A(y), \ \forall a,b \in K \text{ and } x,y \in M. \\ F_A(ax+by) \leq F_A(x) \vee F_A(y) \end{array}$$

$$(ii) \quad \begin{array}{l} T_A(gm) \geq T_A(m) \\ I_A(gm) \geq I_A(m), \ \forall g \in G, m \in M. \\ F_A(gm) \leq F_A(m) \end{array}$$

**Theorem 3.1** *Let $M$ be a $G$ -module over $K$ , and $A = (T, I, F)$ be a neutrosophic $G$ -module on $M$ . Then $Supp_M(A)$ is a $G$ -submodule of $M$.*

**Proof.** Let $x, y \in \text{supp}_M(A)$ and $a, b \in K$, then $T_A(x) > 0$, $I_A(x) > 0$, $F_A(x) < 1$, and

$$\begin{array}{l} T_A(y) > 0, \ I_A(y) > 0, \ F_A(y) < 1 \Rightarrow \\ T_A(ax+by) \geq T_A(x) \wedge T_A(y) > 0 \\ I_A(ax+by) \geq I_A(x) \wedge I_A(y) > 0, \\ F_A(ax+by) \leq F_A(x) \vee F_A(y) < 1. \end{array}$$

Therefore, $ax + by \in Supp_M(A)$.

Also, $T_A(gx) \geq T_A(x) > 0$, $I_A(gx) \geq I_A(x) > 0$, $F_A(gx) \leq F_A(x) < 1$, for every $g \in G$ and $x \in M$. Therefore, $gx \in Supp_M(A)$. Hence $Supp_M(A)$ is a $G$ -submodule of $M$.

**Theorem 3.2** *Let $M$ be a $G$ -module over $K$ and $A = (T, I, F)$, $B = (T', I', F')$ be two neutrosophic $G$ -modules on $M$ . Then $A \cap B$ is also a neutrosophic G-module on $M$.*

**Proof.** Let $a, b \in K$ and $x, y \in M$, then

$$T_{A\cap B}(ax+by) = T_A(ax+by) \wedge T_B(ax+by) \geq \{T_A(x) \wedge T_A(y)\} \wedge$$

$$\wedge \{T_B(x) \wedge T_B(y)\} = \{T_A(x) \wedge T_B(x)\} \wedge \{T_A(y) \wedge T_B(y)\} = T_{A\cap B}(x) \wedge T_{A\cap B}(y)$$

Thus, $T_{A\cap B}(ax+by) \geq T_{A\cap B}(x) \wedge T_{A\cap B}(y)$.

Similarly, we can show that $I_{A\cap B}(ax+by) \geq I_{A\cap B}(x) \wedge I_{A\cap B}(y)$.

$$F_{A\cap B}(ax+by) = F_A(ax+by) \vee F_B(ax+by) \leq$$

$$\leq \{F_A(y) \vee F_B(y)\} \vee \{F_B(x) \vee F_B(y)\} =$$

$$= \{F_A(x) \vee F_B(x)\} \vee \{F_A(y) \vee F_B(y)\} = F_{A\cap B}(x) \vee F_{A\cap B}(y)$$

Thus, $F_{A\cap B}(ax+by) \leq F_{A\cap B}(x) \vee F_{A\cap B}(y)$.

For, $g \in G$ and $z \in M$ , we have $T_{A\cap B}(gz) = T_A(gz) \wedge T_B(gz) \geq T_A(z) \wedge T_B(z)$ $= T_{A\cap B}(z)$, i.e., $T_{A\cap B}(gz) \geq T_{A\cap B}(z)$.

Similarly, we can show that

$I_{A\cap B}(gz) \geq I_{A\cap B}(z)$.

$F_{A\cap B}(gz) = F_A(gz) \vee F_B(gz) \leq F_A(z) \vee F_B(z) = F_{A\cap B}(z)$, i.e.,

$F_{A\cap B}(gz) \leq F_{A\cap B}(z)$.

Hence $A \cap B$ is a neutrosophic $G$ -module on $M$.

**Theorem 3.3** *Let $M$ be a $G$ -module over $K$ and $\{A_i = (T_i, I_i, F_i); i = 1, 2, ...\}$ be a family of neutrosophic G-modules on $M$. Then $\bigcap_{i=1}^{\infty} A_i$ is also neutrosophic $G$ -module on $M$.*

**Theorem 3.4** *Let $M_1, M_2$ be a G-modules over $K$ and $A, B$ be neutrosophic G-modules on $M_1$ and $M_2$ respectively. Then $A \times B$ is also a neutrosophic G-module on $M_1 \times M_2$.*

**Proof.** Let $a, b \in K$ and $x = (x_1, y_1), \; y = (x_2, y_2) \in M_1 \times M_2$, then
$T_{A\times B}(ax+by) = T_{A\times B}\{a(x_1,y_1)+b(x_2,y_2)\} = T_{A\times B}\{(ax_1+bx_2),(ay_1+by_2)\}$
$= T_A(ax_1+bx_2) \wedge T_B(ay_1+by_2) \geq \{T_A(x_1)\wedge T_A(x_2)\} \wedge \{T_B(y_1)\wedge T_B(y_2)\}$
$= \{T_A(x_1)\wedge T_B(y_1)\} \wedge \{T_A(x_2)\wedge T_B(y_2)\} = T_{A\times B}(x_1,y_1) \wedge T_{A\times B}(x_2,y_2)$
Thus, $T_{A\times B}(ax+by) \geq T_{A\times B}(x) \wedge T_{A\times B}(y)$.
Similarly, we can show that $I_{A\times B}(ax+by) \geq I_{A\times B}(x) \wedge I_{A\times B}(y)$.

$$F_{A\times B}(ax+by) = F_{A\times B}\{a(x_1,y_1)+b(x_2,y_2)\} = F_{A\times B}\{(ax_1+bx_2),(ay_1+by_2)\}$$

$$= F_A(ax_1+bx_2) \vee F_b(ay_1+by_2) \leq \{F_A(x_1)\vee F_A(x_2)\} \vee \{F_B(y_1)\vee F_b(y_2)\}$$

$$= \{F_A(x_1)\vee F_B(y_1)\} \vee \{F_A(x_2)\vee F_B(y_2)\} = F_{A\times B}(x_1,y_1) \vee F_{A\times B}(x_2,y_2)$$

Thus, $F_{A\times B}(ax+by) \leq F_{A\times B}(x) \vee F_{A\times B}(y)$.
For, $g \in G$ and $z = (x, y) \in M_1 \times M_2$, we have

$$T_{A\times B}(gz) = T_{A\times B}\{g(x,y)\} = T_{A\times B}(gx,gy)$$

$$= T_A(gx) \wedge T_B(gy) \geq T_A(x) \wedge T_B(y) = T_{A\times B}(z), \text{ i.e.,}$$

$$T_{A\times B}(gz) \geq F_{A\times B}(z).$$

Similarly, we can show that

$$I_{A\times B}(gz) \geq I_{A\times B}(z).$$

$$F_{A\times B}(gz) = F_{A\times B}\{g(x,y)\} = F_{A\times B}(gx,gy)$$

$$= F_A(gx) \vee F_B(gy) \leq F_A(x) \vee F_B(y) = F_{A\times B}(z), \text{ i.e.,}$$

$$F_{A\times B}(gz) \leq F_{A\times B}(z).$$

Hence $A \times B$ is a neutrosophic $G$-module on $M_1 \times M_2$.

**Definition 3.2.** Let $M$ be a $G$-module $K$ over and $A = (T_A, I_A, F_A)$ and $B = (T_B, I_B, F_B)$ be neutrosophic $G$-modules on $M$, then their sum $A+B = (T_{A+B}, I_{A+B}, F_{A+B})$ is defined as

$$T_{A+B}(x) = \bigvee_{x=a+b} \{T_A(a) \wedge T_B(b)\},$$

$$I_{A+B}(x) = \bigvee_{x=a+b} \{I_A(a) \wedge I_B(b)\},$$

$$F_{A+B}(x) = \bigwedge_{x=a+b} \{F_A(a) \vee F_B(b)\},$$

for all $x \in M$.

**Theorem 3.5** *Let $M$ be a G-module over $K$ and $A, B$ be two neutrosophic G-modules on $M$.*
*Then $A + B$ is also a neutrosophic G-module on $M$ .*

**Proof.** Let $x, y \in M$ be any two elements and let $\min\{T_{A+B}(x), T_{A+B}(y)\} = \alpha$.

Let $\varepsilon > 0$ be given, then on $\alpha - \varepsilon < T_{A+B}(x) = \underset{x=a+b}{\vee}\{T_A(a) \wedge T_B(b)\}$ and $\alpha - \varepsilon < T_{A+B}(y) = \underset{y=c+d}{\vee}\{T_A(c) \wedge T_B(d)\}$ so there exists a representation $x = a + b, y = c + d$, where $a, b, c, d \in M$ such that $\alpha - \varepsilon < T_A(a) \wedge T_B(b)$ and $\alpha - \varepsilon < T_A(c) \wedge T_B(d) \Longrightarrow \alpha - \varepsilon < T_A(a)$, $\alpha - \varepsilon < T_B(b)$ and $\alpha - \varepsilon < T_A(c), \alpha - \varepsilon < T_B(d) \Longrightarrow \alpha - \varepsilon < T_A(a) \wedge T_B(c) \leq T_A(a + c)$ and $\alpha - \varepsilon < T_A(b) \wedge T_B(d) \leq T_B(b + d)$.

Thus, we get $x + y = (a + b) + (c + d) = (a + c) + (b + d)$ such that
$\Longrightarrow \alpha - \varepsilon < T_A(a + c) \wedge T_B(b + d)$
$\Longrightarrow \alpha - \varepsilon < \underset{x+y=(a+c)+(b+d)}{\vee}\{T_A(a + c) \wedge T_B(b + d)\} = T_{A+B}(x + y)$.

Since $\varepsilon$ is arbitrary, it follows that $T_{A+B}(x + y) \geq \alpha = T_{A+B}(x) \wedge T_{A+B}(y)$.

Similarly, we can show that $I_{A+B}(x + y) \geq I_{A+B}(x) \wedge I_{A+B}(y)$ and $F_{A+B}(x + y) \leq F_{A+B}(x) \vee F_{A+B}(y)$.

Further, let $\beta = T_{A+B}(x) \vee T_{A+B}(y) = T_{A+B}(x)$, and let $\varepsilon > 0$, then
$\beta - \varepsilon < T_{A+B}(x) = \underset{x=a+b}{\vee}\{T_A(a) \wedge T_B(b)\}$, so there exists a representation $x = a+b$ such that
$\beta - \varepsilon < T_A(a) \wedge T_B(b) \Longrightarrow \beta - \varepsilon < T_A(a)$, $\beta - \varepsilon < T_B(b) \Longrightarrow$
$\beta - \varepsilon < T_A(a) \leq T_A(ka)$, $\beta - \varepsilon < T_B(b) \leq T_B(kb)$ for any $k \in K \Longrightarrow \beta - \varepsilon < T_A(ka) \wedge T_B(kb)$ for any $k \in K$.

Now, $kx = k(a + b) = ka + kb$ so that
$\beta - \varepsilon < T_A(ka) \wedge T_B(kb) \Longrightarrow \beta - \varepsilon < \underset{kx=k(a+b)}{\vee}\{T_A(ka) \wedge T_B(kb)\} = T_{A+B}(kx)$.

Since $\varepsilon$ is arbitrary, it follows that $T_{A+B}(kx) \geq \beta = T_{A+B}(x)$.

Similarly, we can show that
$I_{A+B}(kx) \geq \beta = I_{A+B}(x)$ and $F_{A+B}(kx) \leq F_{A+B}(x)$.

Further, let $g \in G$ and $x \in M$ be any element, then $T_{A+B}(x) = \underset{x=a+b}{\vee}\{T_A(a) \wedge T_B(b)\}$

Now, $T_A(a) \leq T_A(ga)$, $T_B(b) \leq T_B(gb) \Longrightarrow T_A(a) \wedge T_B(b) \leq T_A(ga) \wedge T_B(gb)$.

Also, $gx = g(a + b) = ga + gb$

$$\beta - \varepsilon < T_{A+B}(x) = \underset{x=a+b}{\vee}\{T_A(a) \wedge T_B(b)\} \leq$$

$$\leq \underset{gx=g(a+b)}{\vee}\{T_A(ga) \wedge T_B(gb)\} = T_{A+B}(gx) \text{ i.e., } T_{A+B}(gx) \geq T_{A+B}(x).$$

Similarly, we can show that $I_{A+B}(gx) \geq I_{A+B}(x)$ and $F_{A+B}(gx) \leq F_{A+B}(x)$.

Hence $A + B$ is a neutrosophic $G$-module on $M$.

**Definition 3.3.** Let $M$ be a $G$-module over $K$ and $A = (T_A, I_A, F_A)$ and $B = (T_B, I_B, F_B)$ be neutrosophic $G$-modules on $M$, then their product is $AB = (T_{AB}, I_{AB}, F_{AB})$ defined as

$$T_{AB} = \underset{x=\sum\limits_{i<\infty}(a_i+b_i)}{\vee}\left\{\underset{i}{\wedge}\left(T_A(a_i) \wedge T_B(b_i)\right)\right\},$$

$$I_{AB} = \underset{x=\sum\limits_{i<\infty}(a_i+b_i)}{\vee}\left\{\underset{i}{\wedge}\left(I_A(a_i) \wedge I_B(b_i)\right)\right\},$$

$$F_{AB} = \underset{x=\sum\limits_{i<\infty}(a_i+b_i)}{\wedge}\left(\left\{\underset{i}{\vee}F_A(a_i) \vee F_B(b_i)\right\}\right),$$

for all $x \in M$.

**Theorem 3.6** *Let $M$ be a G-module over $K$ and $A, B$ be two neutrosophic G-modules on $M$ . Then $AB$ is also a neutrosophic $G$–module on $M$ .*

**Proof.** Let $x, y \in M$ be any two elements and let $T_{AB}(x) \wedge T_{AB}(y) = \alpha$.

Let $\varepsilon > 0$ be given, then on $\alpha - \varepsilon < T_{AB}(x) = \bigvee_{x=\sum\limits_{i<\infty}(a_i+b_i)} \left\{ \bigwedge_i (T_A(a_i) \wedge T_B(b_i)) \right\}$

and

$$\alpha-\varepsilon < T_{AB}(y) = \bigvee_{y=\sum\limits_{i<\infty}(p_i+q_i)} \left\{ \bigwedge_i (T_A(p_i) \wedge T_B(q_i)) \right\} \Longrightarrow \alpha-\varepsilon < \bigwedge_i \{(T_A(a_i) \wedge T_B(b_i))\}$$

and

$\alpha - \varepsilon < \bigwedge_i \{(T_A(p_i) \wedge T_B(q_i))\}$ , for all $i$.

$\Longrightarrow \alpha - \varepsilon < T_A(a_i), \wedge T_B(b_i)$ and $\alpha - \varepsilon < T_A(p_i) \wedge T_B(q_i)$ for all $i$.

$\Longrightarrow \alpha - \varepsilon < T_A(a_i),\ \alpha - \varepsilon < T_B(b_i)$ and $\alpha - \varepsilon < T_A(p_i), \alpha - \varepsilon < T_B(q_i)$ for all $i$.

$\Longrightarrow \alpha-\varepsilon < T_A(a_i) \wedge T_A(p_i) \leq T_A(a_i+p_i)$ and $\alpha-\varepsilon < T_B(b_i) \wedge T_B(q_i) \leq T_B(b_i+q_i)$, for all $i$.

Thus , we get $x + y = \sum((a_i + b_i) + (p_i + q_i))$, where $a_i, b_i, p_i, q_i \in M$ such that $\alpha - \varepsilon < T_A(a_i + p_i) \wedge T_B(b_i + q_i)$, for all

$i \Longrightarrow \alpha - \varepsilon < \bigwedge_i \{(T_A(a_i + p_i) \wedge T_B(b_i + q_i))\}$

$\Longrightarrow \alpha - \varepsilon < \bigwedge_{x+y=\sum\limits_{i<\infty}((a_i b_i)+(p_i q_i))} \bigwedge_i \{T_A(a_i + p_i) \wedge T_B(b_i + q_i)\} = T_{AB}(x + y)$

Since $\varepsilon > 0$ is arbitrary, so we have $T_{AB}(x + y) \geq \alpha = T_{AB}(x) \wedge T_{AB}(y)$.

Similarly, we can show that $I_{AB}(x + y) \geq \alpha = I_{AB}(x) \wedge I_{AB}(y)$ and $F_{AB}(x + y) \leq F_{AB}(x) \vee F_{AB}(y)$.

Further, let $\beta = T_{AB}(x) \vee T_{AB}(y) = T_{AB}(x)$, and let $\varepsilon > 0$, then

$\beta - \varepsilon < T_{AB}(x) = \bigwedge_{x=\sum\limits_{i<\infty}(a_i+b_i)} \bigwedge_i \{T_A(a_i) \wedge T_B(b_i)\}$ ,so there exists a representation $x = \sum\limits_{i<\infty} (a_i + b_i)$ such that

$\beta - \varepsilon < \bigwedge_i \{T_A(a_i) \wedge T_B(b_i)\} \Longrightarrow \beta - \varepsilon < T_A(a_i) \wedge T_B(b_i)$ for all $i$

$\Longrightarrow \beta - \varepsilon < T_A(a_i), \beta - \varepsilon < T_B(b_i) \Longrightarrow \beta - \varepsilon < T_A(a_i) \leq T_A(ka_i), \beta - \varepsilon < T_B(b_i) \leq T_B(kb_i)$ for all $k \in K \Longrightarrow \beta - \varepsilon < T_A(ka_i) \wedge T_B(kb_i)$ for all $i$.

Hence

$\beta - \varepsilon < \bigwedge_i \{T_A(ka_i) \wedge T_B(b_i)\} < \bigwedge_{kx=\sum\limits_{i<\infty} k(a_i+b_i)} \bigwedge_i \{T_A(ka_i) \wedge T_B(b_i)\} = T_{AB}(kx)$.

Since $\varepsilon > 0$ is arbitrary, so we have $T_{AB}(kx) \geq \beta = T_{AB}(x)$.

Similarly, we can show that $I_{AB}(kx) \geq \beta = I_{AB}(x)$ and $F_{AB}(kx) \leq F_{AB}(x)$.

Further, $g \in G$ let and $x \in M$ be any element, then

$T_{AB}(x) = \bigwedge_{kx=\sum\limits_{i<\infty}(a_i+b_i)} \left\{ \bigwedge_i (T_A(a_i) \wedge T_B(b_i)) \right\}$.

Now, $T_A(a_i) \leq T_A(ga_i) \Longrightarrow T_A(a_i) \wedge T_A(b_i) \leq T_A(ga_i) \wedge T_A(gb_i)$, for all $i$

$\Longrightarrow \bigwedge_i T_A(a_i) \wedge T_B(b_i) \leq \bigwedge_i (T_A(ga_i) \wedge T_B(gb_i))$ , for all $i$

$$\Longrightarrow \bigwedge_{x=\sum\limits_{i<\infty}(a_i+b_i)} \left\{ \bigwedge_i (T_A(a_i) \wedge T_B(b_i)) \right\}$$

$$\leq \bigvee_{gx=\sum\limits_{i<\infty} g(a_i+b_i)} \left\{ \bigwedge_i T_A(ga_i) \wedge T_B(gb_i) \right\} \text{ i.e,}$$

$$T_{AB}(x) = \bigwedge_{x=\sum\limits_{i<\infty}(a_i+b_i)} \left\{ \bigwedge_i (T_A(a_i) \wedge T_B(b_i)) \right\}$$

$$\leq \bigwedge_{gx=\sum\limits_{i<\infty} g(a_i+b_i)} \left\{ \bigwedge_i (T_A(ga_i) \wedge T_B(gb_i)) \right\} = T_{AB}(gx).$$

Similarly, we can show that $I_{AB}(gx) \geq I_{AB}(x)$ and $F_{AB}(gx) \leq F_{AB}(x)$.

Hence $AB$ is a neutrosophic $G$-module on $M$.

**Definition 3.4.** Let $M$ be a $G$-module over $K$ and $A$ be neutrosophic $G-$module on $M$. Let $N$ be a $G$-submodule of $M$. Then the restriction of $A$ on $N$ is denoted by $A|_N$ is a neutrosophic set on $N$ defined as

$(A|_N)(x) = \left(T_{A|_N}(x), I_{A|_N}(x), F_{A|_N}(x)\right)$, where , $T_{A|_N}(x) = T_A(x)$, $I_{A|_N}(x) = I_A(x)$, $F_{A|_N}(x) = F_A(x)$, $\forall x \in N$.

**Proposition 3.1.** If $A$ be neutrosophic $G$-module of a $G$-module $M$ over $K$ and let $N$ be a $G$ -submodule of $M$. Then $A|_N$ is a neutrosophic $G$-module of $N$.

**Proof.** Let $a, b \in K$ and $x, y \in N$ , then

$T_{A|_N}(ax+by) = T_A(ax+by) \geq T_A(x) \wedge T_A(y), T_{A|_N}(x) \vee T_{A|_N}(y)$, $\forall (ax+by) \in N$.

Thus, $T_{A|_N}(ax+by) \geq T_{A|_N}(x) \wedge T_{A|_N}(y)$.

Similarly, we can show that $I_{A|_N}(ax+by) \geq I_{A|_N}(x) \wedge I_{A|_N}(y)$.

$F_{A|_N}(ax+by) = F_A(ax+by) \leq F_A(x) \vee F_A(y) = F_{A|_N}(x) \vee F_{A|_N}(y)$, $\forall (ax+by) \in N$.

Thus, $F_{A|_N}(ax+by) \leq F_{A|_N}(x) \vee F_{A|_N}(y)$.

For, $g \in G$ and $z \in N$, we have , $T_{A|_N}(gz) = T_A(gz) \geq T_A(z)$, $\forall gz \in N$.

Similarly, we can show that $I_{A|_N}(gz) \geq I_A(z)$ .

$F_{A|_N}(gz) = F_A(gz) \leq F_A(z)$.

Hence $A|_N$ is a neutrosophic $G$-module on $N$ .

**Proposition 3.2.** Let $M$ be a $G$ -module over $K$ and $K$ be a $G$-submodule of $M$. Then the neutrosophic set $A_{M/N}$ on $M/N$ defined by

$$T_{A_N}(x+N) = \vee \{T_A(x+n) : n \in N\},\ I_{A_N}(x+N) = \vee \{I_A(x+n) : n \in N\} \text{ and}$$

$$F_{A_N}(x+N) = \wedge \{F_A(x+n) : n \in N\},\ \forall x \in M,$$

is a neutrosophic $G$-module on $M/N$ .

**Proof.** For $a, b \in K$ and $x, y \in M$, we have

$$T_{A_N}\{a(x+N) + b(y+N)\} = T_{A_N}\{(ax+by)+N)\}$$

$$= \vee \{T_A(\{ax+by\}+n) : n \in N\}$$

$$= \vee \{T_A(\{ax+by\}+an_1+bn_2) : n_1, n_2 \in N\}$$

$$= \vee \{T_A(\{a(x+n_1)+b(y+n_2)\}) : n_1, n_2 \in N\}$$

$$\geq \vee\{T_A\{a(x+n_1)\} \wedge T_A\{b(y+n_2)\} : n_1, n_2 \in N\}$$

$$\geq \vee \{T_A(x+n_1) \wedge T_A(y+n_2) : n_1, n_2 \in N\} \geq [\vee \{T_A(x+n_1) : n_1 \in N\}] \wedge$$

$$[\vee \{T_A(y+n_2) : n_2 \in N\}] = T_A(x+N) \wedge T_A(y+N)$$

where $n = an_1 + bn_2$ , for some $n_1, n_2 \in N$.

Thus, $T_{A_N}\{a(x+N)+b(y+N)\} \geq T_A(x+N) \wedge T_A(y+N)$.

Similarly, we can show that $I_{A_N}\{a(x+N)+b(y+N)\} \geq I_A(x+N) \wedge I_A(y+N)$.

$$F_{A_N}\{a(x+N)+b(y+N)\} = F_{A_N}\{(ax+by)+N\} = \wedge \{F_A(\{ax+by\}+n) : n \in N\}$$

$$= \wedge \{F_A(\{ax+by\}+an_1+bn_2) : n_1, n_2 \in N\}$$
$$= \wedge \{F_A(\{a(x+n_1)+b(y+n_2)\}) : n_1, n_2 \in N\}$$
$$\leq \wedge \{F_A\{a(x+n_1)\} \vee F_A : \{b(y+n_2)\} : n_1, n_2 \in N\}$$
$$\leq \wedge \{F_A(x+n_1) \vee F_A(y+n_2) : n_1, n_2 \in N\}$$
$$\leq [\wedge \{F_A(x+n_1) : n_1 \in N\}] \vee [\wedge \{F_A(y+n_2) : n_2 \in N\}] = F_A(x+N) \vee F_A(y+N)$$

where $n = an_1 + bn_2$, for some $n_1, n_2 \in N$.

Thus $F_{A_N} \{a(x+N)+b(y+N)\} \leq F_A(x+N) \vee F_A(y+N)$.

Also,

$$T_{A_N}[g(x+N)] = T_{A_N}(gx+N) = \vee \{T_A(gx+N) : n \in N\}$$
$$= \vee \{T_A(gx+gn_3) : n_3 \in N\} = \vee \{T_A(g(x+n_3)) : n_3 \in N\}$$
$$\geq \vee \{T_A(x+n_3) : n_3 \in N\} = T_{A_N}(x+N).$$

Thus $T_{A_N}[g(x+N)] \geq T_{A_N}(x+N)$.

Similarly, we can show that $I_{A_N}[g(x+N)] \geq I_{A_N}(x+N)$.

$$F_{A_N}[g(x+N)] = F_{A_N}(gx+N)$$
$$= \wedge \{F_{A_N}(gx+n) : n \in N\} = \wedge \{F_A(gx+gn_3) : n_3 \in N\}$$
$$= \wedge \{F_A(g(x+n_3)) : n_3 \in N\} \leq \wedge \{F_A(x+n_3) : n_3 \in N\} = F_{A_N}(x+N).$$

Thus $F_{A_N}[g(x+N)] \leq F_{A_N}(x+N)$.

Therefore, $A_N = (T_{A_N}, I_{A_N}, F_{A_N})$ is neutrosophic $G$-module on $M/N$.

**Remark 3.1.** The neutrosophic $G$-module $A_{M/N}$ defined on $M/N$, as defined above, is called the quotient neutrosophic $G$-module or factor neutrosophic $G$-module of $A$ on $M$ relative to $G$-submodule $N$ .

## References


1. Atanassov K.T.: *Intuitionistic fuzzy sets*, Fuzzy Sets and Systems, **20** (1), 87-96 (1986).
2. Bayramov S.A.: *Fuzzy and Fuzzy Soft structure in Algebra*, Lambert Academic Publishing, (2012).
3. Bera T.: Mahapatra NK.: *On neutrosophic soft function*, Ann Fuzzy Math Inform, **12**, 101-119 (2016).
4. Biswas R.: *Intuitionistic fuzzy subgroups*, Mathematical Forum, **10**, 37-46 (1989).
5. Davvaz B., Dudek W. A., Jun Y. B.: *Intuitionistic fuzzy-submodules*, Information Sciences, **176**, 285-300 (2006).
6. Deli I, Broumi S.: *Neutrosophic soft relations and some properties*, Ann Fuzzy Math Inform, **9**, 169-182 (2015).
7. Fernadez, S.: *Fuzzy-modules and fuzzy representations*, TAJOPAM, **1**, 107-114 (2002).
8. Gunduz(Aras) C., Davvaz B.: *The universal coefficient theorem in the category of intuitionistic fuzzy modules,* Utilitas Math. **81**, 131-156 (2010).
9. Gunduz(Aras) C., Bayramov S.: *Intuitionistic fuzzy soft modules,* Computers and Mathematics with Applications, **62**, 2480-2486 (2011).
10. Gunduz(Aras) C., Bayramov S.: *Fuzzy soft modules*, International Mathematical Forum, **6** (11), 517-527 (2011).
11. Maji P.K.: *Neutrosophic soft set*, Ann Fuzzy Math Inform, **5**, 157-168 (2013).
12. Naegoita, C.V., Ralescu, D.A.: Application of Fuzzy Sets in System Analysis, *Birkhauser, Basel*, (1975).

13. Permouth S.R. L., Malik D.S.: *On categories of fuzzy modules*, Information Sciences **52**, 211-220 (1990).
14. Sharma P.K., Tarandeep K.: *Intuitionistic fuzzy G-modules*, Notes on intuitionistic Fuzzy Sets, **21** (1), 6-23 (2015).
15. Sharma P.K.: *Direct Sum of Intuitionistic Fuzzy Submodules of G-module*, Asian Journal of Fuzzy and Applied Mathematics, **4** (4) 36-46, (2016).
16. Sharma P.K.: *Exact Sequence of intuitionistic fuzzy G-modules*, Notes on intuitionistic Fuzzy Sets, **23** (5), 66-84 (2017).
17. Smarandache, F.: *Neutrosophic set, a generalisation of the intuitionistic fuzzy sets,* Int. J. Pure Appl. Math. **24**, 287-297 (2005).
18. Veliyeva, K., Bayramov, S.: *Inverse system of neutrosophic modules*, Lankaran State University, Scientific News Journal, Department of Natural Sciences, **1**, 18-32, (2018).
19. Zadeh, L.A.: *Fuzzy sets*, Information and Control, **8**, 338-353 (1965).
20. Zahedi, M.M.: *Some results on fuzzy modules*, Fuzzy Sets and Sustems **55**, 355-361 (1993).