Neutrosophic H-Ideal on BCK-Algebras

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Abstract. In this paper, we introduce the notion of neutrosophic H-ideals in BCK-algebras and study their fundamental properties. Then we investigate the relation between neutrosophic H-ideals and intuitionistic H-ideals and fuzzy H-ideals.

Keywords: BCK-Algebra, BCI-Algebra, Neutrosophic, H-Ideal.

AMS Mathematical Subject Classification [2010]: 03B47, 03G25, 06D99.

1. Introduction

Fuzzy set theory, which was introduced by Zadeh in 1965, is the oldest and most widely reported component of present day soft computing, allowing the design of more flexible information processing systems, with applications in different areas, such as artificial intelligence, multiagent systems, machine learning, knowledge discovery, information processing, statistics and data analysis, system modeling, control system, decision sciences, economics, medicine and engineering. The notion of BCK-algebra was introduced first in 1966 by Iseki, Japanese mathematician [2], and has been extensively investigated by many researchers. Ideal theory play an important rule in studying these algebras. From logical point of view, various ideals correspond to various sets of provable formulas. Iseki in 1975 introduced the concept of ideals in BCK-algebras [3]. In 1999, Khalid and Ahmad introduced fuzzy H-ideals in BCI-algebras [5]. In 2003, Zhan and Tan introduced the fuzzy H-ideals in BCK-algebras [10]. In 2010, Satyanarayana et al. introduced intuitionistic fuzzy H-ideals in BCK-algebras [7].
Neutrosophic set theory was introduced by Smarandache in 1998 ([9]). Neutrosophic sets are a new mathematical tool for dealing with uncertainties which are free from many difficulties that have troubled the usual theoretical approaches. Research works on neutrosophic set theory for many applications such as information fussion, probability theory, control theory, decision making, measurement theory, etc. In 2006, Kandasamy and Smarandache introduced the concept of neutrosophic algebraic structures [4]. Since then, several researchers have studied the concepts and a great deal of literature has been produced. Agboola et al. continued the study of some types of neutrosophic algebraic structures [1].
The aim of this paper is to study the concept of neutrosophic H-ideals in BCK-algebras and obtain some results. Then we introduce the consept of $(\alpha, \beta, \gamma)-$level subset of a neutrosophic H-ideal and prove that $(\alpha, \beta, \gamma)-$level subset is a H-ideal of X if and only if $A = (\mu_A, \sigma_A, \nu_A)$ is a neutrosophic H-ideal of X.

Definition 1.1. [6]. Let X be a set with a binary operation $*$ and a constant 0. Then $(X, *, 0)$ of type $(2, 0)$ is called a BCK-algebra if it satisfies the following axioms:
$(BCK1)$ $((x * y) * (x * z)) * (z * y) = 0$,
$(BCK2)$ $(x * (x * y)) * y = 0$,
$(BCK3)$ $x * x = 0$,

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$(BCK4)$ $x * y = 0$ and $y * x = 0$ imply $x = y$,
$(BCK5)$ $0 * x = 0$, for all $x, y, z \in X$.

Definition 1.2. [6]. A nonempty subset I of BCK-algebra X is called an ideal if it satisfies the following conditions:
$(I1)$ $0 \in I$, $(I2)$ $x * y \in I$ and $y \in I$ implies $x \in I$, for all $x, y \in X$.

Definition 1.3. [10]. A fuzzy set μ in X is called a fuzzy H-ideal of X if it satisfies
(1) $\mu(0) \geq \mu(x)$,
(2) $\mu(x * z) \geq min\{\mu(x * (y * z)), \mu(y)\}$, for all $x, y, z \in X$.

Definition 1.4. [8]. Let X be a non-empty set. A neutrosophic set (NS) of X is a structure of the form

$$A := \{< x; \mu_A(x), \sigma_A(x), \nu_A(x) >| \, x \in X\}$$

where $\mu_A : X \to [0,1]$ is a truth membership function, $\sigma_A : X \to [0,1]$ is an indeterminate membership function, and $\nu_A : X \to [0,1]$ is a false membership function. We shall use the symbol $A = (\mu_A, \sigma_A, \nu_A)$ for the neutrosophic set

$$A := \{< x; \mu_A(x), \sigma_A(x), \nu_A(x) >| \, x \in X\}.$$

2. Main results

In this section we denote BCK-algebra $(X, *, 0)$ by X.

Definition 2.1. A neutrosophic set $A = (\mu_A, \sigma_A, \nu_A)$ of X is called a neutrosophic ideal of X, if it satisfies the following conditions:
$(NI1)$ $\mu_A(0) \geq \mu_A(x)$, $\sigma_A(0) \leq \sigma_A(x)$ and $\nu_A(0) \leq \nu_A(x)$,
$(NI2)$ $\mu_A(x) \geq min\{\mu_A(x * y), \mu_A(y)\}$,
$(NI3)$ $\sigma_A(x) \leq max\{\sigma_A(x * y), \sigma_A(y)\}$,
$(NI4)$ $\nu_A(x) \leq max\{\nu_A(x * y), \nu_A(y)\}$, for all $x, y \in X$.

Definition 2.2. A neutrosophic set $A = (\mu_A, \sigma_A, \nu_A)$ of X is called a neutrosophic H-ideal, if it satisfies the following conditions:
$(NHI1)$ $\mu_A(0) \geq \mu_A(x)$, $\sigma_A(0) \leq \sigma_A(x)$ and $\nu_A(0) \leq \nu_A(x)$,
$(NHI2)$ $\mu_A(x * z) \geq min\{\mu_A(x * (y * z)), \mu_A(y)\}$,
$(NHI3)$ $\sigma_A(x * z) \leq max\{\sigma_A(x * (y * z)), \sigma_A(y)\}$,
$(NHI4)$ $\nu_A(x * z) \leq max\{\nu_A(x * (y * z)), \nu_A(y)\}$, for all $x, y, z \in X$.

The set of all neutrosophic H-ideals of X is denoted by $NHI(X)$.

Example 2.3. Let $X = \{0, 1, 2, 3, 4\}$ be a BCK-algebra with the following cayley table we

Table 1. BCK-algebra

$*$	0	1	2	3	4
0	0	0	0	0	0
1	1	0	0	1	0
2	2	2	0	2	0
3	3	3	3	0	3
4	4	4	4	4	0

define neutrosophic set $A = (\mu_A, \sigma_A, \nu_A)$ as the following
then $A = (\mu_A, \sigma_A, \nu_A)$ is a neutrosophic H-ideal of X.

Proposition 2.4. *Every neutrosophic H-ideal of X is a neutrosophic ideal.*

TABLE 2.

A	0	1	2	3	4
μ_A	0.9	0.8	0.5	0.1	0.3
σ_A	0.2	0.6	0.7	0.4	0.8
ν_A	0.1	0.2	0.5	0.6	0.9

THEOREM 2.5. *If X is an associative BCK-algebra, then every neutrosophic ideal is a neutrosophic H-ideal of X.*

EXAMPLE 2.6. In the Example 2.3, X is a non associative BCK-algebra. We define $B = (\mu_B, \sigma_B, \nu_B)$ as the following

TABLE 3.

B	0	1	2	3	4
μ_B	0.9	0.8	0.5	0.4	0.3
σ_B	0.2	0.6	0.7	0.5	0.8
ν_B	0.1	0.2	0.5	0.8	0.9

then B is a neutrosophic ideal but is not a neutrosophic H-ideal of X.

COROLLARY 2.7. *Let X be an associative BCK-algebra. Then A is a neutrosophic ideal of X if and only if A is a neutrosophic H-ideal.*

PROPOSITION 2.8. *Let $A = (\mu_A, \sigma_A, \nu_A)$ be a neutrosophic H-ideal of X. Then (μ_A, ν_A) is an intuitionistic H-ideal of X.*

PROPOSITION 2.9. *Let $A = (\mu_A, \sigma_A, \nu_A)$ be a neutrosophic H-ideal of X. Then μ_A is a fuzzy H-ideal of X.*

PROPOSITION 2.10. *Let $A = (\mu_A, \sigma_A, \nu_A)$ be a neutrosophic ideal of X.*
(i) If $x \leq y$, then $\mu_A(x) \geq \mu_A(y)$, $\sigma_A(x) \leq \sigma_A(y)$ and $\nu_A(x) \leq \nu_A(y)$.
(ii) $\mu_A(x * y) \geq \mu_A(x)$, $\sigma_A(x * y) \leq \sigma_A(x)$ and $\nu_A(x * y) \leq \nu_A(x)$.
(iii) $min\{\mu_A(x), \mu_A(y)\} \leq \mu_A(x * y)$, $min\{\sigma_A(x), \sigma_A(y)\} \geq \sigma_A(x * y)$, $min\{\nu_A(x), \nu_A(y)\} \geq \nu_A(x * y)$.

COROLLARY 2.11. *Let $A = (\mu_A, \sigma_A, \nu_A)$ be a neutrosophic H-ideal of X. If $x \leq y$, then $\mu_A(x) \geq \mu_A(y)$, $\sigma_A(x) \leq \sigma_A(y)$ and $\nu_A(x) \leq \nu_A(y)$.*

THEOREM 2.12. *Let $A = (\mu_A, \sigma_A, \nu_A)$ be a neutrosophic H-ideal of X. Then $A^c = (\nu_A, 1 - \sigma_A, \mu_A)$ is a neutrosophic H-ideal of X if and only if $\mu_A = \sigma_A = \nu_A = 0$.*

THEOREM 2.13. *Let A and B be two neutrosophic H-ideals of X. Then $A \wedge B$ is too.*

COROLLARY 2.14. *Let $\{A_i\}_{i \in I}$ be a family of neutrosophic H-ideals in X. Then $\bigwedge_{i \in I} A_i$ is too.*

THEOREM 2.15. *Let $A = (\mu_A, \sigma_A, \nu_A)$ be a neutrosophic H-ideal of X. Then the sets $\chi_{\mu_A} := \{x \in X : \mu_A(x) = \mu_A(0)\}$, $\chi_{\sigma_A} := \{x \in X : \sigma_A(x) = \sigma_A(0)\}$, $\chi_{\nu_A} := \{x \in X : \nu_A(x) = \nu_A(0)\}$, are H-ideals of X.*

DEFINITION 2.16. Given a neutrosophic set $A = (\mu_A, \sigma_A, \nu_A)$ on X and $\alpha, \beta, \gamma \in [0, 1]$ with $0 \leq \alpha + \beta + \gamma \leq 3$. We define the following sets $\mu^\alpha := \{x \in X : \ \mu_A(x) \geq \alpha\}$, $\sigma^\beta := \{x \in X : \ \sigma_A(x) \leq \beta\}$, $\nu^\gamma := \{x \in X : \ \nu_A(x) \leq \gamma\}$. Then we say that the set

$$A(\alpha, \beta, \gamma) := \{x \in X : \ \mu_A(x) \geq \alpha, \sigma_A(x) \leq \beta, \nu_A(x) \leq \gamma\}$$

is the $(\alpha, \beta, \gamma)-$level subset of A. Obviously, we have

$$A(\alpha, \beta, \gamma) = \mu^\alpha \cap \sigma^\beta \cap \nu^\gamma.$$

THEOREM 2.17. *If $A = (\mu_A, \sigma_A, \nu_A)$ is a neutrosophic H-ideal of X, then μ^α, σ^β and ν^γ are H-ideals of X, for all $\alpha, \beta, \gamma \in [0, 1]$ with $0 \leq \alpha + \beta + \gamma \leq 3$, whenever they are nonemoty.*

COROLLARY 2.18. *Let A be a neutrosophic set on X and $\alpha, \beta, \gamma \in [0, 1]$ be such that $0 \leq \alpha + \beta + \gamma \leq 3$. If A is a neutrosophic H-idea of X, then the nonemoty $(\alpha, \beta, \gamma)-$level subset of A is a H-ideal of X.*

Let us give an example to illustrate the Theorem 2.17.

EXAMPLE 2.19. In Example 2.3, we have

$$\mu^\alpha = \begin{cases} \emptyset, & \text{if } \alpha \in (0.9, 1], \\ \{0\}, & \text{if } \alpha \in (0.8, 0.9], \\ \{0, 1\}, & \text{if } \alpha \in (0.5, 0.8] \\ \{0, 1, 2\}, & \text{if } \alpha \in (0.3, 0.5], \\ \{0, 1, 2, 4\}, & \text{if } \alpha \in (0.1, 0.3], \\ X, & \text{if } \alpha \in [0, 0.1], \end{cases} \tag{1}$$

$$\sigma^\beta = \begin{cases} \emptyset, & \text{if } \beta \in [0, 0.2), \\ \{0\}, & \text{if } \beta \in [0.2, 0.4), \\ \{0, 3\}, & \text{if } \beta \in [0.4, 0.6) \\ \{0, 1, 3\}, & \text{if } \beta \in [0.6, 0.7), \\ \{0, 1, 2, 3\}, & \text{if } \beta \in [0.7, 0.8), \\ X, & \text{if } \beta \in [0.8, 1], \end{cases} \tag{2}$$

$$\nu^\gamma = \begin{cases} \emptyset, & \text{if } \gamma \in [0, 0.1), \\ \{0\}, & \text{if } \gamma \in [0.1, 0.2), \\ \{0, 1\}, & \text{if } \gamma \in [0.2, 0.5) \\ \{0, 1, 2\}, & \text{if } \gamma \in [0.5, 0.6), \\ \{0, 1, 2, 3\}, & \text{if } \gamma \in [0.6, 0.9), \\ X, & \text{if } \gamma \in [0.9, 1], \end{cases} \tag{3}$$

which are $H-$ideals of X.

THEOREM 2.20. *Let for all $\alpha, \beta, \gamma \in [0, 1]$, μ^α, σ^β and ν^γ be H-ideals of X. Then $A = (\mu_A, \sigma_A, \nu_A)$ is a neutrosophic H-ideal of X.*

Acknowledgement

The authors would like to express their thanks to the referees for their comments and suggestions which improved the paper.

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