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Neutrosophic Multigroups and Applications

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Abstract: In recent years, fuzzy multisets and neutrosophic sets have become a subject of great interest for researchers and have been widely applied to algebraic structures include groups, rings, fields and lattices. Neutrosophic multiset is a generalization of multisets and neutrosophic sets. In this paper, we proposed a algebraic structure on neutrosophic multisets is called neutrosophic multigroups which allow the truth-membership, indeterminacy-membership and falsity-membership sequence have a set of real values between zero and one. This new notation of group as a bridge among neutrosophic multiset theory, set theory and group theory and also shows the effect of neutrosophic multisets on a group structure. We finally derive the basic properties of neutrosophic multigroups and give its applications to group theory.

Keywords: neutrosophic sets; neutrosophic multisets; neutrosophic multigroups; neutrosophic multisubgroups

1. Introduction

In the real world, there are much uncertainty information which cannot be handled by crisp values. The fuzzy set theory [1] has been an age old and effective tool to tackle uncertainty information by introduced Zadeh but it can be applied only on random process. Therefore, on the basis of fuzzy set theory, Sebastian and Ramakrishnan [2] introduced Multi-Fuzzy Sets, Atanassov [3] proposed intuitionistic fuzzy set theory, Shinoj and John [4] initiated intuitionistic fuzzy multisets. Recently, the above theories have developed in many directions and found its applications in a wide variety of fields including algebraic structures. For example, on fuzzy sets [5–7], on fuzzy multi sets [8–10], on intuitionistic fuzzy sets [11–19], on intuitionistic fuzzy multi sets [20] are some of the selected works.

But these theories cannot manage the all types of uncertainties, such as indeterminate and inconsistent information some decision-making problems. For instance, "when we ask the opinion of an expert about certain statement, he or she may that the possibility that the statement is true is 0.5 and the statement is false is 0.6 and the degree that he or she is not sure is 0.2" [21]. In order to overcome this shortage, Smarandache [22] introduced neutrosophic set theory to makes the theory of Atanassov [3] very convenient and easily applicable in practice. Then, Wang et al. [21] gave the some operations and results of single valued neutrosophic set theory. In order to establish the algebraic structures of neutrosophic sets, some authors gave definition of neutrosophic groups [23–26] that is actually a example of a group. To develop the neutrosophic set theory, the concept of neutrosophic multi sets was initiated by Deli et al. [27] and Ye [28,29] for modeling vagueness and uncertainty. Using their definitions, in this paper, we define a new type of neutrosophic group on a neutrosophic multi set, which we call neutrosophic multi set group. Since this new concept a brings the neutrosophic multi set theory, set theory and the group theory together, it is very functional in the sense of improving the neutrosophic multi set theory with respect to group structure. Rosenfeld [30] extended the classical group theory to fuzzy set. By using the definitions and results on fuzzy sets in [6,30] and on intuitionistic fuzzy multiset in [20], we applied the definitions and results to neutrosophic multi set theory.The above set theories have been applied to many different areas including neutrosophic environments have been studied by many researchers in [31–39]. In this paper the notion of neutrosophic multigroup along with some related properties have been introduced by follow the results of intuitionistic fuzzy group theory. This concept will bring a new opportunity in research and development of neutrosophic sets theory.

The paper is organized as follows. In Section 2, we briefly review some preliminary concepts that will be used in the paper. In Section 3, we introduce the concept of neutrosophic multi group and give several basic properties and operations. In Section 4, we give some applications to the group theory with respect to neutrosophic multi groups. In Section 5, we make some concluding remarks and suggest.

2. Preliminary

In this section, we present basic definitions of fuzzy set theory, multi fuzzy set theory, intuitionistic fuzzy set theory, intuitionistic fuzzy multi set theory, neutrosophic set theory and neutrosophic multi set theory. For more detailed explanations related to this section, we refer to the earlier studies [1,2,4,6,20,22,27,30].

Definition 1 ([1])**.** *Let E be a universe.*

Then, a fuzzy set X over E is defined by

$$X = \{(\mu_X(x)/x) : x \in E\}, \tag{1}$$

where μ_X is called membership function of X and defined by $\mu_X : E \to [0,1]$. For each $x \in E$, the value $\mu_X(x)$ represents the degree of x belonging to the fuzzy set X.

Definition 2 ([2])**.** *Let X be a non-empty set. A multi-fuzzy set A on X is defined as:*

$$A = \{< x, \mu_1(x), \mu_2(x), \mu_3(x), ..., \mu_i ... : x \in E\}, \tag{2}$$

where $\mu_i : X \to [0,1]$ for all $i \in \{1,2,...,p\}$ and $x \in E$.

Definition 3 ([4])**.** *Let X be a nonempty set. An Intuitionistic Fuzzy Multi-set A denoted by IFMS drawn from X is characterized by two functions: 'count membership' of $A(CM_A)$ and 'count non membership' of $A(CN_A)$ given respectively by $A(CM_A) : X \to Q$ and $A(CN_A) : X \to Q$ where Q is the set of all crisp multi-sets drawn from the unit interval $[0,1]$ such that, for each $x \in X$, the membership sequence is defined as a decreasingly ordered sequence of elements in $CM_A(x)$, which is denoted by $(\mu^1_A(x), \mu^2_A(x), ..., \mu^P_A(x))$ where $\mu^1_A(x) \geq \mu^2_A(x) \geq ... \geq \mu^P_A(x)$ and the corresponding non membership sequence will be denoted by $(\nu^1_A(x), \nu^2_A(x), ..., \nu^P_A(x))$ such that $0 \leq \mu^i_A(x) + \nu^i_A(x) \leq 1$ for every $x \in X$ and $i = (1,2,3,...,p)$. An IFMS A is denoted by*

$$A = \{\langle x : (\mu^1_A(x), \mu^2_A(x), ..., \mu^P_A(x)), (\nu^1_A(x), \nu^2_A(x), ..., \nu^P_A(x))\rangle : x \in X\}. \tag{3}$$

Definition 4 ([4])**.** *Length of an element x in an IFMS. A defined as the Cardinality of $CM_A(x)$ or $CN_A(x)$ for which $0 \leq \mu^j_A(x) + \nu^j_A(x) \leq 1$ and it is denoted by $L(x : A)$. That is,*

$$L(x : A) = |CM_A(x)| = |CN_A(x)|. \tag{4}$$

Proposition 1 ([20])**.** *Let $A, B, A_i \in IFMS(X)$; then, the following results hold:*

1. $[A^{-1}]^{-1} = A$.
2. $A \subseteq B \Rightarrow A^{-1} \subseteq B^{-1}$.

3. $[\bigcup_{i=1}^{n} A_i]^{-1} = \bigcup_{i=1}^{n}[A_i^{-1}]$.
4. $[\bigcap_{i=1}^{n} A_i]^{-1} = \bigcap_{i=1}^{n}[A_i^{-1}]$.
5. $(A \circ B)^{-1} = B^{-1} \circ A^{-1}$.
6.
$$\begin{aligned} CM_{A\circ B}(x) &= \vee_{y\in X}\{CM_A(y) \wedge CM_B(y^{-1}x)\} \ \ \forall\, x \in X \\ &= \vee_{y\in X}\{CM_A(xy^{-1}) \wedge CM_B(y)\} \ \ \forall\, x \in X. \end{aligned}$$
$$\begin{aligned} CN_{A\circ B}(x) &= \wedge_{y\in X}\{CN_A(y) \vee CN_B(y^{-1}x)\} \ \ \forall\, x \in X \\ &= \wedge_{y\in X}\{CN_A(xy^{-1}) \vee CN_B(y)\} \ \ \forall\, x \in X. \end{aligned}$$

Definition 5 ([20]). *Let X be a group. An intuitionistic fuzzy multiset G over X is an intuitionistic fuzzy multi group $(IFMG)$ over X if the counts(count membership and non membership) of G satisfies the following four conditions:*

1. $CM_G(xy) \geq CM_G(x) \wedge CM_G(y)\ \forall\, x, y \in X$.
2. $CM_G(x^{-1}) \geq CT_G(x)\ \forall\, x \in X$.
3. $CN_G(xy) \leq CN_G(x) \wedge CI_G(y)\ \forall\, x, y \in X$.
4. $CN_G(x^{-1}) \leq CN_G(x)\ \forall\, x \in X$.

Definition 6 ([22]). *Let X be a space of points (objects), with a generic element in X denoted by x. A neutrosophic set(N-set) A in X is characterized by a truth-membership function T_A, a indeterminacy-membership function I_A and a falsity-membership function F_A. $T_A(x)$, $I_A(x)$ and $F_A(x)$ are real standard or nonstandard subsets of $[^-0, 1^+]$.*

It can be written as

$$A = \{< x, (T_A(x), I_A(x), F_A(x)) >: x \in X,\ T_A(x), I_A(x), F_A(x) \in [0,1]\}. \tag{5}$$

There is no restriction on the sum of $T_A(x)$; $I_A(x)$ and $F_A(x)$, so $^-0 \leq supT_A(x) + supI_A(x) + supF_A(x) \leq 3^+$.

Here, 1^+ = 1+ε, where 1 is its standard part and ε its non-standard part. Similarly, $^-0$ = 1+ε, where 0 is its standard part and ε its non-standard part.

Definition 7 ([27]). *Let E be a universe. A neutrosophic multiset set(Nms) A on E can be defined as follows:*

$$\begin{aligned} A &= \{< x, (T_A^1(x), T_A^2(x), ..., T_A^P(x)), (I_A^1(x), I_A^2(x), ..., I_A^P(x)), \\ &\qquad (F_A^1(x), F_A^2(x), ..., F_A^P(x)) >:\ x \in E\}, \end{aligned} \tag{6}$$

where

$$T_A^1(x), T_A^2(x), ..., T_A^P(x) : E \to [0,1],$$

$$I_A^1(x), I_A^2(x), ..., I_A^P(x) : E \to [0,1],$$

and

$$F_A^1(x), F_A^2(x), ..., F_A^P(x) : E \to [0,1]$$

such that

$$0 \leq supT_A^i(x) + supI_A^i(x) + supF_A^i(x) \leq 3$$

$(i = 1, 2, ..., P)$ and

$$T_A^1(x) \leq T_A^2(x) \leq ... \leq T_A^P(x)$$

for any $x \in E$.

$(T_A^1(x), T_A^2(x), ..., T_A^P(x))$, $(I_A^1(x), I_A^2(x), ..., I_A^P(x))$ and $(F_A^1(x), F_A^2(x), ..., F_A^P(x))$ is the truth-membership sequence, indeterminacy-membership sequence and falsity-membership sequence of

the element x, respectively. In addition, P is called the dimension(cardinality) of Nms A, denoted d(A). We arrange the truth-membership sequence in decreasing order, but the corresponding indeterminacy-membership and falsity-membership sequence may not be in decreasing or increasing order.

Definition 8 ([27,28])**.** *Let A, B be two Nms. Then,*

1. *A is said to be Nm-subset of B is denoted by $A \widetilde{\subseteq} B$ if $T_A^i(x) \leq T_B^i(x)$, $I_A^i(x) \geq I_B^i(x)$, $F_A^i(x) \geq F_B^i(x)$, $\forall x \in E$ and $i = 1, 2, ..., P$.*
2. *A is said to be neutrosophic equal of B is denoted by $A = B$ if $T_A^i(x) = T_B^i(x)$, $I_A^i(x) = I_B^i(x)$, $F_A^i(x) = F_B^i(x)$, $\forall x \in E$ and $i = 1, 2, ..., P$.*
3. *The union of A and B is denoted by $A \widetilde{\cup} B = C$ and is defined by*

$$C = \{< x, (T_C^1(x), T_C^2(x), ..., T_C^P(x)), (I_C^1(x), I_C^2(x), ..., I_C^P(x)), (F_C^1(x), F_C^2(x), ..., F_C^P(x)) >: \ x \in E\},$$

where $T_C^i = T_A^i(x) \vee T_B^i(x)$, $I_C^i = I_A^i(x) \wedge I_B^i(x)$, $F_C^i = F_A^i(x) \wedge F_B^i(x)$, $\forall x \in E$ and $i = 1, 2, ..., P$.

4. *The intersection of A and B is denoted by $A \widetilde{\cap} B = D$ and is defined by*

$$D = \{< x, (T_D^1(x), T_D^2(x), ..., T_D^P(x)), (I_D^1(x), I_D^2(x), ..., I_D^P(x)), (F_D^1(x), F_D^2(x), ..., F_D^P(x)) >: \ x \in E\},$$

where $T_D^i = T_A^i(x) \wedge T_B^i(x)$, $I_D^i = I_A^i(x) \vee I_B^i(x)$, $F_D^i = F_A^i(x) \vee F_B^i(x)$, $\forall x \in E$ and $i = 1, 2, ..., P$.

3. Neutrosophic Multigroups

In this section, we introduce neutrosophic multigroups and investigate their basic properties. Throughout this section,

1. Let X be a group with a binary operation and the identity element is e.
2. $NMS(X)$ denotes the set of all neutrosophic multisets over the X.
3. $NMG(X)$ denotes the set of all neutrosophic multi groups NMG over the group X.

Definition 9. *Let X be a group $A \in NMS(X)$. Then, A^{-1} is defined as*

$$\begin{aligned} A^{-1} &= \{< x, ({T_A^1}^{-1}(x), {T_A^2}^{-1}(x), ..., T_A^P(x))^{-1}, (I_A^1(x)^{-1}, {I_A^2}^{-1}(x), ..., {I_A^P}^{-1}(x)), \\ &\qquad ({F_A^1}^{-1}(x), {F_A^2}^{-1}(x), ..., {F_A^P}^{-1}(x)) >: \ x \in E\}, \end{aligned} \tag{7}$$

where ${T^i}_A^{-1}(x) = {T^i}_A(x^{-1})$, ${I^i}_A^{-1}(x) = {I^i}_A(x^{-1})$ and ${F^i}_A^{-1}(x) = {F^i}_A(x^{-1})$ for all $i = 1, 2, ..., P$.

Definition 10. *Let X be a classical group $A \in NMS(X)$. Then, A is called a neutrosophic multi groupoid over X if*

1. ${T^i}_G(xy) \geq {T^i}_G(x) \wedge {T^i}_G(y)$,
2. ${I^i}_G(xy) \leq {I^i}_G(x) \vee {I^i}_G(y)$,
3. ${F^i}_G(xy) \leq {F^i}_G(x) \vee {F^i}_G(y)$,

for all $x, y \in X$ and $i = 1, 2, ..., P$.

A is called a neutrosophic multi group(NM-group) over X if the neutrosophic multi groupoid satisfies

1. ${T^i}_G(x^{-1}) \geq {T^i}_G(x)$,
2. ${I^i}_G(x^{-1}) \leq {I^i}_G(x)$,
3. ${F^i}_G(x^{-1}) \leq {F^i}_G(x)$,

for all $x \in X$ and $i = 1, 2, ..., P$.

Example 1. *Assume that* $(Z_3, +)$ *is a classical group. Then,*

$$A = \{\langle 0; (0.8, 0.7, 0.6, 0.4), (0.1, 0.1, 0.2, 0.3), (0.2, 0.3, 0.4, 0.5)\rangle, \langle 1; (0.7, 0.6, 0.4, 0.3), (0.2, 0.3, 0.2, 0.3), (0.3, 0.4, 0.5, 0.6)\rangle, \langle 2; (0.8, 0.6, 0.6, 0.4), (0.1, 0.2, 0.2, 0.3), (0.2, 0.4, 0.4, 0.5)\rangle\}$$

is a NM-group. However,

$$B = \{\langle 0; (0.8, 0.7, 0.6, 0.4), (0.1, 0.1, 0.2, 0.3), (0.2, 0.3, 0.4, 0.5)\rangle, \langle 1; (0.9, 0.5, 0.4, 0.3), (0.2, 0.1, 0.2, 0.3), (0.3, 0.3, 0.5, 0.4)\rangle, \langle 2; (0.8, 0.7, 0.6, 0.4), (0.1, 0.3, 0.2, 0.3), (0.2, 0.4, 0.4, 0.6)\rangle\}$$

is not a NM-group because ${T^i}_B(1^{-1})$ *is not greater than or equal to* ${T^i}_B(1)$.

From the Definition 10 and Example 1, it is clear that a NM-group is a generalized case of fuzzy group and intuitionistic fuzzy multi group.

Proposition 2. *Let X be a classical group and* $A \in NMS(X)$. *If* $A \in NMG(X)$; *then,*

1. ${T^i}_A(e) \geq {T^i}_A(x)\ \forall\, x \in X$,
2. ${I^i}_A(e) \leq {I^i}_A(x)\ \forall\, x \in X$,
3. ${F^i}_A(e) \leq {F^i}_A(x)\ \forall\, x \in X$,

for all $x \in X$ *and* $i = 1, 2, ..., P$.

Proof. Since A an $NM - group$ over X, then

1. $$\begin{aligned} {T^i}_A(e) &= {T^i}_A(x.x^{-1}) \\ &\geq {T^i}_A(x) \wedge {T^i}_A(x^{-1}) \\ &\geq {T^i}_A(x) \wedge {T^i}_A(x) \\ &= {T^i}_A(x) \end{aligned}$$

   for all $x \in X$ and $i = 1, 2, ..., P$.
2. $$\begin{aligned} {I^i}_A(e) &= {I^i}_A(x.x^{-1}) \\ &\leq {I^i}_A(x) \vee {I^i}_A(x^{-1}) \\ &\leq {I^i}_A(x) \vee {I^i}_A(x) \\ &= {I^i}_A(x) \end{aligned}$$

   for all $x \in X$ and $i = 1, 2, ..., P$.
3. $$\begin{aligned} {F^i}_A(e) &= {F^i}_A(x.x^{-1}) \\ &\leq {F^i}_A(x) \vee {F^i}_A(x^{-1}) \\ &\leq {F^i}_A(x) \vee {F^i}_A(x) \\ &= {F^i}_A(x) \end{aligned}$$

   for all $x \in X$ and $i = 1, 2, ..., P$.

□

Proposition 3. *Let X be a classical group and* $A \in NMS(X)$. *If* $A \in NMG(X)$, *then*

1. ${T^i}_A(x^n) \geq {T^i}_A(x)\ \forall\, x \in X$,
2. ${I^i}_A(x^n) \leq {I^i}_A(x)\ \forall\, x \in X$,
3. ${F^i}_A(x^n) \leq {F^i}_A(x)\ \forall\, x \in X$,

for all $x \in X$ *and* $i = 1, 2, ..., P$.

Proof. Since A an $NM - group$ over X, then

1.
$$\begin{aligned} T^i{}_A(x^n) &\geq T^i{}_A(x) \wedge T^i{}_A(x^{n-1}) \\ &\geq T^i{}_A(x) \wedge T^i{}_A(x) \wedge ... \wedge T^i{}_A(x) \\ &= T^i{}_A(x) \end{aligned}$$
for all $x \in X$ and $i = 1, 2, ..., P$.

2.
$$\begin{aligned} I^i{}_A(x^n) &\leq I^i{}_A(x) \vee I^i{}_A(x^{n-1}) \\ &\leq I^i{}_A(x) \vee I^i{}_A(x) \vee ... \vee I^i{}_A(x) \\ &= I^i{}_A(x) \end{aligned}$$
for all $x \in X$ and $i = 1, 2, ..., P$.

3.
$$\begin{aligned} F^i{}_A(x^n) &\leq F^i{}_A(x) \vee F^i{}_A(x^{n-1}) \\ &\leq F^i{}_A(x) \vee F^i{}_A(x) \vee ... \vee F^i{}_A(x) \\ &= F^i{}_A(x) \end{aligned}$$
for all $x \in X$ and $i = 1, 2, ..., P$.

□

Definition 11. *Let Y be a subgroup of X, $B \in NMG(Y)$, $B \tilde{\subseteq} A$ and $A \in NMG(X)$. If $B \in NMG(Y)$, then B is called a neutrosophic multi subgroup of A over X and denoted by $B \tilde{\leq} A$.*

Example 2. *Assume that $(Z_3, +)$ is a classical group. We define A and B neutrosophic multi group over $(Z_3, +)$ by*
$$\begin{aligned} A &= \{\langle 0; (0.4, 0.3, 0.3, 0.2), (0.1, 0.1, 0.2, 0.3), (0.2, 0.3, 0.4, 0.6)\rangle, \\ &\quad \langle 1; (0.6, 0.5, 0.3, 0.2), (0.2, 0.4, 0.2, 0.3), (0.3, 0.2, 0.5, 0.6)\rangle, \\ &\quad \langle 2; (0.8, 0.7, 0.5, 0.4), (0.1, 0.3, 0.2, 0.3), (0.2, 0.1, 0.4, 0.5)\rangle\}. \end{aligned}$$
$$\begin{aligned} B &= \{\langle 0; (0.4, 0.3, 0.3, 0.2), (0.1, 0.1, 0.2, 0.3), (0.2, 0.3, 0.4, 0.6)\rangle, \\ &\quad \langle 1; (0.6, 0.5, 0.3, 0.2), (0.2, 0.4, 0.2, 0.3), (0.3, 0.2, 0.5, 0.6)\rangle\}. \end{aligned}$$
Then, B is a neutrosophic multi subgroup of A over $(Z_3, +)$ and denoted by $B \tilde{\leq} A$.

Theorem 1. *Let X be a group $A \in NMS(X)$. Then, A is an $NM - group$ if and only if $T^i{}_A(xy^{-1}) \geq T^i{}_A(x) \wedge T^i{}_A(y)$, $I^i{}_A(xy^{-1}) \leq I^i{}_A(x) \vee I^i{}_A(y)$ and $F^i{}_A(xy^{-1}) \leq F^i{}_A(x) \vee F^i{}_A(y)$ for all $x, y \in X$.*

Proof. Assume that A is an $NM - group$ over X. Then,
$$\begin{aligned} T^i{}_A(xy^{-1}) &\geq T^i{}_A(x) \wedge T^i{}_A(y^{-1}) \\ &\geq T^i{}_A(x) \wedge T^i{}_A(y) \end{aligned}$$
for all $x, y \in X$ and $i = 1, 2, ..., P$.
$$\begin{aligned} I^i{}_A(xy^{-1}) &\leq I^i{}_A(x) \vee I^i{}_A(y^{-1}) \\ &\leq I^i{}_A(x) \vee I^i{}_A(y) \end{aligned}$$
for all $x, y \in X$ and $i = 1, 2, ..., P$.
$$\begin{aligned} F^i{}_A(xy^{-1}) &\leq F^i{}_A(x) \vee F^i{}_A(y^{-1}) \\ &\leq F^i{}_A(x) \vee F^i{}_A(y) \end{aligned}$$
for all $x, y \in X$ and $i = 1, 2, ..., P$.

Conversely, the given condition be satisfied. Firstly,

$$\begin{aligned} T^i{}_A(x^{-1}) &= T^i{}_A(e.x^{-1}) \\ &\geq T^i{}_A(e) \wedge T^i{}_A(x) \\ &= T^i{}_A(x) \end{aligned}$$

$$\begin{aligned} T^i{}_A(xy) &\geq T^i{}_A(x) \wedge T^i{}_A(y^{-1}) \\ &\geq T^i{}_A(x) \wedge T^i{}_A(y). \end{aligned}$$

Secondly,

$$\begin{aligned} I^i{}_A(x^{-1}) &= I^i{}_A(e.x^{-1}) \\ &\leq I^i{}_A(e) \vee I^i{}_A(x) \\ &= I^i{}_A(x) \end{aligned}$$

$$\begin{aligned} I^i{}_A(xy) &\leq I^i{}_A(x) \vee I^i{}_A(y^{-1}) \\ &\leq I^i{}_A(x) \vee I^i{}_A(y). \end{aligned}$$

Thirdly,

$$\begin{aligned} F^i{}_A(x^{-1}) &= F^i{}_A(e.x^{-1}) \\ &\leq F^i{}_A(e) \vee F^i{}_A(x) \\ &= F^i{}_A(x) \end{aligned}$$

$$\begin{aligned} F^i{}_A(xy) &\leq F^i{}_A(x) \vee F^i{}_A(y^{-1}) \\ &\leq F^i{}_A(x) \vee F^i{}_A(y) \end{aligned}$$

so the proof is complete. □

Definition 12. *Let $A, B \in NMS(X)$. Then, their "AND" operation is denoted by $A\tilde{\wedge}B$ and is defined by*

$$A\tilde{\wedge}B = \{(x,y),\ T^i{}_{A\tilde{\wedge}B}(x,y),\ I^i{}_{A\tilde{\wedge}B}(x,y),\ F^i{}_{A\tilde{\wedge}B}(x,y) : (x,y) \in X \times X\}, \tag{8}$$

where $T^i{}_{A\tilde{\wedge}B}(x,y) = T^i{}_A(x) \wedge T^i{}_B(y)$, $I^i{}_{A\tilde{\wedge}B}(x,y) = I^i{}_A(x) \vee I^i{}_B(y)$, $F^i{}_{A\tilde{\wedge}B}(x,y) = F^i{}_A(x) \vee F^i{}_B(y)$.

Theorem 2. *Let $A, B \in NMG(X)$. Then, $A\tilde{\wedge}B$ is a neutrosophic multi group over X.*

Proof. Let $(x_1, y_1), (x_2, y_2) \in X \times X$. Then,

$$\begin{aligned} T^i{}_{A\tilde{\wedge}B}((x_1,y_1),(x_2,y_2)^{-1}) &= T^i{}_{A\tilde{\wedge}B}(x_1x_2^{-1}, y_1y_2^{-1}) \\ &= T^i{}_A(x_1x_2^{-1}) \wedge T^i{}_B(y_1y_2^{-1}) \\ &\geq (T^i{}_A(x_1) \wedge T^i{}_A(x_2)) \wedge (T^i{}_B(y_1) \wedge T^i{}_B(y_2)) \\ &= (T^i{}_A(x_1) \wedge T^i{}_B(y_1)) \wedge (T^i{}_A(x_2) \wedge T^i{}_B(y_2)) \\ &= T^i{}_{A\tilde{\wedge}B}(x_1,y_1) \wedge T^i{}_{A\tilde{\wedge}B}(x_2,y_2) \end{aligned}$$

$$\begin{aligned} I^i{}_{A\tilde{\wedge}B}((x_1,y_1),(x_2,y_2)^{-1}) &= I^i{}_{A\tilde{\wedge}B}(x_1x_2^{-1}, y_1y_2^{-1}) \\ &= I^i{}_A(x_1x_2^{-1}) \vee I^i{}_B(y_1y_2^{-1}) \\ &\leq (I^i{}_A(x_1) \vee I^i{}_A(x_2)) \vee (I^i{}_B(y_1) \vee I^i{}_B(y_2)) \\ &= (I^i{}_A(x_1) \vee I^i{}_B(y_1)) \vee (I^i{}_A(x_2) \vee I^i{}_B(y_2)) \\ &= I^i{}_{A\tilde{\wedge}B}(x_1,y_1) \vee I^i{}_{A\tilde{\wedge}B}(x_2,y_2) \end{aligned}$$

and

$$\begin{aligned} F^i{}_{A\tilde{\wedge}B}((x_1,y_1),(x_2,y_2)^{-1}) &= F^i{}_{A\tilde{\wedge}B}(x_1x_2^{-1}, y_1y_2^{-1}) \\ &= F^i{}_A(x_1x_2^{-1}) \vee F^i{}_B(y_1y_2^{-1}) \\ &\leq (F^i{}_A(x_1) \vee F^i{}_A(x_2)) \vee (F^i{}_B(y_1) \vee F^i{}_B(y_2)) \\ &= (F^i{}_A(x_1) \vee F^i{}_B(y_1)) \vee (F^i{}_A(x_2) \vee F^i{}_B(y_2)) \\ &= F^i{}_{A\tilde{\wedge}B}(x_1,y_1) \vee F^i{}_{A\tilde{\wedge}B}(x_2,y_2) \end{aligned}$$

for all $(x_1, y_1), (x_2, y_2) \in X$ and $i = 1, 2, ..., P$. Therefore, $A \tilde{\wedge} B$ is a neutrosophic multi group over X, hence the proof.

□

Example 3. *Let us take into consideration the classical group $(Z_3, +)$. Define the neutrosophic multiset A, B on $(Z_3, +)$ as follows:*

$$\begin{aligned}
A &= \{\langle 0; (0.5, 0.3, 0.2, 0.1), (0.1, 0.1, 0.2, 0.3), (0.2, 0.3, 0.4, 0.5)\rangle, \\
&\langle 1; (0.6, 0.4, 0.3, 0.2), (0.1, 0.3, 0.3, 0.4), (0.1, 0.2, 0.4, 0.6)\rangle, \\
&\langle 2; (0.7, 0.5, 0.3, 0.2), (0.2, 0.2, 0.3, 0.4), (0.3, 0.3, 0.4, 0.6)\rangle, \\
B &= \{\langle 0; (0.6, 0.5, 0.4, 0.2), (0.2, 0.2, 0.3, 0.4), (0.3, 0.3, 0.4, 0.6)\rangle, \\
&\langle 1; (0.8, 0.6, 0.4, 0.3), (0.1, 0.1, 0.2, 0.3), (0.2, 0.2, 0.3, 0.4)\rangle\} \\
&are\ NM-groups. \\
A \tilde{\wedge} B &= \{\langle (0, 0); (0.5, 0.3, 0.2, 0.1), (0.2, 0.2, 0.2, 0.4), (0.2, 0.3, 0.4, 0.6)\rangle, \\
&\langle (0, 1); (0.5, 0.3, 0.2, 0.1), (0.1, 0.1, 0.2, 0.3), (0.2, 0.2, 0.3, 0.5)\rangle, \\
&\langle (1, 0); (0.6, 0.4, 0.3, 0.2), (0.2, 0.2, 0.3, 0.4), (0.3, 0.3, 0.4, 0.6)\rangle, \\
&\langle (1, 1); (0.6, 0.4, 0.3, 0.2), (0.1, 0.3, 0.3, 0.4), (0.2, 0.2, 0.4, 0.6)\rangle, \\
&\langle (2, 0); (0.6, 0.5, 0.3, 0.2), (0.2, 0.2, 0.3, 0.4), (0.3, 0.3, 0.4, 0.6)\rangle, \\
&\langle (2, 1); (0.7, 0.5, 0.3, 0.2), (0.2, 0.2, 0.3, 0.4), (0.3, 0.3, 0.4, 0.6)\rangle\}.
\end{aligned}$$

Then, $A \tilde{\wedge} B \in NMG(X)$.

Definition 13. *Let X be a classical group and $A, B \in NMS(X)$. Then, their "OR" operation is denoted by $A \tilde{\vee} B$ and is defined by*

$$A \tilde{\vee} B = \{(x, y), T^i{}_{A \tilde{\vee} B}(x, y), I^i{}_{A \tilde{\vee} B}(x, y), F^i{}_{A \tilde{\vee} B}(x, y) : (x, y) \in X \times X\} \tag{9}$$

where $T^i{}_{A \tilde{\vee} B}(x, y) = T^i{}_A(x) \vee T^i{}_B(y)$, $I^i{}_{A \tilde{\vee} B}(x, y) = I^i{}_A(x) \wedge I^i{}_B(y)$, $F^i{}_{A \tilde{\vee} B}(x, y) = F^i{}_A(x) \wedge F^i{}_B(y)$.

Proposition 4. *Let $A, B \in NMG(X)$. Then, $T^i{}_{A \tilde{\vee} B}(x) \leq T^i{}_{A \tilde{\vee} B}(x^{-1})$, $I^i{}_{A \tilde{\vee} B}(x) \geq I^i{}_{A \tilde{\vee} B}(x^{-1})$, $F^i{}_{A \tilde{\vee} B}(x) \geq F^i{}_{A \tilde{\vee} B}(x^{-1})$.*

Proof. Let $(x_1, y_1), (x_2, y_2) \in X \times X$. Then,

$$\begin{aligned}
T^i{}_{A \tilde{\vee} B}((x_1, y_1), (x_2, y_2)^{-1}) &= T^i{}_{A \tilde{\vee} B}(x_1 x_2^{-1}, y_1 y_2^{-1}) \\
&= T^i{}_A(x_1 x_2^{-1}) \vee T^i{}_B(y_1 y_2^{-1}) \\
&\leq (T^i{}_A(x_1) \vee T^i{}_A(x_2)) \vee (T^i{}_B(y_1) \vee T^i{}_B(y_2)) \\
&= (T^i{}_A(x_1) \vee T^i{}_B(y_1)) \vee (T^i{}_A(x_2) \vee T^i{}_B(y_2)) \\
&= T^i{}_{A \tilde{\vee} B}(x_1, y_1) \vee T^i{}_{A \tilde{\vee} B}(x_2, y_2)
\end{aligned}$$

$$\begin{aligned}
I^i{}_{A \tilde{\vee} B}((x_1, y_1), (x_2, y_2)^{-1}) &= I^i{}_{A \tilde{\vee} B}(x_1 x_2^{-1}, y_1 y_2^{-1}) \\
&= I^i{}_A(x_1 x_2^{-1}) \wedge I^i{}_B(y_1 y_2^{-1}) \\
&\geq (I^i{}_A(x_1) \wedge I^i{}_A(x_2)) \wedge (I^i{}_B(y_1) \wedge I^i{}_B(y_2)) \\
&= (I^i{}_A(x_1) \wedge I^i{}_B(y_1)) \wedge (I^i{}_A(x_2) \wedge I^i{}_B(y_2)) \\
&= I^i{}_{A \tilde{\vee} B}(x_1, y_1) \wedge I^i{}_{A \tilde{\vee} B}(x_2, y_2)
\end{aligned}$$

and

$$\begin{aligned}
F^i{}_{A \tilde{\vee} B}((x_1, y_1), (x_2, y_2)^{-1}) &= F^i{}_{A \tilde{\vee} B}(x_1 x_2^{-1}, y_1 y_2^{-1}) \\
&= F^i{}_A(x_1 x_2^{-1}) \wedge F^i{}_B(y_1 y_2^{-1}) \\
&\geq (F^i{}_A(x_1) \wedge F^i{}_A(x_2)) \wedge (F^i{}_B(y_1) \wedge F^i{}_B(y_2)) \\
&= (F^i{}_A(x_1) \wedge F^i{}_B(y_1)) \wedge (F^i{}_A(x_2) \wedge F^i{}_B(y_2)) \\
&= F^i{}_{A \tilde{\vee} B}(x_1, y_1) \wedge F^i{}_{A \tilde{\vee} B}(x_2, y_2)
\end{aligned}$$

for all $(x_1, y_1), (x_2, y_2) \in X$ and $i = 1, 2, ..., P$—hence the proof.

From this, it is clear that, if $A, B \in NMG(X)$, then $A\tilde{\vee}B \in NMG(X)$ iff $T^i{}_{A\tilde{\vee}B}(x,y) \geq T^i{}_{A\tilde{\vee}B}(x) \wedge T^i{}_{A\tilde{\vee}B}(y), I^i{}_{A\tilde{\vee}B}(x,y) \leq I^i{}_{A\tilde{\vee}B}(x) \vee I^i{}_{A\tilde{\vee}B}(y), F^i{}_{A\tilde{\vee}B}(x,y) \leq F^i{}_{A\tilde{\vee}B}(x) \vee F^i{}_{A\tilde{\vee}B}(y)$. □

Corollary 1. *Let $A, B \in NMG(X)$. Then, $A\tilde{\vee}B$ need not be an element of $NMG(X)$.*

Example 4. *Let us take into consideration the classical group $(Z_4, +)$. Define the neutrosophic multiset A, B on $(Z_4, +)$ as follows:*

$$
\begin{aligned}
A \; &= \{\langle 0; (0.5, 0.3, 0.2, 0.1), (0.1, 0.1, 0.2, 0.3), (0.2, 0.3, 0.4, 0.5)\rangle, \\
&\quad \langle 1; (0.6, 0.4, 0.3, 0.2), (0.1, 0.3, 0.3, 0.4), (0.1, 0.2, 0.4, 0.6)\rangle, \\
&\quad \langle 2; (0.7, 0.5, 0.3, 0.2), (0.2, 0.2, 0.3, 0.4), (0.3, 0.3, 0.4, 0.6)\rangle, \\
&\quad \langle 3; (0.7, 0.6, 0.4, 0.3), (0.2, 0.1, 0.2, 0.3), (0.3, 0.2, 0.1, 0.3)\rangle\} \\
B \; &= \{\langle 0; (0.6, 0.5, 0.4, 0.2), (0.2, 0.2, 0.3, 0.4), (0.2, 0.3, 0.4, 0.6)\rangle, \\
&\quad \langle 1; (0.8, 0.6, 0.4, 0.3), (0.1, 0.1, 0.2, 0.3), (0.2, 0.2, 0.3, 0.4)\rangle\} \\
&\quad are\ NM - groups. \\
A\tilde{\vee}B \; &= \{\langle (0,0); (0.6, 0.5, 0.4, 0.2), (0.1, 0.1, 0.2, 0.3), (0.2, 0.3, 0.4, 0.5)\rangle, \\
&\quad \langle (0,1); (0.8, 0.6, 0.4, 0.3), (0.1, 0.1, 0.2, 0.3), (0.2, 0.2, 0.3, 0.4)\rangle, \\
&\quad \langle (1,0); (0.6, 0.5, 0.4, 0.2), (0.1, 0.2, 0.3, 0.4), (0.1, 0.2, 0.4, 0.6)\rangle, \\
&\quad \langle (1,1); (0.8, 0.6, 0.4, 0.3), (0.1, 0.1, 0.2, 0.3), (0.1, 0.2, 0.3, 0.4)\rangle, \\
&\quad \langle (2,0); (0.7, 0.5, 0.4, 0.2), (0.1, 0.2, 0.3, 0.4), (0.2, 0.3, 0.4, 0.6)\rangle, \\
&\quad \langle (2,1); (0.8, 0.6, 0.4, 0.3), (0.1, 0.1, 0.2, 0.3), (0.2, 0.2, 0.3, 0.4)\rangle, \\
&\quad \langle (3,0); (0.7, 0.6, 0.4, 0.3), (0.2, 0.1, 0.2, 0.3), (0.2, 0.2, 0.1, 0.3)\rangle, \\
&\quad \langle (3,1); (0.8, 0.6, 0.4, 0.3), (0.1, 0.1, 0.2, 0.3), (0.2, 0.2, 0.1, 0.3)\rangle\}.
\end{aligned}
$$

However, $T^i{}_{A\tilde{\vee}B}(3,0) \geq T^i{}_{A\tilde{\vee}B}(1,0)$. Then, $A\tilde{\vee}B \notin NMG(X)$.

Theorem 3. *Let X be a classical group and $A \in NMG(X)$. Then, the followings are equivalent:*

1. $T^i_A(yx) = T^i_A(xy), I^i_A(yx) = I^i_A(xy)$ *and* $F^i_A(yx) = F^i_A(xy)$ *for all* $x, y \in X$.
2. $T^i_A(xyx^{-1}) = T^i_A(y), I^i_A(xyx^{-1}) = I^i_A(y)$ *and* $F^i_A(xyx^{-1}) = F^i_A(y)$ *for all* $x, y \in X$.
3. $T^i_A(xyx^{-1}) \geq T^i_A(y), I^i_A(xyx^{-1}) \leq I^i_A(y)$ *and* $F^i_A(xyx^{-1}) \leq F^i_A(y)$ *for all* $x, y \in X$.
4. $T^i_A(xyx^{-1}) \leq T^i_A(y), I^i_A(xyx^{-1}) \geq I^i_A(y)$ *and* $F^i_A(xyx^{-1}) \geq F^i_A(y)$ *for all* $x, y \in X$.

Proof.
1. (1) ⇒ (2): Let $x, y \in X$. Then,

$$
\begin{aligned}
T^i_A(xyx^{-1}) = T^i_A(x^{-1}xy) = T^i_A(y), \\
I^i_A(xyx^{-1}) = I^i_A(x^{-1}xy) = I^i_A(y), \\
F^i_A(xyx^{-1}) = F^i_A(x^{-1}xy) = F^i_A(y).
\end{aligned}
$$

2. (2) ⇒ (3): Immediate.
3. (3) ⇒ (4)

$$
\begin{aligned}
T^i_A(xyx^{-1}) \leq T^i_A(x^{-1}xy(x^{-1})^{-1}) = T^i_A(y), \\
I^i_A(xyx^{-1}) \geq I^i_A(x^{-1}xy(x^{-1})^{-1}) = I^i_A(y), \\
F^i_A(xyx^{-1}) \geq F^i_A(x^{-1}xy(x^{-1})^{-1}) = F^i_A(y).
\end{aligned}
$$

4. (4) ⇒ (1): Let $x, y \in X$. Then,

$$
\begin{aligned}
T^i_A(xy) \; &= T^i_A(x.yx.x^{-1}) \\
&\leq T^i_A(yx) \\
&= T^i_A(y.xy.y^{-1}) \\
&\leq T^i_A(xy),
\end{aligned}
$$

$$
\begin{aligned}
I^i_A(xy) &= I^i_A(x.yx.x^{-1}) \\
&\leq I^i_A(yx) \\
&= I^i_A(y.xy.y^{-1}) \\
&\leq I^i_A(xy), \\
F^i_A(xy) &= F^i_A(x.yx.x^{-1}) \\
&\leq F^i_A(yx) \\
&= F^i_A(y.xy.y^{-1}) \\
&\leq F^i_A(xy)
\end{aligned}
$$

Hence, $T^i_A(yx) = T^i_A(xy), I^i_A(yx) = I^i_A(xy), F^i_A(yx) = F^i_A(xy)$.
□

Definition 14. *Let X be a group,* $A \in NMS(X)$ *and B is a nonempty neutrosophic multi subset of A over X. Then, B is called an abelian neutrosophic multi subset of A if* $T^i_A(yx) = T^i_A(xy), I^i_A(yx) = I^i_A(xy)$ *and* $F^i_A(yx) = F^i_A(xy)$ *for all* $x, y \in X$.

Example 5. 1_X *and* 1_e *are normal neutrosophic multi subgroup of X. If X is a commutative group, every neutrosophic multi subgroup of X is normal.*

Definition 15. *Let X be a group,* $A \in NMG(X)$ *and B is a neutrosophic multi subgroup of A over X. Then, B is called an a normal neutrosophic multi subgroup of A, denoted by* $B \tilde{\bigtriangledown} A$ *if it is an abelian neutrosophic multi subset of A over X.*

Example 6. *Assume that* $(Z_3, +)$ *is a classiccal group. Define the neutrosophic multisets A and B on* $(Z_3, +)$ *as follows:*

$$
\begin{aligned}
A = \{ &\langle 0; (0.6, 0.5, 0.4, 0.2), (0.1, 0.1, 0.2, 0.3), (0.2, 0.3, 0.4, 0.6) \rangle, \\
&\langle 1; (0.5, 0.4, 0.4, 0.3), (0.2, 0.1, 0.2, 0.3), (0.3, 0.4, 0.5, 0.6) \rangle, \\
&\langle 2; (0.9, 0.7, 0.6, 0.5), (0.1, 0.1, 0.2, 0.3), (0.2, 0.2, 0.3, 0.5) \rangle \}
\end{aligned}
$$

is a NM-group. If

$$
\begin{aligned}
B = \{ &\langle 0; (0.6, 0.5, 0.4, 0.2), (0.1, 0.1, 0.2, 0.3), (0.2, 0.3, 0.4, 0.6) \rangle, \\
&\langle 1; (0.5, 0.4, 0.4, 0.3), (0.2, 0.1, 0.2, 0.3), (0.3, 0.3, 0.5, 0.4) \rangle \},
\end{aligned}
$$

then B is a neutrosophic multi subgroup of A over $(Z_3, +)$ *and denoted by* $B \tilde{\leq} A$. *Therefore,* $B \tilde{\bigtriangledown} A$.

Corollary 2. *Let* $A \in NMG(X)$ *and B be a neutrosophic multi subgroup of A over X. If X is an abelian group, then B is a normal neutrosophic multi subgroup of A over X.*

4. Applications of Neutrosophic Multi Groups

In this section, we give some applications to the group theory with respect to neutrosophic multi groups.

Definition 16. *Let A be a neutrosophic multiset on X and* $\alpha \in [0, 1]$. *Define the* α*-level sets of A as follows:*

$$
\begin{aligned}
(T^i{}_A)_\alpha &= \{x \in X : T^i{}_A(x) \geq \alpha\}, \\
(I^i{}_A)^\alpha &= \{x \in X : I^i{}_A(x) \leq \alpha\}, \\
(F^i{}_A)^\alpha &= \{x \in X : F^i{}_A(x) \leq \alpha\}.
\end{aligned}
$$

It is easy to verify that

$$
\begin{array}{ll}
(1) & If A \tilde{\subseteq} B \ \ and \ \ \alpha \in [0,1], then \\
& (T^i{}_A)_\alpha \subseteq (T^i{}_B)_\alpha, (I^i{}_A)^\alpha \supseteq (I^i{}_B)^\alpha \ \ and \ \ (F^i{}_A)^\alpha \supseteq (F^i{}_B)^\alpha. \\
(2) & \alpha \leq \beta \ \ implies \ \ (T^i{}_A)_\alpha \supseteq (T^i{}_A)_\beta, (I^i{}_A)^\alpha \subseteq (I^i{}_A)^\beta \ \ and \ \ (F^i{}_A)^\alpha \subseteq (F^i{}_A)^\beta.
\end{array}
$$

Proposition 5. *A is a neutrosophic multi group of a classical group X if and only if for all $\alpha \in [0,1]$, α-level sets of A, $(T^i{}_A)_\alpha$, $(I^i{}_A)_\alpha$ and $(F^i{}_A)^\alpha$ are classical subgroups of X.*

Proof. Let A be a neutrosophic multi subgroup of X, $\alpha \in [0,1]$ and $x.y \in (T^i{}_A)_\alpha$ (similarly $x.y \in (I^i{}_A)_\alpha, (F^i{}_A)^\alpha$). By the assumption, $T^i{}_A(xy^{-1}) \geq T^i{}_A(x) \wedge T^i{}_A(y) \geq \alpha \wedge \alpha = \alpha$ (and similarly, $I^i{}_A(xy^{-1}) \leq \alpha$ and $F^i{}_A(xy^{-1}) \leq \alpha$). Hence, $xy^{-1} \in (T^i{}_A)_\alpha$ (and similarly $xy^{-1} \in (I^i{}_A)^\alpha, (F^i{}_A)^\alpha$) for each $\alpha \in [0,1]$. This means that $(T^i{}_A)_\alpha$ (and similarly $(I^i{}_A)^\alpha, (F^i{}_A)^\alpha$) is a classical subgroup of X for each $\alpha \in [0,1]$.

Conversely, let $(T^i{}_A)_\alpha$ be a classical subgroup of X, for each $\alpha \in [0,1]$. Let $x, y \in X, \alpha = T^i{}_A(x) \wedge T^i{}_A(y)$ and $\beta = T^i{}_A(x)$. Since $(T^i{}_A)_\alpha$ and $(T^i{}_A)_\beta$ are classical subgroups of X, $x.y \in (T^i{}_A)_\alpha$ and $x^{-1} \in (T^i{}_A)_\beta$. Thus, $T^i{}_A(xy^{-1}) \geq \alpha = T^i{}_A(x) \wedge T^i{}_A(y)$ and $T^i{}_A(x^{-1}) \geq \beta = T^i{}_A(x)$. Similarly, $I^i{}_A(xy^{-1}) \leq I^i{}_A(x) \vee I^i{}_A(y)$ and $F^i{}_A(xy^{-1}) \leq F^i{}_A(x) \vee F^i{}_A(y)$. □

Theorem 4. *Let X_1, X_2 be the classical groups and $g : X_1 \to X_2$ be a group homomorphism. If A is a neutrosophic multi subgroup of X_1, then the image of A, $g(A)$ is a neutrosophic multi subgroup of X_2.*

Proof. Let $A \in NMS(X_1)$ and $y_1, y_2 \in X_2$. If $g^{-1}(y_1) = \emptyset$ or $g^{-1}(y_2) = \emptyset$, then it is clear that $g(A) \in NMS(X_2)$. Let us assume that there exists $x_1, x_2 \in X_1$ such that $g(x_1) = y_1$ and $g(x_2) = y_2$. Since g is a group homomorphism,

$$
\begin{array}{l}
g(T^i{}_A)(y_1y_2^{-1}) = \bigvee_{y_1y_2^{-1}=g(x)} T^i{}_A(x) \geq T^i{}_A(x_1x_2^{-1}), \\
g(I^i{}_A)(y_1y_2^{-1}) = \bigwedge_{y_1y_2^{-1}=g(x)} I^i{}_A(x) \leq I^i{}_A(x_1x_2^{-1}), \\
g(F^i{}_A)(y_1y_2^{-1}) = \bigwedge_{y_1y_2^{-1}=g(x)} F^i{}_A(x) \leq F^i{}_A(x_1x_2^{-1}).
\end{array}
$$

By using the above inequalities, let us prove that $g(A)(y_1y_2^{-1}) \geq g(A)(y_1) \wedge g(A)(y_2)$:

$$
\begin{array}{ll}
g(A)(y_1y_2^{-1}) & = g(T^i{}_A)(y_1y_2^{-1}), g(I^i{}_A)(y_1y_2^{-1}), g(F^i{}_A)(y_1y_2^{-1}) \\
& = \bigvee_{y_1y_2^{-1}=g(x)} T^i{}_A(x), \bigwedge_{y_1y_2^{-1}=g(x)} I^i{}_A(x), \bigwedge_{y_1y_2^{-1}=g(x)} F^i{}_A(x) \\
& \geq (T^i{}_A(x_1x_2^{-1}), I^i{}_A(x_1x_2^{-1}), F^i{}_A(x_1x_2^{-1})) \\
& \geq (T^i{}_A(x_1) \wedge T^i{}_A(x_2), I^i{}_A(x_1) \vee I^i{}_A(x_2), F^i{}_A(x_1) \vee F^i{}_A(x_2) \\
& = (T^i{}_A(x_1), I^i{}_A(x_1), F^i{}_A(x_1)) \wedge (T^i{}_A(x_2), I^i{}_A(x_2), F^i{}_A(x_2)).
\end{array}
$$

This is satisfied for each $x_1, x_2 \in X_1$ with $g(x_1) = y_1$ and $g(x_2) = y_2$, then it is obvious that

$$
\begin{array}{ll}
g(A)(y_1y_2^{-1}) & \geq (\bigvee_{y_1=g(x_1)} T^i{}_A(x_1), \bigwedge_{y_1=g(x_1)} I^i{}_A(x_1), \bigwedge_{y_1=g(x_1)} F^i{}_A(x_1)) \\
& \wedge(\bigvee_{y_2=g(x_2)} T^i{}_A(x_2), \bigwedge_{y_2=g(x_2)} I^i{}_A(x_2), \bigwedge_{y_2=g(x_2)} F^i{}_A(x_2)) \\
& = (g(T^i{}_A)(y_1), g(I^i{}_A)(y_1), g(F^i{}_A)(y_1)) \wedge (g(T^i{}_A)(y_2), g(I^i{}_A)(y_2), g(F^i{}_A)(y_2)) \\
& = g(A)(y_1) \wedge g(A)(y_2).
\end{array}
$$

Hence, the image of a neutrosophic multi subgroup is also a neutrosophic multi subgroup. □

Theorem 5. *Let X_1, X_2 be the classical groups and $g : X_1 \to X_2$ be a group homomorphism. If B is a neutrosophic multi subgroup of X_2, then the preimage $g^{-1}(B)$ is a neutrosophic multi subgroup of X_1.*

Proof. Let $B \in NMS(X_2)$ and $x_1, x_2 \in X_1$. Since g is a group homomorphism, the following inequality is obtained:

$$\begin{aligned} g^{-1}(B)(x_1x_2^{-1}) \quad & (T^i{}_B(g(x_1x_2^{-1})), I^i{}_B(g(x_1x_2^{-1})), F^i{}_B(g(x_1x_2^{-1}))) \\ &= (T^i{}_B(g(x_1)g(x_2)^{-1}), I^i{}_B(g(x_1)g(x_2)^{-1}), F^i{}_B(g(x_1)g(x_2)^{-1})) \\ &\geq (T^i{}_B(g(x_1)) \wedge T^i{}_B(g(x_2)), I^i{}_B(g(x_1)) \vee I^i{}_B(g(x_2)), F^i{}_B(g(x_1)) \vee F^i{}_B(g(x_2))) \\ &= (T^i{}_B(g(x_1)), I^i{}_B(g(x_1)), F^i{}_B(g(x_1))) \wedge (T^i{}_B(g(x_2)), I^i{}_B(g(x_2)), F^i{}_B(g(x_2))) \\ &= g^{-1}(B)(x_1) \wedge g^{-1}(B)(x_2). \end{aligned}$$

Therefore, $g^{-1}(B) \in NMS(X_1)$. □

Definition 17. *Let X be a classical group. $A \in NMG(X)$; then, the compound function of A and A is defined as*

$$A\tilde{o}A(z) = \{z, T^i{}_{A\tilde{o}A}(z), I^i{}_{A\tilde{o}A}(z), F^i{}_{A\tilde{o}A}(z) : \forall z \in X\}, \tag{10}$$

where $T^i{}_{A\tilde{o}A}(z) = (\vee_{xy=z} T^i{}_A(y) \wedge T^i{}_A(zy^{-1}))$, $I^i{}_{A\tilde{o}A}(z) = (\wedge_{xy=z} I^i{}_A(y) \vee I^i{}_A(zy^{-1}))$ and $F^i{}_{A\tilde{o}A}(z) = (\wedge_{xy=z} F^i{}_A(y) \vee F^i{}_A(zy^{-1}))$.

Theorem 6. *Let $A \in NMS(X)$. Then, $A \in NMG(X)$ iff $A\tilde{o}A \tilde{\subseteq} A$ and $A \tilde{\subseteq} A^{-1}$.*

Proof. Let $A \in NMS(X)$ and $x, y, z \in X$.

$$\begin{aligned} &\Rightarrow T^i{}_A(xy) \geq T^i{}_A(x) \wedge T^i{}_A(y) \\ &\Rightarrow T^i{}_A(z) \geq \vee\{T^i{}_A(x) \wedge T^i{}_A(y); xy = z\} \\ &= T^i{}_{A\tilde{o}A}(z) \end{aligned}$$

$$\begin{aligned} &\Rightarrow I^i{}_A(xy) \leq T^i{}_A(x) \vee I^i{}_A(y) \\ &\Rightarrow I^i{}_A(z) \leq \wedge\{I^i{}_A(x) \vee I^i{}_A(y); xy = z\} \\ &= I^i{}_{A\tilde{o}A}(z) \end{aligned}$$

$$\begin{aligned} &\Rightarrow F^i{}_A(xy) \leq F^i{}_A(x) \vee T^i{}_A(y) \\ &\Rightarrow F^i{}_A(z) \leq \wedge\{F^i{}_A(x) \vee F^i{}_A(y); xy = z\} \\ &= F^i{}_{A\tilde{o}A}(z) \end{aligned}$$

$\Rightarrow A\tilde{o}A \tilde{\subseteq} A$.

Now, by Proposition 2, we get the conditions. Conversely, suppose $A\tilde{o}A \tilde{\subseteq} A$ and $A \tilde{\subseteq} A^{-1}$

$$\begin{aligned} &\Rightarrow T^{i}{}^{-1}_A(x) \geq T^i{}_A(x) \text{ but } T^{i}{}^{-1}_A(x) = T^i{}_A(x^{-1}) \Rightarrow T^i{}_A(x^{-1}) \geq T^i{}_A(x) \\ &\Rightarrow I^{i}{}^{-1}_A(x) \leq T^i{}_A(x) \text{ but } I^{i}{}^{-1}_A(x) = I^i{}_A(x^{-1}) \Rightarrow I^i{}_A(x^{-1}) \leq I^i{}_A(x) \\ &\Rightarrow F^{i}{}^{-1}_A(x) \leq F^i{}_A(x) \text{ but } F^{i}{}^{-1}_A(x) = F^i{}_A(x^{-1}) \Rightarrow F^i{}_A(x^{-1}) \leq F^i{}_A(x) \end{aligned}$$

since $A \in NMS(X)$; then, to prove $A \in NMG(X)$, it enough to prove that $T^i{}_A(xy^{-1}) \geq T^i{}_A(x) \wedge T^i{}_A(y)$, $I^i{}_A(xy^{-1}) \leq I^i{}_A(x) \vee I^i{}_A(y)$ and $F^i{}_A(xy^{-1}) \leq F^i{}_A(x) \vee F^i{}_A(y)$ $\forall\, x, y \in X$.

Now,

$$\begin{aligned} T^i{}_A(xy^{-1}) \quad &\geq T^i{}_{A\tilde{o}A}(xy^{-1}) \\ &= \vee_{z\in X}\{T^i{}_A(z) \wedge T^i{}_A(z^{-1}xy^{-1})\} \\ &\geq \{T^i{}_A(x) \wedge T^i{}_A(y^{-1}); z = x\} \\ &\geq T^i{}_A(x) \wedge T^i{}_A(y) \end{aligned}$$

$$\begin{aligned} I^i{}_A(xy^{-1}) \quad &\leq I^i{}_{A\tilde{o}A}(xy^{-1}) \\ &= \wedge_{z\in X}\{I^i{}_A(z) \vee I^i{}_A(z^{-1}xy^{-1})\} \\ &\leq \{I^i{}_A(x) \vee I^i{}_A(y^{-1}); z = x\} \\ &\leq I^i{}_A(x) \vee I^i{}_A(y) \end{aligned}$$

$$\begin{aligned} F^i{}_A(xy^{-1}) &\leq F^i{}_{A\tilde{\circ}A}(xy^{-1}) \\ &= \wedge_{z\in X}\{F^i{}_A(z) \vee F^i{}_A(z^{-1}xy^{-1})\} \\ &\leq \{F^i{}_A(x) \vee F^i{}_A(y^{-1}); z = x\} \\ &\leq F^i{}_A(x) \vee F^i{}_A(y), \end{aligned}$$

hence the proof. □

Corollary 3. *Let $A \in NMS(X)$. Then, $A \in NMG(X)$ iff $A\tilde{\circ}A = A$ and $A\tilde{\subseteq}A^{-1}$.*

Proof. Let $A \in NMG(X)$. Then,

$$\begin{aligned} T^i{}_{A\tilde{\circ}A}(x) &= \vee\{T^i{}_A(y) \wedge T^i{}_A(z); y, z \in X \text{ and } yz = x\} \\ &\geq \{T^i{}_A(e) \wedge T^i{}_A(e^{-1}x)\} \\ &= T^i{}_A(x) \end{aligned}$$

$$\begin{aligned} I^i{}_{A\tilde{\circ}A}(x) &= \wedge\{I^i{}_A(y) \vee I^i{}_A(z); y, z \in X \text{ and } yz = x\} \\ &\leq \{I^i{}_A(e) \vee I^i{}_A(e^{-1}x)\} \\ &= I^i{}_A(x) \end{aligned}$$

$$\begin{aligned} F^i{}_{A\tilde{\circ}A}(x) &= \wedge\{F^i{}_A(y) \vee F^i{}_A(z); y, z \in X \text{ and } yz = x\} \\ &\leq \{F^i{}_A(e) \vee F^i{}_A(e^{-1}x)\} \\ &= F^i{}_A(x). \end{aligned}$$

Therefore, $A\tilde{\subseteq}A\tilde{\circ}A$.
Hence, by the above theorem, the proof is complete. □

Theorem 7. *Let X be a classical group and $A, B \in NMS(X)$. If $A, B \in NMG(X)$, then $A\tilde{\cap}B \in NMG(X)$.*

Proof. Let $x, y \in X$ be arbitrary:

$$\Rightarrow T^i{}_A(xy^{-1}) \geq T^i{}_A(x) \wedge T^i{}_A(y^{-1}), T^i{}_B(xy^{-1}) \geq T^i{}_B(x) \wedge T^i{}_B(y^{-1})$$

$$I^i{}_A(xy^{-1}) \leq I^i{}_A(x) \vee I^i{}_A(y^{-1}), I^i{}_B(xy^{-1}) \leq I^i{}_B(x) \vee T^i{}_B(y^{-1})$$

$$F^i{}_A(xy^{-1}) \leq F^i{}_A(x) \vee F^i{}_A(y^{-1}), F^i{}_B(xy^{-1}) \leq F^i{}_B(x) \vee F^i{}_B(y^{-1}).$$

Now,

$$\begin{aligned} T^i{}_{A\tilde{\cap}B}(xy^{-1}) &= T^i{}_{A\tilde{\cap}B}(x) \wedge T^i{}_{A\tilde{\cap}B}(y^{-1}) \text{ by definition intersection} \\ &\geq [T^i{}_A(x) \wedge T^i{}_A(y^{-1})] \wedge [T^i{}_B(x) \wedge T^i{}_B(y^{-1})] \\ &= [T^i{}_A(x) \wedge T^i{}_B(x)] \wedge [T^i{}_A(y^{-1}) \wedge T^i{}_B(y^{-1})] \text{ by commutative property of minimum} \\ &\geq [T^i{}_A(x) \wedge T^i{}_B(x)] \wedge [T^i{}_A(y) \wedge T^i{}_B(y)] \text{ since } A,\ B \in NMG(X) \\ &= T^i{}_{A\tilde{\cap}B}(x) \wedge T^i{}_{A\tilde{\cap}B}(y) \text{ by definition intersection} \\ &\Rightarrow T^i{}_{A\tilde{\cap}B}(xy^{-1}) \geq T^i{}_{A\tilde{\cap}B}(x) \wedge T^i{}_{A\tilde{\cap}B}(y) \ \ (1) \end{aligned}$$

$$\begin{aligned} I^i{}_{A\tilde{\cap}B}(xy^{-1}) &= I^i{}_{A\tilde{\cap}B}(x) \vee I^i{}_{A\tilde{\cap}B}(y^{-1}) \text{ by definition intersection} \\ &\leq [I^i{}_A(x) \vee I^i{}_A(y^{-1})] \vee [I^i{}_B(x) \vee I^i{}_B(y^{-1})] \\ &= [I^i{}_A(x) \vee I^i{}_B(x)] \vee [I^i{}_A(y^{-1}) \vee I^i{}_B(y^{-1})] \text{ by commutative property of maximum} \\ &\leq [I^i{}_A(x) \vee I^i{}_B(x)] \vee [I^i{}_A(y) \vee I^i{}_B(y)] \text{ since } A,\ B \in NMG(X) \\ &= I^i{}_{A\tilde{\cap}B}(x) \vee I^i{}_{A\tilde{\cap}B}(y) \text{ by definition intersection} \\ &\Rightarrow I^i{}_{A\tilde{\cap}B}(xy^{-1}) \leq I^i{}_{A\tilde{\cap}B}(x) \wedge I^i{}_{A\tilde{\cap}B}(y) \ \ (2) \end{aligned}$$

$$\begin{aligned} F^i{}_{A\tilde{\cap}B}(xy^{-1}) &= F^i{}_{A\tilde{\cap}B}(x) \vee F^i{}_{A\tilde{\cap}B}(y^{-1}) \text{ by definition intersection} \\ &\leq [F^i{}_A(x) \vee F^i{}_A(y^{-1})] \vee [F^i{}_B(x) \vee F^i{}_B(y^{-1})] \\ &= [F^i{}_A(x) \vee F^i{}_B(x)] \vee [F^i{}_A(y^{-1}) \vee F^i{}_B(y^{-1})] \text{ by commutative property of maximum} \\ &\leq [F^i{}_A(x) \vee F^i{}_B(x)] \vee [F^i{}_A(y) \vee F^i{}_B(y)] \text{ since } A,\ B \in NMG(X) \\ &= F^i{}_{A\tilde{\cap}B}(x) \vee F^i{}_{A\tilde{\cap}B}(y) \text{ by definition intersection} \\ &\Rightarrow F^i{}_{A\tilde{\cap}B}(xy^{-1}) \leq F^i{}_{A\tilde{\cap}B}(x) \wedge F^i{}_{A\tilde{\cap}B}(y) \ \ (3) \end{aligned}$$

From (1), (2) and (3), $A\tilde{\cap}B \in NMG(X)$, hence the proof. □

Remark 1. *Let X be a classical group and $\{A_i; i \in I\}$ be neutrosophic multiset on X. If $\{A_i; i \in I\}$ is a family of $NMG(X)$ over X, then their intersection $\tilde{\bigcap}_{i\in I} A_i$ is also a $NMG(X)$ over X.*

Proposition 6. *Let $A, B \in NMG(X)$. Then, $T^i{}_{A\tilde{\cup}B}(x) \leq T^i{}_{A\tilde{\cup}B}(x^{-1}), I^i{}_{A\tilde{\cup}B}(x) \geq I^i{}_{A\tilde{\cup}B}(x^{-1}),\ F^i{}_{A\tilde{\cup}B}(x) \geq F^i{}_{A\tilde{\cup}B}(x^{-1})$.*

Proof. Let $x, y \in X$. Now,

$$\begin{aligned} T^i{}_{A\tilde{\cup}B}(x^{-1}) &= \vee\{T^i{}_A(x^{-1}), T^i{}_B(x^{-1})\} \\ &\geq \vee\{T^i{}_A(x), T^i{}_B(x)\} \ \textit{since}\ A,\ B \in NMG(X) \\ &= T^i{}_{A\tilde{\cup}B}(x) \end{aligned}$$

$$\begin{aligned} I^i{}_{A\tilde{\cup}B}(x^{-1}) &= \wedge\{I^i{}_A(x^{-1}), I^i{}_B(x^{-1})\} \\ &\leq \wedge\{I^i{}_A(x), I^i{}_B(x)\} \ \textit{since}\ A,\ B \in NMG(X) \\ &= I^i{}_{A\tilde{\cup}B}(x) \end{aligned}$$

$$\begin{aligned} F^i{}_{A\tilde{\cup}B}(x^{-1}) &= \wedge\{F^i{}_A(x^{-1}), F^i{}_B(x^{-1})\} \\ &\leq \wedge\{F^i{}_A(x), F^i{}_B(x)\} \ \textit{since}\ A,\ B \in NMG(X) \\ &= F^i{}_{A\tilde{\cup}B}(x), \end{aligned}$$

hence the proof.

From this, it is clear that, if $A, B \in NMG(X)$, then $A\tilde{\cup}B \in NMG(X)$ iff $T^i{}_{A\tilde{\cup}B}(xy) \geq T^i{}_{A\tilde{\cup}B}(x) \wedge T^i{}_{A\tilde{\cup}B}(y), I^i{}_{A\tilde{\cup}B}(xy) \leq T^i{}_{A\tilde{\cup}B}(x) \vee I^i{}_{A\tilde{\cup}B}(y), F^i{}_{A\tilde{\cup}B}(xy) \leq F^i{}_{A\tilde{\cup}B}(x) \vee F^i{}_{A\tilde{\cup}B}(y)$. □

Corollary 4. *Let $A, B \in NMG(X)$. Then, $A\tilde{\cup}B$ need not be an element of $NMG(X)$.*

Example 7. *Assume that $X = \{1, -1, i, -i\}$ is a classical group. Then,*

$$\begin{aligned} A &= \{\langle 1; (0.5, 0.3, 0.2, 0.1), (0.1, 0.1, 0.2, 0.3), (0.2, 0.3, 0.4, 0.5)\rangle, \\ &\quad \langle -1; (0.7, 0.6, 0.4, 0.3), (0.1, 0.2, 0.2, 0.4), (0.2, 0.5, 0.4, 0.3)\rangle\}, \\ &\quad \langle i; (0.6, 0.4, 0.3, 0.2), (0.1, 0.3, 0.3, 0.4), (0.1, 0.2, 0.4, 0.6)\rangle, \\ &\quad \langle -i; (0.6, 0.4, 0.3, 0.2), (0.1, 0.3, 0.3, 0.4), (0.1, 0.2, 0.4, 0.6)\rangle\}, \\ B &= \{\langle 1; (0.5, 0.6, 0.6, 0.4), (0.1, 0.2, 0.2, 0.3), (0.2, 0.4, 0.4, 0.5)\rangle, \\ &\quad \langle -1; (0.7, 0.6, 0.4, 0.3), (0.2, 0.1, 0.2, 0.3), (0.3, 0.4, 0.5, 0.3)\rangle\} \\ &\quad \textit{are } NM - \textit{groups}. \\ A \cup B &= \{\langle 1; (0.5, 0.6, 0.6, 0.4), (0.1, 0.1, 0.2, 0.3), (0.2, 0.3, 0.4, 0.5)\rangle, \\ &\quad \langle -1; (0.7, 0.6, 0.4, 0.3), (0.1, 0.2, 0.2, 0.3), (0.2, 0.4, 0.4, 0.3)\rangle, \\ &\quad \langle i; (0.6, 0.4, 0.3, 0.2), (0.1, 0.3, 0.3, 0.4), (0.1, 0.2, 0.4, 0.6)\rangle, \\ &\quad \langle -i; (0.6, 0.4, 0.3, 0.2), (0.1, 0.3, 0.3, 0.4), (0.1, 0.2, 0.4, 0.6)\rangle\}. \end{aligned}$$

However, $T^i{}_{A\tilde{\cup}B}(1) \geq T^i{}_{A\tilde{\cup}B}(i) \wedge T^i{}_{A\tilde{\cup}B}(-i)$ as $i.(-i) = 1$. Then, $A\tilde{\cup}B \notin NMG(X)$.

Proposition 7. *If $A \in NMG(X)$ and X_1 is a subgroup of X, then $A_{|_{X_1}}$ (i.e., A restricted to X_1) $\in NM - group(X_1)$ and is a neutrosophic multi subgroup of A.*

Proof. Let $x, y \in X_1$. Then, $xy^{-1} \in X_1$. Now,

$$T^i{}_{A_{|X_1}}(xy^{-1}) = T^i{}_A(xy^{-1}) \geq T^i{}_A(x) \wedge T^i{}_A(y) = T^i{}_{A_{|X_1}}(x) \wedge T^i{}_{A_{|X_1}}(y),$$

$$I^i{}_{A_{|X_1}}(xy^{-1}) = I^i{}_A(xy^{-1}) \leq I^i{}_A(x) \vee I^i{}_A(y) = I^i{}_{A_{|X_1}}(x) \vee I^i{}_{A_{|X_1}}(y),$$

$$F^i{}_{A_{|X_1}}(xy^{-1}) = F^i{}_A(xy^{-1}) \leq F^i{}_A(x) \vee F^i{}_A(y) = F^i{}_{A_{|X_1}}(x) \vee F^i{}_{A_{|X_1}}(y).$$

The second part is trivial. □

Definition 18. *Let $A \in NMG(X)$ and $B \in NMG(Y)$ be two neutrosophic multi groups over the groups X and Y, respectively. Then, the Cartesian product of A and B is defined as $(A\tilde{\times}B)(x,y) = A(x)\tilde{\times}B(y)$ where*

$$A\tilde{\times}B = \{(x,y), T^i{}_{A\tilde{\times}B}(x,y), I^i{}_{A\tilde{\times}B}(x,y), F^i{}_{A\tilde{\times}B}(x,y) : (x,y) \in X \times Y\}, \tag{11}$$

where $T^i{}_{A\tilde{\times}B}(x,y) = T^i{}_A(x) \vee T^i{}_B(y)$, $I^i{}_{A\tilde{\times}B}(x,y) = I^i{}_A(x) \wedge I^i{}_B(y)$, $F^i{}_{A\tilde{\times}B}(x,y) = F^i{}_A(x) \wedge F^i{}_B(y)$.

Example 8. *Assume that $(Z_2, +)$ and $(Z_3, +)$ are classiccal groups. Define the neutrosophic multi group A on $(Z_2, +)$ and B on $(Z_3, +)$ as follows:*

$$\begin{aligned} A \;&= \{\langle 0; (0.6, 0.5, 0.4, 0.2), (0.2, 0.2, 0.3, 0.4), (0.2, 0.3, 0.4, 0.6)\rangle, \\ &\langle 1; (0.6, 0.5, 0.4, 0.3), (0.2, 0.1, 0.2, 0.3), (0.1, 0.2, 0.3, 0.4)\rangle\} \end{aligned}$$

$$\begin{aligned} B \;&= \{\langle 0; (0.7, 0.6, 0.5, 0.4), (0.2, 0.2, 0.3, 0.4), (0.3, 0.4, 0.5, 0.5)\rangle, \\ &\langle 1; (0.7, 0.6, 0.5, 0.3), (0.3, 0.4, 0.3, 0.4), (0.3, 0.2, 0.5, 0.5)\rangle, \\ &\langle 2; (0.8, 0.7, 0.5, 0.4), (0.1, 0.3, 0.2, 0.3), (0.2, 0.1, 0.3, 0.5)\rangle\}. \end{aligned}$$

$$\begin{aligned} A\tilde{\times}B \;&= \{\langle (0,0); (0.6, 0.5, 0.4, 0.2), (0.2, 0.2, 0.3, 0.4), (0.2, 0.3, 0.4, 0.6)\rangle, \\ &\langle (0,1); (0.6, 0.5, 0.4, 0.2), (0.3, 0.4, 0.3, 0.4), (0.3, 0.3, 0.5, 0.6)\rangle, \\ &\langle (0,2); (0.6, 0.5, 0.4, 0.2), (0.2, 0.3, 0.3, 0.4), (0.2, 0.3, 0.4, 0.6)\rangle, \\ &\langle (1,0); (0.6, 0.5, 0.4, 0.3), (0.2, 0.2, 0.3, 0.4), (0.3, 0.4, 0.5, 0.5)\rangle, \\ &\langle (1,1); (0.6, 0.5, 0.4, 0.3), (0.3, 0.4, 0.3, 0.4), (0.3, 0.2, 0.5, 0.5)\rangle, \\ &\langle (1,2); (0.6, 0.5, 0.4, 0.3), (0.2, 0.3, 0.2, 0.3), (0.2, 0.2, 0.3, 0.5)\rangle\}. \end{aligned}$$

Then, $A\tilde{\times}B$ is a neutrosophic multi group.

Theorem 8. *Let $A, B \in NMG(X)$. The cartesian product of A and B is denoted by $A\tilde{\times}B \in NMG(X)$.*

Proof. From the Theorem 1, it is clear that a $NMG(X)$ is a neutrosophic multi group:

$$\begin{aligned} T^i{}_{A\tilde{\times}B}((x_1,y_1),(x_2,y_2)^{-1}) \;&= T^i{}_{A\tilde{\times}B}(x_1x_2^{-1}, y_1y_2^{-1}) \\ &= T^i{}_A(x_1x_2^{-1}) \wedge T^i{}_B(y_1y_2^{-1}) \\ &\geq (T^i{}_A(x_1) \wedge T^i{}_A(x_2)) \wedge (T^i{}_B(y_1) \wedge T^i{}_B(y_2)) \\ &= (T^i{}_A(x_1) \wedge T^i{}_B(y_1)) \wedge (T^i{}_A(x_2) \wedge T^i{}_B(y_2)) \\ &= T^i{}_{A\tilde{\times}B}(x_1,y_1) \wedge T^i{}_{A\tilde{\times}B}(x_2,y_2) \end{aligned}$$

$$\begin{aligned} I^i{}_{A\tilde{\times}B}((x_1,y_1),(x_2,y_2)^{-1}) \;&= I^i{}_{A\tilde{\times}B}(x_1x_2^{-1}, y_1y_2^{-1}) \\ &= I^i{}_A(x_1x_2^{-1}) \vee I^i{}_B(y_1y_2^{-1}) \\ &\leq (I^i{}_A(x_1) \vee I^i{}_A(x_2)) \vee (I^i{}_B(y_1) \vee I^i{}_B(y_2)) \\ &= (I^i{}_A(x_1) \vee I^i{}_B(y_1)) \vee (I^i{}_A(x_2) \vee I^i{}_B(y_2)) \\ &= I^i{}_{A\tilde{\times}B}(x_1,y_1) \vee I^i{}_{A\tilde{\times}B}(x_2,y_2) \end{aligned}$$

and

$$\begin{aligned} F^i{}_{A\tilde{\times}B}((x_1,y_1),(x_2,y_2)^{-1}) &= F^i{}_{A\tilde{\times}B}(x_1x_2^{-1},y_1y_2^{-1}) \\ &= F^i{}_A(x_1x_2^{-1}) \vee F^i{}_B(y_1y_2^{-1}) \\ &\leq (F^i{}_A(x_1) \vee F^i{}_A(x_2)) \vee (F^i{}_B(y_1) \vee F^i{}_B(y_2)) \\ &= (F^i{}_A(x_1) \vee F^i{}_B(y_1)) \vee (F^i{}_A(x_2) \vee F^i{}_B(y_2)) \\ &= F^i{}_{A\tilde{\times}B}(x_1,y_1) \vee F^i{}_{A\tilde{\times}B}(x_2,y_2) \end{aligned}$$

for all $x, y \in X$ and $i = 1, 2, ..., P$—hence the proof. □

5. Conclusions

The concept of a group is of fundamental importance in the study of algebra. In this paper, the algebraic structure of neutrosophic multiset is introduced as a neutrosophic multigroup. The neutrosophic multigroup is a generalized case of intuitionistic fuzzy multigroup and fuzzy multigroup. The various basic operations, definitions and theorems related to neutrosophic multigroup have been discussed. The foundations which we made through this paper can be used to get an insight into the higher order structures of group theory.

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Abbreviations

The following abbreviations are used in this manuscript:

NMG	Neutrosophic Multigroup
NMS	Neutrosophic Multiset
IFMS	Intuitionistic Fuzzy Multiset

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