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Neutrosophic $\mathcal{N}$-structures and their applications in semigroups

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Abstract. The notion of neutrosophic $\mathcal{N}$-structure is introduced, and applied it to semigroup. The notions of neutrosophic $\mathcal{N}$-subsemigroup, neutrosophic $\mathcal{N}$-product and ε-neutrosophic $\mathcal{N}$-subsemigroup are introduced, and several properties are investigated. Conditions for neutrosophic $\mathcal{N}$-structure to be neutrosophic $\mathcal{N}$-subsemigroup are provided. Using neutrosophic $\mathcal{N}$-product, characterization of neutrosophic $\mathcal{N}$-subsemigroup is discussed. Relations between neutrosophic $\mathcal{N}$-subsemigroup and ε-neutrosophic $\mathcal{N}$-subsemigroup are discussed. We show that the homomorphic preimage of neutrosophic $\mathcal{N}$-subsemigroup is a neutrosophic $\mathcal{N}$-subsemigroup, and the onto homomorphic image of neutrosophic $\mathcal{N}$-subsemigroup is a neutrosophic $\mathcal{N}$-subsemigroup.

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1. Introduction

Zadeh [?] introduced the degree of membership/truth (t) in 1965 and defined the fuzzy set. As a generalization of fuzzy sets, Atanassov [?] introduced the degree of nonmembership/falsehood (f) in 1986 and defined the intuitionistic fuzzy set. Smarandache proposed the term "neutrosophic" because "neutrosophic" etymologically comes from "neutrosophy" [French *neutre*, Latin *neuter*, neutral, and Greek *sophia*, skill/wisdom] which means knowledge of neutral thought, and this third/neutral represents the main distinction between "fuzzy"/"intuitionistic fuzzy" logic/set and "neutrosophic" logic/set, i.e. the *included middle* component (Lupasco-Nicolescu's logic in philosophy), i.e. the neutral/indeterminate/unknown part (besides the "truth"/"membership" and "falsehood"/"non-membership" components that both appear in fuzzy logic/set). Smarandache introduced the degree of indeterminacy/neutrality (i) as independent component in 1995 (published in 1998) and defined the neutrosophic set on three components

$$(\text{t, i, f}) = (\text{truth, indeterminacy, falsehood}).$$

For more detail, refer to the site

http://fs.gallup.unm.edu/FlorentinSmarandache.htm.

The concept of neutrosophic set (NS) developed by Smarandache [?] and Smarandache [?] is a more general platform which extends the concepts of the classic set and fuzzy set, intuitionistic fuzzy set and interval valued intuitionistic fuzzy set. Neutrosophic set theory is applied to various part (refer to the site

http://fs.gallup.unm.edu/neutrosophy.htm).

A (crisp) set A in a universe X can be defined in the form of its characteristic function $\mu_A : X \to \{0, 1\}$ yielding the value 1 for elements belonging to the set A and the value 0 for elements excluded from the set A. So far most of the generalization of the crisp set have been conducted on the unit interval $[0, 1]$ and they are consistent with the asymmetry observation. In other words, the generalization of the crisp set to fuzzy sets relied on spreading positive information that fit the crisp point $\{1\}$ into the interval $[0, 1]$. Because no negative meaning of information is suggested, we now feel a need to deal with negative information. To do so, we also feel a need to supply mathematical tool. To attain such object, Jun et al. [?] introduced a new function which is called negative-valued function, and constructed $\mathcal{N}$-structures. This structure is applied to BE-algebra, BCK/BCI-algebra and BCH-algebra etc. (see [?], [?], [?], [?]).

In this paper, we introduce the notion of neutrosophic $\mathcal{N}$-structure and applied it to semigroup. We introduce the notion of neutrosophic $\mathcal{N}$-subsemi-group and investigate several properties. We provide conditions for neutrosophic $\mathcal{N}$-structure to be neutrosophic $\mathcal{N}$-subsemigroup. We define neutrosophic $\mathcal{N}$-product, and give characterization of neutrosophic $\mathcal{N}$-subsemigroup by using neutrosophic $\mathcal{N}$-product. We also introduce ε-neutrosophic subsemigroup, and investigate relations between neutrosophic subsemigroup and ε-neutrosophic subsemigroup. We show that the homomorphic preimage of neutrosophic $\mathcal{N}$-subsemigroup is a neutrosophic $\mathcal{N}$-subsemigroup, and the onto homomorphic image of neutrosophic $\mathcal{N}$-subsemigroup is a neutrosophic $\mathcal{N}$-subsemigroup.

2. Preliminaries

Let X be a semigroup. Let A and B be subsets of X. Then the multiplication of A and B is defined as follows:

$$AB = \{ab \in X \mid a \in A,\ b \in B\}.$$

By a subsemigroup of X, we mean a nonempty subset A of X such that $A^2 \subseteq A$. We consider the empty set $\varnothing$ is always a subsemigroup of X.

We refer the reader to the book [?] for further information regarding fuzzy semigroups.

For any family $\{a_i \mid i \in \Lambda\}$ of real numbers, we define:

$$\bigvee\{a_i \mid i \in \Lambda\} := \begin{cases} \max\{a_i \mid i \in \Lambda\} & \text{if } \Lambda \text{ is finite,} \\ \sup\{a_i \mid i \in \Lambda\} & \text{otherwise} \end{cases}$$

and

$$\bigwedge\{a_i \mid i \in \Lambda\} := \begin{cases} \min\{a_i \mid i \in \Lambda\} & \text{if } \Lambda \text{ is finite,} \\ \inf\{a_i \mid i \in \Lambda\} & \text{otherwise.} \end{cases}$$

For any real numbers a and b, we also use $a \vee b$ and $a \wedge b$ instead of $\bigvee\{a, b\}$ and $\bigwedge\{a, b\}$, respectively.

3. Neutrosophic $\mathcal{N}$-structures

Denote by $\mathcal{F}(X, [-1, 0])$ the collection of functions from a set X to $[-1, 0]$. We say that an element of $\mathcal{F}(X, [-1, 0])$ is a negative-valued function from X to $[-1, 0]$ (briefly, $\mathcal{N}$-function on X). By an $\mathcal{N}$-structure, we mean an ordered pair (X, f) of X and an $\mathcal{N}$-function f on X. In what follows, let X denote the nonempty universe of discourse unless otherwise specified.

Definition 3.1. A neutrosophic $\mathcal{N}$-structure over X is defined to be the structure:

$$X_{\mathbf{N}} := \frac{X}{(T_N, I_N, F_N)} = \left\{ \frac{x}{(T_N(x), I_N(x), F_N(x))} \mid x \in X \right\} \tag{3.1}$$

where T_N, I_N and F_N are $\mathcal{N}$-functions on X which are called the negative truth membership function, the negative indeterminacy membership function and the negative falsity membership function, respectively, on X.

Note that every neutrosophic $\mathcal{N}$-structure $X_{\mathbf{N}}$ over X satisfies the condition:

$$(\forall x \in X) \left(-3 \leq T_N(x) + I_N(x) + F_N(x) \leq 0\right).$$

Example 3.2. Consider a universe of discourse $X = \{x, y, z\}$. We know that

$$X_{\mathbf{N}} = \left\{ \frac{x}{(-0.7, -0.5, -0.1)}, \frac{y}{(-0.2, -0.3, -0.4)}, \frac{z}{(-0.3, -0.6, -0.1)} \right\}$$

is a neutrosophic $\mathcal{N}$-structure over X.

Definition 3.3. Let $X_{\mathbf{N}} := \frac{X}{(T_N, I_N, F_N)}$ and $X_{\mathbf{M}} := \frac{X}{(T_M, I_M, F_M)}$ be neutrosophic $\mathcal{N}$-structures over X. We say that $X_{\mathbf{M}}$ is a neutrosophic $\mathcal{N}$-substructure over X, denoted by $X_{\mathbf{N}} \subseteq X_{\mathbf{M}}$, if it satisfies:

$$(\forall x \in X)(T_N(x) \geq T_M(x), I_N(x) \leq I_M(x), F_N(x) \geq F_M(x)).$$

If $X_{\mathbf{N}} \subseteq X_{\mathbf{M}}$ and $X_{\mathbf{M}} \subseteq X_{\mathbf{N}}$, we say that $X_{\mathbf{N}} = X_{\mathbf{M}}$.

Definition 3.4. Let $X_{\mathbf{N}} := \frac{X}{(T_N, I_N, F_N)}$ and $X_{\mathbf{M}} := \frac{X}{(T_M, I_M, F_M)}$ be neutrosophic $\mathcal{N}$-structures over X.

(1) The union of $X_{\mathbf{N}}$ and $X_{\mathbf{M}}$ is defined to be a neutrosophic $\mathcal{N}$-structure

$$X_{\mathbf{N} \cup \mathbf{M}} = (X; T_{N \cup M}, I_{N \cup M}, F_{N \cup M}),$$

where

$$T_{N\cup M}(x) = \bigwedge\{T_N(x), T_M(x)\},$$

$$I_{N\cup M}(x) = \bigvee\{I_N(x), I_M(x)\},$$

$$F_{N\cup M}(x) = \bigwedge\{F_N(x), F_M(x)\},$$

for all $x \in X$.

(2) The intersection of $X_{\mathbf{N}}$ and $X_{\mathbf{M}}$ is defined to be a neutrosophic $\mathcal{N}$-structure

$$X_{\mathbf{N}\cap\mathbf{M}} = (X; T_{N\cap M}, I_{N\cap M}, F_{N\cap M}),$$

where

$$T_{N\cap M}(x) = \bigvee\{T_N(x), T_M(x)\},$$

$$I_{N\cap M}(x) = \bigwedge\{I_N(x), I_M(x)\},$$

$$F_{N\cap M}(x) = \bigvee\{F_N(x), F_M(x)\},$$

for all $x \in X$.

Definition 3.5. Given a neutrosophic $\mathcal{N}$-structure $X_{\mathbf{N}} := \frac{X}{(T_N, I_N, F_N)}$ over X, the complement of $X_{\mathbf{N}}$ is defined to be a neutrosophic $\mathcal{N}$-structure

$$X_{\mathbf{N}^c} := \frac{X}{(T_{N^c}, I_{N^c}, F_{N^c})}$$

over X, where

$$T_{N^c}(x) = -1 - T_N(x),\ I_{N^c}(x) = -1 - I_N(x) \text{ and } F_{N^c}(x) = -1 - F_N(x),$$

for all $x \in X$.

Example 3.6. Let $X = \{a, b, c\}$ be a universe of discourse and let $X_{\mathbf{N}}$ be the neutrosophic $\mathcal{N}$-structure over X in Example **??**. Let $X_{\mathbf{M}}$ be a neutrosophic $\mathcal{N}$-structure over X which is given by

$$X_{\mathbf{M}} = \left\{\frac{x}{(-0.3, -0.5, -0.2)}, \frac{y}{(-0.4, -0.2, -0.2)}, \frac{z}{(-0.5, -0.7, -0.8)}\right\}.$$

The union and intersection of $X_{\mathbf{N}}$ and $X_{\mathbf{M}}$ are given as follows respectively:

$$X_{\mathbf{N}\cup\mathbf{M}} = \left\{\frac{x}{(-0.7, -0.5, -0.2)}, \frac{y}{(-0.4, -0.3, -0.4)}, \frac{z}{(-0.5, -0.7), -0.8)}\right\}$$

and

$$X_{\mathbf{N}\cap\mathbf{M}} = \left\{\frac{x}{(-0.3, -0.5, -0.1)}, \frac{y}{(-0.2, -0.2, -0.2)}, \frac{z}{(-0.3, -0.6, -0.1)}\right\}.$$

The complement of $X_{\mathbf{N}}$ is given by

$$X_{\mathbf{M}^c} = \left\{\frac{x}{(-0.7, -0.5, -0.8)}, \frac{y}{(-0.6, -0.8, -0.8)}, \frac{z}{(-0.5, -0.3, -0.2)}\right\}.$$

4. Applications in semigroups

In this section, we take a semigroup X as the universe of discourse unless otherwise specified.

Definition 4.1. A neutrosophic $\mathcal{N}$-structure $X_{\mathbf{N}}$ over X is called a neutrosophic $\mathcal{N}$-subsemigroup of X if the following condition is valid:

$$(\forall x, y \in X) \left(\begin{array}{l} T_N(xy) \leq \bigvee\{T_N(x), T_N(y)\} \\ I_N(xy) \geq \bigwedge\{I_N(x), I_N(y)\} \\ F_N(xy) \leq \bigvee\{F_N(x), F_N(y)\} \end{array} \right). \tag{4.1}$$

Let $X_{\mathbf{N}}$ be a neutrosophic $\mathcal{N}$-structure over X and let $\alpha, \beta, \gamma \in [-1, 0]$ be such that $-3 \leq \alpha + \beta + \gamma \leq 0$. Consider the following sets:

$$\begin{aligned} T_N^{\alpha} &:= \{x \in X \mid T_N(x) \leq \alpha\}, \\ I_N^{\beta} &:= \{x \in X \mid I_N(x) \geq \beta\}, \\ F_N^{\gamma} &:= \{x \in X \mid F_N(x) \leq \gamma\}. \end{aligned} \tag{4.2}$$

The set

$$X_{\mathbf{N}}(\alpha, \beta, \gamma) := \{x \in X \mid T_N(x) \leq \alpha, I_N(x) \geq \beta, F_N(x) \leq \gamma\}$$

is called a (α, β, γ)-level set of $X_{\mathbf{N}}$. Note that

$$X_{\mathbf{N}}(\alpha, \beta, \gamma) = T_N^{\alpha} \cap I_N^{\beta} \cap F_N^{\gamma}.$$

Theorem 4.2. *Let $X_{\mathbf{N}}$ be a neutrosophic $\mathcal{N}$-structure over X and let $\alpha, \beta, \gamma \in [-1, 0]$ be such that $-3 \leq \alpha+\beta+\gamma \leq 0$. If $X_{\mathbf{N}}$ is a neutrosophic $\mathcal{N}$-subsemigroup of X, then the (α, β, γ)-level set of $X_{\mathbf{N}}$ is a subsemigroup of X whenever it is nonempty.*

Proof. Assume that $X_{\mathbf{N}}(\alpha, \beta, \gamma) \neq \varnothing$ for $\alpha, \beta, \gamma \in [-1, 0]$ with $-3 \leq \alpha + \beta + \gamma \leq 0$. Let $x, y \in X_{\mathbf{N}}(\alpha, \beta, \gamma)$. Then $T_N(x) \leq \alpha$, $I_N(x) \geq \beta$, $F_N(x) \leq \gamma$, $T_N(y) \leq \alpha$, $I_N(y) \geq \beta$ and $F_N(y) \leq \gamma$. Thus it follows from (**??**) that

$$T_N(xy) \leq \bigvee\{T_N(x), T_N(y)\} \leq \alpha,$$

$$I_N(xy) \geq \bigwedge\{I_N(x), I_N(y)\} \geq \beta,$$

$$F_N(xy) \leq \bigvee\{F_N(x), F_N(y)\} \leq \gamma.$$

So $xy \in X_{\mathbf{N}}(\alpha, \beta, \gamma)$. Hence $X_{\mathbf{N}}(\alpha, \beta, \gamma)$ is a subsemigroup of X. □

Theorem 4.3. *Let $X_{\mathbf{N}}$ be a neutrosophic $\mathcal{N}$-structure over X and let $\alpha, \beta, \gamma \in [-1, 0]$ be such that $-3 \leq \alpha + \beta + \gamma \leq 0$. If T_N^{α}, I_N^{β} and F_N^{γ} are subsemigroups of X, then $X_{\mathbf{N}}$ is a neutrosophic $\mathcal{N}$-subsemigroup of X.*

Proof. Assume that there are $a, b \in X$ such that $T_N(ab) > \bigvee\{T_N(a), T_N(b)\}$. Then $T_N(ab) > t_{\alpha} \geq \bigvee\{T_N(a), T_N(b)\}$ for some $t_{\alpha} \in [-1, 0)$. Thus $a, b \in T_N^{t_{\alpha}}$ but $ab \notin T_N^{t_{\alpha}}$, which is a contradiction. So

$$T_N(xy) \leq \bigvee\{T_N(x), T_N(y)\},$$

for all $x, y \in X$.

Assume that $I_N(ab) < \bigwedge\{I_N(a), I_N(b)\}$, for some $a, b \in X$. Then $a, b \in I_N^{t_\beta}$ and $ab \notin I_N^{t_\beta}$, for $t_\beta := \bigwedge\{I_N(a), I_N(b)\}$. This is a contradiction. Thus

$$I_N(xy) \geq \bigwedge\{I_N(x), I_N(y)\},$$

for all $x, y \in X$.

Now, suppose that there exist $a, b \in X$ and $t_\gamma \in [-1, 0)$ such that

$$F_N(ab) > t_\gamma \geq \bigvee\{F_N(a), F_N(b)\}.$$

Then $a, b \in F_N^{t_\gamma}$ and $ab \notin F_N^{t_\gamma}$, which is a contradiction. Thus

$$F_N(xy) \leq \bigvee\{F_N(x), F_N(y)\},$$

for all $x, y \in X$. Hence $X_{\mathbf{N}}$ is a neutrosophic $\mathcal{N}$-subsemigroup of X. □

Theorem 4.4. *The intersection of two neutrosophic $\mathcal{N}$-subsemigroups is also a neutrosophic $\mathcal{N}$-subsemigroup.*

Proof. Let $X_{\mathbf{N}} := \frac{X}{(T_N, I_N, F_N)}$ and $X_{\mathbf{M}} := \frac{X}{(T_M, I_M, F_M)}$ be neutrosophic $\mathcal{N}$-subsemigroups of X. For any $x, y \in X$, we have

$$\begin{aligned}
T_{N\cap M}(xy) &= \bigvee\{T_N(xy), T_M(xy)\} \\
&\leq \bigvee\left\{\bigvee\{T_N(x), T_N(y)\}, \bigvee\{T_M(x), T_M(y)\}\right\} \\
&= \bigvee\left\{\bigvee\{T_N(x), T_M(x)\}, \bigvee\{T_N(y), T_M(y)\}\right\} \\
&= \bigvee\{T_{N\cap M}(x), T_{N\cap M}(y)\},
\end{aligned}$$

$$\begin{aligned}
I_{N\cap M}(xy) &= \bigwedge\{I_N(xy), I_M(xy)\} \\
&\geq \bigwedge\left\{\bigwedge\{I_N(x), I_N(y)\}, \bigwedge\{I_M(x), I_M(y)\}\right\} \\
&= \bigwedge\left\{\bigwedge\{I_N(x), I_M(x)\}, \bigwedge\{I_N(y), I_M(y)\}\right\} \\
&= \bigwedge\{I_{N\cap M}(x), I_{N\cap M}(y)\}
\end{aligned}$$

and

$$\begin{aligned}
F_{N\cap M}(xy) &= \bigvee\{F_N(xy), F_M(xy)\} \\
&\leq \bigvee\left\{\bigvee\{F_N(x), F_N(y)\}, \bigvee\{F_M(x), F_M(y)\}\right\} \\
&= \bigvee\left\{\bigvee\{F_N(x), F_M(x)\}, \bigvee\{F_N(y), F_M(y)\}\right\} \\
&= \bigvee\{F_{N\cap M}(x), F_{N\cap M}(y)\},
\end{aligned}$$

for all $x, y \in X$. Then $X_{\mathbf{N}\cap\mathbf{M}}$ is a neutrosophic $\mathcal{N}$-subsemigroup of X. □

Corollary 4.5. *If $\{X_{N_i} \mid i \in \mathbb{N}\}$ is a family of neutrosophic $\mathcal{N}$-subsemigroups of X, then so is $X_{\cap N_i}$.*

Let $X_{\mathbf{N}} := \frac{X}{(T_N, I_N, F_N)}$ and $X_{\mathbf{M}} := \frac{X}{(T_M, I_M, F_M)}$ be neutrosophic $\mathcal{N}$-structures over X. The neutrosophic $\mathcal{N}$-product of $X_{\mathbf{N}}$ and $X_{\mathbf{M}}$ is defined to be a neutrosophic $\mathcal{N}$-structure over X

$$\begin{aligned} X_{\mathbf{N}} \odot X_{\mathbf{M}} &= \frac{X}{T_{N\circ M},\, I_{N\circ M},\, F_{N\circ M}} \\ &= \left\{ \frac{x}{T_{N\circ M}(x),\, I_{N\circ M}(x),\, F_{N\circ M}(x)} \mid x \in X \right\}, \end{aligned}$$

where

$$T_{N\circ M}(x) = \begin{cases} \bigwedge\limits_{x=yz} \{T_N(y) \vee T_M(z)\} & \text{if } \exists\, y, z \in X \text{ such that } x = yz \\ 0 & \text{otherwise,} \end{cases}$$

$$I_{N\circ M}(x) = \begin{cases} \bigvee\limits_{x=yz} \{I_N(y) \wedge I_M(z)\} & \text{if } \exists\, y, z \in X \text{ such that } x = yz \\ -1 & \text{otherwise} \end{cases}$$

and

$$F_{N\circ M}(x) = \begin{cases} \bigwedge\limits_{x=yz} \{F_N(y) \vee F_M(z)\} & \text{if } \exists\, y, z \in X \text{ such that } x = yz \\ 0 & \text{otherwise.} \end{cases}$$

For any $x \in X$, the element $\frac{x}{T_{N\circ M}(x),\, I_{N\circ M}(x),\, F_{N\circ M}(x)}$ is simply denoted by

$$(X_{\mathbf{N}} \odot X_{\mathbf{M}})(x) := (T_{N\circ M}(x),\, I_{N\circ M}(x),\, F_{N\circ M}(x))$$

for the sake of convenience.

Theorem 4.6. *A neutrosophic $\mathcal{N}$-structure $X_{\mathbf{N}}$ over X is a neutrosophic $\mathcal{N}$-subsemigroup of X if and only if $X_{\mathbf{N}} \odot X_{\mathbf{N}} \subseteq X_{\mathbf{N}}$.*

Proof. Assume that $X_{\mathbf{N}}$ is a neutrosophic $\mathcal{N}$-subsemigroup of X and let $x \in X$. If $x \neq yz$ for all $x, y \in X$, then clearly $X_{\mathbf{N}} \odot X_{\mathbf{N}} \subseteq X_{\mathbf{N}}$. Assume that there exist $a, b \in X$ such that $x = ab$. Then

$$T_{N\circ N}(x) = \bigwedge_{x=ab} \{T_N(a) \vee T_N(b)\} \geq \bigwedge_{x=ab} T_N(ab) = T_N(x),$$

$$I_{N\circ N}(x) = \bigvee_{x=ab} \{I_N(a) \wedge I_N(b)\} \leq \bigvee_{x=ab} I_N(ab) = I_N(x),$$

and

$$F_{N\circ N}(x) = \bigwedge_{x=ab} \{F_N(a) \vee F_N(b)\} \geq \bigwedge_{x=ab} F_N(ab) = F_N(x).$$

Thus $X_{\mathbf{N}} \odot X_{\mathbf{N}} \subseteq X_{\mathbf{N}}$.

Conversely, let $X_{\mathbf{N}}$ be any neutrosophic $\mathcal{N}$-structure over X such that $X_{\mathbf{N}} \odot X_{\mathbf{N}} \subseteq X_{\mathbf{N}}$. Let x and y be any elements of X and let $a = xy$. Then

$$T_N(xy) = T_N(a) \leq T_{N\circ N}(a) = \bigwedge_{a=bc} \{T_N(b) \vee T_N(c)\} \leq T_N(x) \vee T_N(y),$$

$$I_N(xy) = I_N(a) \geq I_{N\circ N}(a) = \bigvee_{a=bc} \{I_N(b) \wedge I_N(c)\} \geq I_N(x) \wedge I_N(y),$$

and

$$F_N(xy) = F_N(a) \leq F_{N \circ N}(a) = \bigwedge_{a=bc} \{F_N(b) \vee F_N(c)\} \leq F_N(x) \vee F_N(y).$$

Thus $X_{\mathbf{N}}$ is a neutrosophic $\mathcal{N}$-subsemigroup of X. $\square$

Since $[-1, 0]$ is a completely distributive lattice with respect to the usual ordering, we have the following theorem.

Theorem 4.7. *If $\{X_{N_i} \mid i \in \mathbb{N}\}$ is a family of neutrosophic $\mathcal{N}$-subsemigroups of X, then $(\{X_{N_i} \mid i \in \mathbb{N}\}, \subseteq)$ forms a complete distributive lattice.*

Theorem 4.8. *Let X be a semigroup with identity e and let $X_{\mathbf{N}} := \frac{X}{(T_N, I_N, F_N)}$ be a neutrosophic $\mathcal{N}$-structure over X such that*

$$(\forall x \in X)\left(X_{\mathbf{N}}(e) \geq X_{\mathbf{N}}(x)\right),$$

that is, $T_N(e) \leq T_N(x)$, $I_N(e) \geq I_N(x)$ and $F_N(e) \leq F_N(x)$ for all $x \in X$. If $X_{\mathbf{N}}$ is a neutrosophic $\mathcal{N}$-subsemigroup of X, then $X_{\mathbf{N}}$ is neutrosophic idempotent, that is, $X_{\mathbf{N}} \odot X_{\mathbf{N}} = X_{\mathbf{N}}$.

Proof. For any $x \in X$, we have

$$T_{N \circ N}(x) = \bigwedge_{x=yz} \{T_N(y) \vee T_N(z)\} \leq T_N(x) \vee T_N(e) = T_N(x),$$

$$I_{N \circ N}(x) = \bigvee_{x=yz} \{I_N(y) \wedge I_N(z)\} \geq I_N(x) \wedge I_N(e) = I_N(x)$$

and

$$F_{N \circ N}(x) = \bigwedge_{x=yz} \{F_N(y) \vee F_N(z)\} \leq F_N(x) \vee F_N(e) = F_N(x).$$

This shows that $X_{\mathbf{N}} \subseteq X_{\mathbf{N}} \odot X_{\mathbf{N}}$. Since $X_{\mathbf{N}} \supseteq X_{\mathbf{N}} \odot X_{\mathbf{N}}$, by Theorem **??**, we know that $X_{\mathbf{N}}$ is neutrosophic idempotent. $\square$

Definition 4.9. A neutrosophic $\mathcal{N}$-structure $X_{\mathbf{N}}$ over X is called an ε-neutrosophic $\mathcal{N}$-subsemigroup of X, if the following condition is valid:

$$(\forall x, y \in X) \begin{pmatrix} T_N(xy) \leq \bigvee\{T_N(x), T_N(y), \varepsilon_T\} \\ I_N(xy) \geq \bigwedge\{I_N(x), I_N(y), \varepsilon_I\} \\ F_N(xy) \leq \bigvee\{F_N(x), F_N(y), \varepsilon_F\} \end{pmatrix}, \tag{4.3}$$

where $\varepsilon_T, \varepsilon_I, \varepsilon_F \in [-1, 0]$ such that $-3 \leq \varepsilon_T + \varepsilon_I + \varepsilon_F \leq 0$.

Example 4.10. Let $X = \{e, a, b, c\}$ be a semigroup with the Cayley table which is given in Table **??**.
Let $X_{\mathbf{N}}$ be a neutrosophic $\mathcal{N}$-structure over X which is given as follows:

$$X_{\mathbf{N}} = \left\{ \frac{e}{(-0.4, -0.3, -0.25)}, \frac{a}{(-0.3, -0.5, -0.25)}, \frac{b}{(-0.2, -0.3, -0.2)}, \frac{c}{(-0.1, -0.7, -0.1)} \right\}.$$

Then $X_{\mathbf{N}}$ is an ε-neutrosophic $\mathcal{N}$-subsemigroup of X with $\varepsilon = (-0.4\ -0.2,\ -0.3)$.

Table 1. Cayley table for the binary operation "$\cdot$"

$\cdot$	e	a	b	c
e	e	e	e	e
a	e	a	e	a
b	e	e	b	b
c	e	a	b	c

Proposition 4.11. *Let $X_{\mathbf{N}}$ be an ε-neutrosophic $\mathcal{N}$-subsemigroup of X. If $X_{\mathbf{N}}(x) \leq (\varepsilon_T, \varepsilon_I, \varepsilon_F)$, that is, $T_N(x) \geq \varepsilon_T$, $I_N(x) \leq \varepsilon_I$ and $F_N(x) \geq \varepsilon_F$, for all $x \in X$, then $X_{\mathbf{N}}$ is a neutrosophic $\mathcal{N}$-subsemigroup of X.*

Proof. Straightforward. □

Theorem 4.12. *Let $X_{\mathbf{N}}$ be a neutrosophic $\mathcal{N}$-structure over X and let $\alpha, \beta, \gamma \in [-1, 0]$ be such that $-3 \leq \alpha+\beta+\gamma \leq 0$. If $X_{\mathbf{N}}$ is an ε-neutrosophic $\mathcal{N}$-subsemigroup of X, then the (α, β, γ)-level set of $X_{\mathbf{N}}$ is a subsemigroup of X whenever $(\alpha, \beta, \gamma) \leq (\varepsilon_T, \varepsilon_I, \varepsilon_F)$, that is, $\alpha \geq \varepsilon_T$, $\beta \leq \varepsilon_I$ and $\gamma \geq \varepsilon_F$.*

Proof. Assume that $X_{\mathbf{N}}(\alpha, \beta, \gamma) \neq \varnothing$ for $\alpha, \beta, \gamma \in [-1, 0]$ with $-3 \leq \alpha + \beta + \gamma \leq 0$. Let $x, y \in X_{\mathbf{N}}(\alpha, \beta, \gamma)$. Then $T_N(x) \leq \alpha$, $I_N(x) \geq \beta$, $F_N(x) \leq \gamma$, $T_N(y) \leq \alpha$, $I_N(y) \geq \beta$ and $F_N(y) \leq \gamma$. Thus it follows from (**??**) that

$$T_N(xy) \leq \bigvee\{T_N(x), T_N(y), \varepsilon_T\} \leq \bigvee\{\alpha, \varepsilon_T\} = \alpha,$$

$$I_N(xy) \geq \bigwedge\{I_N(x), I_N(y), \varepsilon_I\} \geq \bigwedge\{\beta, \varepsilon_I\} = \beta,$$

$$F_N(xy) \leq \bigvee\{F_N(x), F_N(y), \varepsilon_F\} \leq \bigvee\{\gamma, \varepsilon_F\} = \gamma.$$

So $xy \in X_{\mathbf{N}}(\alpha, \beta, \gamma)$. Hence $X_{\mathbf{N}}(\alpha, \beta, \gamma)$ is a subsemigroup of X. □

Theorem 4.13. *Let $X_{\mathbf{N}}$ be a neutrosophic $\mathcal{N}$-structure over X and let $\alpha, \beta, \gamma \in [-1, 0]$ be such that $-3 \leq \alpha + \beta + \gamma \leq 0$. If T_N^{α}, I_N^{β} and F_N^{γ} are subsemigroups of X for all $\varepsilon_T, \varepsilon_I, \varepsilon_F \in [-1, 0]$ with $-3 \leq \varepsilon_T + \varepsilon_I + \varepsilon_F \leq 0$ and $(\alpha, \beta, \gamma) \leq (\varepsilon_T, \varepsilon_I, \varepsilon_F)$, then $X_{\mathbf{N}}$ is an ε-neutrosophic $\mathcal{N}$-subsemigroup of X.*

Proof. Assume that there are $a, b \in X$ such that

$$T_N(ab) > \bigvee\{T_N(a), T_N(b), \varepsilon_T\}.$$

Then $T_N(ab) > t_\alpha \geq \bigvee\{T_N(a), T_N(b), \varepsilon_T\}$, for some $t_\alpha \in [-1, 0)$. It follows that $a, b \in T_N^{t_\alpha}$, $ab \notin T_N^{t_\alpha}$ and $t_\alpha \geq \varepsilon_T$. This is a contradiction, since $T_N^{t_\alpha}$ is a subsemigroup of X by hypothesis. Thus

$$T_N(xy) \leq \bigvee\{T_N(x), T_N(y), \varepsilon_T\},$$

for all $x, y \in X$. Suppose that $I_N(ab) < \bigwedge\{I_N(a), I_N(b), \varepsilon_I\}$, for some $a, b \in X$. If we take $t_\beta := \bigwedge\{I_N(a), I_N(b), \varepsilon_I\}$, then $a, b \in I_N^{t_\beta}$, $ab \notin I_N^{t_\beta}$ and $t_\beta \leq \varepsilon_I$. This is a contradiction. So

$$I_N(xy) \geq \bigwedge\{I_N(x), I_N(y), \varepsilon_I\},$$

for all $x, y \in X$. Now, suppose that there exist $a, b \in X$ and $t_\gamma \in [-1, 0)$ such that

$$F_N(ab) > t_\gamma \geq \bigvee\{F_N(a), F_N(b), \varepsilon_F\}.$$

Then $a, b \in F_N^{t_\gamma}$, $ab \notin F_N^{t_\gamma}$ and $t_\gamma \geq \varepsilon_F$, which is a contradiction. Thus

$$F_N(xy) \leq \bigvee\{F_N(x), F_N(y), \varepsilon_F\},$$

for all $x, y \in X$. Hence $X_{\mathbf{N}}$ is an ε-neutrosophic $\mathcal{N}$-subsemigroup of X. □

Theorem 4.14. *For any $\varepsilon_T, \varepsilon_I, \varepsilon_F, \delta_T, \delta_I, \delta_F \in [-1, 0]$ with $-3 \leq \varepsilon_T + \varepsilon_I + \varepsilon_F \leq 0$ and $-3 \leq \delta_T + \delta_I + \delta_F \leq 0$, if $X_{\mathbf{N}}$ and $X_{\mathbf{M}}$ are an ε-neutrosophic $\mathcal{N}$-subsemigroup and a δ-neutrosophic $\mathcal{N}$-subsemigroup, respectively, of X, then their intersection is a ξ-neutrosophic $\mathcal{N}$-subsemigroup of X for $\xi := \varepsilon \wedge \delta$, that is, $(\xi_T, \xi_I, \xi_F) = (\varepsilon_T \vee \delta_T, \varepsilon_I \wedge \delta_I, \varepsilon_F \vee \delta_F)$.*

Proof. For any $x, y \in X$, we have

$$\begin{aligned}
T_{N\cap M}(xy) &= \bigvee\left\{T_N(xy), T_M(xy)\right\} \\
&\leq \bigvee\left\{\bigvee\{T_N(x), T_N(y), \varepsilon_T\}, \bigvee\{T_M(x), T_M(y), \delta_T\}\right\} \\
&\leq \bigvee\left\{\bigvee\{T_N(x), T_N(y), \xi_T\}, \bigvee\{T_M(x), T_M(y), \xi_T\}\right\} \\
&= \bigvee\left\{\bigvee\{T_N(x), T_M(x), \xi_T\}, \bigvee\{T_N(y), T_M(y), \xi_T\}\right\} \\
&= \bigvee\left\{\bigvee\{T_N(x), T_M(x)\}, \bigvee\{T_N(y), T_M(y)\}, \xi_T\right\} \\
&= \bigvee\left\{T_{N\cap M}(x), T_{N\cap M}(y), \xi_T\right\},
\end{aligned}$$

$$\begin{aligned}
I_{N\cap M}(xy) &= \bigwedge\left\{I_N(xy), I_M(xy)\right\} \\
&\geq \bigwedge\left\{\bigwedge\{I_N(x), I_N(y), \varepsilon_I\}, \bigwedge\{I_M(x), I_M(y), \delta_I\}\right\} \\
&\geq \bigwedge\left\{\bigwedge\{I_N(x), I_N(y), \xi_I\}, \bigwedge\{I_M(x), I_M(y), \xi_I\}\right\} \\
&= \bigwedge\left\{\bigwedge\{I_N(x), I_M(x), \xi_I\}, \bigwedge\{I_N(y), I_M(y), \xi_I\}\right\} \\
&= \bigwedge\left\{\bigwedge\{I_N(x), I_M(x)\}, \bigwedge\{I_N(y), I_M(y)\}, \xi_I\right\} \\
&= \bigwedge\left\{I_{N\cap M}(x), I_{N\cap M}(y), \xi_I\right\},
\end{aligned}$$

and

$$\begin{aligned}
F_{N\cap M}(xy) &= \bigvee\{F_N(xy), F_M(xy)\}\\
&\leq \bigvee\left\{\bigvee\{F_N(x), F_N(y), \varepsilon_F\}, \bigvee\{F_M(x), F_M(y), \delta_F\}\right\}\\
&\leq \bigvee\left\{\bigvee\{F_N(x), F_N(y), \xi_F\}, \bigvee\{F_M(x), F_M(y), \xi_F\}\right\}\\
&= \bigvee\left\{\bigvee\{F_N(x), F_M(x), \xi_F\}, \bigvee\{F_N(y), F_M(y), \xi_F\}\right\}\\
&= \bigvee\left\{\bigvee\{F_N(x), F_M(x)\}, \bigvee\{F_N(y), F_M(y)\}, \xi_F\right\}\\
&= \bigvee\{F_{N\cap M}(x), F_{N\cap M}(y), \xi_F\}.
\end{aligned}$$

Then $X_{N\cap M}$ is a ξ-neutrosophic $\mathcal{N}$-subsemigroup of X. □

Theorem 4.15. *Let $X_{\mathbf{N}}$ be an ε-neutrosophic $\mathcal{N}$-subsemigroup of X. If*

$$\kappa := (\kappa_T, \kappa_I, \kappa_F) = \left(\bigvee_{x\in X}\{T_N(x)\}, \bigwedge_{x\in X}\{I_N(x)\}, \bigvee_{x\in X}\{F_N(x)\}\right),$$

then the set

$$\Omega := \{x \in X \mid T_N(x) \leq \kappa_T \vee \varepsilon_T,\ I_N(x) \geq \kappa_I \wedge \varepsilon_I,\ F_N(x) \leq \kappa_F \vee \varepsilon_F\}$$

is a subsemigroup of X.

Proof. Let $x, y \in \Omega$ for any $x, y \in X$. Then

$$T_N(x) \leq \kappa_T \vee \varepsilon_T = \bigvee_{x\in X}\{T_N(x)\} \vee \varepsilon_T,$$

$$I_N(x) \geq \kappa_I \wedge \varepsilon_I = \bigwedge_{x\in X}\{I_N(x)\} \wedge \varepsilon_I,$$

$$F_N(x) \leq \kappa_F \vee \varepsilon_F = \bigvee_{x\in X}\{F_N(x)\} \vee \varepsilon_F,$$

$$T_N(y) \leq \kappa_T \vee \varepsilon_T = \bigvee_{y\in X}\{T_N(y)\} \vee \varepsilon_T,$$

$$I_N(y) \geq \kappa_I \wedge \varepsilon_I = \bigwedge_{y\in X}\{I_N(y)\} \wedge \varepsilon_I,$$

$$F_N(y) \leq \kappa_F \vee \varepsilon_F = \bigvee_{y\in X}\{F_N(y)\} \vee \varepsilon_F.$$

Thus it follows from (**??**) that

$$\begin{aligned}
T_N(xy) &\leq \bigvee\{T_N(x), T_N(y), \varepsilon_T\}\\
&\leq \bigvee\{\kappa_T \vee \varepsilon_T, \kappa_T \vee \varepsilon_T, \varepsilon_T\}\\
&= \kappa_T \vee \varepsilon_T,
\end{aligned}$$

$$\begin{aligned} I_N(xy) &\geq \bigwedge\{I_N(x), I_N(y), \varepsilon_I\} \\ &\geq \bigwedge\{\kappa_I \wedge \varepsilon_I, \kappa_I \wedge \varepsilon_I, \varepsilon_I\} \\ &= \kappa_I \wedge \varepsilon_I \end{aligned}$$

and

$$\begin{aligned} F_N(xy) &\leq \bigvee\{F_N(x), F_N(y), \varepsilon_F\} \\ &\leq \bigvee\{\kappa_F \vee \varepsilon_F, \kappa_F \vee \varepsilon_F, \varepsilon_F\} \\ &= \kappa_F \vee \varepsilon_F. \end{aligned}$$

So $xy \in \Omega$. Hence Ω is a subsemigroup of X. □

For a map $f : X \to Y$ of semigroups and a neutrosophic $\mathcal{N}$-structure $X_{\mathbf{N}} := \frac{Y}{(T_N, I_N, F_N)}$ over Y and $\varepsilon = (\varepsilon_T, \varepsilon_I, \varepsilon_F)$ with $-3 \leq \varepsilon_T + \varepsilon_I + \varepsilon_F \leq 0$, define a neutrosophic $\mathcal{N}$-structure $X_{\mathbf{N}}^{\varepsilon} := \frac{X}{(T_N^{\varepsilon}, I_N^{\varepsilon}, F_N^{\varepsilon})}$ over X by:

$$T_N^{\varepsilon} : X \to [-1, 0],\ x \mapsto \bigvee \{T_N(f(x)), \varepsilon_T\},$$

$$F_N^{\varepsilon} : X \to [-1, 0],\ x \mapsto \bigwedge \{I_N(f(x)), \varepsilon_I\},$$

$$F_N^{\varepsilon} : X \to [-1, 0],\ x \mapsto \bigvee \{F_N(f(x)), \varepsilon_F\}.$$

Theorem 4.16. *Let $f : X \to Y$ be a homomorphism of semigroups. If a neutrosophic $\mathcal{N}$-structure $X_{\mathbf{N}} := \frac{Y}{(T_N, I_N, F_N)}$ over Y is an ε-neutrosophic $\mathcal{N}$-subsemigroup of Y, then $X_{\mathbf{N}}^{\varepsilon} := \frac{X}{(T_N^{\varepsilon}, I_N^{\varepsilon}, F_N^{\varepsilon})}$ is an ε-neutrosophic $\mathcal{N}$-subsemigroup of X.*

Proof. For any $x, y \in X$, we have

$$\begin{aligned} T_N^{\varepsilon}(xy) &= \bigvee \{T_N(f(xy)), \varepsilon_T\} \\ &= \bigvee \{T_N(f(x)f(y)), \varepsilon_T\} \\ &\leq \bigvee \left\{\bigvee\{T_N(f(x)), T_N(f(y)), \varepsilon_T\}, \varepsilon_T\right\} \\ &= \bigvee \left\{\bigvee\{T_N(f(x)), \varepsilon_T\}, \bigvee\{T_N(f(y)), \varepsilon_T\}, \varepsilon_T\right\} \\ &= \bigvee \{T_N^{\varepsilon}(x), T_N^{\varepsilon}(y), \varepsilon_T\}, \end{aligned}$$

$$\begin{aligned} I_N^{\varepsilon}(xy) &= \bigwedge \{I_N(f(xy)), \varepsilon_I\} \\ &= \bigwedge \{I_N(f(x)f(y)), \varepsilon_I\} \\ &\geq \bigwedge \left\{\bigwedge\{I_N(f(x)), I_N(f(y)), \varepsilon_I\}, \varepsilon_I\right\} \\ &= \bigwedge \left\{\bigwedge\{I_N(f(x)), \varepsilon_I\}, \bigwedge\{I_N(f(y)), \varepsilon_I\}, \varepsilon_I\right\} \\ &= \bigwedge \{I_N^{\varepsilon}(x), I_N^{\varepsilon}(y), \varepsilon_I\}, \end{aligned}$$

and

$$\begin{aligned} F_N^{\varepsilon}(xy) &= \bigvee \{F_N(f(xy)), \varepsilon_F\} \\ &= \bigvee \{F_N(f(x)f(y)), \varepsilon_F\} \\ &\leq \bigvee \left\{ \bigvee\{F_N(f(x)), F_N(f(y)), \varepsilon_F\}, \varepsilon_F \right\} \\ &= \bigvee \left\{ \bigvee\{F_N(f(x)), \varepsilon_F\}, \bigvee\{F_N(f(y)), \varepsilon_F\}, \varepsilon_F \right\} \\ &= \bigvee \{F_N^{\varepsilon}(x), F_N^{\varepsilon}(y), \varepsilon_F\}. \end{aligned}$$

Then $X_{\mathbf{N}}^{\varepsilon} := \frac{X}{(T_N^{\varepsilon}, I_N^{\varepsilon}, F_N^{\varepsilon})}$ is an ε-neutrosophic $\mathcal{N}$-subsemigroup of X. □

Let $f : X \to Y$ be a function of sets. If $Y_{\mathbf{M}} := \frac{Y}{(T_M, I_M, F_M)}$ is a neutrosophic $\mathcal{N}$-structures over Y, then the preimage of $Y_{\mathbf{M}}$ under f is defined to be a neutrosophic $\mathcal{N}$-structures

$$f^{-1}(Y_{\mathbf{M}}) = \frac{X}{(f^{-1}(T_M),\ f^{-1}(I_M),\ f^{-1}(F_M))}$$

over X, where $f^{-1}(T_M)(x) = T_M(f(x))$, $f^{-1}(I_M)(x) = I_M(f(x))$ and $f^{-1}(F_M)(x) = F_M(f(x))$ for all $x \in X$.

Theorem 4.17. *Let $f : X \to Y$ be a homomorphism of semigroups. If $Y_{\mathbf{M}} := \frac{Y}{(T_M, I_M, F_M)}$ is a neutrosophic $\mathcal{N}$-subsemigroup of Y, then the preimage of $Y_{\mathbf{M}}$ under f is a neutrosophic $\mathcal{N}$-subsemigroup of X.*

Proof. Let

$$f^{-1}(Y_{\mathbf{M}}) = \frac{X}{(f^{-1}(T_M),\ f^{-1}(I_M),\ f^{-1}(F_M))}$$

be the preimage of $Y_{\mathbf{M}}$ under f. For any $x, y \in X$, we have

$$\begin{aligned} f^{-1}(T_M)(xy) = T_M(f(xy)) &= T_M(f(x)f(y)) \\ &\leq \bigvee \{T_M(f(x)), T_M(f(y))\} \\ &= \bigvee \{f^{-1}(T_M)(x), f^{-1}(T_M)(y)\}, \end{aligned}$$

$$\begin{aligned} f^{-1}(I_M)(xy) = I_M(f(xy)) &= I_M(f(x)f(y)) \\ &\geq \bigwedge \{I_M(f(x)), I_M(f(y))\} \\ &= \bigwedge \{f^{-1}(I_M)(x), f^{-1}(I_M)(y)\} \end{aligned}$$

and

$$\begin{aligned} f^{-1}(F_M)(xy) = F_M(f(xy)) &= F_M(f(x)f(y)) \\ &\leq \bigvee \{F_M(f(x)), F_M(f(y))\} \\ &= \bigvee \{f^{-1}(F_M)(x), f^{-1}(F_M)(y)\}. \end{aligned}$$

Then $f^{-1}(Y_{\mathbf{M}})$ is a neutrosophic $\mathcal{N}$-subsemigroup of X. □

Let $f : X \to Y$ be an onto function of sets. If $X_{\mathbf{N}} := \frac{X}{(T_N, I_N, F_N)}$ is a neutrosophic $\mathcal{N}$-structures over X, then the image of $X_{\mathbf{N}}$ under f is defined to be a neutrosophic $\mathcal{N}$-structures

$$f\left(X_{\mathbf{N}}\right) = \frac{Y}{\left(f(T_N),\ f(I_N),\ f(F_N)\right)}$$

over Y, where

$$f(T_N)(y) = \bigwedge_{x \in f^{-1}(y)} T_N(x),$$

$$f(I_N)(y) = \bigvee_{x \in f^{-1}(y)} I_N(x),$$

$$f(F_N)(y) = \bigwedge_{x \in f^{-1}(y)} F_N(x).$$

Theorem 4.18. *For an onto homomorphism $f : X \to Y$ of semigroups, let $X_{\mathbf{N}} := \frac{X}{(T_N, I_N, F_N)}$ be a neutrosophic $\mathcal{N}$-structure over X such that*

$$(\forall T \subseteq X)\,(\exists x_0 \in T) \begin{pmatrix} T_N(x_0) = \bigwedge\limits_{z \in T} T_N(z) \\ I_N(x_0) = \bigvee\limits_{z \in T} I_N(z) \\ F_N(x_0) = \bigwedge\limits_{z \in T} F_N(z) \end{pmatrix}. \tag{4.4}$$

If $X_{\mathbf{N}}$ is a neutrosophic $\mathcal{N}$-subsemigroup of X, then the image of $X_{\mathbf{N}}$ under f is a neutrosophic $\mathcal{N}$-subsemigroup of Y.

Proof. Let

$$f\left(X_{\mathbf{N}}\right) = \frac{Y}{\left(f(T_N),\ f(I_N),\ f(F_N)\right)}$$

be the image of $X_{\mathbf{N}}$ under f. Let $a, b \in Y$. Then $f^{-1}(a) \neq \varnothing$ and $f^{-1}(a) \neq \varnothing$ in X, which imply from (**??**) that there are $x_a \in f^{-1}(a)$ and $x_b \in f^{-1}(b)$ such that

$$T_N(x_a) = \bigwedge_{z \in f^{-1}(a)} T_N(z), = \bigvee_{z \in f^{-1}(a)} I_N(z),\ F_N(x_a) = \bigwedge_{z \in f^{-1}(a)} F_N(z),$$

$$T_N(x_b) = \bigwedge_{w \in f^{-1}(b)} T_N(w), = \bigvee_{w \in f^{-1}(b)} I_N(w),\ F_N(x_b) = \bigwedge_{w \in f^{-1}(b)} F_N(w).$$

Thus

$$\begin{aligned} f(T_N)(ab) &= \bigwedge_{x \in f^{-1}(ab)} T_N(x) \leq T_N(x_a x_b) \\ &\leq \bigvee\{T_N(x_a), T_N(x_b)\} \\ &= \bigvee \left\{ \bigwedge_{z \in f^{-1}(a)} T_N(z),\ \bigwedge_{w \in f^{-1}(b)} T_N(w) \right\} \\ &= \bigvee \left\{ f(T_N)(a), f(T_N)(b) \right\}, \end{aligned}$$

$$\begin{aligned} f(I_N)(ab) &= \bigvee_{x\in f^{-1}(ab)} I_N(x) \geq I_N(x_a x_b) \\ &\geq \bigwedge\{I_N(x_a), I_N(x_b)\} \\ &= \bigwedge\left\{\bigvee_{z\in f^{-1}(a)} I_N(z), \bigvee_{w\in f^{-1}(b)} I_N(w)\right\} \\ &= \bigwedge\left\{f(I_N)(a), f(I_N)(b)\right\}, \end{aligned}$$

and

$$\begin{aligned} f(F_N)(ab) &= \bigwedge_{x\in f^{-1}(ab)} F_N(x) \leq F_N(x_a x_b) \\ &\leq \bigvee\{F_N(x_a), F_N(x_b)\} \\ &= \bigvee\left\{\bigwedge_{z\in f^{-1}(a)} F_N(z), \bigwedge_{w\in f^{-1}(b)} F_N(w)\right\} \\ &= \bigvee\left\{f(F_N)(a), f(F_N)(b)\right\}. \end{aligned}$$

So $f\left(X_{\mathbf{N}}\right)$ is a neutrosophic $\mathcal{N}$-subsemigroup of Y. □

Conclusions

In order to deal with the negative meaning of information, Jun et al. [**?**] have introduced a new function which is called negative-valued function, and constructed $\mathcal{N}$-structures. The concept of neutrosophic set (NS) has been developed by Smarandache in [**?**] and [**?**] as a more general platform which extends the concepts of the classic set and fuzzy set, intuitionistic fuzzy set and interval valued intuitionistic fuzzy set. In this article, we have introduced the notion of neutrosophic $\mathcal{N}$-structure and applied it to semigroup. We have introduced the notion of neutrosophic $\mathcal{N}$-subsemigroup and investigated several properties. We have provided conditions for neutrosophic $\mathcal{N}$-structure to be neutrosophic $\mathcal{N}$-subsemigroup. We have defined neutrosophic $\mathcal{N}$-product, and gave characterization of neutrosophic $\mathcal{N}$-subsemigroup by using neutrosophic $\mathcal{N}$-product. We also have introduced ε-neutrosophic subsemigroup, and investigated relations between neutrosophic subsemigroup and ε-neutrosophic subsemigroup. We have shown that the homomorphic preimage of neutrosophic $\mathcal{N}$-subsemigroup is a neutrosophic $\mathcal{N}$-subsemigroup, and the onto homomorphic image of neutrosophic $\mathcal{N}$-subsemigroup is a neutrosophic $\mathcal{N}$-subsemigroup.

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