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ORIGINAL RESEARCH PAPER

Neutrosophic normed spaces and statistical convergence

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Abstract
We define the neutrosophic normed space and the statistical convergence in neutrosophic normed space. We give the statistically Cauchy sequence in neutrosophic normed space and present the statistically completeness in connection with a neutrosophic normed space.

Keywords Neutrosophic normed spaced · t-Norm · t-Conorm · Statistical convergence · Statistical Cauchy · Statistically completeness

Mathematics Subject Classification 46S40 · 11B39 · 03E72 · 40G15

1 Introduction

Fuzzy Sets (FSs) put forward by Zadeh [28] has influenced deeply all the scientific fields since the publication of the paper. It is seen that this concept, which is very important for real-life situations, had not enough solution to some problems in time. New quests for such problems have been coming up. Atanassov [1] initiated Intuitionistic fuzzy sets (IFSs) for such cases. Neutrosophic set (NS) is a new version of the idea of the classical set which is defined by Smarandache [24]. The first world publication related to the concept of neutrosophy was published in 1998 and included in the literature [22]. Examples of other generalizations are FS [28] interval-valued FS [26], IFS [1], interval-valued IFS [2], the sets paraconsistent, dialetheist, paradoxist, and tautological [23], Pythagorean fuzzy sets [27].

Using the concepts Probabilistic metric space and fuzzy, fuzzy metric space (FMS) is introduced in [14]. Kaleva and Seikkala [9] have defined the FMS as a

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distance between two points to be a non-negative fuzzy number. In [7] some basic properties of FMS studied and the Baire Category Theorem for FMS proved. Further, some properties such as separability, countability are given and Uniform Limit Theorem is proved in [8]. Afterward, FMS has used in the applied sciences such as fixed point theory, image and signal processing, medical imaging, decision-making et al. After defined of the intuitionistic fuzzy set (IFS), it was used in all areas where FS theory was studied. Park [21] defined IF metric space (IFMS), which is a generalization of FMSs. Park used George and Veeramani's [7] thought of applying t-norm and t-conorm to FMS meanwhile defining IFMS and studying its basic features.

Bera and Mahapatra defined the neutrosophic soft linear spaces (NSLSs) [3]. Later, neutrosophic soft normed linear spaces(NSNLS) has been defined by Bera and Mahapatra [4]. In [4], neutrosophic norm, Cauchy sequence in NSNLS, convexity of NSNLS, metric in NSNLS were studied. In future studies on this subject, it is also possible to work with the idea of "Probabilistic metric space" using neutrosophic probability [25].

In this paper was organized as follows:

1. Introduction
2. Preliminaries
3. Method
4. Neutrosophic Normed Spaces
5. Statistical convergence on NNS
6. Statistical complete NNS
7. Conclusion

2 Preliminaries

Choose $\mathcal{A} \subset Z^+$. Randomly select an integer in $J = [1, n]$. Likely, if we calculate the ratio of the number of elements of set $\mathcal{A}$ in the interval J and the number of all elements in the interval J, it will be seen that this ratio belongs to set $\mathcal{A}$. If this possibility consists (for $n \to \infty$), then we can say that this limit is used to as the asymptotic density(AD) of $\mathcal{A}$. This refers us that the AD is a type of possibility of selection a number from the set $\mathcal{A}$. For $\mathcal{A}, \mathcal{B} \subset Z^+$, if $\mathcal{A} \Delta \mathcal{B}$ is finite, then the set $\mathcal{A}$ is as asymptotically equal to the set $\mathcal{B}$ ($\mathcal{A} \sim \mathcal{B}$). The notions of a lower AD and of convergence in density was given by Freedman in [6] as follows:

If the following conditions are hold, then for every values of J, the function g is called lower AD such that g is a function that characterize for all sets of N:

i. If $\mathcal{A} \sim \mathcal{B}$, then $g(\mathcal{A}) = g(\mathcal{B})$,
ii. If $\mathcal{A} \cap \mathcal{B} = \emptyset$, then $g(\mathcal{A}) + g(\mathcal{B}) \leq g(\mathcal{A} \cup \mathcal{B})$,
iii. $g(\mathcal{A}) + g(\mathcal{B}) \leq 1 + g(\mathcal{A} \cap \mathcal{B})$, for all $\mathcal{A}$,
iv. $g(Z^+) = 1$.

According to the definition of lower AD, we can give the definition upper AD as follows:

Let g be any density and A be any set of N. The function $\overline{g}$ is called upper AD, if $\overline{g}(\mathcal{A}) = 1 - g(Z^{+} \backslash \mathcal{A})$.

If $g(\mathcal{A}) = \overline{g}(\mathcal{A})$, then $\mathcal{A}$ has natural density(ND) according to g, for $\mathcal{A} \subset Z^{+}$. The term AD is often used for the function

$$d(\mathcal{A}) = \liminf_{k\to\infty} \frac{\mathcal{A}(k)}{k},$$

where $\mathcal{A} \subset N$ and $\mathcal{A}(k) = \sum_{x \leq k, x\in\mathcal{A}} 1$. Also the ND of $\mathcal{A}$ is given by $d(\mathcal{A}) = \lim_k k^{-1}|\mathcal{A}(k)|$ where $|\mathcal{A}(k)|$ denotes the number of elements in $\mathcal{A}(k)$.

A real numbers sequence (a_k) is statistically convergent(SC) to $\mathcal{L}$ if for every $\varepsilon > 0$ the set $\{k \in N : |a_k - \mathcal{L}| \geq \varepsilon\}$ has ND zero and denoted by S. In this case, we write $S - \lim a_k = L$ or $a_k \to \mathcal{L}(S)$.

The sequence (a_k) is statistically Cauchy sequence(SCa) if for every $\varepsilon > 0$ there is a positive integer $N = N(\varepsilon)$ such that

$$d\big(\{k \in N : |a_k - a_{N(\varepsilon)}| \geq \varepsilon\}\big) = 0.$$

Triangular norms (t-norms) (TN) were initiated by Menger [16]. In the problem of computing the distance between two elements in space, Menger offered using probability distributions instead of using numbers for distance. TNs are used to generalize with the probability distribution of triangle inequality in metric space conditions. Triangular conorms (t-conorms) (TC) know as dual operations of TNs. TNs and TCs are very significant for fuzzy operations(intersections and unions).

Definition 1 Give an operation $\circ : [0,1] \times [0,1] \to [0,1]$. If the operation $\circ$ is satisfying the following conditions, then it is called that the operation $\circ$ is *continuous TN*: For $s, t, u, v \in [0,1]$,

i. $s \circ 1 = s$
ii. If $s \leq u$ and $t \leq v$, then $s \circ t \leq u \circ v$,
iii. $\circ$ is continuous,
iv. $\circ$ is commutative and associative.

Definition 2 Give an operation $\bullet : [0,1] \times [0,1] \to [0,1]$. If the operation $\bullet$ is satisfying the following conditions, then it is called that the operation $\bullet$ is *continuous TC*:

i. $s \bullet 0 = s$,
ii. If $s \leq u$ and $t \leq v$, then $s \bullet t \leq u \bullet v$,
iii. $\bullet$ is continuous,
iv. $\bullet$ is commutative and associative.

Form above definitions, we note that if we choose $0 < \varepsilon_1, \varepsilon_2 < 1$ for $\varepsilon_1 > \varepsilon_2$, then there exist $0 < \varepsilon_3, \varepsilon_4 < 0, 1$ such that $\varepsilon_1 \circ \varepsilon_3 \geq \varepsilon_2, \quad \varepsilon_1 \geq \varepsilon_4 \bullet \varepsilon_2$. Further, if we choose $\varepsilon_5 \in (0,1)$, then there exist $\varepsilon_6, \varepsilon_7 \in (0,1)$ such that $\varepsilon_6 \circ \varepsilon_6 \geq \varepsilon_5$ and $\varepsilon_7 \bullet \varepsilon_7 \leq \varepsilon_5$.

Definition 3 [13] Take F be an arbitrary set, $N = \{<u, G(u), B(u), Y(u)> : u \in F\}$ be a NS such that $N : F \times F \times R^+ \to [0,1]$. Let $\circ$ and $\bullet$ show the continuous TN and continuous TC, respectively. If the following conditions are hold, then the four-tuple $(F, N, \circ, \bullet)$ is called neutrosophic metric space(NMS):

i. $0 \leq G(u,v,\lambda) \leq 1, \quad 0 \leq B(u,v,\lambda) \leq 1, \quad 0 \leq Y(u,v,\lambda) \leq 1 \quad \forall \lambda \in R^+$,
ii. $G(u,v,\lambda) + B(u,v,\lambda) + Y(u,v,\lambda) \leq 3$, (for $\lambda \in R^+$),
iii. $G(u,v,\lambda) = 1 \quad$ (for $\lambda > 0$) iff $u = v$,
iv. $G(u,v,\lambda) = G(v,u,\lambda) \quad$ (for $\lambda > 0$),
v. $G(u,v,\lambda) \circ G(v,y,\mu) \leq G(u,y,\lambda+\mu) \quad (\forall \lambda, \mu > 0)$,
vi. $G(u,v,.) : [0,\infty) \to [0,1]$ is continuous,
vii. $lim_{\lambda \to \infty} G(u,v,\lambda) = 1 \quad (\forall \lambda > 0)$,
viii. $B(u,v,\lambda) = 0 \quad$ (for $\lambda > 0$) iff $u = v$,
ix. $B(u,v,\lambda) = B(v,u,\lambda) \quad$ (for $\lambda > 0$),
x. $B(u,v,\lambda) \bullet B(v,y,\mu) \geq B(u,y,\lambda+\mu) \quad (\forall \lambda, \mu > 0)$,
xi. $B(u,v,.) : [0,\infty) \to [0,1]$ is continuous,
xii. $lim_{\lambda \to \infty} B(u,v,\lambda) = 0 \quad (\forall \lambda > 0)$,
xiii. $Y(u,v,\lambda) = 0 \quad$ (for $\lambda > 0$) iff $u = v$,
xiv. $Y(u,v,\lambda) = Y(v,u,\lambda) \quad (\forall \lambda > 0)$,
xv. $Y(u,v,\lambda) \bullet Y(v,y,\mu) \geq Y(u,y,\lambda+\mu) \quad (\forall \lambda, \mu > 0)$,
xvi. $Y(u,v,.) : [0,\infty) \to [0,1]$ is continuous,
xvii. $lim_{\lambda \to \infty} Y(u,v,\lambda) = 0 \quad$ (for $\lambda > 0$),
xviii. If $\lambda \leq 0$, then $G(u,v,\lambda) = 0$, $B(u,v,\lambda) = 1$ and $Y(u,v,\lambda) = 1$.

$\forall u, v, y \in F$. Then $\mathcal{N} = (G, B, Y)$ is called Neutrosophic metric(NM) on F.

3 Method

Density is an important concept in the Number Theory and has many variations. The reason for this variation is that some density definitions do not apply to all sequences. Asymptotic density (AD) is an example of this case. The emergence of new density definitions is meant to fill these and similar gaps.

The AD is one of the opportunities to evaluate how large a subset of N. It is intuitively known that Z^+ are much more than perfect squares. Simply put, each perfect square is positive and there are more positive numbers than perfect squares. The set of positive integers is not, actually, larger than the set of perfect squares. Because both sets are infinite and countable. Then, these are put in one-to-one correspondence. However, if one goes through N, the squares become increasingly infrequent. It is precisely in this instance, natural density (ND) helps us and makes this intuition precise.

The Theory of FS has submitted to employ imprecise, vagueness and inexact data [28]. FSs, have been widely implemented in different disciplines and technologies. The Theory of FS cannot always cope with the lack of knowledge of membership degrees. That's why Atanassov [1] introduced the theory of IFS which the extension of FSs.

The statistical convergence is a generalization of the notion of ordinary convergence. The concept of statistical convergence was defined in IF normed spaces by Karakus et al. [11]. After Karakus et al. [11] work, various statistical convergence definitions were used with IFNS [5, 10, 12, 15, 17–20].

In this paper, neutrosophic normed space (NNS) is defined and the definition statistical convergence with respect to NNS is given. The fundamental properties of NNS and statistical convergence with respect to NNS are investigated.

4 Neutrosophic normed spaces

Definition 4 Take F as a vector space, $\mathcal{N} = \{ <u, \mathcal{G}(u), \mathcal{B}(u), \mathcal{Y}(u)> \ : u \in F\}$ be a normed space(NS) such that $\mathcal{N} : F \times R^+ \to [0, 1]$. Let $\circ$ and $\bullet$ show the continuous TN and continuous TC, respectively. If the following conditions are hold, then the four-tuple $V = (F, \mathcal{N}, \circ, \bullet)$ is called NNS: For all $u, v \in F$ and $\lambda, \mu > 0$ and for each $\sigma \neq 0$,

i. $0 \leq \mathcal{G}(u,\lambda) \leq 1, \quad 0 \leq \mathcal{B}(u,\lambda) \leq 1, \quad 0 \leq \mathcal{Y}(u,\lambda) \leq 1 \quad \forall \lambda \in R^+$,
ii. $\mathcal{G}(u,\lambda) + \mathcal{B}(u,\lambda) + \mathcal{Y}(u,\lambda) \leq 3$, (for $\lambda \in R^+$),
iii. $\mathcal{G}(u,\lambda) = 1$ (for $\lambda > 0$) iff $u = 0$,
iv. $\mathcal{G}(\sigma u,\lambda) = \mathcal{G}(u, \frac{\lambda}{|\sigma|})$,
v. $\mathcal{G}(u,\mu) \circ \mathcal{G}(v,\lambda) \leq \mathcal{G}(u + v, \lambda + \mu)$,
vi. $\mathcal{G}(u, .)$ is continuous non-decreasing function
vii. $lim_{\lambda \to \infty} \mathcal{G}(u,\lambda) = 1$,
viii. $\mathcal{B}(u,\lambda) = 0$ (for $\lambda > 0$) iff $u = 0$,
ix. $\mathcal{B}(\sigma u,\lambda) = \mathcal{B}(u, \frac{\lambda}{|\sigma|})$,
x. $\mathcal{B}(u,\mu) \bullet \mathcal{B}(v,\lambda) \geq \mathcal{B}(u + v, \lambda + \mu)$,
xi. $\mathcal{B}(u, .)$ is continuous non-increasing function,
xii. $lim_{\lambda \to \infty} \mathcal{B}(u,\lambda) = 0$,
xiii. $\mathcal{Y}(u,\lambda) = 0$ (for $\lambda > 0$) iff $u = 0$,
xiv. $\mathcal{Y}(\sigma u,\lambda) = \mathcal{Y}(u, \frac{\lambda}{|\sigma|})$,
xv. $\mathcal{Y}(u,\mu) \bullet \mathcal{Y}(v,\lambda) \geq \mathcal{Y}(u + v, \lambda + \mu)$,
xvi. $\mathcal{Y}(u, .)$ is continuous non-increasing function
xvii. $lim_{\lambda \to \infty} \mathcal{Y}(u,\lambda) = 0$,
xviii. If $\lambda \leq 0$, then $\mathcal{G}(u,\lambda) = 0$, $\mathcal{B}(u,\lambda) = 1$ and $\mathcal{Y}(u,\lambda) = 1$.

Then $\mathcal{N} = (\mathcal{G}, \mathcal{B}, \mathcal{Y})$ is called Neutrosophic norm(NN).

Example 1 Let $(F, \|.\|)$ be a NS. Give the operations $\circ$ and $\bullet$ as TN $u \circ v = uv$; TC $u \bullet v = u + v - uv$. For $\lambda > \|u\|$,

$$\mathcal{G}(u,\lambda) = \frac{\lambda}{\lambda + \|u\|}, \quad \mathcal{B}(u,\lambda) = \frac{\|u\|}{\lambda + \|u\|} \quad \mathcal{Y}(u,\lambda) = \frac{\|u\|}{\lambda},$$

$\forall u, v \in F$ and $\lambda > 0$. If we take $\lambda \leq \|u\|$, then $\mathcal{G}(u,\lambda) = 0$, $\mathcal{B}(u,\lambda) = 1$ and $\mathcal{Y}(u,\lambda) = 1$. Then, $(F, \mathcal{N}, \circ, \bullet)$ is NNS such that $\mathcal{N} : F \times R^+ \to [0, 1]$.

Definition 5 Let V be a NNS, the sequence (a_n) in V, $0<\varepsilon<1$ and $\lambda > 0$. Then, the sequence (a_n) is converges to a if and only if there exists $N \in N$ such that $\mathcal{G}(a_n - a, \lambda) > 1 - \varepsilon$, $\mathcal{B}(a_n - a, \lambda) < \varepsilon$ and $\mathcal{Y}(a_n - a, \lambda) < \varepsilon$. That is $\lim_{n\to\infty} \mathcal{G}(a_n - a, \lambda) = 1$, $\lim_{n\to\infty} \mathcal{B}(a_n - a, \lambda) = 0$ and $\lim_{n\to\infty} \mathcal{Y}(a_n - a, \lambda) = 0$ as $\lambda > 0$. In that case, the sequence (a_n) is called a convergent sequence in V. The convergent in NNS is denoted by $\mathcal{N} - \lim a_n = \mathcal{L}$.

Theorem 1

i. *If (a_n) in V is convergent, then the limit point is unique.*
ii. *In V, if $\lim_{n\to\infty} a_n = a$ and $\lim_{n\to\infty} b_n = b$, then $\lim_{n\to\infty} a_n + b_n = a + b$.*
iii. *In V, if $\lim_{n\to\infty} a_n = a$ and $\alpha \neq 0$, then $\lim_{n\to\infty} \alpha a_n = \alpha a$.*

Since the theorem can be proved straightforward, we omitted it.

Definition 6 Let V be a NNS, the sequence (a_n) in V, $0<\varepsilon<1$ and $\lambda > 0$. Then, the sequence (a_n) is *Cauchy* in NNS V if there is a $N \in N$ such that $\mathcal{G}(a_n - a_m, \lambda) > 1 - \varepsilon$, $\mathcal{B}(a_n - a_m, \lambda) < \varepsilon$ and $\mathcal{Y}(a_n - a_m, \lambda) < \varepsilon$ for $n, m \geq N$. A NNS V is called *complete* iff every Cauchy sequence (a_n) is converges to a in NNS V.

Example 2 Consider the $\mathcal{G}, \mathcal{B}, \mathcal{Y}$ in Example 1. Then the V is a NNS. Further,

$$\lim_{m,n\to\infty} \frac{\lambda}{\lambda + \|a_n - a_m\|} = 1, \quad \lim_{m,n\to\infty} \frac{\|a_n - a_m\|}{\lambda + \|a_n - a_m\|} = 0, \quad \lim_{m,n\to\infty} \frac{\|a_n - a_m\|}{\lambda} = 0,$$

that is

$$\lim_{m,n\to\infty} \mathcal{G}(a_n - a_m, \lambda) = 1, \quad \lim_{m,n\to\infty} \mathcal{B}(a_n - a_m, \lambda) = 0, \quad \lim_{m,n\to\infty} \mathcal{Y}(a_n - a_m, \lambda) = 0.$$

Hence, we say that the sequence (a_n) is a Cauchy in NNS V.

Note that every convergent sequence in V is a Cauchy. But the inverse of this expression is not be true. For example, choose $U = \{\frac{1}{n} : n \in N\} \subset \mathbf{R}$ and $\|a\| = |a|$. From the $\mathcal{G}, \mathcal{B}, \mathcal{Y}$ and NN in Example 1, we can say that $(U, N, \circ, \bullet)$ is a NNS. The sequence (a_n) is also Cauchy, but

$$\lim_{n\to\infty} \mathcal{B}(a_n - a_m, \lambda) = \lim \frac{|\frac{1}{n} - \frac{1}{m}|}{\lambda + |\frac{1}{n} - \frac{1}{m}|} \neq 0. \tag{1}$$

That is the Cauchy sequence (a_n) is not convergent in NMS.

Theorem 2

i. *If for $u, v \in [0, 1]$, we take the continuous TN $u \circ v = \min\{u, v\}$ and the continuous TC $u \bullet v = \max\{u, v\}$, then every Cauchy sequence is bounded in NMS V.*
ii. *Let the sequences (a_n) and (b_n) be Cauchy and the sequence (α_n) is scalars in NMS V. Then, the sequences $(a_n + b_n)$ and $(\alpha_n a_n)$ are also Cauchy in NMS V.*

iii. *V is a complete NMS, if every Cauchy sequence has a convergent subsequence in NMS V.*

From the definitions of NMS, $\mathcal{G}, \mathcal{B}, \mathcal{Y}$, Cauchy sequence in V, completeness, it can be easily proved.

Example 3 let V be a NNS. If we take $G(u,v,\lambda) = \mathcal{G}(u-v,\lambda)$, $B(u,v,\lambda) = \mathcal{B}(u-v,\lambda)$ and $Y(u,v,\lambda) = \mathcal{Y}(u-v,\lambda)$, then $\mathcal{N} = (G,B,Y)$ is a NMS on F, which is induced by the NNM $\mathcal{N}$. Further, $\mathcal{G}(u-v,\lambda) = \mathcal{G}(v-u,\lambda)$, $\mathcal{B}(u-v,\lambda) = \mathcal{B}(v-u,\lambda)$ and $\mathcal{Y}(u-v,\lambda) = \mathcal{Y}(v-u,\lambda)$.

Definition 7 Let V be a NNS. For $\lambda > 0$, $u \in F$ and $0 < \varepsilon < 1$,

$$O(u,\varepsilon,\lambda) = \{v \in F : \mathcal{G}(u-v,\lambda) > 1-\varepsilon, \quad \mathcal{B}(u-v,\lambda) < \varepsilon, \quad \mathcal{Y}(u-v,\lambda) < \varepsilon\}$$

is called open ball (OB) with center u, radius ε.

A subset $A \subseteq F$ is called open if for each $u \in A$, there exist $\lambda > 0$, $0 < \varepsilon < 1$ such that $O(u,\varepsilon,\lambda) \subseteq A$.

If we take $\tau_{\mathcal{N}}$ as the family of all open subset of F, then $\tau_{\mathcal{N}} = \{foreach \quad u \in A, thereexist \quad \lambda > 0, 0 < \varepsilon < 1 suchthat \quad O(u,\varepsilon,\lambda) \subseteq A\}$ is called the topology induced by NN.

Definition 8 The set $A \subset F$ is called neutrosophic-bounded(NB) in NNS V, if there exist $\lambda > 0$, and $\varepsilon \in (0,1)$ such that $\mathcal{G}(u,\lambda) > 1-\varepsilon$, $\mathcal{B}(u,\lambda) < \varepsilon$ and $\mathcal{Y}(u,\lambda) < \varepsilon$ for each $u \in A$.

Theorem 3 *Every compact subset A of a NNS is NB.*

This theorem can be proved similar to Theorem 3.9 in [13].

If V is NNS induced by NN and $A \subset F$, then A is NB if and only if it is bounded. Then, using the Theorem 3, we have:

Corollary 1 *In a NNS V, every compact set is closed and NB.*

5 Statistical convergence on NNS

Definition 9 Take a NNS V. A sequence (a_k) is called statistical convergence with respect to NN (SC-NN), if there exist $\mathcal{L} \in F$ such that the set

$$K_\varepsilon := \left\{k \leq n : \mathcal{G}(a_k - \mathcal{L},\lambda) \leq 1-\varepsilon \; or \quad \mathcal{B}(a_k - \mathcal{L},\lambda) \geq \varepsilon, \quad \mathcal{Y}(a_k - \mathcal{L},\lambda) \geq \varepsilon\right\}$$

or equivalently

$$K_{\varepsilon} := \left\{ k \leq n : \mathcal{G}(a_k - \mathcal{L}, \lambda) > 1 - \varepsilon \ or \quad \mathcal{B}(a_k - \mathcal{L}, \lambda) < \varepsilon, \quad \mathcal{Y}(a_k - \mathcal{L}, \lambda) < \varepsilon \right\}.$$

has ND zero, for every $\varepsilon > 0$ and $\lambda > 0$. That is $d(K_{\varepsilon}) = 0$ or equivalently,

$$\lim_n \frac{1}{n} \left| \left\{ k \leq n : \mathcal{G}(a_k - \mathcal{L}, \lambda) \leq 1 - \varepsilon \ or \quad \mathcal{B}(a_k - \mathcal{L}, \lambda) \geq \varepsilon, \quad \mathcal{Y}(a_k - \mathcal{L}, \lambda) \geq \varepsilon \right\} \right| = 0.$$

Therefore, we write $S_{\mathcal{N}} - \lim a_k = \mathcal{L}$ or $a_k \to \mathcal{L}(S_{\mathcal{N}})$. The set of SC-NN will be denoted by $S_{\mathcal{N}}$. If $\mathcal{L} = 0$, then we will write $S_0\mathcal{N}$.

Example 4 Let $(F, \|.\|)$ be a NS. For all $u, v \in [0, 1]$, define the TN $u \circ v = uv$ and the TC $u \bullet v = \min\{u + v, 1\}$. We take $\mathcal{G}, \mathcal{B}, \mathcal{Y}$ in Example 1, for all $a \in F$, $\lambda > 0$. Then V is a NNS. Define,

$$a_k = \begin{cases} 1, & k = m^2 \quad (m \in N) \\ 0, & otherwise \end{cases}$$

Consider

$$K_n(\varepsilon, \lambda) = \{k \leq n : \mathcal{G}(a_k, \lambda) \leq 1 - \varepsilon \quad or \quad \mathcal{B}(a_k, \lambda) \geq \varepsilon, \quad \mathcal{Y}(a_k, \lambda) \geq \varepsilon\}$$

for every $\varepsilon \in (0, 1)$ and for any $\lambda > 0$. Then we have,

$$\begin{aligned} K_n(\varepsilon, \lambda) &= \left\{ k \leq n : \frac{\lambda}{\lambda + \|a_k\|} \leq 1 - \varepsilon \quad or \quad \frac{\|a_k\|}{\lambda + \|a_k\|} \geq \varepsilon, \quad \frac{\|a_k\|}{\lambda} \geq \varepsilon \right\} \\ &= \left\{ k \leq n : \|a_k\| \geq \frac{\lambda\varepsilon}{1 - \varepsilon}, \quad or \quad \|a_k\| \geq \lambda\varepsilon \right\} \\ &= \{k \leq n : a_k = 1\} = \{k \leq n : k = m^2 \quad and \quad m \in N\}. \end{aligned}$$

Then,

$$\frac{1}{n}|K_n(\varepsilon, \lambda)| = \frac{1}{n}\left|\{k \leq n : k = m^2 \quad and \quad m \in N\}\right| \leq \frac{\sqrt{n}}{n}$$

That is, when n becomes sufficiently large, the quantity $\mathcal{G}(a_k, \lambda)$ becomes less than $1 - \varepsilon$ and similarly the quantities $\mathcal{B}(a_k, \lambda)$ and $\mathcal{Y}(a_k, \lambda)$ become larger than ε. So, $\frac{1}{n}|K_n(\varepsilon, \lambda)| = 0$ for $\varepsilon > 0$ and $\lambda > 0$.

Lemma 1 may be easily obtained by using the definitions and properties of density dedicated in Sect. 2 and Definition 9.

Lemma 1 *Choose a NNS V. The following statements are equivalent, for every* $\varepsilon > 0$ *and* $\lambda > 0$:

- i. $S_{\mathcal{N}} - \lim a_k = \mathcal{L}$,

- ii. $\lim_n \frac{1}{n}\left|\left\{k \leq n : \mathcal{G}(a_k - \mathcal{L}, \lambda) \leq 1 - \varepsilon\right\}\right| = \lim_n \frac{1}{n}\left|\left\{\mathcal{B}(a_k - \mathcal{L}, \lambda) \geq \varepsilon\right\}\right|$
  $= \lim_n \frac{1}{n}\left|\left\{\mathcal{Y}(a_k - \mathcal{L}, \lambda) \geq \varepsilon\right\}\right| = 0$
- iii. $\lim_n \frac{1}{n}\left|\left\{k \leq n : \mathcal{G}(a_k - \mathcal{L}, \lambda) > 1 - \varepsilon \quad \textit{and}, \quad \mathcal{B}(a_k - \mathcal{L}, \lambda) < \varepsilon,\right.\right.$
  $\left.\left.\mathcal{Y}(a_k - \mathcal{L}, \lambda) < \varepsilon\right\}\right| = 1,$
- iv. $\lim_n \frac{1}{n}\left|\left\{k \leq n : \mathcal{G}(a_k - \mathcal{L}, \lambda) > 1 - \varepsilon\right\}\right| = \lim_n \frac{1}{n}\left|\left\{k \leq n : \mathcal{B}(a_k - \mathcal{L}, \lambda) < \varepsilon\right\}\right|$
  $= \lim_n \frac{1}{n}\left|\left\{k \leq n : \mathcal{Y}(a_k - \mathcal{L}, \lambda) < \varepsilon\right\}\right| = 1.$
- v. $S - \lim \mathcal{G}(a_k - \mathcal{L}, \lambda) = 1,$ *and* $S - \lim \mathcal{B}(a_k - \mathcal{L}, \lambda) = 0,$ $S - \lim \mathcal{Y}(a_k - \mathcal{L}, \lambda) = 0.$

Theorem 4 *Let V be a NNS. If (a_k) is SC-NN, then $S_{\mathcal{N}} - \lim a_k = \mathcal{L}$ is unique.*

Proof Suppose that $S_{\mathcal{N}} - \lim a_k = \mathcal{L}_1$ and $S_{\mathcal{N}} - \lim a_k = \mathcal{L}_2$ for $\mathcal{L}_1 \neq \mathcal{L}_2$. Choose $\varepsilon > 0$. Then, for a given $\mu > 0$, $(1 - \varepsilon) \circ (1 - \varepsilon) > 1 - \mu$ and $\varepsilon \bullet \varepsilon < \mu$. For any $\lambda > 0$, let's write the following sets:

$$K_{\mathcal{G}1}(\varepsilon, \lambda) := \left\{k \leq n : \mathcal{G}\left(a_k - \mathcal{L}_1, \frac{\lambda}{2}\right) \leq 1 - \varepsilon\right\},$$
$$K_{\mathcal{G}2}(\varepsilon, \lambda) := \left\{k \leq n : \mathcal{G}\left(a_k - \mathcal{L}_2, \frac{\lambda}{2}\right) \leq 1 - \varepsilon\right\}$$
$$K_{\mathcal{B}1}(\varepsilon, \lambda) := \left\{k \leq n : \mathcal{B}\left(a_k - \mathcal{L}_1, \frac{\lambda}{2}\right) \geq \varepsilon\right\}$$
$$K_{\mathcal{B}2}(\varepsilon, \lambda) := \left\{k \leq n : \mathcal{B}\left(a_k - \mathcal{L}_2, \frac{\lambda}{2}\right) \geq \varepsilon\right\}$$
$$K_{\mathcal{Y}1}(\varepsilon, \lambda) := \left\{k \leq n : \mathcal{Y}\left(a_k - \mathcal{L}_1, \frac{\lambda}{2}\right) \geq \varepsilon\right\}$$
$$K_{\mathcal{Y}2}(\varepsilon, \lambda) := \left\{k \leq n : \mathcal{Y}\left(a_k - \mathcal{L}_2, \frac{\lambda}{2}\right) \geq \varepsilon\right\}.$$

We know that $S_{\mathcal{N}} - \lim a_k = \mathcal{L}_1$. Then, using the Lemma 1, for all $\lambda > 0$,

$$d(K_{\mathcal{G}1}(\mu, \lambda)) = d(K_{\mathcal{B}1}(\mu, \lambda)) = d(K_{\mathcal{Y}1}(\mu, \lambda)) = 0.$$

Further, since we get $S_{\mathcal{N}} - \lim a_k = \mathcal{L}_2$, using the Lemma 1, for $\lambda > 0$,

$$d(K_{\mathcal{G}2}(\mu, \lambda)) = d(K_{\mathcal{B}2}(\mu, \lambda)) = d(K_{\mathcal{Y}2}(\mu, \lambda)) = 0.$$

Let

$$K_{\mathcal{N}}(\mu,\lambda) := \left\{K_{\mathcal{G}1}(\mu,\lambda)\cup K_{\mathcal{G}2}(\mu,\lambda)\right\}\cap\left\{K_{\mathcal{B}1}(\mu,\lambda)\cup K_{\mathcal{B}2}(\mu,\lambda)\right\}$$
$$\cap\left\{K_{\mathcal{Y}1}(\mu,\lambda)\cup K_{\mathcal{Y}2}(\mu,\lambda)\right\}.$$

Then observe that $d(K_{\mathcal{N}}(\mu,\lambda))=0$ which implies $d(N/K_{\mathcal{N}}(\mu,\lambda))=1$. Then, we have three possible situations, when take $k\in N/K_{\mathcal{N}}(\mu,\lambda)$:

i. $k\in N/\left(K_{\mathcal{G}1}(\mu,\lambda)\cup K_{\mathcal{G}2}(\mu,\lambda)\right)$,
ii. $k\in N/\left(K_{\mathcal{B}1}(\mu,\lambda)\cup K_{\mathcal{B}2}(\mu,\lambda)\right)$,
ii. $k\in N/\left(K_{\mathcal{Y}1}(\mu,\lambda)\cup K_{\mathcal{Y}2}(\mu,\lambda)\right)$.

Firstly, consider (i). Then we have

$$\mathcal{G}(\mathcal{L}_1-\mathcal{L}_2,\lambda)\geq\mathcal{G}\left(a_k-\mathcal{L}_1,\frac{\lambda}{2}\right)\circ\mathcal{G}\left(a_k-\mathcal{L}_2,\frac{\lambda}{2}\right)>(1-\varepsilon)\circ(1-\varepsilon).$$

and so since $(1-\varepsilon)\circ(1-\varepsilon)>1-\mu$,

$$\mathcal{G}(\mathcal{L}_1-\mathcal{L}_2,\lambda)>1-\mu. \tag{2}$$

Using the (2), for all $\lambda>0$, we obtain $\mathcal{G}(\mathcal{L}_1-\mathcal{L}_2,\lambda)=1$, where $\mu>0$ is arbitrary. That is, $\mathcal{L}_1=\mathcal{L}_2$ is obtained.

For the situation (ii.), if we take $k\in N/\left(K_{\mathcal{B}1}(\mu,\lambda)\cup K_{\mathcal{B}2}(\mu,\lambda)\right)$, then we can write

$$\mathcal{B}(\mathcal{L}_1-\mathcal{L}_2,\lambda)\leq\mathcal{B}\left(a_k-\mathcal{L}_1,\frac{\lambda}{2}\right)\bullet\mathcal{B}\left(a_k-\mathcal{L}_2,\frac{\lambda}{2}\right)<\varepsilon\bullet\varepsilon.$$

Using $\varepsilon\bullet\varepsilon<\mu$, we can see that $\mathcal{B}(\mathcal{L}_1-\mathcal{L}_2,\lambda)<\mu$. For all $\lambda>0$, we obtain $\mathcal{B}(\mathcal{L}_1-\mathcal{L}_2,\lambda)=0$, where $\mu>0$ is arbitrary. Thus $\mathcal{L}_1=\mathcal{L}_2$. Again, for the situation (iii.), if we take $k\in N/\left(K_{\mathcal{Y}1}(\mu,\lambda)\cup K_{\mathcal{Y}2}(\mu,\lambda)\right)$, then we can write

$$\mathcal{Y}(\mathcal{L}_1-\mathcal{L}_2,\lambda)\leq\mathcal{Y}\left(a_k-\mathcal{L}_1,\frac{\lambda}{2}\right)\bullet\mathcal{Y}\left(a_k-\mathcal{L}_2,\frac{\lambda}{2}\right)<\varepsilon\bullet\varepsilon.$$

Using $\varepsilon\bullet\varepsilon<\mu$, we can see that $\mathcal{Y}(\mathcal{L}_1-\mathcal{L}_2,\lambda)<\mu$. For all $\lambda>0$, we obtain $\mathcal{Y}(\mathcal{L}_1-\mathcal{L}_2,\lambda)=0$. Thus $\mathcal{L}_1=\mathcal{L}_2$. This step completes the proof. □

Theorem 5 *If $\mathcal{N}-\lim a_k=\mathcal{L}$ for NNS V, then $S_{\mathcal{N}}-\lim a_k=\mathcal{L}$.*

Proof If $\mathcal{N}-\lim a_k=\mathcal{L}$, then, for every $\varepsilon>0$ and $\lambda>0$, there exist a number $N\in N$ such that $\mathcal{G}(a_k-\mathcal{L},\lambda)>1-\varepsilon$, *and* $\mathcal{B}(a_k-\mathcal{L},\lambda)<\varepsilon$, $\mathcal{Y}(a_k-\mathcal{L},\lambda)<\varepsilon$, for all $k\geq N$. Therefore, the set

$$\{k\leq n:\mathcal{G}(a_k-\mathcal{L},\lambda)\leq 1-\varepsilon,\quad or\quad\mathcal{B}(a_k-\mathcal{L},\lambda)\geq\varepsilon,\quad\mathcal{Y}(a_k-\mathcal{L},\lambda)\geq\varepsilon\}$$

has at most finitely many terms. Hence, since every finite subset of N has density zero,

$$\lim_{n} \frac{1}{n} |\{k \leq n : \mathcal{G}(a_k - \mathcal{L}, \lambda) \leq 1 - \varepsilon, \quad or \quad \mathcal{B}(a_k - \mathcal{L}, \lambda) \geq \varepsilon,$$
$$\mathcal{Y}(a_k - \mathcal{L}, \lambda) \geq \varepsilon\}| = 0.$$

This completes the proof. □

Theorem 6 *Let V be an NNS. $S_{\mathcal{N}} - \lim a_k = \mathcal{L}$ iff there exists a increasing index sequence $J = \{j_1, j_2, \ldots\} \subseteq N$, while $d(J) = 1, \quad \mathcal{N} - \lim_{n\to\infty} a_{j_n} = \mathcal{L}$.*

Proof Suppose that $S_{\mathcal{GN}} - \lim a_k = \mathcal{L}$. For any $\lambda > 0$ and $\mu = 1, 2, \ldots$,

$$P_{\mathcal{N}}(\mu, \lambda) = \left\{k \leq N : \mathcal{G}(a_k - \mathcal{L}, \lambda) > 1 - \frac{1}{\mu} \quad and \quad \mathcal{B}(a_k - \mathcal{L}, \lambda) < \frac{1}{\mu},\right.$$
$$\left.\mathcal{Y}(a_k - \mathcal{L}, \lambda) < \frac{1}{\mu}\right\}$$

and

$$R_{\mathcal{N}}(\mu, \lambda) = \left\{k \leq N : \mathcal{G}a_k - \mathcal{L}, \lambda) \leq 1 - \frac{1}{\mu} \quad or\right.$$
$$\left.\mathcal{B}(a_k - \mathcal{L}, \lambda) \geq \frac{1}{\mu}, \quad \mathcal{Y}(a_k - \mathcal{L}, \lambda) \geq \frac{1}{\mu}\right\}.$$

Then, $d(R_{\mathcal{N}}(\mu, \lambda)) = 0$, since $S_{\mathcal{N}} - \lim a_k = \mathcal{L}$. Further, for $\lambda > 0$ and $\mu = 1, 2, \ldots$, $P_{\mathcal{N}}(\mu, \lambda) \supset P_{\mathcal{N}}(\mu + 1, \lambda)$ and so,

$$d(P_{\mathcal{GN}}(\mu, \lambda)) = 1. \tag{3}$$

Now, we will show that for $k \in P_{\mathcal{N}}(\mu, \lambda), \quad \mathcal{N} - \lim a_k = \mathcal{L}$. Assume that $\mathcal{N} - \lim a_k \neq \mathcal{L}$, for some $k \in P_{\mathcal{N}}(\mu, \lambda)$. Then, there is $\rho > 0$ and a positive integer N such that $\mathcal{G}(a_k - \mathcal{L}, \lambda) \leq 1 - \rho$ or $\mathcal{B}(a_k - \mathcal{L}, \lambda) \geq \rho$, $\mathcal{Y}(a_k - \mathcal{L}, \lambda) \geq \rho$, for all $k \geq N$. Let $\mathcal{G}(a_k - \mathcal{L}, \lambda) > 1 - \rho$ and $\mathcal{B}(a_k - \mathcal{L}, \lambda) < \rho$, $\mathcal{Y}(a_k - \mathcal{L}, \lambda) < \rho$ for all $k < N$. Hence

$$\lim_{n} \frac{1}{n} \left| \{k \leq N : \mathcal{G}(a_k - \mathcal{L}, \lambda) > 1 - \rho \quad and \right.$$
$$\left. \mathcal{B}(a_k - \mathcal{L}, \lambda) < \rho, \quad \mathcal{Y}(a_k - \mathcal{L}, \lambda) < \tau\} \right| = 0.$$

Since $\rho > 1/\mu$, we obtain $d(P_{\mathcal{N}}(\mu, \lambda)) = 0$, which contradicts (3). That's why, $\mathcal{N} - \lim a_k = \mathcal{L}$.

Assume that there exists a subset $J = \{j_1, j_2, \ldots\} \subseteq N$ such that $d(J) = 1$ and $\mathcal{N} - \lim_{n\to\infty} a_{j_n} = \mathcal{L}$, i.e. there exists $N \in N$ such that $\mathcal{G}(a_k - \mathcal{L}, \lambda) > 1 - \mu$ and $\mathcal{B}(a_k - \mathcal{L}, \lambda) < \mu, \quad \mathcal{Y}(a_k - \mathcal{L}, \lambda) < \mu$, for every $\mu > 0$ and $\lambda > 0$. In that case,

$$
\begin{aligned}
R_{\mathcal{N}}(\mu,\lambda) := \{k \leq N : \mathcal{G}(a_k - \mathcal{L},\lambda) \leq 1-\mu \quad or \quad & \mathcal{B}(a_k - \mathcal{L},\lambda) \\
\geq \mu, \quad \mathcal{Y}(a_k - \mathcal{L},\lambda) \geq \mu\} & \\
\subseteq N - \{j_{N+1}, j_{N+2,\ldots}\}. &
\end{aligned}
$$

Therefore $d(R_{\mathcal{N}}(\mu,\lambda)) \leq 1-1=0$. Hence $S_{\mathcal{N}} - \lim a_k = \mathcal{L}$. □

6 Statistical complete NNS

Definition 10 The sequence (a_k) is called statistically Cauchy with respect to NN $\mathcal{N}$ ($SCa - NN$) in NNS V, if there exists $N = N(\varepsilon)$, for every $\varepsilon > 0$ and $\lambda > 0$ such that

$$
KC_{\varepsilon} := \{k \leq n : \mathcal{G}(a_k - a_N,\lambda) \leq 1-\varepsilon \quad or \quad \mathcal{B}(a_k - a_N,\lambda) \geq \varepsilon, \quad \mathcal{Y}(a_k - a_N,\lambda) \geq \varepsilon\}
$$

has ND zero. That is, $d(KC_{\varepsilon}) = 0$.

Theorem 7 *If a sequence (a_k) is $SC - NN$ in NNS V, then it is $SCa - NN$.*

Proof Let (a_k) be SC-NN. We get $(1-\varepsilon) \circ (1-\varepsilon) > 1-\mu$ and $\varepsilon \bullet \varepsilon < \mu$, for a given $\varepsilon > 0$, choose $\mu > 0$. Then we have,

$$
\begin{aligned}
d(A(\varepsilon,\lambda)) = d\Bigg(\Bigg\{k \leq n : \mathcal{G}\left(a_k - \mathcal{L}, \frac{\lambda}{2}\right) \leq 1-\varepsilon \quad or \\
\mathcal{B}\left(a_k - \mathcal{L}, \frac{\lambda}{2}\right) \geq \varepsilon, \mathcal{Y}\left(a_k - \mathcal{L}, \frac{\lambda}{2}\right) \geq \varepsilon\Bigg\}\Bigg)
\end{aligned} \tag{4}
$$

and so

$$
\begin{aligned}
d\big(A^C(\varepsilon,\lambda)\big) = d\Bigg(\Bigg\{k \leq n : \mathcal{G}\left(a_k - \mathcal{L}, \frac{\lambda}{2}\right)) > 1-\varepsilon \quad and \\
\mathcal{B}(a_k - L), \frac{\lambda}{2}\bigg) < \varepsilon, \quad \mathcal{Y}(a_k - L), \frac{\lambda}{2}) < \varepsilon\Bigg\}) = 1,
\end{aligned}
$$

for $\lambda > 0$. Let $p \in A^C(\varepsilon,\lambda)$. Then,

$$
\mathcal{G}(a_p - \mathcal{L},\lambda) > 1-\varepsilon \quad and \quad \mathcal{B}(a_p - \mathcal{L},\lambda) < \varepsilon, \quad \mathcal{Y}(a_p - \mathcal{L},\lambda) < \varepsilon.
$$

Let

$$
\begin{aligned}
B(\varepsilon,\lambda) = \big\{k \leq n : \mathcal{G}(a_k - a_p,\lambda) \leq 1-\mu \quad or \\
\mathcal{B}(a_k - a_p,\lambda) \geq \mu, \quad \mathcal{Y}(a_k - a_p,\lambda) \geq \mu\big\}.
\end{aligned}
$$

We claim that $B(\varepsilon,\lambda) \subset A(\varepsilon,\lambda)$. Let $q \in B(\varepsilon,\lambda)/A(\varepsilon,\lambda)$. Then,

$$
\mathcal{G}(a_q - a_p,\lambda) \leq 1-\mu \quad and \quad \mathcal{G}\left(a_q - \mathcal{L}, \frac{\lambda}{2}\right) > 1-\mu,
$$

in particular $\mathcal{G}(a_p - \mathcal{L}, \frac{\lambda}{2}) > 1-\varepsilon$. Then,

$$1-\mu \geq \mathcal{G}(a_q - a_p, \lambda) \geq \mathcal{G}\left(a_q - \mathcal{L}, \frac{\lambda}{2}\right) \circ \mathcal{G}\left(a_p - \mathcal{L}, \frac{\lambda}{2}\right) > (1-\varepsilon)\circ(1-\varepsilon) > 1-\mu,$$

which is not possible. Moreover,

$$\mathcal{B}(a_q - a_p, \lambda) \geq \mu \quad and \quad \mathcal{B}\left(a_q - \mathcal{L}, \frac{\lambda}{2}\right) < \mu,$$

in particular $\mathcal{B}(a_p - \mathcal{L}, \frac{\lambda}{2}) < \varepsilon$. Then,

$$\mu \leq \mathcal{B}(a_q - a_p, \lambda) \leq \mathcal{B}\left(a_q - \mathcal{L}, \frac{\lambda}{2}\right) \circ \mathcal{B}\left(a_p - \mathcal{L}, \frac{\lambda}{2}\right) < \varepsilon \bullet \varepsilon < \mu,$$

which is not possible. Similarly,

$$\mathcal{Y}(a_q - a_p, \lambda) \geq \mu \quad and \quad \mathcal{Y}\left(a_q - \mathcal{L}, \frac{\lambda}{2}\right) < \mu,$$

in particular $\mathcal{Y}(a_p - \mathcal{L}, \frac{\lambda}{2}) < \varepsilon$. Then,

$$\mu \leq \mathcal{Y}(a_q - a_p, \lambda) \leq \mathcal{Y}\left(a_q - \mathcal{L}, \frac{\lambda}{2}\right) \circ \mathcal{Y}\left(a_p - \mathcal{L}, \frac{\lambda}{2}\right) < \varepsilon \bullet \varepsilon < \mu,$$

which is not possible. In that case, $B(\varepsilon, \lambda) \subset A(\varepsilon, \lambda)$. Then, by (4), $d(\varepsilon, \lambda) = 0$ and (a_k) is SCa-NN. □

Definition 11 The NNS V is called statistically $(SC - NN)$ complete, if every $SCa - NN$ is $SC - NN$.

Theorem 8 *Every NNS V is $(SC - NN)-$complete.*

Proof Let (a_k) be $SCa - NN$ but not $SC - NN$. Choose $\mu > 0$. We get $(1-\varepsilon)\circ(1-\varepsilon) > 1-\mu$ and $\varepsilon \bullet \varepsilon < \mu$, for a given $\varepsilon > 0$ and $\lambda > 0$,. Since (a_k) is not $SC - NN$,

$$\mathcal{G}(a_k - a_N, \lambda) \geq \mathcal{G}\left(a_k - \mathcal{L}, \frac{\lambda}{2}\right) \circ \mathcal{G}\left(a_N - \mathcal{L}, \frac{\lambda}{2}\right) > (1-\varepsilon)\circ(1-\varepsilon) > 1-\mu,$$
$$\mathcal{B}(a_k - a_N, \lambda) \leq \mathcal{B}\left(a_k - \mathcal{L}, \frac{\lambda}{2}\right) \bullet \mathcal{B}\left(a_N - \mathcal{L}, \frac{\lambda}{2}\right) < \varepsilon \bullet \varepsilon < \mu,$$
$$\mathcal{Y}(a_k - a_N, \lambda) \leq \mathcal{Y}\left(a_k - \mathcal{L}, \frac{\lambda}{2}\right) \bullet \mathcal{Y}\left(a_N - \mathcal{L}, \frac{\lambda}{2}\right) < \varepsilon \bullet \varepsilon < \mu.$$

For

$$T(\varepsilon, \lambda) = \{k \leq N : \mathcal{B}_{a_k - a_N}(\varepsilon) \leq 1 - \mu\},$$

$d(T^C(\varepsilon, \lambda)) = 0$ and so $d(T(\varepsilon, \lambda)) = 1$, which is a contradiction, since (a_k) was $SCa - NN$. So that (a_k) must be $SC - NN$. Hence every NNS is $(SC - NN)-$complete. □

From Theorems 6, 7, 8, we have:

Theorem 9 *Let V be an NNS. Then, for any sequence $(a_k) \in F$, the following conditions are equivalent:*

i. (a_k) *is* $SC - NN$.
ii. (a_k) *is* $SCa - NN$.
iii. *NNS* V *is* $(SC - NN)-$*complete.*
iv. *There exists an increasing index sequence* $J = (j_n)$ of *natural numbers such that* $d(J) = 1$ *and the subsequence* (a_{j_n}) *is a* $SCa - NN$.

7 Conclusion

We have defined to neutrosophic normed space and statistical convergence in neutrosophic normed space. The structural characteristic properties of NNSs have been established and examples are given. Further, statistical convergence with respect to neutrosophic norm is introduced and some fundamental properties are examined. Statistical Cauchy sequence and statistically completeness for neutrosophic norm are defined.

Compliance with ethical standards

Conflict of interest The authors declare that they have no conflict of interest.

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