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Abstract: In this paper, we introduce neutrosophic upper and neutrosophic lower almost pre-continuous-multifunctions as a generalization of neutrosophic multifunctions. Some characterizations and several properties concerning neutrosophic upper and neutrosophic lower almost pre-continuous- multifunctions are obtained. further characterizations and several properties concerning neutrosophic upper (lower) pre-continuous continuous multifunctions are obtained. The relationship between these multifunctions and their graphs are investigated.

Keywords: neutrosophic topology, neutrosophic pre-continuous multifunctions, neutrosophic pre open set, neutrosophic continuous multifunctions, neutrosophic upper (lower) pre-continuous.

1 Introduction

The fundamental concept of the fuzzy sets was first introduced by Zadeh in his classical paper [12] of 1965. The idea of "intuitionistic fuzzy sets" was first published by Atanassov [7] and many works by the same author and his colleagues appeared in the literature [15, 16]. The theory of fuzzy topological spaces was introduced and developed by Chang [6] and since then various notions in classical topology have been extended ta fuzzy topological spaces. In 1997, Coker [5] introduced the concept intuitionistic fuzzy multifunctions and studied their lower and upper intuitionistic fuzzy semi continuity from a topological space to an intuitionistic fuzzy topological space. F. Smarandache defined the notion of neutrosophic topology on the non-standard interval [13, 14, 18, 19, 20]. Also in various recent papers, F. Smarandache generalizes intuitionistic fuzzy sets (IFSs) and other kinds of sets to neutrosophic sets (NSs). Also, (Zhang, Smarandache, and Wang, 2005) introduced the notion of interval neutrosophic set which is an instance of neutrosophic set and studied various properties. Recently, Wadei Al-Omeri and Smarandache [9, 10, 14, 21, 22] introduce and study a number of the definitions of neutrosophic continuity, neutrosophic open sets, and obtain several preservation properties and some characterizations concerning neutrosophic functions and neutrosophic connectedness. the theory of multifunctions plays an important role in functional analysis and fixed point theory. It also has a wide range of applications in artificial intelligence, economic theory, decision theory,non-cooperative games.

The concepts of the upper and lower pre-continuous multifunctions was introduced in [17].In this paper we introduce and study the neutrosophic version of upper and lower pre-continuous multifunctions. Inspired by the research works of Smarandache [13, 2], we introduce and study the notions of neutrosophic upper pre-continuous and neutrosophic upper pre-continuous multifunctions in this paper. Further, we present some characterizations and properties.

This paper is arranged as follows. In Section 2, we will recall some notions which will be used throughout this paper. In Section 3, neutrosophic upper pre-continuous (resp. neutrosophic lower pre-continuous) are introduced and investigate its basic properties. In Section 4, we study upper almost neutrosophic pre-continuous (lower almost neutrosophic pre-continuous) and study some of their properties. Finally, the applications are vast and the researchers in the field are exploring these realms of research and proved.

2 Preliminaries

Definition 2.1. [4] Let $\mathscr{R}$ be a non-empty fixed set. A neutrosophic set (NS for short) $\tilde{S}$ is an object having the form $\tilde{S} = \{\langle r, \mu_{\tilde{S}}(r), \sigma_{\tilde{S}}(r), \gamma_{\tilde{S}}(r)\rangle : r \in \mathscr{R}\}$, where $\mu_{\tilde{S}}(r), \sigma_{\tilde{S}}(r)$, and $\gamma_{\tilde{S}}(r)$ are represent the degree of member ship function, the degree of indeterminacy, and the degree of non-membership, respectively, of each element $r \in \mathscr{R}$ to the set $\tilde{S}$.

Neutrosophic sets in $\mathscr{S}$ will be denoted by $\tilde{S}, \lambda, \psi, \mathcal{W}, B, G$, etc., and although subsets of $\mathscr{R}$ will be denoted by $\tilde{R}, \tilde{B}, T, B, p_0, r$, etc.

A neutrosophic set $\tilde{S} = \{\langle r, \mu_{\tilde{S}}(r), \sigma_{\tilde{S}}(r), \gamma_{\tilde{S}}(r)\rangle : r \in \mathscr{R}\}$ can be identified to an ordered triple $\langle \mu_{\tilde{S}}(r), \sigma_{\tilde{S}}(r)$ $, \gamma_{\tilde{S}}(r)\rangle$ in $\rfloor 0^-, 1^+ \lfloor$ on $\mathscr{R}$.

Remark 2.2. [4] A neutrosophic set $\tilde{S}$ is an object having the form $\tilde{S} = \{r, \mu_{\tilde{S}}(r),$ $\sigma_{\tilde{S}}(r), \gamma_{\tilde{S}}(r)\}$ for the NS $\tilde{S} = \{\langle r, \mu_{\tilde{S}}(r), \sigma_{\tilde{S}}(r), \gamma_{\tilde{S}}(r)\rangle : r \in \mathscr{R}\}$.

Definition 2.3. [1] Let $\tilde{S} = \langle \mu_{\tilde{S}}(r), \sigma_{\tilde{S}}(r), \gamma_{\tilde{S}}(r)\rangle$ be an NS on $\mathscr{R}$. Maybe the complement of the set $\tilde{S}(C(\tilde{S})$, for short) definitionned as follows.

(i) $C(\tilde{S}) = \{\langle r, 1 - \mu_{\tilde{S}}(r), 1 - \gamma_{\tilde{S}}(r)\rangle : r \in \mathscr{R}\}$,

(ii) $C(\tilde{S}) = \{\langle r, \gamma_{\tilde{S}}(r), \sigma_{\tilde{S}}(r), \mu_{\tilde{S}}(r)\rangle : r \in \mathscr{R}\}$

(iii) $C(\tilde{S}) = \{\langle r, \gamma_{\tilde{S}}(r), 1 - \sigma_{\tilde{S}}(r), \mu_{\tilde{S}}(r)\rangle : r \in \mathscr{R}\}$

Definition 2.4. [4] Let r be a non-empty set, and $GNSs$ $\tilde{S}$ and B be in the form $\tilde{S} = \{r, \mu_{\tilde{S}}(r), \sigma_{\tilde{S}}(r), \gamma_{\tilde{S}}(r)\}$, $B = \{r, \mu_B(r), \sigma_B(r), \gamma_B(r)\}$. Then $(\tilde{S} \subseteq B)$ definitionned as follows.

(i) Type 1: $\tilde{S} \subseteq B \Leftrightarrow \mu_{\tilde{S}}(r) \leq \mu_B(r), \sigma_{\tilde{S}}(r) \geq \sigma_B(r),$ and $\gamma_{\tilde{S}}(r) \leq \gamma_B(r)$ or

(ii) Type 2: $\tilde{S} \subseteq B \Leftrightarrow \mu_{\tilde{S}}(r) \leq \mu_B(r), \sigma_{\tilde{S}}(r) \geq \sigma_B(r),$ and $\gamma_{\tilde{S}}(r) \geq \gamma_B(r)$.

Definition 2.5. [4] Let $\{\tilde{S}_j : j \in J\}$ be an arbitrary family of an NSs in $\mathscr{R}$. Then

(i) $\cap \tilde{S}_j$ definitionned as:
-Type 1: $\cap \tilde{S}_j = \langle r, \underset{j\in J}{\wedge} \mu_{\tilde{S}j}(r), \underset{j\in J}{\wedge} \sigma_{\tilde{S}j}(r), \underset{j\in J}{\vee} \gamma_{\tilde{S}j}(r)\rangle$
-Type 2: $\cap \tilde{S}_j = \langle r, \underset{j\in J}{\wedge} \mu_{\tilde{S}j}(r), \underset{j\in J}{\vee} \sigma_{\tilde{S}j}(r), \underset{j\in J}{\vee} \gamma_{\tilde{S}j}(r)\rangle$.

(ii) $\cup\tilde{S}_j$ definitionned as:
-Type 1: $\cup\tilde{S}_j = \langle r, \underset{j\in J}{\vee}\mu_{\tilde{S}j}(r), \underset{j\in J}{\vee}\sigma_{\tilde{S}j}(r), \underset{j\in J}{\wedge}\gamma_{\tilde{S}j}(r)\rangle$
-Type 2: $\cup\tilde{S}_j = \langle r, \underset{j\in J}{\vee}\mu_{\tilde{S}j}(r), \underset{j\in J}{\wedge}\sigma_{\tilde{S}j}(r), \underset{j\in J}{\wedge}\gamma_{\tilde{S}j}(r)\rangle$

Definition 2.6. [2] A neutrosophic topology (NT for short) and a non empty set $\mathscr{R}$ is a family $\mathcal{T}$ of neutrosophic subsets of $\mathscr{R}$ satisfying the following axioms

(i) $0_N, 1_N \in \mathcal{T}$

(ii) $G_1 \cap G_2 \in \mathcal{T}$ for any $G_1, G_2 \in \mathcal{T}$

(iii) $\cup G_i \in \mathcal{T}, \forall \{G_i|j \in J\} \subseteq \mathcal{T}$.

The pair $(\mathscr{R}, \mathcal{T})$ is called a neutrosophic topological space (NTS for short).

Definition 2.7. [4] Let $\tilde{S}$ be an NS and $(\mathscr{R}, \mathcal{T})$ an NT where $\tilde{S} = \{r, \mu_{\tilde{S}}(r), \sigma_{\tilde{S}}(r), \gamma_{\tilde{S}}(r)\}$. Then,

(i) $NCL(\tilde{S}) = \cap\{K : K$ is an NCS in $\mathscr{R}$ and $\tilde{S} \subseteq K\}$

(ii) $NInt(\tilde{S}) = \cup\{G : G$ is an NOS in $\mathscr{R}$ and $G \subseteq \tilde{S}\}$

It can be also shown that $NCl(\tilde{S})$ is an NCS and $NInt(\tilde{S})$ is an NOS in $\mathscr{R}$. We have

(i) $\tilde{S}$ is in $\mathscr{R}$ iff $NCl(\tilde{S})$.

(ii) $\tilde{S}$ is an NCS in $\mathscr{R}$ iff $NInt(\tilde{S}) = \tilde{S}$.

Definition 2.8. [4] Let $\tilde{S} = \{\mu_{\tilde{S}}(r), \sigma_{\tilde{S}}(r), \gamma_{\tilde{S}}(r)\}$ be a neutrosophic open sets and $B = \{\mu_B(r), \sigma_B(r), \gamma_B(r)\}$ a neutrosophic set on a neutrosophic topological space $(\mathscr{R}, \mathcal{T})$. Then

(i) $\tilde{S}$ is called neutrosophic regular open iff $\tilde{S} = NInt(NCl(\tilde{S}))$.

(ii) The complement of neutrosophic regular open is neutrosophic regular closed.

Definition 2.9. [9] Let $\tilde{S}$ be an NS and $(\mathscr{R}, \mathcal{T})$ an NT. Then

(i) Neutrosophic semiopen set $(NSOS)$ if $\tilde{S} \subseteq NCl(NInt(\tilde{S}))$,

(ii) Neutrosophic preopen set $(NPOS)$ if $\tilde{S} \subseteq NInt(NCl(\tilde{S}))$,

(iii) Neutrosophic α-open set $(N\alpha OS)$ if $\tilde{S} \subseteq NInt(NCl(NInt(\tilde{S})))$

(iv) Neutrosophic β-open set $(N\beta OS)$ if $\tilde{S} \subseteq NCl(NInt(NCl(\tilde{S})))$

Definition 2.10. [11] Let $(\mathscr{R}, \mathcal{T})$ be a topological space in the classical sense and $(\mathscr{S}, \mathcal{T}_1)$ be a neutrosophic topological space. $F : (\mathscr{R}, \mathcal{T}) \longrightarrow (\mathscr{S}, \mathcal{T}_1)$ is called a neutrosophic multifunction if and only if for each $r \in \mathscr{R}$, $F(r)$ is a neutrosophic set in $\mathscr{S}$.

Definition 2.11. [11] For a neutrosophic multifunction $F : (\mathscr{R}, \mathcal{T}) \longrightarrow (\mathscr{S}, \mathcal{T}_1)$, the upper inverse $F^+(\lambda)$ and lower inverse $F^-(\lambda)$ of a neutrosophic set λ in $\mathscr{S}$ are dened as follows:

$$F^+(\lambda) = \{r \in \mathscr{R} | F(r) \leq \lambda\} \tag{2.1}$$

and

$$F^-(\lambda) = \{r \in \mathscr{R} | F(r) q \lambda\} \tag{2.2}$$

Lemma 2.12. *[11] In a neutrosophic multifunction $F : (\mathscr{R}, \mathcal{T}) \longrightarrow (\mathscr{S}, \mathcal{T}_1)$, we have $F^-(1 - \lambda) = \mathscr{R} - F^+(\lambda)$, for any neutrosophic set λ in $\mathscr{S}$.*

A neutrosophic set $\mathscr{S}$ in $\mathscr{S}$ is said to be q-coincident with a neutrosophic set ψ, denoted by $Sq\psi$, if and only if there exists $p \in \mathscr{S}$ such that $S(p) + \psi(p) > 1$. A neutrosophic set $\mathscr{S}$ of $\mathscr{S}$ is called a neutrosophic neighbourhood of a fuzzy point p_ϵ in $\mathscr{S}$ if there exists a neutrosophic open set ψ in $\mathscr{S}$ such that $p_\epsilon \in \psi \leq S$.

3 Neutrosophic Pre-continuous multifunctions

Definition 3.1. In a neutrosophic multifunction $F : (\mathscr{R}, \mathcal{T}) \longrightarrow (\mathscr{S}, \mathcal{T}_1)$ is said to be

(i) neutrosophic lower pre-continuous at a point $p_0 \in \mathscr{R}$, if for any neutrosophic open set $\mathcal{W} \leq \mathscr{S}$ such that $F(p_0)q\mathcal{W}$ there exists $\tilde{R} \in NPO(\mathscr{R})$ containing p_0 such that $F(\tilde{R})q\mathcal{W}$, $\forall r \in \tilde{R}$.

(ii) neutrosophic upper pre-continuous at a point $p_0 \in \mathscr{R}$, if for any neutrosophic open set $\mathcal{W} \leq \mathscr{S}$ such that $F(p_0) \leq \mathcal{W}$ there exists $\tilde{R} \in NPO(\mathscr{R})$ containing p_0 such that $F(\tilde{R}) \leq \mathcal{W}$.

(iii) neutrosophic upper pre-continuous (resp. neutrosophic lower pre-continuous) if it is neutrosophic upper pre-continuous (resp. neutrosophic lower pre-continuous) at every point of $\mathscr{R}$.

A subset $\tilde{R}$ of a neutrosophic topological space $(\mathscr{R}, \mathcal{T})$ is said to be neutrosophic neighbourhood (resp. neutrosophic-preneighbourhood) of a point $r \in \mathscr{R}$ if there exists a neutrosophic-open (resp. neutrosophic-preopen) set $\tilde{S}$ such that $r \in \tilde{S} \subseteq \tilde{R}$, neutrosophic neighbourhood (resp. neutrosophic pre-neighbourhood) write briefly neutrosophic nbh (resp. neutrosophic pre-nbh).

Theorem 3.2. *A neutrosophic multifunction $F : (\mathscr{R}, \mathcal{T}) \longrightarrow (\mathscr{S}, \mathcal{T}_1)$, the the following statements are equivalent:*

(i) F is neutrosophic upper pre-continuous at p_0;

(ii) $F^+(\tilde{S}) \in NPO(X)$ for any neutrosophic open set $\tilde{S}$ of $\mathscr{S}$,

(iii) $F^-(T)$ is neutrosophic pre-closed in $\mathscr{R}$ for any neutrosophic closed set T of $\mathscr{S}$,

(iv) $pNCl(F^-(\mathcal{W})) \subseteq F^-(NCl(\mathcal{W}))$ for each neutrosophic set $\mathcal{W}$ of $\mathscr{S}$.

(v) for each point $p_0 \in \mathscr{R}$ and each neutrosophic nbh $\tilde{S}$ of $F(r)$, $F^+(\tilde{S})$ is a neutrosophic pre-nbh of p_0,

(vi) for each point $p_0 \in \mathscr{R}$ and each neutrosophic nbh $\tilde{S}$ of $F(r)$, there exists a neutrosophic pre-nbh of p_0 such that $F(\tilde{R}) \leq \tilde{S}$,

(vii) $F^+(NInt(\mathcal{W})) \subseteq pNInt(F^+(\mathcal{W}))$ *for every neutrosophic subset* $\mathcal{W}$ *of* $I^{\mathscr{S}}$*,*

(viii) $F^+(\tilde{S}) \subseteq NInt(NCl(F^+(\tilde{S})))$ *for every neutrosophic open subset* $\tilde{S}$ *of* $I^{\mathscr{S}}$*,*

(ix) for each point $p_0 \in \mathscr{R}$ *and each neutrosophic nbh* $\tilde{S}$ *of* $F(r)$*,* $Cl(F^+(\tilde{S}))$ *is a neighbourhood of* r*.*

Proof. $(i) \Rightarrow (ii)$: Let $\tilde{S}$ be any arbitrary NOS of $\mathscr{S}$ and $p_0 \in F^+(\tilde{S})$. Then $F(p_0) \in \mathscr{S}$. There exists an NPO set $\tilde{R}$ of $\mathscr{R}$ containing p_0 such that $F(\tilde{R}) \subseteq \tilde{S}$. Since

$$p_0 \in \tilde{R} \subseteq NInt(NCl(\tilde{R})) \subseteq NInt(NCl(F^+(\tilde{S}))) \tag{3.1}$$

and so we have

$$F^+(\tilde{S}) \subseteq NInt(NCl(F^+(\tilde{S}))). \tag{3.2}$$

Hence $F^+(\tilde{S})$ is an NPO in $\mathscr{R}$.

$(ii) \Rightarrow (iii)$: It follows from the fact that $F^+(\mathscr{S} - B) = \mathscr{R} - F^-(\mathcal{W})$ for any subset $\mathcal{W}$ of $\mathscr{S}$.

$(iii) \Rightarrow (iv)$: For any subset $\mathcal{W}$ of $\mathscr{S}$, $NCl(\mathcal{W})$ is an NCS in $\mathscr{S}$ and then $F^-(NCl(\mathcal{W}))$ is neutrosophic pre-closed in $\mathscr{R}$. Hence,

$$pNCl(F^-(\mathcal{W})) \subseteq pNCl(F^-(NCl(\tilde{R}))) \subseteq F^-(NCl(\tilde{R})). \tag{3.3}$$

$(iv) \Rightarrow (iii)$: Let β be any arbitrary NCS of $\mathscr{S}$. Then

$$pNCl(F^-(M)) \subseteq F^-(NCl(\beta)) = F^-(\beta), \tag{3.4}$$

and hence $F^-(\beta)$ is NPC in $\mathscr{R}$.

$(ii) \Rightarrow (v)$: Let $p_0 \in \mathscr{R}$ and $\tilde{S}$ be a nbh of $F(p_0)$. There exists an NOS $\tilde{B}$ of $\mathscr{S}$ such that

$$F(p_0) \subseteq \tilde{B} \subseteq \tilde{S}. \tag{3.5}$$

Then we have $p_0 \in F^+(\tilde{B}) \subseteq F^+(\mathscr{S})$ and since $F^+(\mathscr{S})$ is neutrosophic pre-open in $\mathscr{R}$, $F^+(\tilde{S})$ is a neutrosophic pre-nbh of p_0.

$(v) \Rightarrow (vi)$: Let $p_0 \in \mathscr{R}$ and $\tilde{S}$ be any neutrosophic nbh of $F(p_0)$. Put $\tilde{R} = F^+(\tilde{S})$. By (v) $\tilde{R}$ is a neutrosophic pre-nbh of p_0 and $F(\tilde{R}) \subseteq \tilde{S}$.

$(vi) \Rightarrow (i)$: Let $p_0 \in \mathscr{R}$ and $\tilde{S}$ be any neutrosophic open set of $\mathscr{S}$ such that $F(p_0) \subseteq \tilde{S}$. Then $\tilde{S}$ is a neutrosophic nbh of $F(p_0)$ and there exists a neutrosophic pre-nbh $\tilde{R}$ of p_0 such that $F(\tilde{R}) \subseteq \tilde{S}$. Therefore, there exists an NPO $\tilde{B}$ in $\mathscr{R}$ such that $p_0 \in \tilde{B} \subseteq \tilde{R}$ and so $F(\tilde{B}) \subseteq \tilde{S}$.

$(ii) \Rightarrow (vii)$: Let $\mathcal{W}$ be an NOs set of $\mathscr{S}$, $NInt(\mathcal{W})$ is an NO in $\mathscr{S}$ and then $F^+(NInt(\mathcal{W}))$ is NPO in $\mathscr{R}$. Hence,

$$F^+(NInt(\mathcal{W})) \subseteq pNInt(F^+(\mathcal{W})). \tag{3.6}$$

$(vii) \Rightarrow (ii)$: Let $\tilde{S}$ be any neutrosophic open set of $\mathscr{S}$. By (vii) $F^+(\tilde{S}) = F^+(Int(\tilde{S})) \subseteq pNInt(F^+(\tilde{S}))$ and hence $F^+(\tilde{S})$ is an NPO in $\mathscr{R}$.

$(viii) \Rightarrow (ix)$: Let $p_0 \in \mathscr{R}$ and $\tilde{S}$ be any neutrosophic nbh of $F(r)$. Then

$$p_0 \in F^+(\tilde{S}) \subseteq NInt(NCl(F^+(\tilde{S}))) \subseteq NCl(F^+(\tilde{S})), \tag{3.7}$$

and hence $NCl(F^+(\tilde{S}))$ is a neutrosophic nbh of p_0.

$(viii) \Rightarrow (ix)$: Let $\tilde{S}$ be any open set of $\mathscr{S}$ and

$$p_0 \in F^+(\tilde{S}). \tag{3.8}$$

Then

$$NCl(F^+(\tilde{S})) \tag{3.9}$$

is a neutrosophic nbh of p_0 and thus

$$NInt(NCl(F^+(\tilde{S}))). \tag{3.10}$$

Hence,

$$F^+(\tilde{S}) \subseteq NInt(NCl(F+(\tilde{S}))). \tag{3.11}$$

□

Theorem 3.3. *For a neutrosophic multifunction $F : (\mathscr{R}, \mathcal{T}) \longrightarrow (\mathscr{S}, \mathcal{T}_1)$, the following statements are equivalent:*

(i) F is neutrosophic lower pre-continuous at p_0;

(ii) $F^+(\tilde{S}) \in NPO(X)$ for any NOs $\tilde{S}$ of $\mathscr{S}$,

(iii) $F^+(T) \in NPC(X)$ for any neutrosophic closed set T of $\mathscr{S}$,

(iv) for each $p_0 \in \mathscr{R}$ and each neutrosophic nbh $\tilde{S}$ which intersects $F(r)$, $F^-(\tilde{S})$ is a neutrosophic pre-nbh of p_0,

(v) for each $p_0 \in \mathscr{R}$ and each neutrosophic nbh $\tilde{S}$ which intersects $F(r)$, there exists a neutrosophic pre-neighbourhood $\tilde{R}$ of p_0 such that $F(u) \cap \tilde{S} \neq \phi$ or any $u \in \tilde{R}$,

(vi) $pNCl(F^+(\mathcal{W})) \subseteq F^+(NCl(\mathcal{W}))$ for any neutrosophic set $\mathcal{W}$ of $\mathscr{S}$.

(vii) $F^-(NInt(\mathcal{W})) \subseteq pNInt(F^-(\mathcal{W}))$ for every neutrosophic subset $\mathcal{W}$ of $I^{\mathscr{S}}$,

(viii) $F^-(\tilde{S}) \subseteq NInt(NCl(F^-(\tilde{S})))$ for any NOs subset $\tilde{S}$ of $I^{\mathscr{S}}$,

(ix) for each point $p_0 \in \mathscr{R}$ and each neutrosophic nbh $\tilde{S}$ of $F(r)$, $Cl(F^-(\tilde{S}))$ is a neighbourhood of r.

Proof. $(i) \Rightarrow (ii)$: Let $\tilde{S}$ be any arbitrary NOS of $\mathscr{S}$ and $p_0 \in F^+(\tilde{S})$. Then by (a), there exists an NPO set $\tilde{R}$ of $\mathscr{R}$ containing p_0 such that $F(\tilde{R}) \subseteq \tilde{S}$. Since

$$p_0 \in \tilde{R} \subseteq NInt(NCl(\tilde{R})) \subseteq NInt(NCl(F^-(\tilde{S}))) \tag{3.12}$$

and so we have

$$F^-(\tilde{S}) \subseteq NInt(NCl(F^-(\tilde{S}))), \tag{3.13}$$

and hence

$$F^-(\tilde{S}) \in NPO(\mathscr{R}). \tag{3.14}$$

$(ii) \Rightarrow (iii)$: It follows from the fact that

$$F^+(\mathscr{S} \setminus B) = \mathscr{R} \setminus F^-(\mathcal{W}) \tag{3.15}$$

for any $\mathcal{W} \in \mathscr{S}$.

$(iii) \Rightarrow (vi)$: Let $\mathcal{W}$ in $\mathscr{S}$, $NCl(\mathcal{W})$ is an NCS in $\mathscr{S}$. By (iii) $F^{+}(NCl(\mathcal{W}))$ is neutrosophic pre-closed in $\mathscr{R}$. Hence,

$$pNCl(F^{+}(\mathcal{W})) \subseteq pNCl(F^{+}(NCl(\tilde{R}))) \subseteq F^{+}(NCl(\tilde{R})). \tag{3.16}$$

$(iv) \Rightarrow (iii)$: Let β be NCs of $\mathscr{S}$. Then

$$pNCl(F^{+}(M)) \subseteq pNCl(F^{+}(F^{+}(NCl(\beta))) \subseteq F^{+}(NCl(\beta)) = F^{+}(\beta) \Rightarrow F^{-}(\beta) \tag{3.17}$$

is NPC in $\mathscr{R}$.

$(ii) \Rightarrow (v)$: Let $p_0 \in \mathscr{R}$ and $\tilde{S}$ be a neutrosophic nbh of $F(p_0)$. There exists an NOS $\tilde{B}$ of $\mathscr{S}$ such that

$$F(p_0) \subseteq \tilde{B} \subseteq \tilde{S}. \tag{3.18}$$

Then we have

$$p_0 \in F^{-}(\tilde{B}) \subseteq F^{-}(\mathscr{S}), \tag{3.19}$$

and since $F^{-}(V)$ is neutrosophic pre-open in $\mathscr{R}$, by (ii) $F^{-}(\tilde{S})$ is a neutrosophic pre-nbh of p_0.

$(v) \Rightarrow (vi)$: Let $p_0 \in \mathscr{R}$ and $\tilde{S}$ be any neutrosophic nbh of $F(p_0)$. Put $\tilde{R} = F^{-}(\tilde{S})$. By (v) $\tilde{R}$ is a neutrosophic pre-nbh of p_0 and $F(\tilde{R}) \subseteq \tilde{S}$.

$(vi) \Rightarrow (i)$: Let $p_0 \in \mathscr{R}$ and $\tilde{S}$ be any NOs of $\mathscr{S}$ such that $F(p_0) \subseteq \tilde{S}$. Then $\tilde{S}$ is a neutrosophic nbh of $F(p_0)$ by (vi) there exists a neutrosophic pre-nbh $\tilde{R}$ of p_0 such that $F(\tilde{R}) \subseteq \tilde{S}$. Therefore, there exists an NPO $\tilde{B}$ in $\mathscr{R}$ such that

$$p_0 \in \tilde{B} \subseteq \tilde{R} \tag{3.20}$$

and so

$$\tilde{S} \subseteq F^{-}(\tilde{B}). \tag{3.21}$$

$(ii) \Rightarrow (vii)$: Let $\mathcal{W}$ be an NOs set of $\mathscr{S}$, $NInt(\mathcal{W})$ is an NO in $\mathscr{S}$ and then $F^{+}(NInt(\mathcal{W}))$ is NPO in $\mathscr{R}$. Hence, $F^{+}(NInt(\mathcal{W})) \subseteq pNInt(F^{+}(\mathcal{W}))$.

$(vii) \Rightarrow (ii)$: Let $\tilde{S}$ be any NOs of $\mathscr{S}$. By (vii)

$$F^{+}(\tilde{S}) = F^{-}(Int(\tilde{S})) \subseteq pNInt(F^{-}(\tilde{S})), \tag{3.22}$$

and hence $F^{-}(\tilde{S})$ is an NPO in $\mathscr{R}$.

$(vi) \Rightarrow (vii)$: Let $\mathcal{W}$ be any neutrosophic open set of $\mathscr{S}$, then

$$[F^{-}(NInt(\mathcal{W}))]^{c} = F^{+}(NCl(\mathcal{W}^{c})) \supset pNCl(F^{+}(NInt(NCl(\mathcal{W}^{c})))) \tag{3.23}$$

$$= pNCl(F^{+}(NCl(NInt(\mathcal{W})))^{c}) = pNCl(F^{-}(NCl(NInt(\mathcal{W}))))^{c} \tag{3.24}$$

$$= [pNInt(F^{-}(NCl(NInt(\mathcal{W}))))]^{c}. \tag{3.25}$$

Thus we obtained

$$F^{-}(NInt(\mathcal{W})) \supset pNInt(F^{-}(NCl(NInt(\mathcal{W})))). \tag{3.26}$$

$(vi) \Rightarrow (vii)$: Obvious. □

We now show by means of the following examples that
lower neutrosophic pre-continuous $\not\Rightarrow$ upper neutrosophic pre-continuous.

Example 3.1. Let $\mathscr{R} = \{u, v, w\}$ and $\mathscr{S} = [0,1]$. Let $\mathcal{T}$ and $\mathcal{T}_1$, be respectively the topology on $\mathscr{R}$ and neutrosophic topology on $\mathscr{S}$,given by $\mathcal{T}\{\mathscr{R}_N, \phi_N, \{u,w\}\}$, $\mathcal{T}_1 = \{C_o, C, \mu_{\tilde{S}}, \sigma_{\tilde{S}}, \gamma_{\tilde{S}}, (\mu_{\tilde{S}} \cup \sigma_{\tilde{S}}), (\mu_{\tilde{S}} \cap \sigma_{\tilde{S}})\}$. Where $\mu_{\tilde{S}}(r) = r$, $\sigma_{\tilde{S}}(r) = I - r$, for $r \in \mathscr{S}$, and

$$\mu(r) = \begin{cases} r, & if \quad 0 \leq r \leq \dfrac{1}{2}, \\ 0, & if \quad \dfrac{1}{2} \leq r \leq 1, \end{cases} \tag{3.27}$$

We definitionne a neutrosophic multifunction $F : (\mathscr{R}, \mathcal{T}) \longrightarrow (\mathscr{S}, \mathcal{T}_1)$ be letting $F(u) = (\mu_{\tilde{S}} \cap \sigma_{\tilde{S}})$, $F(v) = \sigma_{\tilde{S}}$ and $F(w) = \gamma_{\tilde{S}}$.

$\{u, w\}$ is neutrosophic open set in $\mathscr{R}$ and therefore $\{u, w\}$ is neutrosophic pre-open set. The other neutrosophic pre-open set in $(\mathscr{R}, \mathcal{T})$ are $\{u, v\}$, $\{v, w\}$, $\{u\}$ and $\{w\}$. Then $\{u\}$ is not neutrosophic pre-open set in $(\mathscr{R}, \mathcal{T})$. From definitionnition of $\mu_{\tilde{S}}$ and $\sigma_{\tilde{S}}$ we find that,

$$(\mu_{\tilde{S}} \cup \sigma_{\tilde{S}})(r) = \begin{cases} 1 - r, & if \quad 0 \leq r \leq \dfrac{1}{2}, \\ r, & if \quad \dfrac{1}{2} \leq r \leq 1, \end{cases} \tag{3.28}$$

$$(\mu_{\tilde{S}} \cap \sigma_{\tilde{S}})(r) = \begin{cases} r, & if \quad 0 \leq r \leq \dfrac{1}{2}, \\ 1 - r, & if \quad \dfrac{1}{2} \leq r \leq 1, \end{cases} \tag{3.29}$$

Now $\sigma_{\tilde{S}} \in \mathcal{T}_1$ but $F^+(\sigma_{\tilde{S}}) = \{v\}$ which is not neutrosophic pre-open set in $(\mathscr{R}, \mathcal{T})$. Hence F is not upper neutrosophic pre-continuous. Then $F^-(\sigma_{\tilde{S}}) = \{v\}$ which is not neutrosophic pre-open set in $(\mathscr{R}, \mathcal{T})$. Therefore F is not lower neutrosophic pre-continuous

Remark 3.4. [11] A subset μ of a topological space $(\mathscr{R}, \mathcal{T})$ can be considered as a neutrosophic set with characteristic function definitionned by

$$\mu(r) = \begin{cases} 1, & if \quad u \in \mu, \\ 0, & if \quad v \notin \mu, \end{cases} \tag{3.30}$$

Let $(\mathscr{S}, \mathcal{T}_1)$ be a neutrosophic topological space. The neutrosophic sets of the form $\mu \times \nu$ with $\mu \in \mathcal{T}$ and $\nu \in \mathcal{T}_1$ make a basis for the product neutrosophic topology $\mathcal{T} \times \mathcal{T}_1$ on $\mathscr{R} \times \mathscr{S}$, where for any $(u, v) \in \mathscr{R} \times \mathscr{S}$, $(\mu \times \nu)(u, v) = min\{\mu(u), \nu(v)\}$.

Definition 3.5. [11] For a neutrosophic multifunction $F : (\mathscr{R}, \mathcal{T}) \longrightarrow (\mathscr{S}, \mathcal{T}_1)$, the neutrosophic graph multifunction $F_G : \mathscr{R} \times \mathscr{R} \longrightarrow \mathscr{S}$ of F is definitionned by $F_G(r) = r_1 \times F(r)$ for every $r \in \mathscr{R}$.

Lemma 3.6. *In a neutrosophic multifunction $F : (\mathscr{R}, \mathcal{T}) \longrightarrow (\mathscr{S}, \mathcal{T}_1)$, the following hold: a) $F_G^+(\tilde{R} \times \tilde{S}) = \tilde{R} \cap F^+(\tilde{S})$*
b) $F_G^-(\tilde{R} \times \tilde{S}) = \tilde{R} \cap F^+(\tilde{S})$ for all subsets $\tilde{R} \in \mathscr{R}$ and $\tilde{S} \in \mathscr{S}$.

Theorem 3.7. *If the neutrosophic graph multifunction F_G of a neutrosophic multifunction $F : (\mathscr{R}, \mathcal{T}) \longrightarrow (\mathscr{S}, \mathcal{T}_1)$ is neutrosophic lower precontinuous, then F is neutrosophic lower precontinuous.*

Proof. Suppose that F_G is neutrosophic lower precontinuous and $s \in \mathscr{R}$. Let B be an $NOs \in \mathscr{S}$ such that $F(r)qB$. Then there exists $r \in \mathscr{S}$ such that $(F(r))(r) + A(r) > 1$. Then

$$(F_G(r))(r, r) + (\mathscr{R} \times B)(r, r) = (F(r))(r) + B(r) > 1. \tag{3.31}$$

Hence, $F_G(r)q(\mathscr{R} \times B)$. Since F_G is neutrosophic lower precontinuous, there exists an open set $A \in \mathscr{R}$ such that $r \in A$ and $F_G(b)q(\mathscr{R} \times B)\ \forall a \in A$. Let there exists $a_0 \in A$ such that $F(a_0)qB$. Then $\forall r \in \mathscr{S}$, $(F(a_0))(r) + B(r) < 1$. For any $(b, c) \in \mathscr{R} \times \mathscr{S}$, we have

$$(F_G(a_0))(b, c) \subseteq (F(a_0))(c), \tag{3.32}$$

and

$$(\mathscr{R} \times B)(b, c) \subseteq B(c). \tag{3.33}$$

Since $\forall r \in \mathscr{S}$, $(F(a_0))(r) + B(r) < 1$,

$$(F_G(a_0))(b, c) + (\mathscr{R} \times B)(b, c) < 1. \tag{3.34}$$

Thus, $F_G(a_0)q(\mathscr{R} \times B)$, where $a_0 \in A$. This is a contradiction since

$$F_G(a)q(\mathscr{R} \times B),\ \forall a \in A, \tag{3.35}$$

Therefore, F is neutrosophic lower precontinuous. □

Definition 3.8. A neutrosophic space $(\mathscr{R}, \mathcal{T})$ is said to be neutrosophic pre-regular (NP-regular) if for every NCs F and a point $u \in F$, there exist disjoint neutrosophic-preopen sets $\tilde{R}$ and $\tilde{S}$ such that $F \subseteq \tilde{R}$ and $u \in \tilde{S}$.

Theorem 3.9. *Let $F : (\mathscr{R}, \mathcal{T}) \longrightarrow (\mathscr{S}, \mathcal{T}_1)$ be a neutrosophic multifunction and $F_G : \mathscr{R} \longrightarrow \mathscr{R} \times \mathscr{S}$ the graph multifunction of F. If F_G is neutrosophic upper pre-continuous (neutrosophic lower pre-continuous, then F is neutrosophic upper pre-continuous. (neutrosophic lower pre-continuous.) and $\mathscr{R}$ is NP-regular.*

Proof. Let F_G be a neutrosophic upper pre-continuous multifunction and $\tilde{S}$ be a neutrosophic open set containing $F(r)$ such that $r \in F^+(\tilde{S})$. Then $\mathscr{R} \times \tilde{S}$ is a neutrosophic open set of $\mathscr{R} \times \mathscr{S}$ containing $F_G(r)$. Since F_G is neutrosophic upper pre-continuous, there exists an $NPOs$ $\tilde{R}$ of $\mathscr{R}$ containing r such that $\tilde{R}_p^- \subseteq F_G^+(\mathscr{R} \times \tilde{S})$. Therefore we obtain

$$\tilde{R}_p^- \subseteq F^+(\tilde{S}). \tag{3.36}$$

Now we show that $\mathscr{R}$ is NP-regular. Let $\tilde{R}$ be any $NPOs$ of $\mathscr{R}$ containing r. Since

$$F_G(r) \in \tilde{R} \times \mathscr{S}, \tag{3.37}$$

and $\tilde{R} \times \mathscr{S}$ is neutrosophic open in $\mathscr{R} \times \mathscr{S}$, there exists an $NPOs$ set U of $\mathscr{R}$ such that

$$U_p^- \subseteq F_G^+(\tilde{R} \times \mathscr{S}). \tag{3.38}$$

Therefore we have

$$r \in U \subseteq U_p^- \subseteq \tilde{R}. \tag{3.39}$$

This shows that $\mathscr{R}$ is NP-regular.

The proof for neutrosophic upper lower-continuous is similar. □

Theorem 3.10. *Let $\mathscr{R}$ is NP-regular. A neutrosophic multifunction $F : (\mathscr{R}, \mathcal{T}) \longrightarrow (\mathscr{S}, \mathcal{T}_1)$ is neutrosophic lower pre-continuous iff $F_G(r)$ is neutrosophic lower pre-continuous.*

Proof. $\Longrightarrow$ Let $r \in \mathscr{R}$ and A be any $NPOs$ of $\mathscr{R} \times \mathscr{S}$ such that $r \in F_G(A)$. Since

$$A \cap (\{r\} \times F(r)) \neq \phi, \tag{3.40}$$

there exists $s \in F(r)$ such that $(r, s) \in A$. Hence

$$(r, s) \in \tilde{R} \times \tilde{S} \subseteq A \tag{3.41}$$

for some NOs $\tilde{R} \subseteq \mathscr{R}$ and $\tilde{S} \subseteq \mathscr{S}$. Since $\mathscr{R}$ is NP-regular, there exists $B \in NPO(\mathscr{R}, r)$ such that

$$r \in B \subseteq B_p^- \subseteq \tilde{R}. \tag{3.42}$$

Since F is neutrosophic lower pre-continuous, there exists $W \in NPO(\mathscr{R}, r)$ such that

$$W_p^- \subseteq F^-(\tilde{S}). \tag{3.43}$$

By Lemma 3.6, we have

$$W_p^- \cap B_p^- \subseteq \tilde{R} \cap F^-(\tilde{S}) = F_G^-(\tilde{R} \times \tilde{S}) \subseteq F_G^-(B). \tag{3.44}$$

Moreover, we have $B \cap W \in NPO(\mathscr{R}, r)$ and hence $F_G(r)$ is neutrosophic lower pre-continuous.
$\Longleftarrow$ Let $r \in \mathscr{R}$ and $\tilde{S}$ be any $NOs \in \mathscr{S}$ such that $r \in F^-(\tilde{S})$. Then $\mathscr{R} \times \tilde{S}$ is

$$NO \in \mathscr{R} \times \mathscr{S}. \tag{3.45}$$

Since F_G is neutrosophic lower pre-continuous and lemma,

$$F_G^-(\mathscr{R} \times \tilde{S}) = \mathscr{R} \cap F^-(\tilde{S}) = F^-(\tilde{S}) \tag{3.46}$$

is $NPOs \in \mathscr{R}$. This shows that F is neutrosophic lower pre-continuous. □

Definition 3.11. [11] A neutrosophic set Δ of a neutrosophic topological space $\mathscr{S}$ is said to be neutrosophic compact relative to $\mathscr{S}$ if every cover $\{\Delta_\lambda\}_{\lambda \in \Lambda}$ of Δ by neutrosophic open sets of $\mathscr{S}$ has a finite subcover $\{\Delta_i\}_{i=1}^n$ of Δ.

Theorem 3.12. *Let $F : (\mathscr{R}, \mathcal{T}) \longrightarrow (\mathscr{S}, \mathcal{T}_1)$ be a neutrosophic multifunction such that $F(r)$ is compact for each $r \in \mathscr{R}$. And $\mathscr{R}$ is a NP-regular space. If F is neutrosophic upper pre-continuous then F_G is neutrosophic upper pre-continuous.*

Proof. Let $r \in \mathscr{R}$ and A be any $NOs \in \mathscr{R} \times \mathscr{S}$ containing $F_G(r)$. For each $s \in F(r)$, there exist open sets $\tilde{R}(s) \subseteq \mathscr{R}$ and $\tilde{S}(s) \subseteq \mathscr{S}$ such that

$$(r, s) \in \tilde{R}(s) \times \tilde{S}(s) \subseteq A. \tag{3.47}$$

The family $\{\tilde{S}(s) : s \in F(r)\}$ is a neutrosophic open cover of $F(r)$. Since $F(r)$ is compact, there exists a finite number of points $\{s_j\}_{j=1}^n$ in $F(r)$ such that

$$F(r) \subseteq \cup\{\tilde{S}(s_j) : j = 1, ...n\}. \tag{3.48}$$

Use $\tilde{R} = \cap\{\tilde{R}(s_j) : j = 1, ..., n\}$ and $\tilde{S} = \{\tilde{S}(s_j) : j = 1, ..., n\}$. Then $\tilde{R}$ and $\tilde{S}$ are $NOs \in \mathscr{R}$ and $\mathscr{S}$, respectively and

$$\{r\} \times F(r) \subseteq \tilde{R} \times \tilde{S} \subseteq A. \tag{3.49}$$

Since F is neutrosophic upper pre-continuous, there exists $S \in NPO(\mathscr{R}, r)$ such that

$$S_p^- \subseteq F^+(\tilde{S}). \tag{3.50}$$

Since $\mathscr{R}$ is NP-regular, there exists $G \in NPO(\mathscr{R}, r)$ such that

$$r \in G \subseteq G_p^- \subseteq \tilde{R}. \tag{3.51}$$

Hence, we have

$$\{r\} \times F(r) \subseteq G_p^- \times \tilde{S} \subseteq \tilde{R} \times \tilde{S} \subseteq A. \tag{3.52}$$

Then we have

$$(S \cap G)_p^- \subseteq SG_p^- \cap G_p^- \subseteq F^+(\tilde{S}) \cap G_p^- = F_G^+(G_p^- \times \tilde{S}) \subseteq F_G^+(A). \tag{3.53}$$

Moreover, we obtain $S \cap G \in NPO(\mathscr{R}, r)$ and hence F_G is neutrosophic upper pre-continuous. □

Proposition 3.2. Let B and $\mathscr{R}_o$ be subsets of neutrosophic topological space $(\mathscr{R}, \mathcal{T})$.

(i) If $B \in NPO(\mathscr{R})$ and $\mathscr{R}_o$ is NSO in $\mathscr{R}$, then $B \cap \mathscr{R}_o \in NPO(\mathscr{R}_o)$.

(ii) If $B \in NPO(\mathscr{R}_o)$ and $\mathscr{R}_o \in NPO(\mathscr{R})$, then $B \in NPO(\mathscr{R})$.

Proposition 3.3. Let B and $\mathscr{R}$ be subsets of neutrosophic topological space $(\mathscr{R}, \mathcal{T})$, $B \subseteq \mathscr{R} \subseteq \mathscr{R}$. Let the neutrosophic pre-closure $(B_p^-)_{\mathscr{R}}$ of B in the neutrosophic subspace $\mathscr{R}_o$:

(i) If $\mathscr{R}$ is NSO in $\mathscr{R}$, then $(B_p^-)_{\mathscr{R}_o} \subseteq (B_p^-)_{\mathscr{R}}$.

(ii) If B in $NPO(\mathscr{R}_o)$ and $\mathscr{R}_o$ in $NPO(\mathscr{R})$, then $B_p^- \subseteq (B_p^-)_{\mathscr{R}_o}$.

Theorem 3.13. *A neutrosophic multifunction $F : (\mathscr{R}, \mathcal{T}) \longrightarrow (\mathscr{S}, \mathcal{T}_1)$ is upper almost neutrosophic pre-continuous. (lower almost neutrosophic pre-continuous) if $\forall\, r \in \mathscr{R}$ there exists an $NPOs$ $\mathscr{R}_o$ containing r such that $F|\mathscr{R}_o : (\mathscr{R}, \mathcal{T}) \longrightarrow (\mathscr{S}, \mathcal{T}_1)$ is upper almost neutrosophic pre-continuous. (lower almost neutrosophic pre-continuous).*

Proof. Let $r \in \mathscr{R}$ and $\tilde{S}$ be an neutrosophic open set of $\mathscr{S}$ containing $F(r)$ such that $r \in F^+(\tilde{S})$ and there exists $\mathscr{R}_o \in NPO(\mathscr{R}, r)$ such that

$$F|\mathscr{R}_o : (\mathscr{R}, \mathcal{T}) \longrightarrow (\mathscr{S}, \mathcal{T}_1), \tag{3.54}$$

is upper almost neutrosophic pre-continuous. Therefore, there exists $\tilde{R}$ in $NPO(\mathscr{R}_o, r)$ such that

$$(\tilde{R}_p^-)_{\mathscr{R}_o} \subseteq (F|\mathscr{R}_o)^+(\tilde{S}). \tag{3.55}$$

By Proposition 3.2 and Proposition 3.3, $\tilde{R}$ in $NPO(X, r)$ and

$$\tilde{R}_p^- \subseteq (\tilde{R}_p^-)_{\mathscr{R}_o}. \tag{3.56}$$

Therefore

$$F(\tilde{R}_p^-) = (F|\mathscr{R}_o)(\tilde{R}_p^-) \subseteq (F|\mathscr{R}_o)((\tilde{R}_p^-)_{\mathscr{R}_o}) \subseteq \tilde{S}. \tag{3.57}$$

This shows that F is upper almost neutrosophic pre-continuous. □

4 Almost Neutrosophic pre-continuous multifunctions

Definition 4.1. Let $(\mathscr{R}, \mathcal{T})$ be a neutrosophic topological space and $(\mathscr{S}, \mathcal{T}_1)$ a topological space. A neutrosophic multifunction $F : (\mathscr{R}, \mathcal{T}) \longrightarrow (\mathscr{S}, \mathcal{T}_1)$ is said to be:

(i) upper almost neutrosophic pre-continuous at a point $r \in \mathscr{R}$ if for each open set $\tilde{S}$ of $I^{\mathscr{S}}$ such that $r \in F^+(\tilde{S})$, there exists a neutrosophic pre-open set $\tilde{R}$ of $\mathscr{R}$ containing r such that $\tilde{R} \subseteq F^+(NInt(NCl(\tilde{S})))$;

(ii) lower almost neutrosophic pre-continuous at a point $r \in \mathscr{R}$ if for each neutrosophic open set $\tilde{S} \in \mathscr{S}$ such that $r \in F(\tilde{S})$, there exists a neutrosophic pre-open $\tilde{R}$ of $\mathscr{R}$ containing r such that $\tilde{R} \subseteq F^-(NInt(NCl(\tilde{S})))$;

(iii) upper (resp. lower) almost neutrosophic pre-continuous if F has this property at each point of $\mathscr{R}$.

Theorem 4.2. *Let $(\mathscr{R}, \mathcal{T})$ be a neutrosophic topological space and $F : (\mathscr{R}, \mathcal{T}) \longrightarrow (\mathscr{S}, \mathcal{T}_1)$ a neutrosophic multifunction from a neutrosophic topological space $(\mathscr{R}, \mathcal{T})$ to a topological space $(\mathscr{S}, \mathcal{T}_1)$. Then the following properties are equivalent:*

(i) F is upper almost neutrosophic-pre-continuous;

(ii) for any $r \in \mathscr{R}$ and for all NOs $\tilde{S}$ of $\mathscr{S}$ such that $F(r) \subseteq \tilde{S}$, there exists a neutrosophic pre-open $\tilde{R}$ of $\mathscr{R}$ containing r such that if $z \in \tilde{R}$, then $F(z) \subseteq NInt(NCl(\tilde{S}))$;

(iii) for any $r \in \mathscr{R}$ and for all $NROs$ G of $\mathscr{S}$ such that $F(r) \subseteq G$, there exists a neutrosophic per-open $\tilde{R}$ of $\mathscr{R}$ containing r such that $F(\tilde{R}) \subseteq G$;

(iv) for any $r \in \mathscr{R}$ and for all closed set $F^+(\mathscr{S} - M)$, there exists a neutrosophic per-closed N of $\mathscr{R}$ such that $r \in \mathscr{R} - N$ and $F^-(NInt(NCl(M))) \subseteq N$;

(v) $F^+(NInt(NCl(\tilde{S})))$ is neutrosophic pre-open in $\mathscr{R}$ for any NOs $\tilde{S}$ of $\mathscr{S}$;

(vi) $F^-(NCl(NInt(F)))$ is neutrosophic pre-closed in $\mathscr{R}$ for each closed set F of $\mathscr{S}$;

(vii) $F^+(G)$ is neutrosophic pre-open in $\mathscr{R}$ for each regular open G of $\mathscr{S}$;

(viii) $F^+(H)$ is neutrosophic pre-closed in $\mathscr{R}$ for each regular closed H of $\mathscr{S}$;

(ix) for any point $r \in \mathscr{R}$ and each neutrosophic-nbh $\tilde{S}$ of $F(r)$, $F^+(NInt(NCl(\tilde{S})))$ is a neutrosophic pre-nbh of r;

(x) for any point $r \in \mathscr{R}$ and each neutrosophic-nbh $\tilde{S}$ of $F(r)$, there exists a neutrosophic pre-nbh $\tilde{R}$ of r such that $F(\tilde{R}) \subseteq NInt(NCl(\tilde{S}))$;

(xi) $pNCl(F^-(NCl(NInt(A)))) \subseteq F^-(NCl(NInt(NCl(A))))$ for any subset A of $\mathscr{S}$;

(xii) $F^+(NInt(NCl(NInt(A)))) \subseteq pNInt(F^+(NInt(NCl(A))))$ for any subset A of $\mathscr{S}$.

Proof. $(i) \Rightarrow (ii)$: Obvious.

$(ii) \Rightarrow (iii)$: Let $r \in \mathscr{R}$ and G be a regular open set of $\mathscr{S}$ such that $F(r) \subseteq G$. By (ii), there exists an $NPOs$ $\tilde{R}$ containing r such that if $z \in \tilde{R}$, then $F(z) \subseteq NInt(NCl(G)) = G$. We obtain $F(\tilde{R}) \subseteq G$.

$(iii) \Rightarrow (ii)$: Let $r \in \mathscr{R}$ and $\tilde{S}$ be an NOs set of $\mathscr{S}$ such that $F(r) \subseteq \tilde{S}$. Then, $NInt(NCl(\tilde{S}))$ is $NROs$ in $\mathscr{S}$. By (iii), there exists an $NPOS$ of $\mathscr{R}$ containing r such that

$$F(\tilde{R}) \subseteq NInt(NCl(\tilde{S})). \tag{4.1}$$

$(ii) \Rightarrow (iv)$: Let $r \in \mathscr{R}$ and M be an NCs of $\mathscr{S}$ such that $r \in F^{+}(\mathscr{S} - M)$. By (ii), there exists an $NPOs$ $\tilde{R}$ of $\mathscr{R}$ containing r such that

$$F(\tilde{R}) \subseteq NInt(NCl(\mathscr{S}M)). \tag{4.2}$$

We have

$$NInt(NCl(\mathscr{S} - M)) = \mathscr{S} - NCl(NInt(M)) \tag{4.3}$$

and

$$\tilde{R} \subseteq F^{+}(\mathscr{S} - NCl(NInt(M))) = \mathscr{R} - F^{-}(NCl(NInt(M))). \tag{4.4}$$

We get

$$F^{-}(NCl(NInt(M))) \subseteq \mathscr{R} - \tilde{R}. \tag{4.5}$$

Let $N = \mathscr{R} - \tilde{R}$. Then, $r \in \mathscr{R} - N$ and N is an $NPCs$.

$(iv) \Rightarrow (ii)$: The proof is similar to $(ii) \Rightarrow (iv)$.

$(i) \Rightarrow (v)$: Let $\tilde{S}$ be any neutrosophic open set of $\mathscr{S}$ and $r \in F^{+}(NInt(NCl(\tilde{S})))$. By (i), there exists an $NPOs$ $\tilde{R}_r$ of $\mathscr{R}$ containing r such that

$$\tilde{R}_r \subseteq F^{+}(NInt(NCl(\tilde{S}))). \tag{4.6}$$

Hence, we obtain

$$F^{+}(NInt(NCl(\tilde{S}))) = \cup_r \in F^{+}(NInt(NCl(\tilde{S})))\tilde{R}_r. \tag{4.7}$$

Therefore, $F^{+}(NInt(NCl(\tilde{S})))$ is an $NPOs$ of $\mathscr{R}$.

$(v) \Rightarrow (i)$: Let $\tilde{S}$ be any neutrosophic open set of $\mathscr{S}$ and $r \in F^{+}(\tilde{S})$. By (v), $F^{+}(NInt(NCl(\tilde{S})))$ is $NPOs$ in $\mathscr{R}$. Let $\tilde{R} = F^{+}(NInt(NCl(\tilde{S})))$. Then,

$$F(\tilde{R}) \subseteq NInt(NCl(\tilde{S})). \tag{4.8}$$

Therfore, F is upper neutrosophic pre-continuous.

$(v) \Rightarrow (vi)$: Let F be any neutrosophic closed set of $\mathscr{S}$. Then, $\mathscr{S} - F$ is an NOs of $\mathscr{S}$. By (v), $F^{+}(NInt(NCl(\mathscr{S} - F)))$ is $NPOs \in \mathscr{R}$. Since $NInt(NCl(\mathscr{S} - F)) = \mathscr{S} - NCl(NInt(F))$, it follows that $F^{+}(NInt(NCl(\mathscr{S} - F))) = F^{+}(\mathscr{S} - NCl(NInt(F))) = \mathscr{R} - F^{-}(NCl(NInt(F)))$. We obtain that $F^{-}(NCl(NInt(F)))$ is $NPCs \in \mathscr{R}$.

$(vi) \Rightarrow (v)$: The proof is similar to $(v) \Rightarrow (vi)$.

$(v) \Rightarrow (vii)$: Let G be any $NROs$ of $\mathscr{S}$. By (v),

$$F^{+}(NInt(NCl(G))) = F^{+}(G) \tag{4.9}$$

is $NPOs \in \mathscr{R}$.

$(vii) \Rightarrow (v)$: Let $\tilde{S}$ be any neutrosophic-open set of $\mathscr{S}$. Then, $NInt(NCl(\tilde{S}))$ is $NROsin\mathscr{S}$. By (vii),

$$F^{+}(NInt(NCl(\tilde{S}))) \tag{4.10}$$

is $NPOs \in \mathscr{R}$.

$(vi) \Rightarrow (viii)$: The proof is similar to $(v) \Rightarrow (vii)$.

$(viii) \Rightarrow (vi)$: The proof is similar to $(vii) \Rightarrow (v)$.

$(v) \Rightarrow (ix)$: Let $r \in \mathscr{R}$ and $\tilde{S}$ be a neutrosophic nbh of $F(r)$. Then, there exists an open set G of $\mathscr{S}$ such that

$$F(r) \subseteq G \subseteq \tilde{S}. \tag{4.11}$$

Hence, we obtain $r \in F^{+}(G) \subseteq F^{+}(\tilde{S})$. Since $F^{+}(NInt(NCl(G)))$ is $NPOs \in \mathscr{R}$, $F^{+}(NInt(NCl(\tilde{S})))$ is a neutrosophic pre-nbh of r.

$(ix) \Rightarrow (x)$: Let $r \in \mathscr{R}$ and $\tilde{S}$ be a neutrosophic nbh of $F(r)$. By (ix), $F^{+}(NInt(NCl(\tilde{S})))$ is a neutrosophic-nbh of r. Let $\tilde{R} = F^{+}(NInt(NCl(\tilde{S})))$. Then,

$$F(\tilde{R}) \subseteq NInt(NCl(\tilde{S})). \tag{4.12}$$

$(x) \Rightarrow (i)$: Let $r \in \mathscr{R}$ and $\tilde{S}$ be any NOs of $\mathscr{S}$ such that $F(r) \subseteq \tilde{S}$. Then, $\tilde{S}$ is a neutrosophic-nbh of $F(r)$. By (r), there exists a neutrosophic pre-nbh $\tilde{R}$ of r such that

$$F(\tilde{R}) \subseteq NInt(NCl(\tilde{S})). \tag{4.13}$$

Hence, there exists an $NPOs$ G of $\mathscr{R}$ such that

$$r \in G \subseteq \tilde{R}, \tag{4.14}$$

and hence

$$F(G) \subseteq F(\tilde{R}) \subseteq NInt(NCl(\tilde{S})). \tag{4.15}$$

We obtain that F is upper almost neutrosophic pre-continuous.

$(vi) \Rightarrow (xi)$: For every subset A of $\mathscr{S}$, $NCl(A)$ is NCs in $\mathscr{S}$. By (vi), $F^{-}(NCl(NInt(NCl(A))))$ is $NPCs \in \mathscr{R}$. Hence, we obtain

$$pNCl(F^{-}(NCl(NInt(A)))) \subseteq F^{-}(NCl(NInt(NCl(A)))). \tag{4.16}$$

$(xi) \Rightarrow (vi)$: For any NCs F of $\mathscr{S}$. Then we have

$$pNCl(F^{-}F)))) \subseteq F^{-}(NCl(NInt(NCl(F)))) = F^{-}(NCl(NInt(F))). \tag{4.17}$$

Thus, $F^{-}(NCl(NInt(F)))$ is $NPCs \in \mathscr{R}$.

$(v) \Rightarrow (xii)$: For every subset A of $\mathscr{S}$, $NInt(A)$ is $NO \in \mathscr{S}$. By (v),

$$F^{+}(NInt(NCl(NInt(A)))) \tag{4.18}$$

is $NPOs$ in $\mathscr{R}$. Therefore, we obtain

$$F^{+}(NInt(NCl(NInt(A)))) \subseteq pNInt(F^{+}(NInt(NCl(A)))). \tag{4.19}$$

$(xii) \Rightarrow (v)$: Let $\tilde{S}$ be any subset of $\mathscr{S}$. Then

$$F^{+}(NInt(NCl(\tilde{S}))) \subseteq pNInt(F^{+}(NInt(NCl(\tilde{S})))). \tag{4.20}$$

Therefore, $F^{+}(NInt(NCl(\tilde{S})))$ is $NPOs \in \mathscr{R}$. □

Remark 4.3. If $F : (\mathscr{R}, \mathcal{T}) \longrightarrow (\mathscr{S}, \mathcal{T}_1)$ are neutrosophic upper pre-continuous multifunctions, then F is a neutrosophic upper almost pre-continuous multifunction.

The implication is not reversible.

Example 4.1. Let $\mathscr{R} = \{\mu, \nu, \omega\}$ and $\mathscr{S} = \{u, v, w, t, h\}$. Let $(\mathscr{R}, \mathcal{T})$ be a neutrosophic topology on $\mathscr{R}$ and $\sigma_{\tilde{S}}$ a topology on $\mathscr{S}$ given by $\mathcal{T} = \{\phi_N, \{v\}, \{w\}, \{v, w\}, \mathscr{R}_N\}$ and $\sigma_{\tilde{S}} = \{\phi_N, \{u, v, w, t\}, \mathscr{S}_N\}$. Definitionne the multifunction $F : (\mathscr{R}, \mathcal{T}) \longrightarrow (\mathscr{S}, \mathcal{T}_1)$ by $F(\mu) = \{w\}$, $F(\nu) = \{v, t\}$ and $F(\omega) = \{u, h\}$. Then F is upper almost neutrosophic precontinuous but not upper neutrosophic precontinuous, since $\{u, v, w, t, h\} \in \sigma_{\tilde{S}}$ and $F^{+}(\{u, v, w, t, h\}) = \{\mu, \nu\}$ is not neutrosophic pre-open in $\mathscr{R}$.

Theorem 4.4. *Let $F : (\mathscr{R}, \mathcal{T}) \longrightarrow (\mathscr{S}, \mathcal{T}_1)$ be a multifunction from a neutrosophic topological space $(\mathscr{R}, \mathcal{T})$ to a topological space $(\mathscr{S}, \mathcal{T}_1)$. Then the following properties are equivalent:*

(i) F is lower almost neutrosophic-precontinuous;

(ii) for each $r \in \mathscr{R}$ and for each open set $\tilde{S}$ of $\mathscr{S}$ such that $F(r) \cap \tilde{S} \neq \phi$, there exists a neutrosophic-preopen $\tilde{R}$ of $\mathscr{R}$ containing r such that if $z \in \tilde{R}$, then $F(z) \cap NInt(NCl(\tilde{S})) \neq \phi$;

(iii) for each $r \in \mathscr{R}$ and for each regular open set G of $\mathscr{S}$ such that $F(r) \cap G \neq \phi$, there exists a neutrosophic-peropen $\tilde{R}$ of $\mathscr{R}$ containing r such that if $z \in \tilde{R}$, then $F(z) \cap G \neq \phi$;

(iv) for each $r \in \mathscr{R}$ and for each closed set M of $\mathscr{S}$ such that $r \in F^{+}(\mathscr{S} - M)$, there exists a neutrosophic-perclosed N of $\mathscr{R}$ such that $r \in \mathscr{R} - N$ and $F^{+}(NCl(NInt(M))) \subseteq N$;

(v) $F^{-}(NInt(NCl(\tilde{S})))$ is neutrosophic-pre-open in $\mathscr{R}$ for any NOs $\tilde{S}$ of $\mathscr{S}$;

(vi) $F^{+}(NCl(NInt(F)))$ is neutrosophic-pre-closed in $\mathscr{R}$ for any NCs F of $\mathscr{S}$;

(vii) $F^{-}(G)$ is neutrosophic-per-open in $\mathscr{R}$ for any $NROs$ G of $\mathscr{S}$;

(viii) $F^{+}(H)$ is neutrosophic-perclosed in $\mathscr{R}$ for any $NRCs$ H of $\mathscr{S}$;

(ix) $pNCl(F^{+}(NCl(NInt(B)))) \subseteq F^{+}(NCl(NInt(NCl(B))))$ for every subset B of $\mathscr{S}$;

(x) $F^{-}(NInt(NCl(NInt(B)))) \subseteq pNInt(F^{-}(NInt(NCl(B))))$ for every subset B of $\mathscr{S}$.

Proof. It is similar to that of Theoremark 4.2. □

5 Conclusions and/or Discussions

Topology on lattice is a type of theory developed on lattice which involves many problems on ordered structure. For instance, complete distributivity of lattices is a pure algebraic problem that establishes a connection between algebra and analysis. neutrosophic topology is a generalization of fuzzy topology in classical mathematics, but it also has its own marked characteristics. Some scholars used tools for examining neutrosophic topological spaces and establishing new types from existing ones. Attention has been paid to define and characterize new weak forms of continuity.

We have introduced neutrosophic upper and neutrosophic lower almost pre-continuous-multifunctions as a generalization of neutrosophic multifunctions over neutrosophic topology space. Many results have been established to show how far topological structures are preserved by these neutrosophic upper pre-continuous (resp. neutrosophic lower pre-continuous). We also have provided examples where such properties fail to be preserved. In this paper we have introduced the concept of upper and lower pre-continuous multifunction and study some properties of these functions together with the graph of upper and lower pre-continuous as well as upper and lower weakly pre-continuous multifunction.

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