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NEUTROSOPHIC QUADRUPLE BCI-COMMUTATIVE IDEALS

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ABSTRACT. The notion of a neutrosophic quadruple BCI-commutative ideal in a neutrosophic quadruple BCI-algebra is introduced, and several properties are investigated. Relations between a neutrosophic quadruple ideal and a neutrosophic quadruple BCI-commutative ideal are discussed, and conditions for the neutrosophic quadruple ideal to be a neutrosophic quadruple BCI-commutative ideal are given. Conditions for the neutrosophic quadruple set to be a neutrosophic quadruple BCI-commutative ideal are provided, and the extension property of a neutrosophic quadruple BCI-commutative ideal is considered.

1. INTRODUCTION

Smarandache [22, 23, 24] have introduced the notion of neutrosophic sets which are a more general platform that extends the notions of classic set, (intuitionistic) fuzzy set and interval valued (intuitionistic) fuzzy set. Jun and his colleagues have considered the application of neutrosophic set in BCK/BCI-algebras [3, 4, 5, 6, 8, 9, 10, 11, 13, 14, 18, 21, 26].

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In neutrosophic logic, each proposition is approximated to represent respectively the truth (T), the falsehood (F), and the indeterminacy (I), whereT, I, F are standard or non-standard subsets of the non-standard unit interval $]0^-, 1^+[= 0^- \cup [0, 1] \cup 1^+$. The notion of neutrosophic quadruple number, which is represented by a known part and an unknown part to describe a neutrosophic logic proposition, was introduced by Florentin Smarandache in [25]. The algebra system $(NQ, *)$ based on neutrosophic quadruple numbers are introduced and the properties have discussed [25].

Neutrosophic quadruple algebraic structures and hyperstructures have been discussed in [1, 2]. Jun and his colleagues have studied neutrosophic quadruple BCK/BCI-algebraic structures [12, 19, 20].

In this paper, we introduce the notion of a neutrosophic quadruple BCI-commutative ideal in a neutrosophic quadruple BCI-algebra, and investigate several properties. We discuss relations between a neutrosophic quadruple ideal and a neutrosophic quadruple BCI-commutative ideal. We give conditions for the neutrosophic quadruple ideal to be a neutrosophic quadruple BCI-commutative ideal. We provide conditions for the neutrosophic quadruple set to be a neutrosophic quadruple BCI-commutative ideal. We consider the extension property of a neutrosophic quadruple BCI-commutative ideal.

2. Preliminaries

A *BCI-algebra* is a set X with a special element 0 and a binary operation "$*$" that satisfies the following conditions:

(I) $(\forall x, y, z \in X)\ (((x * y) * (x * z)) * (z * y) = 0)$,

(II) $(\forall x, y \in X)\ ((x * (x * y)) * y = 0)$,

(III) $(\forall x \in X)\ (x * x = 0)$,

(IV) $(\forall x, y \in X)\ (x * y = 0,\ y * x = 0\ \Rightarrow\ x = y)$.

If a BCI-algebra X satisfies the following identity:

(V) $(\forall x \in X)\ (0 * x = 0)$,

then X is called a *BCK-algebra*. Any BCK/BCI-algebra X satisfies the following conditions:

$$(\forall x \in X)\,(x * 0 = x)\,, \tag{1}$$

$$(\forall x, y, z \in X)\,(x \leq y\ \Rightarrow\ x * z \leq y * z,\ z * y \leq z * x)\,, \tag{2}$$

$$(\forall x, y, z \in X)\,((x * y) * z = (x * z) * y)\,, \tag{3}$$

$$(\forall x, y, z \in X)\,((x * z) * (y * z) \leq x * y) \tag{4}$$

where $x \leq y$ if and only if $x * y = 0$.

Any BCI-algebra X satisfies the following conditions (see [7]):

$$(\forall x, y \in X)(x * (x * (x * y)) = x * y), \tag{5}$$

$$(\forall x, y \in X)(0 * (x * y) = (0 * x) * (0 * y)), \tag{6}$$

$$(\forall x, y \in X)(0 * (0 * (x * y)) = (0 * y) * (0 * x)). \tag{7}$$

A nonempty subset S of a BCK/BCI-algebra X is called a *subalgebra* of X if $x * y \in S$ for all $x, y \in S$. A subset G of a BCK/BCI-algebra X is called an *ideal* of X if it satisfies:

$$0 \in G, \tag{8}$$

$$(\forall x \in X)\,(\forall y \in G)\,(x * y \in G \;\Rightarrow\; x \in G)\,. \tag{9}$$

Any ideal G of a BCK/BCI-algebra X satisfies:

$$(\forall x, y \in X)(y \leq x,\ x \in G \;\Rightarrow\; y \in G). \tag{10}$$

A subset G of a BCI-algebra X is called

- a *closed ideal* of X (see [7]) if it is an ideal of X which satisfies:

$$(\forall x \in X)(x \in G \;\Rightarrow\; 0 * x \in G). \tag{11}$$

- a *BCI-commutative ideal* of X (see [16]) if it satisfies (8) and

$$(\forall x, y, z \in X)\left(\begin{array}{l} (x * y) * z \in G,\ z \in G \\ \;\Rightarrow\; x * ((y * (y * x)) * (0 * (0 * (x * y)))) \in G \end{array}\right), \tag{12}$$

We refer the reader to the books [7, 17] for further information regarding BCK/BCI-algebras, and to the site "http://fs.gallup.unm.edu/neutrosophy.htm" for further information regarding neutrosophic set theory.

We consider neutrosophic quadruple numbers based on a set instead of real or complex numbers.

Let X be a set. A *neutrosophic quadruple X-number* is an ordered quadruple (a, xT, yI, zF) where $a, x, y, z \in X$ and T, I, F have their usual neutrosophic logic meanings (see [12]).

The set of all neutrosophic quadruple X-numbers is denoted by $\mathcal{N}(X)$, that is,

$$\mathcal{N}(X) := \{(a, xT, yI, zF) \mid a, x, y, z \in X\},$$

and it is called the *neutrosophic quadruple set* based on X. If X is a BCK/BCI-algebra, a neutrosophic quadruple X-number is called a *neutrosophic quadruple BCK/BCI-number* and we say that $\mathcal{N}(X)$ is the *neutrosophic quadruple BCK/BCI-set*.

Given $x_1, x_2, x_3, x_4 \in X$, the neutrosophic quadruple BCK/BCI-number (x_1, x_2T, x_3I, x_4F) is denoted by $\widetilde{x}$, that is,

$$\widetilde{x} = (x_1, x_2T, x_3I, x_4F),$$

and the zero neutrosophic quadruple BCK/BCI-number $(0, 0T, 0I, 0F)$ is denoted by $\widetilde{0}$, that is,

$$\widetilde{0} = (0, 0T, 0I, 0F).$$

Then $(\mathcal{N}(X); \boxdot, \widetilde{0})$ is a BCK/BCI-algebra (see [12]), which is called *neutrosophic quadruple BCK/BCI-algebra*, and it is simply denoted by $\mathcal{N}(X)$.

We define an order relation "$\ll$" and the equality "$=$" on the neutrosophic quadruple BCK/BCI-algebra $\mathcal{N}(X)$ as follows:

$$\widetilde{x} \ll \widetilde{y} \Leftrightarrow x_i \leq y_i \text{ for } i = 1, 2, 3, 4,$$
$$\widetilde{x} = \widetilde{y} \Leftrightarrow x_i = y_i \text{ for } i = 1, 2, 3, 4$$

for all $\widetilde{x}$, $\widetilde{y} \in \mathcal{N}(X)$. It is easy to verify that "$\ll$" is an equivalence relation on $\mathcal{N}(X)$.

Let X be a BCK/BCI-algebra. Given nonempty subsets J and K of X, consider the set

$$\mathcal{N}(J, K) := \{(a, xT, yI, zF) \in \mathcal{N}(X) \mid a, x \in J \ \&\ y, z \in K\},$$

which is called the neutrosophic quadruple (J, K)-set.

The neutrosophic quadruple (J, K)-set with $J = K$ is called the neutrosophic quadruple J-set, and is denoted by $\mathcal{N}(J)$.

Lemma 2.1 ([15, 17])**.** *A nonempty subset G of X is an ideal of X if and only if it satisfies* (8) *and*

$$(\forall x, y, z \in X)(y, z \in G,\ x * y \leq z \ \Rightarrow\ x \in G). \tag{13}$$

Lemma 2.2 ([12])**.** *If J and K are (closed) ideals of X, then the neutrosophic quadruple (J, K)-set is a neutrosophic quadruple (closed) ideal of $\mathcal{N}(X)$.*

3. Neutrosophic quadruple BCI-commutative ideals

In what follows, let X and $\mathcal{N}(X)$ be a BCI-algebra and a neutrosophic quadruple BCI-algebra, respectively, unless otherwise specified. Let $\mathcal{P}^*(X)$ be the class of all nonempty subsets of X.

Definition 3.1. Given $J, K \in \mathcal{P}^*(X)$, if the neutrosophic quadruple (J, K)-set is a BCI-commutative ideal of $\mathcal{N}(X)$, we say $\mathcal{N}(J, K)$ is a *neutrosophic quadruple BCI-commutative ideal* of $\mathcal{N}(X)$.

Example 3.2. Consider a BCI-algebra $(\mathbb{Z}, -, 0)$ where $\mathbb{Z}$ is the set of integers. Let $J = \{x \in \mathbb{Z} \mid x \geq 0\}$ and $K = \{x \in \mathbb{Z} \mid x \leq 0\}$ be subsets of $\mathbb{Z}$. Then $\mathcal{N}(J,K)$ is a neutrosophic quadruple BCI-commutative ideal of $\mathcal{N}(\mathbb{Z})$.

Example 3.3. Let $X = \{0, 1, 2, 3, 4\}$ be a set and we define a binary operation "$*$" by Table 1.

Table 1. Cayley table for the binary operation "$*$"

$*$	0	1	2	3	4
0	0	0	0	0	4
1	1	0	0	0	4
2	2	2	0	0	4
3	3	3	2	0	4
4	4	4	4	4	0

Then X is a proper BCI-algebra (see [16]), and the neutrosophic quadruple BCI-algebra $\mathcal{N}(X)$ has 625 elements. If we take $J = \{0, 1\}$ and $K = \{0, 1\}$, then the neutrosophic quadruple (J,K)-set is given as follows:

$$\mathcal{N}(J,K) = \{\widetilde{0}, \widetilde{\beta_1}, \widetilde{\beta_2}, \widetilde{\beta_3}, \widetilde{\beta_4}, \widetilde{\beta_5}, \widetilde{\beta_6}, \widetilde{\beta_7}, \widetilde{\beta_8}, \widetilde{\beta_9}, \widetilde{\beta_{10}}, \widetilde{\beta_{11}}, \widetilde{\beta_{12}}, \widetilde{\beta_{13}}, \widetilde{\beta_{14}}, \widetilde{\beta_{15}}\}$$

and it is a neutrosophic quadruple BCI-commutative ideal of $\mathcal{N}(X)$ where

$$\begin{aligned}
&\widetilde{0} = (0, 0T, 0I, 0F)\ ,\ \widetilde{\beta_1} = (0, 0T, 0I, 1F)\ ,\ \widetilde{\beta_2} = (0, 0T, 1I, 0F) \\
&\widetilde{\beta_3} = (0, 0T, 1I, 1F)\ ,\ \widetilde{\beta_4} = (0, 1T, 0I, 0F)\ ,\ \widetilde{\beta_5} = (0, 1T, 0I, 1F) \\
&\widetilde{\beta_6} = (0, 1T, 1I, 0F)\ ,\ \widetilde{\beta_7} = (0, 1T, 1I, 1F)\ ,\ \widetilde{\beta_8} = (1, 0T, 0I, 0F) \\
&\widetilde{\beta_9} = (1, 0T, 0I, 1F)\ ,\ \widetilde{\beta_{10}} = (1, 0T, 1I, 0F)\ ,\ \widetilde{\beta_{11}} = (1, 0T, 1I, 1F) \\
&\widetilde{\beta_{12}} = (1, 1T, 0I, 0F)\ ,\ \widetilde{\beta_{13}} = (1, 1T, 0I, 1F) \\
&\widetilde{\beta_{14}} = (1, 1T, 1I, 0F)\ ,\ \widetilde{\beta_{15}} = (1, 1T, 1I, 1F).
\end{aligned}$$

Theorem 3.4. *For any $J, K \in \mathcal{P}^*(X)$, if J and K are BCI-commutative ideals of X, then the neutrosophic quadruple (J,K)-set is a neutrosophic quadruple BCI-commutative ideal of $\mathcal{N}(X)$.*

Proof. Assume that J and K are BCI-commutative ideals of X. Since $0 \in J \cap K$, it is clear that $\widetilde{0} = (0, 0T, 0I, 0F) \in \mathcal{N}(J,K)$. Let $\widetilde{x}, \widetilde{y}, \widetilde{z} \in \mathcal{N}(X)$ be such that $\widetilde{z} \in \mathcal{N}(J,K)$ and

$(\widetilde{x} \boxdot \widetilde{y}) \boxdot \widetilde{z} \in \mathcal{N}(J,K)$. Then

$$\widetilde{z} = (z_1, z_2T, z_3I, z_4F) \in \mathcal{N}(J,K),$$

and so $z_1, z_2 \in J$ and $z_3, z_4 \in K$. Also

$$\begin{aligned}(\widetilde{x} \boxdot \widetilde{y}) \boxdot \widetilde{z} &= ((x_1, x_2T, x_3I, x_4F) \boxdot (y_1, y_2T, y_3I, y_4F)) \boxdot (z_1, z_2T, z_3I, z_4F) \\ &= ((x_1 * y_1) * z_1, ((x_2 * y_2) * z_2)T, ((x_3 * y_3) * z_3)I, ((x_4 * y_4) * z_4)F) \\ &\in \mathcal{N}(J,K),\end{aligned}$$

and thus $(x_i * y_i) * z_i \in J$ and $(x_j * y_j) * z_j \in K$ for $i = 1,2$ and $j = 3,4$. Since J and K are BCI-commutative ideals of X, it follows that

$$x_i * ((y_i * (y_i * x_i)) * (0 * (0 * (x_i * y_i)))) \in J$$

and

$$x_j * ((y_j * (y_j * x_j)) * (0 * (0 * (x_j * y_j)))) \in K$$

for $i = 1,2$ and $j = 3,4$. Therefore

$$\begin{aligned}&\widetilde{x} \boxdot ((\widetilde{y} \boxdot (\widetilde{y} \boxdot \widetilde{x})) \boxdot (\widetilde{0} \boxdot (\widetilde{0} \boxdot (\widetilde{x} \boxdot \widetilde{y})))) \\ = \;& (x_1, x_2T, x_3I, x_4F) \boxdot (((y_1, y_2T, y_3I, y_4F) \boxdot ((y_1, y_2T, y_3I, y_4F) \boxdot \\ & (x_1, x_2T, x_3I, x_4F))) \boxdot ((0, 0T, 0I, 0F) \boxdot ((0, 0T, 0I, 0F) \boxdot \\ & ((x_1, x_2T, x_3I, x_4F) \boxdot (y_1, y_2T, y_3I, y_4F))))) \\ = \;& (x_1 * ((y_1 * (y_1 * x_1)) * (0 * (0 * (x_1 * y_1)))), \\ & (x_2 * ((y_2 * (y_2 * x_2)) * (0 * (0 * (x_2 * y_2)))))T, \\ & (x_3 * ((y_3 * (y_3 * x_3)) * (0 * (0 * (x_3 * y_3)))))I, \\ & (x_4 * ((y_4 * (y_4 * x_4)) * (0 * (0 * (x_4 * y_4)))))F) \\ \in \;& \mathcal{N}(J,K).\end{aligned}$$

Therefore $\mathcal{N}(J,K)$ is a neutrosophic quadruple BCI-commutative ideal of $\mathcal{N}(X)$. □

Proposition 3.5. *For any $J, K \in \mathcal{P}^*(X)$, every neutrosophic quadruple BCI-commutative ideal $\mathcal{N}(J,K)$ of $\mathcal{N}(X)$ satisfies the following implication.*

$$\widetilde{x} \boxdot \widetilde{y} \in \mathcal{N}(J,K) \;\Rightarrow\; \widetilde{x} \boxdot ((\widetilde{y} \boxdot (\widetilde{y} \boxdot \widetilde{x})) \boxdot (\widetilde{0} \boxdot (\widetilde{0} \boxdot (\widetilde{x} \boxdot \widetilde{y})))) \in \mathcal{N}(J,K) \tag{14}$$

for all $\widetilde{x}, \widetilde{y} \in \mathcal{N}(X)$.

Proof. Let $\widetilde{x}, \widetilde{y} \in \mathcal{N}(X)$ be such that $\widetilde{x} \boxdot \widetilde{y} \in \mathcal{N}(J,K)$. Then $\widetilde{0} \in \mathcal{N}(J,K)$ and

$$\begin{aligned}
(\widetilde{x} \boxdot \widetilde{y}) \boxdot \widetilde{0} &= ((x_1, x_2T, x_3I, x_4F) \boxdot (y_1, y_2T, y_3I, y_4F)) \boxdot (0, 0T, 0I, 0F) \\
&= ((x_1 * y_1) * 0, ((x_2 * y_2) * 0)T, ((x_3 * y_3) * 0)I, ((x_4 * y_4) * 0)F) \\
&= (x_1 * y_1), (x_2 * y_2)T, (x_3 * y_3)I, (x_4 * y_4)F) \\
&= \widetilde{x} \boxdot \widetilde{y} \\
&\in \mathcal{N}(J,K).
\end{aligned}$$

Therefore $\widetilde{x} \boxdot ((\widetilde{y} \boxdot (\widetilde{y} \boxdot \widetilde{x})) \boxdot (\widetilde{0} \boxdot (\widetilde{0} \boxdot (\widetilde{x} \boxdot \widetilde{y})))) \in \mathcal{N}(J,K)$. □

Corollary 3.6. *If J and K are BCI-commutative ideals of X, then the neutrosophic quadruple (J,K)-set satisfies the condition* (14).

Proposition 3.7. *For any $J, K \in \mathcal{P}^*(X)$, every neutrosophic quadruple BCI-commutative and closed ideal $\mathcal{N}(J,K)$ of $\mathcal{N}(X)$ satisfies the following implication.*

$$\widetilde{x} \boxdot \widetilde{y} \in \mathcal{N}(J,K) \;\Rightarrow\; \widetilde{x} \boxdot (\widetilde{y} \boxdot (\widetilde{y} \boxdot \widetilde{x})) \in \mathcal{N}(J,K) \tag{15}$$

for all $\widetilde{x}, \widetilde{y} \in \mathcal{N}(X)$.

Proof. Assume that $\mathcal{N}(J,K)$ is a neutrosophic quadruple BCI-commutative and closed ideal of $\mathcal{N}(X)$. Let $\widetilde{x}, \widetilde{y} \in \mathcal{N}(X)$ be such that $\widetilde{x} \boxdot \widetilde{y} \in \mathcal{N}(J,K)$. Then $\widetilde{0} \boxdot (\widetilde{x} \boxdot \widetilde{y}) \in \mathcal{N}(J,K)$ and

$$\widetilde{x} \boxdot ((\widetilde{y} \boxdot (\widetilde{y} \boxdot \widetilde{x})) \boxdot (\widetilde{0} \boxdot (\widetilde{0} \boxdot (\widetilde{x} \boxdot \widetilde{y})))) \in \mathcal{N}(J,K).$$

Since

$$\begin{aligned}
&(\widetilde{x} \boxdot (\widetilde{y} \boxdot (\widetilde{y} \boxdot \widetilde{x}))) \boxdot (\widetilde{x} \boxdot ((\widetilde{y} \boxdot (\widetilde{y} \boxdot \widetilde{x})) \boxdot (\widetilde{0} \boxdot (\widetilde{0} \boxdot (\widetilde{x} \boxdot \widetilde{y}))))) \\
&\ll ((\widetilde{y} \boxdot (\widetilde{y} \boxdot \widetilde{x})) \boxdot (\widetilde{0} \boxdot (\widetilde{0} \boxdot (\widetilde{x} \boxdot \widetilde{y})))) \boxdot (\widetilde{y} \boxdot (\widetilde{y} \boxdot \widetilde{x})) \\
&= ((\widetilde{y} \boxdot (\widetilde{y} \boxdot \widetilde{x})) \boxdot (\widetilde{y} \boxdot (\widetilde{y} \boxdot \widetilde{x}))) \boxdot (\widetilde{0} \boxdot (\widetilde{0} \boxdot (\widetilde{x} \boxdot \widetilde{y}))) \\
&= \widetilde{0} \boxdot (\widetilde{0} \boxdot (\widetilde{0} \boxdot (\widetilde{x} \boxdot \widetilde{y}))) \\
&= \widetilde{0} \boxdot (\widetilde{x} \boxdot \widetilde{y}).
\end{aligned}$$

It follows from Lemma 2.1 that $\widetilde{x} \boxdot (\widetilde{y} \boxdot (\widetilde{y} \boxdot \widetilde{x})) \in \mathcal{N}(J,K)$. □

Corollary 3.8. *If J and K are BCI-commutative and closed ideals of X, then the neutrosophic quadruple (J,K)-set satisfies the condition* (15).

Theorem 3.9. *For any $J, K \in \mathcal{P}^*(X)$, if the neutrosophic quadruple (J, K)-set is a neutrosophic quadruple BCI-commutative ideal of $\mathcal{N}(X)$, then it is a neutrosophic quadruple ideal of $\mathcal{N}(X)$.*

Proof. Assume that $\mathcal{N}(J, K)$ is a neutrosophic quadruple BCI-commutative ideal of $\mathcal{N}(X)$. Then $\widetilde{0} = (0, 0T, 0I, 0F) \in \mathcal{N}(J, K)$. Assume that $(z_1, z_2T, z_3I, z_4F) \in \mathcal{N}(J, K)$ and

$$(x_1, x_2T, x_3I, x_4F) \boxdot (z_1, z_2T, z_3I, z_4F) \in \mathcal{N}(J, K).$$

for any elements (x_1, x_2T, x_3I, x_4F) and (z_1, z_2T, z_3I, z_4F) of $\mathcal{N}(X)$. Then

$$((x_1, x_2T, x_3I, x_4F) \boxdot (0, 0T, 0I, 0F)) \boxdot (z_1, z_2T, z_3I, z_4F) \in \mathcal{N}(J, K).$$

It follows that

$$\begin{aligned}
(x_1, x_2T, x_3I, x_4F) &= (x_1 * ((0 * (0 * x_1)) * (0 * (0 * (x_1 * 0)))), \\
&\quad (x_2 * ((0 * (0 * x_2)) * (0 * (0 * (x_2 * 0)))))T, \\
&\quad (x_3 * ((0 * (0 * x_3)) * (0 * (0 * (x_3 * 0)))))I, \\
&\quad (x_4 * ((0 * (0 * x_4)) * (0 * (0 * (x_4 * 0)))))F) \\
&= (x_1, x_2T, x_3I, x_4F) \boxdot (((0, 0T, 0I, 0F) \boxdot ((0, 0T, 0I, 0F) \boxdot \\
&\quad (x_1, x_2T, x_3I, x_4F))) \boxdot ((0, 0T, 0I, 0F) \boxdot ((0, 0T, 0I, 0F) \boxdot \\
&\quad ((x_1, x_2T, x_3I, x_4F) \boxdot (0, 0T, 0I, 0F))))) \\
&\in \mathcal{N}(J, K).
\end{aligned}$$

Therefore $\mathcal{N}(J, K)$ is an ideal of $\mathcal{N}(X)$. □

Corollary 3.10. *If J and K are BCI-commutative ideals of X, then the neutrosophic quadruple (J, K)-set is a neutrosophic quadruple ideal.*

The converse of Theorem 3.9 is not true, as seen in the following example.

Example 3.11. Let $X = \{0, 1, 2, 3, 4\}$ be a set and we define a binary operation "$*$" by Table 2.

Then X is a BCI-algebra (see [17]), and the neutrosophic quadruple BCI-algebra $\mathcal{N}(X)$ has 625 elements. If we take $J = \{0, 1\}$ and $K = \{0, 2\}$, then the neutrosophic quadruple (J, K)-set is given as follows:

$$\mathcal{N}(J, K) = \{\widetilde{0}, \widetilde{\beta}_i \mid i = 1, 2, \cdots, 15\}$$

Table 2. Cayley table for the binary operation "$*$"

$*$	0	1	2	3	4
0	0	0	0	0	0
1	1	0	1	0	0
2	2	2	0	0	0
3	3	3	3	0	0
4	4	4	4	3	0

where

$$\widetilde{0} = (0, 0T, 0I, 0F)\ ,\ \widetilde{\beta}_1 = (0, 0T, 0I, 2F)\ ,\ \widetilde{\beta}_2 = (0, 0T, 2I, 0F)$$
$$\widetilde{\beta}_3 = (0, 0T, 2I, 2F)\ ,\ \widetilde{\beta}_4 = (0, 1T, 0I, 0F)\ ,\ \widetilde{\beta}_5 = (0, 1T, 0I, 2F)$$
$$\widetilde{\beta}_6 = (0, 1T, 2I, 0F)\ ,\ \widetilde{\beta}_7 = (0, 1T, 2I, 2F)\ ,\ \widetilde{\beta}_8 = (1, 0T, 0I, 0F)$$
$$\widetilde{\beta}_9 = (1, 0T, 0I, 2F)\ ,\ \widetilde{\beta}_{10} = (1, 0T, 2I, 0F)\ ,\ \widetilde{\beta}_{11} = (1, 0T, 2I, 2F)$$
$$\widetilde{\beta}_{12} = (1, 1T, 0I, 0F)\ ,\ \widetilde{\beta}_{13} = (1, 1T, 0I, 2F)$$
$$\widetilde{\beta}_{14} = (1, 1T, 2I, 0F)\ ,\ \widetilde{\beta}_{15} = (1, 1T, 2I, 2F)$$

It is routine to verify that $\mathcal{N}(J, K)$ is a neutrosophic quadruple ideal of $\mathcal{N}(X)$. If we take $\widetilde{\gamma}_1 = (1, 2T, 3I, 3F)$ and $\widetilde{\gamma}_2 = (2, 3T, 4I, 3F)$ in $\mathcal{N}(X)$, then

$$\begin{aligned}(\widetilde{\gamma}_1 \boxdot \widetilde{\gamma}_2) \boxdot \widetilde{\beta}_7 &= ((1, 2T, 3I, 3F) \boxdot (2, 3T, 4I, 3F)) \boxdot (0, 1T, 2I, 2F)\\ &= ((1 * 2) * 0, (2 * 3) * 1T, (3 * 4) * 2I, (3 * 3) * 2F)\\ &= (1, 0T, 0I, 0F) = \widetilde{\beta}_8 \in \mathcal{N}(J, K),\end{aligned}$$

but

$$\begin{aligned}&\widetilde{\gamma}_1 \boxdot ((\widetilde{\gamma}_2 \boxdot (\widetilde{\gamma}_2 \boxdot \widetilde{\gamma}_1)) \boxdot (\widetilde{0} \boxdot (\widetilde{0} \boxdot (\widetilde{\gamma}_1 \boxdot \widetilde{\gamma}_2))))\\ = \;&(1, 2T, 3I, 3F) \boxdot (((2, 3T, 4I, 3F) \boxdot ((2, 3T, 4I, 3F) \boxdot (1, 2T, 3I, 3F))) \boxdot\\ &((0, 0T, 0I, 0F) \boxdot ((0, 0T, 0I, 0F) \boxdot ((1, 2T, 3I, 3F) \boxdot (2, 3T, 4I, 3F)))))\\ = \;&(0, 0T, 3I, 3F) \notin \mathcal{N}(J, K).\end{aligned}$$

Therefore $\mathcal{N}(J, K)$ is not a neutrosophic quadruple BCI-commutative ideal of $\mathcal{N}(X)$.

We provide conditions for a neutrosophic quadruple ideal to be a neutrosophic quadruple BCI-commutative ideal.

Theorem 3.12. *Let J and K be ideals of X which satisfies the following assertion:*

$$x * ((y * (y * x)) * (0 * (0 * (x * y)))) \in J \text{ (resp., } K) \tag{16}$$

for all $x, y \in X$ with $x * y \in J$ (resp., K). Then the neutrosophic quadruple (J, K)-set is a neutrosophic quadruple BCI-commutative ideal of $\mathcal{N}(X)$.

Proof. Let $x, y, z \in X$ be such that $z \in J$ (resp., K) and $(x * y) * z \in J$ (resp., K). Since J and K are ideals of X, it follows that $x * y \in J$ (resp., K). Hence the condition (16) is valid by hypothesis. This shows that J and K are BCI-commutative ideals of X. Therefore, $\mathcal{N}(J, K)$ is a neutrosophic quadruple BCI-commutative ideal of $\mathcal{N}(X)$ by Theorem 3.4. □

The converse of Theorem 3.12, is not true in general, as seen in the following example.

Example 3.13. in Example 3.11, Let $J = \{0, 1\}$ and $K = \{0, 2\}$. We can see that, for $x = 1$ and $y = 2$,

$$x * ((y * (*x)) * (0 * (x * y))) = 1 \notin K, x * y = 1 \notin K$$

Moreover, $\mathcal{N}(J, K)$ is not a neutrosophic quadruple BCI-commutative ideal of $\mathcal{N}(X)$.

Corollary 3.14. *For any ideals J and K of X, if the neutrosophic quadruple (J, K)-set satisfies*

$$\widetilde{x} \boxdot ((\widetilde{y} \boxdot (\widetilde{y} \boxdot \widetilde{x})) \boxdot (\widetilde{0} \boxdot (\widetilde{0} \boxdot (\widetilde{x} \boxdot \widetilde{y})))) \in \mathcal{N}(J, K) \tag{17}$$

for all $\widetilde{x}$, $\widetilde{y} \in \mathcal{N}(X)$ with $\widetilde{x} \boxdot \widetilde{y} \in \mathcal{N}(J, K)$, then the neutrosophic quadruple (J, K)-set is a neutrosophic quadruple BCI-commutative ideal of $\mathcal{N}(X)$.

Theorem 3.15. *Let J and K be ideals of X, which satisfies the following assertion:*

$$x * y \in J \text{ (resp., } K) \;\Rightarrow x * (y * (y * x)) \in J \text{ (resp., } K) \tag{18}$$

for all $x, y \in X$. Then the neutrosophic quadruple (J, K)-set is a neutrosophic quadruple BCI-commutative ideal of $\mathcal{N}(X)$.

Proof. Let $x, y, z \in X$ be such that $z \in J$ (resp., K) and $(x * y) * z \in J$ (resp., K). Since J and K are ideals of X, it follows that $x * y \in J$ (resp., K). Hence $x * (y * (y * x)) \in J$ (resp., K) by (18). Since $0 * (0 * (x * y)) \leq x * y$ for all $x, y \in X$, we have $0 * (0 * (x * y)) \in J$ (resp., K) by (10). Note that

$$\begin{aligned} & (x * ((y * (y * x)) * (0 * (0 * (x * y))))) * (x * (y * (y * x))) \\ \leq\; & (y * (y * x)) * ((y * (y * x)) * (0 * (0 * (x * y)))) \\ \leq\; & 0 * (0 * (x * y)) \end{aligned}$$

for all $x, y \in X$ by (I) and (II). It follows from Lemma 2.1 that

$$x * ((y * (y * x)) * (0 * (0 * (x * y)))) \in J \text{ (resp., } K).$$

Hence J and K are BCI-commutative ideals of X, and therefore $\mathcal{N}(J,K)$ is a neutrosophic quadruple BCI-commutative ideal of $\mathcal{N}(X)$ by Theorem 3.4. □

The converse of Theorem 3.15, is not true in general, as seen in the following example.

Example 3.16. Consider Example 3.11. For $x = 1$ and $y = 2$, we can see that $1 * 2 = 1 \in \{0,1\} = J$ but $1 * (2 * (2 * 1)) = 1 \notin \{0,2\} = K$ and $\mathcal{N}(J,K)$ is not a neutrosophic quadruple BCI-commutative ideal of $\mathcal{N}(X)$.

Theorem 3.17. *For any $J, K \in \mathcal{P}^*(X)$, if the neutrosophic quadruple (J,K)-set is a neutrosophic quadruple ideal of $\mathcal{N}(X)$ that satisfies the following assertion:*

$$\widetilde{x} \boxdot \widetilde{y} \in \mathcal{N}(J,K) \ \Rightarrow \widetilde{x} \boxdot (\widetilde{y} \boxdot (\widetilde{y} \boxdot \widetilde{x})) \in \mathcal{N}(J,K) \tag{19}$$

for all $\widetilde{x}, \widetilde{y} \in \mathcal{N}(X)$, then the neutrosophic quadruple (J,K)-set is a neutrosophic quadruple BCI-commutative ideal of $\mathcal{N}(X)$.

Proof. Let $\mathcal{N}(J,K)$ be a neutrosophic quadruple ideal of $\mathcal{N}(X)$ which satisfies the condition (19). Let $\widetilde{x}$, $\widetilde{y}$, $\widetilde{z} \in \mathcal{N}(X)$ be such that $(\widetilde{x} \boxdot \widetilde{y}) \boxdot \widetilde{z} \in \mathcal{N}(J,K)$ and $\widetilde{z} \in \mathcal{N}(J,K)$. Then $\widetilde{x} \boxdot \widetilde{y} \in \mathcal{N}(X)$ since $\mathcal{N}(J,K)$ is a neutrosophic quadruple ideal of $\mathcal{N}(X)$, and thus $\widetilde{x} \boxdot (\widetilde{y} \boxdot (\widetilde{y} \boxdot \widetilde{x})) \in \mathcal{N}(J,K)$ by (19). Also, we have

$$\begin{aligned}&((\widetilde{x} \boxdot ((\widetilde{y} \boxdot (\widetilde{y} \boxdot \widetilde{x})) \boxdot (\widetilde{0} \boxdot (\widetilde{0} \boxdot (\widetilde{x} \boxdot \widetilde{y}))))) \boxdot (\widetilde{x} \boxdot (\widetilde{y} \boxdot (\widetilde{y} \boxdot \widetilde{x}))))\boxdot \\ &(\widetilde{0} \boxdot (\widetilde{0} \boxdot (\widetilde{x} \boxdot \widetilde{y}))) = \widetilde{0} \in \mathcal{N}(J,K)\end{aligned}$$

and $\widetilde{0} \boxdot (\widetilde{0} \boxdot (\widetilde{x} \boxdot \widetilde{y})) \in \mathcal{N}(J,K)$ since $\widetilde{0} \boxdot (\widetilde{0} \boxdot (\widetilde{x} \boxdot \widetilde{y})) \ll \widetilde{x} \boxdot \widetilde{y}$. It follows that

$$\widetilde{x} \boxdot ((\widetilde{y} \boxdot (\widetilde{y} \boxdot \widetilde{x})) \boxdot (\widetilde{0} \boxdot (\widetilde{0} \boxdot (\widetilde{x} \boxdot \widetilde{y})))) \in \mathcal{N}(J,K).$$

Hence $\mathcal{N}(J,K)$ is a neutrosophic quadruple BCI-commutative ideal of $\mathcal{N}(X)$. □

We consider the extension property of a neutrosophic quadruple BCI-commutative ideal.

Theorem 3.18. *For any J, K, G, $H \in \mathcal{P}^*(X)$ with $J \subseteq G$ and $K \subseteq H$, if J and K (resp., G and H) are BCI-commutative ideals (resp., closed ideals) of X, then the neutrosophic quadruple (G,H)-set is a neutrosophic quadruple BCI-commutative ideal of $\mathcal{N}(X)$ which contains the neutrosophic quadruple (J,K)-set.*

Proof. By Theorem 3.4 and Lemma 2.2, $\mathcal{N}(J,K)$ is a neutrosophic quadruple BCI-commutative ideal of $\mathcal{N}(X)$ and $\mathcal{N}(G,H)$ is a neutrosophic quadruple ideal of $\mathcal{N}(X)$, respectively. It is clear that $\mathcal{N}(J,K) \subseteq \mathcal{N}(G,H)$. Let $x, y \in X$ be such that $x * y \in G$ (resp., H). Then $0 * (x * y) \in G$ (resp., H) since G and H are closed ideals of X, and $((x * (x * y)) * y) * 0 = 0 \in J$ (resp., K). It follows from (1), (3) and (12) that

$$\begin{aligned}
& (x * ((y * (y * (x * (x * y)))))) * (x * y) \\
= \ & (x * (x * y)) * ((y * (y * (x * (x * y))))) \\
= \ & (x * (x * y)) * ((y * (y * (x * (x * y)))) * (0 * (0 * ((x * (x * y)) * y)))) \\
\in \ & J \text{ (resp., } K) \\
\subseteq \ & G \text{ (resp., } H).
\end{aligned}$$

Hence $x * ((y * (y * (x * (x * y))))) \in G$ (resp., H). On the other hand,

$$\begin{aligned}
& (x * (y * (y * x))) * (x * ((y * (y * (x * (x * y)))))) \\
\leq \ & (y * (y * (x * (x * y)))) * (y * (y * x)) \\
\leq \ & (y * x) * (y * (x * (x * y))) \\
\leq \ & (x * (x * y)) * x \\
= \ & 0 * (x * y) \\
\in \ & G \text{ (resp., } H).
\end{aligned}$$

Hence we have $x*(y*(y*x)) \in G$ (resp., H). Therefore the neutrosophic quadruple (G,H)-set is a neutrosophic quadruple BCI-commutative ideal of $\mathcal{N}(X)$ by Theorem 3.15. □

Theorem 3.19. *Given four ideals J, K, G and H of X with $J \subseteq G$ and $K \subseteq H$, assume that the neutrosophic quadruple (G,H)-set is a neutrosophic quadruple closed ideal of $\mathcal{N}(X)$. If the neutrosophic quadruple (J,K)-set is a neutrosophic quadruple BCI-commutative ideal of $\mathcal{N}(X)$, then so is the neutrosophic quadruple (G,H)-set and $\mathcal{N}(J,K) \subseteq \mathcal{N}(G,H)$.*

Proof. It is clear that $\mathcal{N}(J,K) \subseteq \mathcal{N}(G,H)$. Suppose that $\tilde{x} \boxdot \tilde{y} \in \mathcal{N}(G,H)$ for all $\tilde{x}, \tilde{y} \in \mathcal{N}(X)$. Then $\tilde{0} \boxdot (\tilde{x} \boxdot \tilde{y}) \in \mathcal{N}(G,H)$ and $(\tilde{x} \boxdot (\tilde{x} \boxdot \tilde{y})) \boxdot \tilde{y} = \tilde{0} \in \mathcal{N}(J,K)$. It follows from Proposition

3.5 that

$$\begin{aligned}
& (\widetilde{x} \boxdot (\widetilde{y} \boxdot (\widetilde{y} \boxdot (\widetilde{x} \boxdot (\widetilde{x} \boxdot \widetilde{y}))))) \boxdot (\widetilde{x} \boxdot \widetilde{y}) \\
= \ & (\widetilde{x} \boxdot (\widetilde{x} \boxdot \widetilde{y})) \boxdot (\widetilde{y} \boxdot (\widetilde{y} \boxdot (\widetilde{x} \boxdot (\widetilde{x} \boxdot \widetilde{y})))) \\
= \ & (\widetilde{x} \boxdot (\widetilde{x} \boxdot \widetilde{y})) \boxdot ((\widetilde{y} \boxdot (\widetilde{y} \boxdot (\widetilde{x} \boxdot (\widetilde{x} \boxdot \widetilde{y})))) \boxdot (0 \boxdot (0 \boxdot ((\widetilde{x} \boxdot (\widetilde{x} \boxdot \widetilde{y})) \boxdot \widetilde{y})))) \\
\in \ & \mathcal{N}(J, K) \\
\subseteq \ & \mathcal{N}(G, H).
\end{aligned}$$

Hence $\widetilde{x} \boxdot (\widetilde{y} \boxdot (\widetilde{y} \boxdot (\widetilde{x} \boxdot (\widetilde{x} \boxdot \widetilde{y})))) \in \mathcal{N}(G, H)$. Now, since

$$\begin{aligned}
& (\widetilde{x} \boxdot (\widetilde{y} \boxdot (\widetilde{y} \boxdot \widetilde{x}))) \boxdot (\widetilde{x} \boxdot (\widetilde{y} \boxdot (\widetilde{y} \boxdot (\widetilde{x} \boxdot (\widetilde{x} \boxdot \widetilde{y}))))) \\
\ll \ & (\widetilde{y} \boxdot (\widetilde{y} \boxdot (\widetilde{x} \boxdot (\widetilde{x} \boxdot \widetilde{y})))) \boxdot (\widetilde{y} \boxdot (\widetilde{y} \boxdot \widetilde{x})) \\
\ll \ & (\widetilde{y} \boxdot \widetilde{x}) \boxdot (\widetilde{y} \boxdot (\widetilde{x} \boxdot (\widetilde{x} \boxdot \widetilde{y}))) \\
\ll \ & (\widetilde{x} \boxdot (\widetilde{x} \boxdot \widetilde{y})) \boxdot \widetilde{x} \\
= \ & \widetilde{0} \boxdot (\widetilde{x} \boxdot \widetilde{y}) \\
\in \ & \mathcal{N}(G, H),
\end{aligned}$$

we have $\widetilde{x} \boxdot (\widetilde{y} \boxdot (\widetilde{y} \boxdot \widetilde{x})) \in \mathcal{N}(G, H)$. Using Theorem 3.17, we know that the neutrosophic quadruple (G, H)-set is a neutrosophic quadruple BCI-commutative ideal of $\mathcal{N}(X)$. $\square$

Corollary 3.20. *Given four ideals J, K, G and H of X with $J \subseteq G$ and $K \subseteq H$, if G and H are closed ideals of X and if the neutrosophic quadruple (J, K)-set satisfies the condition* (19)*, then the neutrosophic quadruple (G, H)-set is a neutrosophic quadruple BCI-commutative ideal of $\mathcal{N}(X)$ and $\mathcal{N}(J, K) \subseteq \mathcal{N}(G, H)$.*

4. Conclusion

We know that the notion of neutrosophic set is a more general platform that extend the notions of classic set, (intuitionistic) fuzzy set and interval valued (intuitionistic) fuzzy set. So, in this paper, we defined the notion of neutrosophic quadruple BCI-commutative ideal in a neutrosophic quadruple BCI-algebra. Then we find the relations between neutrosophic quadruple ideals and a neutrosophic quadruple BCI-commutative ideals. Finally, we provided some conditions that the neutrosophic quadruple set to be a neutrosophic quadruple BCI-commutative ideal. In the future, we can apply this structure to the other logical algebras such as BL-algebras and MV-algebras.

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