Neutrosophic sets applied to mighty filters in BE-algebras

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Abstract. The notion of a neutrosophic subalgebra of a BE-algebra is introduced and consider characterizations of a neutrosophic subalgebra and a neutrosophic filter. We defined the notion of a neutrosophic mighty filter of a BE-algebra, and investigated some properties of it. We provide conditions for a neutrosophic filter to be a neutrosophic mighty filter.

1. Introduction

In 2007, Kim and Kim [6] introduced the notion of a BE-algebra, and investigated several properties. In [1], Ahn and So introduced the notion of ideals in BE-algebras. They gave several descriptions of ideals in BE-algebras. Y. B. Jun et. al [4] introduced the notions of hesitant fuzzy subalgebras and hesitant fuzzy filters of BE-algebras and investigated their relations and properties. J. S. Han et. al [3] defined the notion of hesitant fuzzy implicative filter of a BE-algebra, and considered some properties of it.

Zadeh [11] introduced the degree of membership/truth (t) in 1965 and defined the fuzzy set. As a generalization of fuzzy sets, Atanassov [2] introduced the degree of nonmembership/falsehood (f) in 1986 and defined the intuitionistic fuzzy set. Smarandache introduced the degree of indeterminacy/neutrality (i) as independent component in 1995 (published in 1998) and defined the neutrosophic set on three components (t, i, f) = (truth, indeterminacy, falsehood). In 2015, neutrosophic set theory is applied to BE-algebra, and the notion of neutrosophic filter is introduced [9]. A new definition of neutrosopic filter is established and some basic properties are presented [12].

In this paper, we introduce the notion of a neutrosophic subalgebra of a BE-algebra and consider characterizations of a neutrosophic subalgebra and a neutrosophic filter. We defined the notion of a neutrosophic mighty filter of a BE-algebra, and investigated some properties of it. We provide conditions for a neutrosophic filter to be a neutrosophic mighty filter.

2. Preliminaries

By a *BE-algebra* ([6]) we mean a system $(X;*,1)$ of type $(2,0)$ which the following axioms hold:

(BE1) $(\forall x \in X)\,(x * x = 1)$,

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(BE2) $(\forall x \in X)\,(x * 1 = 1)$,

(BE3) $(\forall x \in X)\,(1 * x = x)$,

(BE4) $(\forall x, y, z \in X)\,(x * (y * z) = y * (x * z)$ (exchange).

We introduce a relation " $\leq$ " on X by $x \leq y$ if and only if $x * y = 1$.

A BE-algebra $(X; *, 1)$ is said to be *transitive* if it satisfies: for any $x, y, z \in X$, $y * z \leq (x * y) * (x * z)$. A BE-algebra $(X; *, 1)$ is said to be *self distributive* if it satisfies: for any $x, y, z \in X$, $x * (y * z) = (x * y) * (x * z)$. Note that every self distributive BE-algebra is transitive, but the converse is not true in general ([6]).

Every self distributive BE-algebra $(X; *, 1)$ satisfies the following properties:

(2.1) $(\forall x, y, z \in X)\,(x \leq y \Rightarrow z * x \leq z * y$ and $y * z \leq x * z)$,

(2.2) $(\forall x, y \in X)\,(x * (x * y) = x * y)$,

(2.3) $(\forall x, y, z \in X)\,(x * y \leq (z * x) * (z * y))$,

Definition 2.1. Let $(X; *, 1)$ be a BE-algebra and let F be a non-empty subset of X. Then F is a *filter* of X ([6]) if

(F1) $1 \in F$;

(F2) $(\forall x, y \in X)(x * y, x \in F \Rightarrow y \in F)$.

F is a *mighty filter* ([8]) of X if it satisfies (F1) and

(F3) $(\forall x, y, z \in X)(z * (y * x), z \in F \Rightarrow ((x * y) * y) * x \in F)$.

Theorem 2.2. ([8]) *A filter F of a BE-algebra X is mighty if and only if*

(2.4) $(\forall x, y \in X)(y * x \in F \Rightarrow ((x * y) * y) * x \in F)$.

Definition 2.3. Let X be a space of points (objects) with generic elements in X denoted by x. A simple valued neutrosophic set A in X is characterized by a truth-membership function $T_A(x)$, an indeterminacy-membership function $I_A(x)$, and a falsity-membership function $F_A(x)$. Then a simple valued neutrosopic set A can be denoted by

$$A := \{\langle x, T_A(x), I_A(x), F_A(x)\rangle | x \in X\},$$

where $T_A(x), I_A(x), F_A(x) \in [0, 1]$ for each point x in X. Therefore the sum of $T_A(x), I_A(x)$, and $F_A(x)$ satisfies the condition $0 \leq T_A(x) + I_A(x) + F_A(x) \leq 3$.

For convenience, "simple valued neutrosophic set" is abbreviated to "neutrosophic set" later.

Definition 2.4. ([10]) A neutrosophic set A is contained in the other neutrosophic B, denoted by $A \subseteq B$, if and only if $T_A(x) \leq T_B(x), I_A(x) \geq I_B(x)$, and $F_A(x) \geq F_B(x)$ for any $x \in X$. Two neutrosophic sets A and B are equal, written as $A = B$, if and only if $A \subseteq B$ and $B \subseteq A$.

Definition 2.5. ([12]) Let A be a neutrosophic set in a BE-algebra X and $\alpha, \beta, \gamma \in [0, 1]$ with $0 \leq \alpha + \beta + \gamma \leq 3$ and an (α, β, γ)-level set of X denoted by $A^{(\alpha,\beta,\gamma)}$ is defined as

$$A^{(\alpha,\beta,\gamma)} = \{x \in X | T_A(x) \geq \alpha, I_A(x) \leq \beta, F_A(x) \leq \gamma\}.$$

3. Neutrosophic subalgebras in BE-algebras

Definition 3.1. A neutrosophic set A in a BE-algebra X is called a *neutrosophic subalgebra* of X if it satisfies:

(NSS) $\min\{T_A(x), T_A(y)\} \leq T_A(x*y), \max\{I_A(x), I_A(y)\} \geq I_A(x*y),$ and $\max\{F_A(x), F_A(y)\} \geq F_A(x*y)$, for any $x, y \in X$.

Example 3.2. Let $X := \{1, a, b, c\}$ be a BE-algebra ([4]) with the following table:

$*$	1	a	b	c
1	1	a	b	c
a	1	1	a	a
b	1	1	1	a
c	1	1	a	1

Define a neutrosophic set A in X as follows:

$$T_A(x) = \begin{cases} 0.83, & \text{if } x \in \{1, a\} \\ 0.13, & \text{otherwise,} \end{cases}$$

$$I_A(x) = \begin{cases} 0.15, & \text{if } x \in \{1, a\} \\ 0.82, & \text{otherwise,} \end{cases}$$

$$F_A(x) = \begin{cases} 0.15, & \text{if } x \in \{1, a\} \\ 0.82, & \text{otherwise.} \end{cases}$$

It is easy to check that A is a neutrosophic subalgebra of X.

Definition 3.3. ([12]) A neutrosophic set A in a BE-algebra X is called a *neutrosophic filter* of X if it satisfies:

(NSF1) $T_A(x) \leq T_A(1), I_A(x) \geq I_A(1),$ and $F_A(x) \geq F_A(1)$, for any $x \in X$;

(NSF2) $\min\{T_A(x), T_A(x*y)\} \leq T_A(y), \max\{I_A(x), I_A(x*y)\} \geq I_A(y),$ and $\max\{F_A(x), F_A(x*y)\} \geq F_A(y)$, for any $x, y \in X$.

Proposition 3.4. *Every neutrosophic filter of a BE-algebra X is a neutrosophic subalgebra of X.*

Proof. Let A be a neutrosophic filter of X. For any $x, y \in X$, we have $\min\{T_A(x), T_A(y)\} \leq \min\{T_A(1), T_A(y)\} = \min\{T_A(y*(x*y)), T_A(y)\} \leq T_A(x*y)$, $\max\{I_A(x), I_A(y)\} \geq \max\{I_A(1), I_A(y)\} = \max\{I_A(y*(x*y)), I_A(y)\} \geq I_A(x*y)$, and $\max\{F_A(x), F_A(y)\} \geq \max\{F_A(1), F_A(y)\} = \max\{F_A(y*(x*y)), F_A(y)\} \geq F_A(x*y)$. Hence A is a neutrosophic subalgebra of X. □

The converse of Proposition 3.4 may not be true in general (see Example 3.5).

Example 3.5. Let $X := \{1, a, b\}$ be a BE-algebra with the following table:

$*$	1	a	b
1	1	a	b
a	1	1	a
b	1	1	1

Define a neutrosophic set A in X as follows: $T_A = \{(1, 0.83), (a, 0.13), (b, 0.16)\}, I_A = \{(1, 0.15), (a, 0.15), (b, 0.82)\}$, and $F_A = \{(1, 0.15), (a, 0.15), (b, 0.82)\}$. It is easy to check that A is a neutrosophic subalgebra of X. But it is not a neutrosophic filter of X, since $\min\{T_A(b * a), T_A(b)\} = \min\{T_A(1), T_A(b)\} = 0.16 \nleq 0.13 = T_A(a)$.

Theorem 3.6. *Let A be a neutrosophic set in a BE-algebra X and let $\alpha, \beta, \gamma \in [0, 1]$ with $0 \leq \alpha + \beta + \gamma \leq 3$. Then A is a neutrosophic subalgebra of X if and only if all of (α, β, γ)-level set $A^{(\alpha,\beta,\gamma)}$ are subalgebras of X when $A^{(\alpha,\beta,\gamma)} \neq \emptyset$.*

Proof. Assume that A is a neutrosophic subalgebra of X. Let $\alpha, \beta, \gamma \in [0, 1]$ be such that $0 \leq \alpha + \beta + \gamma \leq 3$ and $A^{(\alpha,\beta,\gamma)} \neq \emptyset$. Let $x, y \in A^{(\alpha,\beta,\gamma)}$. Then $T_A(x) \geq \alpha, T_A(y) \geq \alpha, I_A(x) \leq \beta, I_A(y) \leq \beta$ and $F_A(x) \leq \gamma, F_A(y) \leq \gamma$. Using (NSS), we have $\alpha \leq \min\{T_A(x), T_A(y)\} \leq T_A(x * y)$, $\beta \geq \max\{I_A(x), I_A(y)\} \geq I_A(x * y)$, and $\gamma \geq \max\{F_A(x), F_A(y)\} \geq F_A(x * y)$. Hence $x * y \in A^{(\alpha,\beta,\gamma)}$. Therefore $A^{(\alpha,\beta,\gamma)}$ is a subalgebra of X.

Conversely, all of (α, β, γ)-level set $A^{(\alpha,\beta,\gamma)}$ are subalgebras of X when $A^{(\alpha,\beta,\gamma)} \neq \emptyset$. Assume that there exist $a_t, b_t, a_i, b_i \in X$ and $a_f, b_f \in X$ such that $\min\{T_A(a_t), T_A(b_t)\} > T_A(a_t * b_t), \max\{I_A(a_i), I_A(b_i)\} < I_A(a_i * b_i)$, and $\max\{F_A(a_f), F_A(b_f)\} < F_A(a_f * b_f)$. Then $\min\{T_A(a_t), T_A(b_t)\} \geq t_{\alpha_1} > T_A(a_t * b_t), \max\{I_A(a_i), I_A(b_i)\} \leq t_{\alpha_2} < I_A(a_i * b_i)$, and $\max\{F_A(a_f), F_A(b_f)\} \leq t_{\alpha_3} < F_A(a_f * b_f)$ for some $t_{\alpha_1} \in (0, 1]$, and $t_{\alpha_2}, t_{\alpha_3} \in [0, 1)$. Hence $a_t, b_t, a_i, b_i, a_f, b_f \in A^{(t_{\alpha_1},t_{\alpha_2},t_{\alpha_3})}$, but $a_t * b_t, a_i * b_i, a_f * b_f \notin A^{(t_{\alpha_1},t_{\alpha_2},t_{\alpha_3})}$, which is a contradiction. Hence $\min\{T_A(x), T_A(y)\} \leq T_A(x * y), \max\{I_A(x), I_A(y)\} \geq I_A(x * y)$, and $\max\{F_A(x), F_A(y)\} \geq F_A(x * y)$ for any $x, y \in X$. Therefore A is a neutrosophic subalgebra of X. □

Since $[0, 1]$ is a completely distributive lattice with respect to the usual ordering, we have the following theorem.

Theorem 3.7. *If $\{A_i | i \in \mathbb{N}\}$ is a family of neutrosopic subalgebras of a BE-algebra X, then $(\{A_i | i \in \mathbb{N}\}, \subseteq)$ forms a complete distributive lattice.*

Proposition 3.8. *If A is a neutrosopic subalgebra of a BE-algebra X, then $T_A(x) \leq T_A(1), I_A(x) \geq I_A(1)$, and $F_A(x) \geq F_A(1)$ for all $x \in X$.*

Proof. Straightforward. □

Theorem 3.9. *Let A be a neutrosophic subalgebra of a BE-algebra X. If there exists a sequence $\{a_n\}$ in X such that $\lim_{n\to\infty} T_A(a_n) = 1, \lim_{n\to\infty} I_A(a_n) = 0$, and $\lim_{n\to\infty} F_A(a_n) = 0$, then $T_A(1) = 1, I_A(1) = 0$, and $F_A(1) = 0$.*

Proof. By Proposition 3.8, we have $T_A(x) \leq T_A(1), I_A(x) \geq I_A(1)$, and $F_A(x) \geq F_A(1)$ for all $x \in X$. Hence we have $T_A(a_n) \leq T_A(1), I_A(a_n) \geq I_A(1)$, and $F_A(a_n) \geq F_A(1)$ for every positive integer n. Therefore $1 = \lim_{n\to\infty} T_A(a_n) \leq T_A(1) \leq 1, 0 = \lim_{n\to\infty} I_A(a_n) \geq I_A(1) \geq 0$, and $0 = \lim_{n\to\infty} F_A(a_n) \geq F_A(1) \geq 0$. Thus we have $T_A(1) = 1, T_A(1) = 0$, and $F_A(1) = 0$. □

Proposition 3.10. *If every neutrosophic subalgebra A of a BE-algebra X satisfies the condition*

(3.1) $T_A(x * y) \geq T_A(x), I_A(x * y) \leq I_A(x), F_A(x * y) \leq F_A(x)$, *for any $x, y \in X$,*

then T_A, I_A, and F_A are constant functions.

Proof. It follows from (3.1) that $T_A(x) = T_A(1*x) \geq T_A(1), I_A(x) = I_A(1*x) \leq I_A(1)$, and $F_A(x) = F_A(1*x) \leq F_A(1)$ for any $x \in X$. By Proposition 3.8, we have $T_A(x) = T_A(1), I_A(x) = I_A(1)$, and $F_A(x) = F_A(1)$ for any $x \in X$. Hence T_A, I_A, and F_A are constant functions. □

Proposition 3.11. *Let A be a neutrosophic filter of a BE-algebra X. Then*

(i) $\min\{T_A(x*(y*z)), T_A(y)\} \leq T_A(x*z), \max\{I_A(x*(y*z)), I_A(y)\} \geq I_A(x*z)$, *and* $\max\{F_A(x*(y*z)), F_A(y)\} \geq F_A(x*z)$ *for any* $x, y \in X$.

(ii) $T_A(a) \leq T_A((a*x)*x), I_A(a) \geq I_A((a*x)*x)$, *and* $F_A(a) \geq F_A((a*x)*x)$ *for any* $a, x \in X$.

Proof. (i) Using (BE4) and (NSF2), we have $T_A(x*z) \geq \min\{T_A(y*(x*z)), T_A(y)\} = \min\{T_A(x*(y*z)), T_A(y)\}, I_A(x*z) \leq \max\{I_A(y*(x*z)), I_A(y)\} = \max\{I_A(x*(y*z)), I_A(y)\}$, and $F_A(x*z) \leq \max\{F_A(y*(x*z)), F_A(y)\} = \max\{F_A(x*(y*z)), F_A(y)\}$ for any $x, y \in X$.
(ii) Taking $y := (a*x)*x$ and $x := a$ in (NSF2), we have $T_A((a*x)*x) \geq \min\{T_A(a*((a*x)*x)), T_A(a)\} = \min\{T_A((a*x)*(a*x)), T_A(a)\} = \min\{T_A(1), T_A(a)\} = T_A(a), I_A((a*x)*x) \leq \max\{I_A(a*((a*x)*x)), I_A(a)\} = \max\{I_A((a*x)*(a*x)), I_A(a)\} = \max\{I_A(1), I_A(a)\} = I_A(a)$, and $F_A((a*x)*x) \leq \max\{F_A(a*((a*x)*x)), F_A(a)\} = \max\{F_A((a*x)*(a*x)), F_A(a)\} = \max\{F_A(1), F_A(a)\} = F_A(a)$ for any $a, x \in X$. □

Theorem 3.12. ([12]) *Let A be a neutrosophic set in a BE-algebra. Then A is a neutrosophic filter of X if and only if it satisfies (NSF1) and*

(3.2) *if* $x \leq y*z$ *for any* $x, y \in X$, *then* $\min\{T_A(x), T_A(y)\} \leq T_A(z), \max\{I_A(x), I_A(y)\} \geq I_A(z)$, *and* $\max\{F_A(x), F_A(y)\} \geq F_A(z)$.

Theorem 3.13. *If every neutrosophic set of a BE-algebra X satisfies (NSF1) and Proposition 3.11(i), then it is a neutrosophic filter of X.*

Proof. Taking $x := 1$ in Proposition 3.11(i) and using (BE3), we get $T_A(z) = T_A(1*z) \geq \min\{T_A(1*(y*z)), T_A(y)\} = \min\{T_A(y*z), T_A(y)\}, I_A(z) = I_A(1*z) \leq \max\{I_A(1*(y*z)), T_A(y)\} = \max\{I_A(y*z), I_A(y)\}$, and $F_A(z) = F_A(1*z) \leq \max\{F_A(1*(y*z)), F_A(y)\} = \max\{F_A(y*z), F_A(y)\}$ for any $y, z \in X$. Hence A is a neutrosophic filter of X. □

Corollary 3.14. *Let A be a neutrosophic set of a BE-algebra X. Then A is a neutrosophic filter of X if and only if it satisfies (NSF1) and Proposition 3.11(i).*

Theorem 3.15. *Let A be a neutrosophic set of a BE-algebra X. Then A is a neutrosophic filter of X if and only if it satisfies the following conditions:*

(i) $T_A(y*x) \geq T_A(x), I_A(y*x) \leq I_A(x)$, *and* $F_A(y*x) \leq F_A(x)$;

(ii) $T_A((a*(b*x))*x) \geq \min\{T_A(a), T_A(b)\}, I_A((a*(b*x))*x) \leq \max\{I_A(a), I_A(b)\}$, *and* $F_A((a*(b*x))*x) \leq \max\{F_A(a), F_A(b)\}$ *for any* $a, b, x \in X$.

Proof. Assume that A is a neutrosophic filter of X. Using (NSF2), we have $T_A(y*x) \geq \min\{T_A(x*(y*x)), T_A(x)\} = \min\{T_A(1), T_A(x)\} = T_A(x), I_A(y*x) \leq \max\{I_A(x*(y*x)), I_A(x)\} = \max\{I_A(1), I_A(x)\} = I_A(x)$, and $F_A(y*x) \leq \max\{F_A(x*(y*x)), F_A(x)\} = \max\{F_A(1), F_A(x)\} = F_A(x)$, for any $x, y \in X$. It follows from Proposition 3.11 that $T_A((a*(b*x))*x) \geq \min\{T_A((a*(b*x))*(b*x)), T_A(b)\} \geq \min\{T_A(a), T_A(b)\}$, $I_A((a*(b*x))*x) \leq \max\{I_A((a*(b*x))*(b*x)), I_A(b)\} \leq \max\{I_A(a), I_A(b)\}$, and $F_A((a*(b*x))*x) \leq \max\{F_A((a*(b*x))*(b*x)), F_A(b)\} \leq \max\{F_A(a), F_A(b)\}$ for any $x, a, b \in X$.

Conversely, assume that A is a neutrosophic set of X satisfying conditions (i) and (ii). Taking $y := x$ in (i), we have $T_A(1) = T_A(x*x) \geq T_A(x), I_A(1) = I_A(x*x) \leq I_A(x)$ and $F_A(1) = F_A(x*x) \leq F_A(x)$ for any $x \in X$. Using (ii), we get $T_A(y) = T_A(1*y) = T_A(((x*y)*(x*y))*y) \geq \min\{T_A(x*y), T_A(x)\}, I_A(y) = I_A(1*y) = I_A(((x*y)*(x*y))*y) \leq \max\{I_A(x*y), I_A(x)\}, F_A(y) = F_A(1*y) = F_A(((x*y)*(x*y))*y) \leq \max\{F_A(x*y), F_A(x)\}$ for any $x, y \in X$. Hence A is a neutrosophic filter of X. □

4. **Neutrosophic mighty filters in BE-algebras**

Definition 4.1. A neutrosophic set A in a BE-algebra X is called a *neutrosophic mighty filter* of X if it satisfies (NSF1) and

(NSF3) $\min\{T_A(z*(y*x)), T_A(z)\} \leq T_A(((x*y)*y)*x)), \max\{I_A(z*(y*x)), I_A(z)\} \geq I_A(((x*y)*y)*x)$, and $\max\{F_A(z*(y*x)), F_A(z)\} \geq F_A(((x*y)*y)*x)$ for any $x, y, z \in X$.

Example 4.2. Let $X := \{1, a, b, c, d, 0\}$ be a BE-algebra ([8]) with the following table:

$*$	1	a	b	c	d	0
1	1	a	b	c	d	0
a	1	1	b	c	b	c
b	1	a	1	b	a	d
c	1	a	1	1	a	a
d	1	1	1	b	1	b
0	1	1	1	1	1	1

Define a neutrosophic set A in X as follows:

$$T_A(x) = \begin{cases} 0.83, & \text{if } x \in \{1, b, c\} \\ 0.12, & \text{otherwise,} \end{cases}$$

$$I_A(x) = \begin{cases} 0.14, & \text{if } x \in \{1, b, c\} \\ 0.81, & \text{otherwise,} \end{cases}$$

$$F_A(x) = \begin{cases} 0.14, & \text{if } x \in \{1, b, c\} \\ 0.81, & \text{otherwise.} \end{cases}$$

It is easy to check that A is a neutrosophic mighty filter of X.

Proposition 4.3. *Every neutrosophic mighty filter of a BE-algebra X is a neutrosophic filter of X.*

Proof. Let A be a neutrosophic mighty filter of X. Putting $y := 1$ in (NSF3), we obtain $\min\{T_A(z*(1*x)), T_A(z)\} = \min\{T_A(z*x), T_A(z)\} \leq T_A(((x*1)*1)*x) = T_A(x), \max\{I_A(z*(1*x)), I_A(z)\} = \max\{I_A(z*x), I_A(z)\} \geq I_A(((x*1)*1)*x) = I_A(x)$, and $\max\{F_A(z*(1*x)), F_A(z)\} = \max\{F_A(z*x), F_A(z)\} \geq F_A(((x*1)*1)*x) = F_A(x)$ for any $x, y, z \in X$. Hence A is a neutrosophic filter of X. □

The converse of Proposition 4.3 may be not true in general (see Example 4.4).

Example 4.4. Let $X := \{1, a, b, c, d\}$ be a BE-algebra ([5]) with the following table:

$*$	1	a	b	c	d
1	1	a	b	c	d
a	1	1	b	c	d
b	1	a	1	c	c
c	1	1	b	1	b
d	1	1	1	1	1

Define a neutrosophic set A in X as follows:

$$T_A(x) = \begin{cases} 0.84, & \text{if } x = 1 \\ 0.11, & \text{otherwise,} \end{cases}$$

$$I_A(x) = \begin{cases} 0.13, & \text{if } x = 1 \\ 0.81, & \text{otherwise,} \end{cases}$$

$$F_A(x) = \begin{cases} 0.13, & \text{if } x = 1 \\ 0.81, & \text{otherwise.} \end{cases}$$

Then A is a neutrosophic filter of X, but not a neutrosophic mighty filter of X, since $\min\{T_A(1*(c*a)), T_A(1)\} = T_A(1) = 0.84 \nleq T_A(((a*c)*c)*a) = T_A(a) = 0.11$.

Theorem 4.5. *Any neutrosophic filter A of a BE-algebra X is mighty if and only if it satisfies the following conditions:*

(4.1) $T_A(y*x) \leq T_A(((x*y)*y)*x), I_A(y*x) \geq I_A(((x*y)*y)*x)$, *and* $F_A(y*x) \geq F_A(((x*y)*y)*x)$ *for any* $x, y \in X$.

Proof. Suppose that a neutrosophic filter A of a BE-algebra X satisfies the condition (4.1). Using (NSF2) and (4.1), we have $\min\{T_A(z*(y*x)), T_A(z)\} \leq T_A(y*x) \leq T_A(((x*y)*y)*x), \max\{I_A(z*(y*x)), I_A(z)\} \geq I_A(y*x) \geq I_A(((x*y)*y)*x)$, and $\max\{F_A(z*(y*x)), F_A(z)\} \geq F_A(y*x) \geq F_A(((x*y)*y)*x)$ for any $x, y \in X$. Hence A is a neutrosophic mighty filter of X.

Conversely, assume that the neutrosophic filter A of X is mighty. Setting $z := 1$ in (NSF3), we have $\min\{T_A(1*(y*x)), T_A(1)\} = T_A(y*x) \leq T_A(((x*y)*y)*x), \max\{I_A(1*(y*x)), I_A(1)\} = I_A(y*x) \geq I_A(((x*y)*y)*x)$, and $\max\{F_A(1*(y*x)), F_A(1)\} = F_A(y*x) \geq F_A(((x*y)*y)*x)$ for any $x, y \in X$. Hence (4.1) holds. □

Proposition 4.6. *Let A be a neutrosophic mighty filter of a BE-algebra X. Denote that $X_T := \{x \in X | T_A(x) = T_A(1)\}, X_I := \{x \in X | I_A(x) = I_A(1)\}$, and $X_F := \{x \in X | F_A(x) = F_A(1)\}$. Then X_T, X_I, and X_F are mighty filters of X.*

Proof. Clearly, $1 \in X_T, X_I, X_F$. Let $z*(y*x), z \in X_T$. Then $T_A(z*(y*x)) = T_A(1), T_A(z) = T_A(1)$. Hence $\min\{T_A(z*(y*x)), T_A(z)\} = T_A(1) \leq T_A(((x*y)*y)*x)$ and so $T_A((x*y)*y)*x) = T_A(1)$. Therefore $((x*y)*y)*x \in X_T$. Thus X_T is a mighty filter of X. Similarly, X_I, X_F are mighty filters of X. □

Theorem 4.7. *Let A, B be neutrosophic filters of a transitive BE-algebra X such that $A \subseteq B$ and $T_A(1) = T_B(1), I_A(1) = I_B(1), F_A(1) = F_B(1)$. If A is mighty, then B is mighty.*

Proof. Let $x, y \in X$. Since A is a neutrosophic mighty filter of a BE-algebra X, by (4.1) and $A\subseteq B$ we have $T_A(1) = T_A(y*((y*x)*x)) \le T_A(((((y*x)*x)*y)*y)*((y*x)*x)) \le T_B(((((y*x)*x)*y)*y)*((y*x)*x))$. Since $T_A(1) = T_B(1)$, we get $T_B((y*x)*((((y*x)*x)*y)*y)*x)) = T_B(((((y*x)*x)*y)*y)*((y*x)*x)) = T_B(1)$. It follows from (NSF1) and (NSF2) that

$$\begin{aligned} T_B(y*x) =& \min\{T_B(1), T_B(y*x)\} \\ =& \min\{T_B((y*x)*(((((y*x)*x)*y)*y)*x)), T_B(y*x)\} \\ \le& T_B(((((y*x)*x)*y)*y)*x). \end{aligned} \tag{4.2}$$

Since X is transitive, we get

$$\begin{aligned} &[((((y*x)*x)*y)*y)*x]*[((x*y)*y)*x] \\ &\ge ((x*y)*y)*((((y*x)*x)*y)*y) \\ &\ge (((y*x)*x)*y)*(x*y) \\ &\ge x*((y*x)*x) \\ &= (y*x)*(x*x) \\ &= (y*x)*1 = 1. \end{aligned}$$

It follows from Theorem 3.12 that $\min\{T_B(((((y*x)*x)*y)*y)*x), T_B(1)\} = T_B(((((y*x)*x)*y)*y)*x) \le T_B(((x*y)*y)*x)$. Using (4.2), we have $T_B(y*x) \le T_B(((((y*x)*x)*y)*y)*x) \le T_B(((x*y)*y)*x)$. Therefore $T_B(y*x) \le T_B(((x*y)*y)*x)$. Similarly, we have $I_B(y*x) \ge T_B(((x*y)*y)*x)$ and $F_B(y*x) \ge F_B(((x*y)*y)*x)$. By Theorem 4.5, B is a neutrosophic mighty filter of X. □

Theorem 4.8. *Let A be a neutrosophic set in a BE-algebra X and let $\alpha, \beta, \gamma \in [0, 1]$ with $0 \le \alpha + \beta + \gamma \le 3$. Then A is a neutrosophic mighty filter of X if and only if all of (α, β, γ)-level set $A^{(\alpha,\beta,\gamma)}$ are mighty filters of X when $A^{(\alpha,\beta,\gamma)} \ne \emptyset$.*

Proof. Assume that A is a neutrosophic mighty filter of X. Let $\alpha, \beta, \gamma \in [0, 1]$ be such that $0 \le \alpha + \beta + \gamma \le 3$ and $A^{(\alpha,\beta,\gamma)} \ne \emptyset$. Let $z*(y*x), z \in A^{(\alpha,\beta,\gamma)}$. Then $T_A(z*(y*x)) \ge \alpha, T_A(z) \ge \alpha, I_A(z*(y*x)) \le \beta, I_A(z) \le \beta$, and $F_A(z*(y*x)) \le \gamma, F_A(z) \le \gamma$. By Definition 4.1, we have $T_A(1) \ge T_A(((x*y)*y)*x) \ge \min\{T_A(z*(y*x)), T_A(z)\} \ge \alpha, I_A(1) \le I_A(((x*y)*y)*x) \le \max\{I_A(z*(y*x)), I_A(z)\} \le \beta$, and $F_A(1) \le F_A(((x*y)*y)*x) \le \max\{F_A(z*(y*x)), F_A(z)\} \le \gamma$. Hence $1, ((x*y)*y)*x \in A^{(\alpha,\beta,\gamma)}$. Therefore $A^{(\alpha,\beta,\gamma)}$ are mighty filters of X.

Conversely, suppose that there exist $a, b, c \in X$ such that $T_A(a) > T_A(1), I_A(b) < I_A(1)$, and $F_A(c) < F_A(1)$. Then there exist $a_t \in (0, 1]$ and $b_t, c_t \in [0, 1)$ such that $T_A(a) \ge a_t > T_A(1), I_A(b) \le b_t < I_A(1)$ and $F_A(c) \le c_t < F_A(1)$. Hence $1 \notin A^{(a_t,b_t,c_t)}$, which is a contradiction. Therefore $T_A(x) \le T_A(1), I_A(x) \ge I_A(1)$ and $F_A(x) \ge F_A(1)$ for all $x \in X$. Assume that there exist $a_t, b_t, c_t, a_i, b_i, c_i \in X$ and $a_f, b_f, c_f \in X$ such that $T_A(((a_t*b_t)*b_t)*a_t) < \min\{T_A(c_t*(b_t*a_t)), T_A(c_t)\}, I_A(((a_i*b_i)*b_i)*a_i) > \max\{I_A(c_i*(b_i*a_i)), I_A(c_i)\}$, and $F_A(((a_f*b_f)*b_f)*a_f) > \max\{F_A(c_f*(b_f*a_f)), F_A(c_f)\}$. Then there exist $s_t \in (0, 1]$ and $s_i, s_f \in [0, 1)$ such that $T_A(((a_t*b_t)*b_t)*a_t) < s_t \le \min\{T_A(c_t*(b_t*a_t)), T_A(c_t)\}, I_A(((a_i*b_i)*b_i)*a_i) > s_i \ge \max\{I_A(c_i*(b_i*a_i)), I_A(c_i)\}$, and $F_A(((a_f*b_f)*b_f)*a_f) > s_f \ge \max\{F_A(c_f*(b_f*a_f)), F_A(c_f)\}$. Hence $c_t*(b_t*a_t), c_t, c_i*(b_i*a_i), c_i \in A^{(s_t,s_i,s_f)}$ and $c_f*(b_f*a_f), c_f \in A^{(s_t,s_i,s_f)}$ but $((a_t*b_t)*b_t)*a_t, ((a_i*b_i)*b_i)*a_i \notin A^{(s_t,s_i,s_f)}$, and $((a_f*b_f)*b_f)*a_f \notin A^{(s_t,s_i,s_f)}$, which is a contradiction. Therefore $\min\{T_A(z*(y*x)), T_A(z)\} \leq T_A(((x*y)*y)*x)), \max\{I_A(z*(y*x)), I_A(z)\} \geq I_A(((x*y)*y)*x))$, and $\max\{F_A(z*(y*x)), F_A(z)\} \geq F_A(((x*y)*y)*x))$ for any $x, y, z \in X$. Thus A is a neutrosophic mighty filter of X □

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