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Neutrosophic Triplet Group (revisited)

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Abstract. We have introduced for the first time the notion of neutrosophic triplet since 2014, which has the form (x, neut(x), anti(x)) with respect to a given binary well-defined law, where neut(x) is the neutral of x, and anti(x) is the opposite of x. Then we define the neutrosophic triplet group (2016), prove several theorems about it, and give some examples. This paper is an improvement and a development of our 2016 published paper.
Groups are the most fundamental and rich algebraic structure with respect to some binary operation in the study of algebra. In this paper, for the first time, we introduced the notion of neutrosophic triplet, which is a collection of three elements that satisfy certain axioms with respect to a binary operation. These neutrosophic triplets highly depend on the defined binary operation. Further, in this paper, we used these neutrosophic triplets to introduce the innovative notion of neutrosophic triplet group, which is a completely different from the classical group in the structural properties. A big advantage of neutrosophic triplet is that it gives a new group (neutrosophic triplet group) structure to those algebraic structures, which are not group with respect to some binary operation in the classical group theory. In neutrosophic triplet group, we apply the fundamental law of Neutrosophy that for an idea A, we have the neutral of A denoted as neut(a) and the opposite of A dented as anti(A) to capture this beautiful picture of neutrosophic triplet group in algebraic structures. We also studied some interesting properties of this newly born structure. We further defined neutro-homomorphisms for neutrosophic triplet groups. A neutro-homomorphism is the generalization of the classical homomorphism with two extra conditions. As a further generalization, we gave rise to a new field or research called Neutrosophic Triplet Structures (such as neutrosophic triplet ring, neutrosophic triplet field, neutrosophic triplet vector space, etc.). In the end, we gave main distinctions and comparison of neutrosophic triplet group with the Molaei's generalized group as well as the possible application areas of the neutrosophic triplet groups. In this paper we improve our [13] results on neutrosophic triplet groups.

Keywords: Groups, homomorphism, neutrosophic triplet, neutrosophic triplet group, neutro-homomorphism t.

1 Introduction

Neutrosophy is a new branch of philosophy that studies the nature, origin and scope of neutralities as well as their interaction with ideational spectra. Florentin Smarandache [8] in 1995, first introduced the concept of neutrosophic logic and neutrosophic set where each proposition in neutrosophic logic is approximated to have the percentage of truth in a subset T, the percentage of indeterminacy in a subset I, and the percentage of falsity in a subset F so that this neutrosophic logic is called an extension of fuzzy logic especially of the intuitionistic fuzzy logic. In fact neutrosophic set is the generalization of classical sets[9], fuzzy set[12], intuitionistic fuzzy set[1,9], and interval valued fuzzy set[9] etc. This mathematical tool is used to handle problems consisting uncertainty, imprecision, indeterminacy, inconsistency, incompleteness and falsity. By utilizing the idea of neutrosophic theory, Vasantha Kandasamy and Florentin Smarandache studied neutrosophic algebraic structures in [4,5,6] by inserting an indeterminate element "I" in the algebraic structure and then combine "I" with each element of the structure with respect to corresponding binary operation *. They call it neutrosophic number { $a + bI$, with a, b real numbers, and I = literal indeterminacy, $I^2 = I$ } and the generated algebraic structure is then termed as neutrosophic algebraic structure. They further study several neutrosophic algebraic structures such as neutrosophic fields, neutrosophic vector spaces, neutrosophic groups, neutrosophic bigroups, neutrosophic N-groups, neutrosophic semigroups, neutrosophic bisemigroups, neutrosophic N-semigroup, neutrosophic loops, neutrosophic biloops, neutrosophic N-loop, neutrosophic groupoids, and neutrosophic bigroupoids and so on.

Groups [2,3,11] are very important in algebraic structures because they play the role of a backbone in almost all algebraic structures theory. Groups are thought as old algebra due to its rich structure than any other notion. In many algebraic structures, groups provide concrete foundation such as, rings, fields, vector spaces, etc. Groups are also important in many other areas like physics, chemistry, combinatorics, biology etc. to study the symmetries and other behavior among their elements. The most important aspect of a group is group action. There are many types of groups, such as: permutation groups, matrix groups, transformation groups, Lie-groups etc. that are highly used as a practical perspective in our daily life. Generalized groups [7] are important in this aspect.

In this paper, for the first time, we introduced the idea of neutrosophic triplet. The newly born neutrosophic triplets are highly dependable on the proposed binary operation. These neutrosophic triplets have been discussed by Smarandache and Ali in Physics [10]. Moreover, we used these neutrosophic triplets to introduce neutrosophic triplet group, which is different from the classical group both in structural and foundational properties from all aspects. Furthermore, we gave some interesting and fundamental properties and notions with illustrative examples. We also introduced a new type of homomorphism called as neutro-homomorphism, which is in fact a generalization of the classical homomorphism under some conditions. We also study neutro-homomorphism for neutrosophic triplet groups. The rest of the paper is organized as follows. After the literature review in section 1, we introduced neutrosophic triplets in section 2. Section 3 is dedicated to the introduction of neutrosophic triplet groups with some of its interesting properties. In section 4, we developed neutro-homomorphism and in section 5, we gave distinction and comparison of neutrosophic triplet group with the Molaei's generalized group. We also draw a brief sketch of the possible applications of neutrosophic triplet group in other research areas. Conclusion is given in section 6.

2 Neutrosophic Triplet

Remark 2.1. All below theorems and propositions in a Neutrosophic Triplet Set (NTS) and Neutrosophic Triplet Group (NTG) are true when the multipliers are non-zero and cancellable multipliers.

An element $a \in (S, *)$, where * is a binary law, *is cancellable to the left* if:

$$\forall\, b, c \in S, \text{ from } a*b = a*c \text{ one gets only } b = c.$$

The element a is *cancellable to the right* if:

$$\forall\, b, c \in S, \text{ from } b*a = c*a \text{ one gets only } b = c.$$

And, the element a is *cancellable (in general)* if the element a is both cancelable to the left and to the right.

Definition 2.1.1. Let N be a set together with a binary operation *. Then N is called a *neutrosophic triplet set* if for any $a \in N$, there is a neutral of "a" called $neut(a)$, different from the classical algebraic unitary element, and an opposite of "a" called $anti(a)$, with $neut(a)$ and $anti(a)$ belonging to N, such that:

$$a * neut(a) = neut(a) * a = a$$

and

$$a * anti(a) = anti(a) * a =$$

The elements a, $neut(a)$, and $anti(a)$ are collectively called as neutrosophic triplet and we denote it by (a, $neut(a)$, $anti(a)$). By $neut(a)$, we mean $neutral$ of a and apparently, a is just the first coordinate of a neutrosophic triplet and not a neutrosophic triplet.

For the same element a in N, there may be more neutrals to it *neut(a)* and more opposites of it *anti(a)*.

Remark 2.2

If a well-defined binary law * on the set N has a classical algebraic unitary element e in N, then no other triplet of the form (e, b, c) can be formed, except the (e, e, e), i.e. when $b = c = e$, which is not accepted as neutrosophic triplet.

Consequently, the set $(N, *)$ with a classical unitary element cannot be a neutrosophic triplet set.

Remark 2.2.

It is important that there are at least two different neutral elements with respect to all set elements into a neutrosophic triplet set.

Definition 2.1.3. A *Zero Neutrosophic Triplet* on the neutrosophic triplet set N, is a neutrosophic triplet of the form $(0, 0, a)$, where $0, a \in N$ {of course, the triplet $(0, 0, a)$ must satisfy the axioms of the neutrosophic triplet}.

Example 2.1.3.1. Let N=be a set with respect to multiplication × modulo 10 in $a \in N, 6 \times a = a \times 6 = a(mod\ 10)$.
It should be remarked, to this example, that *6* is a classical algebraic unitary element on *N,* with respect to the multiplication × modulo *10*, because for any *a* ∈ *N, 6 × a = a × 6 = a (mod 10)*.
But *6* cannot be a neutral for the element *0* ∈ *N*, because *(0, 6, ?)* cannot form a neutrosophic triplet since there is no *anti(0)* such that:
0 × anti(0) = anti(0) × 0 = 6.
Therefore, the neutrosophic triplets of *0* [called *Zero Neutrosophic Triplets*] are
(0, 0, 0), (0, 0, 2), (0, 0, 4), (0, 0, 6), (0, 0, 8).
N is not a neutrosophic triplet set since, except element *0*, the other elements *2, 4, 6,* and *8* do not have neutral elements different from the classical unitary element *6.*
Theorem 2.1. Let *N* be a set endowed with the binary law *, which is well-defined and has the classical algebraic unitary element *e* ∈ *N,*

$$\forall x \in N,\ e * x = x * e = x.$$

If *(e, b, c)* is a neutrosophic triplet, with *b, c* ∈ *N,* then *b = c = e.*
{In other words, if a set *N* has a classical algebraic unitary element *e,* with respect to the binary well-defined law *, then the only neutrosophic triplet of *e* is *(e, e, e),* which is mutually called trivial neutrosophic triplet, the only triplet that makes exception from the definition of neutrosophic triplets.}
***Proof*.**
Let *(e, b, c)* be a neutrosophic triplet. Since *neut(e) = b,* one has:
*e*neut(e) = neut(e)*e = e,*
but *e*b = b* and *b*e = b* too (since *e* is the classical algebraic unitary element on the set *N*),
whence *b = e.*
And, because *anti(e) = c,* one has:
*e*c = c*e = e,*
but *e*c = c* and *c*e = c* too (since *e* is the classical algebraic unitary element on the set *N*),
whence *c = e.*
Therefore, the only triplet of the classical algebraic unitary (identity) element is
(e, e, e), but it cannot be considered a neutrosophic triplet.
Definition 2.2: The element b in (N, *) is the second component , denoted as $neut(\cdot)$ of a neutrosophic triplet, if there exist other elements a and c in N such that $a * b = b * a = a$ and $a * c = c * a = b$. The formed neutrosophic triplet is (a, b, c) .
Definition 2.3: The element c in $(N,*)$ is the third component, denoted as $neut(\cdot)$, of a neutrosophic triplet, if there exist other elements a and c in N such that $a * b = b * a = a$ and $a * c = c * a = b$. The formed neutrosophic triplet is (a, b, c).
Example 2.2. Consider Z under multiplication modulo 6 , where

$$Z_6 = \{0,1,2,3,4,5\}$$

The classical unitary element is *e = 1.*
Then 2 gives rise to a neutrosophic triplet because $neut(2) = 4$, as $2 \times 4 = 8$. Also $anti(2) = 2$ because $2 \times 4 = 4$. Thus $(2,4,2)$ is a neutrosophic triplet. Similarly 4 gives rise to a neutrosophic triplet because $neut(4) = anti(4) = 4$. So $(4,4,4)$ is a neutrosophic triplet. 3 has two neutrals, $neut(3) = \{3, 5\}$, and forms one neutrosophic triplet (3, 3, 3), but 3 does not give rise to a neutrosophic triplet for $neut(3) = 5$ since $anti(3)$ does not exist in Z_6 for this neutral,
5 has no *neut(5)* so no neutrosophic triplet related to *5*, and last but not the least 0 gives rise to a trivial neutrosophic triplet as $neut(0) = anti(0) = 0$. The zero neutrosophic triplets are denoted by(0,0,0), (0, 0, 1), (0, 0, 2), (0, 0, 3), (0, 0, 4), (0, 0, 5).
Z_6 is not a neutrosophic set, since *1* and *5* have no corresponding neutrosophic triplets, but
$M_6 = \{0, 2, 3, 4\} \subset Z_6$ *is a commutative neutrosophic group* [whose definition will be provided below].

Theorem 2.3. If $(a, neut(a), anti(0))$ form a neutrosophic triplet, then

1. $(anti(a), neut(a), a)$ also form a neutrosophic triplet, and similarly

2. $(neut(a), neut(a), neut(a))$ form a neutrosophic triplet.

Proof. We prove both 1 and 2.
1. Of course, $anti(a) * a = neut(a)$.

We need to prove that:

$$anti(a) * neut(a) = anti(a)$$

Multiply by a to the left and we get:
$a * anti(a) * neut(a) = a * anti(a)$ Or
$[a * anti(a)] * neut(a) = neut(a)$Or

$$neut(a) * neut(a) = neut(a)$$

Again multiply by a to the left and we get:

$$a * neut(a) * neut(a) = a * neut(a)$$

Or

$$[a * neut(a)] * neut(a) = a$$

Or

$$a * neut(a) = a$$

2. To show that $(neut(a), neut(a), neut(a))$ is a neutrosophic triplet, it results from the fact that

$$neut(a) * neut(a) = neut(a).$$

3 Neutrosophic Triplet Group

Definition 3.1: Let (N,*) be a neutrosophic triplet set (which includes the trivial neutrosophic triplet too, if any). Then N is called a neutrosophic triplet group, if the following conditions are satisfied.
1) If (N,*) is well-defined, i.e. for any $a, b \in N$, one has $a * b \in N$.
2) If (N,*) is associative, i.e. (a*b)*c= a*(b*c) for all $a, b, c \in N$.

The neutrosophic triplet group, in general, is not a group in the classical algebraic way.

We consider, as the neutrosophic neutrals replacing the classical unitary element, and the neutrosophic opposites as replacing the classical inverse elements.

Example 3.2. *Consider*$(Z_{10}, \#)$*, where # is defined as* $a\#b = 3ab$

Let $M_{10} = \{0, 2, 4, 5, 6, 8\} \subset Z_{10}$. *Then* $(M_{10},\#)$ *is a neutrosophic triplet group under the binary #.*
It is also associative, i.e.

$$(a\#b)\#c=a\#(b\#c).$$

Now take L. H. S to prove the R. H. S, so

$$a\#(b\#c)=3ab\#c.$$

,

3(3ab)c=9abc,
3a(3bc)=3a(b#c),
a#(b#c).

The classical unitary element on Z_{10} with respect to the law # is $e = 7$, *since:*

$a \# e = e \# a = 3ae = 3a(7) = 21a = a \pmod{10}$ for any $a \in Z_{10}$.

Therefore, we choose all triplets whose neutral elements are different from *7*, and we get the following neutrosophic triplets:

(0, 0, 0), (0, 0, 2), (0, 0, 4), (0, 0, 5), (0, 0, 6), (0, 0, 8), (2, 2, 2), (4, 2, 6), (5, 5, 5), (6, 2, 4), and *(8, 2, 8).*

All above neutrals *neut(.) = 0, 2,* and *5* are different from the classical unitary element *7.*

Z_{10} is not a neutrosophic triplet group, nor even a neutrosophic triplet set.

But its subset $M_{10} = \{0, 2, 4, 5, 6, 8\}$ is a commutative neutrosophic triplet group, since the law # is well-defined, commutative, associative, and each element belonging to *M* has a corresponding neutrosophic triplet.

Definition 3.3: Let $(N,*)$ be a neutrosophic triplet group. Then N is called a commutative neutrosophic triplet group if for all $a, b \in N$ we have $a * b = b * a$.

Example 3.4. Consider *(M, *),* where $M = \{0, 1\}$, and the binary law * is defined as

$a*b = a + b - ab \pmod 4$ for all $a, b \in M$.

Then *(M, *)* is a not a neutrosophic triplet group, not even a neutrosophic triplet set.

Proof.

The law * has a classical algebraic unitary element $e = 0$, since:

For any $a \in M$, $a*0 = 0*a = a + 0 - a \times 0 = a \pmod 4$.

Therefore, (M, *) cannot be a neutrosophic triplet group.

Theorem 3.5. Every idempotent element gives rise to a neutrosophic triplet.

Proof. Let a be an idempotent element. Then by definition $a^2 = a$. Since $a^2 = a$, which clearly implies that $neut(a) = a$ and $anti(a) = a$. Hence a gives rise to a neutrosophic triplet (a, a, a) .

Theorem 3.6. There are no neutrosophic triplets in Z_n with respect to multiplication modulo n if n is a prime, except the zero neutrosophic triplets $(0, 0, 0), (0, 0, 1), \ldots, (0, 0, n-1)$.

Proof. It is obvious. The multiplication modulo n is well-defined, associative, and commutative.

For $n = 2$ (even prime), $Z_2 = \{0, 1\}$ has the classical algebraic unitary element, with respect to multiplication modulo 2, $e = 1$, and Z_2 has the zero neutrosophic triplets $(0, 0, 0), (0, 0, 1)$.

Whence Z_2 is a not a neutrosophic triplet group, not even a neutrosophic triplet set.

Let $Z_n = \{0, 1, 2, \ldots, n-1\}$, for n odd prime. The classical algebraic unitary element of Z_n is 1, and the zero neutrosophic triplets are $(0, 0, 0), (0, 0, 1), \ldots, (0, 0, n-1)$.

Let's compute the neutral of $2 \leq p \leq p - 1$, if any, let $neut(p) = x$. We need to find x.

$px = p \pmod n$, or $px - p = 0 \pmod n$, or $p(x-1) = 0 \pmod n$,

whence $x-1 = 0 \pmod n$ since n is an odd prime, and n and p are relatively prime numbers,

or $x = 1 \pmod n$, therefore there is no neutral of the elements $p \in \{2, 3, \ldots, n-1\}$, since 1 is excluded as classical algebraic unitary element. Thus no neutrosophic triplets corresponding to the elements $2, 3, \ldots, n-1 \in Z_n$.

Remark 3.6.1. Let $(N,*)$ be a neutrosophic triplet group under $*$ and let $a \in N$. Then $neut(a)$ is not the same for all elements in N (as in classical group), but $neut(a)$ depends on the a and on the operation $*$.

(In example 3.8, $neut(0) = 0$, $neut(4) = 4$, $neut(8) = 4$ and $neut(9) = 9$, so we have three different neutral elements: 0, 4, and 9.)

Theorem number 3.6.2. Let $(N,*)$ be a neutrosophic triplet group under $*$ that satisfies the cancellation law for all its elements. Then, for any a in N, the $neut(a)$ and $anti(a)$ are unique and depend on a.

Proof: Suppose $neut^{(1)}(a)$ and $neut^{(2)}(a)$ be two neutrals of a.

Then since $neut^{(1)}(a)*a = neut^{(2)}(a)*a$, and by cancellation law to the right-hand side, we have $neut^{(1)}(a) = neut^{(2)}(a)$.

Similarly, if $a*neut^{(1)}(a) = a*neut^{(2)}(a)$, by cancellation law to the left-hand side, we have $neut^{(1)}(a) = neut^{(2)}(a)$.

In the same way, since $anti^{(1)}(a)*a = anti^{(2)}(a)*a$, we get $anti^{(1)}(a) = anti^{(2)}(a)$, by cancellation law to the right-hand side.

Again, since $a*anti^{(1)}(a) = a*anti^{(2)}(a)$, we get $anti^{(1)}(a) = anti^{(2)}(a)$, by cancellation law to the left-hand side.

Theorem 3.6.3. If the elements a of NTG do not satisfy the cancellation law, then still for each a in NTG the $neut(a)$ is unique and depending on a, but the $anti(a)$ may not be unique.

Proof.

Let's suppose that $neut^{(1)}(a)$ and $neut^{(2)}(a)$ are two neutrals of a, we have

$$
\begin{aligned}
neut^{(1)}(a) = a*anti^{(1)}(a) &= (neut^{(2)}(a)*a)*anti^{(1)}(a) \\
&= neut^{(2)}(a)*(a*anti^{(1)}(a)) \\
&= neut^{(2)}(a)*neut^{(1)}(a) \\
&= (anti^{(2)}(a)*a)*neut^{(1)}(a) \\
&= anti^{(2)}(a)*(a*neut^{(1)}(a)) \\
&= anti^{(2)}(a)*a \\
&= neut^{(2)}(a).
\end{aligned}
$$

Yet, $anti(a)$ is not unique.

To prove this, let's take a look at the following example.

Example 3.8. Let N=(0,4,8) be a commutative neutrosophic triplet group under multiplication modulo 12 in

(Z_{12}, N). N does not have a classical unitary element.
Then $neut(4) = 4$, $neut(8) = 4$ and $anti(0) ==$ {0,4,8,9}, neut(0) = 0 and *anti(0) = {0, 4, 8, 9}*. This shows that $neut(a)$ is not the same for all elements as in classical group theory; further, each element has only one neutral. Also, the element *0* has four *anti(0)'*s.
The neutrosophic triplets are: *(0, 0, 0), (0, 0, 4), (0, 0, 8), (0, 0, 9), (4, 4, 4), (8, 4, 8), (9, 9, 9).*
Remark 3.9. Let $(N,*)$ be a neutrosophic triplet group with respect to * and let $a \in N$. Then $anti(a)$ is not the same for all elements in N and also anti(N) depends on the element a and on the operation *, and some elements may have many anti's unlike classical group and generalized group.
To prove the above remark, let's take a look to the following example.
Example 3.10. Let N be the commutative neutrosophic triplet group in the above Example 3.8. Then *anti(0) = {0, 4, 8, 9}, anti(4)=4* , anti(8)=8 and anti(9)=9. Therefore, the element 0 has four anti's, and the anti's of the elements are different from each other: 4, 8, 9, and 0.
Proposition 3.11. Let (N, *) be a neutrosophic triplet group with respect to * and let
A
$a, b, c \in N$. Then
1) a*b=a*c if and only if neut(a)*b=neut(a)*c.
2) b*a = c*a if and only if b*neut(a)=c*neut(a).
Proof. 1. Suppose that a*b=a*c. Since N is a neutrosophic triplet group, so $anti(a) \in N$. Multiply anti(a) to the left side of a*b=a*c.

anti(a)*a*b=anti(a)*a*c
[anti(a)*a]*b=[anti(a)*a]*c
neut(a)*b=neut(a)*c

Conversely suppose that neut(a)*b=neut(a)*c.
Multiply a to the left side, we get:

$$[a * neut(a)] * b = [a * neut(a)] * c.$$
$$a * b = a * c$$

2. The proof is similar to 1.
Proposition 3.12. Let (N,*) be a neutrosophic triplet group with respect to * and let
$a, b, c \in N$.
1) If anti(a)*b=anti(b)*c, then neut(a)*b=neut(a)*c.
2) If b*anti(a), then c*anti(a), then b*neut(a)=c*neut(a).
Proof. 1-. Suppose that anti(a)*b=anti(a)*c. Since N is a neutrosophic triplet group with respect to *, so $a \in N$. Multiply a to the left side of anti(a)*b=anti(a)*c, we get:

a*anti(a)*b=a*anti(a)*c
[a*anti(a)]*b=[a*anti(a)]*c
neut(a)*b=neut(a)*c.

2. The proof is the same as (1).
Theorem 3.13. Let (N,*) be a commutative neutrosophic triplet group with respect to * and $a, b, n \in N$. Then

neut(a)*neut(b)=neut(a*b).

Proof. Consider left hand side,neut(a)*neut(b)=neut(a*b).
Now multiply to the left with a and to the right with b , we get:

a*neut(a*b)*b=[a*b]*[neut(a*b)], as * in asociative
=a*b.

Now consider right hand side, we have neut(a*b).
Again multiply to the left with a and to the right with b, we get:

a*neut(a*b)*b , as is associative,
=a*b.

This completes the proof.
Theorem 3.14. Let (N,*) be a commutative neutrosophic triplet group with respect to * and $a, b \in N$. Then

anti(a)*anti(b)=anti(a*b).

Proof. Consider left hand side, anti(a)*anti(b).
Multiply to the left with a and to the right with b, we get:

a*anti(a)*anti(b)*b=[a*anti(a)]*[anti(b)*b]
=neut(a)*neut(b)
=neut(a*b), from the above theorem.

Now consider right hand side, which is anti(a*b).
Multiply to the left with a and to the right with b, we get:

a*anti(a*b), since * is associative.

neut(a*b).

This shows that anti(a)*anti(b) is true for all $a, b \in N$.

Theorem 3.15. Let (N,*) be a commutative neutrosophic triplet group under * and $a, b \in N$. Then

1) neut(a)*neut(b)=neut(b)*neut(a).
2) anti(a)*anti(b)=anti(b)*anti(a).

Proof 1. Consider right hand side neut(b)*neut(a). By Theorem 3, we have

neut(b)*neut(a)=neut(b)*neut(a), as N is commutative,

neut(a)*neut(b), again by Theorem 3.

Hence neut(a)*neut(b)=neut(b)*neut(a).

2) On similar lines, one can easily obtained the proof of (2).

{Actually, both proofs could also result straightforwardly from the commutative property of the neutrosophic triplet group.}

Definition 3.16. Let (N,*) be a neutrosophic triplet group under * and let H be a subset of N. Then H is called a neutrosophic triplet subgroup of N if H itself is a neutrosophic triplet group with respect to*.

Proposition 3.18. Let (N,*) be a neutrosophic triplet group and let $a \in N$ be a subset ofN. Then H is a neutrosophic triplet subgroup of N if and only if the following conditions hold.

1) $a * b \in H$ for all $a, b \in H$.
2) $neut(a) \in H$ for all $a \in H$.
3) $anti(a) \in H$ for all $a \in H$.

Proof. The proof is straightforward.

Definition 3.19. Let N be a neutrosophic triplet group and let $a \in N$. A smallest positive integer $n \geq 1$ such that is called neutrosophic triplet order {with respect to a given *neut(a)*, when the case when there are many neutrals of a}. It is denoted by nto(a).

Theorem 3.21. Let (N,*) be a neutrosophic triplet group with respect to * and let $a \in N$. Then

1) neut(a)*neut(a)=neut(a).

In general $(neut(a))^2 = neut(a)$, where n is a non-zero positive integer.

2) neut(a)*anti(a)=anti(a)*neut(a)=anti(a).

(Theorem 3.21 (1) and (2) were proven in Theorem 2.3. (1) and (2), except $(neut(a))^n = neut(a)$.)

Proof. Consider neut(a)*neut(a)=neut(a).
Multiply a to the left side, we get;

a* neut(a)*neut(a)=a*neut(a)

[a* neut(a)]*neut(a)=[a*neut(a)]

a*neut(a)=a

a=a.

On the same lines, we can see that $(neut(a))^2 = neut(a)$ for a non-zero positive integer n.

2) Consider neut(a)*anti(a)=anti(a).

Multiply to the left with a, we get;

a* neut(a)*anti(a)=a*anti(a)

a* anti(a)=neut(a)

neut(a)=neut(a)

a=a.

Similarly anti(a)*neut(a)=anti(a).

Definition 3.22. Let N be a NTG. If $N = \langle a \rangle$ for some $a \in N$, then N is called a neutro-cyclic triplet group" is better for the definition.

We say that a is a generator part of the neutrosophic triplet group.

Theorem 3.24. Let N be a neutro-cyclic triplet group and let a be a generator part of the neutrosophic triplet. Then

1) $\langle neut(a)\rangle$ generates neutro-cyclic triplet subgroup of N.
2) $\langle anti(a)\rangle$ generates neutro-cyclic triplet subgroup of N.

Proof. Straightforward.

4 Neutro-Homomorphism

In this section, we introduce the neutro-homomorphism for the neutrosophic triplet groups. We also study some of their properties. Further, we defined neutro-isomorphisms.

Definition 4.1. Let $(N_1,*_1)$ and $(N_2,*_2)$be two neutrosophic triplet groups. Let

$$f\colon N_1 \to N_2$$

be a mapping. Then f is called neutro-homomorphism if for all $a,b \in N_1$we have

1) $f(a *_1 b) = f(a) *_1 f(b)$,
2) $f\big(neut_2(a)\big) = neut_2(f(a))$, and
3) $f(anti_{*_1}{}^{(k_1)}(a)) = anti_{*_2}{}^{(k_2)} f(a)$,

where $k_1 = 1, 2, \ldots$ is the order of the neutrosophic triplet $(a, neut_{*1}(a), anti_{*1}(a))$ in the case when the element a has more opposites, and one uses the notations:

$$(a,neut_{*_1}(a),anti_{*_1}{}^{(1)}(a)),(a,neut_{*_1}(a),anti_{*_1}{}^{(2)}(a)),(a,neut_{*_1}(a),anti_{*_1}{}^{(3)}(a)), \text{ etc.}$$

and similar notations for the second law $*_2$:

$$(b,neut_{*_2}(b),anti_{*_2}{}^{(1)}(b)),(b,neut_{*_2}(b),anti_{*_2}{}^{(2)}(b)),(b,neut_{*_2}(b),anti_{*_2}{}^{(3)}(b)), \text{ etc.}$$

Theorem 4.1.

Axioms (1), (2), and (3) are equivalent to extending the one-variable (neutro-homomorphism) function $f(x)$ to three-variable (neutro-homomorphism) function $F(x, y, z)$, defined as follows:

$$F\colon N_1^3 \rightarrow N_2^3$$

if $(a, b, c) \in N_1^3$ is a neutrosophic triplet, then

$$F(a, b, c) = (f(a), f(b), f(c)) \in N_2^3$$

is also a neutrosophic triplet.

Hence in general, for $k_1, k_2 = 1, 2, \ldots,$ one has:

$$F(a,neut_{*_1}(a),anti_{*_1}{}^{(k_1)}(a)) = (f(a), f(neut_{*_1}(a)), f(anti_{*_1}{}^{(k_1)}(a)))$$
$$= (f(a),neut_{*_2} f(a),anti_{*_2}{}^{(k_2)} f(a)).$$

Proof. Almost straightforwardly.

We construct a well-defined law of neutrosophic triplets $\#_1$ on N_1^3 as follows:

for any two neutrosophic triplets (a, b, c) and (α, β, γ) from N_1^3, one has:

$(a, b, c) \#_1 (\alpha, \beta, \gamma) = (a *_1 \alpha, b *_1 \beta, c *_1 \gamma)$,

and a well-defined law of neutrosophic triplets $\#_2$ on N_2^3 as follows:

for any two neutrosophic triplets (u, v, w) and $(\delta, \varepsilon, \zeta)$ from N_2^3, one has:

$(u, v, w) \#_2 (\delta, \varepsilon, \zeta) = (u *_2 \delta, v *_2 \varepsilon, w *_2 \zeta)$.

Whence,

$F((a, b, c) \#_1 (\alpha, \beta, \gamma)) = F(a *_1 \alpha, b *_1 \beta, c *_1 \gamma) = (f(a *_1 \alpha), f(b *_1 \beta), f(c *_1 \gamma))$

$= (f(a) *_2 f(\alpha), f(b) *_2 f(\beta), f(c) *_2 f(\gamma))$.

And further, for $b = neut_{*1}(a)$, $c = anti_{*1}{}^{(k11)}(a)$, and respectively $\beta = neut_{*1}(\alpha)$, $\gamma = anti_{*2}{}^{(k12)}(\alpha)$, one gets:

$F(\, a, neut_{*1}(a), anti_{*1}{}^{(k11)}(a)) \#_1 (\alpha, neut_{*2}(\alpha), anti_{*2}{}^{(k12)}(\alpha))\,) =$

$F(\, (a *_1 \alpha), (neut_{*1}(a) *_1 neut_{*1}(\alpha)), (anti_{*1}{}^{(k11)}(a)) *_1 (anti_{*1}{}^{(k12)}(\alpha))\,) =$

$(\, f(a *_1 \alpha), f(neut_{*1}(a) *_1 neut_{*1}(\alpha)), f(anti_{*1}{}^{(k11)}(a) *_1 anti_{*1}{}^{(k12)}(\alpha))\,) =$

$(\, f(a) *_2 f(\alpha), f(neut_{*1}(a)) *_2 f(neut_{*1}(\alpha)), f(anti_{*1}{}^{(k11)}(a)) *_2 f(anti_{*1}{}^{(k12)}(\alpha))\,) =$

$(\, f(a) *_2 f(\alpha), neut_{*2}(f(a)) *_2 neut_{*2}(f(\alpha)), anti_{*2}{}^{(k21)}(f(a)) *_2 anti_{*2}{}^{(k22)}(f(\alpha))\,) =$

$F(\, a, neut_{*1}(a), anti_{*1}{}^{(k11)}(a)\,) \#_2\ F(\, \alpha, neut_{*1}(\alpha), anti_{*1}{}^{(k12)}(\alpha)\,)$.

Therefore $F(x, y, z)$, *for* (x, y, z) neutrosophic triplets in N_1^3, is its self a neutro-homomorphism.

Proposition 4.3. Every neutro-homomorphism is a classical homomorphism by neglecting the classical unitary element in classical homomorphism.

Proof. First, we neglect the classical unitary element that classical homomorphism maps unitary element to the corresponding unitary element. Now suppose that f is a neutro-homomorphism from a neutrosophic triplet group N_1 to a neutrosophic triplet group N_2. Then by condition (1), it follows that f is a classical homomorphism.

Definition 4.4. A neutro-homomorphism is called neutro-isomorphism if it is one-to-one and onto.

5 Distinctions and Comparison

The distinctions between Molaei's Generalized Group [7] and Neutrosophic Triplet Group are:

I. - in MGG for each element there exists a unique neutral element, which can be the classical group unitary element; while in NTG each element may have a unique neutral element but which is different form the classical element;

II. - in MGG there exists a unique inverse of an element, while in NTG there may be many inverses for the same given element;

III. - MGG has a weaker structure than NTG.

IV. - Smarandache (2016-2017) has generalized the NTG to Neutrosophic Extended Triplet Group (NETG), where the *neut(x)* is allowed to be equal to the classical unitary algebraic element of the group theory [14-16].

So far the applications of neutrosophic triplet sets are in Z, modulo n, n ≥ 2.

But new applications can be found, for example in social science:

One person <A> that has an enemy $\langle anti(A_{d_1}\rangle$(enemy in a degree d_1 of enemy-city), and a neutral person $\langle neut(A_{d_1}\rangle$ with respect to$\langle anti(A_{d_1}\rangle$. Then another enemy $\langle anti(A_{d_2}\rangle$ in a different degree of enemy-city, and a neutral $\langle neut(A_{d_2}\rangle$, and so on. Hence one has the neutrosophic triplets:

$$\langle A, \langle neut\ (A_{d_1})\rangle, \langle anti\ (A_{d_1})\rangle \rangle,$$

$$\langle A, \langle neut\ (A_{d_2})\rangle, \langle anti\ (A_{d_2})\rangle \rangle \text{, and so on.}$$

Then we take another person B in the same way...

$$\langle A, \langle neut\ (B_{d_1})\rangle, \langle anti\ (B_{d_1})\rangle \rangle,$$

$$\langle A, \langle neut\ (B_{d_2})\rangle, \langle anti\ (B_{d_2})\rangle \rangle \text{ etc.}$$

More applications may be found, if we deeply think about cases where we have neutrosophic triplets $\langle A, \langle neut\ (A)\rangle, \langle anti\ (A)\rangle \rangle$ in technology and in science.

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Conclusion

Inspired on the Neutrosophic philosophy, we defined for the first time the neutrosophic triplet. Basically, a neutrosophic triplet is a triad of certain elements, which satisfy certain axioms, which highly depend upon the proposed binary operation. The main theme of this paper is first to introduce the neutrosophic triplets, which are completely new notions, and then apply these neutrosophic triplets to introduce the neutrosophic triplet groups. This neutrosophic triplet group has several extra-ordinary properties as compared to the classical group. We also studied some interesting properties of this newly born structure. We further defined neutro-homomorphisms for neutrosophic triplet groups. A neutro-homomorphism is the generalization of the classical homomorphism with two extra conditions. As a further generalization, we gave rise to a new field or research called Neutrosophic Triplet Structures (such as neutrosophic triplet ring, neutrosophic triplet field, neutrosophic triplet vector space, etc.). In the end, we offered main distinctions and comparison of neutrosophic triplet group with the Molaei's generalized group as well as the possible application areas for the neutrosophic triplet groups.

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