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On Optimizing Neutrosophic Complex Programming Using Lexicographic Order

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Abstract: Neutrosophic sets are considered as a generalization of the crisp set, fuzzy set, and intuitionistic fuzzy set for representing the uncertainty, inconsistency, and incomplete knowledge about the real world problems. This paper aims to characterize the solution of complex programming (CP) problem with imprecise data instead of its prices information. The neutrosophic complex programming (NCP) problem is considered by incorporating single valued trapezoidal neutrosophic numbers in all the parameters of objective function and constraints. The score function corresponding to the neutrosophic number is used to transform the problem into the corresponding crisp CP. Here, lexicographic order is applied for the comparison between any two complex numbers. The comparison is developed between the real and imaginary parts separately. Through this manner, the CP problem is divided into two real sub-problems. In the last, a numerical example is solved for the illustration that shows the applicability of the proposed approach. The advantage of this approach is more flexible and makes a real-world situation more realistic.

Keywords: Complex programming; Neutrosophic numbers; Score function; Lexicographic order; Lingo software; Kuhn- Tucker conditions; Neutrosophic optimal solution.

1. Introduction

In many earlier works in complex programming, the researchers considered the real part only of the complex objective function as the objective function. The constraints of the problem considered as a cone in complex space $\mathbb{C}^n$. Since the concept of complex fuzzy numbers was first introduced [17], many researchers studied the problems of the concept of fuzzy complex numbers. This branch subject will be widely applied in fuzzy system theory, especially in fuzzy mathematical programming, and also in complex programming too.

Complex programming problem was studied first by Levinson who studied the linear programming (LP) in complex space [39]. The duality theorem was extended to the quadratic complex programming by an adaption of the technique which was introduced by Dorn [27, 22]. The linear fractional programming in complex space was proposed [45]. Linear and nonlinear complex programming problems were treated by numerous authors [24, 33- 37, 41]. In applications, many practical problems related to complex variables, for instance, electrical engineering, filter theory, statistical signal processing, etc., were studied.

Some more general minimax fractional programming problem with complex variables was proposed with the establishment of the necessary and sufficient optimality conditions [36, 37]. A certain kind of linear programming with fuzzy complex numbers in the objective function coefficients also considered as complex fuzzy numbers [52]. The hyper complex neutrosophic similarity measure was proposed by numerous authors [29]. Also, they discussed its application in multicriteria decision making problem. There was proposed an interval neutrosophic multiple attribute decision-making method with credibility information [50]. Later, the multiple attribute group decision making based on interval neutrosophic uncertain linguistic variables was studied [51].

An extended TOPSIS for multi-attribute decision making problems with neutrosophic cubic information was proposed [42]. A single valued neutrosophic hesitant fuzzy computational algorithm was developed for multiple objective nonlinear optimization problem [9]. A computational algorithm was developed for the neutrosophic optimization model with an application to determine the optimal shale gas water management under uncertainty [10]. The interval complex neutrosophic set was studied by the formulation and applications in decision-making [11]. A group decision-making method was proposed under hesitant interval neutrosophic uncertain linguistic environment [40]. The neutrosophic complex topological spaces was studied, and introduced the concept of neutrosophic complex $\alpha\psi$ connectedness in neutrosophic complex topological spaces [30].

A computational algorithm based on the single-valued neutrosophic hesitant fuzzy was developed for multiple objective nonlinear optimization problems [9]. A neutrosophic optimization model was formulated and presented a computational algorithm for optimal shale gas water management under uncertainty [10]. A multiple objective programming approach was proposed to solve integer valued neutrosophic shortest path problems [32].

Some linguistic approaches were developed to study the interval complex neutrosophic sets in decision making applications [39].

Neutrosophic sets were studied to search some applications in the area of transportations and logistics. A multi-objective transportation model was studied under neutrosophic environment [43]. The multi-criteria decision making based on generalized prioritized aggregation operators was presented under simplified neutrosophic uncertain linguistic environment [46]. Some dynamic interval valued neutrosophic set were proposed by modeling decision making in dynamic environments [48]. A hybrid plithogenic decision-making approach was proposed with quality function deployment for selecting supply chain sustainability metrics [1]. Some applications of neutrosophic theory were studied to solve transition difficulties of IT-based enterprises [2].

Based on plithogenic sets, a novel model for the evaluation of hospital medical care systems was presented [3]. Some decision making applications of soft computing and IoT were proposed for a novel intelligent medical decision support model [4]. A novel neutrosophic approach was applied to evaluate the green supply chain management practices [5]. Numerous researchers studied the under type-2 neutrosophic numbers. An application of under type-2 neutrosophic number was presented for developing supplier selection with group decision making by using TOPSIS [6]. An application of hybrid neutrosophic multiple criteria group decision making approach for project selection was presented [7]. The Resource levelling problem was studied in construction projects under neutrosophic environment [8].

The N-valued interval neutrosophic sets with their applications in the field of medical diagnosis was presented [16]. Based on the pentagonal neutrosophic numbers, the de-neutrosophication technique was proposed with some applications in determining the minimal spanning tree [18]. The pentagonal fuzzy numbers were studied with their different representations, properties, ranking, defuzzification. The concept of pentagonal fuzzy neutrosophic numbers was proposed with some applications in game and transportation models [19- 20]. Various forms of linear as well as non-linear form of trapezoidal neutrosophic numbers, de-neutrosophication techniques were studied. Their application were also presented in time cost optimization technique and sequencing problems [21]. The parametric divergence measure of neutrosophic sets was studied with its application in decision-making situations [25]. A technique for reducing dimensionality of data in decision-making utilizing neutrosophic soft matrices was proposed [26].

In this paper, we aim to characterize the solution of complex programming (NCP) neutrosophic numbers. The score function corresponding to the neutrosophic number is used to convert the problem into the corresponding crisp CP, and hence lexicographic order used for comparing between any two complex numbers. The comparison developed between the real and imaginary parts separately. Through this manner, the CP problem is divided into two real sub-problems.

The outlay of the paper is organized as follows: In section 2; some preliminaries are presented. In section 3, a NCP problem is formulated. Section 4 characterizes a solution to the NCP problem to obtain neutrosophic optimal solution. In section 5, two numerical examples are given for illustration. Finally some concluding remarks are reported in section 6.

2. Preliminaries

In order to discuss our problem conveniently, basic concepts and results related to fuzzy numbers, trapezoidal fuzzy numbers, intuitionistic trapezoidal fuzzy numbers, neutrosophic set, and complex mathematical programming are recalled.

Definition 1. (Trapezoidal fuzzy numbers, Kaur and Kumar [31]).

A fuzzy number $\widetilde{A} = (r, s, t, u)$. is a trapezoidal fuzzy numbers where $r, s, t, u \in \mathbb{R}$ and its membership function is defined as

$$\mu_{\widetilde{A}}(x) = \begin{cases} \frac{x-r}{s-r}, & r \le x \le s, \\ 1, & s \le x \le t, \\ \frac{u-x}{u-t}, & t \le x \le u, \\ 0, & \text{otherwise}, \end{cases}$$

Definition 2. (Intuitionistic fuzzy set, Atanassov, [12]).

A fuzzy set $\widetilde{A}$ is said to be an intuitionistic fuzzy set $\widetilde{A}^{IN}$of a non empty set X if $\widetilde{A}^{IN} = \{\langle x, \mu_{\widetilde{B}^{IN}}, \rho_{\widetilde{B}^{IN}} \rangle : x \in X\}$, where $\mu_{\widetilde{A}^{IN}}$, and $\rho_{\widetilde{B}^{IN}}$ are membership and nonmembership functions such that $\mu_{\widetilde{A}^{IN}}, \rho_{\widetilde{A}^{IN}} : X \to [0, 1]$ and $0 \le \mu_{\widetilde{A}^{IN}} + \rho_{\widetilde{A}^{IN}} \le 1$, for all $x \in X$.

Definition 3. (Intuitionistic fuzzy number, Atanassov, [13]).

An intuitionistic fuzzy set $\widetilde{A}^{IN}$of a $\mathbb{R}$ is called an Intuitionistic fuzzy number if the following conditions hold:

1. There exists $c \in \mathbb{R}$: $\mu_{\widetilde{A}^{IN}}(c) = 1$, and $\rho_{\widetilde{A}^{IN}}(c) = 0$.
2. $\mu_{\widetilde{A}^{IN}} : \mathbb{R} \to [0, 1]$ is continuous function such that

   $0 \le \mu_{\widetilde{A}^{IN}} + \rho_{\widetilde{B}^{IN}} \le 1$, for all $x \in X$.

3. The membership and non-membership functions of $\widetilde{B}^{IN}$ are

$$\mu_{\tilde{B}^{IN}}(x) = \begin{cases} 0, & -\infty < x < r \\ h(x), & r \leq x \leq s \\ 1, & x = s \\ l(x), & s \leq x \leq t \\ 0, & t \leq x < \infty, \end{cases}$$

$$\rho_{\tilde{B}^{IN}}(x) = \begin{cases} 0, & -\infty < x < a \\ f(x), & a \leq x \leq s \\ 1, & x = s \\ g(x), & s \leq x \leq b \\ 0, & b \leq x < \infty, \end{cases}$$

Where $f, g, h, l: \mathbb{R} \rightarrow [0, 1]$, h and g are strictly increasing functions, l and f are strictly decreasing functions with the conditions $0 \leq f(x) + f(x) \leq 1$, and $0 \leq l(x) + g(x) \leq 1$.

Definition 4. (Trapezoidal intuitionistic fuzzy number, Jianqiang and Zhong, [28]).

A trapezoidal intuitionistic fuzzy number is denoted by $\tilde{B}^{IN} = (r, s, t, u), (a, s, t, b)$, where $a \leq r \leq s \leq t \leq u \leq b$ with membership and non-membership functions are defined as:

$$\mu_{\tilde{B}^{INT}}(x) = \begin{cases} \frac{x-r}{s-r}, & r \leq x < s \\ 1, & s \leq x \leq t \\ \frac{u-x}{u-t}, & t \leq x \leq u \\ 0, & \text{otherwise}, \end{cases}$$

$$\rho_{\tilde{B}^{INT}}(x) = \begin{cases} \frac{s-x}{s-a}, & a \leq x < s \\ 0, & s \leq x \leq t \\ \frac{x-t}{b-t}, & t \leq x \leq b \\ 1, & \text{otherwise}, \end{cases}$$

Definition 5 (Neutrosophic set, Smarandache, [44]).

A neutrosophic set $\bar{B}^N$ of non-empty set X is defined as

$$\bar{B}^N = \{\langle x, I_{\bar{B}^N}(x), J_{\bar{B}^N}(x), V_{\bar{B}^N}(x)\rangle : x \in X, I_{\bar{B}^N}(x), J_{\bar{B}^N}(x), V_{\bar{B}^N}(x) \in]0_-, 1^+[\},$$

where $I_{\overline{B}^N}(x), J_{\overline{B}^N}(x)$, and $V_{\overline{B}^N}(x)$ are truth membership function, an indeterminacy- membership function, and a falsity- membership function and there is no restriction on the sum of $I_{\overline{B}^N}(x), J_{\overline{B}^N}(x)$, and $V_{\overline{B}^N}(x)$, so $0^- \leq I_{\overline{B}^N}(x) + J_{\overline{B}^N}(x) + V_{\overline{B}^N}(x) \leq 3^+$, and $]0_-, 1^+[$ is a nonstandard unit interval.

Definition 6. (Single-valued neutrosophic set, Wang et al., [49]).

A Single-valued neutrosophic set $\overline{B}^{SVN}$of a non empty set X is defined as: $\overline{B}^{SVN} = \{\langle x, I_{\overline{B}^N}(x), J_{\overline{B}^N}(x), V_{\overline{B}^N}(x)\rangle : x \in X\}$,

where $I_{\overline{B}^N}(x), J_{\overline{B}^N}(x)$, and $V_{\overline{B}^N}(x) \in [0,1]$ for each $x \in X$ and $0 \leq I_{\overline{B}^N}(x) + J_{\overline{B}^N}(x) + V_{\overline{B}^N}(x) \leq 3$.

Definition 7. (Single-valued neutrosophic number, Thamariselvi and Santhi, [47]).

Let $\tau_{\tilde{b}}, \varphi_{\tilde{b}}, \omega_{\tilde{b}} \in [0,1]$ and $r, s, t, u \in \mathbb{R}$ such that$r \leq s \leq t \leq u$. Then a single valued trapezoidal neutrosophic number, $\tilde{b}^N = \langle (r, s, t, u) : \tau_{\tilde{b}}, \varphi_{\tilde{b}}, \omega_{\tilde{b}} \rangle$ is a special neutrosophic set on $\mathbb{R}$, whose truth-membership, indeterminacy-membership, and falsity-membership functions are

$$\mu_{\tilde{b}}{}^N(x) = \begin{cases} \tau_{\tilde{b}^N}\left(\frac{x-r}{s-r}\right), & r \leq x < s \\ \tau_{\tilde{b}}, & s \leq x \leq t \\ \tau_{\tilde{b}^N}\left(\frac{u-x}{u-t}\right), & t \leq x \leq u \\ 0, & \text{otherwise,} \end{cases}$$

$$\rho_{\tilde{b}}{}^N(x) = \begin{cases} \frac{s-x+\varphi_{\tilde{b}^N}(x-r)}{s-r}, & r \leq x < s \\ \varphi_{\tilde{b}^N}, & s \leq x \leq t \\ \frac{x-t+\varphi_{\tilde{b}^N}(u-x)}{u-t}, & t \leq x \leq u \\ 1, & \text{otherwise,} \end{cases}$$

$$\sigma_{\tilde{b}}{}^N(x) = \begin{cases} \frac{s-x+\omega_{\tilde{b}^N}(x-r)}{s-r}, & r \leq x < s \\ \omega_{\tilde{b}^N}, & s \leq x \leq t \\ \frac{x-t+\omega_{\tilde{b}^N}(u-x)}{u-t}, & t \leq x \leq u \\ 1, & \text{otherwise.} \end{cases}$$

Where $\tau_{\tilde{b}}, \varphi_{\tilde{b}}$, and $\omega_{\tilde{b}}$ denote the maximum truth, minimum-indeterminacy, and minimum falsity membership degrees, respectively. A single-valued trapezoidal neutrosophic number $\tilde{b}^N = \langle (r, s, t, u): \tau_{\tilde{b}^N}, \varphi_{\tilde{b}^N}, \omega_{\tilde{b}^N} \rangle$ may express in ill-defined quantity about b, which is approximately equal to $[s, t]$.

Definition 8.

Let $\tilde{b}^N = \langle (r, s, t, u): \tau_{\tilde{b}^N}, \varphi_{\tilde{b}^N}, \omega_{\tilde{b}^N} \rangle$, and $\tilde{d}^N = \langle (r', s', t', u'): \tau_{\tilde{d}^N}, \varphi_{\tilde{d}^N}, \omega_{\tilde{d}^N} \rangle$ be two single-valued trapezoidal neutrosophic numbers and $v \neq 0$. The arithematic operations on $\tilde{b}^N$, and $\tilde{d}^N$ are

1. $\tilde{b}^N \oplus \tilde{d}^N = \langle (r + r', s + s', t + t', u + u');\ \tau_{\tilde{b}^N} \wedge \tau_{\tilde{d}^N}, \varphi_{\tilde{b}^N} \vee \varphi_{\tilde{d}^N}, \omega_{\tilde{b}^N} \vee \omega_{\tilde{d}^N} \rangle$,

2. $\tilde{b}^N \ominus \tilde{d}^N = \langle (r - u', s - t', t - s', u' - r);\ \tau_{\tilde{b}^N} \wedge \tau_{\tilde{d}^N}, \varphi_{\tilde{b}^N} \vee \varphi_{\tilde{d}^N}, \omega_{\tilde{b}^N} \vee \omega_{\tilde{d}^N} \rangle$,

3. $$\tilde{b}^N \otimes \tilde{d}^N = \begin{cases} \langle (rr', ss', tt', uu');\ \tau_{\tilde{b}^N} \wedge \tau_{\tilde{d}^N}, \varphi_{\tilde{b}^N} \vee \varphi_{\tilde{d}^N}, \omega_{\tilde{b}^N} \vee \omega_{\tilde{d}^N} \rangle, u,\ u' > 0 \\ \langle (ru', st', st', ru');\ \tau_{\tilde{b}^N} \wedge \tau_{\tilde{d}^N}, \varphi_{\tilde{b}^N} \vee \varphi_{\tilde{d}^N}, \omega_{\tilde{b}^N} \vee \omega_{\tilde{d}^N} \rangle, u < 0,\ u' > 0 \\ \langle (uu', ss', tt', rr');\ \tau_{\tilde{b}^N} \wedge \tau_{\tilde{d}^N}, \varphi_{\tilde{b}^N} \vee \varphi_{\tilde{d}^N}, \omega_{\tilde{b}^N} \vee \omega_{\tilde{d}^N} \rangle, u < 0,\ u' < 0, \end{cases}$$

4. $$\tilde{b}^N \oslash \tilde{d}^N = \begin{cases} \langle (r/u', s/t', t/s', u/r');\ \tau_{\tilde{b}^N} \wedge \tau_{\tilde{d}^N}, \varphi_{\tilde{b}^N} \vee \varphi_{\tilde{d}^N}, \omega_{\tilde{b}^N} \vee \omega_{\tilde{d}^N} \rangle, u,\ u' > 0 \\ \langle (u/u', t/t', s/s', r/r');\ \tau_{\tilde{b}^N} \wedge \tau_{\tilde{d}^N}, \varphi_{\tilde{b}^N} \vee \varphi_{\tilde{d}^N}, \omega_{\tilde{b}^N} \vee \omega_{\tilde{d}^N} \rangle, u < 0,\ u' > 0 \\ \langle (u/r', t/s', s/t', r/u');\ \tau_{\tilde{b}^N} \wedge \tau_{\tilde{d}^N}, \varphi_{\tilde{b}^N} \vee \varphi_{\tilde{d}^N}, \omega_{\tilde{b}^N} \vee \omega_{\tilde{d}^N} \rangle, u < 0,\ u' < 0, \end{cases}$$

5. $$k\tilde{d}^N = f(x) = \begin{cases} \langle (kr, ks, kt, k);\ \tau_{\tau_{\tilde{d}^N}},\ \varphi_{\tau_{\tilde{d}^N}}, \omega_{\tau_{\tilde{d}^N}} \rangle, k > 0, \\ \langle (ku, kt, ks, k\,r);\ \tau_{\tau_{\tilde{d}^N}},\ \varphi_{\tau_{\tilde{d}^N}}, \omega_{\tau_{\tilde{d}^N}} \rangle, k < 0, \end{cases}$$

6. $\tilde{d}^{N^{-1}} = \langle (1/u', 1/t', 1/s', 1/r');\ \tau_{\tau_{\tilde{d}^N}},\ \varphi_{\tau_{\tilde{d}^N}}, \omega_{\tau_{\tilde{d}^N}} \rangle, \tilde{d}^N \neq 0.$

Definition 9 (Score function of single-valued trapezoidal neutrosophic number, Thamaraiselvi and Santhi [47]).

A two single-valued trapezoidal neutrosophic numbers $\tilde{b}$, and $\tilde{d}$ can be compared based on the score function as

$$\text{Score function } SC(\tilde{b}^N) = \left(\frac{1}{16}\right)[r+s+t+u] * [\mu_{\tilde{b}^N} + (1-\rho_{\tilde{b}^N}(x) + (1-\sigma_{\tilde{b}^N}(x)].$$

Definition 10. (Thamaraiselvi and Santhi, [47]).

The order relations between $\tilde{b}^N$ and $\tilde{d}^N$ based on $SC(\tilde{b}^N)$ are defined as

1. If $SC(\tilde{b}^N) < SC(\tilde{d}^N)$, then $\tilde{b}^N \prec \tilde{d}^N$
2. If $SC(\tilde{b}^N) = SC(\tilde{d}^N)$, then $\tilde{b}^N \approx \tilde{d}^N$,
3. If $SC(\tilde{b}^N) > SC(\tilde{d}^N)$, then $\tilde{b}^N \succ \tilde{d}^N$,

3. Problem definition and solution concepts

Consider the following single -valued trapezoidal neutrosophic (NCP) problem

$$\text{(NCP)} \qquad \min \ \tilde{F}^N(x) = \tilde{v}^N(x) + i\,\tilde{w}^N(x)$$

Subject to (1)

$$x \in \tilde{X}^N = \left\{ \begin{array}{c} x \in \Re^n : \tilde{f}_r^{\,N}(x) = \tilde{p}_r^{\,N}(x) + i\,\tilde{q}_r^{\,N}(x) \le \tilde{l}_r^{\,N} + i\,\tilde{h}_r^{\,N}, \\ r = 1, 2, \dots, m \end{array} \right\}$$

Where, $\tilde{v}^N(x) = \sum_{j=1}^n \tilde{c}_j^{\,N} x_j$, $\tilde{w}^N(x) = \sum_{j=1}^n \tilde{d}_j^{\,N} x_j, j = 1, 2, \dots, n;$ $\tilde{p}_r^{\,N}(x) = \sum_{j=1}^n x_j^T \tilde{a}_{rj}^{\,N} x_j$,

$\tilde{q}_r^{\,N}(x) = \sum_{j=1}^n \tilde{e}_{rj}^{\,N} x_j$, are convex functions on $\tilde{X}^N$.

All of $\tilde{c}_j^{\,N}, \tilde{d}_j^{\,N}$, $\tilde{a}_{rj}^{\,N}, \tilde{e}_{rj}^{\,N}$ $\tilde{l}_r^{\,N} = (\tilde{l}_1^{\,N}, \tilde{l}_2^{\,N}, \dots, \tilde{l}_m^{\,N})^T$, $\tilde{h}_r^{\,N} = (\tilde{h}_1^{\,N}, \tilde{h}_2^{\,N}, \dots, \tilde{h}_m^{\,N})^T$ are single-valued trapezoidal neutrosophic numbers.

Definition 11.

Lexicographic order of two complex numbers $z_1 = a + i\,b$, and $z_2 = c + i\,d$ is defined as $z_1 \le z_2 \Leftrightarrow a \le c$ and $b \le d$.

Definition 12.

A neutrosophic feasible point x° is called single-valued trapezoidal neutrosophic optimal solution to NCP problem if

$$\tilde{v}^{N}(x^{\circ}) \le \tilde{v}^{N}(x), \text{ and } \tilde{w}^{N}(x^{\circ}) \le \tilde{w}^{N}(x) \text{ for each } x \in \tilde{X}^{N}.$$

According to the score function in Definition 9, the NCP problem is converted into the following crisp CP problem as

$$\text{(CP)} \quad \text{Min } F(x) = v(x) + i\, w(x)$$

Subject to (2)

$$x \in X = \{x \in \mathbb{R}: f_r(x) = p_r(x) + i\, q_r(x) \le l_r + i\, h_r,\ r = 1, 2, \dots, m\}.$$

4. Characterization of neutrosophic optimal solution for NCP problem

To characterize the neutrosophic optimal solution of NCP problem, let us divide the CP problem into the following two sub-problems

$$(P_v) \quad \text{Min } v(x)$$

Subject to (3)

$$x \in X = \{x \in \mathbb{R}^n: f_r(x) = p_r(x) + i\, q_r(x) \le l_r + i\, h_r,\ r = 1, 2, \dots, m\}, \text{ and}$$

$$(P_w) \quad \text{Min } w(x)$$

Subject to (4)

$$x \in X = \{x \in \mathbb{R}^n: f_r(x) = p_r(x) + i\, q_r(x) \le l_r + i\, h_r,\ r = 1, 2, \dots, m\}.$$

Definition 13.

$x^{\circ} \in X$ is said to be an optimal solution for P_{CP} if and only if $v(x^{\circ}) \le v(x)$, and $w(x^{\circ}) \le w(x)$ for each $x \in X$.

Let us denote S_v and S_w be the set of solution for P_v and P_w, respectively, i.e.,

$$S_v = \{x^{\circ} \in X: v(x^{\circ}) \le v(x);\ \text{for all } x \in X\}, \text{ and} \tag{5}$$

$$S_w = \{x^{*} \in X: v(x^{*}) \le v(x);\ \text{for all } x \in X\} \tag{6}$$

Lemma 1. For $S_v \cap S_w \neq \emptyset$, the solution of CP problem is embedded into $S_v \cap S_w$.

Proof. Assume that x° be a solution of CP, this leads to $v(x^{\circ}) \leq v(x);\ \forall\, x \in X$ (i.e., $x^{\circ} \in S_v$). Similarly, $w(x^{\circ}) \leq w(x);\ \forall\, x \in X$ (i. e., $x^{\circ} \in S_w$). Then, $x^{\circ} \in S_v \cap S_w$.

In this paper, we focus on the case $S_v \cap S_w = \emptyset$.

Lemma 2. If S_v and S_w are open, $S_v \cap S_w = \varnothing$, and v, w are strictly convex functions on X then $\check{x} \in S_v$ is a solution of a conjugate function $\bar{F}(x) = v(x) - i\, w(x)$.

Proof. Since $x^{\circ} \in S_v$, then $v(x^{\circ}) \leq v(x)$ for all $x \in X$. Also,

$$v(x^{\circ}) \leq v(x^{*}) \text{ for all } x^{*} \in S_v \subset X \tag{7}$$

But $x^{*} \in S_w$ which means that $w(x^{*}) \leq v(x^{\circ})$, for all $\check{x} \in S_v \subset X$ and $-i\,w(x^{*}) \geq -i\,w(x^{\circ})$ i. e.,

$$-i\,w(x^{\circ}) \leq -i\,w(x^{*}) \tag{8}$$

From (7) and (8), we get

$v(x^{\circ}) - i\,w(x^{\circ}) \leq v(x^{*}) - i\,w(x^{*})$, for all $x^{*} \in S_w$, (.i.e., $x^{\circ} \in S_v$)is a solution of a conjugate function $\bar{F}(x) = v(x) - i\,w(x)$. Now we will prove that there is no $\hat{x} \in X$ and $\hat{x} \notin S_v$ such that

$$\bar{F}(\hat{x}) = v(\hat{x}) - i\,w(\hat{x}) \leq \bar{F}(x^{\circ}) = v(x^{\circ}) - i\,w(x^{\circ}) \tag{9}$$

There are two cases:

Case 1: Assume that $x \in X$ $\hat{x} \notin S_v$, $\acute{x} \in S_v$ and $v(\hat{x}) - i\,w(\hat{x}) \leq v(x^{\circ}) - i\,w(x^{\circ})$ i.e.,

$$w(x^{\circ}) \leq w(\hat{x})$$

Since the function $w(x)$ is strictly convex and S_w is open, then

$$w(\tau\hat{x} + (1-\tau)x^{\circ}) < \tau w(\hat{x}) + (1-\tau)w(x^{\circ}),\ 0 < \tau < 1. \text{ This leads to}$$

$$w(\tau\hat{x} + (1-\tau)x^{\circ}) < \tau w(\hat{x}) + (1-\tau)w(\hat{x}) \text{ i. e.,}$$

For certain τ such that $\tau\hat{x} + (1-\tau)x^{\circ} \in S_w$, we obtain

$$w\left(\tau\hat{x}+(1-\tau)x^{\circ}\right)<w\left(\hat{x}\right),$$

which contradicts to $\hat{x}\in S_w$ i.e., there is no $\hat{x}\in X,\ \hat{x}\notin S_v,\ \hat{x}\in S_w$ such that

$$\overline{F}(\hat{x})=v(\hat{x})-i\,w(\hat{x})\leq\overline{F}\left(x^{\circ}\right)=v\left(x^{\circ}\right)-i\,w\left(x^{\circ}\right)$$

Case 2: Assume that $\hat{x}\in X,\hat{x}\notin S_v,\hat{x}\notin S_w$ and $v(\hat{x})-i\,w(\hat{x})<v\left(x^{\circ}\right)-i\,w\left(x^{\circ}\right)$ i.e.,

$$v(\hat{x})<v\left(x^{\circ}\right)\ \text{ and }\ w\left(x^{\circ}\right)<w(\hat{x}).$$

Since the function $v(x)$ is strictly convex and S_v is open, then,

$$v\left(\tau x^{\circ}+(1-\tau)\hat{x}\right)<\tau v\left(x^{\circ}\right)+(1-\tau)v(\hat{x}),\ 0\leq\tau\leq1.$$ This leads to

$$v\left(\tau x^{\circ}+(1-\tau)\hat{x}\right)<\tau v\left(x^{\circ}\right)(1-\tau)v(\hat{x}),$$ i.e., for certain τ, we have

$\tau x^{\circ}+(1-\tau)\hat{x}\in S_v$, such that $\tau x^{\circ}+(1-\tau)\hat{x}\in S_v$, we have

$$v\left(\tau x^{\circ}+(1-\tau)\hat{x}\right)<v\left(x^{\circ}\right),$$

which contradicts $x^{\circ}\in S_v$. Thus, there is no $\hat{x}\in X$ such that

$$v(\hat{x})-i\,w(\hat{x})<v\left(x^{\circ}\right)-i\,w\left(x^{\circ}\right)$$

5. Numerical examples

Example1. (Illustration of Lemma 1)

Consider the following complex problem

$$\min(\cos x+i\sin x)$$

Subject to (10)

$$x\in X=\{x\in\mathbb{R}:0\leq x\leq\pi\}.$$

Problem (10) is divided into the following two problems as:

(P_v) $\quad$ $\text{Min}\cos x$

Subject to (11)

$x\in X=\{x\in\mathbb{R}:0\leq x\leq\pi\}$, and

(P_w) $\quad$ $\text{Min}\sin x$

Subject to (12)

$$x \in X = \{x \in \mathbb{R}: 0 \leq x \leq \pi\},$$

The optimal solutions of problem (11) and (12) are $x = \pi$ (i. e., $S_v = \{\pi\}$), and $x = (0, \pi)$, (i. e., $S_w = \{0, \pi\}$), respectively. Thus, the optimal solution of problem (10) is $x = \pi \in S_v \cap S_w$.

Example2. (Illustration of Lemma 2)

Consider the following NCP problem:

$$\text{Min} \quad \tilde{F}^N(x) = \left(\tilde{c}_1{}^N x_1 + \tilde{c}_2{}^N x_2\right) + i \left(\tilde{d}_1{}^N x_1 - \tilde{d}_2{}^N x_2\right)$$

Subject to (13)

$$\left(\tilde{p}_{11}{}^N x_1^2 + \tilde{p}_{22}{}^N x_2^2\right) + i \left(\tilde{q}_1{}^N x_1 + \tilde{q}_2{}^N x_2\right) = \tilde{e}^N + i\tilde{g}^N.$$

Where,

$$\tilde{c}_1{}^N = \langle 5, 8, 10, 14; 0.3, 0.6, 0.6\rangle,$$

$$\tilde{c}_2{}^N = \langle 0, 1, 3, 6; 0.7, 0.5, 0.3\rangle,$$

$$\tilde{d}_1{}^N = \langle 4, 8, 11, 15; 0.6, 0.3, 0.2\rangle,$$

$$\tilde{d}_2{}^N = \langle 16, 18, 22, 30; 0.6, 0.2, 0.4\rangle,$$

$$\tilde{p}_{11}{}^N = \tilde{p}_{22}{}^N = \langle 0, 1, 3, 6; 0.7, 0.5, 0.3\rangle,$$

$$\tilde{q}_1{}^N = \langle 0, 1, 3, 6; 0.7, 0.5, 0.3\rangle = \tilde{q}_2{}^N,$$

$$\tilde{e}^N = \langle 4, 8, 11, 15; 0.6, 0.3, 0.2\rangle,$$

$$\tilde{g}^N = \langle 0, 1, 3, 6; 0.7, 0.5, 0.3\rangle.$$

Using the score function of the single- valued trapezoidal neutrosophic number introduced in Definition 9, the above problem become:

$$\text{Min} \quad F(x) = (3x_1 + x_2) + i(5x_1 - 11x_2)$$

Subject to (14)

$$x_1^2 + x_2^2 + i(x_1 - x_2) \leq 5 + i\,.$$

According to Lexicographic order, the problem is divided into the following two sub-problems as:

(P_v) Min $v(x) = 3x_1 + x_2$

Subject to (15)

$x_1^2 + x_2^2 \leq 5,$

$x_1 - x_2 \leq 1,$ and

(P_w) Min $w(x) = 5x_1 - 11x_2$

Subject to (16)

$x_1^2 + x_2^2 \leq 5,$

$x_1 - x_2 \leq 1.$

By applying the Kuhn-Tucker optimality conditions [14, 22], the optimal solutions of problems (15), and (16) are illustrated as in the following Tables 1 -2.

Table 1. The set of solution of $(\mathbf{P_v})$

S_v	Optimum value
$\{(-2,-1)\}$	$P_v = -7$ $\tilde{P}_v{}^N = \langle -34, -23, -17, -10; 0.3, 0.6, .06 \rangle$

Table 2. The set of solution of $(\mathbf{P_w})$

S_w	Optimum value
$\{(-2,1)\}$	$P_w = -21$ $\tilde{P}_w{}^N = \langle -60, -44, -34, -24; 0.6, 0.3, 0.4\rangle$

Therefore, $S_v \cap S_w = \emptyset$ and S_v is not a solution of the conjugate function $v(x) - i\, w(x)$, because of $v(x)$, and $w(x)$ are not strictly convex functions.

6. Concluding Remarks

In this paper, the solution of complex programming (NCP) with single valued trapezoidal neutrosophic numbers in all the parameters of objective function and constraints has been characterized. Through the use of the score function, the NCP has converted into the corresponding crisp CP problem and hence Lexicographic order has been used for comparing between any two complex numbers. The comparison was developed between the real and imaginary parts separately. Through this manner, the CP problem has divided into two real sub-problems. The main contribution of this approach is more flexible and makes a situation realistic to real world application. The obtained results are more significant to enhance the applicability of single-valued trapezoidal neutrosophic number in various new fields of decision-making situations. The future research scope is to apply the proposed approach to more complex and new applications. Another possibility is to work on the interval type complex neutrosophic sets for the applications in forecasting field.

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