

Article

# On Product of Smooth Neutrosophic Topological Spaces

Kalaivani Chandran <sup>1,\*</sup>, Swathi Sundari Sundaramoorthy <sup>2</sup>, Florentin Smarandache <sup>3</sup>   
and Saeid Jafari <sup>4</sup>

<sup>1</sup> Department of Mathematics, Sri Sivasubramaniya Nadar College of Engineering, Kalavakkam 603110, India

<sup>2</sup> Department of Mathematics, V. V. Vanniaperumal College for Women, Virudhunagar 626001, India;  
swathisundari@vvvcollege.org

<sup>3</sup> Department of Mathematics and Science, University of New Mexico, Gallup, NM 87301, USA;  
smarand@unm.edu

<sup>4</sup> Department of Mathematics, College of Vestsjaelland South, Herrestrade 11, 4200 Slagelse, Denmark;  
sj@vucvklar.dk

\* Correspondence: kalaivanic@ssn.edu.in

Received: 20 August 2020; Accepted: 17 September 2020; Published: 21 September 2020

**Abstract:** In this paper, we develop the notion of the basis for a smooth neutrosophic topology in a more natural way. As a sequel, we define the notion of symmetric neutrosophic quasi-coincident neighborhood systems and prove some interesting results that fit with the classical ones, to establish the consistency of theory developed. Finally, we define and discuss the concept of product topology, in this context, using the definition of basis.

**Keywords:** neutrosophic sets; smooth neutrosophic topology; basis; subbasis; smooth neutrosophic product topology

**MSC:** 18B30

## 1. Introduction

The idea of neutrosophy was initiated and developed by Smarandache [1] in 1999. In recent decades the theory was used at various junctions of mathematics. More precisely, the theory made an outstanding advancement in the field of topological spaces. Salama et al. and Hur et al. [2–6] are some who posted their works of neutrosophic topological spaces, following the approach of Chang [7] in the context of fuzzy topological spaces. One can easily observe that the fuzzy topology introduced by Chang is a crisp collection of fuzzy subsets.

Šostak [8] observed that Chang's approach is crisp in nature and so he redefined the notion of fuzzy topology, often referred as smooth fuzzy topology, as a function from the collection of all fuzzy subsets of  $X$  to  $[0, 1]$ ; Fang Jin-ming et al. and Vembu et al. [9,10] are some who discussed the concept of basis as a function from a suitable collection of fuzzy subsets of  $X$  to  $[0, 1]$ . Yan, Wang, Nanjing, Liang and Yan [11,12] developed a parallel theory in the context of intuitionistic  $I$ -fuzzy topological spaces.

The notion of a single-valued neutrosophic set was proposed by Wang [13] in 2010. In 2016, Gayyar [14] introduced the concept of smooth neutrosophic topological spaces. The notion of the basis for an ordinary single-valued neutrosophic topology was defined and discussed by Kim [15]. Salama, Alblowi, Shumrani, Muhammed Gulisten, Smarandache, Saber, Alsharari, Zhang and Sunderraman [4,16,17] are some others who posted their work in the context of single-valued neutrosophic topological spaces.

In Section 2, we give all basic definitions and results, which are important prerequisites that are needed to go through the theory developed in this paper. In Section 3, we define the notion of the basis

and subbasis for a smooth neutrosophic topology; further, we develop the theory using the concept of neutrosophic quasi-coincident neighborhood systems. In addition, we prove some results which are similar to the classical ones, to establish the consistency of theory developed. Finally, in Section 4, we define and discuss the product of smooth neutrosophic spaces using our definition of basis.

## 2. Preliminaries

In this section, we give all basic definitions and results which we need to go through our work. As usual  $\mathbb{R}$  and  $\mathbb{Q}$  denote the sets of all real numbers and rationals respectively. First we give the definition of a neutrosophic set [1,4].

**Definition 1.** Let  $X$  be a non-empty set. A neutrosophic set in  $X$  is an object having the form

$$\mathbb{N} = \{ \langle x, T_{\mathbb{N}}, I_{\mathbb{N}}, F_{\mathbb{N}} \rangle : x \in X \}$$

where

$$T_{\mathbb{N}} : X \rightarrow ]^{-}0, 1^{+}[ , I_{\mathbb{N}} : X \rightarrow ]^{-}0, 1^{+}[ , F_{\mathbb{N}} : X \rightarrow ]^{-}0, 1^{+}[$$

and

$$^{-}0 \leq T_{\mathbb{N}}(x) + I_{\mathbb{N}}(x) + F_{\mathbb{N}}(x) \leq 3^{+},$$

represent the degree of membership (namely,  $T_{\mathbb{N}}(x)$ ), the degree of indeterminacy (namely,  $I_{\mathbb{N}}(x)$ ) and the degree of non-membership (namely,  $F_{\mathbb{N}}(x)$ ), for all  $x \in X$  to the set to the set  $\mathbb{N}$ .

Here  $^{-}0 = 1 - \epsilon$  and  $1^{+} = 1 + \epsilon$  where  $\epsilon$  is infinitesimal number and  $\epsilon > 0$ ; further,  $1$  and  $\epsilon$  denote standard part and non-standard part of  $1 + \epsilon$ ;  $0$  and  $\epsilon$  denote the standard part and non-standard part of  $0 - \epsilon$ . While dealing with scientific and engineering problems in real life applications, it is difficult to use a neutrosophic set with values from  $]^{-}0, 1^{+}[$ . In order to overcome this draw back, Wang et al. [13] defined the single-valued neutrosophic set, which is a particular case of the neutrosophic set.

**Definition 2.** [13] Let  $X$  be a space of points (objects) with a generic element in  $X$  denoted by  $x$ . Then  $\mathbb{N}$  is called a single-valued neutrosophic set in  $X$  if  $\mathbb{N}$  has of the form  $\mathbb{N} = \langle T_{\mathbb{N}}, I_{\mathbb{N}}, F_{\mathbb{N}} \rangle$ , where  $T_{\mathbb{N}}, I_{\mathbb{N}}, F_{\mathbb{N}} : X \rightarrow [0, 1]$ . In this case,  $T_{\mathbb{N}}, I_{\mathbb{N}}, F_{\mathbb{N}}$  are called the truth membership function, indeterminacy membership function and falsity membership function, respectively.

For conventional reasons and as there is no ambiguity, we refer a single-valued neutrosophic set simply as a neutrosophic set throughout this paper; we also restate the definition, in order to view it explicitly as a function from a non-empty set  $X$  to  $\zeta = [0, 1]^3$ , in the following way:

Let  $X$  be a nonempty set and  $I = [0, 1]$ . A neutrosophic set  $\mathbb{N}$  on  $X$  is a mapping defined as  $\mathbb{N} = \langle T_{\mathbb{N}}, I_{\mathbb{N}}, F_{\mathbb{N}} \rangle : X \rightarrow \zeta$ , where  $\zeta = I^3$  and  $T_{\mathbb{N}}, I_{\mathbb{N}}, F_{\mathbb{N}} : X \rightarrow I$  such that  $0 \leq T_{\mathbb{N}} + I_{\mathbb{N}} + F_{\mathbb{N}} \leq 3$ .

We denote the set of all neutrosophic sets of  $X$  by  $\zeta^X$  and the neutrosophic sets  $\langle 0, 1, 1 \rangle$  and  $\langle 1, 0, 0 \rangle$  by  $0_X$  and  $1_X$  respectively. Let  $(r, s, t), (l, m, n) \in \zeta$ ; then

- $(r, s, t) \sqcup (l, m, n) = (r \vee l, s \wedge m, t \wedge n)$ ;
- $(r, s, t) \sqcap (l, m, n) = (r \wedge l, s \vee m, t \vee n)$ ;
- $(r, s, t) \sqsubseteq (l, m, n) = (r \leq l, s \geq m, t \geq n)$ ;
- $(r, s, t) \sqsupseteq (l, m, n) = (r \geq l, s \leq m, t \leq n)$ .

**Definition 3.** [1,4] Let  $X$  be a non-empty set and let  $\mathbb{N}, \mathbb{M} \in \zeta^X$  be given by  $\mathbb{N} = \langle T_{\mathbb{N}}, I_{\mathbb{N}}, F_{\mathbb{N}} \rangle$  and  $\mathbb{M} = \langle T_{\mathbb{M}}, I_{\mathbb{M}}, F_{\mathbb{M}} \rangle$ . Then

- The complement of  $\mathbb{N}$  denoted by  $\mathbb{N}^c$  is given by

$$\mathbb{N}^c = \langle 1 - T_{\mathbb{N}}, 1 - I_{\mathbb{N}}, 1 - F_{\mathbb{N}} \rangle .$$

- The union of  $N$  and  $M$  denoted by  $N \sqcup M$  is a neutrosophic set in  $X$  given by

$$N \sqcup M = \langle T_N \vee T_M, I_N \wedge I_M, F_N \wedge F_M \rangle.$$

- The intersection of  $N$  and  $M$  denoted by  $N \sqcap M$  is a neutrosophic set in  $X$  given by

$$N \sqcap M = \langle T_N \wedge T_M, I_N \vee I_M, F_N \vee F_M \rangle.$$

- The product of  $N$  and  $M$  denoted by  $N \times M$  is given by

$$(N \times M)(x, y) = N(x) \sqcap M(y), \forall (x, y) \in X \times Y.$$

- We say that  $N \sqsubseteq M$  if  $T_N \leq T_M, I_N \geq I_M, F_N \geq F_M$ .

For an any arbitrary collection  $\{N_i\}_{i \in J} \subseteq \zeta^X$  of neutrosophic sets the union and intersection are given by

- $\sqcup_{i \in J} N_i = \left\langle \bigvee_{i \in J} T_{N_i}, \bigwedge_{i \in J} I_{N_i}, \bigwedge_{i \in J} F_{N_i} \right\rangle$
- $\sqcap_{i \in J} N_i = \left\langle \bigwedge_{i \in J} T_{N_i}, \bigvee_{i \in J} I_{N_i}, \bigvee_{i \in J} F_{N_i} \right\rangle.$

**Definition 4.** Let  $X$  be a nonempty set and  $x \in X$ . If  $r \in (0, 1], s \in [0, 1)$  and  $t \in [0, 1)$ , then a neutrosophic point  $x_{r,s,t}$  in  $X$  given by

$$x_{r,s,t}(z) = \begin{cases} (r, s, t), & \text{if } z = x, \\ (0, 1, 1), & \text{otherwise.} \end{cases}$$

We say  $x_{r,s,t} \in N$  if  $x_{r,s,t} \sqsubseteq N$ . To avoid the ambiguity, we denote the set of all neutrosophic points by  $pt(\zeta^X)$ .

**Definition 5.** A neutrosophic set  $N$  is said to be quasi-coincident with another neutrosophic set  $M$ , denoted by  $N[q]M$ , if there exists an element  $x \in X$  such that

$$T_N(x) + T_M(x) > 1 \text{ or } I_N(x) + I_M(x) < 1 \text{ or } F_N(x) + F_M(x) < 1.$$

If  $M$  is not quasi-coincident with  $N$ , then we write  $M[\bar{q}]N$ .

**Definition 6.** [14] Let  $X$  be a nonempty set. Then a neutrosophic set  $\mathfrak{T} = \langle T_{\mathfrak{T}}, I_{\mathfrak{T}}, F_{\mathfrak{T}} \rangle : \zeta^X \rightarrow \zeta$  is said to be a smooth neutrosophic topology on  $X$  if it satisfies the following conditions:

- C1**  $\mathfrak{T}(0_X) = \mathfrak{T}(1_X) = (1, 0, 0)$ .
- C2**  $\mathfrak{T}(N \sqcap M) \supseteq \mathfrak{T}(N) \sqcap \mathfrak{T}(M), \forall N, M \in \zeta^X$ .
- C3**  $\mathfrak{T}(\sqcup_{i \in J} N_i) \supseteq \sqcap_{i \in J} \mathfrak{T}(N_i), \forall N_i \in \zeta^X, i \in J$ .

The pair  $(X, \mathfrak{T})$  is called a smooth neutrosophic topological space.

### 3. The Basis for a Smooth Neutrosophic Topology

The main objective of this section is to define and discuss the concept of basis for a neutrosophic topology. Many fundamental classical statements and theories describe ways to obtain a topology from a basis; every topology is a basis for itself; characterizations of a set to form a basis; comparison of two topologies is a way to get a basis from a subbasis; quasi-neighborhood systems are discussed. Though the structural development of the theory is same as the ones followed in the context of classical and fuzzy topological spaces, the strategies following the proofs of the statements are entirely different. We start with the definition of a basis for a smooth neutrosophic topology.

**Definition 7.** Let  $\mathfrak{B} : \zeta^X \rightarrow \zeta$  be a function that satisfies:

**B1** If  $x \in X$  and  $\epsilon, \delta > 0$ , then there exists  $M \in \zeta^X$  such that

$$M(x) \supseteq 1_X(x) - (\delta, 0, 0) \text{ and } \mathfrak{B}(M) \supseteq (1, 0, 0) - (\epsilon, 0, 0).$$

**B2** If  $x \in X, M, N \in \zeta^X$  and  $\epsilon, \delta > 0$ , then there exists  $L \in \zeta^X$  such that  $L \sqsubseteq M \sqcap N$ ,

$$L(x) \supseteq (M(x) \sqcap N(x)) - (\delta, 0, 0) \text{ and } \mathfrak{B}(L) \supseteq (\mathfrak{B}(M) \sqcap \mathfrak{B}(N)) - (\epsilon, 0, 0).$$

Then  $\mathfrak{B}$  is called a basis for a smooth neutrosophic topology on  $X$ .

Any function  $\mathfrak{S} : \zeta^X \rightarrow \zeta$  satisfying **B1** is called a subbasis of a smooth neutrosophic topology on  $X$ . A collection  $\{M_\lambda\}_{\lambda \in \Lambda}$  of neutrosophic sets is said to be an inner cover for a neutrosophic set  $M$  if  $M = \sqcup M_\lambda$ .

**Definition 8.** Let  $\mathfrak{B}$  be a basis for a smooth neutrosophic topology on  $X$ . Then the smooth neutrosophic topology  $\mathfrak{T} : \zeta^X \rightarrow \zeta$  generated by  $\mathfrak{B}$  is defined as follows:

$$\mathfrak{T}(M) = \begin{cases} (1, 0, 0) & \text{if } M = 0_x \\ \sqcup_{\Lambda \in \Gamma} \{ \sqcap_{M_\lambda \in \mathcal{L}_\Lambda} \{ \mathfrak{B}(M_\lambda) \} \} & \text{if } M \neq 0_x \end{cases}$$

where  $\{ \mathcal{L}_\Lambda \}_{\Lambda \in \Gamma}$  is the collection of all inner covers  $\mathcal{L}_\Lambda = \{ M_\lambda \}_{\lambda \in \Lambda}$  of  $M$ .

It is clear to see that  $\mathfrak{T}(M) \supseteq \mathfrak{B}(M)$ ; the strict inequality may hold; in fact, it may happen that  $\mathfrak{B}(M) = (0, 1, 1)$  and  $\mathfrak{T}(M) = (1, 0, 0)$ ; however, this is not unnatural as even in the crisp theory a subset that is not an element of a basis may be an element of the topology generated by it. However, we have a question: "If  $\mathfrak{B}(M) \sqsupset (0, 1, 1)$ , can  $\mathfrak{T}(M) \sqsupset \mathfrak{B}(M)$ ?" Of course this may happen, as seen in the following example.

**Example 1.** Let  $X = [0, 1]$ . For any subset  $A \subseteq [0, 1]$ , let  $\xi_A$  denote the neutrosophic set in  $X$  defined by

$$\xi_A(x) = \begin{cases} (1, 0, 0) & \text{if } x \in A \\ (0, 1, 1) & \text{otherwise.} \end{cases}$$

Define  $\mathfrak{B} : \zeta^X \rightarrow \zeta$  by

$$\mathfrak{B}(M) = \begin{cases} (1, 0, 0) & \text{if } M = 1_X \\ (1, 0, 0) & \text{if } M = \xi_{\{(q,1)\}}, \text{ where } q \text{ is rational} \\ (\frac{1}{2}, 0, 0) & \text{if } M = \xi_{\{1\}} \\ (\frac{n}{n+1}, 0, 0) & \text{if } M = \xi_{\{\frac{n}{n+1}\}}, \text{ where } n \in \mathbb{N} \\ (0, 1, 1) & \text{otherwise.} \end{cases}$$

Then  $\mathfrak{B}$  is a basis for a smooth neutrosophic topology  $\mathfrak{T}$  on  $X$ . We note that  $\mathfrak{B}(\xi_{\{(a,1)\}}) = (0, 1, 1)$ , whereas  $\mathfrak{T}(\xi_{\{(a,1)\}}) = (1, 0, 0) \forall a \in [0, 1] \cap \mathbb{Q}^c$  and  $\mathfrak{B}(\xi_{\{1\}}) = (\frac{1}{2}, 0, 0)$ , whereas  $\mathfrak{T}(\xi_{\{1\}}) = (1, 0, 0) \sqsupset \mathfrak{B}(\xi_{\{1\}})$ .

**Theorem 1.** Let  $\mathfrak{B}$  be a basis and  $\mathfrak{T}$  be as defined in Definition 8; then  $\mathfrak{T}$  is a smooth neutrosophic topology on  $X$ .

**Proof.** From the definition of  $\mathfrak{T}$  it directly follows that  $\mathfrak{T}(0_x) = (1, 0, 0)$ . Next we wish to show that  $\mathfrak{T}(1_X) = (1, 0, 0)$ . Indeed, let  $x \in X$  and  $\delta, \epsilon > 0$ ; then by the definition of a basis for a smooth neutrosophic topology, there exists  $M_{x,\delta,\epsilon} \in \zeta^X$  such that  $M_{x,\delta,\epsilon}(x) \supseteq 1_X(x) - (\delta, 0, 0)$  and  $\mathfrak{B}(M_{x,\delta,\epsilon}) \supseteq (1, 0, 0) - (\epsilon, 0, 0)$ , which in turn implies that  $T_{M_{x,\delta,\epsilon}}(x) \geq 1 - \delta$ ,  $I_{M_{x,\delta,\epsilon}}(x) \leq 0$  and  $F_{M_{x,\delta,\epsilon}}(x) \leq 0$ . Thus it follows that

$$\begin{aligned} \sqcup_{x,\delta} M_{x,\delta,\epsilon} &= \left\langle \bigvee_{x,\delta} T_{M_{x,\delta,\epsilon}}, \bigwedge_{x,\delta} I_{M_{x,\delta,\epsilon}}, \bigwedge_{x,\delta} F_{M_{x,\delta,\epsilon}} \right\rangle \\ &= (1, 0, 0) \\ &= 1_X. \end{aligned}$$

If we let  $\mathcal{L}_\epsilon = \{M_{x,\delta,\epsilon}\}_{x,\delta}$ , then it is easy to see that  $\mathcal{L}_\epsilon$  is an inner cover for  $1_X$ . However, since  $\mathfrak{B}(M_{x,\delta,\epsilon}) \supseteq (1, 0, 0) - (\epsilon, 0, 0)$ , we have

$$T_{\mathfrak{B}(M_{x,\delta,\epsilon})} \geq 1 - \epsilon, I_{\mathfrak{B}(M_{x,\delta,\epsilon})} \leq 0 \text{ and } F_{\mathfrak{B}(M_{x,\delta,\epsilon})} \leq 0.$$

Therefore

$$\begin{aligned} \prod_{x \in X, \delta > 0} \{\mathfrak{B}(M_{x,\delta,\epsilon})\} &= \left\langle \bigwedge_{x,\delta} T_{\mathfrak{B}(M_{x,\delta,\epsilon})}, \bigvee_{x,\delta} I_{\mathfrak{B}(M_{x,\delta,\epsilon})}, \bigvee_{x,\delta} F_{\mathfrak{B}(M_{x,\delta,\epsilon})} \right\rangle \\ &\supseteq (1 - \epsilon, 0, 0). \end{aligned}$$

Thus for every  $\epsilon > 0$ , there exists an inner cover  $\mathcal{L}_\epsilon = \{M_{x,\delta,\epsilon}\}_{x,\delta}$  of  $1_X$  such that

$$\prod_{x \in X, \delta > 0} \{\mathfrak{B}(M_{x,\delta,\epsilon})\} \supseteq (1 - \epsilon, 0, 0).$$

Therefore

$$\mathfrak{T}(1_X) \supseteq \sqcup_{\mathcal{L}_\epsilon} \left\{ \prod_{x \in X, \delta > 0} \{\mathfrak{B}(M_{x,\delta,\epsilon})\} \right\} \supseteq (1, 0, 0)$$

and hence  $\mathfrak{T}(1_X) = (1, 0, 0)$ .

Next we claim that  $\mathfrak{T}(M \sqcap N) \supseteq \mathfrak{T}(M) \wedge \mathfrak{T}(N)$  for any two neutrosophic sets  $M, N$  in  $\zeta^X$ . Suppose  $M \sqcap N = 0_x$ ; then there is nothing to prove. Let  $M \sqcap N \neq 0_x$  and let  $\epsilon > 0$ . Then there exist inner covers  $\{M_\lambda\}_{\lambda \in \Lambda_1}$  and  $\{N_\gamma\}_{\gamma \in \Lambda_2}$ , such that  $\prod_{\lambda \in \Lambda_1} \{\mathfrak{B}(M_\lambda)\} = \mathfrak{T}(M) - (\frac{\epsilon}{2}, 0, 0)$  and

$$\prod_{\gamma \in \Lambda_2} \{\mathfrak{B}(N_\gamma)\} \supseteq \mathfrak{T}(N) - (\frac{\epsilon}{2}, 0, 0).$$

Let  $L_{\lambda,\gamma} = M_\lambda \sqcap N_\gamma$  for  $\lambda \in \Lambda_1$  and  $\gamma \in \Lambda_2$  and let  $\Lambda$  denote the set of all pairs  $(\lambda, \gamma)$  for which  $L_{\lambda,\gamma} \neq 0_x$ . Now since  $M \sqcap N \neq 0_x$  there exists an  $x \in X$  such that  $M(x) \sqcap N(x) \neq (0, 1, 1)$ , which implies  $M(x) \neq (0, 1, 1)$  and  $N(x) \neq (0, 1, 1)$ ; then by the definition of an inner cover there exist  $M_{\lambda_0}$  and  $N_{\gamma_0}$  in the corresponding inner covers, such that  $M_{\lambda_0}(x) \sqcap N_{\gamma_0}(x) \neq (0, 1, 1)$  and hence  $(\lambda_0, \gamma_0) \in \Lambda$ . Thus we have  $\Lambda \neq \emptyset$ . Now for any  $(\lambda, \gamma) \in \Lambda$ ,  $x \in X$  and  $\delta > 0$ , let  $D_{\lambda,\gamma,x,\delta} \in \zeta^X$  be such that

$$\begin{aligned} D_{\lambda,\gamma,x,\delta}(x) &\supseteq M_\lambda(x) \sqcap N_\gamma(x) - (\delta, 0, 0) \\ &= L_{\lambda,\gamma}(x) - (\delta, 0, 0), \end{aligned}$$

$D_{\lambda,\gamma,x,\delta} \sqsubseteq L_{\lambda,\gamma}$  and

$$\mathfrak{B}(D_{\lambda,\gamma,x,\delta}) \supseteq (\mathfrak{B}(M_\lambda) \sqcap \mathfrak{B}(N_\gamma)) - (\frac{\epsilon}{2}, 0, 0).$$

Then the collection  $\{D_{\lambda,\gamma,x,\delta}\}_{x,\delta}$  is an inner cover for  $L_{\lambda,\gamma}$  and hence the collection  $\{D_{\lambda,\gamma,x,\delta}\}_{\lambda,\gamma,x,\delta}$  is an inner cover for  $M \sqcap N$ .

Additionally, we have,

$$\begin{aligned}
 \bigcap_{\substack{x \in X, \delta > 0 \\ (\lambda, \gamma) \in \Lambda}} \{ \mathfrak{B}(D_{\lambda, \gamma, x, \delta}) \} P &\supseteq \bigcap_{(\lambda, \gamma) \in \Lambda} \{ \mathfrak{B}(M_\lambda) \sqcap \mathfrak{B}(N_\gamma) - (\frac{\epsilon}{2}, 0, 0) \} \\
 &= \bigcap_{(\lambda, \gamma) \in \Lambda} \{ \mathfrak{B}(M_\lambda) \sqcap \mathfrak{B}(N_\gamma) \} - (\frac{\epsilon}{2}, 0, 0) \\
 &\supseteq \left\{ \bigcap_{(\lambda, \gamma) \in \Lambda} \{ \mathfrak{B}(M_\lambda) \} \sqcap \bigcap_{(\lambda, \gamma) \in \Lambda} \{ \mathfrak{B}(N_\gamma) \} \right\} - (\frac{\epsilon}{2}, 0, 0) \\
 &\supseteq \left\{ \bigcap_{\lambda \in \Lambda_1} \{ \mathfrak{B}(M_\lambda) \} \sqcap \bigcap_{\gamma \in \Lambda_2} \{ \mathfrak{B}(N_\gamma) \} \right\} - (\frac{\epsilon}{2}, 0, 0) \\
 &\supseteq (\mathfrak{T}(M) \sqcap \mathfrak{T}(N)) - (\frac{\epsilon}{2}, 0, 0) - (\frac{\epsilon}{2}, 0, 0) \\
 &= (\mathfrak{T}(M) \sqcap \mathfrak{T}(N)) - (\epsilon, 0, 0).
 \end{aligned}$$

Since this is true for every  $\epsilon > 0$  and

$$\mathfrak{T}(M \sqcap N) \supseteq \bigcap_{\substack{x \in X, \delta > 0 \\ (\lambda, \gamma) \in \Lambda}} \{ \mathfrak{B}(D_{\lambda, \gamma, x, \delta}) \}$$

we have  $\mathfrak{T}(M \sqcap N) \supseteq (\mathfrak{T}(M) \sqcap \mathfrak{T}(N))$  for any  $M, N$  in  $\zeta^X$ .

Finally we prove that  $\mathfrak{T}(\bigsqcup_{\lambda \in \Lambda} M_\lambda) \supseteq \bigcap_{\lambda \in \Lambda} \mathfrak{T}(M_\lambda)$  for any collection  $\{M_\lambda\}_{\lambda \in \Lambda} \subseteq \zeta^X$ . For each  $\epsilon > 0$  and for each  $M_\lambda$ , let  $\{M_{\lambda, \gamma}\}_{\gamma \in \Gamma_\lambda}$  be an inner cover for  $M_\lambda$  such that  $\bigcap_{\gamma \in \Gamma_\lambda} \{ \mathfrak{B}(M_{\lambda, \gamma}) \} \supseteq \mathfrak{T}(M_\lambda) - (\epsilon, 0, 0)$ . Since  $\{M_{\lambda, \gamma}\}_{\gamma \in \Gamma_\lambda}$  is an inner cover for  $M_\lambda$ , we have  $\{M_{\lambda, \gamma}\}_{\lambda \in \Lambda, \gamma \in \Gamma_\lambda}$  is an inner cover for  $\bigsqcup_{\lambda \in \Lambda} M_\lambda$ . Thus it follows that

$$\begin{aligned}
 \mathfrak{T}(\bigsqcup_{\lambda \in \Lambda} M_\lambda) &\supseteq \bigcap_{\lambda \in \Lambda, \gamma \in \Gamma_\lambda} \{ \mathfrak{B}(M_{\lambda, \gamma}) \} \\
 &= \bigcap_{\lambda \in \Lambda} \left\{ \bigcap_{\gamma \in \Gamma_\lambda} \{ \mathfrak{B}(M_{\lambda, \gamma}) \} \right\} \\
 &\supseteq \bigcap_{\lambda \in \Lambda} \{ \mathfrak{T}(M_\lambda) - (\epsilon, 0, 0) \} \\
 &= \bigcap_{\lambda \in \Lambda} \{ \mathfrak{T}(M_\lambda) \} - (\epsilon, 0, 0),
 \end{aligned}$$

which implies  $\mathfrak{T}(\bigsqcup_{\lambda \in \Lambda} M_\lambda) \supseteq \bigcap_{\lambda \in \Lambda} \mathfrak{T}(M_\lambda)$  for any collection  $\{M_\lambda\}_{\lambda \in \Lambda} \subseteq \zeta^X$  as desired.  $\square$

**Definition 9.** Let  $(X, \mathfrak{T})$  be smooth neutrosophic topological space. For all  $x_{r,s,t} \in pt(\zeta^X)$  and  $N \in \zeta^X$ , the mapping  $Q_{x_{r,s,t}}^{\mathfrak{T}} : \zeta^X \rightarrow \zeta$  is defined as follows:

$$Q_{x_{r,s,t}}^{\mathfrak{T}}(N) = \begin{cases} \bigsqcup_{x_{r,s,t}[q]M \subseteq N} \mathfrak{T}(M); & \text{if } x_{r,s,t}[q]N \\ (0, 1, 1) & \text{otherwise.} \end{cases}$$

The set  $Q^{\mathfrak{T}} = \{Q_{x_{r,s,t}}^{\mathfrak{T}} : x_{r,s,t} \in pt(\zeta^X)\}$  is called a neutrosophic quasi-coincident neighborhood system. Further, a neutrosophic quasi-coincident neighborhood system  $Q^{\mathfrak{T}}$  is said to be symmetric if for any  $x_{r,s,t}, y_{l,m,n} \in pt(\zeta^X)$ ,  $Q_{x_{r,s,t}}^{\mathfrak{T}}(M) \supseteq (0, 1, 1)$ ,  $Q_{y_{l,m,n}}^{\mathfrak{T}}(N) \supseteq (0, 1, 1)$ ,  $x_{r,s,t}[q]N$  implies  $y_{l,m,n}[q]M$ .

**Theorem 2.** Let  $(X, \mathfrak{T})$  be neutrosophic topological space. Then for all  $M, N \in \zeta^X$ ,

- (i)  $Q_{x_{r,s,t}}^{\mathfrak{T}}(0_X) = (0, 1, 1)$ ;
- (ii)  $Q_{x_{r,s,t}}^{\mathfrak{T}}(1_X) = (1, 0, 0)$ ;

- (iii)  $Q_{x_{r,s,t}}^{\mathfrak{I}}(M) \supseteq (0, 1, 1)$  implies  $x_{r,s,t}[q]M$ ;
- (iv)  $Q_{x_{r,s,t}}^{\mathfrak{I}}(M \cap N) = Q_{x_{r,s,t}}^{\mathfrak{I}}(M) \cap Q_{x_{r,s,t}}^{\mathfrak{I}}(N)$ ;
- (v)  $Q_{x_{r,s,t}}^{\mathfrak{I}}(M) = \bigsqcup_{x_{r,s,t}[q]N \subseteq M} \bigcap_{y_{l,m,n}[q]N} Q_{y_{l,m,n}}^{\mathfrak{I}}(N)$ .

**Proof.** As (i), (ii) and (iii) follow directly from the definition of  $Q_{x_{r,s,t}}^{\mathfrak{I}}$ , we skip their proof. To prove (iv), first we observe that

$$\begin{aligned} Q_{x_{r,s,t}}^{\mathfrak{I}}(M \cap N) &= \bigsqcup_{x_{r,s,t}[q]L \subseteq M \cap N} \mathfrak{I}(L) \\ &\subseteq \bigsqcup_{x_{r,s,t}[q]L \subseteq M} \mathfrak{I}(L) \\ &= Q_{x_{r,s,t}}^{\mathfrak{I}}(M). \end{aligned}$$

Similarly, it follows that  $Q_{x_{r,s,t}}^{\mathfrak{I}}(M \cap N) \subseteq Q_{x_{r,s,t}}^{\mathfrak{I}}(N)$ , which implies

$$Q_{x_{r,s,t}}^{\mathfrak{I}}(M \cap N) \subseteq Q_{x_{r,s,t}}^{\mathfrak{I}}(M) \cap Q_{x_{r,s,t}}^{\mathfrak{I}}(N).$$

To prove the reverse inequality, consider

$$\begin{aligned} Q_{x_{r,s,t}}^{\mathfrak{I}}(M) \cap Q_{x_{r,s,t}}^{\mathfrak{I}}(N) &= \bigsqcup_{x_{r,s,t}[q]A \subseteq M} \mathfrak{I}(A) \cap \bigsqcup_{x_{r,s,t}[q]B \subseteq N} \mathfrak{I}(B) \\ &= \bigsqcup_{x_{r,s,t}[q](A \cap B) \subseteq (M \cap N)} (\mathfrak{I}(A) \cap \mathfrak{I}(B)) \\ &\subseteq \bigsqcup_{x_{r,s,t}[q](A \cap B) \subseteq (M \cap N)} (\mathfrak{I}(A \cap B)) \\ &\subseteq \bigsqcup_{x_{r,s,t}[q]L \subseteq (M \cap N)} (\mathfrak{I}(L)) \\ &= Q_{x_{r,s,t}}^{\mathfrak{I}}(M \cap N). \end{aligned}$$

To prove (v), for any  $N \in \zeta^X$  with  $x_{r,s,t}[q]N \subseteq M$ , we have  $Q_{x_{r,s,t}}^{\mathfrak{I}}(N) \supseteq \mathfrak{I}(N)$ , and therefore,

$$\mathfrak{I}(N) \subseteq \bigcap_{y_{l,m,n}[q]N} Q_{y_{l,m,n}}^{\mathfrak{I}}(N) \subseteq Q_{x_{r,s,t}}^{\mathfrak{I}}(N) \subseteq Q_{x_{r,s,t}}^{\mathfrak{I}}(M).$$

Hence, we have

$$\begin{aligned} Q_{x_{r,s,t}}^{\mathfrak{I}}(M) &= \bigsqcup_{x_{r,s,t}[q]N \subseteq M} \mathfrak{I}(N) \\ &\subseteq \bigsqcup_{x_{r,s,t}[q]N \subseteq M} \bigcap_{y_{l,m,n}[q]N} Q_{y_{l,m,n}}^{\mathfrak{I}}(N) \\ &\subseteq Q_{x_{r,s,t}}^{\mathfrak{I}}(M) \end{aligned}$$

as desired.  $\square$

**Theorem 3.** Let  $\mathfrak{B} : \zeta^X \rightarrow \zeta$  be a mapping. Then  $\mathfrak{B}$  is a basis of a smooth neutrosophic topology  $\mathfrak{I}$  if and only if  $\mathfrak{B} \subseteq \mathfrak{I}$  and for all  $M \in \zeta^X$ ,  $x_{r,s,t} \in pt(\zeta^X)$ ,  $Q_{x_{r,s,t}}^{\mathfrak{I}}(M) \subseteq \bigsqcup_{x_{r,s,t}[q]N \subseteq M} \mathfrak{B}(N)$ .

**Proof.** Let  $\mathfrak{B}$  be a basis for given smooth neutrosophic topology; then clearly  $\mathfrak{B} \sqsubseteq \mathfrak{T}$ . Let  $M \in \zeta^X$  and  $x_{r,s,t} \in pt(\zeta^X)$ ; then  $Q_{x_{r,s,t}}^{\mathfrak{T}}(M) = \sqcup_{x_{r,s,t}[q]N \sqsubseteq M} \mathfrak{T}(N)$ . Let  $\mathcal{N} = \{N : x_{r,s,t}[q]N \sqsubseteq M\}$ ; then for every  $N \in \mathcal{N}$ , we have

$$\mathfrak{T}(N) = \sqcup_{\lambda \in \Lambda} \sqcup_{N_\lambda = N} \sqcap_{\lambda \in \Lambda} \mathfrak{B}(N_\lambda).$$

Let  $\epsilon > 0$ ; then there exists  $\{N_\lambda : \lambda \in \Lambda\}$  with  $\sqcup_{\lambda \in \Lambda} N_\lambda = N$  such that

$$\sqcap_{\lambda \in \Lambda} \mathfrak{B}(N_\lambda) \supseteq \mathfrak{T}(N) - (\epsilon, 0, 0).$$

Thus there exists an  $N_{\lambda_0}$  such that  $x_{r,s,t}[q]N_{\lambda_0}$  and  $\mathfrak{B}(N_{\lambda_0}) \supseteq \mathfrak{T}(N) - (\epsilon, 0, 0)$ . Hence for every  $N \in \mathcal{N}$ , there exists an  $N_{\lambda_0}$  such that  $\mathfrak{B}(N_{\lambda_0}) \supseteq \mathfrak{T}(N) - (\epsilon, 0, 0)$ , which in turn implies that

$$\sqcup_{N \in \mathcal{N}} \mathfrak{B}(N_{\lambda_0}) \supseteq \sqcup_{N \in \mathcal{N}} \mathfrak{T}(N) - (\epsilon, 0, 0).$$

Thus it follows that,

$$\begin{aligned} \sqcup_{x_{r,s,t}[q]N \sqsubseteq M} \mathfrak{B}(N) &\supseteq \sqcup_{N \in \mathcal{N}} \mathfrak{B}(N_{\lambda_0}) \\ &\supseteq \sqcup_{N \in \mathcal{N}} \mathfrak{T}(N) - (\epsilon, 0, 0) \\ &= \sqcup_{x_{r,s,t}[q]N \sqsubseteq M} \mathfrak{T}(N) - (\epsilon, 0, 0). \end{aligned}$$

This implies that

$$\sqcup_{x_{r,s,t}[q]N \sqsubseteq M} \mathfrak{B}(N) \supseteq \sqcup_{x_{r,s,t}[q]N \sqsubseteq M} \mathfrak{T}(N) = Q_{x_{r,s,t}}^{\mathfrak{T}}(M)$$

as desired.

Conversely, let  $x \in X$  and  $\epsilon, \delta > 0$ ; then clearly  $x_{\delta,0,0} \in pt(\zeta^X)$ . However, since

$$Q_{x_{\delta,0,0}}^{\mathfrak{T}}(1_X) \sqsubseteq \sqcup_{x_{\delta,0,0}[q]N \sqsubseteq 1_X} \mathfrak{B}(N)$$

and  $(1, 0, 0) = Q_{x_{\delta,0,0}}^{\mathfrak{T}}(1_X)$ , it is possible to find an  $N \in \zeta^X$  such that  $N(x) \supseteq 1_X(x) - (\delta, 0, 0)$ , such that  $\mathfrak{B}(N) \supseteq (1, 0, 0) - (\epsilon, 0, 0)$ . Thus, **B1** of Definition 8 follows.

Let  $x \in X, M, N \in \zeta^X$  and  $\epsilon, \delta > 0$ . First we claim that,  $Q_{x_{\delta,0,0}}^{\mathfrak{T}}(M \sqcap N) \supseteq [\mathfrak{B}(M) \sqcap \mathfrak{B}(N)]$ ; consider

$$\begin{aligned} Q_{x_{\delta,0,0}}^{\mathfrak{T}}(M \sqcap N) &= [Q_{x_{\delta,0,0}}^{\mathfrak{T}}(M) \sqcap Q_{x_{\delta,0,0}}^{\mathfrak{T}}(N)] \\ &= \left[ \sqcup_{x_{\delta,0,0}[q]L \sqsubseteq M} \mathfrak{T}(L) \sqcap \sqcup_{x_{\delta,0,0}[q]L \sqsubseteq N} \mathfrak{T}(L) \right] \\ &\supseteq [\mathfrak{T}(M) \sqcap \mathfrak{T}(N)] \\ &\supseteq [\mathfrak{B}(M) \sqcap \mathfrak{B}(N)]. \end{aligned}$$

If  $x_{\delta,0,0}[q]M \sqcap N$ , then for every  $L \in \zeta^X$  with  $L \sqsubseteq M \sqcap N$  and  $x_{\delta,0,0}[q]L$ , we have

$$L(x) \supseteq 1_X(x) - (\delta, 0, 0) \supseteq (M \sqcap N)(x) - (\delta, 0, 0).$$

Let  $\epsilon > 0$ ; then there exists  $L$  such that

$$\begin{aligned} \mathfrak{B}(L) &\supseteq \sqcup_{x_{\delta,0,0}[q]L \sqsubseteq (M \sqcap N)} \mathfrak{B}(L) - (\epsilon, 0, 0) \\ &\supseteq Q_{x_{\delta,0,0}}^{\mathfrak{T}}(M \sqcap N) - (\epsilon, 0, 0) \\ &\supseteq [\mathfrak{B}(M) \sqcap \mathfrak{B}(N)] - (\epsilon, 0, 0). \end{aligned}$$



Suppose  $x_{\delta,0,0}[\bar{q}]M \sqsupseteq N$ . Let

$$\mathcal{L}_x = \{L \in \zeta^X : L(x) \sqsupseteq (M \sqcap N)(x) - (\delta, 0, 0)\}.$$

Then there exists  $L \in \mathcal{L}_x$  such that

$$\begin{aligned} \mathfrak{B}(L) &\sqsupseteq \bigsqcup_{L \in \mathcal{L}_x} \mathfrak{B}(L) - (\epsilon, 0, 0) \\ &\sqsupseteq Q_{x_{\delta,0,0}}^{\mathfrak{T}}(M \sqcap N) - (\epsilon, 0, 0) \\ &\sqsupseteq [\mathfrak{B}(M) \sqcap \mathfrak{B}(N)] - (\epsilon, 0, 0) \end{aligned}$$

Thus, **B2** of Definition 8 follows in both cases.  $\square$

Here we note that, “If  $(X, \mathfrak{T})$  is a smooth neutrosophic topological space, then  $\mathfrak{T}$  is a basis for a smooth fuzzy topology on  $X$  and the smooth fuzzy topology generated by  $\mathfrak{T}$  is itself.” In the following, we give certain theorems which can be proved in a similar fashion to Theorems 3.8, 3.9 and 3.10 in [10].

**Theorem 4.** Let  $\mathfrak{T}$  be a smooth neutrosophic topology on  $X$ . Let  $\mathfrak{B} : \zeta^X \rightarrow \zeta$  be a function satisfying

- i.  $\mathfrak{T}(M) \sqsupseteq \mathfrak{B}(M)$  for all  $M \in \zeta^X$ ;
- ii. If  $M \in \zeta^X$ ,  $x \in X$ ,  $\delta > 0$  and  $\epsilon > 0$ , then there exists  $N \in \zeta^X$  such that  $N(x) \sqsupseteq M(x) - (\delta, 0, 0)$ ,  $N \sqsubseteq M$  and  $\mathfrak{B}(N) \sqsupseteq \mathfrak{T}(M) - (\epsilon, 0, 0)$ .

Then  $\mathfrak{B}$  is a basis for the smooth neutrosophic topology  $\mathfrak{T}$  on  $X$ .

**Theorem 5.** If  $\mathfrak{B}$  is a basis for the smooth fuzzy topological space  $(X, \mathfrak{T})$ , then

- i.  $\mathfrak{T}(M) \sqsupseteq \mathfrak{B}(M)$  for all  $M \in \zeta^X$ .
- ii. If  $x \in X$ ,  $M \in \zeta^X$ ,  $\delta > 0$  and  $\epsilon > 0$ , then there exists  $N \in \zeta^X$  such that  $N(x) \sqsupseteq M(x) - (\delta, 0, 0)$ ,  $N \sqsubseteq M$  and  $\mathfrak{B}(N) \sqsupseteq \mathfrak{T}(M) - (\epsilon, 0, 0)$ .

**Theorem 6.** Let  $\mathfrak{B}$  and  $\mathfrak{B}'$  be bases for the smooth neutrosophic topologies  $\mathfrak{T}$  and  $\mathfrak{T}'$ , respectively, on  $X$ . Then the following conditions are equivalent.

- i.  $\mathfrak{T}'$  is finer than  $\mathfrak{T}$ .
- ii. If  $M \in \zeta^X$ ,  $x \in X$ ,  $\delta > 0$  and  $\epsilon > 0$ , there exists  $N \in \zeta^X$  such that  $N(x) \sqsupseteq M(x) - (\delta, 0, 0)$ ,  $N \sqsubseteq M$  and  $\mathfrak{B}'(N) \sqsupseteq \mathfrak{B}(M) - (\epsilon, 0, 0)$ .

To end this section, we present a theorem which gives a way to get a basis from a subbasis, from which a smooth neutrosophic topology can be generated.

**Theorem 7.** Let  $\mathfrak{S} : \zeta^X \rightarrow \zeta$  be a subbasis for a smooth neutrosophic topology on  $X$ . Define  $\mathfrak{B} : \zeta^X \rightarrow \zeta$  as

$$\mathfrak{B}(M) = \bigsqcup_{D \in \mathcal{D}} \{ \sqcap_{i \in I_D} \{\mathfrak{S}(M_i)\} \},$$

where  $\mathcal{D}$  is the family of all finite collections  $D = \{M_i\}_{i \in I_D}$  of members of  $\zeta^X$  such that  $M = \sqcap_{i \in I_D} M_i$ . Then the  $\mathfrak{B}$  is a basis for a smooth neutrosophic topology on  $X$ .

**Proof.** Since  $\mathcal{D} \neq \emptyset$ , every  $M \in \zeta^X$ , and by the definition of  $\mathfrak{S}$ ,  $\mathfrak{B}$  is well defined. As  $\mathfrak{B}$  clearly satisfies **B1** of Definition 7, it is enough to prove **B2**. Let  $x \in X$ ,  $M, N$  in  $\zeta^X$  and  $\delta, \epsilon > 0$ . Then by the definition of  $\mathfrak{B}$  there exist collections  $\{M_i\}_{i=1,2,\dots,n}$  and  $\{N_j\}_{j=1,2,\dots,m}$  such that

$$M = \sqcap_{i=1}^n M_i, \quad \sqcap_i \{\mathfrak{B}(M_i)\} \sqsupseteq \mathfrak{B}(M) - (\epsilon, 0, 0)$$

and

$$N = \prod_{j=1}^m N_j, \quad \prod_j \{\mathfrak{B}(N_j)\} \supseteq \mathfrak{B}(N) - (\epsilon, 0, 0).$$

Now let us define a collection of neutrosophic sets  $L_k \sqsubseteq 1_X$ , for  $k = 1, 2, \dots, n + m$ , as

$$L_k = \begin{cases} M_k & \text{if } k \leq n \\ N_{k-n} & \text{if } k > n. \end{cases}$$

If we let  $L = \prod_{k=1}^{n+m} L_k$ , then  $L = M \sqcap N$  and therefore

$$L(x) \supseteq (M \sqcap N)(x) - (\delta, 0, 0).$$

Now by definition of  $\mathfrak{B}$ , we have

$$\mathfrak{B}(L) = \sqcup_{D \in \mathcal{D}} \{ \prod_{i \in I_D} \{\mathfrak{S}(L_i)\} \},$$

where  $\mathcal{D}$  is the family of all finite collections  $D = \{L_i\}_{i \in I_D}$  of members of  $\zeta^X$  such that  $L = \prod_{i \in I_D} L_i$ .

Thus it follows that

$$\begin{aligned} \mathfrak{B}(L) &= \sqcup_{D \in \mathcal{D}} \{ \prod_{i \in I_D} \{\mathfrak{S}(L_i)\} \} \\ &\supseteq \prod_k \{\mathfrak{S}(L_k)\} \\ &= \prod_i \{\mathfrak{S}(M_i)\} \sqcap \prod_j \{\mathfrak{S}(N_j)\} \\ &\supseteq (\mathfrak{B}(M) - (\epsilon, 0, 0)) \sqcap (\mathfrak{B}(N) - (\epsilon, 0, 0)) \\ &= (\mathfrak{B}(M) \sqcap \mathfrak{B}(N)) - (\epsilon, 0, 0) \end{aligned}$$

as desired.  $\square$

#### 4. Product of Neutrosophic Topologies

In this section, we first define the concept of a finite product of smooth neutrosophic topologies, using the notion of basis defined in the previous section. We present a way to obtain the product topology from the given bases; in the following we present a subbasis for a product topology. Later, we generalize the discussed contents in the context of an arbitrary product of smooth neutrosophic topologies.

**Definition 10.** Let  $(X, \mathfrak{T}_1)$  and  $(Y, \mathfrak{T}_2)$  be smooth neutrosophic topological spaces. Let  $\mathfrak{B} : \zeta^{X \times Y} \rightarrow \zeta$  be defined as follows:

Let  $A \in \zeta^{X \times Y}$ . If  $A \neq M \times N$  for any  $M \in \zeta^X$  and  $N \in \zeta^Y$ , then define  $\mathfrak{B}(A) = (0, 1, 1)$ .  
Otherwise, define

$$\mathfrak{B}(A) = \sqcup_{\lambda \in \Lambda} \{ \mathfrak{T}_1(M_\lambda) \sqcap \mathfrak{T}_2(N_\lambda) \},$$

where  $\{M_\lambda \times N_\lambda\}_{\lambda \in \Lambda}$  is the collection of all possible ways of writing  $A$  as  $A = M_\lambda \times N_\lambda$ , where  $M_\lambda \in \zeta^X$ ,  $N_\lambda \in \zeta^Y$ .

Then  $\mathfrak{B}$  is a basis for the smooth neutrosophic topology called the smooth neutrosophic product topology on  $X \times Y$ .

**Example 2.** Let  $X_1 = X_2 = \mathbb{R}$  and let  $M_1$  and  $N_1$  be defined by

$$M_1(x) = \left( \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \right) \text{ for all } x \in X_1$$

and

$$N_1(x) = \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right) \text{ for all } x \in X_2.$$

Let  $\mathfrak{T}_1 : \zeta^{X_1} \rightarrow \zeta$  and  $\mathfrak{T}_2 : \zeta^{X_2} \rightarrow \zeta$  be the functions defined by

$$\mathfrak{T}_1(M) = \begin{cases} (1, 0, 0) & \text{if } M = 1_{X_1} \text{ or } M = 0_{X_1} \\ \left(\frac{1}{4}, \frac{1}{2}, \frac{1}{2}\right) & \text{if } M = M_1 \\ (0, 1, 1) & \text{otherwise} \end{cases}$$

and

$$\mathfrak{T}_2(N) = \begin{cases} (1, 0, 0) & \text{if } N = 1_{X_2} \text{ or } N = 0_{X_2} \\ \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right) & \text{if } N = N_1 \\ (0, 1, 1) & \text{otherwise.} \end{cases}$$

Then clearly  $\mathfrak{T}_1$  and  $\mathfrak{T}_2$  are smooth neutrosophic topologies on  $X_1$  and  $X_2$ . From the above definition, we get  $\mathfrak{B} : \zeta^{X_1 \times X_2} \rightarrow \zeta$  given by

$$\mathfrak{B}(E) = \begin{cases} (1, 0, 0) & \text{if } E = 1_{X_1 \times X_2} \text{ or } E = 0_{X_1 \times X_2} \\ \left(\frac{1}{4}, \frac{1}{2}, \frac{1}{2}\right) & \text{if } E = M_1 \times 1_{X_2} \\ \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right) & \text{if } E = 1_{X_1} \times N_1 \\ \left(\frac{1}{4}, \frac{1}{2}, \frac{1}{2}\right) & \text{if } E = M_1 \times N_1 \\ (0, 1, 1) & \text{otherwise} \end{cases}$$

which is a basis for a smooth neutrosophic topology  $\mathfrak{T}$  on  $X_1 \times X_2$  and the smooth neutrosophic topology (product topology) generated by  $\mathfrak{B}$  is given by

$$\mathfrak{T}(E) = \begin{cases} (1, 0, 0) & \text{if } E = 1_{X_1 \times X_2} \text{ or } E = 0_{X_1 \times X_2} \\ \left(\frac{1}{4}, \frac{1}{2}, \frac{1}{2}\right) & \text{if } E = M_1 \times 1_{X_2} \\ \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right) & \text{if } E = 1_{X_1} \times N_1 \\ \left(\frac{1}{4}, \frac{1}{2}, \frac{1}{2}\right) & \text{if } E = M_1 \times N_1 \\ (0, 1, 1) & \text{otherwise.} \end{cases}$$

**Theorem 8.** Let  $\mathfrak{B} : \zeta^{X \times Y} \rightarrow \zeta$  be the function defined in Definition 10. Then  $\mathfrak{B}$  is a basis for a smooth neutrosophic topology on  $X \times Y$ .

**Proof.** If we let  $M = 1_{X \times Y}$ , then clearly **B1** of Definition 7 follows.

Let  $(x, y) \in X \times Y$ ,  $M, N$  in  $\zeta^{X \times Y}$  and  $\delta, \epsilon > 0$ . We wish to show that there exists  $L \in \zeta^{X \times Y}$  such that  $L \sqsubseteq M \sqcap N$ ,

$$L(x, y) \sqsupseteq (M(x, y) \sqcap N(x, y)) - (\delta, 0, 0)$$

and

$$\mathfrak{B}(L) \sqsupseteq (\mathfrak{B}(M) \sqcap \mathfrak{B}(N)) - (\epsilon, 0, 0).$$

Suppose any one of  $M$  and  $N$ , say  $M$  cannot be written as  $M_1 \times M_2$  for any  $M_1 \in \zeta^X$  and  $M_2 \in \zeta^Y$ ; then by letting  $L = M \sqcap N$ , we have  $L(x, y) \sqsupseteq (M(x, y) \sqcap N(x, y)) - (\delta, 0, 0)$ . However, by the definition of  $\mathfrak{B}$ , it follows that  $\mathfrak{B}(M) = (0, 1, 1)$  and therefore  $\mathfrak{B}(L) \sqsupseteq (\mathfrak{B}(M) \sqcap \mathfrak{B}(N)) - (\epsilon, 0, 0)$  as desired. If both  $M$  and  $N$  can be written as  $A \times A'$  and  $B \times B'$  for some  $A, B \in \zeta^X$  and  $A', B' \in \zeta^Y$ , then by the definition of  $\mathfrak{B}$ , there exist  $M_1, N_1 \in \zeta^X$  and  $M_2, N_2 \in \zeta^Y$  such that  $M = M_1 \times M_2$ ,  $N = N_1 \times N_2$ ,

$$\mathfrak{T}_1(M_1) \sqcap \mathfrak{T}_2(M_2) \sqsupseteq \mathfrak{B}(M) - (\epsilon, 0, 0)$$

and

$$\mathfrak{T}_1(N_1) \sqcap \mathfrak{T}_2(N_2) \supseteq \mathfrak{B}(N) - (\epsilon, 0, 0).$$

Now if we let  $L = M \sqcap N$ , then  $L(x, y) = (M(x, y) \sqcap N(x, y)) - (\delta, 0, 0)$  and

$$\begin{aligned} (M \sqcap N)(x, y) &= ((M_1 \times M_2) \sqcap (N_1 \times N_2))(x, y) \\ &= (M_1(x) \sqcap M_2(y)) \sqcap (N_1(x) \sqcap N_2(y)) \\ &= (M_1(x) \sqcap N_1(x)) \sqcap (M_2(y) \sqcap N_2(y)) \\ &= (M_1(x) \sqcap N_1(x)) \times (M_2(y) \sqcap N_2(y)) \\ &= ((M_1 \sqcap N_1) \times (M_2 \sqcap N_2))(x, y). \end{aligned}$$

Now consider,

$$\begin{aligned} \mathfrak{B}(L) &= \mathfrak{B}(M \sqcap N) \\ &= \mathfrak{B}((M_1 \sqcap N_1) \times (M_2 \sqcap N_2)) \\ &\supseteq \mathfrak{T}_1(M_1 \sqcap N_1) \sqcap \mathfrak{T}_2(M_2 \sqcap N_2) \\ &\supseteq (\mathfrak{T}_1(M_1) \sqcap \mathfrak{T}_1(N_1)) \sqcap (\mathfrak{T}_2(M_2) \sqcap \mathfrak{T}_2(N_2)) \\ &= \{\mathfrak{T}_1(M_1) \sqcap \mathfrak{T}_2(M_2)\} \sqcap \{\mathfrak{T}_1(N_1) \sqcap \mathfrak{T}_2(N_2)\} \\ &\supseteq (\mathfrak{B}(M) - (\epsilon, 0, 0)) \sqcap (\mathfrak{B}(N) - (\epsilon, 0, 0)) \\ &= (\mathfrak{B}(M) \sqcap \mathfrak{B}(N)) - (\epsilon, 0, 0) \end{aligned}$$

and hence **B2** of Definition 7 follows in this case also.  $\square$

**Theorem 9.** Let  $\mathfrak{B}_1, \mathfrak{B}_2$  be bases for the smooth neutrosophic topologies  $\mathfrak{T}_1, \mathfrak{T}_2$  respectively. Define  $\mathfrak{B} : \zeta^{X \times Y} \rightarrow \zeta$  as follows:

If  $A \in \zeta^{X \times Y}$  cannot be written as  $M \times N$  for any  $M \in \zeta^X$  and  $N \in \zeta^Y$ , then define  $\mathfrak{B}(A) = (0, 1, 1)$ .  
Otherwise define

$$\mathfrak{B}(A) = \bigsqcup_{\lambda \in \Lambda} \{\mathfrak{B}_1(M_\lambda) \sqcap \mathfrak{B}_1(N_\lambda)\}$$

where  $\{M_\lambda \times N_\lambda\}_{\lambda \in \Lambda}$  is the collection of all possible ways of writing  $A$  as  $A = M_\lambda \times N_\lambda$ , where  $M_\lambda \in \zeta^X, N_\lambda \in \zeta^Y$ .

Then  $\mathfrak{B}$  is a basis for the product topology on  $X \times Y$ .

**Proof.** First we claim that  $\mathfrak{B}$  is a basis for a smooth neutrosophic topology on  $X \times Y$ . Let  $(x, y) \in X \times Y$ ,  $\delta > 0$  and  $\epsilon > 0$ . Now since  $\mathfrak{B}_1$  and  $\mathfrak{B}_2$  are bases for the smooth neutrosophic topologies  $\mathfrak{T}_1$  and  $\mathfrak{T}_2$ , there exist  $M \in \zeta^X$  and  $N \in \zeta^Y$  such that

$$M(x) \supseteq 1_X(x) - (\delta, 0, 0), \quad \mathfrak{B}_1(M) \supseteq (1, 0, 0) - (\epsilon, 0, 0)$$

and

$$N(y) \supseteq 1_Y(y) - (\delta, 0, 0), \quad \mathfrak{B}_2(N) \supseteq (1, 0, 0) - (\epsilon, 0, 0).$$

Let  $A = M \times N$ ; then we have

$$\begin{aligned} A(x, y) &= (M \times N)(x, y) \\ &= M(x) \sqcap N(y) \\ &\supseteq (1_X(x) - (\delta, 0, 0)) \sqcap (1_Y(y) - (\delta, 0, 0)) \\ &= (1_X(x) \sqcap 1_Y(y)) - (\delta, 0, 0) \\ &\supseteq 1_{X \times Y}(x, y) - (\delta, 0, 0) \end{aligned}$$

and

$$\mathfrak{B}(A) \supseteq \mathfrak{B}_1(M) \cap \mathfrak{B}_1(N) \supseteq (1, 0, 0) - (\epsilon, 0, 0).$$

Thus **B1** of Definition 7 follows.

To prove **B2**, let  $(x, y) \in X \times Y$ ,  $M, N \in \zeta^{X \times Y}$  and  $\delta, \epsilon > 0$ . If any one of  $M$  and  $N$ , say  $M$ , cannot be written as  $M_1 \times M_2$  for any  $M_1 \in \zeta^X$  and  $M_2 \in \zeta^Y$ , then by letting  $L = M \cap N$ , as in the above theorem, **B2** of Definition 7 follows. On the other hand, suppose both  $M$  and  $N$  can be written as  $A \times A'$  and  $B \times B'$  for some  $A, B \in \zeta^X$  and  $A', B' \in \zeta^Y$ ; then by definition of  $\mathfrak{B}$ , there exist  $M_1, N_1 \in \zeta^X$ , and  $M_2, N_2 \in \zeta^Y$  such that  $M = M_1 \times M_2$ ,  $N = N_1 \times N_2$ ,

$$\mathfrak{B}_1(M_1) \cap \mathfrak{B}_2(M_2) \supseteq \mathfrak{B}(M) - \left(\frac{\epsilon}{2}, 0, 0\right)$$

and

$$\mathfrak{B}_1(N_1) \cap \mathfrak{B}_2(N_2) \supseteq \mathfrak{B}(N) - \left(\frac{\epsilon}{2}, 0, 0\right).$$

Here it is easy to see that there exists  $L_1 \in \zeta^X$  such that  $L_1 \sqsubseteq M_1 \cap N_1$ ,  $L_1(x) \supseteq (M_1 \cap N_1)(x) - (\delta, 0, 0)$  and  $\mathfrak{B}_1(L_1) \supseteq (\mathfrak{B}_1(M_1) \cap \mathfrak{B}_1(N_1)) - (\frac{\epsilon}{2}, 0, 0)$ , as  $x \in X$  and  $M_1, N_1$  are in  $\zeta^X$ .

Analogously, since  $y \in Y$  and  $M_2, N_2$  are in  $\zeta^Y$ , there exists  $L_2$  in  $\zeta^Y$  such that  $L_2 \sqsubseteq M_2 \cap N_2$ ,  $L_2(y) \supseteq (M_2 \cap N_2)(y) - (\delta, 0, 0)$  and  $\mathfrak{B}_2(L_2) \supseteq (\mathfrak{B}_2(M_2) \cap \mathfrak{B}_2(N_2)) - (\frac{\epsilon}{2}, 0, 0)$ .

Let  $L = L_1 \times L_2$ ; then we have

$$\begin{aligned} L(x, y) &= (L_1 \times L_2)(x, y) \\ &= L_1(x) \cap L_2(y) \\ &\supseteq \{(M_1(x) \cap N_1(x)) \cap (M_2(y) \cap N_2(y))\} - (\delta, 0, 0) \\ &= \{(M_1(x) \times M_2(y)) \cap (N_1(x) \times N_2(y))\} - (\delta, 0, 0) \\ &= (M(x, y) \cap N(x, y)) - (\delta, 0, 0) \end{aligned}$$

and

$$\begin{aligned} \mathfrak{B}(L) &= \mathfrak{B}(L_1 \times L_2) \\ &\supseteq \mathfrak{B}_1(L_1) \cap \mathfrak{B}_2(L_2) \\ &\supseteq \left\{ (\mathfrak{B}_1(M_1) \cap \mathfrak{B}_1(N_1)) - \left(\frac{\epsilon}{2}, 0, 0\right) \right\} \\ &\quad \cap \left\{ (\mathfrak{B}_2(M_2) \cap \mathfrak{B}_2(N_2)) - \left(\frac{\epsilon}{2}, 0, 0\right) \right\} \\ &= \left\{ (\mathfrak{B}_1(M_1) \cap \mathfrak{B}_1(N_1)) \cap (\mathfrak{B}_2(M_2) \cap \mathfrak{B}_2(N_2)) \right\} - \left(\frac{\epsilon}{2}, 0, 0\right) \\ &= \left\{ (\mathfrak{B}_1(M_1) \cap \mathfrak{B}_2(M_2)) \cap (\mathfrak{B}_1(N_1) \cap \mathfrak{B}_2(N_2)) \right\} - \left(\frac{\epsilon}{2}, 0, 0\right) \\ &\supseteq \left( \mathfrak{B}(M) - \left(\frac{\epsilon}{2}, 0, 0\right) \right) \cap \left( \mathfrak{B}(N) - \left(\frac{\epsilon}{2}, 0, 0\right) \right) - \left(\frac{\epsilon}{2}, 0, 0\right) \\ &= (\mathfrak{B}(M) \cap \mathfrak{B}(N)) - \left(\frac{\epsilon}{2}, 0, 0\right) - \left(\frac{\epsilon}{2}, 0, 0\right) \\ &= (\mathfrak{B}(M) \cap \mathfrak{B}(N)) - (\epsilon, 0, 0). \end{aligned}$$

Thus **B2** of Definition 7 follows in this case also. Hence  $\mathfrak{B}$  is a basis for a smooth neutrosophic topology on  $X \times Y$ . Thus, proving that the smooth neutrosophic topology generated by this basis coincides with the smooth neutrosophic product topology remains.

Let  $\mathfrak{T}$  be the smooth fuzzy topology generated by  $\mathfrak{B}$ . Let  $\mathfrak{T}_p$  be the product topology on  $X \times Y$  and  $\mathfrak{B}_p$  be the basis for  $\mathfrak{T}_p$  as described in Definition 10. Now we prove that  $\mathfrak{T}_p = \mathfrak{T}$ . Let  $A \in \zeta^{X \times Y}$ ; then

$$\mathfrak{T}_p(A) = \bigsqcup_{\Lambda \in \Gamma} \left\{ \bigcap_{A_\lambda \in \mathcal{L}_\Lambda} \{ \mathfrak{B}_p(A_\lambda) \} \right\},$$

where  $\{\mathcal{L}_\Lambda\}_{\Lambda \in \Gamma}$  is the collection of all inner covers  $\mathcal{L}_\Lambda = \{A_\lambda\}_{\lambda \in \Lambda}$  of  $A$ . Now we divide the collection  $\{\mathcal{L}_\Lambda\}_{\Lambda \in \Gamma}$ , say  $\mathcal{L}$ , into two subcollections  $\mathcal{L}'$  and  $\mathcal{L}''$  where  $\mathcal{L}'$  is the collection all possible inner covers  $\{A_\lambda\}_{\lambda \in \Lambda}$  of  $A$  so that for all  $\lambda \in \Lambda$ ,  $A_\lambda$  is of the form  $M_\lambda \times N_\lambda$  for at least one  $M_\lambda \in \zeta^X$  and one  $N_\lambda \in \zeta^Y$ , and  $\mathcal{L}''$  is the complement of  $\mathcal{L}'$  in  $\mathcal{L}$ .

If an inner cover  $\mathcal{L}_\Lambda = \{A_\lambda\}_{\lambda \in \Lambda}$  of  $A$  is in  $\mathcal{L}''$ , then for at least one  $\lambda_0 \in \Lambda$ ,  $A_{\lambda_0}$  is not of the form  $M \times N$  for any  $M \in \zeta^X$  and  $N \in \zeta^Y$ ; hence  $\mathfrak{B}_p(A_{\lambda_0}) = (0, 1, 1)$  and therefore

$$\bigcap_{A_\lambda \in \mathcal{L}_\Lambda} \{\mathfrak{B}_p(A_\lambda)\} = (0, 1, 1)$$

and

$$\bigcap_{A_\lambda \in \mathcal{L}_\Lambda} \{\mathfrak{B}(A_\lambda)\} = (0, 1, 1).$$

If  $\mathcal{L}' = \emptyset$ , then  $\mathfrak{T}_p(A) = \mathfrak{T}(A) = (0, 1, 1)$  and hence it is enough to consider the case  $\mathcal{L}' \neq \emptyset$ . Now consider

$$\begin{aligned} \mathfrak{T}_p(A) &= \bigsqcup_{\mathcal{L}} \left\{ \bigcap_{A_\lambda \in \mathcal{L}_\Lambda} \{\mathfrak{B}_p(A_\lambda)\} \right\} \\ &= \bigsqcup_{\mathcal{L}'} \left\{ \bigcap_{A_\lambda \in \mathcal{L}_\Lambda} \{\mathfrak{B}_p(A_\lambda)\} \right\} \\ &= \bigsqcup_{\mathcal{L}'} \left\{ \bigcap_{A_\lambda \in \mathcal{L}_\Lambda} \left\{ \bigsqcup_{A_\lambda = M_\lambda \times N_\lambda} \{\mathfrak{T}_1(M_\lambda) \cap \mathfrak{T}_2(N_\lambda)\} \right\} \right\} \\ &\supseteq \bigsqcup_{\mathcal{L}'} \left\{ \bigcap_{A_\lambda \in \mathcal{L}_\Lambda} \left\{ \bigsqcup_{A_\lambda = M_\lambda \times N_\lambda} \{\mathfrak{B}_1(M_\lambda) \cap \mathfrak{B}_2(N_\lambda)\} \right\} \right\} \\ &= \bigsqcup_{\mathcal{L}'} \left\{ \bigcap_{A_\lambda \in \mathcal{L}_\Lambda} \{\mathfrak{B}(A_\lambda)\} \right\} \\ &= \bigsqcup_{\mathcal{L}} \left\{ \bigcap_{A_\lambda \in \mathcal{L}_\Lambda} \{\mathfrak{B}(A_\lambda)\} \right\} \\ &= \mathfrak{T}(A). \end{aligned}$$

This implies that,  $\mathfrak{T}_p \supseteq \mathfrak{T}$ .

To prove the reverse inequality, let  $A \in \zeta^{X \times Y}$ ,  $\epsilon > 0$  and  $\mathcal{L}, \mathcal{L}', \mathcal{L}''$  be as above. Let  $\mathcal{L}_\Lambda = \{A_\lambda\}_{\lambda \in \Lambda}$  be an inner cover for  $A$ . As above it is enough to consider the case  $\mathcal{L}' \neq \emptyset$ . Now let  $\mathcal{L}_\Lambda \in \mathcal{L}'$ . Then for all  $\lambda \in \Lambda$ , we have  $A_\lambda = M \times N$  for at least one  $M \in \zeta^X$  and one  $N \in \zeta^Y$ . Fix a  $\lambda \in \Lambda$ . Let  $\mathcal{B}_\lambda$  denote the set of all pairs  $(M, N)$  such that  $A_\lambda = M \times N$ . Let  $(M, N) \in \mathcal{B}_\lambda$ . Since  $\mathfrak{B}_1, \mathfrak{B}_2$  are bases for  $\mathfrak{T}_1, \mathfrak{T}_2$ , by Theorem 5, for any  $x \in X, y \in Y$  and  $\delta > 0$  there exist  $M_{x,\delta} \in \zeta^X$  and  $N_{y,\delta} \in \zeta^Y$  such that

$$M_{x,\delta}(x) \supseteq M(x) - (\delta, 0, 0), M_{x,\delta} \sqsubseteq M$$

and

$$N_{y,\delta}(y) \supseteq N(y) - (\delta, 0, 0), N_{y,\delta} \sqsubseteq N$$

with

$$\mathfrak{B}_1(M_{x,\delta}) + (\epsilon, 0, 0) \supseteq \mathfrak{T}_1(M)$$

and

$$\mathfrak{B}_2(N_{y,\delta}) + (\epsilon, 0, 0) \supseteq \mathfrak{T}_2(N).$$

Clearly the collection  $\{M_{x,\delta}\}_{x \in X, \delta > 0}$  is an inner cover for  $M$  and the collection  $\{N_{y,\delta}\}_{y \in Y, \delta > 0}$  is an inner cover for  $N$ . Therefore, the collection  $\{M_{x,\delta} \times N_{y,\delta}\}_{x \in X, y \in Y, \delta > 0}$  is an inner cover for  $M \times N$  which is equal to  $A_\lambda$ . Thus for any pair  $(M, N) \in \mathcal{B}_\lambda$  with  $M \times N = A_\lambda$ , we have an inner cover  $\{M_{x,\delta} \times N_{y,\delta}\}_{x \in X, y \in Y, \delta > 0}$  of  $A_\lambda$  such that

$$\mathfrak{B}_1(M_{x,\delta}) + (\epsilon, 0, 0) \supseteq \mathfrak{T}_1(M) \tag{1}$$

and

$$\mathfrak{B}_2(N_{y,\delta}) + (\epsilon, 0, 0) \sqsupseteq \mathfrak{T}_2(N) \tag{2}$$

for all  $x \in X, y \in X$  and  $\delta > 0$ .

Now since

$$\begin{aligned} \mathfrak{T}_p(A) &= \bigsqcup_{\mathcal{L}} \left\{ \bigcap_{A_\lambda \in \mathcal{L}_\Lambda} \{ \mathfrak{B}_p(A_\lambda) \} \right\} \\ &= \bigsqcup_{\mathcal{L}'} \left\{ \bigcap_{A_\lambda \in \mathcal{L}'_\Lambda} \left\{ \bigsqcup_{(M,N) \in \mathfrak{B}_\lambda} \{ \mathfrak{T}_1(M) \cap \mathfrak{T}_2(N) \} \right\} \right\}, \end{aligned}$$

using (1) and (2), we have

$$\begin{aligned} \mathfrak{T}_p(A) &\sqsubseteq \bigsqcup_{\mathcal{L}'} \left\{ \bigcap_{A_\lambda \in \mathcal{L}'_\Lambda} \{ \mathfrak{T}(A_\lambda) \} \right\} + (\epsilon, 0, 0) \\ &= \bigsqcup_{\mathcal{L}} \left\{ \bigcap_{A_\lambda \in \mathcal{L}_\Lambda} \{ \mathfrak{T}(A_\lambda) \} \right\} + (\epsilon, 0, 0) \\ &= \mathfrak{T}(A) + (\epsilon, 0, 0). \end{aligned}$$

Since this is true for every  $\epsilon > 0$ , it follows that  $\mathfrak{T}_p(A) \sqsubseteq \mathfrak{T}(A)$  and hence we get  $\mathfrak{T}_p \sqsubseteq \mathfrak{T}$  as desired.  $\square$

**Theorem 10.** Let  $(X, \mathfrak{T}_1)$  and  $(Y, \mathfrak{T}_2)$  be smooth neutrosophic topological spaces. Let

$$\mathcal{A}_1 = \{ M / A = M \times 1_Y, M \in \zeta^X \}$$

and

$$\mathcal{A}_2 = \{ N / A = 1_X \times N, N \in \zeta^Y \}.$$

Let  $\mathcal{A} = \mathcal{A}_1 \cup \mathcal{A}_2$ . Define  $\mathfrak{S} : \zeta^{X \times Y} \rightarrow \zeta$  as

$$\mathfrak{S}(A) = \begin{cases} \bigsqcup_{\mathcal{A}} \{ \mathfrak{T}_1(M), \mathfrak{T}_2(N) \} & \text{if } A \neq \emptyset \\ (0, 1, 1) & \text{otherwise.} \end{cases}$$

Then  $\mathfrak{S}$  is a subbasis for the smooth neutrosophic product topology on  $X \times Y$ .

**Proof.** Since  $\mathfrak{S}(1_{X \times Y}) = (1, 0, 0)$ , by letting  $M = 1_{X \times Y}$ , it clearly follows that  $\mathfrak{S}$  is a subbasis for a smooth neutrosophic topology on  $X \times Y$ . Thus all that remains is to show the smooth neutrosophic topology induced by this subbasis is the same as the product topology on  $X \times Y$ . We do this by proving that the basis induced by this subbasis is the same as the basis defined in Definition 10.

Let  $\mathfrak{B}'$  be the basis generated by  $\mathfrak{S}$ . Then for any  $A$  in  $\zeta^{X \times Y}$ , we have

$$\mathfrak{B}'(A) = \bigsqcup_{D \in \mathcal{D}} \left\{ \bigcap_{i \in I_D} \{ \mathfrak{S}(A_i) \} \right\},$$

where  $\mathcal{D}$  is the family of all finite collections  $D = \{A_i\}_{i \in I_D}$  of neutrosophic sets in  $\zeta^{X \times Y}$  for some finite indexing set  $I_D$  such that  $A = \bigcap_{i \in I_D} A_i$ , where each  $A_i \in \zeta^{X \times Y}$ . Let  $\mathfrak{B}$  be the basis for the smooth neutrosophic product topology on  $X \times Y$  as in Definition 10. Let  $A \in \zeta^{X \times Y}$ ; then we claim that  $\mathfrak{B}(A) = \mathfrak{B}'(A)$ . Suppose  $A$  is not of form  $M \times N$  for any  $M \sqsubseteq 1_X$  and  $N \sqsubseteq 1_Y$ . Then by Definition 10 we have,  $\mathfrak{B}(A) = (0, 1, 1)$ . Now let us compute  $\mathfrak{B}'(A)$ . Let  $A = A_1 \cap A_2 \cap \dots \cap A_n$  be any representation of  $A$  as a finite intersection of neutrosophic sets of  $X \times Y$ . First we claim that  $A_i$  is neither of the form  $(M_i \times 1_Y)$  nor of the form  $(1_X \times N_i)$  for at least one  $i$ . If  $A_i = (M_i \times 1_Y)$  or  $A_i = (1_X \times N_i)$  for all  $i$ .

Without loss of generality, let us assume that  $A_i = (M_i \times 1_Y)$  for  $i = 1, 2, \dots, m$  and  $A_i = (1_X \times N_i)$  for  $i = m + 1, m + 2, \dots, n$ , then we have

$$\begin{aligned} A &= A_1 \sqcap A_2 \sqcap \dots \sqcap A_n \\ &= \{(M_1 \times 1_Y) \sqcap \dots \sqcap (M_m \times 1_Y)\} \\ &\quad \sqcap \{(1_X \times N_{m+1}) \sqcap \dots \sqcap (1_X \times N_n)\} \\ &= \{(M_1 \sqcap M_2 \sqcap \dots \sqcap M_m) \times 1_Y\} \\ &\quad \sqcap \{1_X \times (N_{m+1} \sqcap N_{m+2} \sqcap \dots \sqcap N_n)\}. \end{aligned}$$

Now if we let  $M = M_1 \sqcap M_2 \sqcap \dots \sqcap M_m$  and  $N = N_{m+1} \sqcap N_{m+2} \sqcap \dots \sqcap N_n$ , then it follows that  $A = (M \times 1_Y) \sqcap (1_X \times N) = M \times N$ , which is a contradiction to our assumption that  $A$  is not of the form  $M \times N$ . This proves the claim and hence  $\sqcap\{\mathfrak{S}(A_i)\} = (0, 1, 1)$ . Since this is true for any representation of  $A$  as a finite intersection, by the definition of  $\mathfrak{B}'$  we have  $\mathfrak{B}'(A) = (0, 1, 1)$ . Thus  $\mathfrak{B} = \mathfrak{B}'$  in this case.

If  $A$  is of the form  $M \times N$  for some  $M \sqsubseteq 1_X, N \sqsubseteq 1_Y$ . First we claim that  $\mathfrak{B}(A) \supseteq \mathfrak{B}'(A)$ . For, let  $A = A_1 \sqcap A_2 \sqcap \dots \sqcap A_n$  be a representation of  $A$  as a finite intersection of neutrosophic sets in  $\zeta^{X \times Y}$ . If  $A_i$  is neither of the form  $(M_i \times 1_Y)$  nor of the form  $(1_X \times N_i)$ , for at least one  $j$ , then it follows that  $\sqcap\{\mathfrak{S}(A_i)\} = (0, 1, 1)$  and hence  $\mathfrak{B}(A) \supseteq \sqcap\{\mathfrak{S}(A_i)\}$ . Suppose all  $A_i$ 's are either of the form  $(M_i \times 1_Y)$  or of the form  $(1_X \times N_i)$  for some  $M_i \in \zeta^X$  and  $N_i \in \zeta^Y$ ; then we have  $\mathfrak{S}(A_i) \supseteq (0, 1, 1)$  for all  $i$ . Let  $\epsilon > 0$ ; then there exist  $M_i \in \zeta^X$  and  $N_i \in \zeta^Y$  such that

$$A_i = (M_i \times 1_Y), \mathfrak{T}_1(M_i) \supseteq \mathfrak{S}(A_i) - (\epsilon, 0, 0)$$

for  $i = 1, 2, \dots, m$  and

$$A_i = (1_X \times N_i), \mathfrak{T}_1(N_i) \supseteq \mathfrak{S}(A_i) - (\epsilon, 0, 0).$$

for  $i = m + 1, m + 2, \dots, n$ . Then,

$$\begin{aligned} A &= A_1 \sqcap A_2 \sqcap \dots \sqcap A_n \\ &= \{(M_1 \sqcap M_2 \sqcap \dots \sqcap M_m) \times 1_Y\} \\ &\quad \sqcap \{1_X \times (N_{m+1} \sqcap N_{m+2} \sqcap \dots \sqcap N_n)\}. \end{aligned}$$

Let  $M' = M_1 \sqcap \dots \sqcap M_m$  and  $N' = N_{m+1} \sqcap \dots \sqcap N_n$ . Then we have  $A = (M' \times 1_Y) \sqcap (1_X \times N') = M' \times N'$ .

Now consider

$$\begin{aligned} \mathfrak{B}(A) &\supseteq \sqcap\{\mathfrak{T}_1(M'), \mathfrak{T}_2(N')\} \\ &\supseteq \sqcap\left\{\prod_{j=1}^m \mathfrak{T}_1(M_j), \prod_{j=m+1}^n \mathfrak{T}_2(N_j)\right\} \\ &= \sqcap\{\mathfrak{T}_1(M_1), \dots, \mathfrak{T}_1(M_m), \mathfrak{T}_2(N_{m+1}), \dots, \mathfrak{T}_2(N_n)\} \\ &\supseteq \sqcap\{\mathfrak{S}(A_1) - (\epsilon, 0, 0), \dots, \mathfrak{S}(A_n) - (\epsilon, 0, 0)\} \\ &= \sqcap\{\mathfrak{S}(A_i)\} - (\epsilon, 0, 0). \end{aligned}$$

Since this is true for any representation of  $A$  as a finite intersection of neutrosophic sets in  $\zeta^{X \times Y}$ , we have

$$\mathfrak{B}(A) \supseteq \mathfrak{B}'(A).$$

To prove the reverse inequality, let  $\epsilon > 0$ ; then by Definition 10, there exist  $M \in \zeta^X$  and  $N \in \zeta^Y$  such that  $A = M \times N$  and

$$\mathfrak{T}_1(M) \sqcap \mathfrak{T}_2(N) \supseteq \mathfrak{B}(A) - (\epsilon, 0, 0).$$



However,  $M \times N = (M \times 1_Y) \sqcap (1_X \times N)$ ; thus, we have,

$$\begin{aligned} \mathfrak{B}'(A) &= \mathfrak{B}'(M \times N) \\ &\supseteq \sqcap \{ \mathfrak{S}(M \times 1_Y), \mathfrak{S}(1_X \times N) \} \\ &\supseteq \sqcap \{ \mathfrak{T}_1(M), \mathfrak{T}_2(N) \} \\ &\supseteq \mathfrak{B}(A) - (\epsilon, 0, 0) \end{aligned}$$

which implies  $\mathfrak{B}'(A) \supseteq \mathfrak{B}(A)$  as desired.  $\square$

**Definition 11.** Let  $\{(X_i, \mathfrak{T}_i)\}_{i \in J}$  be a collection of smooth neutrosophic topological spaces, for some indexing set  $J$ . Now define a function  $\mathfrak{B} : \zeta^{\prod_{i \in J} X_i} \rightarrow \zeta$  as follows:

Let  $A \in \zeta^{\prod_{i \in J} X_i}$ . If  $A \neq \prod_{i \in J} A_i$  where  $A_i \in \zeta^{X_i}$  and  $A_i = 1_{X_i}$  except for finitely many  $i \in J$ , then define  $\mathfrak{B}(A) = (0, 1, 1)$ . Otherwise define

$$\mathfrak{B}(A) = \sqcup_{\mathcal{A}_A} \{ \sqcap_{i \in J} \{ \mathfrak{T}_i(A_i) \} \},$$

where  $\mathcal{A}_A$  is the collection of all  $\{ \prod_{i \in J} A_i \}$  such that  $A = \prod_{i \in J} A_i$ ,  $A_i \in \zeta^{X_i}$  and  $A_i = 1_{X_i}$  except for finitely many  $i \in J$ .

Then  $\mathfrak{B}$  is a basis for a smooth neutrosophic topology called the smooth product topology on  $\prod_{i \in J} X_i$ .

**Theorem 11.** Let  $\{(X_i, \mathfrak{T}_i)\}_{i \in J}$  be a collection of smooth neutrosophic topological spaces, for some indexing set  $J$ . Let  $\mathfrak{B}$  be as defined in Definition 11; then  $\mathfrak{B}$  is a basis for a smooth neutrosophic topology on  $\prod_{i \in J} X_i$ .

**Proof.** Since  $\mathfrak{B}(1_{\prod_{i \in J} X_i}) = (1, 0, 0)$ , **B1** of Definition 7 follows trivially.

To prove **B2**, let  $M, N \in \zeta^{\prod_{i \in J} X_i}$ ,  $\mathbf{x} \in \prod_{i \in J} X_i$  and  $\epsilon, \delta > 0$ . Let  $\mathcal{A}_M$  be the collection of all  $\{ \prod_{i \in J} M_i \}$  such that  $M = \prod_{i \in J} M_i$ ,  $M_i \in \zeta^{X_i}$  and  $M_i = 1_{X_i}$  except for finitely many  $i \in J$  and let  $\mathcal{A}_N$  be the collection of all  $\{ \prod_{i \in J} N_i \}$  such that  $N = \prod_{i \in J} N_i$ ,  $N_i \in \zeta^{X_i}$  and  $N_i = 1_{X_i}$  except for finitely many  $i \in J$ .

Suppose any one of the collections  $\mathcal{A}_M$  and  $\mathcal{A}_N$ , say  $\mathcal{A}_M$ , is empty. Then by the definition of  $\mathfrak{B}$ , we get that  $\mathfrak{B}(M) = (0, 1, 1)$ . Thus **B2** of Definition 7 follows in this case. If both collections  $\mathcal{A}_M$  and  $\mathcal{A}_N$  are nonempty, then there exist  $A_M = \prod M_i$  in  $\mathcal{A}_M$  and  $A_N = \prod N_i$  in  $\mathcal{A}_N$  such that

$$\sqcap \{ \mathfrak{T}_i(M_i) \} \supseteq \mathfrak{B}(M) - (\epsilon, 0, 0)$$

and

$$\sqcap \{ \mathfrak{T}_i(N_i) \} \supseteq \mathfrak{B}(N) - (\epsilon, 0, 0).$$

Let  $L = M \sqcap N$ ; then clearly

$$L(\mathbf{x}) \supseteq (M(\mathbf{x}) \sqcap N(\mathbf{x})) - (\delta, 0, 0), \forall \mathbf{x} \in \prod X_i$$

and

$$\begin{aligned}
 \mathfrak{B}(\mathbb{L}) &= \mathfrak{B}(\mathbb{M} \cap \mathbb{N}) \\
 &= \mathfrak{B}(\prod \mathbb{M}_i \cap \prod \mathbb{N}_i) \\
 &= \mathfrak{B}(\prod (\mathbb{M}_i \cap \mathbb{N}_i)) \\
 &\supseteq \prod \{\mathfrak{T}_i(\mathbb{M}_i \cap \mathbb{N}_i)\} \\
 &\supseteq \prod \{\mathfrak{T}_i(\mathbb{M}_i) \cap \mathfrak{T}_i(\mathbb{N}_i)\} \\
 &\supseteq \prod \{\mathfrak{T}_i(\mathbb{M}_i)\} \cap \prod \{\mathfrak{T}_i(\mathbb{N}_i)\} \\
 &\supseteq (\mathfrak{B}(\mathbb{M}) - (\epsilon, 0, 0)) \cap (\mathfrak{B}(\mathbb{N}) - (\epsilon, 0, 0)) \\
 &\supseteq (\mathfrak{B}(\mathbb{M}) \cap \mathfrak{B}(\mathbb{N})) - (\epsilon, 0, 0).
 \end{aligned}$$

Thus, **B2** of Definition 7 follows in this case also, and hence  $\mathfrak{B}$  is a basis for a smooth neutrosophic topology on  $\prod X_i$ .  $\square$

**Theorem 12.** Let  $\{(X_i, \mathfrak{T}_i)\}_{i \in J}$  be a collection of smooth neutrosophic topological spaces. For any  $\mathbb{A} \in \zeta^{\prod_{i \in J} X_i}$ , let  $\mathcal{A}_{\mathbb{A}}$  be the collection of all  $\{\prod \mathbb{A}_i\}$  such that  $\mathbb{A} = \prod \mathbb{A}_i$ ,  $\mathbb{A}_i \in \zeta^{X_i}$  and  $\mathbb{A}_i = 1_{X_i}$  except for finitely many  $i \in J$ . Let  $\mathfrak{S} : \zeta^{\prod_{i \in J} X_i} \rightarrow \zeta$  be defined as follows:

$$\mathfrak{S}(\mathbb{A}) = \begin{cases} (1, 0, 0) & \text{if } \mathbb{A} = 1_{\prod X_i} \\ \sqcup_{\mathcal{A}_{\mathbb{A}}} \{ \sqcup_{\mathbb{A}_i \neq 1_{X_i}} \{\mathfrak{T}_i(\mathbb{A}_i)\} \} & \text{if } \mathbb{A} \neq 1_{\prod X_i}, \mathcal{A}_{\mathbb{A}} \neq \emptyset \\ (0, 1, 1) & \text{if } \mathcal{A}_{\mathbb{A}} = \emptyset. \end{cases}$$

Then  $\mathfrak{S}$  is a subbasis for a smooth neutrosophic product topology on  $\prod X_i$ .

**Proof.** Since  $\mathfrak{S}(1_{\prod X_i}) = (1, 0, 0)$ , **B1** of Definition 7 follows. Thus  $\mathfrak{S}$  is a subbasis for a smooth neutrosophic topology on  $\prod X_i$ . Thus, proving that the smooth neutrosophic topology generated from  $\mathfrak{S}$  is the smooth neutrosophic product topology on  $\prod X_i$  needs proving.

Now let  $\mathfrak{B}'$  be the basis generated by  $\mathfrak{S}$  and let  $\mathfrak{B}$  be the basis for the smooth neutrosophic product topology defined in Definition 11. To prove the topologies generated by  $\mathfrak{B}$  and  $\mathfrak{B}'$  are same, we prove the stronger result that  $\mathfrak{B} = \mathfrak{B}'$ .

As  $\mathfrak{B}(1_{\prod X_i}) = \mathfrak{B}'(1_{\prod X_i}) = (1, 0, 0)$  follows trivially, we prove the other cases. Let  $\mathbb{A} \in \zeta^{\prod X_i}$  and let  $\mathcal{A}_{\mathbb{A}}$  be the collection of all  $\{\prod \mathbb{A}_i\}$  such that  $\mathbb{A} = \prod \mathbb{A}_i$ ,  $\mathbb{A}_i \in \zeta^{X_i}$  and  $\mathbb{A}_i = 1_{X_i}$  except for finitely many  $i \in J$ . If  $\mathcal{A}_{\mathbb{A}} = \emptyset$ , then by the definition of  $\mathfrak{B}$ , we have  $\mathfrak{B}(\mathbb{A}) = (0, 1, 1)$ . Now to compute  $\mathfrak{B}'(\mathbb{A})$ , let  $\mathbb{A} = \mathbb{A}_1 \cap \mathbb{A}_2 \cap \dots \cap \mathbb{A}_n$ ; we claim that there must exist at least one  $\mathbb{A}_k$  which is not of the form  $\prod \mathbb{A}_{ki}$  where  $\mathbb{A}_{ki} \in \zeta^{X_i}$  and  $\mathbb{A}_{ki} = 1_{X_i}$  except for finitely many  $i \in J$ . Suppose not; instead, let  $\mathbb{A}_j = \prod \mathbb{A}_{ji}$  where  $\mathbb{A}_{ji} \in \zeta^{X_i}$  and  $\mathbb{A}_{ji} = 1_{X_i}$  except for finitely many  $i \in J$ , for all  $j = 1, 2, \dots, n$ . Then using these finitely many  $\mathbb{A}_{ji}$ 's,  $\mathbb{A}$  can be written in the form  $\prod \mathbb{A}_i$  where  $\mathbb{A}_i \in \zeta^{X_i}$  and  $\mathbb{A}_i = 1_{X_i}$  except for finitely many  $i \in J$ , which is a contradiction to our assumption that  $\mathcal{A}_{\mathbb{A}} = \emptyset$ . Thus there exists at least one  $\mathbb{A}_k$  which is not of the form  $\prod \mathbb{A}_{ki}$  where  $\mathbb{A}_{ki} \in \zeta^{X_i}$  and  $\mathbb{A}_{ki} = 1_{X_i}$  except for finitely many  $i \in J$  and hence  $\mathfrak{S}(\mathbb{A}_k) = (0, 1, 1)$ . Thus we have

$$\prod \{\mathfrak{S}(\mathbb{A}_j) / j = 1, 2, \dots, n\} = (0, 1, 1).$$

Since this is true for any possible finite representation  $\mathbb{A}_1 \cap \mathbb{A}_2 \cap \dots \cap \mathbb{A}_n$  of  $\mathbb{A}$ , we have  $\mathfrak{B}'(\mathbb{A}) = (0, 1, 1)$  and hence  $\mathfrak{B}'(\mathbb{A}) = \mathfrak{B}(\mathbb{A})$  in this case.

If  $\mathcal{A}_{\mathbb{A}} \neq \emptyset$ , then there must exist a representation  $\mathbb{A}_1 \cap \mathbb{A}_2 \cap \dots \cap \mathbb{A}_n$  of  $\mathbb{A}$  such that  $\mathcal{A}_{\mathbb{A}_j} \neq \emptyset$  for all  $j = 1, 2, \dots, n$ , where  $\mathcal{A}_{\mathbb{A}_j}$  is the collection of all  $\{\prod \mathbb{A}_{ji}\}$  such that  $\mathbb{A}_j = \prod \mathbb{A}_{ji}$ ,  $\mathbb{A}_{ji} \in \zeta^{X_i}$  and  $\mathbb{A}_{ji} = 1_{X_i}$

except for finitely many  $i \in J$ . Let  $\epsilon > 0$ . Then for each  $A_j$  we can find a collection  $\{A_{ji}\}_{i \in J}$  such that  $A_j = \Pi A_{ji}$  where  $A_{ji} = 1_{X_i}$  except for finitely many  $i \in J$  and  $\mathfrak{T}_i(A_{ji}) \supseteq \mathfrak{S}(A_j) - (\epsilon, 0, 0)$ . Now since

$$\begin{aligned} A &= A_1 \sqcap A_2 \sqcap \cdots \sqcap A_n \\ &= \Pi A_{1i} \sqcap \Pi A_{2i} \sqcap \cdots \sqcap \Pi A_{ni} \\ &= \Pi(A_{1i} \sqcap A_{2i} \sqcap \cdots \sqcap A_{ni}) \end{aligned}$$

we have

$$\begin{aligned} \mathfrak{B}(A) &= \mathfrak{B}(A_1 \sqcap A_2 \sqcap \cdots \sqcap A_n) \\ &= \mathfrak{B}(\Pi A_{1i} \sqcap \Pi A_{2i} \sqcap \cdots \sqcap \Pi A_{ni}) \\ &= \mathfrak{B}(\Pi(A_{1i} \sqcap A_{2i} \sqcap \cdots \sqcap A_{ni})) \\ &\supseteq \sqcap \{\mathfrak{T}_i(A_{1i} \sqcap A_{2i} \sqcap \cdots \sqcap A_{ni})\} \\ &\supseteq \sqcap \{\mathfrak{T}_i(A_{1i}) \sqcap \mathfrak{T}_i(A_{2i}) \sqcap \cdots \sqcap \mathfrak{T}_i(A_{ni})\} \\ &\supseteq \sqcap \{\mathfrak{S}(A_1) - (\epsilon, 0, 0) \sqcap \mathfrak{S}(A_2) - (\epsilon, 0, 0) \\ &\quad \sqcap \cdots \sqcap \mathfrak{S}(A_n) - (\epsilon, 0, 0)\} \\ &= \sqcap \{\mathfrak{S}(A_1) \sqcap \mathfrak{S}(A_2) \sqcap \cdots \sqcap \mathfrak{S}(A_n)\} - (\epsilon, 0, 0) \\ &= \sqcap \{\mathfrak{S}(A_j)\} - (\epsilon, 0, 0). \end{aligned}$$

Since this is true for any representation of  $A$  as a finite intersection of neutrosophic sets in  $\zeta^{\Pi X_i}$ , we have

$$\mathfrak{B}(A) \supseteq \mathfrak{B}'(A).$$

To prove the reverse inequality, let  $\epsilon > 0$ . Since  $\mathcal{A}_A \neq \emptyset$ , we can find a collection  $\{A_i\}_{i \in J}$  such that  $\Pi A_i \in \mathcal{A}_A$  and

$$\sqcap \{\mathfrak{T}_i(A_i)\} \supseteq \mathfrak{B}(A) - (\epsilon, 0, 0).$$

Thus it follows that

$$\mathfrak{B}'(A) = \mathfrak{B}'(\Pi A_i) \supseteq \sqcap \{\mathfrak{T}_i(A_i)\} \supseteq \mathfrak{B}(A) - (\epsilon, 0, 0)$$

and hence  $\mathfrak{B}'(A) \supseteq \mathfrak{B}(A)$ . Thus  $\mathfrak{B}'(A) = \mathfrak{B}(A)$  in this case also and hence in all the cases.  $\square$

## 5. Conclusions

In this paper, we have defined the notion of a basis and subbasis for a neutrosophic topology as a neutrosophic set from a suitable collection of neutrosophic sets of  $X$  to  $[0, 1]^3$ . Using this idea of considering a basis as a neutrosophic set, we developed a theory of smooth neutrosophic topological spaces that fits exactly with the theory of classical and fuzzy topological spaces. Next, we introduced and investigated the concept of quasi-coincident neighborhood systems in this context. Finally, we defined and discussed the notion of both finite and infinite products of smooth neutrosophic topologies.

## 6. A Discussion for Future Works

The theory can be extended in the following natural ways. One may

- Study the properties of neutrosophic metric topological spaces using the concept of basis defined in this paper;
- Investigate the products of Hausdorff, regular, compact and connected spaces in the context of neutrosophic topological spaces.

**Author Contributions:** Conceptualization, K.C. and S.S.S.; writing, original draft, K.C. and S.S.S.; writing, review and editing, K.C. and S.S.S.; Supervision and Validation F.S. and S.J. All authors have read and agreed to the published version of the manuscript.

**Funding:** This research received no external funding.

**Acknowledgments:** The authors would like to express their sincere thanks the the referees for the useful comments and suggestion in preparing this manuscript.

**Conflicts of Interest:** The authors declare no conflict of interest.

## References

1. Smarandache, F. *A Unifying Field in Logics, Neutrosophy: Neutrosophic Probability, Set and Logic*; American Research Press: Rehoboth, NM, USA, 1999.
2. Hur, K.; Lim, P.K.; Lee, J.G.; Kim, J. The category of neutrosophic sets. *Neutrosophic Sets Syst.* **2016**, *14*, 12–20.
3. Hur, K.; Lim, P.K.; Lee, J.G.; Kim, J. The category of neutrosophic crisp sets. *Ann. Fuzzy Math. Inform.* **2017**, *14*, 43–54. [[CrossRef](#)]
4. Salama, A.A.; Alblowi, S.A. Neutrosophic set and neutrosophic topological spaces. *IOSR J. Math.* **2012**, *3*, 31–35. [[CrossRef](#)]
5. Salama, A.A.; Smarandache, F.; Kroumov, V. Neutrosophic crisp sets and neutrosophic crisp topological spaces. *Neutrosophic Sets Syst.* **2014**, *2*, 25–30.
6. Salama, A.A.; Smarandache, F. *Neutrosophic Crisp Set Theory*; Educational Publisher: Columbus, OH, USA, 2015.
7. Chang, C.L. Fuzzy Topological Spaces. *J. Math. Anal. Appl.* **1968**, *24*, 182–190. [[CrossRef](#)]
8. Šostak A.P. On a Fuzzy Topological Structure. *Rend. Circ. Mat. Palermo Ser. II* **1985**, *11*, 89–103.
9. Fang, J.M.; Yue, Y.L. Base and Subbase in *I*-fuzzy Topological Spaces. *J. Math. Res. Expos.* **2006**, *26*, 89–95.
10. Shakthiganesan, M.; Vembu, R. On the product of smooth fuzzy topological spaces. *Novi Sad J. Math.* **2016**, *46*, 13–31. [[CrossRef](#)]
11. Liang, C.; Yan, C. Base and subbase in intuitionistic *I*-fuzzy topological spaces. *Hacet. J. Math. Stat.* **2014**, *43*, 231–247.
12. Yan, C.H.; Wang, X.K. Intuitionistic *I*-fuzzy topological spaces. *Czechoslov. Math. J.* **2010**, *60*, 233–252. [[CrossRef](#)]
13. Wang, H.; Smarandache, F.; Zhang, Y.Q.; Sunderraman, R. Single valued neutrosophic sets. *Multispace Multistruct.* **2010**, *4*, 410–413.
14. EL Gayyar, M.K. Smooth Neutrosophic Topological Spaces. *Neutrosophic Sets Syst.* **2016**, *12*, 65–72.
15. Kim, J.; Smarandache, F.; Lee J.G.; Hur, K. Ordinary Single Valued Neutrosophic Topological Spaces. *Symmetry* **2019**, *11*, 1075. [[CrossRef](#)]
16. Al Shumrani, M.A.; Gulisten, M.; Smarandache, F. Further theory of neutrosophic triplet topology and applications. *Symmetry* **2020**, *12*, 1207. [[CrossRef](#)]
17. Saber, Y.; Alsharari, F.; Smarandache, F. On single-valued neutrosophic ideals in Šostak's sense. *Symmetry* **2020**, *12*, 193. [[CrossRef](#)]



© 2020 by the authors. Licensee MDPI, Basel, Switzerland. This article is an open access article distributed under the terms and conditions of the Creative Commons Attribution (CC BY) license (<http://creativecommons.org/licenses/by/4.0/>).