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To cite this article: Samed Özkan (2021): On Single-Valued Neutrosophic Proximity Spaces, Fuzzy Information and Engineering, DOI: [10.1080/16168658.2020.1822082](https://doi.org/10.1080/16168658.2020.1822082)

To link to this article: <https://doi.org/10.1080/16168658.2020.1822082>



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Published online: 29 Jan 2021.



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On Single-Valued Neutrosophic Proximity Spaces

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ABSTRACT

In this paper, the notion of single-valued neutrosophic proximity spaces which is a generalisation of fuzzy proximity spaces [Katsaras AK. Fuzzy proximity spaces. *Anal and Appl.* 1979;68(1):100–110.] and intuitionistic fuzzy proximity spaces [Lee SJ, Lee EP. Intuitionistic fuzzy proximity spaces. *Int J Math Math Sci.* 2004;49:2617–2628.] was introduced and some of their properties were investigated. Then, it was shown that a single-valued neutrosophic proximity on a set X induced a single-valued neutrosophic topology on X . Furthermore, the existence of initial single-valued neutrosophic proximity structure is proved. Finally, based on this fact, the product of single-valued neutrosophic proximity spaces was introduced.

ARTICLE HISTORY

Received 5 December 2018
Revised 23 June 2020
Accepted 20 July 2020

KEYWORDS

Single-valued neutrosophic; proximity space; initial structure; product

AMS CLASSIFICATIONS

54E05; 54B10

1. Introduction

In 1998, F. Smarandache [1] introduced the concept of neutrosophic set which is a mathematical tool for handling problems involving incomplete, indeterminate and inconsistent information in real world. The neutrosophic sets are characterised by three membership functions independently: truth, indeterminacy and falsity, which are within the real standard or nonstandard unit interval $]^{-}0, 1^{+}[$. Therefore, this notion is a generalisation of the theory of fuzzy sets [2] and intuitionistic fuzzy sets [3]. Salama and Alblowi [4] introduced and studied neutrosophic topological spaces and its continuous functions.

The neutrosophic set generalises the sets from a philosophical point of view. But, from a scientific or an engineering point of view, the neutrosophic set operators need to be specified. Because, it is not convenient to apply neutrosophic sets to practical problems in the real-life applications. So, Wang et al. [5] introduced the single-valued neutrosophic sets (SVNSs) by simplifying neutrosophic sets (NSs). SVNSs are also a generalisation of intuitionistic fuzzy sets, in which three membership functions are independent and their value belong to the unit interval $[0, 1]$.

Neutrosophic set theory is widely studied by many researchers. It is used in many real application area, such as medical diagnosis [6], image processing [7], fault diagnosis [8] and multi-criteria decision making [9], which are over-cited research topics in various fields.

Proximity spaces were introduced by Efremovich during the first part of 1930s and later axiomatised [10, 11]. He characterised the proximity relation ‘A is near B’ for subsets A and

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B of any set X . Efremovich [11] defined the closure of a subset A of X to be the collection of all points of X 'close' A . In this way, he showed that a topology (completely regular) can be introduced in a proximity space. This theory was improved by Smirnov [12]. He was the first to discover relationship between proximities and uniformities.

In 2007, Peters [13] extended the standard spatial proximity space to a descriptive proximity space that examined the descriptive nearness of objects. The concept of descriptive proximity is useful in identifying, analysing and classifying the parts of a digital image. In recent years, several practical applications in many fields such as cytology (cell biology), criminology, digital image processing have been published [14, 15].

The most comprehensive work on the proximity spaces theory was done by Naimpally and Warrack [16]. All preliminary information about proximity spaces can be found in this source.

The main objective of this paper is

- (1) to introduce the concept of single-valued neutrosophic proximity spaces and investigated some of their properties,
- (2) to show that a single-valued neutrosophic proximity on a set X induced a single-valued neutrosophic topology on X , and
- (3) to define the initial single-valued neutrosophic proximity structure and the product of single-valued neutrosophic proximity spaces.

2. Preliminaries

The following are some basic definitions and notations which we will use throughout the paper.

Definition 2.1 ([4]): Let X be a nonempty fixed set. A neutrosophic set (NS for short) A is an object having the form $A = \{\langle x, T_A(x), I_A(x), F_A(x) \rangle : x \in X\}$ where the functions T_A , I_A and F_A which represent the degree of membership (namely $T_A(x)$), the degree of indeterminacy (namely $I_A(x)$) and the degree of nonmembership (namely $F_A(x)$) respectively, of each element $x \in X$ to the set A with the condition

$$-0 \leq T_A(x) + I_A(x) + F_A(x) \leq 3^+$$

A neutrosophic set $A = \{\langle x, T_A(x), I_A(x), F_A(x) \rangle : x \in X\}$ can be identified to an ordered triple $\langle T_A, I_A, F_A \rangle$ in $]^{-}0, 1^+[$ (nonstandard unit interval) on X .

Remark 2.1 ([4]): Every intuitionistic fuzzy set (IFS for short) A is a nonempty set in X is obviously on NS having the form $A = \{\langle x, T_A(x), 1 - (T_A(x) + F_A(x)), F_A(x) \rangle : x \in X\}$.

Definition 2.2 ([4]): The neutrosophic set 0_N in X may be defined as

$$(0_1) \quad 0_N = \{\langle x, 0, 0, 1 \rangle : x \in X\}$$

$$(0_2) \quad 0_N = \{\langle x, 0, 1, 1 \rangle : x \in X\}$$

$$(0_3) \quad 0_N = \{\langle x, 0, 1, 0 \rangle : x \in X\}$$

$$(0_4) \quad 0_N = \{\langle x, 0, 0, 0 \rangle : x \in X\}.$$

The neutrosophic set 1_N in X may be defined as

- (1₁) $1_N = \{ \langle x, 1, 0, 0 \rangle : x \in X \}$
- (1₂) $1_N = \{ \langle x, 1, 0, 1 \rangle : x \in X \}$
- (1₃) $1_N = \{ \langle x, 1, 1, 0 \rangle : x \in X \}$
- (1₄) $1_N = \{ \langle x, 1, 1, 1 \rangle : x \in X \}$.

Definition 2.3 ([4]): Let $A = \langle x, T_A, I_A, F_A \rangle$ be a NS on X , then the complement of the set A ($C(A)$ for short) may be defined as three kinds of complements:

- (C₁) $C(A) = \{ \langle x, 1 - T_A(x), 1 - I_A(x), 1 - F_A(x) \rangle : x \in X \}$
- (C₂) $C(A) = \{ \langle x, F_A(x), I_A(x), T_A(x) \rangle : x \in X \}$
- (C₃) $C(A) = \{ \langle x, F_A(x), 1 - I_A(x), T_A(x) \rangle : x \in X \}$.

One can define several relations and operations between neutrosophic sets follows:

Definition 2.4 ([4]): Let X be a nonempty set, and neutrosophic sets A and B in the form $A = \{ \langle x, T_A(x), I_A(x), F_A(x) \rangle : x \in X \}$ and $B = \{ \langle x, T_B(x), I_B(x), F_B(x) \rangle : x \in X \}$. Then we may consider two possible definitions for subsets.

$A \subseteq B$ may be defined as

- (1) $A \subseteq B \iff T_A(x) \leq T_B(x), I_A(x) \leq I_B(x)$ and $F_A(x) \geq F_B(x), \forall x \in X$
- (2) $A \subseteq B \iff T_A(x) \leq T_B(x), I_A(x) \geq I_B(x)$ and $F_A(x) \geq F_B(x), \forall x \in X$.

Proposition 2.5 ([4]): For any neutrosophic set A , then the following conditions hold:

- (1) $0_N \subseteq A, 0_N \subseteq 0_N$
- (2) $A \subseteq 1_N, 1_N \subseteq 1_N$.

Definition 2.6 ([4]): Let X be a nonempty set, $A = \langle x, T_A(x), I_A(x), F_A(x) \rangle$ and $B = \langle x, T_B(x), I_B(x), F_B(x) \rangle$ are neutrosophic sets. Then

$A \cap B$ may be defined as

- (1) $A \cap B = \langle x, T_A(x) \wedge T_B(x), I_A(x) \wedge I_B(x), F_A(x) \vee F_B(x) \rangle$
- (2) $A \cap B = \langle x, T_A(x) \wedge T_B(x), I_A(x) \vee I_B(x), F_A(x) \vee F_B(x) \rangle$

$A \cup B$ may be defined as

- (1) $A \cup B = \langle x, T_A(x) \vee T_B(x), I_A(x) \vee I_B(x), F_A(x) \wedge F_B(x) \rangle$
- (2) $A \cup B = \langle x, T_A(x) \vee T_B(x), I_A(x) \wedge I_B(x), F_A(x) \wedge F_B(x) \rangle$

where \vee and \wedge denote the maximum and minimum, respectively.

Since it is not convenient to apply neutrosophic sets to practical problems in the real applications, Wang et al. [5] introduced the concept of single-valued neutrosophic sets

(SVNSs for short), which is an instance of a neutrosophic set. SVNSs can be used in real scientific and engineering applications.

Definition 2.7 ([5]): Let X be a space of points (objects), with a generic element in X denoted by x . A single-valued neutrosophic set (SVNS) \tilde{A} in X is characterised by three membership functions, a truth-membership function $T_{\tilde{A}}$, an indeterminacy-membership function $I_{\tilde{A}}$, and a falsity-membership function $F_{\tilde{A}}$. For each point $x \in X$, $T_{\tilde{A}}, I_{\tilde{A}}, F_{\tilde{A}} \in [0, 1]$.

A SVNS \tilde{A} can be denoted by

$$\tilde{A} = \{(x, T_{\tilde{A}}(x), I_{\tilde{A}}(x), F_{\tilde{A}}(x)) : x \in X\}.$$

Remark 2.2: For the sake of simplicity, we shall use the symbol $\tilde{A} = \langle T_{\tilde{A}}, I_{\tilde{A}}, F_{\tilde{A}} \rangle$ for the single-valued neutrosophic set $\tilde{A} = \{(x, T_{\tilde{A}}(x), I_{\tilde{A}}(x), F_{\tilde{A}}(x)) : x \in X\}$.

Remark 2.3: In SVNSs, we consider the neutrosophic set which takes the value from the subset of the classical unit interval $[0, 1]$ to apply neutrosophic set to science and technology. But the neutrosophic set generalises the sets from a philosophical point of view to deal with incomplete, indeterminate and inconsistent information in real world. Therefore, some neutrosophic components are off the interval $[0, 1]$, i.e. some neutrosophic components > 1 and some neutrosophic components < 0 .

Example 2.8: In a factory a full-time worker works 40 hours per week. Considering the period of last week, worker A worked only 32 hours part-time, worker B worked full time 40 hours, and worker C worked 48 hours, working 8 hours overtime. So the degrees of membership of workers A , B and C are $32/40 = 0.8 < 1$, $40/40 = 1$ and $48/40 = 1.2 > 1$, respectively. We need to make distinction between workers who work overtime, and those who work full-time or part-time. That's why we need to associate a degree of membership greater than 1 to the overtime workers. Similarly, worker D was absent without pay for the whole week, and worker E did not come to work last week, but also caused damage which was estimated at a value half of the weekly salary. The membership degree of worker E has to be less than the worker D 's. So the degrees of membership of workers D and E are $0/40 = 0$ and $-20/40 = -0.5 < 0$, respectively. Consequently, the membership degrees can be off the interval $[0, 1]$ in NSs.

An empty SVNS $\tilde{0}$, a full SVNS $\tilde{1}$ and the operators such as complement, containment, union, intersection in the single-valued neutrosophic sets can be defined in different forms as given in the above definitions for neutrosophic sets. We use the following definitions for these concepts:

Definition 2.9: Let $\tilde{A} = \langle T_{\tilde{A}}, I_{\tilde{A}}, F_{\tilde{A}} \rangle$ and $\tilde{B} = \langle T_{\tilde{B}}, I_{\tilde{B}}, F_{\tilde{B}} \rangle$ be SVNSs on a nonempty set X . Then Empty SVNS $\tilde{0}$ and full SVNS $\tilde{1}$ are defined as

- $\tilde{0} = \{(x, 0, 0, 1) : x \in X\}$
- $\tilde{1} = \{(x, 1, 1, 0) : x \in X\}$

Complement of the SVN \tilde{A} ($C(\tilde{A})$ for short) is defined as

- $C(\tilde{A}) = \{\langle x, 1 - T_{\tilde{A}}(x), 1 - I_{\tilde{A}}(x), 1 - F_{\tilde{A}}(x) \rangle : x \in X\}$

$\tilde{A} \subseteq \tilde{B}$ is defined as

- $\tilde{A} \subseteq \tilde{B} \iff T_{\tilde{A}}(x) \leq T_{\tilde{B}}(x), I_{\tilde{A}}(x) \leq I_{\tilde{B}}(x) \text{ and } F_{\tilde{A}}(x) \geq F_{\tilde{B}}(x), \forall x \in X.$

Union and intersection operators are defined as

- $\tilde{A} \cup \tilde{B} = \{\langle x, T_{\tilde{A}}(x) \vee T_{\tilde{B}}(x), I_{\tilde{A}}(x) \vee I_{\tilde{B}}(x), F_{\tilde{A}}(x) \wedge F_{\tilde{B}}(x) \rangle : x \in X\}$
- $\tilde{A} \cap \tilde{B} = \{\langle x, T_{\tilde{A}}(x) \wedge T_{\tilde{B}}(x), I_{\tilde{A}}(x) \wedge I_{\tilde{B}}(x), F_{\tilde{A}}(x) \vee F_{\tilde{B}}(x) \rangle : x \in X\}.$

Example 2.10: Let $X = \{x_1, x_2\}$, $\tilde{A} = \{\langle x_1, 0.3, 1, 0.6 \rangle, \langle x_2, 0.8, 0.3, 0.5 \rangle\}$ and $\tilde{B} = \{\langle x_1, 0.4, 0.5, 0.9 \rangle, \langle x_2, 0, 0.7, 0.2 \rangle\}$ be SVN S s on X . Then,

$$C(\tilde{A}) = \{\langle x_1, 0.7, 0, 0.4 \rangle, \langle x_2, 0.2, 0.7, 0.5 \rangle\}$$

$$C(\tilde{B}) = \{\langle x_1, 0.6, 0.5, 0.1 \rangle, \langle x_2, 1, 0.3, 0.8 \rangle\}$$

$$\tilde{A} \cup \tilde{B} = \{\langle x_1, 0.4, 1, 0.6 \rangle, \langle x_2, 0.8, 0.7, 0.2 \rangle\}$$

$$\tilde{A} \cap \tilde{B} = \{\langle x_1, 0.3, 0.5, 0.9 \rangle, \langle x_2, 0, 0.3, 0.5 \rangle\}.$$

Proposition 2.11: For any SVN S s \tilde{A} and \tilde{B} , then the following conditions hold:

- (1) $\tilde{0} \subseteq \tilde{A} \subseteq \tilde{1}$
- (2) $\tilde{A} \cup \tilde{0} = \tilde{A}, \tilde{A} \cup \tilde{1} = \tilde{1}$ and $\tilde{A} \cap \tilde{0} = \tilde{0}, \tilde{A} \cap \tilde{1} = \tilde{A}$
- (3) $\tilde{A} \cup \tilde{B} = \tilde{B} \cup \tilde{A}$ and $\tilde{A} \cap \tilde{B} = \tilde{B} \cap \tilde{A}$
- (4) $\tilde{A} = \tilde{B} \iff \tilde{A} \subseteq \tilde{B} \text{ and } \tilde{B} \subseteq \tilde{A}$
- (5) $\tilde{A} \subseteq \tilde{A} \cup \tilde{B}, \tilde{B} \subseteq \tilde{A} \cup \tilde{B}$ and $\tilde{A} \cap \tilde{B} \subseteq \tilde{A}, \tilde{A} \cap \tilde{B} \subseteq \tilde{B}$
- (6) $\tilde{A} \subseteq \tilde{B} \iff \tilde{A} \cup \tilde{B} = \tilde{B}$ and $\tilde{A} \subseteq \tilde{B} \iff \tilde{A} \cap \tilde{B} = \tilde{A}$
- (7) $C(\tilde{0}) = \tilde{1}$ and $C(\tilde{1}) = \tilde{0}$
- (8) $C(C(\tilde{A})) = \tilde{A}$
- (9) $C(\tilde{A} \cup \tilde{B}) = C(\tilde{A}) \cap C(\tilde{B})$
- (10) $C(\tilde{A} \cap \tilde{B}) = C(\tilde{A}) \cup C(\tilde{B}).$

Definition 2.12: Let X and Y be two nonempty sets and $f: X \rightarrow Y$ a function.

- (i) If $\tilde{B} = \{\langle y, T_{\tilde{B}}(y), I_{\tilde{B}}(y), F_{\tilde{B}}(y) \rangle : y \in Y\}$ is an SVN S in Y , then the preimage of \tilde{B} under f is defined by

$$f^{-1}(\tilde{B}) = \{\langle x, f^{-1}(T_{\tilde{B}})(x), f^{-1}(I_{\tilde{B}})(x), f^{-1}(F_{\tilde{B}})(x) \rangle : x \in X\},$$

where $f(x) = y$ and $f^{-1}(T_{\tilde{B}})(x) = T_{\tilde{B}}(f(x)), f^{-1}(I_{\tilde{B}})(x) = I_{\tilde{B}}(f(x)), f^{-1}(F_{\tilde{B}})(x) = F_{\tilde{B}}(f(x)).$

- (ii) If $\tilde{A} = \{\langle x, T_{\tilde{A}}(x), I_{\tilde{A}}(x), F_{\tilde{A}}(x) \rangle : x \in X\}$ is a SVN S in X , then the image of \tilde{A} under f is defined by

$$f(\tilde{A}) = \{\langle y, f(T_{\tilde{A}})(y), f(I_{\tilde{A}})(y), (1 - f(1 - F_{\tilde{A}}))(y) \rangle : y \in Y\},$$

where

$$f(T_{\tilde{A}})(y) = \begin{cases} \sup_{x \in f^{-1}(y)} T_{\tilde{A}}(x), & \text{if } f^{-1}(y) \neq \emptyset \\ 0, & \text{otherwise} \end{cases},$$

$$f(I_{\tilde{A}})(y) = \begin{cases} \sup_{x \in f^{-1}(y)} I_{\tilde{A}}(x), & \text{if } f^{-1}(y) \neq \emptyset \\ 0, & \text{otherwise} \end{cases}$$

and

$$(1 - f(1 - F_{\tilde{A}}))(y) = \begin{cases} \inf_{x \in f^{-1}(y)} F_{\tilde{A}}(x), & \text{if } f^{-1}(y) \neq \emptyset \\ 0, & \text{otherwise.} \end{cases}$$

The concept of single-valued neutrosophic topological space is defined as follows:

Definition 2.13 ([17]): A single-valued neutrosophic topology (SVNT for short) on a nonempty set X is a family $\tilde{\tau}$ of SVNOSs in X satisfying the following axioms:

- ($\tilde{\tau}_1$) $\tilde{0}, \tilde{1} \in \tilde{\tau}$
- ($\tilde{\tau}_2$) $\tilde{G}_1 \cap \tilde{G}_2 \in \tilde{\tau}$ for any $\tilde{G}_1, \tilde{G}_2 \in \tilde{\tau}$
- ($\tilde{\tau}_3$) $\bigcup \tilde{G}_i \in \tilde{\tau}$ for every $\{\tilde{G}_i : i \in J\} \subseteq \tilde{\tau}$.

In this case, the pair $(X, \tilde{\tau})$ is called a single-valued neutrosophic topological space (SVNTS for short). The elements of $\tilde{\tau}$ are called single-valued neutrosophic open sets (SVNOSs for short).

Example 2.14: Let $X = \{x_1, x_2, x_3\}$ and

$$\begin{aligned} \tilde{A} &= \{\langle x_1, 0.4, 0.2, 0.5 \rangle, \langle x_2, 0.5, 0.8, 0.3 \rangle, \langle x_3, 0.7, 0.5, 1 \rangle\} \\ \tilde{B} &= \{\langle x_1, 0.8, 0.8, 0.3 \rangle, \langle x_2, 0.7, 1, 0.1 \rangle, \langle x_3, 0.9, 0.9, 0.7 \rangle\} \\ \tilde{C} &= \{\langle x_1, 0.2, 0.1, 0.6 \rangle, \langle x_2, 0.3, 0.7, 0.4 \rangle, \langle x_3, 0.6, 0.3, 1 \rangle\} \\ \tilde{D} &= \{\langle x_1, 0.6, 0.4, 0.4 \rangle, \langle x_2, 0.5, 1, 0.2 \rangle, \langle x_3, 0.8, 0.7, 0.8 \rangle\} \end{aligned}$$

Then the family $\tilde{\tau} = \{\tilde{0}, \tilde{1}, \tilde{A}, \tilde{B}, \tilde{C}, \tilde{D}\}$ of single-valued neutrosophic sets in X is SVNT on X .

Definition 2.15: The complement of \tilde{A} of SVNOS is called a single-valued neutrosophic closed set (SVNCS for short) in X .

Definition 2.16: Let $(X, \tilde{\tau})$ and $(Y, \tilde{\sigma})$ be SVNTSs. A map $f: X \rightarrow Y$ is said to be continuous if $f^{-1}(\tilde{B})$ is an SVNOS in X , for each SVNOS \tilde{B} in Y , or equivalently, $f^{-1}(\tilde{B})$ is an SVNCS in X , for each SVNCS \tilde{B} in Y .

Single-valued neutrosophic interior and closure operations in SVNTSs are defined as follows:

Definition 2.17 ([17]): Let $(X, \tilde{\tau})$ be an SVNTS and \tilde{A} be an SVN in X . Then the single-valued neutrosophic interior and closure of \tilde{A} are defined by

$$\begin{aligned} \text{int}(\tilde{A}) &= \bigcup \{ \tilde{G} : \tilde{G} \text{ is an SVNOS in } X \text{ and } \tilde{G} \subseteq \tilde{A} \} \\ \text{cl}(\tilde{A}) &= \bigcap \{ \tilde{F} : \tilde{F} \text{ is an SVNCS in } X \text{ and } \tilde{A} \subseteq \tilde{F} \}. \end{aligned}$$

It can be also shown that $\text{int}(\tilde{A})$ is an SVNOS and $\text{cl}(\tilde{A})$ is an SVNCS in X .

- (1) \tilde{A} is SVNOS if and only if $\tilde{A} = \text{int}(\tilde{A})$
- (2) \tilde{A} is SVNCS if and only if $\tilde{A} = \text{cl}(\tilde{A})$.

Proposition 2.18: For any SVN \tilde{A} in $(X, \tilde{\tau})$, we have

- (1) $\text{int}(C(\tilde{A})) = C(\text{cl}(\tilde{A}))$
- (2) $\text{cl}(C(\tilde{A})) = C(\text{int}(\tilde{A}))$.

Proof: Let \tilde{A} be an SVN in $(X, \tilde{\tau})$.

$$\begin{aligned} (1) \quad \text{int}(C(\tilde{A})) &= \bigcup \{ \tilde{G} : \tilde{G} \text{ is an SVNOS in } X \text{ and } \tilde{G} \subseteq C(\tilde{A}) \} \\ &= \bigcup \{ \tilde{G} : \tilde{G} \text{ is an SVNOS in } X \text{ and } \tilde{A} \subseteq C(\tilde{G}) \} \\ &= \bigcup \{ C(\tilde{F}) : C(\tilde{F}) \text{ is an SVNOS in } X \text{ and } \tilde{A} \subseteq \tilde{F} \} \\ &= \bigcup \{ C(\tilde{F}) : \tilde{F} \text{ is an SVNCS in } X \text{ and } \tilde{A} \subseteq \tilde{F} \} \\ &= C(\bigcap \{ \tilde{F} : \tilde{F} \text{ is an SVNCS in } X \text{ and } \tilde{A} \subseteq \tilde{F} \}) \\ &= C(\text{cl}(\tilde{A})). \end{aligned}$$

- (2) Similarly to (1). ■

Definition 2.19 ([16]): A proximity (Efremovich proximity) space is a pair (X, δ) , where X is a set and δ is a binary relation on the power set of X such that

- (P1) $A \delta B$ iff $B \delta A$;
- (P2) $A \delta (B \cup C)$ iff $A \delta B$ or $A \delta C$;
- (P3) $A \delta B$ implies $A, B \neq \emptyset$;
- (P4) $A \cap B \neq \emptyset$ implies $A \delta B$;
- (P5) $A \bar{\delta} B$ implies there is an $E \subseteq X$ such that $A \bar{\delta} E$ and $(X - E) \bar{\delta} B$,

where $A \bar{\delta} B$ means it is not true that $A \delta B$.

A function $f: (X, \delta) \rightarrow (Y, \delta')$ between two proximity spaces is called a *proximity mapping* (or a *p-map*) if and only if $f(A) \delta' f(B)$ whenever $A \delta B$. It can easily be shown that f is a *p-map* if and only if, for subsets C and D of Y , $f^{-1}(C) \bar{\delta} f^{-1}(D)$ whenever $C \bar{\delta}' D$.

3. Single-Valued Neutrosophic Proximity Spaces

In this section, we introduce the concept of neutrosophic proximity spaces as a generalisation of fuzzy proximity spaces [18] and intuitionistic fuzzy proximity spaces [19].

Definition 3.1: Let X be a nonempty set and $t, i, f \in [0, 1]$. The single-valued neutrosophic set $\tilde{x}_{t,i,f}$ is called a single-valued neutrosophic point (SVNP for short) in X given by

$$\tilde{x}_{t,i,f}(\tilde{x}_p) = \begin{cases} (t, i, f) & , \text{if } \tilde{x} = \tilde{x}_p \\ (0, 0, 1) & , \text{if } \tilde{x} \neq \tilde{x}_p \end{cases}$$

for $\tilde{x}_p \in X$ is called the support of $\tilde{x}_{t,i,f}$, where t denotes the degree of membership value, i denotes the degree of indeterminacy and f denotes the degree of non-membership value of $\tilde{x}_{t,i,f}$.

Theorem 3.2 ([17]): Let $(X, \tilde{\tau})$ be an SVNTS, $SVNS(X)$ denote the set of all single-valued neutrosophic sets in X and $cl: SVNS(X) \rightarrow SVNS(X)$ the SVN closure in $(X, \tilde{\tau})$. Then for any $\tilde{A}, \tilde{B} \in SVNS(X)$ the following properties hold:

- (1) $cl(\tilde{0}) = \tilde{0}$
- (2) $\tilde{A} \subseteq cl(\tilde{A})$
- (3) $cl(cl(\tilde{A})) = cl(\tilde{A})$
- (4) $cl(\tilde{A} \cup \tilde{B}) = cl(\tilde{A}) \cup cl(\tilde{B})$
- (5) If $\tilde{A} \subseteq \tilde{B}$, then $cl(\tilde{A}) \subseteq cl(\tilde{B})$.

Theorem 3.3 ([17]): Let $(X, \tilde{\tau})$ be an SVNTS and the single-valued neutrosophic operator $cl: SVNS(X) \rightarrow SVNS(X)$ satisfies the properties (1) – (4) in Theorem 3.2. Then there exists a single-valued neutrosophic topology $\tilde{\tau}_{cl}$ on X such that $cl_{\tilde{\tau}_{cl}} = cl$.

Definition 3.4: Let X be a nonempty set and $SVNS(X)$ denote the set of all single-valued neutrosophic sets in X . A single-valued neutrosophic proximity space (SVNPS for short) is a pair (X, δ) , where δ is a relation on $SVNS(X)$ such that

- (P1) $\tilde{A} \delta \tilde{B}$ iff $\tilde{B} \delta \tilde{A}$;
- (P2) $\tilde{A} \delta (\tilde{B} \cup \tilde{C})$ iff $\tilde{A} \delta \tilde{B}$ or $\tilde{A} \delta \tilde{C}$;
- (P3) $\tilde{A} \delta \tilde{B}$ implies $\tilde{A} \neq \tilde{0}$ and $\tilde{B} \neq \tilde{0}$;
- (P4) $\tilde{A} \cap \tilde{B} \neq \tilde{0}$ implies $\tilde{A} \delta \tilde{B}$;
- (P5) $\tilde{A} \bar{\delta} \tilde{B}$ implies there is an $\tilde{E} \in SVNS(X)$ such that $\tilde{A} \bar{\delta} \tilde{E}$ and $C(\tilde{E}) \bar{\delta} \tilde{B}$,

where $\tilde{A} \bar{\delta} \tilde{B}$ means it is not true that $\tilde{A} \delta \tilde{B}$.

Definition 3.5: A map $f: (X, \delta) \rightarrow (Y, \delta')$ between two single-valued neutrosophic proximity spaces is called a *single-valued neutrosophic proximity mapping* (or a \tilde{p} -map) if and only if $f(\tilde{A}) \delta' f(\tilde{B})$ whenever $\tilde{A} \delta \tilde{B}$.

It can easily be shown that f is a \tilde{p} -map if and only if, for each $\tilde{C}, \tilde{D} \in SVNS(Y)$, $f^{-1}(\tilde{C}) \delta f^{-1}(\tilde{D})$ whenever $\tilde{C} \delta' \tilde{D}$.

We have easily the following lemma, which follow directly from axioms $(\tilde{P}1)$, $(\tilde{P}2)$ and $(\tilde{P}4)$ of Definition 3.4.

Lemma 3.6: *Let (X, δ) be a SVNPS. Then the following properties hold:*

- (1) If $\tilde{A} \delta \tilde{B}, \tilde{A} \subseteq \tilde{C}$ and $\tilde{B} \subseteq \tilde{D}$, then $\tilde{C} \delta \tilde{D}$
- (2) $\tilde{A} \delta \tilde{A}$ and $\tilde{A} \delta \tilde{1}$ for each $\tilde{A} \neq \tilde{0}$

Theorem 3.7: *Let (X, δ) be an SVNPS and define a map $cl: SVNS(X) \rightarrow SVNS(X)$ by $cl(\tilde{A}) = \bigcap \{C(\tilde{B}) \in SVNS(X) \mid \tilde{A} \delta \tilde{B}\}$ for each $\tilde{A} \in SVNS(X)$. Then the following properties hold:*

- (1) $cl(\tilde{0}) = \tilde{0}$
- (2) $\tilde{A} \subseteq cl(\tilde{A})$
- (3) $cl(cl(\tilde{A})) = cl(\tilde{A})$
- (4) $cl(\tilde{A} \cup \tilde{B}) = cl(\tilde{A}) \cup cl(\tilde{B})$

Proof: (1) $cl(\tilde{0}) = \tilde{0}$

$$cl(\tilde{0}) = \bigcap \{C(\tilde{B}) \mid \tilde{B} \delta \tilde{0}\} = (0, 0, 1) = \tilde{0} \text{ since } \tilde{1} \delta \tilde{0}.$$

- (2) $\tilde{A} \subseteq cl(\tilde{A})$

Let $\tilde{A} = (T_{\tilde{A}}, I_{\tilde{A}}, F_{\tilde{A}}) \in SVNS(X)$. Take any $\tilde{B} = (T_{\tilde{B}}, I_{\tilde{B}}, F_{\tilde{B}}) \in SVNS(X)$ such that $\tilde{A} \delta \tilde{B}$. Then $\tilde{A} \cap \tilde{B} = \tilde{0} = (0, 0, 1)$ and hence $\min\{T_{\tilde{A}}, T_{\tilde{B}}\} = 0$, $\min\{I_{\tilde{A}}, I_{\tilde{B}}\} = 0$ and $\max\{F_{\tilde{A}}, F_{\tilde{B}}\} = 1$. So $T_{\tilde{A}} + T_{\tilde{B}} \leq 1$, $I_{\tilde{A}} + I_{\tilde{B}} \leq 1$ and $F_{\tilde{A}} + F_{\tilde{B}} \geq 1$. Thus $T_{\tilde{A}} \leq 1 - T_{\tilde{B}}$, $I_{\tilde{A}} \leq 1 - I_{\tilde{B}}$ and $F_{\tilde{A}} \geq 1 - F_{\tilde{B}}$. Hence $C(\tilde{B}) = (1 - T_{\tilde{B}}, 1 - I_{\tilde{B}}, 1 - F_{\tilde{B}}) \supseteq (T_{\tilde{A}}, I_{\tilde{A}}, F_{\tilde{A}}) = \tilde{A}$. Therefore $\tilde{A} \subseteq \bigcap \{C(\tilde{B}) \mid \tilde{A} \delta \tilde{B}\} = cl(\tilde{A})$.

- (3) $cl(cl(\tilde{A})) = cl(\tilde{A})$

It is sufficient to show that $cl(\tilde{A}) \delta \tilde{B}$ iff $\tilde{A} \delta \tilde{B}$ by the definition of closure.

If $\tilde{A} \delta \tilde{B}$, then $cl(\tilde{A}) \delta \tilde{B}$ obviously.

Conversely, suppose that $\tilde{A} \delta \tilde{B}$ and $cl(\tilde{A}) \delta \tilde{B}$. Then there exists an $\tilde{E} \in SVNS(X)$ such that $\tilde{B} \delta \tilde{E}$ and $C(\tilde{E}) \delta \tilde{A}$. Since $cl(\tilde{A}) \delta \tilde{B}$ and $\tilde{B} \delta \tilde{E}$, $cl(\tilde{A}) \not\subseteq \tilde{E}$ and $T_{cl(\tilde{A})} \not\leq T_{\tilde{E}}$ or $I_{cl(\tilde{A})} \not\leq I_{\tilde{E}}$ or $F_{cl(\tilde{A})} \not\geq F_{\tilde{E}}$. So there exists an $x \in X$ such that (i) $T_{cl(\tilde{A})}(x) > T_{\tilde{E}}(x)$ or (ii) $I_{cl(\tilde{A})}(x) > I_{\tilde{E}}(x)$ or (iii) $F_{cl(\tilde{A})}(x) < F_{\tilde{E}}(x)$.

- (i) If $T_{cl(\tilde{A})}(x) > T_{\tilde{E}}(x)$, we choose $a \in [0, 1]$ such that $T_{\tilde{E}}(x) < a < T_{cl(\tilde{A})}(x)$. Define $\tilde{K}: X \rightarrow [0, 1] \times [0, 1] \times [0, 1]$ by

$$\tilde{K}(x_p) = \begin{cases} (1 - a, 0, 1), & \text{if } x = x_p \\ (0, 0, 1), & \text{if } x \neq x_p \end{cases}$$

Then $\tilde{K} \in SVNS(X)$ and $\tilde{K} \subseteq C(\tilde{E})$ since $T_{\tilde{K}}(x) < T_{C(\tilde{E})}(x)$, $I_{\tilde{K}}(x) \leq I_{C(\tilde{E})}(x)$ and $F_{\tilde{K}}(x) \geq F_{C(\tilde{E})}(x)$. If $\tilde{K} \delta \tilde{A}$, then $cl(\tilde{A}) \subseteq C(\tilde{K})$ by the definition of closure and hence $T_{cl(\tilde{A})}(x) \leq T_{C(\tilde{K})}(x) = a < T_{cl(\tilde{A})}(x)$. This is a contradiction. Thus $\tilde{K} \not\delta \tilde{A}$. Since $\tilde{K} \subseteq C(\tilde{E})$, $\tilde{A} \delta C(\tilde{E})$. This is a contradiction to the fact that $C(\tilde{E}) \delta \tilde{A}$. Hence $T_{cl(\tilde{A})}(x) \leq T_{\tilde{E}}(x)$.

- (ii) If $I_{cl(\tilde{A})}(x) > I_{\tilde{E}}(x)$, we choose $b \in [0, 1]$ such that $I_{\tilde{E}}(x) < b < I_{cl(\tilde{A})}(x)$. Define $\tilde{L}: X \rightarrow [0, 1] \times [0, 1] \times [0, 1]$ by

$$\tilde{L}(x_p) = \begin{cases} (0, 1 - b, 1), & \text{if } x = x_p \\ (0, 0, 1), & \text{if } x \neq x_p. \end{cases}$$

Then $\tilde{L} \in SVNS(X)$ and $\tilde{L} \subseteq C(\tilde{E})$ since $T_{\tilde{L}}(x) \leq T_{C(\tilde{E})}(x)$, $I_{\tilde{L}}(x) < I_{C(\tilde{E})}(x)$ and $F_{\tilde{L}}(x) \geq F_{C(\tilde{E})}(x)$. If $\tilde{L} \delta \tilde{A}$, then $cl(\tilde{A}) \subseteq C(\tilde{L})$ by the definition of closure and hence $I_{cl(\tilde{A})}(x) \leq I_{C(\tilde{L})}(x) = b < I_{cl(\tilde{A})}(x)$. This is a contradiction. Thus $\tilde{L} \delta \tilde{A}$. Since $\tilde{L} \subseteq C(\tilde{E})$, $\tilde{A} \delta C(\tilde{E})$. This is a contradiction to the fact that $C(\tilde{E}) \delta \tilde{A}$. Hence $I_{cl(\tilde{A})}(x) \leq I_{\tilde{E}}(x)$.

- (iii) If $F_{cl(\tilde{A})}(x) < F_{\tilde{E}}(x)$, we choose $c \in [0, 1]$ such that $F_{cl(\tilde{A})}(x) < c < F_{\tilde{E}}(x)$. Define $\tilde{M}: X \rightarrow [0, 1] \times [0, 1] \times [0, 1]$ by

$$\tilde{M}(x_p) = \begin{cases} (0, 0, 1 - c), & \text{if } x = x_p \\ (0, 0, 1), & \text{if } x \neq x_p. \end{cases}$$

Then $\tilde{M} \in SVNS(X)$ and $\tilde{M} \subseteq C(\tilde{E})$ since $T_{\tilde{M}}(x) \leq T_{C(\tilde{E})}(x)$, $I_{\tilde{M}}(x) \leq I_{C(\tilde{E})}(x)$ and $F_{\tilde{M}}(x) > F_{C(\tilde{E})}(x)$. If $\tilde{M} \delta \tilde{A}$, then $cl(\tilde{A}) \subseteq C(\tilde{M})$ by the definition of closure and hence $F_{cl(\tilde{A})}(x) \geq F_{C(\tilde{M})}(x) = c > F_{cl(\tilde{A})}(x)$. This is a contradiction. Thus $\tilde{M} \delta \tilde{A}$. Since $\tilde{M} \subseteq C(\tilde{E})$, $\tilde{A} \delta C(\tilde{E})$.

This is a contradiction to the fact that $C(\tilde{E}) \delta \tilde{A}$. Hence $F_{cl(\tilde{A})}(x) \geq F_{\tilde{E}}(x)$.

(4) $cl(\tilde{A} \cup \tilde{B}) = cl(\tilde{A}) \cup cl(\tilde{B})$

Let $\tilde{A} = (T_{\tilde{A}}, I_{\tilde{A}}, F_{\tilde{A}})$, $\tilde{B} = (T_{\tilde{B}}, I_{\tilde{B}}, F_{\tilde{B}}) \in SVNS(X)$. Since $T_{\tilde{A}} \leq T_{\tilde{A}} \vee T_{\tilde{B}}$, $I_{\tilde{A}} \leq I_{\tilde{A}} \vee I_{\tilde{B}}$ and $F_{\tilde{A}} \leq F_{\tilde{A}} \vee F_{\tilde{B}}$, $\tilde{A} \subseteq \tilde{A} \cup \tilde{B}$. So, from Theorem 3.2 (5), we have $cl(\tilde{A}) \subseteq cl(\tilde{A} \cup \tilde{B})$. Similarly, $cl(\tilde{B}) \subseteq cl(\tilde{A} \cup \tilde{B})$, and hence $cl(\tilde{A} \cup \tilde{B}) \supseteq cl(\tilde{A}) \cup cl(\tilde{B})$.

On the other hand, suppose $cl(\tilde{A} \cup \tilde{B}) \not\subseteq cl(\tilde{A}) \cup cl(\tilde{B})$. Then there exists an $x \in X$ such that (i) $T_{cl(\tilde{A} \cup \tilde{B})}(x) > T_{cl(\tilde{A})}(x) \vee T_{cl(\tilde{B})}(x)$ or (ii) $I_{cl(\tilde{A} \cup \tilde{B})}(x) > I_{cl(\tilde{A})}(x) \vee I_{cl(\tilde{B})}(x)$ or (iii) $F_{cl(\tilde{A} \cup \tilde{B})}(x) < F_{cl(\tilde{A})}(x) \wedge F_{cl(\tilde{B})}(x)$.

- (i) Suppose $T_{cl(\tilde{A} \cup \tilde{B})}(x) > T_{cl(\tilde{A})}(x) \vee T_{cl(\tilde{B})}(x)$. We may assume $T_{cl(\tilde{A})}(x) \geq T_{cl(\tilde{B})}(x)$. Let $T_{cl(\tilde{A} \cup \tilde{B})}(x) = \alpha$. Then $T_{cl(\tilde{A})}(x) < \alpha$ and hence there exists an $\epsilon > 0$ such that $T_{cl(\tilde{A})}(x) < \alpha - \epsilon$. Since $T_{cl(\tilde{A})}(x) = \bigwedge \{1 - T_{\tilde{C}}(x) \mid \tilde{C} \delta \tilde{A}\}$, there exists a $\tilde{C} \in SVNS(X)$ such that $\tilde{C} \delta \tilde{A}$ and $1 - T_{\tilde{C}}(x) < \alpha - \epsilon$. Note that $1 - T_{\tilde{C}}(x) \geq T_{cl(\tilde{A})}(x) \geq T_{cl(\tilde{B})}(x) > T_{cl(\tilde{B})}(x) - \epsilon/2$, and hence $T_{cl(\tilde{B})}(x) < 1 - T_{\tilde{C}}(x) + \epsilon/2$. Since $T_{cl(\tilde{B})}(x) = \bigwedge \{1 - T_{\tilde{D}}(x) \mid \tilde{D} \delta \tilde{B}\}$, there exists a $\tilde{D} \in SVNS(X)$ such that $\tilde{D} \delta \tilde{B}$ and $1 - T_{\tilde{D}}(x) < 1 - T_{\tilde{C}}(x) + \epsilon/2$. Since $(\tilde{C} \cap \tilde{D}) \delta \tilde{A}$ and $(\tilde{C} \cap \tilde{D}) \delta \tilde{B}$, we have $(\tilde{C} \cap \tilde{D}) \delta (\tilde{A} \cup \tilde{B})$. So, from the definition of closure, we have $cl(\tilde{A} \cup \tilde{B}) \subseteq C(\tilde{C} \cap \tilde{D})$. Also $(1 - T_{\tilde{C}}(x)) \vee (1 - T_{\tilde{D}}(x)) < 1 - T_{\tilde{C}}(x) + \epsilon/2$. Hence, by Proposition 2.11(10), $\alpha = T_{cl(\tilde{A} \cup \tilde{B})}(x) \leq T_{C(\tilde{C} \cap \tilde{D})}(x) = T_{C(\tilde{C})}(x) \vee T_{C(\tilde{D})}(x) = (1 - T_{\tilde{C}}(x)) \vee (1 - T_{\tilde{D}}(x)) < 1 - T_{\tilde{C}}(x) + \epsilon/2 < \alpha - \epsilon + \epsilon/2 = \alpha - \epsilon/2$.

This is a contradiction.

- (ii) Suppose $I_{cl(\tilde{A} \cup \tilde{B})}(x) > I_{cl(\tilde{A})}(x) \vee I_{cl(\tilde{B})}(x)$.

We may assume $I_{cl(\tilde{A})}(x) \geq I_{cl(\tilde{B})}(x)$. Let $I_{cl(\tilde{A} \cup \tilde{B})}(x) = \beta$. Then $I_{cl(\tilde{A})}(x) < \beta$ and hence there exists an $\epsilon > 0$ such that $I_{cl(\tilde{A})}(x) < \beta - \epsilon$. Since $I_{cl(\tilde{A})}(x) = \bigwedge \{1 - I_{\tilde{C}}(x) \mid \tilde{C} \delta \tilde{A}\}$, there exists a $\tilde{C} \in SVNS(X)$ such that $\tilde{C} \delta \tilde{A}$ and $1 - I_{\tilde{C}}(x) < \beta - \epsilon$. Note that, $1 - I_{\tilde{C}}(x) \geq I_{cl(\tilde{A})}(x) \geq I_{cl(\tilde{B})}(x) > I_{cl(\tilde{B})}(x) - \epsilon/2$, and hence $I_{cl(\tilde{B})}(x) < 1 - I_{\tilde{C}}(x) + \epsilon/2$.

Since $I_{cl(\tilde{B})}(x) = \bigwedge \{1 - I_{\tilde{D}}(x) \mid \tilde{D} \bar{\delta} \tilde{B}\}$, there exists a $\tilde{D} \in SVNS(X)$ such that $\tilde{D} \bar{\delta} \tilde{B}$ and $1 - I_{\tilde{D}}(x) < 1 - I_{\tilde{C}}(x) + \epsilon/2$. Since $(\tilde{C} \cap \tilde{D}) \bar{\delta} \tilde{A}$ and $(\tilde{C} \cap \tilde{D}) \bar{\delta} \tilde{B}$, we have $(\tilde{C} \cap \tilde{D}) \bar{\delta} (\tilde{A} \cup \tilde{B})$. So, from the definition of closure, we have $cl(\tilde{A} \cup \tilde{B}) \subseteq C(\tilde{C} \cap \tilde{D})$. Also $(1 - I_{\tilde{C}}(x)) \vee (1 - I_{\tilde{D}}(x)) < 1 - I_{\tilde{C}}(x) + \epsilon/2$. Hence $\beta = I_{cl(\tilde{A} \cup \tilde{B})}(x) \leq I_{C(\tilde{C} \cap \tilde{D})}(x) = I_{C(\tilde{C})}(x) \vee I_{C(\tilde{D})}(x) = (1 - I_{\tilde{C}}(x)) \vee (1 - I_{\tilde{D}}(x)) < 1 - I_{\tilde{C}}(x) + \epsilon/2 < \beta - \epsilon + \epsilon/2 = \beta - \epsilon/2$. This is a contradiction.

(iii) Suppose $F_{cl(\tilde{A} \cup \tilde{B})}(x) < F_{cl(\tilde{A})}(x) \wedge F_{cl(\tilde{B})}(x)$.

We may assume $F_{cl(\tilde{A})}(x) \leq F_{cl(\tilde{B})}(x)$. Let $F_{cl(\tilde{A} \cup \tilde{B})}(x) = \gamma$. Then $F_{cl(\tilde{A})}(x) > \gamma$ and hence there exists an $\epsilon > 0$ such that $F_{cl(\tilde{A})}(x) > \gamma + \epsilon$. Since $F_{cl(\tilde{A})}(x) = \bigvee \{1 - F_{\tilde{C}}(x) \mid \tilde{C} \bar{\delta} \tilde{A}\}$, there exists a $\tilde{C} \in SVNS(X)$ such that $\tilde{C} \bar{\delta} \tilde{A}$ and $1 - F_{\tilde{C}}(x) > \gamma + \epsilon$. Note that, $1 - F_{\tilde{C}}(x) \leq F_{cl(\tilde{A})}(x) \leq F_{cl(\tilde{B})}(x) < F_{cl(\tilde{B})}(x) + \epsilon/2$, and hence $F_{cl(\tilde{B})}(x) > 1 - F_{\tilde{C}}(x) - \epsilon/2$. Since $F_{cl(\tilde{B})}(x) = \bigvee \{1 - F_{\tilde{D}}(x) \mid \tilde{D} \bar{\delta} \tilde{B}\}$, there exists a $\tilde{D} \in SVNS(X)$ such that $\tilde{D} \bar{\delta} \tilde{B}$ and $1 - F_{\tilde{D}}(x) > 1 - F_{\tilde{C}}(x) - \epsilon/2$. Since $(\tilde{C} \cap \tilde{D}) \bar{\delta} \tilde{A}$ and $(\tilde{C} \cap \tilde{D}) \bar{\delta} \tilde{B}$, we have $(\tilde{C} \cap \tilde{D}) \bar{\delta} (\tilde{A} \cup \tilde{B})$. So, from the definition of closure, we have $cl(\tilde{A} \cup \tilde{B}) \subseteq C(\tilde{C} \cap \tilde{D})$. Also $(1 - F_{\tilde{C}}(x)) \wedge (1 - F_{\tilde{D}}(x)) > 1 - F_{\tilde{C}}(x) - \epsilon/2$. Hence $\gamma = F_{cl(\tilde{A} \cup \tilde{B})}(x) \geq F_{C(\tilde{C} \cap \tilde{D})}(x) = F_{C(\tilde{C})}(x) \wedge F_{C(\tilde{D})}(x) = (1 - F_{\tilde{C}}(x)) \wedge (1 - F_{\tilde{D}}(x)) > 1 - F_{\tilde{C}}(x) - \epsilon/2 > \gamma + \epsilon - \epsilon/2 = \gamma + \epsilon/2$.

This is a contradiction. ■

Theorem 3.8: For an SVNPS (X, δ) , the family

$$\tilde{\tau}(\delta) = \{\tilde{A} \in SVNS(X) \mid cl(C(\tilde{A})) = C(\tilde{A})\}$$

is an SVN topology on X . This topology is called the SVN topology on X induced by the SVN proximity δ .

Proof: It follows from Theorems 3.3 and 3.7. ■

Theorem 3.9: Let (X, δ_1) and (Y, δ_2) be two single-valued neutrosophic proximity spaces. An SVN proximity mapping $f: (X, \delta_1) \rightarrow (Y, \delta_2)$ is continuous with respect to the SVN topologies $\tilde{\tau}(\delta_1)$ and $\tilde{\tau}(\delta_2)$.

Proof: Let $\tilde{A} \in \tilde{\tau}(\delta_2)$. Then $cl(C(\tilde{A})) = C(\tilde{A})$. We will show that $cl(C(f^{-1}(\tilde{A}))) = C(f^{-1}(\tilde{A}))$. Clearly $C(f^{-1}(\tilde{A})) \subseteq cl(C(f^{-1}(\tilde{A})))$.

Conversely, let $\tilde{B} \bar{\delta}_2 C(\tilde{A})$. Since f is a \tilde{p} -map, $f^{-1}(\tilde{B}) \bar{\delta}_1 f^{-1}(C(\tilde{A})) = C(f^{-1}(\tilde{A}))$. So

$$cl(C(f^{-1}(\tilde{A}))) = \bigcap \{C(\tilde{F}) \mid \tilde{F} \bar{\delta}_1 C(f^{-1}(\tilde{A}))\} \subseteq C(f^{-1}(\tilde{B})).$$

Hence for any $\tilde{B} \bar{\delta}_2 C(\tilde{A})$, $cl(C(f^{-1}(\tilde{A}))) \subseteq C(f^{-1}(\tilde{B}))$. Thus we have

$$\begin{aligned} cl(C(f^{-1}(\tilde{A}))) &\subseteq \bigcap \{C(f^{-1}(\tilde{B})) \mid \tilde{B} \bar{\delta}_2 C(\tilde{A})\} \\ &= \bigcap \{f^{-1}(C(\tilde{B})) \mid \tilde{B} \bar{\delta}_2 C(\tilde{A})\} \\ &= f^{-1}(\bigcap \{C(\tilde{B}) \mid \tilde{B} \bar{\delta}_2 C(\tilde{A})\}) \\ &= f^{-1}(cl(C(\tilde{A}))) = f^{-1}(C(\tilde{A})) \\ &= C(f^{-1}(\tilde{A})). \end{aligned}$$

So $cl(C(f^{-1}(\tilde{A}))) = C(f^{-1}(\tilde{A}))$. Hence $f^{-1}(\tilde{A})$ is open. Therefore, $f: (X, \tilde{\tau}(\delta_1)) \rightarrow (Y, \tilde{\tau}(\delta_2))$ is continuous. ■

4. Initial Structures and Products

We prove the existences of initial single-valued neutrosophic proximity space. Then we define the product of SVNPSs.

Theorem 4.1: Let X be a set, $\{(X_\alpha, \delta_\alpha) \mid \alpha \in \Lambda\}$ be a family of single-valued neutrosophic proximity spaces, and for each $\alpha \in \Lambda$, let $f_\alpha: X \rightarrow X_\alpha$ be a \tilde{p} -map. For any $\tilde{A}, \tilde{B} \in SVN(X)$, define

$\tilde{A} \delta \tilde{B}$ iff for every finite families $\{\tilde{A}_i \mid i = 1, \dots, n\}$ and $\{\tilde{B}_j \mid j = 1, \dots, m\}$ where $\tilde{A} = \bigcup_{i=1}^n \tilde{A}_i$ and $\tilde{B} = \bigcup_{j=1}^m \tilde{B}_j$ (i.e. finite covers of \tilde{A} and \tilde{B} respectively), there exist an \tilde{A}_i and a \tilde{B}_j such that $f_\alpha(\tilde{A}_i) \delta_\alpha f_\alpha(\tilde{B}_j)$ for each $\alpha \in \Lambda$.

Then δ is the coarsest (initial) proximity structure of single-valued neutrosophic spaces on X for which all mappings $f_\alpha: (X, \delta) \rightarrow (X_\alpha, \delta_\alpha)$ ($\alpha \in \Lambda$) are \tilde{p} -map.

Proof: We first prove that δ is an SVN proximity on X .

(P1) $\tilde{A} \delta \tilde{B}$ iff $\tilde{B} \delta \tilde{A}$

Since δ_α is an SVN proximity structure for each $\alpha \in \Lambda$, it is clear that $\tilde{A} \delta \tilde{B}$ iff $\tilde{B} \delta \tilde{A}$.

(P2) $\tilde{A} \delta (\tilde{B} \cup \tilde{C})$ iff $\tilde{A} \delta \tilde{B}$ or $\tilde{A} \delta \tilde{C}$

If $\tilde{A} \delta \tilde{B}$, then $\tilde{A} \delta \tilde{D}$ for each $\tilde{D} \supseteq \tilde{B}$. Because every cover of \tilde{D} is a cover of \tilde{B} . Therefore, $\tilde{A} \delta \tilde{B}$ or $\tilde{A} \delta \tilde{C}$ implies $\tilde{A} \delta (\tilde{B} \cup \tilde{C})$.

Conversely, suppose $\tilde{A} \delta (\tilde{B} \cup \tilde{C})$. Then, there exist finite covers $\{\tilde{A}_i \mid i = 1, \dots, n\}$ and $\{\tilde{B}_j \mid j = 1, \dots, m\}$ of \tilde{A} and \tilde{B} respectively such that $f_\alpha(\tilde{A}_i) \delta_\alpha f_\alpha(\tilde{B}_j)$ for some $\alpha = s_{ij} \in \Lambda$, where $i = 1, \dots, n$ and $j = 1, \dots, m$. Likewise, there are finite covers $\{\tilde{A}'_k \mid k = 1, \dots, p\}$ and $\{\tilde{B}_j \mid j = m+1, \dots, m+q\}$ of \tilde{A} and \tilde{C} respectively such that $f_\alpha(\tilde{A}'_k) \delta_\alpha f_\alpha(\tilde{B}_j)$ for some $\alpha = t_{kj} \in \Lambda$, where $k = 1, \dots, p$ and $j = m+1, \dots, m+q$. Then, $\{\tilde{A}_i \cup \tilde{A}'_k \mid i = 1, \dots, n; k = 1, \dots, p\}$ and $\{\tilde{B}_j \mid j = 1, \dots, m+q\}$ are finite covers of \tilde{A} and $\tilde{B} \cup \tilde{C}$, respectively. Hence, from the fact that $f_\alpha(\tilde{A}_i \cup \tilde{A}'_k) \delta_\alpha f_\alpha(\tilde{B}_j)$ for $\alpha = s_{ij}$ or $\alpha = t_{kj}$, we conclude that $\tilde{A} \delta (\tilde{B} \cup \tilde{C})$.

(P3) $\tilde{A} \delta \tilde{B}$ implies $\tilde{A} \neq \tilde{0}$ and $\tilde{B} \neq \tilde{0}$

It is obvious.

(P4) $\tilde{A} \cap \tilde{B} \neq \tilde{0}$ implies $\tilde{A} \delta \tilde{B}$

We will show that if $\tilde{A} \cap \tilde{B} \neq \tilde{0}$, then $\tilde{A} \delta \tilde{B}$. Suppose $\tilde{A} \cap \tilde{B} \neq \tilde{0}$. Then, there exist finite covers $\{\tilde{A}_i \mid i = 1, \dots, n\}$ and $\{\tilde{B}_j \mid j = 1, \dots, m\}$ of \tilde{A} and \tilde{B} respectively such that $f_\alpha(\tilde{A}_i) \delta_\alpha f_\alpha(\tilde{B}_j)$ for some $\alpha = s_{ij} \in \Lambda$, where $i = 1, \dots, n$ and $j = 1, \dots, m$. Since for each $\alpha \in \Lambda$, δ_α is an SVN proximity structure on X_α , $f_\alpha(\tilde{A}_i) \cap f_\alpha(\tilde{B}_j) = \tilde{0}$. From this, it follows that

$$f_\alpha\left(\bigcup_{i=1}^n \tilde{A}_i\right) \cap f_\alpha\left(\bigcup_{j=1}^m \tilde{B}_j\right) = f_\alpha(\tilde{A}) \cap f_\alpha(\tilde{B}) = \tilde{0}.$$

So we have $\tilde{A} \cap \tilde{B} = \tilde{0}$.

(P5) $\tilde{A} \delta \tilde{B}$ implies there is an $\tilde{E} \in SVN(X)$ such that $\tilde{A} \delta \tilde{E}$ and $C(\tilde{E}) \delta \tilde{B}$.

If $\tilde{A} \tilde{\delta} \tilde{B}$, then there exist finite covers $\{\tilde{A}_i \mid i = 1, \dots, n\}$ and $\{\tilde{B}_j \mid j = 1, \dots, m\}$ of \tilde{A} and \tilde{B} respectively such that $f_\alpha(\tilde{A}_i) \tilde{\delta}_\alpha f_\alpha(\tilde{B}_j)$ for some $\alpha = s_{ij} \in \Lambda$, where $i = 1, \dots, n$ and $j = 1, \dots, m$. Since each $(X_\alpha, \delta_\alpha)$ is a single-valued neutrosophic proximity space, there exist \tilde{E}_{ij} such that $f_\alpha(\tilde{A}_i) \tilde{\delta}_\alpha \tilde{E}_{ij}$ and $C(\tilde{E}_{ij}) \tilde{\delta}_\alpha f_\alpha(\tilde{B}_j)$. Set $\tilde{E}_j = \bigcap_{i=1}^n f_\alpha^{-1}(\tilde{E}_{ij})$ and $\tilde{E} = \bigcup_{j=1}^m \tilde{E}_j$, i.e. $\tilde{E} = \bigcup_{j=1}^m \bigcap_{i=1}^n f_\alpha^{-1}(\tilde{E}_{ij})$. It follows that $f_\alpha(\tilde{E}_j) = f_\alpha(\bigcap_{i=1}^n f_\alpha^{-1}(\tilde{E}_{ij})) \subseteq \bigcap_{i=1}^n f_\alpha(f_\alpha^{-1}(\tilde{E}_{ij})) \subseteq \bigcap_{i=1}^n \tilde{E}_{ij} \subseteq \tilde{E}_{ij}$. Since $f_\alpha(\tilde{E}_j) \subseteq \tilde{E}_{ij}$, we have $f_\alpha(\tilde{A}_i) \tilde{\delta}_\alpha f_\alpha(\tilde{E}_j)$ for $\alpha = s_{ij} \in \Lambda$; that is, $\tilde{A} \tilde{\delta} \tilde{E}$.

Let $\tilde{D}_{ij} = C(f_\alpha^{-1}(\tilde{E}_{ij})) = f_\alpha^{-1}(C(\tilde{E}_{ij}))$ and $\tilde{F}_j = C(\tilde{E}_j) = \bigcup_{i=1}^n \tilde{D}_{ij}$. Then $C(\tilde{E}) = \bigcap_{j=1}^m \tilde{F}_j$, i.e. $C(\tilde{E}) = \bigcap_{j=1}^m \bigcup_{i=1}^n C(f_\alpha^{-1}(\tilde{E}_{ij}))$. Since $C(\tilde{E}_{ij}) \tilde{\delta}_\alpha f_\alpha(\tilde{B}_j)$ and $\tilde{D}_{ij} = f_\alpha^{-1}(C(\tilde{E}_{ij}))$, we have $f_\alpha^{-1}(C(\tilde{E}_{ij})) \tilde{\delta} f_\alpha^{-1}(f_\alpha(\tilde{B}_j))$, i.e. by $\tilde{P}2$ of Definition 3.4, $\tilde{D}_{ij} \tilde{\delta} \tilde{B}_j$ for all i and j . This implies $\tilde{F}_j \tilde{\delta} \tilde{B}_j$ for all j . Hence $C(\tilde{E}) \tilde{\delta} \tilde{B}_j$ for all j , showing that $C(\tilde{E}) \tilde{\delta} \tilde{B}$.

It is clear that all mappings $f_\alpha: (X, \delta) \rightarrow (X_\alpha, \delta_\alpha)$ ($\alpha \in \Lambda$) are \tilde{p} -map. Let δ' be another SVN proximity on X with respect to which each f_α is a \tilde{p} -map. We shall show that δ' is finer than δ , which will complete the proof. Suppose $\tilde{A} \delta' \tilde{B}$ and consider any covers $\{\tilde{A}_i \mid i = 1, \dots, n\}$ and $\{\tilde{B}_j \mid j = 1, \dots, m\}$ of \tilde{A} and \tilde{B} respectively. Since $(\tilde{A}_1 \cup \dots \cup \tilde{A}_n) \delta' \tilde{B}$, by $\tilde{P}2$ of Definition 3.4, there is an $i \in \{1, \dots, n\}$ such that $\tilde{A}_i \delta' \tilde{B}$. Similarly, $\tilde{A}_i \delta' (\tilde{B}_1 \cup \dots \cup \tilde{B}_m)$, by $\tilde{P}2$ of Definition 3.4, there is an $j \in \{1, \dots, m\}$ such that $\tilde{A}_i \delta' \tilde{B}_j$. Since each f_α is a \tilde{p} -map with respect to δ' , it follows that $f_\alpha(\tilde{A}_i) \delta'_\alpha f_\alpha(\tilde{B}_j)$ for each $\alpha \in \Lambda$. Hence, we get $\tilde{A} \tilde{\delta} \tilde{B}$, i.e. δ' is finer than δ . ■

Definition 4.2: Let $\{(X_\alpha, \delta_\alpha) \mid \alpha \in \Lambda\}$ be a family of single-valued neutrosophic proximity spaces, and $X = \prod_{\alpha \in \Lambda} X_\alpha$. The product SVN proximity on X is defined to be the initial proximity structure $\delta = \prod_{\alpha \in \Lambda} \delta_\alpha$ on X with respect to which each projection map $P_\alpha: (X, \delta) \rightarrow (X_\alpha, \delta_\alpha)$ ($\alpha \in \Lambda$) is a \tilde{p} -map. In that case (X, δ) is said to be the product SVNPS.

Corollary 4.3: A mapping f from an SVNPS (Y, δ^*) to $(X = \prod_{\alpha \in \Lambda} X_\alpha, \delta)$, i.e. $f: (Y, \delta^*) \rightarrow (X, \delta)$, is a \tilde{p} -map if and only if the composition $f_\alpha \circ f: (Y, \delta^*) \rightarrow (X_\alpha, \delta_\alpha)$ is a \tilde{p} -map for every $\alpha \in \Lambda$.

Proof: Let (Y, δ^*) be an SVNPS and $f: (Y, \delta^*) \rightarrow (X, \delta)$. It can easily be shown that if f is a \tilde{p} -map, then for each $\alpha \in \Lambda$, $f_\alpha \circ f$ is a \tilde{p} -map.

Conversely, suppose that $f_\alpha \circ f$ is a \tilde{p} -map for each $\alpha \in \Lambda$. We will show that f is a \tilde{p} -map. Let $\tilde{A}, \tilde{B} \subseteq X$, $\tilde{A} \delta^* \tilde{B}$ and $\{\tilde{A}_i \mid i = 1, \dots, n\}$ and $\{\tilde{B}_j \mid j = 1, \dots, m\}$ be finite covers of $f(\tilde{A})$ and $f(\tilde{B})$ respectively. Then $\tilde{A} = \bigcup_{i=1}^n \tilde{A}_i$, $\tilde{B} = \bigcup_{j=1}^m \tilde{B}_j$ and we have $\tilde{A} \subseteq \bigcup_{i=1}^n f^{-1}(\tilde{A}_i)$, $\tilde{B} \subseteq \bigcup_{j=1}^m f^{-1}(\tilde{B}_j)$. Since $\tilde{A} \delta^* \tilde{B}$, we obtain $\bigcup_{i=1}^n f^{-1}(\tilde{A}_i) \delta^* \bigcup_{j=1}^m f^{-1}(\tilde{B}_j)$ and by $\tilde{P}2$ of Definition 3.4, there exist i, j such that $f^{-1}(\tilde{A}_i) \delta^* f^{-1}(\tilde{B}_j)$. Since $f_\alpha \circ f \circ f^{-1}(\tilde{A}_i) \subseteq f_\alpha(\tilde{A}_i)$, $f_\alpha \circ f \circ f^{-1}(\tilde{B}_j) \subseteq f_\alpha(\tilde{B}_j)$ and $f_\alpha \circ f$ is a \tilde{p} -map for each $\alpha \in \Lambda$, it follows that $f_\alpha(\tilde{A}_i) \delta'_\alpha f_\alpha(\tilde{B}_j)$ for each $\alpha \in \Lambda$. This proves that $f(\tilde{A}) \delta f(\tilde{B})$ so that f is a \tilde{p} -map. ■

5. Conclusion

Proximity and uniformity are important concepts close to topology and they have rich topological properties. For this reason, in recent years, these notions constitute a significant research area in the field of topological spaces. Also, these concepts have been studied by many authors on the fuzzy, soft and neutrosophic sets. In this paper, we introduced the single-valued neutrosophic proximity spaces and presented some of their properties. Then,

we showed that each single-valued neutrosophic proximity determines a single-valued neutrosophic topology. Also, we introduced the initial single-valued neutrosophic proximity structure and hence we defined the products. We concluded that all the results of classical proximity spaces are still valid on the single-valued neutrosophic proximity spaces. We believe that these theoretical results will help the researchers to solve practical applications in various areas, to advance and promote other generalisations and the further studies on SVNPSs.

In future studies, the single-valued uniform spaces can be introduced and the relationships among the notions of single-valued uniform, proximity and topological spaces can be investigated. Also, various topological notions such as separation, closedness, connectedness and compactness may be characterised in the single-valued topological spaces. Furthermore, in [20–22], using new types of partial belong and total non-belong relations on soft separation axioms and decision-making problem were investigated. In a similar way, these can be used on the domain of single-valued neutrosophic topological spaces and proximity spaces.

Acknowledgments

I would like to thank the referees for their valuable and helpful suggestions that improved the paper.

Disclosure statement

No potential conflict of interest was reported by the author(s).

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