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On the powers of fuzzy neutrosophic soft matrices

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Abstract. In this paper, The powers of fuzzy neutrosophic soft square matrices (FNSSMs) under the operations $\oplus (= max)$ and $\otimes (= min)$ are studied. We show that the powers of a given FNSM stabilize if and only if its orbits stabilize for each starting fuzzy neutrosophic soft vector (FNSV) and prove a necessary and sufficient condition for this property using the associated graphs of the FNSM. Applications of the obtained results to several spacial classes of FNSMs (including circulants) are given.

Keywords: Fuzzy neutrosophic soft set (FNSS), fuzzy neutrosophic soft matrix (FNSM), fuzzy neutrosophic soft vector (FNSV), fuzzy neutrosophic soft eigenvectors (FNSEvs), circulant fuzzy neutrosophic soft matrix (CFNSM).

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1. Introduction

In 1965, Zadeh [24] used membership degree $\mu_A(x) \in [0, 1]$ to find the belongingness of an element to a set. When $\mu_A(x)$ itself becomes uncertain, then it is hard to define by a crisp value for it. This was solved by using interval-valued fuzzy sets (IVFSs) by Turksen [22]. In some real life applications, one has to consider not only the truth membership supported by the evidence but also the falsity membership against the evidence, which is

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beyond the scope of fuzzy sets and IVFSs. Atanassove [2] introduced intuitionistic fuzzy set (IFS) as a generalization of fuzzy sets to consider the both truth membership and falsity membership. Later IFS was extended to the interval-valued intuitionistic fuzzy stes (IVIFSs) by Atanassov [3] for generalization purpose. A bibliometric analysis on fuzzy decision-related research and a scientometric review on IFS studies can be respectively found in Liu and Liao [13] and Yu and Liao. Due to some restriction on truth and falsity membership values, fuzzy set and its extensions can only handle uncertain information but not the indeterminate and inconsistent information, which may exist in reality. For example where 10 voters are participating in a voting process. In time t_1, three vote "*yes*", two vote "*no*" and five are undecided. In neutrosophic notation, it is expressed as $(0.3\ \ 0.5\ \ 0.2)$. In time t_2, three vote "yes", two vote "no", two give up and three are undecided, then it can be expressed as $(0.3\ \ 0.3\ \ 0.2)$, which is beyond the scope of intuitionistic fuzzy set. This type of situations is well managed by the neutrosophic set (NS), where indeterminancy is quantified explicity an truth, indeterminancy, and falsity membership are independent to each other NS provides a more reasonable mathematical framework to deal with indeterminate and inconsistent information. During the last decade, the concept of NS and inteval neutrosophic set (INS) have been used in various application such as medical diagnosis, database, topology, image processing (see, Guo and Sengur [9]) and decision making problems (see, Broumi and Smarandache [4]).

Smarandache [19] introduced neutrosophy as a branch of philosophy which studies the origin, nature, and scope of neutralities. Smarandache defined indeterminacy explicitly and stated that truth, indeterminacy and falsity membership are independent and lies within $]^{-}0, 1^{+}[$, which is the non-standard unit interval and an extension of the standard interval $[0, 1]$. Maji [16] introduced neutrosophic soft set (NSS) as a combination of NS and soft set defined by Molodtsov [17].

An algebraic structure $(\mathcal{N}, \oplus, \otimes)$ where $\mathcal{N}$ is a bounded linearly ordered set with the upper bound denoted by $\mathbf{1}$ and the lower bound by $\mathbf{0}$, $\oplus = max$, and $\otimes = min$ is called by some authors a bottleneck algebra (Cechlarova [5]) and by others a fuzzy algebra [11, 12]. The important property of the operations $\oplus$ and $\otimes$ is that they are both idempotent. moreover, for every $a, b \in \mathcal{N}$, $a \oplus b \in \{a, b\}$ and $a \otimes b \in \{a, b\}$.

The set of all n-tuples (n-vectors) over $\mathcal{N}$ will be denoted by $\mathcal{N}_n$ and the set of all square matrices of order n by $\mathcal{N}(n, n)$. The operations $\oplus$ and $\otimes$ are extended to vectors (fuzzy vector, intuitionistic fuzzy vector) and matrices (fuzzy matrices, intuitionistic fuzzy matrices) in the usual way, i.e., for $A, B \in \mathcal{N}_{(n,n)}$ we have $A \otimes B = C \in \mathcal{N}_{(n,n)}$, where

$$C_{ij} = \sum_{k=1}^{n} a_{ik} \otimes b_{kj}.$$

If we create the sequence of powers A, $A^2 = A \otimes A$, $A^3 = A^2 \otimes A, ...$ of a given matrix A, then no entry different from the entries in the original A can be obtained. Hence although the sequence of powers is infinite, it contains only a finite number of different members. That means that at some point a repetition must occur, resulting in some periodic behavior or stabilization.

This phenomenon has already been observed in [21], where a condition for stabilization of compact matrices has been derived. In [11, 12], it was shown that the period of a square matrix divides $[n]$, the least common multiple of the integers $1, 2, ..., n$ and an algorithm for computing this period was given. Some other works concentrate on periodicity and stabilization of powers of special matrices; one of the most recent papers is [10], where a sufficient condition for stabilization of powers of circulant matrices is proved.

All the above papers use purely algebraic methods. However, for computations in extremal algebraic structure like the bottleneck /fuzzy algebra, the language of graph theory has been used since, say, the 1960s. It has proved to be useful in both directions: matrix operations can be used for designing some graph-theoretical algorithms, and graphs shed light on some questions formulated for matrices. From an extensive literature on this topic let us mention [5, 7, 25].

Using the idea of intuitionistic fuzzy sets Im and Lee have defined the concept of intuitionistic fuzzy matrix as a natural generalization of fuzzy matrices and they introduced the determinant of square intuitionistic fuzzy matrices. Rajarajeswari and Dhanalakshimi have [18] recently introduced intuitionistic fuzzy soft matrices (IFSMs) that has been an effective tool in the application of medical diagnosis. Arockiarani and Sumathi have [20] recently investigated some new operation on FNSMs. Uma et al [23] introduced FNSMs of Type-I and Type-II. Also, for more details about fuzzy neutrosophic soft matrix, see [8, 14, 15]

The aim of the present paper is to introduce digraphs into the study of powers of FNSMs over bottleneck/ fuzzy algebra and use them to derive a necessary and sufficient condition for stabilization of power sequence. In the final section we illustrate this theory by its application to some special classes of FNSMs.

2. Preliminaries

In this section some basic definitions of neutrosophic set (NS), fuzzy neutrosophic soft set (FNSS), fuzzy neutrosophic soft matrix (FNSM), fuzzy neutrosophic soft matrix of type-I, and a review of graph-theoretical notions are given.

Definition 2.1 [19] A neutrosophic set A on the universe of discourse X is defined as $A = \{\langle x, T_A(x), I_A(x), F_A(x)\rangle, x \in X\}$, where $T, I, F : X \to\ ]^-0, 1^+[$ and

$$^-0 \leqslant T_A(x) + I_A(x) + F_A(x) \leqslant 3^+. \tag{1}$$

From philosophical point of view the neutrosophic set takes the value from real standard or non-standard subsets of $]^-0, 1^+[$. But in real life application especially in scientific and engineering problems it is difficult to use neutrosophic set with value from real standard or non-standard subset of $]^-0, 1^+[$. Hence we consider the neutrosophic set which takes the value from the subset of $[0, 1]$. Therefore we can rewrite equation (1) as $0 \leqslant T_A(x) + I_A(x) + F_A(x) \leqslant 3$. In short an element $\widetilde{a}$ in the neutrosophic set A, can be written as $\widetilde{a} = \langle a^T, a^I, a^F\rangle$, where a^T denotes degree of truth, a^I denotes degree of indeterminacy, a^F denotes degree of falsity such that $0 \leqslant a^T + a^I + a^F \leqslant 3$.

Definition 2.2 [17] Let U be the initial universe set and E be a set of parameter. Consider a non-empty set $A, A \subset E$. Let $P(U)$ denotes the set of all fuzzy neutrosophic sets of U. The collection (F, A) is termed to the fuzzy neutrosophic soft set (FNSS) over U, where F is a mapping given by $F : A \to P(U)$. Here after we simply consider A as FNSS over U instead of (F, A).

Definition 2.3 [1] Let $U = \{c_1, c_2, ..., c_m\}$ be the universal set and E be the set of parameters given by $E = \{e_1, e_2, ..., e_m\}$. Let $A \subset E$. A pair (F, A) be a $FNSS$ over U. Then the subset of $U \times E$ is defined by $R_A = \{(u, e); e \in A, u \in F_A(e)\}$, which is called a relation form of (F_A, E). The membership function, indeterminacy membership function and non membership function are written by $T_{R_A} : U \times E \to [0,1]$, $I_{R_A} : U \times E \to [0,1]$ and $F_{R_A} : U \times E \to [0,1]$, where $T_{R_A}(u,e) \in [0,1], I_{R_A}(u,e) \in [0,1]$ and $F_{R_A}(u,e) \in [0,1]$ are the membership value, indeterminacy value and non membership value respectively of $u \in U$ for each $e \in E$. If $[(T_{ij},\ I_{ij},\ F_{ij})] = [T_{ij}(u_i,e_j),\ I_{ij}(u_i,e_j)\ ,F_{ij}(u_i,e_j)]$ we define a matrix

$$[\langle T_{ij}\ ,I_{ij},\ F_{ij}\rangle]_{m\times n} = \begin{bmatrix} \langle T_{11},\ I_{11},\ F_{11}\rangle & \cdots & \langle T_{1n},\ I_{1n},\ F_{1n}\rangle \\ \langle T_{21},\ I_{21},\ F_{21}\rangle & \cdots & \langle T_{2n},\ I_{2n},\ F_{2n}\rangle \\ \vdots & \vdots & \vdots \\ \langle T_{m1},\ I_{m1},\ F_{m1}\rangle & \cdots & \langle T_{mn},\ I_{mn},\ F_{mn}\rangle \end{bmatrix}$$

which is called an $m \times n$ FNSM of the FNSS (F_A, E) over U.

FNSMs of Type-I

Definition 2.4 [23] Let $A = (\langle a^T_{ij},\ a^I_{ij},\ a^F_{ij}\rangle),\ B = \langle (b^T_{ij},\ b^I_{ij},\ b^F_{ij}\rangle) \in \mathcal{N}_{m\times n}$. The component wise addition and component wise multiplication defined as
$A \oplus B = (sup\{a^T_{ij},\ b^T_{ij}\},\quad sup\{a^I_{ij},\ b^I_{ij}\},\quad inf\{a^F_{ij},\ b^F_{ij}\})$
$A \otimes B = (inf\{a^T_{ij},\ b^T_{ij}\},\quad inf\{a^I_{ij},\ b^I_{ij}\},\quad sup\{a^F_{ij},\ b^F_{ij}\})$

Definition 2.5 [23] Let $A \in \mathcal{N}_{m\times n},\ B \in \mathcal{N}_{n\times p}$, the composition of A and B is defined as

$$A \circ B = \left(\sum_{k=1}^{n}(a^T_{ik} \wedge\ b^T_{kj}),\quad \sum_{k=1}^{n}(a^I_{ik} \wedge\ b^I_{kj}),\quad \prod_{k=1}^{n}(a^F_{ik} \vee\ b^F_{kj})\right)$$

equivalently we can write the same as

$$= \left(\bigvee_{k=1}^{n}(a^T_{ik} \wedge b^T_{kj}),\quad \bigvee_{k=1}^{n}(a^I_{ik} \wedge b^I_{kj}),\quad \bigwedge_{k=1}^{n}(a^F_{ik} \vee b^F_{kj})\right).$$

The product $A \circ B$ is defined if and only if the number of columns of A is same as the number of rows of B. Then A and B are said to be conformable for multiplication. We shall use AB instead of $A \circ B$, Where $\sum(a^T_{ik} \wedge\ b^T_{kj})$ means max-min operation and $\prod_{k=1}^{n}(a^F_{ik} \vee\ b^F_{kj})$ means min-max operation.

A review of graph-theoretical notions [6]
A digraph is a pair $\mathcal{G} = (V, H)$ where V is a finite set, called the node set, and H is a subset of $V \times V$, called the arc set. $\mathcal{G}' = (V', H')$ is a subgraph of $\mathcal{G}$ if $V' \subseteq V$ and $H' \subseteq H$. If each arc (i,j) is assigned a weight $c(i,j)$ (sometimes called its capacity), then the digraph is called a weighted digraph. A sequence of nodes

$$P = (i_0, i_1, ..., i_m) \tag{2}$$

is called a path, if for all $j = 1, ..., m$ the pair $(i_{j-1}, i_j) \in H$; for brevity we shall also say that the arcs (i_{j-i}, i_j) are on the path P. If all the nodes on a path are different, the path is called elementary. If $i_0 = i_m$, then the path is a cycle; if all the nodes (except of the first and last one) on a cycle are different, then the cycle is elementary. The length of a path or a cycle is equal to the number of arcs on it and denoted by $l(P)$. Now, for the present theory the follwing obvious fact is crucial.

Lemma 2.6 [6] Every path of length at least n in a digraph on n nodes contains a cycle.

A digraph that does not contain any cycle is called acyclic. If for each pair of nodes u, v in $\mathcal{G}$ there is a path from u to v and a path from v to u in $\mathcal{G}$, then $\mathcal{G}$ is called strongly connected. A maximal strongly connected subgraph of a given digraph is its strongly connected component (SCC for short). Every SCC can contain either several nodes (in that case it must contain at least one cycle) or a single node u; and in the latter case we shall call it an acyclic SCC if the loop (u, u) is not its arc.

The greatest common divisor (gcd for short) of all cycle lengths in a non-acyclic digraph $\mathcal{G}$ is called the period of $\mathcal{G}$. Note that in digraphs it makes no difference whether we consider all cycles or only the elementary ones.

Finally, if $\mathcal{G}$ is a weighted digraph and P a path in $\mathcal{G}$ of the form (2), then the number $c(P) = c(i_0, i_1) \otimes c(i_1, i_2) \otimes ... \otimes c(i_{m-1}, i_m)$ is called the capacity of the path P.

3. Two types of digraphs

In this section we discuses about two types of digraphs namely associated digraph and associated throshold digraph. Then the connection between the entries in powers of A. Suppose that a square FNSM A of order n is given. We shall define two kinds of digraphs connected with A. First, the associated digraph of A, denoted by $\mathcal{G}(A)$, is the complete weighted digraph on the node set N with each arc (i, j) assigned the capacity $\langle a_{ij}^T \ a_{ij}^I \ a_{ij}^F \rangle$. Conversely, for a given weighted digraph $\mathcal{G}$ the corresponding FNSM can be created in the usal way.

***Example* 3.1** As an illustration, $\mathcal{G}(A)$ for the FNSM

$$A = \begin{bmatrix} \langle 0.1 \ 0.2 \ 0.5 \rangle & \langle 0.8 \ 0.7 \ 0.1 \rangle & \langle 0 \quad 0 \quad 1 \rangle & \langle 0.7 \ 0.6 \ 0.1 \rangle \\ \langle 0.5 \ 0.4 \ 0.2 \rangle & \langle 0.1 \ 0.2 \ 0.5 \rangle & \langle 0.2 \ 0.3 \ 0.4 \rangle & \langle 0.5 \ 0.4 \ 0.2 \rangle \\ \langle 0 \quad 0 \quad 1 \rangle & \langle 0.4 \ 0.3 \ 0.2 \rangle & \langle 0.1 \ 0.2 \ 0.5 \rangle & \langle 0.2 \ 0.3 \ 0.4 \rangle \\ \langle 0.7 \ 0.6 \ 0.1 \rangle & \langle 0 \quad 0 \quad 1 \rangle & \langle 0.4 \ 0.3 \ 0.2 \rangle & \langle 0.5 \ 0.4 \ 0.2 \rangle \end{bmatrix}$$

is given in Fig-1.

If a value $\langle h^T \ h^I \ h^F \rangle$ is given together with the FNSM $A \in \mathcal{N}_{(n,n)}$, then the associated threshold digraph $\mathcal{G}(A, \langle h^T \ h^I \ h^F \rangle) = (V, H(\langle h^T \ h^I \ h^F \rangle))$ is defined by $V = N$ and $(i, j) \in H(\langle h^T \ h^I \ h^F \rangle)$ if and only if $a_{ij}^T \geqslant h^T; \ \ a_{ij}^I \geqslant h^I; \ \ a_{ij}^F \leqslant h^F$. It can be easily seen that $\mathcal{G}(A, \langle h^T \ h^I \ h^F \rangle)$ is a subgraph of $\mathcal{G}(A, \langle h^T \ h^I \ h^F \rangle')$ for $\langle h^T \ h^I \ h^F \rangle' \leqslant \langle h^T \ h^I \ h^F \rangle$. Because as the value of the threshold decreases, some new arcs can be added, but none will disappear. So, $G(A, \langle h^T \ h^I \ h^F \rangle)$ consists of n isolated nodes for $\langle h^T \ h^I \ h^F \rangle$ greater that the maximum entry in A, and as soon as $\langle h^T \ h^I \ h^F \rangle$ is less than or equal to the minimum entry of A, $\mathcal{G}(A, \langle h^T \ h^I \ h^F \rangle)$ is a complete digraph with loops. We shall imagine decreasing the threshold later too, so when we say " the first nontrivial threshold digraph" we mean the threshold digraph $\mathcal{G}(A, \langle h^T \ h^I \ h^F \rangle)$ for $\langle h^T \ h^I \ h^F \rangle$ equal to the maximum entry in A. In Fig-2 (a)-(c), the threshold digraphs $\mathcal{G}(A, \langle 0.8 \ 0.7 \ 0.1 \rangle), \mathcal{G}(A, \langle 0.7 \ 0.6 \ 0.1 \rangle)$, and $\mathcal{G}(A, \langle 0.5 \ 0.4 \ 0.2 \rangle)$ for the above FNSM A are given.

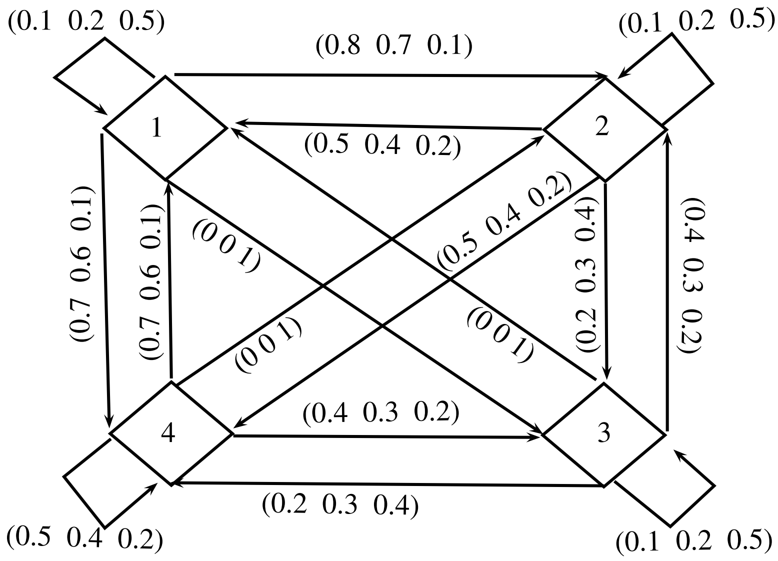
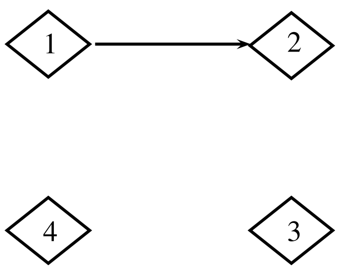
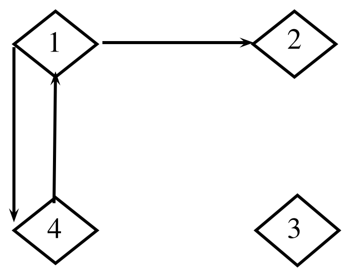


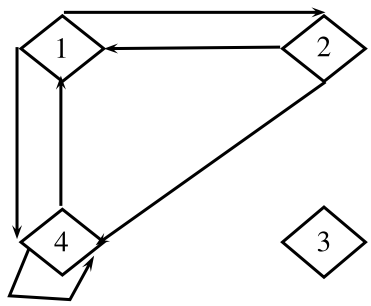
Fig-1



a) $G(A, \langle 0.8\ \ 0.7\ \ 0.1 \rangle)$



b) $G(A, \langle 0.7\ \ 0.6\ \ 0.1 \rangle)$



c) $G(A, \langle 0.5\ \ 0.4\ \ 0.2 \rangle)$

Fig-2

Now we shall examine the connection between the entries in powers of A and the paths in the associated digraph. First of all, the entry $\langle a_{ij}^T\ a_{ij}^I\ a_{ij}^F \rangle$ can be viewed as the capacity of the (unique) path of length 1 from node i to node j. The entries of A^2 are of the form

$$(a_{ij}^T)^2 = \sum_{k=1}^{n} a_{ik}^T \otimes a_{kj}^T,$$
$$(a_{ij}^I)^2 = \sum_{k=1}^{n} a_{ik}^I \otimes a_{kj}^I,$$
$$(a_{ij}^F)^2 = \prod_{k=1}^{n} a_{ik}^F \oplus a_{kj}^F.$$

which can be recognized as the maximum capacity of a path of length 2 beginning at node i and ending at node j. By induction, for A^m we get

$$\begin{aligned}(a_{ij}^T)^m &= \sum_{k=1}^{n} (a_{ik}^T)^{m-1} \otimes a_{kj}^T,\\ (a_{ij}^I)^m &= \sum_{k=1}^{n} (a_{ik}^I)^{m-1} \otimes a_{kj}^I,\\ (a_{ij}^F)^m &= \prod_{k=1}^{n} (a_{ik}^F)^{m-1} \oplus a_{kj}^F,\end{aligned} \tag{3}$$

where $\langle a_{ik}^T\ a_{ik}^I\ a_{ik}^F\rangle^{m-1}$ is the maximum capacity of a path of length $m-1$ from node i to node k. Now, each path P of length m that begins at node i and ends at node j can be split into a path of length $m-1$ from i to some intermediate node k and then a single arc from k to j. Clearly, for P to have the maximum capacity, it is necessary that such a partition will result in a maximum capacity path of length $m-1$ from i to k. We see (3) computes the maximum of such partitions over all possible intermediate nodes k, showing that $\langle a_{ij}^T\ a_{ij}^I\ a_{ij}^F\rangle^m$ is the maximum capacity of a path of length m from i to j.

Lemma 3.2 For a given FNSM $A \in \mathcal{N}_{(n,n)}$ the following conditions are equivalent:

(*i*) $(a_{ij}^T)^m \geqslant h^T$; $(a_{ij}^I)^m \geqslant h^I$; $(a_{ij}^F)^m \leqslant h^F$.

(*ii*) In the threshold digraph $\mathcal{G}(A, \langle h^T\ h^I\ h^F\rangle)$ there exists a path of length m from node i to node j.

If, moreover, $m \geqslant n$, then these conditions are equivalent withthe following

(*iii*) In the threshold digraph $\mathcal{G}(A, \langle h^T\ h^I\ h^F\rangle)$ there is a path from node i to node j containing a cycle.

Proof. If $(a_{ij}^T)^m \geqslant h^T$, $(a_{ij}^I)^m \geqslant h^I$ and $(a_{ij}^F)^m \leqslant h^F$, then there exists a path P from node i to node j with length m and capacity at least $\langle h^T\ h^I\ h^F\rangle$. But, due to the properties of the operation $\otimes$, for each arc (k,l) on P we have $c(k,l) = (a_{kl}^T) \geqslant h^T$, $(a_{kl}^I) \geqslant h^I$ and $(a_{kl}^F) \leqslant h^F$. However, this is exactly the condition for the pair (k,l) to be in the arc set of $\mathcal{G}(A, \langle h^T\ h^I\ h^F\rangle)$. Therefore P itself is the sought path.

Conversely, any path P in $\mathcal{G}(A, \langle h^T\ h^I\ h^F\rangle)$ corresponds to the same path in $\mathcal{G}(A)$, but here its capacity is at least $\langle h^T\ h^I\ h^F\rangle$, since P contains only arcs of weight at least $\langle h^T\ h^I\ h^F\rangle$). So any path P from i to j in $\mathcal{G}(A, \langle h^T\ h^I\ h^F\rangle)$ with length m will ensure $(a_{ij}^T)^m \geqslant h^T$; $(a_{ij}^I)^m \geqslant h^I$; $(a_{ij}^F)^m \leqslant h^F$.

For the equivalence with (iii) simply use Lemma 2.6 ■

4. Orbits of a fuzzy neutrosophic soft matrix and their interpretation in digraphs

In this section we discuss about the orbits of a fuzzy neutrosophic soft matrix and their interpretation in digraphs. Let a FNSM $A \in \mathcal{N}_{(n,n)}$ and a FNSV $b \in \mathcal{N}_n$ be given.

Definition 4.1 The sequence of FNSVs $\langle x^T \;\; x^I \;\; x^F\rangle(0)$, $\langle x^T \;\; x^I \;\; x^F\rangle(1)$, $\langle x^T \;\; x^I \;\; x^F\rangle(2)$,...,$\langle x^T \;\; x^I \;\; x^F\rangle(k)$, ..., where $\langle x^T \;\; x^I \;\; x^F\rangle(0) = \langle b^T \;\; b^I \;\; b^F\rangle$ and $\langle x^T \;\; x^I \;\; x^F\rangle(k+1) = A \otimes \langle x^T \;\; x^I \;\; x^F\rangle(k)$ is called the orbit of the FNSM A generated by the FNSV $\langle b^T \;\; b^I \;\; b^F\rangle$. We shall denote it by $orb(A, \langle b^T \;\; b^I \;\; b^F\rangle)$.

Definition 4.2 The orbit $orb(A, \langle b^T \;\; b^I \;\; b^F\rangle)$ is said to stabilize if there exists an integer k_0 such that for $k \geqslant k_0$ we have $\langle x^T \;\; x^I \;\; x^F\rangle(k+1) = \langle x^T \;\; x^I \;\; x^F\rangle(k)$. $orb(A, \langle b^T \;\; b^I \;\; b^F\rangle)$ oscillates if it does not stabilize but there exists two integers k_0 and $t > 1$ such that $\langle x^T \;\; x^I \;\; x^F\rangle(k+t) = \langle x^T \;\; x^I \;\; x^F\rangle(k)$ for each $k \geqslant k_0$. The smallest t with this property is called the period of the orbit.

***Example* 4.3** If we take the FNSM A from Example3.1 and the starting FNSV $\langle b^T \;\; b^I \;\; b^F\rangle = (\langle 0.5 \;\; 0.4 \;\; 0.2\rangle, \langle 0.4 \;\; 0.3 \;\; 0.2\rangle, \langle 0.8 \;\; 0.7 \;\; 0.1\rangle, \langle 0.3 \;\; 0.2 \;\; 0.5\rangle)^t$, then we obtain the following orbit:

$$\langle x^T \;\; x^I \;\; x^F\rangle(0) = \begin{pmatrix} \langle 0.5 \;\; 0.4 \;\; 0.2\rangle \\ \langle 0.4 \;\; 0.3 \;\; 0.2\rangle \\ \langle 0.8 \;\; 0.7 \;\; 0.1\rangle \\ \langle 0.3 \;\; 0.2 \;\; 0.5\rangle \end{pmatrix}, \langle x^T \;\; x^I \;\; x^F\rangle(1) = \begin{pmatrix} \langle 0.4 \;\; 0.5 \;\; 0.2\rangle \\ \langle 0.5 \;\; 0.4 \;\; 0.2\rangle \\ \langle 0.4 \;\; 0.2 \;\; 0.2\rangle \\ \langle 0.5 \;\; 0.4 \;\; 0.2\rangle \end{pmatrix},$$
$$\langle x^T \;\; x^I \;\; x^F\rangle(2) = \begin{pmatrix} \langle 0.5 \;\; 0.4 \;\; 0.2\rangle \\ \langle 0.5 \;\; 0.4 \;\; 0.2\rangle \\ \langle 0.4 \;\; 0.3 \;\; 0.2\rangle \\ \langle 0.5 \;\; 0.4 \;\; 0.2\rangle \end{pmatrix}, \langle x^T \;\; x^I \;\; x^F\rangle(3) = \begin{pmatrix} \langle 0.5 \;\; 0.4 \;\; 0.2\rangle \\ \langle 0.5 \;\; 0.4 \;\; 0.2\rangle \\ \langle 0.4 \;\; 0.3 \;\; 0.2\rangle \\ \langle 0.5 \;\; 0.4 \;\; 0.2\rangle \end{pmatrix}, ...,$$

which clearly stabilizes; whereas for

$$\langle b^T \;\; b^I \;\; b^F\rangle = (\langle 0.7 \;\; 0.6 \;\; 0.1\rangle, \langle 0 \;\; 0 \;\; 1\rangle, \langle 0 \;\; 0 \;\; 1\rangle, \langle 0 \;\; 0 \;\; 1\rangle)^t$$

we get

$$\langle x^T \;\; x^I \;\; x^F\rangle(0) = \begin{pmatrix} \langle 0.7 \;\; 0.6 \;\; 0.1\rangle \\ \langle 0 \;\; 0 \;\; 1\rangle \\ \langle 0 \;\; 0 \;\; 1\rangle \\ \langle 0 \;\; 0 \;\; 1\rangle \end{pmatrix}, \langle x^T \;\; x^I \;\; x^F\rangle(1) = \begin{pmatrix} \langle 0.1 \;\; 0.2 \;\; 0.6\rangle \\ \langle 0.5 \;\; 0.4 \;\; 0.2\rangle \\ \langle 0 \;\; 0 \;\; 1\rangle \\ \langle 0.7 \;\; 0.6 \;\; 0.1\rangle \end{pmatrix},$$
$$\langle x^T \;\; x^I \;\; x^F\rangle(2) = \begin{pmatrix} \langle 0.7 \;\; 0.6 \;\; 0.1\rangle \\ \langle 0.5 \;\; 0.4 \;\; 0.2\rangle \\ \langle 0.4 \;\; 0.3 \;\; 0.2\rangle \\ \langle 0.5 \;\; 0.4 \;\; 0.2\rangle \end{pmatrix}, \langle x^T \;\; x^I \;\; x^F\rangle(3) = \begin{pmatrix} \langle 0.5 \;\; 0.4 \;\; 0.2\rangle \\ \langle 0.5 \;\; 0.4 \;\; 0.2\rangle \\ \langle 0.4 \;\; 0.3 \;\; 0.2\rangle \\ \langle 0.7 \;\; 0.6 \;\; 0.1\rangle \end{pmatrix},$$
$$\langle x^T \;\; x^I \;\; x^F\rangle(4) = \begin{pmatrix} \langle 0.7 \;\; 0.6 \;\; 0.1\rangle \\ \langle 0.5 \;\; 0.4 \;\; 0.2\rangle \\ \langle 0.4 \;\; 0.3 \;\; 0.2\rangle \\ \langle 0.5 \;\; 0.4 \;\; 0.2\rangle \end{pmatrix},$$

and we see that the period of this orbit is 2.

We can argue, as in the beginning of this paper, that each orbit must either stabilize or oscillate, since no new entries are generated in the process. Moreover, it was show that for each FNSM A there is at least one nonzero starting FNSV $\langle b^T \quad b^I \quad b^F\rangle$ causing $orb(A, \langle b^T \quad b^I \quad b^F\rangle)$ to stabilize, in which case the FNSV we arrive at is clearly an fuzzy neutrosophic soft eigen vector of the FNSM A.

Definition 4.4 A FNSM $A \in \mathcal{N}_{(n,n)}$ is called strongly stable if $orb(A, \langle b^T \quad b^I \quad b^F\rangle)$ stabilizes for each $\langle b^T \quad b^I \quad b^F\rangle \in \mathcal{N}_n$.

Theorem 4.5 For given A FNSM $A \in \mathcal{N}_{(n,n)}$ is strongly stable if and only if the powers of A stabilize.

Proof. The sufficient condition is implied by the definition of the orbit, since $\langle x^T \quad x^I \quad x^F\rangle(k) = A^k \otimes \langle b^T \quad b^I \quad b^F\rangle$. For the converse implication notice that powers of A are nothing but FNSM whose columns are in fact orbits, generated by columns of A as starting FNSVs. Hence if each orbit stabilizes, the powers of A will eventually stabilize too. ■

Now, let us look at the interpretation of orbits in the associated digraph. Let $\langle x^T \quad x^I \quad x^F\rangle(m)_i = (A^m \otimes \langle b^T \quad b^I \quad b^F\rangle)_i$, where

$$\begin{aligned} x^T(m)_i &= \sum_{j=1}^{n} (a_{ij}^T)^m \otimes b_j^T, \\ x^I(m)_i &= \sum_{j=1}^{n} (a_{ij}^I)^m \otimes b_j^I, \\ x^F(m)_i &= \prod_{j=1}^{n} (a_{ij}^F)^m \oplus b_j^F. \end{aligned} \tag{4}$$

We already know that $\langle a_{ij}^T \quad a_{ij}^I \quad a_{ij}^F\rangle^m$ is the maximum capacity of a path of length m from node i to node j. The FNSV $\langle b^T \quad b^I \quad b^F\rangle$ can be viewed as assigning capacities to nodes; let us call the value $\langle b_j^T \quad b_j^I \quad b_j^F\rangle$ the terminal capacity at node j. Then, since in (4) the maximum is taken over all terminal nodes j of the paths, we can interpret this as the maximum capacity of a terminated path of length m, starting at node i. That means that the capacity of a terminated path is computed by multiplying the capacity of the path itself by the capacity of the node it terminates at. The capacity of a terminated path P will be denoted by $ct(P)$.

Now take a fixed $i \in N$ and denote

$$\langle h^T \quad h^I \quad h^F\rangle = \limsup_{k\to\infty}\{\langle x^T \quad x^I \quad x^F\rangle(k)_i\}.$$

That means that $\langle h_j^T \quad h_j^I \quad h_j^F\rangle$ is the maximum number that appears in the sequence $\{\langle x^T \quad x^I \quad x^F\rangle(k)_i\}_{k=1}^{\infty}$ infinitely many times, or that for each $k_0 \in \mathbb{N}$ there exists an integer $k \geqslant k_0$ such that $\langle x^T \quad x^I \quad x^F\rangle(k)_i = \langle h^T \quad h^I \quad h^F\rangle$. The significance of the value $\langle h^T \quad h^I \quad h^F\rangle$ in the associated digraphs is summarized in the following Lemma, whose proof follows easily from the definitions.

Lemma 4.6 Let $A \in \mathcal{N}_{(n,n)}, \langle b^T \quad b^I \quad b^F\rangle \in \mathcal{N}_n$, and $i \in N$ be given. The following conditions are equivalent:

(i) $\quad \langle h^T \quad h^I \quad h^F\rangle = \limsup_{k\to\infty}\{\langle x^T \quad x^I \quad x^F\rangle(k)_i\}$;

(ii) $\langle h^T \;\; h^I \;\; h^F\rangle$ is the greatest value such that for every $k_0 \in \mathbb{N}$ there is $k \geqslant k_0$ such that $G(A)$ contains a path P starting at i with $l(P) = k$ and $ct(P) \geqslant \langle h^T \;\; h^I \;\; h^F\rangle$;

(iii) $\langle h^T \;\; h^I \;\; h^F\rangle$ is the greatest value such that for each $k_0 \in \mathbb{N}, G(A, \langle h^T \;\; h^I \;\; h^F\rangle)$ contains a path P with length $l(P) \geqslant k_0$ begining at node i and ending at some node j with $\langle b_j^T \; b_j^I \; b_j^F\rangle \geqslant \langle h^T \; h^I \; h^F\rangle$;

(iv) $\langle h^T \; h^I \; h^F\rangle$ is the greatest value such that in $\mathcal{G}(A, \langle h^T \; h^I \; h^F\rangle)$ there is a path P from i to some node j with $\langle b_j^T \;\; b_j^I \;\; b_j^F\rangle \geqslant \langle h^T \; h^I \; h^F\rangle$ such that P contains a cycle.

Lemma 4.7 If $c_1, c_2, ..., c_m$ are integers with $gcd(c_1, c_2, ..., c_m) = 1$, then there exists $k_0 \in \mathbb{N}$ such that each integer $k \geqslant k_0$ can be expressed as a nonnegative linear combination of $c_1, c_2, ..., c_m$, *i.e.*, there exist nonnegative integers $\alpha_1, \alpha_2, ..., \alpha_m$, such that $k = \alpha_1 c_1 + \alpha_2 c_2 + ... + \alpha_m c_m$.

Definition 4.8 A digraph $\mathcal{G}$ is strongly stable if each strongly connected component of $\mathcal{G}$ either is acyclic or has period 1.

Theorem 4.9 A FNSM $A \in \mathcal{N}_{(n,n)}$ is strongly stable if and only if each threshold digraph $\mathcal{G}(A, \langle h^T \; h^I \; h^F\rangle)$ is strongly stable.

Proof. Let us fix $\langle b_j^T \;\; b_j^I \;\; b_j^F\rangle$ and i. Take $\langle h^T \;\; h^I \;\; h^F\rangle = \limsup_{k\to\infty}\{\langle x^T \;\; x^I \;\; x^F\rangle(k)_i\}$ and look at the digraph $\mathcal{G}(A, \langle h^T \;\; h^I \;\; h^F\rangle)$. Due to Lemma 4.6, there is a path P beginning at i and ending at some node j with $\langle b_j^T \;\; b_j^I \;\; b_j^F\rangle \geqslant \langle h^T \;\; h^I \;\; h^F\rangle$ such that P contains a cycle, say $\mathcal{G}$. Look at the SCC $\mathcal{G}'$ of $\mathcal{G}(A, \langle h^T \;\; h^I \;\; h^F\rangle)$ containing $\mathcal{G}$. Now $\mathcal{G}(A, \langle h^T \;\; h^I \;\; h^F\rangle)$ is strongly stable by the assumption. Therefore, $\mathcal{G}'$ contains cycles $\mathcal{G}_1, \mathcal{G}_2, ..., \mathcal{G}_m$ with lengths $c_1, c_2, ..., c_m$ such that $gcd(c_1, c_2, ..., c_m) = 1$. As they are all in the same SCC, it is possible to pick a path (say P') that starts at i, meets each of the cycles $\mathcal{G}_1, \mathcal{G}_2, ..., \mathcal{G}_m$ and ends at j. Then, by Lemma 4.7, there exists $k_0 \in \mathbb{N}$ such that one can find in $\mathcal{G}(A, \langle h^T \;\; h^I \;\; h^F\rangle)$ a path from i to j of length k for any $k \geqslant k_0$, traversing the cycles $\mathcal{G}_1, \mathcal{G}_2, ..., \mathcal{G}_m$ suitable numbers of times. That means that $\langle x^T \;\; x^I \;\; x^F\rangle(k)_i \geqslant \langle h^T \;\; h^I \;\; h^F\rangle$ for all $k \geqslant k_0$. However, the same cannot be said about any $\langle h^T \;\; h^I \;\; h^F\rangle' > \langle h^T \;\; h^I \;\; h^F\rangle$, since $\langle h^T \;\; h^I \;\; h^F\rangle$ was chosen to be the greatest value appearing in the sequence $\{\langle x^T \;\; x^I \;\; x^F\rangle(k)_i\}_{k=1}^{\infty}$ infinitely many times. Therefore, $\{\langle x^T \;\; x^I \;\; x^F\rangle(k)_i\}_{k=1}^{\infty}$ stabilizes for each i, implying that $\{\langle x^T \;\; x^I \;\; x^F\rangle(k)\}_{k=1}^{\infty}$ stabilizes too.

Conversely, suppose that the orbit $orb(A, \langle b^T \; b^I \; b^F\rangle)$ stabilizes for each starting FNSV $\langle b^T \, b^I \, b^F\rangle$, but $\mathcal{G}(A, \langle h^T \, h^I \, h^F\rangle)$ is not strongly stable for some $\langle h^T \, h^I \, h^F\rangle > \langle 0 \; 0 \; 1\rangle$. Then there is a SCC $\mathcal{G}'$ of $\mathcal{G}(A, \langle h^T \;\; h^I \;\; h^F\rangle)$, containing a cycle, but such that the gcd of all the cycle lengths in $\mathcal{G}'$ is $d > 1$. Now, take an arbitrary node i from $\mathcal{G}'$ and define a starting FNSV by $\langle b^T \;\; b^I \;\; b^F\rangle_i = \langle h^T \;\; h^I \;\; h^F\rangle, \langle b_j^T \;\; b_j^I \;\; b_j^F\rangle = \langle 0 \;\; 0 \;\; 1\rangle$ for all the remaining nodes. Using $\langle h^T \;\; h^I \;\; h^F\rangle > \langle 0 \;\; 0 \;\; 1\rangle$, it is easy to see that to get $\langle x^T \;\; x^I \;\; x^F\rangle(k)_i = \langle h^T \;\; h^I \;\; h^F\rangle$ for some $k \in \mathbb{N}$, it is necessary and sufficient that there be a cycle $\mathcal{G}$ (not necessarily elementary) of length k from i to i in $\mathcal{G}(A, \langle h^T \; h^I \; h^F\rangle)$. In fact that means that $\mathcal{G}$ is in $\mathcal{G}'$. But since d divides the lengths of all such cycles, d divides k too. We can conclude that $\langle x^T \;\; x^I \;\; x^F\rangle(k)_i = \langle h^T \;\; h^I \;\; h^F\rangle$ will occur infinitely many times, but only for k a multiple of d. Therefore $orb(A, \langle b^T \;\; b^I \;\; b^F\rangle)$ does not stabilize. ■

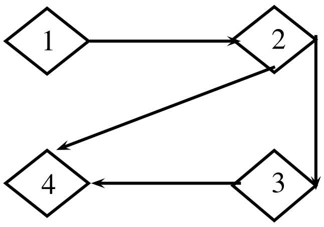
***Example* 4.10** For the FNSM A from Example 3.1 we conclude that it is not strongly stable by looking at its threshold digraphs. $\mathcal{G}(A, \langle 0.7 \; 0.6 \; 0.1\rangle)$ contains a strong component formed by nodes 1 and 4, which consists of the only cycle of length 2, and therefore its period is 2. The associated threshold digraphs $\mathcal{G}(A, \langle 0.7 \; 0.6 \; 0.1\rangle)$, $\mathcal{G}(A, \langle 0.5 \; 0.4 \; 0.2\rangle)$

and $\mathcal{G}(A, \langle 0.4\ 0.3\ 0.2\rangle)$ for

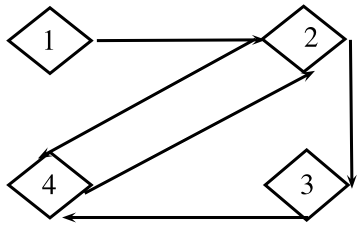
$$A = \begin{bmatrix} \langle 0 \quad 0 \quad 1\rangle & \langle 0.7\ 0.6\ 0.1\rangle & \langle 0 \quad 0 \quad 1\rangle & \langle 0 \quad 0 \quad 1\rangle \\ \langle 0 \quad 0 \quad 1\rangle & \langle 0 \quad 0 \quad 1\rangle & \langle 0.7\ 0.6\ 0.1\rangle & \langle 0.7\ 0.6\ 0.1\rangle \\ \langle 0 \quad 0 \quad 1\rangle & \langle 0 \quad 0 \quad 1\rangle & \langle 0 \quad 0 \quad 1\rangle & \langle 0.7\ 0.6\ 0.1\rangle \\ \langle 0.4\ 0.3\ 0.2\rangle & \langle 0.5\ 0.4\ 0.2\rangle & \langle 0 \quad 0 \quad 1\rangle & \langle 0 \quad 0 \quad 1\rangle \end{bmatrix}$$

are given in Fig-3. Notice that all the other threshold digraphs for this FNSM are either equal to one of those, or complete, or containing no arcs, therefore it is sufficient to examine the above three.

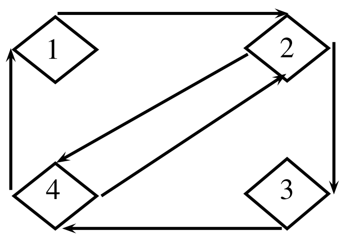
$\mathcal{G}(A,\ \langle 0.7\ 0.6\ 0.1\rangle)$ is acyclic. $\mathcal{G}(A,\ \langle 0.5\ 0.4\ 0.2\rangle)$ separates into two SCCs. The one containing only node 1 is acyclic, while the one formed by nodes 2, 3, 4 contains two elementary cycles: (2, 4) and (2, 3, 4). Their lengths are comprime; therefore $\mathcal{G}(A, \langle 0.5\ 0.4\ 0.2\rangle)$ is strongly stable. $\mathcal{G}(A,\ \langle 0.4\ 0.3\ 0.2\rangle)$ is strongly connected itself with the lengths of its elementary cycles equal to 2,3,4 hence strongly stable. Therefore, we can conclude that A is strongly stable, which also means that its powers will stabilize.



a) $G(A, \langle 0.7\ \ 0.6\ \ 0.1\rangle)$



b) $G(A, \langle 0.5\ \ 0.4\ \ 0.2\rangle)$



c) $G(A, \langle 0.4\ \ 0.3\ \ 0.2\rangle)$

Fig - 3

5. Applications to special classes of fuzzy neutrosophic soft matrices

In this section we show that the powers of an upper triangular FNSM with always stablizes provided necessary and sufficient condition for a symmetric FNSM to be strongly stabilizes and obtain condition under which a circulent FNSM is strongly stable.

Definition 5.1 A FNSM $A \in \mathcal{N}_{(n,n)}$ is called upper triangular if each entry over the main diagonal is not less than any entry below the main diagonal, i.e.,

$$\begin{aligned} min\{a_{ij}^T;\ i \leqslant j\} &\geqslant \ max\{a_{ij}^T;\ i > j\}, \\ min\{a_{ij}^I;\ i \leqslant j\} &\geqslant \ max\{a_{ij}^I;\ i > j\}, \\ max\{a_{ij}^F;\ i \leqslant j\} &\leqslant \ min\{a_{ij}^F;\ i > j\}. \end{aligned}$$

Note that this definition includes the classical upper triangular FNSMs, i.e. such that below the main diagonal there are only zero entries.

Theorem 5.2 The powers of an upper triangular FNSM always stabilize.

Proof. Notice that if any cycle containing at least two nodes appears in a threshold digraph $\mathcal{G}(A, \langle h^T\ h^I\ h^F\rangle)$ of an upper triangular FNSM A, then this means that some entry $\langle a_{ij}^T\ a_{ij}^I\ a_{ij}^F\rangle$ below the main diagonal of the FNSM has been involved as the capacity of an arc on this cycle. But due to the definition of upper triangular FNSMs, that means that all the main-diagonal entries fulfill $\langle a_{kk}^T\ a_{kk}^I\ a_{kk}^F\rangle \geqslant \langle a_{ij}^T\ a_{ij}^I\ a_{ij}^F\rangle \geqslant \langle h^T\ h^I\ h^F\rangle$. Therefore, $\mathcal{G}(A, \langle h^T\ h^I\ h^F\rangle)$ contains all the loops and each SCC in it has period 1. ■

Definition 5.3 A FNSM $A \in \mathcal{N}_{(n,n)}$ is symmetric if $\langle a_{ij}^T\ a_{ij}^I\ a_{ij}^F\rangle = \langle a_{ji}^T\ a_{ji}^I\ a_{ji}^F\rangle$ for all $i, j \in N$.

Note that all the associated digraphs of a symmetric FNSM are symmetric. So, we can consider just their undirected versions and condition for stabilization becomes simpler.

Theorem 5.4 A symmetric FNSM A is strongly stable if and only if in each undirected threshold graph $\mathcal{G}(A, \langle h^T\ h^I\ h^F\rangle)$ each SCC either contains an odd cycle or is an isolated node.

Definition 5.5 A FNSM $A \in \mathcal{N}_{(n,n)}$ is called a circulant FNSM, or simply a circulant, if it is of the form

$$A = \begin{bmatrix} \langle a_1^T\ a_1^I\ a_1^F\rangle & \langle a_2^T\ a_2^I\ a_2^F\rangle & \langle a_3^T\ a_3^I\ a_3^F\rangle & \cdots & \langle a_{n-1}^T\ a_{n-1}^I\ a_{n-1}^F\rangle & \langle a_n^T\ a_n^I\ a_n^F\rangle \\ \langle a_n^T\ a_n^I\ a_n^F\rangle & \langle a_1^T\ a_1^I\ a_1^F\rangle & \langle a_2^T\ a_2^I\ a_2^F\rangle & \cdots & \langle a_{n-2}^T\ a_{n-2}^I\ a_{n-2}^F\rangle & \langle a_{n-1}^T\ a_{n-1}^I\ a_{n-1}^F\rangle \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ \langle a_2^T\ a_2^I\ a_2^F\rangle & \langle a_3^T\ a_3^I\ a_3^F\rangle & \langle a_4^T\ a_4^I\ a_4^F\rangle & \cdots & \langle a_n^T\ a_n^I\ a_n^F\rangle & \langle a_1^T\ a_1^I\ a_1^F\rangle \end{bmatrix}$$

The set of entries with the same index is called a stripe; the entries $\langle a_i^T\ \ a_i^I\ \ a_i^F\rangle$ form the i^{th} stripe. In the associated digraph $\mathcal{G}(A)$ each stripe i defines a set of arcs of the form $(k, k+i-1)$ for $k = 1, 2, ..., n$. Obviously, all the numbers here are considered modulo n. We shall say that the span of an arc in the i^{th} stripe is $i-1$; that is, these arcs fall into a set of disjoint cycles (all with the same length equal to $n/gcd(n, i-1)$ for $i = 2, ..., n$ and 1 for the first stripe).

***Example* 5.6** As an illustration, consider the circulant

$$A = \begin{bmatrix} \langle a_1^T\ \ a_1^I\ \ a_1^F\rangle & \langle a_2^T\ \ a_2^I\ \ a_2^F\rangle & \langle a_3^T\ \ a_3^I\ \ a_3^F\rangle & \langle a_4^T\ \ a_4^I\ \ a_4^F\rangle & \langle a_5^T\ \ a_5^I\ \ a_5^F\rangle & \langle a_6^T\ \ a_6^I\ \ a_6^F\rangle \\ \langle a_6^T\ \ a_6^I\ \ a_6^F\rangle & \langle a_1^T\ \ a_1^I\ \ a_1^F\rangle & \langle a_2^T\ \ a_2^I\ \ a_2^F\rangle & \langle a_3^T\ \ a_3^I\ \ a_3^F\rangle & \langle a_4^T\ \ a_4^I\ \ a_4^F\rangle & \langle a_5^T\ \ a_5^I\ \ a_5^F\rangle \\ \langle a_5^T\ \ a_5^I\ \ a_5^F\rangle & \langle a_6^T\ \ a_6^I\ \ a_6^F\rangle & \langle a_1^T\ \ a_1^I\ \ a_1^F\rangle & \langle a_2^T\ \ a_2^I\ \ a_2^F\rangle & \langle a_3^T\ \ a_3^I\ \ a_3^F\rangle & \langle a_4^T\ \ a_4^I\ \ a_4^F\rangle \\ \langle a_4^T\ \ a_4^I\ \ a_4^F\rangle & \langle a_5^T\ \ a_5^I\ \ a_5^F\rangle & \langle a_6^T\ \ a_6^I\ \ a_6^F\rangle & \langle a_1^T\ \ a_1^I\ \ a_1^F\rangle & \langle a_2^T\ \ a_2^I\ \ a_2^F\rangle & \langle a_3^T\ \ a_3^I\ \ a_3^F\rangle \\ \langle a_3^T\ \ a_3^I\ \ a_3^F\rangle & \langle a_4^T\ \ a_4^I\ \ a_4^F\rangle & \langle a_5^T\ \ a_5^I\ \ a_5^F\rangle & \langle a_6^T\ \ a_6^I\ \ a_6^F\rangle & \langle a_1^T\ \ a_1^I\ \ a_1^F\rangle & \langle a_2^T\ \ a_2^I\ \ a_2^F\rangle \\ \langle a_2^T\ \ a_2^I\ \ a_2^F\rangle & \langle a_3^T\ \ a_3^I\ \ a_3^F\rangle & \langle a_4^T\ \ a_4^I\ \ a_4^F\rangle & \langle a_5^T\ \ a_5^I\ \ a_5^F\rangle & \langle a_6^T\ \ a_6^I\ \ a_6^F\rangle & \langle a_1^T\ \ a_1^I\ \ a_1^F\rangle \end{bmatrix}$$

The corresponding digraphs for its individual stripes are given in Fig-4.

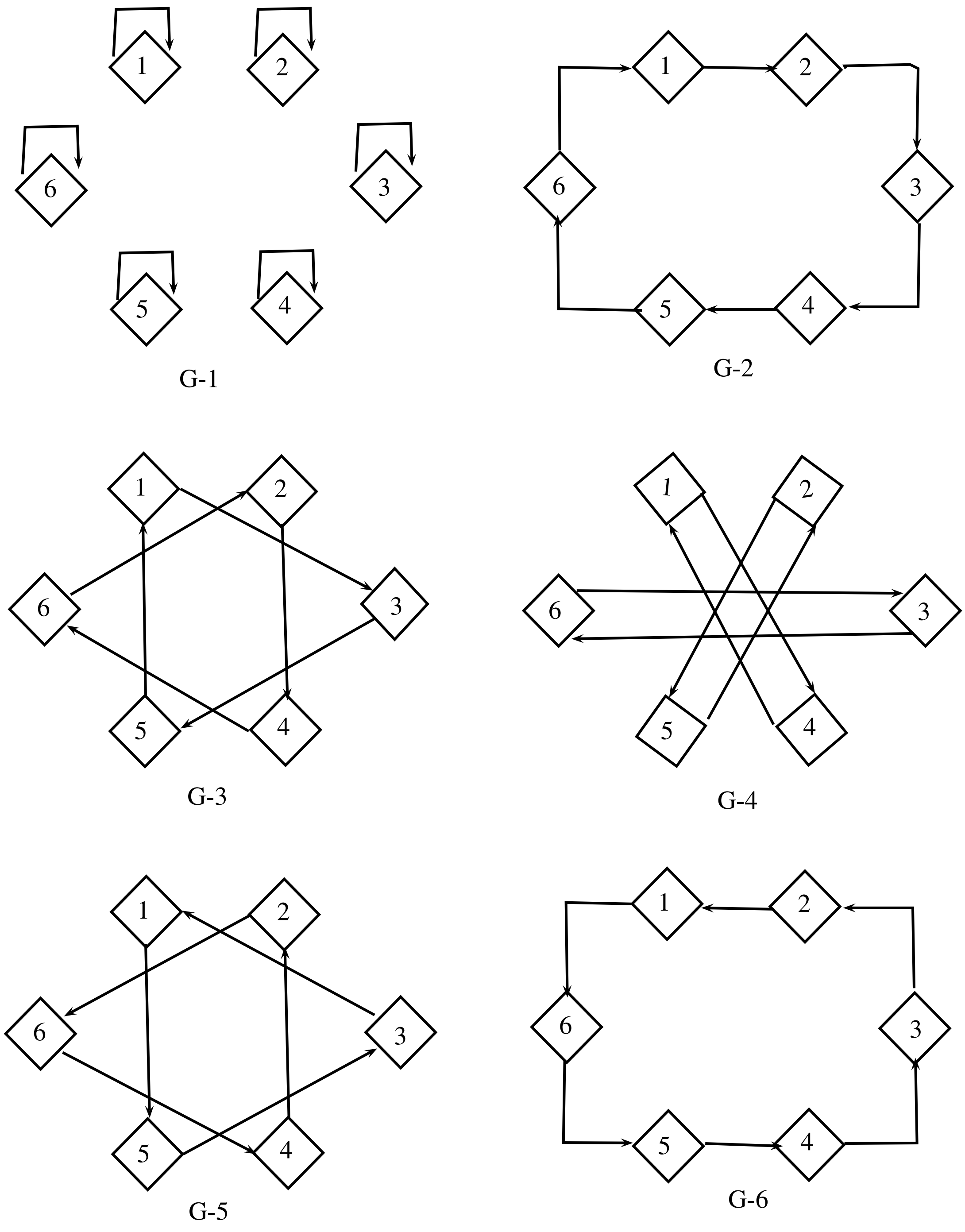


Fig-4

From these pictures we can make the following observations which we later generalize. If $\langle a_1^T \ \ a_1^I \ \ a_1^F \rangle > max\{\langle a_2^T \ \ a_2^I \ \ a_2^F \rangle, ..., \langle a_6^T \ \ a_6^I \ \ a_6^F \rangle\}$, then the first nontrivial threshold digraph consists of six loops. Hence it consists of six SCCs (each with period 1). For a lower value of the threshold, some more arcs will appear, causing some SCCs to merge, but the period of none of them will increase. Thus, in this case, the powers of the circulant will stabilize. However, if e.g. $\langle a_2^T \ \ a_2^I \ \ a_2^F \rangle > max\{\langle a_1^T \ \ a_1^I \ \ a_1^F \rangle, \langle a_3^T \ \ a_3^I \ \ a_3^F \rangle, ..., \langle a_6^T \ \ a_6^I \ \ a_6^F \rangle\}$, then the first nontrivial threshold digraph consists of a single cycle of length 6. So, it has a single SCC with period 6. Thus the powers of A will oscillate. Denote $J(A) = \{j; \langle a_j^T \ a_j^I \ a_j^F\rangle = max\{\langle a_1^T \ a_1^I \ a_1^F\rangle, \langle a_2^T \ a_2^I \ a_2^F\rangle, ..., \langle a_n^T \ a_n^I \ a_n^F\rangle\}\}$ for a given circulant $A \in \mathcal{N}_{(n,n)}$.

Theorem 5.7 If $1 \in J(A)$ for a given circulant $A \in \mathcal{N}_{(n,n)}$, then A is strongly stable. If $gcd(n, J(A)) = d > 1$, then the powers of A oscillate.

Proof. The first assertion follows from the above. For the second one it is sufficient to show that the period of the first threshold digraph $\mathcal{G}$ for A is equal to $d > 1$. Let $J(A) = \{i_1, i_2, ..., i_k\}$ and $1 \notin J(A)$. Then $\mathcal{G}$ contains arcs with spans $i_1-1, i_2-1, ..., i_k-1$. To get a cycle in $\mathcal{G}$ we must return to the starting node, traversing only arcs with these spans. Hence, by adding the spans of the arcs on a cycle together, we must get a multiple of n or

$$0 \equiv m_1(i_1 - 1) + m_2(i_2 - 1) + ... + m_k(i_k - 1)(modn), \tag{5}$$

where $m_1, m_2, ..., m_k$ are the numbers of arcs on the cycle, chosen from strips $i_1, i_2, ..., i_k$. (5) is also equivalent to the following congruence:

$$m_1 + m_2 + ... + m_k \equiv m_1 i_1 + m_2 i_2 + ... + m_k i_k (modn).$$

Notice that $m_1 + m_2 + ... + m_k$ is the length of the obtained cycle. Elementary number theory now implies that if $i_1, i_2, ..., i_k$ have a common divisor d with n, then all the cycle lengths in G will be divisible by d and the proof is complete. ■

6. Conclusion

In this paper, the authors presented two types of digraph, orbits of a fuzzy neutrosophic soft matrices and their interpretation in digraphs and application to special classes of fuzzy neutrosophic soft matrices.

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