Circuits, Systems, and Signal Processing
https://doi.org/10.1007/s00034-019-01248-1

On the Stabilizability for a Class of Linear Time-Invariant Systems Under Uncertainty

Nguyen Thi Kim Son[1,2] · Nguyen Phuong Dong[3] · Le Hoang Son[4] · Mohamed Abdel-Basset[5] · Gunasekaran Manogaran[6] · Hoang Viet Long[7]

Received: 4 April 2019 / Revised: 17 August 2019 / Accepted: 21 August 2019

Abstract
The uncertainty principle is one of the most important features in modeling and solving linear time-invariant (LTI) systems. The neutrality phenomena of some factors in real models have been widely recognized by engineers and scientists. The convenience and flexibility of neutrosophic theory in the description and differentiation of uncertainty terms make it take advantage of modeling and designing of control systems. This paper deals with the controllability and stabilizability of LTI systems containing neutrosophic uncertainty in the sense of both indeterminacy parameters and functional relationships. We define some properties and operators between neutrosophic numbers via horizontal membership function of a relative-distance-measure variable. Results on exponential matrices of neutrosophic numbers are well-defined with the notion e^{tA} deployed in a series of neutrosophic matrices. Moreover, we introduce the concepts of controllability and stabilizability of neutrosophic systems in the sense of Granular derivatives. Sufficient conditions to guarantee the controllability of neutrosophic LTI systems are established. Some numerical examples, related to RLC circuit and DC motor systems, are exhibited to illustrate the effectiveness of theoretical results.

Keywords Neutrosophic numbers · Controllability · Stabilizability · Granular computing

1 Introduction

A recent neutrosophic theory is a unifying field in logics that extends the concept of fuzzy sets using an indeterminacy value. The fundamental concepts of the neutrosophic set were introduced in [26–30]. There are many areas in which the neutrosophic theory is successfully applied [5,7–9,22]. In indeterminate problems, neutrosophic numbers,

✉ Hoang Viet Long
longhv08@gmail.com

Extended author information available on the last page of the article

Published online: 06 September 2019

which appear in form $a+bU$, easily express determinate and/or indeterminate information and are applied to fault diagnosis of gasoline engines and steam turbines using the similarity of neutrosophic numbers [12]. Additionally, some neutrosophic optimization techniques with applications were developed in [33–35]. Some other applications of neutrosophic theory in reliability test, monitoring the variability or sampling plans, were investigated [1–3]. For more details, readers can find in some recent researches, such as [6,11,14,19,31,36,37].

There exist many significant problems that must be investigated. For example, the first problem is to define an appropriate difference operator to make the space of neutrosophic numbers become a linear space, or furthermore, define the concept of the neutrosophic derivative. In [38], the difference was given in the form

$$z_1 - z_2 = (a_1 - a_2) + (b_1 - b_2)U. \tag{1}$$

where $z_1 = a_1 + b_1 U$, $z_2 = a_2 + b_2 U$ are neutrosophic numbers and $U = [U^-, U^+]$ is the indeterminacy. However, we can see that this subtraction cannot be a candidate for defining derivatives or constructing analysis properties on the set of neutrosophic numbers. For example, let $z_1 = 4 + U$ be a neutrosophic number with the indeterminacy $U = [0, 1]$. Then, we can directly see that the difference of $z_1 - z_1$ is different from $z_1 + (-1)z_1$. Indeed, according to the formula (1), we obtain that

$$z_1 - z_2 = (4 - 4) + (1 - 1)U = 0,$$

while

$$z_1 + (-1)z_1 = 4 + U + (-4) + (-1)U = [0, 1] + (-1)[0, 1] = [-1, 1] \neq 0,$$

that follows the space of neutrosophic numbers is not a linear space. This leads to a lot of challenges and difficulties in defining further analysis properties on the space of neutrosophic numbers. For further aim, there are considerable questions such as how to define the derivatives of neutrosophic number functions, how to understand control problems under neutrosophic uncertainty and how to apply the stability of control problems in signal processing under neutrosophic environments.

Besides, the control problems for neutrosophic linear time-invariant systems are interesting and worth being studied. The controllability is one of the fundamental concepts in mathematical control theory, which plays an important role in many engineering control systems. The concept of controllability was firstly introduced by Kalman in 1960, which leads to several important results regarding the behavior of dynamical systems [10]. However, there are only a few studies on the controllability of neutrosophic differential systems, except to recent paper of Ye and Cui [39] for SISO neutrosophic linear systems

$$\frac{\mathrm{d}x}{\mathrm{d}t} = A(U)x + B(U)u, \quad y = C(U)x \tag{2}$$

where the coefficients are neutrosophic numbers and no neutrosophic derivative was considered. In our point of view, because the input of a system may be impacted by neutrosophic uncertainty, the output should be a neutrosophic-valued function. Thus, we need to extend the considered dynamical system in the new setting with neutrosophic differentiability.

This paper deals with the stabilizability of linear time-invariant systems (LTI systems) in the case that the underlying functions are neutrosophic-valued and the neutrosophic calculus is taken into account. We focus on establishing some sufficient and necessary conditions to guarantee the controllability and stabilizability of neutrosophic control systems (14) in connection with granular control systems (15) via horizontal membership functions. The idea of horizontal membership functions was originally introduced by Piegat et al. [13,25] and developed for granular differentiability of fuzzy-valued functions by Mazandarani et al. [16–18] and Son et al. [31,32]. Recently, granular computing has been used in decision making [21], soft computing [13] and signal processing [25].

LTI theory came from applied mathematics, and then, it has been directly employed to circuits, signal processing, control theory and nuclear magnetic resonance spectroscopy, etc [23,40]. Unlike state machines, LTI systems have a memory of past states as well as the ability to predict the long-term behavior of systems. Thus, LTI systems are most popularly applied to the controller in power companies. The main difficulty in studying these problems is the lack of calculus tools in neutrosophic numbers spaces that makes neutrosophic dynamical systems more complex. In this paper, we study the controllability and stabilizability of the following LTI neutrosophic system:

$$\frac{\mathrm{d}_{\mathrm{gr}} x(t)}{\mathrm{d}t} = Ax(t) + Bu(t),$$

where $x(t)$ is the state variables taking values in neutrosophic environment, A, B are neutrosophic matrices with appropriate dimensions, $u(t)$ is the input control, and $\mathrm{d}_{\mathrm{gr}}(.)$ stands for granular derivative of neutrosophic function. The main contributions and detail approach of this paper can be highlighted as follows:

- The first important highlight of this work is based on the indeterminate expression of neutrosophic numbers in [38] and the idea of horizontal membership function approach [16], where we convert each neutrosophic number into a class of real parametric form. The advantage of this approach was represented in our previous work for triangular neutrosophic numbers [31]. The appearance of the relative-distance-measure variable $\mu \in [0, 1]$ helps to convert each neutrosophic number $z = a + b[U^-, U^+]$ into parametric forms $z^{\mathrm{gr}}(\mu) = a + bU^- + b\left(U^+ - U^-\right)\mu,\ \mu \in [0, 1]$. This representation has the advantage that we can further define the arithmetic operations, the derivatives, the integral of neutrosophic number functions as well as build the numerical algorithms.
- We define the granular difference between neutrosophic numbers—one important step to define further differentiability of neutrosophic number functions as well as neutrosophic differential equations and their applications to neutrosophic dynamic systems. The superiority of the proposed method lies on the fact that it does not necessitate the increasing of diameter of neutrosophic-valued function or multicase of solution related to so-called switching points as we often face in fuzzy analysis, see [4,15].

- We also attain the first step in building topological structures of neutrosophic numbers space by introducing granular metric and complete metric space $(\mathcal{E}, \rho_{\text{gr}})$. By ensuring the convergence of Cauchy sequences in $\mathcal{E}$, we can further study some qualitative and quantitative properties of solutions to dynamical systems arising in the fields of science and engineering, for example, experimental approximation solution algorithms can be developed through the convergence of solution sequence in neutrosophic complete metric spaces.
- In Sect. 2.3, we propose the novel concept of the exponential matrix. Firstly, based on arithmetic operations in $\mathcal{E}$, we introduce some neutrosophic matrix operations such as addition, subtraction, multiplication, scalar multiplication, matrix transpose and matrix inverse. Next, in order to define explicitly the solution formula of a linear system

$$\frac{\mathrm{d}_{\text{gr}} x(t)}{\mathrm{d}t} = Ax(t) + Bu(t), \qquad x(0) = x_0, \tag{3}$$

  we introduce the concept of exponential matrix e^{tA} and then, we give some characteristic properties of this matrix. Hence, the explicit formula of solution to the system (3) is given in Corollary 2.3.
- The controllability and stabilizability of neutrosophic linear time-invariant systems are introduced and investigated and in addition, some necessary and sufficient conditions for the controllability of an LTI neutrosophic system are given in Theorems 3.1 and 3.2 . Especially, a criterion of Kalman's criterion type is given in Corollary 3.1, which is an effective tool to ensure the controllability of LTI neutrosophic systems. After representating neutrosophic LTI system by Liénard form, we study the stabilizability criterion for LTI neutrosophic systems.
- We demonstrate the effectiveness and significance of obtained theoretical results by some numerical examples on Liénard equation and some engineering problems related to the RLC circuit control model and DC motor system model.

This paper is organized as follows: Section 2 presents some preliminaries on granular calculus of neutrosophic numbers, such as the neutrosophic limit, neutrosophic gr-derivative and neutrosophic gr-integral. Additionally, we introduce the concept of neutrosophic matrices and matrix operations. The main results on the controllability, stability and stabilizability of the neutrosophic LTI system are presented in Sect. 3. Section 4 illustrates the theoretical results by some numerical examples. Finally, the conclusions and future works are discussed in Sect. 5.

2 Preliminaries

2.1 Space of Neutrosophic Numbers

A neutrosophic number [28] is the number consisting of the determinate part a and the indeterminate part bU and is denoted by $z = a + bU$, where $a \in \mathbb{R}$, $b \in \mathbb{R}^+$ and U is the indeterminacy. We denote $\mathcal{E}$ by the set of all neutrosophic numbers.

Remark 2.1 Assume that the possible changeable range of the determinacy U is given by $\left[U^-, U^+\right]$. Then, a neutrosophic number $z = a + bU$ can be specified as a changeable interval number $z = \left[a + bU^-, a + bU^+\right]$. In particular, if either $b = 0$ or $U^- = U^+$ that means $bU = 0$ or $bU \in \mathbb{R}$ then $z = a$ or $z = a + bU$ can be degenerated to a real number, while in the case $a = 0$, z is degenerated to the indeterminate part $z = bU$.

Definition 2.1 (*Granular representation*) Let $z = a + bU$ be a neutrosophic number for $a, b \in \mathbb{R}$ and the indeterminate part $U = [U^-, U^+]$. Then, by denoting $\ell[U] = U^+ - U^-$—the length of changeable interval, the number z can be rewritten in the horizontal membership function form as follows

$$z^{\mathrm{gr}} : [0, 1] \to \mathbb{R},\ \mu_z \mapsto z^{\mathrm{gr}}(\mu_z) = a + bU^- + b\ell[U]\mu_z$$

where "gr" represents for the granule of information included in $[a+bU^-, a+bU^+]$, $\mu_z \in [0, 1]$ is called relative-distance-measure variable. The horizontal membership function of $z \in \mathcal{E}$ is denoted by $\mathcal{L}(z) \triangleq z^{\mathrm{gr}}(\mu_z)$.

Remark 2.2 The interval representation of $z \in \mathcal{E}$ can be obtained from the granular representation by using the following transformation

$$\mathcal{N}(z^{\mathrm{gr}}(\mu_z)) = \left[\min_{\mu_z \in [0,1]} z^{\mathrm{gr}}(\mu_z),\ \max_{\mu_z \in [0,1]} z^{\mathrm{gr}}(\mu_z)\right]. \tag{4}$$

Definition 2.2 Let z_1 and z_2 be two neutrosophic numbers. Then

i. $z_1 = z_2$ if and only if $\mathcal{L}(z_1) = \mathcal{L}(z_2)$ for all $\mu_{z_1} = \mu_{z_2} \in [0, 1]$.
ii. $z_1 \geq z_2$ if and only if $\mathcal{L}(z_1) \geq \mathcal{L}(z_2)$ for all $\mu_{z_1} = \mu_{z_2} \in [0, 1]$.

Next, based on gr-representation approach, we introduce the concept of arithmetic operations in $\mathcal{E}$.

Definition 2.3 Let z_1 and z_2 be two neutrosophic numbers whose respective horizontal membership functions are $\mathcal{L}(z_1)$ and $\mathcal{L}(z_2)$ and $\otimes$ denotes for one of the four operations in $\mathcal{E}$, i.e., addition, subtraction, multiplication or division operation. Then, we define

$$\mathcal{L}(z_1 \otimes z_2) \triangleq \mathcal{L}(z_1) * \mathcal{L}(z_2),$$

where the notion "$*$" represents for respective operations in $\mathbb{R}$. Especially, the difference in this sense is called granular difference (gr-difference) and denoted by $\ominus^{\mathrm{gr}}$.

Remark 2.3 Based on granular representation approach, the following relations hold for all $z_1, z_2, z_3 \in \mathcal{E}$

i. $z_1 \ominus^{\mathrm{gr}} z_2 = -(z_2 \ominus^{\mathrm{gr}} z_1)$,
ii. $z_1 \ominus^{\mathrm{gr}} z_1 = 0$,
iii. $z_1 + (-1)z_1 = 0$,
iv. $z_1 \ominus^{\mathrm{gr}} (-1)z_2 = z_1 + z_2$,
v. $(z_1 + z_2)z_3 = z_1 z_2 + z_2 z_3$.

Table 1 The different between arithmetic operations in some recent literature

Literature	$v = z - z^2$	$\overline{v} = z(1 - z)$
Moore et al. [20] on Moore interval arithmetic	$-13 + 6U$	$-12 + 6U$
Smarandache [28] on Neutrosophic numbers arithmetic	$-20 + 6U$	$-12 + 6U$
Piegat and Landowski [24] on RDM interval arithmetic	$-12 + 6U$	$-12 + 6U$

Remark 2.4 Conventionally, there exist additive and multiplicative identities such as neutrosophic numbers $0 = 0 + 0U$ and $1 = 1 + 0U$.

The following example will give a comparison of the result obtained by the granular representation approach with those of some previous approaches in their arithmetic operations that show the advantage of granular representation in complex problems.

Example 2.1 In this example, let $z = 3 + U$ be a neutrosophic number with the indeterminacy $U = [0, 1]$. Note that the neutrosophic number z can be rewritten in following changeable interval $[3, 4]$. Now, we will compare the results obtained by Moore arithmetic operation [20], Smarandache arithmetic operation [28] and RDM interval arithmetic [24] for following nonlinear equation

$$v = z - z^2. \tag{5}$$

Note that Eq. (5) can be represented in the following form

$$\overline{v} = z(1 - z).$$

The comparison result is shown in Table 1.

It can be seen that the results of operation obtained by RDM interval arithmetic do not depend on the form of equation, which proves that the granular approach can correctly solve more complicated problem.

2.2 Neutrosophic Number Functions and Their Calculus Properties

Definition 2.4 A function $f : [a, b] \subset \mathbb{R} \to \mathcal{E}$ given by $t \mapsto f(t)$ is said to be a neutrosophic number -valued function or $\mathcal{E}$-valued function. If the $\mathcal{E}$-valued function f includes n distinct neutrosophic numbers $z_1, z_2, \ldots, z_n$ then the horizontal membership function of f at $t \in [a, b]$, denoted by $\mathcal{L}(f(t)) \triangleq f^{\text{gr}}(t, \mu_f)$, can be given as

$$f^{\text{gr}} : [a, b] \times [0, 1] \times \cdots \times [0, 1] \to \mathbb{R},$$

where $\mu_f \triangleq (\mu_1, \mu_2, \ldots, \mu_n)$.

Example 2.2 Let $U = [0, 1]$ and $z_1 = 5 + 2U$, $z_2 = -4 + U$ be two neutrosophic numbers with respective horizontal membership functions $z_1^{\text{gr}}(\mu_1) = 5 + 2\mu_1$,

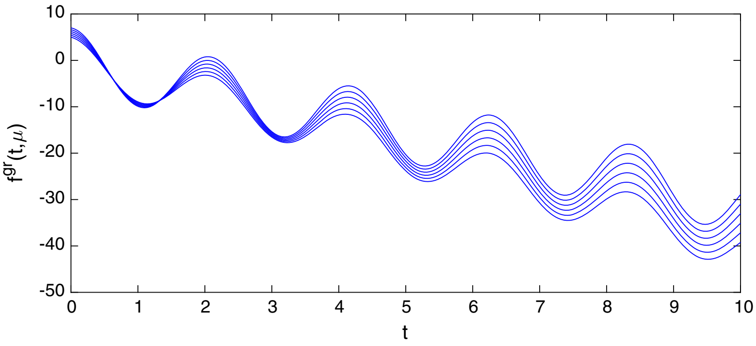

Fig. 1 The $\mathcal{E}$-valued function $f(t) = z_1 \cos 3t + z_2 t$ for $t \in [0, 10]$

$z_2^{\mathrm{gr}}(\mu_2) = -4 + \mu_2$, where $\mu_1, \mu_2 \in [0, 1]$. Here, we define an $\mathcal{E}$-valued function $f(t) = z_1 \cos 3t + z_2 t$ on the interval $[0, 10]$. Then, the horizontal membership function of $f(t)$ is given by

$$f^{\mathrm{gr}}(t, \mu_f) = \mathcal{L}(f(t)) = z_1^{\mathrm{gr}}(\mu_1) \cos 3t + z_2^{\mathrm{gr}}(\mu_2) t = (5 + 2\mu_1) \cos 3t + (-4 + \mu_2) t,$$

for $\mu_f = (\mu_1, \mu_2)$ and the graphical representation of $\mathcal{E}$-valued function $f(t)$ is shown in Fig. 1.

Next, we introduce the concept of metric on the set of neutrosophic numbers.

Definition 2.5 (*Granular metric*) For $z_1 = a_1 + b_1 U_1$, $z_2 = a_2 + b_2 U_2 \in \mathcal{E}$, the distance between these numbers is the function $\rho^{\mathrm{gr}} : \mathcal{E} \times \mathcal{E} \to \mathbb{R}^+ \cup \{0\}$ given as follows:

$$\begin{aligned} \rho^{\mathrm{gr}}(z_1, z_2) &= \max_{\mu_1, \mu_2} \left| z_1^{\mathrm{gr}}(\mu_1) - z_2^{\mathrm{gr}}(\mu_2) \right| \\ &= \max_{\mu_1, \mu_2} \left| \left(a_1 + b_1 U_1^- + b_1 \ell[U_1]\mu_1\right) - \left(a_2 + b_2 U_2^- + b_2 \ell[U_2]\mu_2\right) \right|. \end{aligned} \tag{6}$$

Proposition 2.1 *The function ρ^{gr} defined by (6) is a metric on $\mathcal{E}$, namely granular metric.*

Proof Let z and $\tilde{z}$ be two numbers in $\mathcal{E}$ with respective granular representation

$$z^{\mathrm{gr}}(\mu_1) = a_1 + b_1 U_1^- + b_1 \ell[U_1]\mu_1 \quad \text{and} \quad \tilde{z}^{\mathrm{gr}}(\mu_2) = a_2 + b_2 U_2^- + b_2 \ell[U_2]\mu_2.$$

By definition of ρ^{gr}, we obtain that $\rho^{\mathrm{gr}}(z, \tilde{z}) \geq 0$ and if $\rho^{\mathrm{gr}}(z, \tilde{z}) = 0$ then

$$\left| z^{\mathrm{gr}}(\mu_1) - \tilde{z}^{\mathrm{gr}}(\mu_2) \right| = 0 \Longleftrightarrow z^{\mathrm{gr}}(\mu_1) = \tilde{z}^{\mathrm{gr}}(\mu_2),$$

for all $\mu_1, \mu_2 \in [0, 1]$. Thus, according to Definition 2.2 (i), it follows that $z = \tilde{z}$.

Since the symmetry property of the function ρ^{gr} can be easily seen from its definition, the rest of our proof is to show that

$$\rho^{\mathrm{gr}}(z_1, z_2) \leq \rho^{\mathrm{gr}}(z_1, z_3) + \rho^{\mathrm{gr}}(z_3, z_2) \quad \text{for all } z_1, z_2, z_3 \in \mathcal{E}. \tag{7}$$

Indeed, since the fact that the inequality

$$\left|z_1^{\mathrm{gr}}(\mu_1) - z_2^{\mathrm{gr}}(\mu_2)\right| \leq \left|z_1^{\mathrm{gr}}(\mu_1) - z_3^{\mathrm{gr}}(\mu_3)\right| + \left|z_3^{\mathrm{gr}}(\mu_3) - z_2^{\mathrm{gr}}(\mu_2)\right|$$

holds for all $\mu_1, \mu_2, \mu_3 \in [0, 1]$, one gets

$$\left|z_1^{\mathrm{gr}}(\mu_1) - z_2^{\mathrm{gr}}(\mu_2)\right| \leq \max_{\mu_1,\mu_2} \left\{\left|z_1^{\mathrm{gr}}(\mu_1) - z_3^{\mathrm{gr}}(\mu_3)\right|\right\} + \max_{\mu_1,\mu_2,\mu_3} \left\{\left|z_3^{\mathrm{gr}}(\mu_3) - z_2^{\mathrm{gr}}(\mu_2)\right|\right\}$$

Hence, it follows $\max_{\mu_1,\mu_2} \left|z_1^{\mathrm{gr}}(\mu_1) - z_2^{\mathrm{gr}}(\mu_2)\right| \leq \max_{\mu_1,\mu_3} \left|z_1^{\mathrm{gr}}(\mu_1) - z_3^{\mathrm{gr}}(\mu_3)\right| + \max_{\mu_2,\mu_3} \left|z_3^{\mathrm{gr}}(\mu_3) - z_2^{\mathrm{gr}}(\mu_2)\right|$. This means the inequality (7) holds. The proof is complete. □

Theorem 2.1 *The space $\mathcal{E}$ endowed with ρ^{gr} is a metric space. Moreover, it is a complete metric space.*

Proof Assume that a sequence $\{z_n\}_{n\geq 1} \subset \mathcal{E}$ is Cauchy sequence in $\mathcal{E}$, which means that for all $\epsilon > 0$, there exists $n_0 \in \mathbb{N}^*$ such that for all $n, p \in \mathbb{N}$, $n \geq n_0$ and $p \geq 1$, we have

$$\rho^{\mathrm{gr}}(z_{n+p}, z_n) < \epsilon \Leftrightarrow \max_{\mu_1,\mu_2} \left|z_{n+p}^{\mathrm{gr}}(\mu_{n+p}) - z_n^{\mathrm{gr}}(\mu_n)\right| < \epsilon.$$

Thus, it implies that $\left|z_{n+p}^{\mathrm{gr}}(\mu_{n+p}) - z_n^{\mathrm{gr}}(\mu_n)\right| < \epsilon$ for all $\mu_n, \mu_{n+p} \in [0, 1]$, which means $\left\{z_n^{\mathrm{gr}}(\mu_n)\right\}_{n\geq 1}$ is a Cauchy sequence in $\mathbb{R}$, and hence, it is convergent. In addition, since the sequence $\left\{z_n^{\mathrm{gr}}(\mu_n)\right\}_{n\geq 1}$, given by

$$z_n^{\mathrm{gr}}(\mu) = a_n + b_n\left[U_n^- + \left(U_n^+ - U_n^-\right)\mu\right] = a_n + b_n U_n^- + \left[(a_n + b_n U_n^+) - (a_n + b_n U_n^-)\right]\mu$$

is convergent for all $\mu \in [0, 1]$, which follows that the sequences $\left\{a_n + b_n U_n^-\right\}_{n\geq 1}$, $\left\{a_n + b_n U_n^+\right\}_{n\geq 1}$ are also convergent corresponding to the cases $\mu = 0$ and $\mu = 1$. Here, with no loss of generality, we assume that

$$\lim_{n\to\infty} \left(a_n + b_n U_n^-\right) = c^-, \qquad \lim_{n\to\infty} \left(a_n + b_n U_n^+\right) = c^+.$$

In addition, since $a_n + b_n U_n^- \leq a_n + b_n U_n^+$, $\forall n \geq 1$, it implies $c^- \leq c^+$. Moreover, since $c^-, c^+ \in \mathbb{R}$, there exist $U^-, U^+ \in \mathbb{R}$ such that $c^- = a + bU^-$, $c^+ = a + bU^+$, where a, b are limits of sequences $\{a_n\}$, $\{b_n\}$, respectively. Hence, by denoting $z = [c^-, c^+] = a + b[U^-, U^+]$, we can conclude that z is a neutrosophic number with indeterminate part $U = [U^-, U^+]$ and it is the limit of Cauchy sequence $\{z_n\}_{n\geq 1}$. This completes the proof. □

Definition 2.6 (*Limit of $\mathcal{E}$-valued function*) Let $f : [a, b] \subset \mathbb{R} \to \mathcal{E}$ and $t_0 \in [a, b]$. Then, we say that the function f has the finite limit as t tends to t_0 if and only if there exists $\tau \in \mathcal{E}$ such that $\lim_{t \to t_0} \rho^{\mathrm{gr}}(f(t), \tau) = 0$, that is $\forall \epsilon > 0$, $\exists\, \delta(t_0, \epsilon) > 0$ such that $\forall t \in [a, b] : 0 < |t - t_0| < \delta$, it implies $\rho^{\mathrm{gr}}(f(t), \tau) < \epsilon$.

Definition 2.7 (*The continuity*) An $\mathcal{E}$-valued function $f : (a, b) \subset \mathbb{R} \to \mathcal{E}$ is said to be continuous on (a, b) if for each $t_0 \in (a, b)$, for all $\epsilon > 0$, there exists $\delta > 0$ such that $\forall t \in (a, b) : |t - t_0| < \delta$ then $\rho^{\mathrm{gr}}(f(t), f(t_0)) < \epsilon$, i.e., $\lim_{t \to t_0} \rho^{\mathrm{gr}}(f(t), f(t_0)) = 0$.

Definition 2.8 (*The differentiability*) Let $f : (a, b) \subset \mathbb{R} \to \mathcal{E}$ and $t_0 \in (a, b)$. Then, we say that f is granular differentiable (gr-differentiable) at the point t_0 if there exists $\dfrac{\mathrm{d}_{\mathrm{gr}} f(t_0)}{\mathrm{d}t} \in \mathcal{E}$ such that the limit

$$\lim_{h \to 0} \frac{f(t_0 + h) \ominus^{\mathrm{gr}} f(t_0)}{h} = \frac{\mathrm{d}_{\mathrm{gr}} f(t_0)}{\mathrm{d}t}$$

holds. Then, we call the value $\dfrac{\mathrm{d}_{\mathrm{gr}} f(t_0)}{\mathrm{d}t}$ the granular derivative (or gr-derivative for short) of function f at the point t_0. As a result, the function f is said to be gr-differentiable on the interval (a, b) if and only if the gr-derivative $\dfrac{\mathrm{d}_{\mathrm{gr}} f(t)}{\mathrm{d}t}$ exists for all $t \in (a, b)$. Then, the mapping $t \mapsto \dfrac{\mathrm{d}_{\mathrm{gr}} f(t)}{\mathrm{d}t}$ is called the gr-derivative of f on (a, b) and denoted by $\dfrac{\mathrm{d}_{\mathrm{gr}} f(t)}{\mathrm{d}t}$ or $f'_{\mathrm{gr}}(t)$.

Next, we give a necessary and sufficient condition for the gr-differentiability of neutrosophic function.

Proposition 2.2 *Let $f : (a, b) \subset \mathbb{R} \to \mathcal{E}$ and $t_0 \in (a, b)$. Then, the function f is gr-differentiable at the point t_0 if and only if its horizontal membership function is differentiable at the point t_0. Then, we have*

$$\mathcal{L}\left(\frac{\mathrm{d}_{\mathrm{gr}} f(t_0)}{\mathrm{d}t}\right) = \frac{\partial f^{\mathrm{gr}}(t_0, \mu_f)}{\partial t}.$$

Proof By using the assumption $\mathcal{E}$-valued function f is gr-differentiable at the point $t_0 \in (a, b)$, we have for all $\epsilon > 0$, there exists $\delta > 0$ such that for all $h \in (0, \delta)$,

$$\rho^{\mathrm{gr}}\left(\frac{f(t_0 + h) \ominus^{\mathrm{gr}} f(t_0)}{h}, \frac{\mathrm{d}_{\mathrm{gr}} f(t_0)}{\mathrm{d}t}\right) < \epsilon.$$

Next, by using the definition of granular metric, the above inequality becomes

$$\max_{\mu_f, \mu_{f'}} \left| \frac{1}{h}\left(f^{\mathrm{gr}}(t_0 + h, \mu_f) - f^{\mathrm{gr}}(t_0, \mu_f)\right) - \left(\frac{\mathrm{d}_{\mathrm{gr}} f}{\mathrm{d}t}\right)^{\mathrm{gr}}(t_0, \mu_{f'}) \right| < \epsilon \quad \text{for all } \mu_f,\ \mu_{f'},$$

that is equivalent to

$$\left|\frac{1}{h}\left(f^{\mathrm{gr}}(t_0+h,\mu_f)-f^{\mathrm{gr}}(t_0,\mu_f)\right)-\left(\frac{\mathrm{d}_{\mathrm{gr}} f}{\mathrm{d}t}\right)^{\mathrm{gr}}(t_0,\mu_{f'})\right|<\epsilon,$$

for all $\epsilon > 0$ and $h \in (0, \delta)$. Then, by letting $h \to 0$, we obtain that

$$\lim_{h\to 0}\frac{f^{\mathrm{gr}}(t_0+h,\mu_f)-f^{\mathrm{gr}}(t_0,\mu_f)}{h}=\left(\frac{\mathrm{d}_{\mathrm{gr}} f}{\mathrm{d}t}\right)^{\mathrm{gr}}(t_0,\mu_{f'}),$$

for all μ_f, $\mu_{f'}$, which leads to the differentiability of the horizontal membership function of f. The converse statement can be obtained by using similar arguments. The proof is complete. □

Proposition 2.3 *Assume that f and g are differentiable $\mathcal{E}$-valued functions on (a, b). Then, the following statements are fulfilled:*

i. $\dfrac{\mathrm{d}_{\mathrm{gr}}(z_0)}{\mathrm{d}t}=0$ *for all* $z_0 \in \mathcal{E}$.

ii. $\dfrac{\mathrm{d}_{\mathrm{gr}}\left(\alpha f(t)\pm\beta g(t)\right)}{dt}=\alpha\dfrac{\mathrm{d}_{\mathrm{gr}} f(t)}{\mathrm{d}t}\pm\beta\dfrac{\mathrm{d}_{\mathrm{gr}} g(t)}{\mathrm{d}t}$ *for all* $t \in (a, b)$ *and* $\alpha, \beta \in \mathbb{R}$.

iii. $\dfrac{\mathrm{d}_{\mathrm{gr}}[f(t)g(t)]}{\mathrm{d}t}=g(t)\dfrac{\mathrm{d}_{\mathrm{gr}} f(t)}{\mathrm{d}t}+f(t)\dfrac{\mathrm{d}_{\mathrm{gr}} g(t)}{\mathrm{d}t}$ *for all* $t \in (a, b)$.

Example 2.3 Let $f(t) = z_1 \cos 3t + z_2 t$ be an $\mathcal{E}-$ valued function defined on the interval [0, 10], where $z_1 = 5 + 2U$ and $z_2 = -4 + U$ are two neutrosophic numbers with the indeterminate part $U = [0, 1]$. From Example 2.2, it is easy to see that the horizontal membership function of f, given by $\mathcal{L}(f(t)) = (5 + 2\mu_1) \cos 3t + (-4 + \mu_2)t$, is a differentiable function on [0, 10]. Moreover, its derivative is

$$\frac{\partial f^{\mathrm{gr}}(t,\mu_1,\mu_2)}{\partial t}=-3(5+2\mu_1)\sin 3t+(-4+\mu_2)\quad \text{for all } \mu_1,\mu_2\in[0,1].$$

Thus, we deduce that the function f is gr-differentiable on [0, 10]. In addition, thanks to (4), the gr-derivative of f can be given as follows

$$\begin{aligned}\frac{\mathrm{d}_{\mathrm{gr}} f(t)}{\mathrm{d}t}&=\mathcal{N}\left(\frac{\partial f^{\mathrm{gr}}(t,\mu_1,\mu_2)}{\partial t}\right)\\&=\left[\min_{\mu_1,\mu_2}\{(-15-6\mu_1)\sin 3t+\mu_2-4\},\ \max_{\mu_1,\mu_2}\{(-15-6\mu_1)\sin 3t+\mu_2-4\}\right]\\&=[-21,-15]\sin 3t+[-4,-3].\end{aligned}$$

Therefore, we obtain the gr-derivative $\dfrac{\mathrm{d}_{\mathrm{gr}} f(t)}{\mathrm{d}t}=(-21+6U)\sin 3t+(-4+U)$, whose graphical representation is shown in Fig. 2.

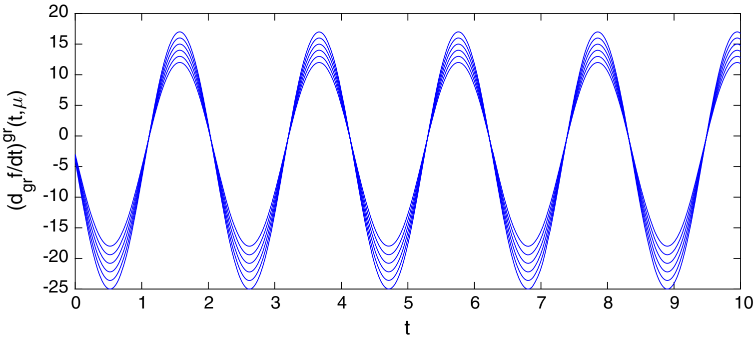

Fig. 2 The gr-derivative $\frac{\mathrm{d}_{\mathrm{gr}} f(t)}{\mathrm{d}t}$ of $\mathcal{E}$-valued function $f(t)$ on $[0, 10]$

Definition 2.9 Assume that $\Phi : [a, b] \to \mathcal{E}$ is a continuous $\mathcal{E}$-valued function and its horizontal membership function $\mathcal{L}(\Phi(t)) := \Phi(t, \overline{\mu})$ is integrable on $[a, b]$, i.e., there exists a number $\mathcal{I}(\overline{\mu}) \in \mathbb{R}$ such that $\mathcal{I}(\overline{\mu}) = \int_a^b \Phi(t, \overline{\mu})\mathrm{d}t$. Then, the neutrosophic number $\mathcal{I}$, obtained by the transformation $\mathcal{I} = \mathcal{N}(\mathcal{I}(\overline{\mu}))$, is said to be the granular integral (gr-integral) of function Φ on $[a, b]$ and denoted by $\mathcal{I} = \int_a^b \Phi(t)\mathrm{d}t$.

Remark 2.5 By analogous arguments as in Proposition 2.2, we can also prove that the granular integrability of neutrosophic number function f and the integrability of its horizontal membership function are equivalent.

Corollary 2.1 *Let $\Phi : [a, b] \to \mathcal{E}$ be an $\mathcal{E}$-valued function and $c \in [a, b]$. Then, if the function Φ is gr-integrable on $[a, b]$ then $\Phi(t)$ is also gr-integrable on each sub-interval $[a, c] \subseteq [a, b]$. Moreover, we have*

$$\int_a^b \Phi(t)\mathrm{d}t = \int_a^c \Phi(t)\mathrm{d}t + \int_c^b \Phi(t)\mathrm{d}t.$$

The following theorem is the extension of the fundamental result of real analysis that determines the form of anti-derivative of an $\mathcal{E}$-valued function.

Theorem 2.2 *Let Φ be a continuous $\mathcal{E}-$ valued function defined on $[a, b]$. Then, for each $t \in [a, b]$, the function Λ, given by $\Lambda(t) = \int_a^t \Phi(s)\mathrm{d}s$, is an anti-derivative of the function $\Phi(t)$.*

Proof Let $t_0 \in [a, b]$ be fixed. By the assumption that the function Φ is continuous at t_0, we deduce that $\forall \epsilon > 0$, $\exists \delta > 0$ such that $\forall t \in [a, b] : |t - t_0| < \delta$ then $\rho^{\mathrm{gr}}(\Phi(t), \Phi(t_0)) < \epsilon$. Next, for h is a number sufficiently near 0, we consider the following quotient

$$\frac{\Delta\Lambda}{\Delta t}\Big|_{t=t_0} = \frac{1}{h}\left[\Lambda(t_0 + h) \ominus^{\mathrm{gr}} \Lambda(t_0)\right] = \frac{1}{h}\left[\int_a^{t_0+h} \Phi(s)\mathrm{d}s \ominus^{\mathrm{gr}} \int_a^{t_0} \Phi(s)\mathrm{d}s\right].$$

Using horizontal membership function approach, we obtain

$$\mathcal{L}\left(\frac{\Delta\Lambda}{\Delta t}\bigg|_{t=t_0}\right) = \frac{1}{h}\left[\int_a^{t_0+h} \Phi^{\mathrm{gr}}(s_0+h,\mu)\mathrm{d}s - \int_a^{t_0} \Phi^{\mathrm{gr}}(s_0,\mu)\mathrm{d}s\right]$$
$$= \frac{1}{h}\int_{t_0}^{t_0+h} \Phi^{\mathrm{gr}}(s_0+h,\mu)\mathrm{d}s.$$

Next, by the use of mean value theorem, we get

$$\mathcal{L}\left(\frac{\Delta\Lambda}{\Delta t}\bigg|_{t=t_0}\right) = \frac{1}{h}\int_{t_0}^{t_0+h} \Phi^{\mathrm{gr}}(s_0+h,\mu)\mathrm{d}s = \Phi^{\mathrm{gr}}(t_0+\tau h,\mu),$$

where $\tau \in (0, 1)$. Here, note that $t_0 + \tau h$ tends to t_0 as $h \to 0$. Thus, it implies that

$$\mathcal{L}\left(\frac{\mathrm{d}_{\mathrm{gr}}\Lambda(t_0)}{\mathrm{d}t}\right) = \lim_{h\to 0}\mathcal{L}\left(\frac{\Delta\Lambda}{\Delta t}\bigg|_{t=t_0}\right) = \lim_{h\to 0}\Phi^{\mathrm{gr}}(t_0+\tau h,\mu) = \Phi^{\mathrm{gr}}(t_0,\mu),$$

which follows $\dfrac{\mathrm{d}_{\mathrm{gr}}\Lambda(t_0)}{\mathrm{d}t} = \Phi(t_0)$. Additionally, since $t_0 \in [a, b]$ is chosen arbitrarily, the proof is complete. □

Corollary 2.2 *As a consequence of Theorem 2.2, we have that*

$$\frac{\mathrm{d}_{\mathrm{gr}}}{\mathrm{d}t}\left(\int_a^t \Phi(s)\mathrm{d}s\right) = \Phi(t), \quad t \in [a, b].$$

Theorem 2.3 *(Newton–Leibniz formula) Assume that $\phi : [a, b] \subseteq \mathbb{R} \to \mathcal{E}$ is gr-differentiable on $[a, b]$ and the function $\Phi(t) := \dfrac{\mathrm{d}_{\mathrm{gr}}\phi(t)}{\mathrm{d}t}$ is continuous on this interval. Then Φ is gr-integrable and*

$$\int_a^b \Phi(t)\mathrm{d}t = \phi(b) \ominus^{\mathrm{gr}} \phi(a).$$

Proof As a result of Theorem 2.2, function $\Lambda(t) = \int_a^t \Phi(s)\mathrm{d}s$ is an anti-derivative of the function f on $[a, b]$ and its granular representation is

$$\Lambda^{\mathrm{gr}}(t,\mu_\Lambda) = \int_a^t \Phi^{\mathrm{gr}}(s,\mu_\Phi)\mathrm{d}s, \quad t \in [a, b].$$

This expression means that $\Lambda^{\mathrm{gr}}(t,\mu_\Lambda)$ is an anti-derivative of the function $\Phi^{\mathrm{gr}}(t,\mu_\Phi)$ on $[a, b]$. Thus, if $\phi^{\mathrm{gr}}(t,\mu_\phi)$ is another anti-derivative of $\Phi^{\mathrm{gr}}(t,\mu_\Phi)$ on $[a, b]$ then

$$\phi^{\mathrm{gr}}(t,\mu_\phi) = \Lambda^{\mathrm{gr}}(t,\mu_\Lambda) + C = \int_a^t \Phi^{\mathrm{gr}}(s,\mu_\Phi)ds + C, \quad t \in [a, b] \tag{8}$$

where C is a real constant. Next, by substituting $t = a$ into the equality (8), we obtain that $C = \phi^{\mathrm{gr}}(a, \mu_\phi)$. Then, the integral equality (8) becomes

$$\phi^{\mathrm{gr}}(t, \mu_\phi) = \int_a^t \Phi^{\mathrm{gr}}(s, \mu_\Phi)\mathrm{d}s + \phi^{\mathrm{gr}}(a, \mu_\phi),$$

or equivalently, $\int_a^t \Phi^{\mathrm{gr}}(s, \mu_\Phi)\mathrm{d}s = \phi^{\mathrm{gr}}(t, \mu_\phi) - \phi^{\mathrm{gr}}(a, \mu_\phi)$. Let $t = b$ then we immediately get

$$\int_a^b \Phi^{\mathrm{gr}}(s, \mu_\Phi)\mathrm{d}s = \phi^{\mathrm{gr}}(b, \mu_\phi) - \phi^{\mathrm{gr}}(a, \mu_\phi),$$

Finally, by using the transformation (4), we can see that

$$\int_a^b \mathcal{N}\left(\Phi^{gr}(s, \mu_\Phi)\right)\mathrm{d}s = \mathcal{N}\left(\phi^{gr}(b, \mu_\phi) - \phi^{gr}(a, \mu_\phi)\right),$$

that means the integral equality $\int_a^b \Phi(t)\mathrm{d}t = \phi(b) \ominus^{gr} \phi(a)$ holds. □

Example 2.4 Let $\Lambda(t) = z_1 e^{-t} + z_2 \cos 2t$ be an $\mathcal{E}$-valued function defined on the interval $[0, 2\pi]$, where $z_1 = 4 + U$, $z_2 = -6 + U \in \mathcal{E}$. Then, the horizontal membership function of the function $\Lambda(t)$ is given by

$$\Lambda^{\mathrm{gr}}(t, \mu_1, \mu_2) = z_1^{\mathrm{gr}}(\mu_1)e^{-t} + z_2^{\mathrm{gr}}(\mu_2)\cos 2t = (4 + \mu_1)\, e^{-t} + (-6 + \mu_2)\cos 2t.$$

By similar method as in Example 2.3, we can prove that the function $\Lambda(t)$ is gr-differentiable on $[0, 2\pi]$ and its derivative is denoted by $f(t)$ whose granular representation is

$$f^{gr}(t, \mu_1, \mu_2) = \frac{\partial \Lambda^{\mathrm{gr}}(t, \mu_1, \mu_2)}{\partial t} = (-4 - \mu_1)\, e^{-t} + (12 - 2\mu_2)\sin 2t,$$

where $\mu_1, \mu_2 \in [0, 1]$. Then, by employing the transformation (4), we get that

$$f(t) = (-5 + U)\, e^{-t} + (10 + 2U)\sin 2t.$$

In addition, it is easy to prove that the function f is continuous on $[0, 2\pi]$. Then, we can see that all assumptions of Theorem 2.3 are fulfilled. Hence, we immediately obtain that

$$\int_0^{2\pi} f(t)\mathrm{d}t = \Lambda(7) \ominus^{gr} \Lambda(0) = \left(z_1 e^{-2\pi} + z_2\right) \ominus^{gr} (z_1 + z_2) = 5(e^{-2\pi} - 1) + \left(1 - e^{-2\pi}\right) U.$$

The graphical representation of two functions $\Lambda(t)$ and $f(t)$ is shown in Fig. 3.

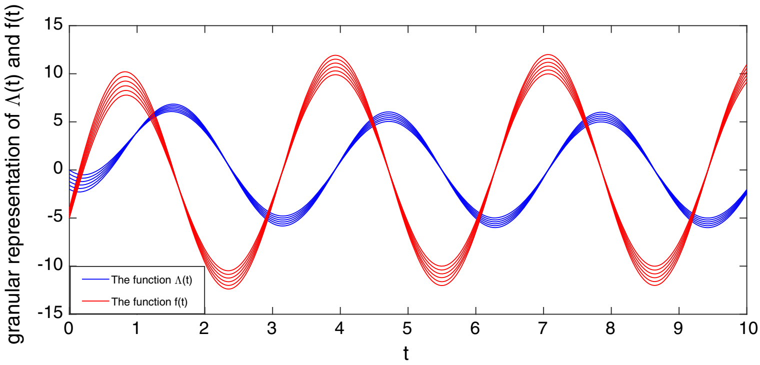

Fig. 3 Representation of the $\mathcal{E}$-valued functions $\Lambda(t)$ and $f(t)$

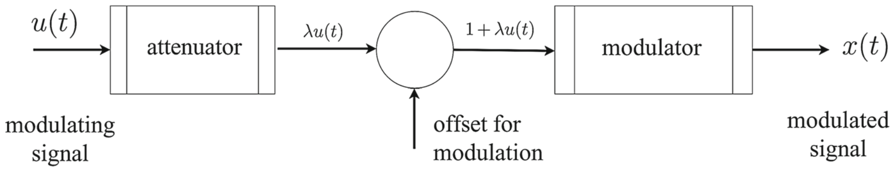

Fig. 4 The amplitude modulation system

Example 2.5 Consider an information-bearing signal $u(t)$ applied as an input to an AM system referred to as an amplitude modulator. In communications, the input $u(t)$ to a modulator is called the modulating signal, while its output $x(t)$ is called the modulated signal. The steps involved in an amplitude modulator are illustrated in Fig. 4, where the modulating signal $u(t)$ is first processed by attenuating it by a factor $\lambda = 0.3$ and adding a DC offset such that the resulting signal $1 + \lambda u(t)$ is positive for all time t. The modulated signal is produced by multiplying the processed input signal $1 + \lambda u(t)$ with a high-frequency carrier $f(t) = A \cos\left(2\pi \omega_f t\right)$. Multiplication by a sinusoidal wave of frequency ω_f shifts the frequency content of the modulating signal $u(t)$ by an additive factor of ω_f. The amplitude modulated signal $x(t)$ is mathematically expressed as follows

$$x(t) = A\left[1 + \lambda u(t)\right] \cos\left(2\pi \omega_f t\right), \qquad t > 0, \tag{9}$$

where A and ω_f are the amplitude and frequency of the sinusoidal carrier, respectively. It should be noted that the amplitude A and frequency ω_f of the carrier signal along with the attenuation factor λ used in the modulator are fixed and thus, Eq. (9) provides a direct relationship between the input and the output signals of an amplitude modulator. However, rather than the particular value, we may have only the vague, imprecise and incomplete information about the amplitude A and frequency ω_f being a result of errors in measurement, observations, experiment, or it may be maintenance-induced errors, which are uncertain in nature. Therefore, to overcome these uncertainties and vagueness, based on approximate realistic measurements, one may present these uncertain values as following neutrosophic numbers

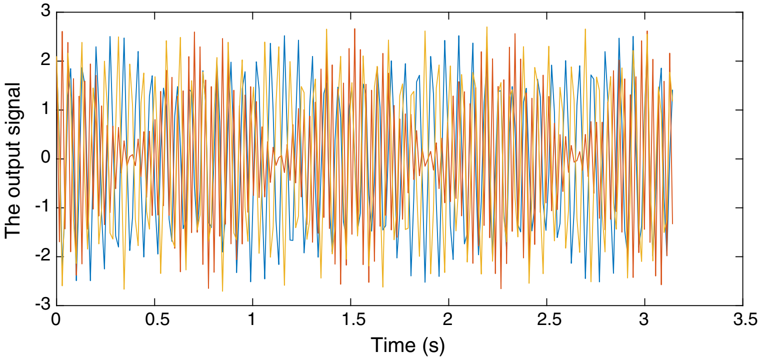

Fig. 5 The amplitude modulated signal $x(t)$ on [0, 10]

$$A = 2 + 0.1U \qquad \omega_f = (1.5 + 0.2U) \times 10^8.$$

In addition, let us assume that the input signal $u(t)$ is given by $u(t) = \sin(10^8 \pi t)$. Then, based on the horizontal membership function approach, the signal Eq. (9) can be represented as follows

$$x^{gr}(t, \mu) = (2 + 0.1\mu_1)\left[1 + 0.3u^{gr}(t, \mu)\right] \cos\left(2 \times 10^8 \pi (1.5 + 0.2\mu_2)t\right), \quad \mu_1, \mu_2 \in [0, 1].$$

The graphical representation of the signal $x(t)$ is given in Fig. 5.

2.3 Neutrosophic Matrices

Definition 2.10 [37] A neutrosophic matrix A of order $m \times n$ is defined as

$$A = \begin{pmatrix} z_{11} & z_{12} & \dots & z_{1n} \\ z_{21} & z_{22} & \dots & z_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ z_{m1} & z_{m2} & \dots & z_{mn} \end{pmatrix}$$

where z_{ij} is a given neutrosophic number for each $i = \overline{1, m}$, $j = \overline{1, n}$. In special case, if $m = n$ then the matrix A is called a square neutrosophic matrix of order n. In addition, the gr-representation of A can be given by

$$A_{gr}(\mu_A) = \begin{pmatrix} z_{11}^{gr}(\mu_{11}) & z_{12}^{gr}(\mu_{12}) & \dots & z_{1n}^{gr}(\mu_{1n}) \\ z_{21}^{gr}(\mu_{21}) & z_{22}^{gr}(\mu_{22}) & \dots & z_{2n}^{gr}(\mu_{2n}) \\ \vdots & \vdots & \ddots & \vdots \\ z_{n1}^{gr}(\mu_{n1}) & z_{n2}^{gr}(\mu_{n2}) & \dots & z_{nn}^{gr}(\mu_{nn}) \end{pmatrix},$$

for each $\mu_{ij} \in [0, 1]$.

Remark 2.6 Based on arithmetic operations in $\mathcal{E}$ introduced in Definition 2.3 and the classical matrix operations (e.g., matrix addition–subtraction, scalar multiplication, matrix transpose, matrix inverse, and so on), we can perform the neutrosophic matrix operations.

Example 2.6 Let A and B be two neutrosophic matrices given as follows

$$A = \begin{pmatrix} 3+U & 2 \\ -1+U & 5 \end{pmatrix} \qquad \text{and} \qquad B = \begin{pmatrix} 2+U & 3 \\ 0 & 3+U \end{pmatrix},$$

with the indeterminacy $U = [0, 1]$. Then, by gr-representation, we can rewrite these matrices in the form

$$A_{gr}(\mu_A) = \begin{pmatrix} 3+\mu_1 & 2 \\ -1+\mu_2 & 5 \end{pmatrix} \qquad\qquad B_{gr}(\mu_B) = \begin{pmatrix} 2+\mu_3 & 3 \\ 0 & 3+\mu_4 \end{pmatrix},$$

where $\mu_i \in [0, 1]$ $(i = \overline{1, 4})$. Then, we can introduce the following matrix operations:

- *Matrix addition* The granular representation of matrices $A + B$ is

$$(A+B)_{gr}(\mu_{A+B}) = A_{gr}(\mu_A) + B_{gr}(\mu_B) = \begin{pmatrix} 5+\mu_1+\mu_3 & 5 \\ -1+\mu_2 & 8+3\mu_4 \end{pmatrix}.$$

  By using the transformation (4), we obtain

$$\mathcal{N}\left(A_{gr}(\mu_A) + B_{gr}(\mu_B)\right)$$
$$= \left(\begin{matrix} \left[\min\limits_{\mu_1,\mu_3} (5+\mu_1+\mu_3), \max\limits_{\mu_1,\mu_3} (5+\mu_1+\mu_3)\right] & 5 \\ \left[\min\limits_{\mu_2,\mu_5} (-1+\mu_2), \max\limits_{\mu_2,\mu_5} (-1+\mu_2)\right] & \left[\min\limits_{\mu_4} (8+3\mu_4), \max\limits_{\mu_4} (8+3\mu_4)\right] \end{matrix}\right).$$

  Therefore, the matrix $A + B = \begin{pmatrix} 5+2U & 5 \\ -1+U & 8+3U \end{pmatrix}$, where $U = [0, 1]$ is the indeterminacy.

- *Matrix subtraction* The granular representation of the gr-difference $A \ominus^{\mathrm{gr}} B$ can be given as follows

$$(A \ominus^{\mathrm{gr}} B)_{\mathrm{gr}}(\mu_{A\ominus^{gr}B}) = A_{\mathrm{gr}}(\mu_A) - B_{gr}(\mu_B) = \begin{pmatrix} 1+\mu_1-\mu_3 & -1 \\ -1+\mu_2 & 2-\mu_4 \end{pmatrix}.$$

  By using the transformation (4), we obtain

$$\mathcal{N}\left(A_{\mathrm{gr}}(\mu_A) - B_{gr}(\mu_B)\right)$$
$$= \left(\begin{matrix} \left[\min\limits_{\mu_1,\mu_3} (1+\mu_1-\mu_3), \max\limits_{\mu_1,\mu_3} (1+\mu_1-\mu_3)\right] & -1 \\ \left[\min\limits_{\mu_2} (-1+\mu_2), \max\limits_{\mu_2} (-1+\mu_2)\right] & \left[\min\limits_{\mu_4} (2-\mu_4), \max\limits_{\mu_4} (2-\mu_4)\right] \end{matrix}\right).$$

Therefore, the gr-difference matrices $A \ominus^{\text{gr}} B = \begin{pmatrix} 2U & -1 \\ -1+U & 1+U \end{pmatrix}$ with the indeterminacy $U = [0, 1]$.

- *Scalar multiplication* The gr-representation of scalar multiplication of A with $\lambda = 2$ is given by

$$(2A)_{gr}(\mu_{2A}) = 2A_{\text{gr}}(\mu_A) = \begin{pmatrix} 6+2\mu_1 & 4 \\ -2+2\mu_2 & 10 \end{pmatrix}.$$

Then, by using the transformation (4), we obtain

$$\mathcal{N}\left(2A_{gr}(\mu_A)\right) = \begin{pmatrix} \left[\min\limits_{\mu_1}(6+2\mu_1), \max\limits_{\mu_1}(6+2\mu_1)\right] & 4 \\ \left[\min\limits_{\mu_2}(-2+2\mu_2), \max\limits_{\mu_2}(-2+2\mu_2)\right] & 10 \end{pmatrix}.$$

Therefore, the matrix $2A = \begin{pmatrix} 6+2U & 4 \\ -2+2U & 10 \end{pmatrix}$ with the indeterminacy $U = [0, 1]$.

- *Multiplying matrices* The gr-representation of the product of A and B can be given by

$$\begin{aligned} (AB)_{\text{gr}}(\mu_{AB}) &= A_{\text{gr}}(\mu_A)B_{\text{gr}}(\mu_B) \\ &= \begin{pmatrix} \mu_1\mu_3 + 2\mu_1 + 3\mu_3 + 6 & 3\mu_1 + 2\mu_4 + 15 \\ \mu_2\mu_3 + 2\mu_2 - \mu_3 - 2 & 3\mu_2 + 5\mu_4 + 12 \end{pmatrix} \\ &= \begin{pmatrix} \Phi_1(\mu_1, \mu_3) & \Phi_2(\mu_1, \mu_4) \\ \Phi_3(\mu_2, \mu_3) & \Phi_4(\mu_2, \mu_4) \end{pmatrix}. \end{aligned}$$

Then, by using the transformation (4), we obtain that

$$\begin{aligned} &\mathcal{N}(A_{\text{gr}}(\mu_A)B_{\text{gr}}(\mu_B)) \\ &= \begin{pmatrix} \left[\min\limits_{\mu_1,\mu_3} \Phi_1(\mu_1, \mu_3), \max\limits_{\mu_1,\mu_3} \Phi_1(\mu_1, \mu_3)\right] & \left[\min\limits_{\mu_1,\mu_4} \Phi_2(\mu_1, \mu_4), \max\limits_{\mu_1,\mu_4} \Phi_2(\mu_1, \mu_4)\right] \\ \left[\min\limits_{\mu_2,\mu_3} \Phi_3(\mu_2, \mu_3), \max\limits_{\mu_2,\mu_3} \Phi_3(\mu_2, \mu_3)\right] & \left[\min\limits_{\mu_2,\mu_4} \Phi_4(\mu_2, \mu_4), \max\limits_{\mu_2,\mu_4} \Phi_4(\mu_2, \mu_4)\right] \end{pmatrix}. \end{aligned}$$

Therefore, the multiplication of two matrices $AB = \begin{pmatrix} 6+6U & 15+5U \\ -3+4U & 12+8U \end{pmatrix}$ with $U = [0, 1]$.

- *Matrix transpose* The granular representation of transpose matrix A^{T} is

$$(A^{\text{T}})_{\text{gr}}(\mu_{A^{\text{T}}}) = (A_{\text{gr}}(\mu_A))^{\text{T}} = \begin{pmatrix} 3+\mu_1 & -1+\mu_2 \\ 2 & 5 \end{pmatrix}.$$

Then, by using the transformation (4), we obtain that

$$\mathcal{N}\left(\left(A_{\mathrm{gr}}(\mu_A)\right)^{\mathrm{T}}\right)=\begin{pmatrix}\left[\min\limits_{\mu_1}(3+\mu_1),\max\limits_{\mu_1}(3+\mu_1)\right] & \left[\min\limits_{\mu_2}(-1+\mu_2),\max\limits_{\mu_2}(-1+\mu_2)\right]\\ 2 & 5\end{pmatrix}.$$

Therefore, the transpose matrix $A^{\mathrm{T}}=\begin{pmatrix}3+U & -1+U\\ 2 & 5\end{pmatrix}$ where $U=[0,1]$ is the indeterminacy.

- *Matrix inverse* The granular representation of inverse matrix A^{-1} is

$$(A^{-1})_{\mathrm{gr}}(\mu_{A^{-1}})=(A_{\mathrm{gr}}(\mu_A))^{-1}=\begin{pmatrix}\frac{5}{17+5\mu_1-2\mu_2} & \frac{-2}{17+5\mu_1-2\mu_2}\\ \frac{1-\mu_2}{17+5\mu_1-2\mu_2} & \frac{3+\mu_1}{17+5\mu_1-2\mu_2}\end{pmatrix}.$$

Then, by using the transformation (4), we obtain that

$$\begin{aligned}&\mathcal{N}\left((A_{\mathrm{gr}}(\mu_A))^{-1}\right)\\ &=\begin{pmatrix}\left[\min\limits_{\mu_1,\mu_2}\left(\frac{5}{17+5\mu_1-2\mu_2}\right),\max\limits_{\mu_1,\mu_2}\left(\frac{5}{17+5\mu_1-2\mu_2}\right)\right] & \left[\min\limits_{\mu_1,\mu_2}\left(\frac{-2}{17+5\mu_1-2\mu_2}\right),\max\limits_{\mu_1,\mu_2}\left(\frac{-2}{17+5\mu_1-2\mu_2}\right)\right]\\ \left[\min\limits_{\mu_1,\mu_2}\left(\frac{1-\mu_2}{17+5\mu_1-2\mu_2}\right),\max\limits_{\mu_1,\mu_2}\left(\frac{1-\mu_2}{17+5\mu_1-2\mu_2}\right)\right] & \left[\min\limits_{\mu_1,\mu_2}\left(\frac{3+\mu_1}{17+5\mu_1-2\mu_2}\right),\max\limits_{\mu_1,\mu_2}\left(\frac{3+\mu_1}{17+5\mu_1-2\mu_2}\right)\right]\end{pmatrix}.\end{aligned}$$

Therefore, the inverse matrix $A^{-1}=\begin{pmatrix}\frac{5}{22}+\frac{7}{66}U & -\frac{2}{15}+\frac{7}{165}U\\ \frac{1}{17}U & \frac{3}{17}+\frac{2}{85}U\end{pmatrix}$ with indeterminacy $U=[0,1]$.

Remark 2.7 We can see that some results of matrix operations on the set of neutrosophic matrices such as the subtraction, matrices multiplication or matrix inverse are quite different from classical results.

Definition 2.11 Let $A=\left[z_{ij}\right]_{n\times n}$ be a square neutrosophic matrix of order n. We call a number $\lambda\in\mathcal{E}$ an eigenvalue of A if and only if $\det\left(\lambda^{gr}(\mu)\mathtt{I}_n-A_{gr}(\mu_A)\right)=0$, where $\det(\cdot)$ and $\mathtt{I}_n$ represent for the determinant and the $n\times n$ identity matrix, respectively.

Next, we define the concept of exponential matrix e^{tA}, $t>0$ for the class of neutrosophic matrix A, which can be considered as a key for presenting solution of LTI neutrosophic differential systems.

Definition 2.12 (exponential matrix) Consider a time-invariant linear system

$$\begin{cases}\dfrac{\mathrm{d_{gr}}x(t)}{\mathrm{d}t} &= Ax(t), \qquad t\geq 0\\ x(0) &= x_0.\end{cases}\tag{10}$$

The exponential matrix of the system (10) is defined as

$$e^{tA} := \sum_{k=0}^{\infty} \frac{1}{k!}(tA)^k = \mathtt{I}_n + (tA) + \frac{1}{2!}(tA)^2 + \frac{1}{3!}(tA)^3 + \ldots + \frac{1}{n!}(tA)^n + \cdots \tag{11}$$

Some common properties of exponential matrix still hold for the case of neutrosophic matrix

Proposition 2.4 *The exponential matrix e^{tA} has following properties*

i. $\frac{\mathrm{d_{gr}}}{\mathrm{d}t}\left(e^{tA}\right) = Ae^{tA}$.
ii. $e^{0A} = \mathtt{I}_n$.
iii. $e^{(t+s)A} = e^{tA}e^{sA}$.
iv. $\left(e^{tA}\right)^{-1} = e^{-tA}$.

Proof Firstly, we can see that the gr-representation of the exponential matrix e^{tA} can be given by

$$e^{tA_{gr}}(\mu_{exp}) = \mathtt{I}_n^{gr}(\mu_{\mathtt{I}}) + tA_{gr}(\mu) + \frac{1}{2!}t^2\left(A_{gr}(\mu)\right)^2 + \cdots + \frac{1}{n!}t^n\left(A_{gr}(\mu)\right)^n + \cdots$$

where $\mu \in [0, 1]$. Therefore, we have following results

i. The gr-derivative of the exponential matrix $e^{tA_{\mathrm{gr}}}(\mu_{\mathrm{exp}})$ can be computed as

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\left[e^{tA_{\mathrm{gr}}}(\mu_{\mathrm{exp}})\right] &= \frac{\mathrm{d}}{\mathrm{d}t}\left[\mathtt{I}_n^{\mathrm{gr}}(\mu_{\mathtt{I}}) + tA_{\mathrm{gr}}(\mu) + \frac{t^2}{2!}\left(A_{\mathrm{gr}}(\mu)\right)^2 + \cdots + \frac{t^n}{n!}\left(A_{\mathrm{gr}}(\mu)\right)^n + \cdots\right]\\ &= A_{\mathrm{gr}}(\mu)\left[\mathtt{I}_n^{gr}(\mu_{\mathtt{I}}) + tA_{gr}(\mu) + \cdots + \frac{t^{n-1}}{(n-1)!}\left(A_{gr}(\mu)\right)^n + \cdots\right]\\ &= A_{\mathrm{gr}}(\mu)e^{tA_{gr}}(\mu_{exp}).\end{aligned}$$

By using the transformation (4), the assertion (i) holds.

ii. For $t = 0$, we have

$$e^{0A_{gr}}(\mu_{exp}) = \mathtt{I}_n^{\mathrm{gr}}(\mu_{\mathtt{I}}) + 0A_{\mathrm{gr}}(\mu) + \frac{1}{2!}\left(0A_{\mathrm{gr}}(\mu)\right)^2 + \cdots + \frac{1}{n!}\left(0A_{\mathrm{gr}}(\mu)\right)^n + \cdots = \mathtt{I}_n^{gr}(\mu_{\mathtt{I}}).$$

Hence, we obtain that the exponential matrix $e^{0A} = \mathtt{I}_n$.

iii. By the formula (11), we have

$$e^{(t+s)A} = \mathtt{I}_n + (t+s)A + \frac{(t+s)^2}{2!}A^2 + \frac{(t+s)^3}{3!}A^3 + \cdots + \frac{(t+s)^n}{n!}A^n + \cdots$$

whose granular representation is given as

$$e^{(t+s)A_{\mathrm{gr}}}(\mu_{exp}) = \mathtt{I}_n^{\mathrm{gr}}(\mu_{\mathtt{I}}) + (t+s)A_{\mathrm{gr}}(\mu) + \frac{(t+s)^2}{2!}\left(A_{\mathrm{gr}}(\mu)\right)^2 + \cdots + \frac{(t+s)^n}{n!}\left(A_{gr}(\mu)\right)^n + \cdots$$

On the other hand,

$$\begin{aligned}
& e^{tA_{\mathrm{gr}}}(\mu_{\mathrm{exp}})e^{sA_{\mathrm{gr}}}(\mu_{\mathrm{exp}}) \\
&= \left[\mathtt{I}_n^{\mathrm{gr}}(\mu_{\mathtt{I}}) + tA_{\mathrm{gr}}(\mu) + \cdots + \frac{t^n}{n!}\left(A_{gr}(\mu)\right)^n + \cdots\right] \\
&\quad \left[\mathtt{I}_n^{\mathrm{gr}}(\mu_{\mathtt{I}}) + sA_{\mathrm{gr}}(\mu) + \cdots + \frac{s^n}{n!}\left(A_{\mathrm{gr}}(\mu)\right)^n + \cdots\right] \\
&= \mathtt{I}_n^{\mathrm{gr}}(\mu_{\mathtt{I}}) + (t+s)A_{\mathrm{gr}}(\mu) + \left(t^2 + ts + s^2\right)\left(A_{gr}(\mu)\right)^2 + \cdots \\
&\quad + \left(\frac{t^n}{n!} + \frac{t^{n-1}s}{(n-1)!} + \cdots + \frac{t^{n-k}s^k}{(n-k)!k!} + \cdots + \frac{ts^n}{(n-1)!} + \frac{s^n}{n!}\right)\left(A_{\mathrm{gr}}(\mu)\right)^n + \cdots \\
&= \mathtt{I}_n^{\mathrm{gr}}(\mu_{\mathtt{I}}) + (t+s)A_{gr}(\mu) + \frac{(t+s)^2}{2!}\left(A_{\mathrm{gr}}(\mu)\right)^2 + \cdots + \frac{(t+s)^n}{n!}\left(A_{\mathrm{gr}}(\mu)\right)^n + \cdots \\
&= e^{(t+s)A_{\mathrm{gr}}}(\mu_{\mathrm{exp}}).
\end{aligned}$$

iv. From the assertion (ii) and (iii), we have

$$\mathtt{I}_n = e^{0A} = e^{(t-t)A} = e^{(t+(-t))A} = e^{tA}e^{-tA},$$

which implies the assertion (iv) is fulfilled. □

Proposition 2.5 *The system* (10) *can be represented as*

$$x(t) = e^{tA}x_0. \tag{12}$$

Proof Firstly, we prove that (12) satisfies the state equation. Indeed, by Proposition 2.4 (i), we have

$$\frac{\mathrm{d_{gr}}x(t)}{\mathrm{d}t} = \frac{\mathrm{d_{gr}}}{\mathrm{d}t}\left(e^{tA}x_0\right) = Ae^{tA}x_0 = Ax(t).$$

In addition, from the assertion (ii) of Proposition 2.4, we have

$$x(0) = e^{0A}x_0 = \mathtt{I}_n x_0 = x_0,$$

that means $x(t)$ satisfies the initial condition $x(0) = x_0$. □

Corollary 2.3 *Consider the linear system* $\frac{\mathrm{d_{gr}}x(t)}{\mathrm{d}t} = Ax(t) + Bu(t)$ *subject to the initial condition* $x(0) = x_0$ *and the input* $u(t)$, $t \geq 0$. *As a consequence of Proposition* 2.5, *it follows that the response is given by*

$$x(t) = e^{tA}x_0 + \int_0^t e^{(t-s)A}Bu(s)|rmds. \tag{13}$$

Proof Indeed, it is easy to see that $x(t)$ given by (13) satisfies the initial condition

$$x(0) = e^{0A}x_0 + \int_0^0 e^{(t-s)A}Bu(s)\mathrm{d}s = x_0.$$

Thanks to Corollary 2.2, we also have

$$\begin{aligned}
\frac{\mathrm{d}_{\mathrm{gr}}x(t)}{\mathrm{d}t} &= \frac{\mathrm{d}_{gr}}{\mathrm{d}t}\left(e^{tA}x_0\right) + \frac{\mathrm{d}_{gr}}{\mathrm{d}t}\left(\int_0^t e^{(t-s)A}Bu(s)\mathrm{d}s\right) \\
&= Ae^{tA}x_0 + Ae^{(t-t)A}Bu(t) + \int_0^t Ae^{(t-s)A}Bu(s)\mathrm{d}s \\
&= A\left(e^{tA}x_0 + A\int_0^t e^{(t-s)A}Bu(s)\mathrm{d}s\right) + Bu(t) \\
&= Ax(t) + Bu(t),
\end{aligned}$$

that means $x(t)$ given by (13) satisfies the state equation $\frac{\mathrm{d}_{\mathrm{gr}}x(t)}{\mathrm{d}t} = Ax(t) + Bu(t)$. □

Remark 2.8 To compute the exponential matrix $e^{tA_{gr}}(\mu_{exp})$, we need to find some more effective methods and one of them is the use of Laplace transformation, which is considered as one of the most effective methods. It is not difficult to see that the Laplace transformation of $e^{tA_{gr}}(\mu_{exp})$, denoted by $\mathtt{L}[e^{tA_{gr}}(\mu_{exp})]$, is given as follows

$$\mathtt{L}[e^{tA_{\mathrm{gr}}}(\mu_{\mathrm{exp}})] = \left(\lambda^{\mathrm{gr}}(\mu)\mathtt{I}_n - A_{\mathrm{gr}}(\mu)\right)^{-1}.$$

Hence, the matrix $e^{tA_{\mathrm{gr}}}(\mu_{\mathrm{exp}})$ can be obtained by using inverse Laplace transform

$$\mathtt{L}^{-1}\left[\left(\lambda^{\mathrm{gr}}(\mu)\mathtt{I}_n - A_{\mathrm{gr}}(\mu)\right)^{-1}\right].$$

Example 2.7 Consider a neutrosophic matrix

$$A = \begin{pmatrix} 3 & 0 \\ 0 & 2+U \end{pmatrix}$$

whose gr-representation is $A_{\mathrm{gr}}(\mu_A) = \begin{pmatrix} 3 & 0 \\ 0 & 2+\mu \end{pmatrix}$ for all $\mu \in [0, 1]$.
We can see that the inverse matrix of $\lambda^{\mathrm{gr}}(\mu)\mathtt{I}_n - A_{\mathrm{gr}}(\mu_A)$ is

$$\left(\lambda^{\mathrm{gr}}(\mu)\mathtt{I}_n^{\mathrm{gr}}(\mu) - A_{\mathrm{gr}}(\mu)\right)^{-1} = \begin{pmatrix} \frac{1}{\lambda^{gr}(\mu)-3} & 0 \\ 0 & \frac{1}{\lambda^{gr}(\mu)-(2+\mu)} \end{pmatrix}.$$

Then, by using inverse Laplace transform, the exponential matrix $e^{tA_{\text{gr}}}(\mu_{\text{exp}})$ can be calculated by

$$e^{tA_{\text{gr}}}(\mu_{\text{exp}}) = \mathtt{L}^{-1}\left[\left(\lambda^{\text{gr}}(\mu)\mathtt{I}_n - A_{\text{gr}}(\mu)\right)^{-1}\right] = \begin{pmatrix} e^{3t} & 0 \\ 0 & e^{(2+\mu)t} \end{pmatrix}.$$

Therefore, we obtain $e^{tA} = \begin{pmatrix} e^{3t} & 0 \\ 0 & e^{(2+U)t} \end{pmatrix}$.

Example 2.8 Let us find the output response $y(t)$ of the following neutrosophic LTI system with the initial condition $x(0) = \begin{pmatrix} 1 \\ 1+U \end{pmatrix}$ and the control input $u(t) = \pi + U$.

$$\begin{cases} \dfrac{\mathrm{d}_{\text{gr}}x(t)}{\mathrm{d}t} & = \begin{pmatrix} 0 & 1 \\ -3 & -4 \end{pmatrix} x(t) + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u(t) \\ y(t) & = \begin{pmatrix} 1 & 0 \end{pmatrix} x(t), \end{cases}$$

where the indeterminate part $U = [0, 1]$. Here, for simplicity, let us denote

$$A = \begin{pmatrix} 0 & 1 \\ -3 & -4 \end{pmatrix}, \qquad B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \qquad C = \begin{pmatrix} 1 & 0 \end{pmatrix}.$$

By similar arguments as in Example 2.7, we have

$$e^{tA_{gr}}(\mu) = \begin{pmatrix} \frac{3}{2}e^{-t} - \frac{1}{2}e^{-3t} & -\frac{3}{2}e^{-t} + \frac{3}{2}e^{-3t} \\ \frac{1}{2}e^{-t} - \frac{1}{2}e^{-3t} & \frac{3}{2}e^{-3t} - \frac{1}{2}e^{-t} \end{pmatrix}.$$

Hence, we have

$$\begin{aligned} x^{gr}(t,\mu) &= e^{tA_{\text{gr}}}(\mu)x_0^{gr}(\mu) + \int_0^t e^{sA_{\text{gr}}}(\mu)B_{gr}(\mu)u^{gr}(t-s,\mu)\mathrm{d}s \\ &= \begin{pmatrix} \frac{2e^{-3t}+3(e^{-3t}-e^{-t})\mu}{2} \\ \frac{2e^{-3t}+(3e^{-3t}-e^{-t})\mu}{2} \end{pmatrix} + (\pi+\mu)\int_0^t \begin{pmatrix} \frac{3e^{-3s}-3e^{-s}}{2} \\ \frac{3e^{-3s}-e^{-s}}{2} \end{pmatrix} \mathrm{d}s \\ &= \begin{pmatrix} \frac{2e^{-3t}+3(e^{-3t}-e^{-t})\mu}{2} \\ \frac{2e^{-3t}+(3e^{-3t}-e^{-t})\mu}{2} \end{pmatrix} + (\pi+\mu)\begin{pmatrix} \frac{3e^{-s}-e^{-3s}}{2} \\ \frac{e^{-s}-e^{-3s}}{2} \end{pmatrix}\Bigg|_0^t \\ &= \begin{pmatrix} e^{-3t} + \frac{\pi}{2}\left(3e^{-t} - e^{-3t}\right) + \left(e^{-3t}+1\right)\mu \\ e^{-3t} + \frac{\pi}{2}\left(e^{-t} - e^{-3t}\right) + e^{-3t}\mu \end{pmatrix}. \end{aligned}$$

Then, we obtain that $y^{gr}(t,\mu)$ can be calculated as

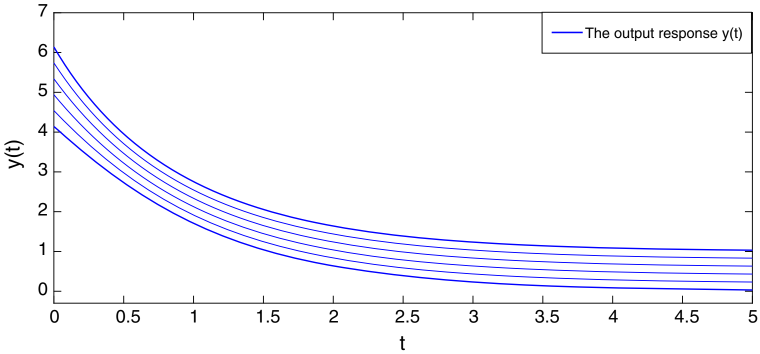

Fig. 6 The graphical representation of $y(t)$ with the indeterminacy $U = [0, 1]$

$$y^{\mathrm{gr}}(t,\mu) = \begin{pmatrix}1 & 0\end{pmatrix} \begin{pmatrix} e^{-3t} + \frac{\pi}{2}\left(3e^{-t} - e^{-3t}\right) + \left(e^{-3t}+1\right)\mu \\ e^{-3t} + \frac{\pi}{2}\left(e^{-t} - e^{-3t}\right) + e^{-3t}\mu \end{pmatrix}$$
$$= e^{-3t} + \frac{\pi}{2}\left(3e^{-t} - e^{-3t}\right) + \left(e^{-3t}+1\right)\mu.$$

By the transformation (4), the output response $y(t) = e^{-3t} + \frac{\pi}{2}\left(3e^{-t} - e^{-3t}\right) + \left(e^{-3t}+1\right)U$. The graphical representation of the output response $y(t)$ is shown in Fig. 6.

3 Main Results

Consider a linear time-invariant neutrosophic control system

$$\begin{cases} \frac{\mathrm{d}_{\mathrm{gr}}x(t)}{\mathrm{d}t} &= Ax(t) + Bu(t) \\ y(t) &= Cx(t), \end{cases} \qquad t \geq 0, \tag{14}$$

where $A \in \mathrm{Mat}_{n\times n}(\mathcal{E})$, $B \in \mathrm{Mat}_{n\times m}(\mathcal{E})$, $C \in \mathrm{Mat}_{p\times n}(\mathcal{E})$ are given neutrosophic matrices, $\frac{\mathrm{d}_{\mathrm{gr}}x(t)}{\mathrm{d}t}$ stands for the gr-derivative of state vector $x(t)$ and $u(t)$ is the control input. By gr-representation approach, the corresponding granular LTI system of the system (14) is

$$\begin{cases} \frac{\partial x^{\mathrm{gr}}(t,\mu)}{\partial t} &= A_{\mathrm{gr}}(\mu)x^{\mathrm{gr}}(t,\mu) + B_{\mathrm{gr}}(\mu)u^{\mathrm{gr}}(t,\mu) \\ y^{\mathrm{gr}}(t,\mu) &= C_{\mathrm{gr}}(\mu)x^{\mathrm{gr}}(t,\mu), \end{cases} \tag{15}$$

where $\mu \in [0, 1]$ and $t \geq 0$.

3.1 The Controllability for Linear Time-Invariant Neutrosophic Systems

Definition 3.1 The LTI neutrosophic system (14) is called controllable if for every state $x_1 \in \mathcal{E}^n$ and for every $\Delta > 0$, there exists an input control $u(t)$, $t \in [0, \Delta]$ such that under the control $u(t)$, the state vector $x(t)$ can be steered from the initial state x_0 to the final state x_1 at the time $t = \Delta$. In particular,

- If for every $\Delta > 0$, there exists an input control $u(t)$, $t \in [0, \Delta]$ such that under this input, the state vector $x(t)$ can be steered from the initial state $x_0 \neq 0$ to the origin $x_1 = 0$ at the time $t = \Delta$ then we say that the system (14) is null controllable.
- If for every $\Delta > 0$, there exists an input control $u(t)$, $t \in [0, \Delta]$ such that under this input, the state vector $x(t)$ can be steered from the origin $x_0 = 0$ to an other state $x_1 \neq 0$ at the time $t = \Delta$ then we say that the system (14) is reachable.

Remark 3.1 The LTI neutrosophic system (14) is controllable (reachable, null controllable) if and only if its granular linear time-invariant system (15) is controllable (reachable, null controllable) for all $\mu \in [0, 1]$.

Remark 3.2 To check the controllability of the system (14), let this system start at the initial state $x(0) = x_0$ and then, by the integral formula (13), we investigate its state response at the time $t = \Delta$:

$$x(\Delta) = e^{A\Delta}x(0) + \int_0^{\Delta} e^{(\Delta-\tau)A}Bu(\tau)d\tau = e^{A\Delta}x(0) + \int_0^{\Delta} e^{\tau A}Bu(\Delta-\tau)d\tau$$

whose granular representation is as

$$x^{gr}(\Delta, \mu) = e^{A_{gr}\Delta}(\mu)x^{gr}(0, \mu) + \int_0^{\Delta} e^{\tau A_{gr}}(\mu)B_{gr}(\mu)u^{gr}(\Delta-\tau, \mu)d\tau \tag{16}$$

for each $\mu \in [0, 1]$. It can be seen that the state response only depends on the matrices A, B. Hence, when we discuss the controllability of the system, we only need discuss the controllability of the pair (A, B).

Definition 3.2 For each input control $u(t)$ and $\Delta > 0$, let the system start at $x(0) = 0$ and then, we have the zero-state response at $t = \Delta$ is given as follows

$$x_u(\Delta, 0) = \int_0^{\Delta} e^{\tau A}Bu(\Delta-\tau)\mathrm{d}\tau.$$

Then, we say that a state $x_0 \neq 0$ is uncontrollable if it is orthogonal to the state $x_u(\Delta, 0)$, for all $\Delta > 0$ and for all control input $u(t)$, $t \in [0, \Delta]$, that is $\left(x_0^{gr}(\mu)\right)^{\mathrm{T}} x_u^{gr}(\Delta, 0, \mu) = 0$ for all $\Delta > 0$ and for all $\mu \in [0, 1]$.

Next, we give a necessary and sufficient condition for the controllability of LTI neutrosophic system (14).

Theorem 3.1 *The LTI neutrosophic system* (14) *is controllable if and only if it has no uncontrollable state.*

Proof As a consequence of Remark 3.1, we only need to prove that the corresponding granular LTI system of the system (14) is controllable for all $\mu \in [0, 1]$ if and only if it has no uncontrollable state.

The necessary condition If the granular linear time-invariant system $\left(A_{\mathrm{gr}}(\mu), B_{\mathrm{gr}}(\mu)\right)$ has no uncontrollable state then we consider following matrix

$$\Pi(\Delta) = \int_0^{\Delta} e^{\tau A_{\mathrm{gr}}}(\mu) B_{gr}(\mu) \left[B_{gr}(\mu)\right]^{\mathrm{T}} \left[e^{\tau A_{gr}}(\mu)\right]^{\mathrm{T}} \mathrm{d}\tau$$

for each $\mu \in [0, 1]$. Our aim is to prove that the matrix $\Pi(\Delta)$ is positively defined for all $\mu \in [0, 1]$. Otherwise, we assume by contrary that there exists $x_0^{\mathrm{gr}}(\mu) \neq 0$ such that

$$\begin{aligned} &\left[x_0^{\mathrm{gr}}(\mu)\right]^{\mathrm{T}} \left(\int_0^{\Delta} e^{\tau A_{\mathrm{gr}}}(\mu) B_{gr}(\mu) \left[B_{\mathrm{gr}}(\mu)\right]^{\mathrm{T}} \left[e^{\tau A_{\mathrm{gr}}}(\mu)\right]^{\mathrm{T}} \mathrm{d}\tau\right) x_0^{\mathrm{gr}}(\mu) = 0 \\ \Rightarrow &\int_0^{\Delta} \left[x_0^{\mathrm{gr}}(\mu)\right]^{\mathrm{T}} e^{\tau A_{\mathrm{gr}}}(\mu) B_{gr}(\mu) \left(\left[x_0^{gr}(\mu)\right]^{\mathrm{T}} e^{\tau A_{\mathrm{gr}}}(\mu) B_{\mathrm{gr}}(\mu)\right)^{\mathrm{T}} \mathrm{d}\tau = 0, \end{aligned}$$

that means $\left[x_0^{gr}(\mu)\right]^{\mathrm{T}} e^{\tau A_{gr}}(\mu) B_{gr}(\mu) = 0$ for a.e $t \in [0, \Delta]$. Thus, it follows

$$\left[x_0^{gr}(\mu)\right]^{\mathrm{T}} \int_0^{\Delta} e^{\tau A_{gr}}(\mu) B_{gr}(\mu) u^{gr}(\Delta - \tau, \mu) d\tau = 0,$$

which means $\left[x_0^{gr}(\mu)\right]^{\mathrm{T}} x_u^{gr}(\Delta, 0, \mu) = 0$ for all $\mu \in [0, 1]$. This leads to a contradiction. Hence, we have $\Pi(\Delta)$ is positive-defined and so, $\Pi(\Delta)^{-1}$ exists.

Now, for every states $x_0^{gr}(\mu)$, $x_1^{gr}(\mu)$ and for $\Delta > 0$, consider the control input $u^{gr}(t, \mu)$

$$u^{\mathrm{gr}}(\Delta - \tau, \mu) = \left[e^{\tau A_{\mathrm{gr}}}(\mu) B_{\mathrm{gr}}(\mu)\right]^{\mathrm{T}} \Pi(\Delta)^{-1} \left(x_1^{\mathrm{gr}}(\mu) - e^{(\Delta-\tau)A_{\mathrm{gr}}}(\mu) x_0^{\mathrm{gr}}(\mu)\right).$$

By this control input, the state of the system $\left(A_{gr}(\mu_A), B_{\mathrm{gr}}(\mu_B)\right)$ is steered from the initial state $x_0^{\mathrm{gr}}(\mu)$ to the final state $x_1^{\mathrm{gr}}(\mu)$ at the time Δ. Indeed, we have

$$\begin{aligned} x^{\mathrm{gr}}(\Delta, \mu) = &\, e^{(\Delta-\tau)A_{gr}}(\mu) x_0^{\mathrm{gr}}(\mu) + \\ &\int_0^{\Delta} e^{\tau A_{gr}}(\mu) B_{gr}(\mu) \left[e^{\tau A_{gr}}(\mu) B_{gr}(\mu)\right]^{\mathrm{T}} \Pi(\Delta)^{-1} \left(x_1^{gr}(\mu) - e^{(\Delta-\tau)A_{gr}}(\mu) x_0^{gr}(\mu)\right) \mathrm{d}\tau \\ = &\, e^{(\Delta-\tau)A_{gr}}(\mu) x_0^{gr}(\mu) + \\ &\left(\int_0^{\Delta} e^{\tau A_{gr}}(\mu) B_{gr}(\mu) \left[e^{\tau A_{gr}}(\mu) B_{gr}(\mu)\right]^{\mathrm{T}} d\tau\right) \Pi(\Delta)^{-1} \left(x_1^{gr}(\mu) - e^{(\Delta-\tau)A_{gr}}(\mu) x_0^{gr}(\mu)\right) \\ = &\, e^{(\Delta-\tau)A_{\mathrm{gr}}}(\mu) x_0^{gr}(\mu) + \Pi(\Delta)\Pi(\Delta)^{-1} \left(x_1^{gr}(\mu) - e^{(\Delta-\tau)A_{gr}}(\mu) x_0^{gr}(\mu)\right) = x_1^{gr}(\mu). \end{aligned}$$

Hence, the system is controllable.

The sufficient condition If there exists a state $x_0 \neq 0$ such that for all $\Delta > 0$ and for all control $u(t)$, $t \in [0, \Delta]$, the state x_0 is orthogonal to the zero-state response

$x_u(\Delta, 0)$, that is $\left(x_0^{\mathrm{gr}}(\mu)\right)^{\mathrm{T}} x_u^{\mathrm{gr}}(\Delta, 0, \mu) = 0$ for all $\Delta > 0$, $\mu \in [0, 1]$ and for all control $u(t)$, then it is easy to see that there has no control that can steer the state $x^{gr}(t, \mu)$ from the initial state $x^{gr}(0, \mu) = 0$ to the final state $x_0^{gr}(\mu)$ in time Δ. This implies the system (15) is not controllable, and thus, the system (14) is also uncontrollable. □

By Theorem 3.1, checking the controllability of the system (14) is equivalent to the existence of an uncontrollable state $\overline{x} \neq 0$, which can be shown by following theorem

Theorem 3.2 *A state* $\overline{x} \neq 0$ *is an uncontrollable state if and only if*

$$\overline{x}^{\mathrm{T}} \left(B \;\; AB \;\; A^2B \;\cdots\; A^{n-1}B\right) = 0.$$

Proof For this proof, we need to show that the state $\overline{x}^{\mathrm{gr}}(\mu)$ satisfies

$$\left(\overline{x}^{\mathrm{gr}}(\mu)\right)^{\mathrm{T}} \left(B_{gr}(\mu) \;\; A_{\mathrm{gr}}(\mu)B_{gr}(\mu) \;\; A_{gr}^2(\mu)B_{gr}(\mu) \;\cdots\; A_{gr}^{n-1}(\mu)B_{gr}(\mu)\right) = 0, \quad \mu \in [0, 1].$$

Indeed, let us consider the zero-state response $x_u^{gr}(\Delta, 0, \mu)$ of the system (14)

$$x_u^{gr}(\Delta, 0, \mu) = \int_0^{\Delta} e^{sA_{gr}}(\mu)B_{gr}(\mu)u^{gr}(\Delta - s, \mu)\mathrm{d}s, \qquad \mu \in [0, 1].$$

Additionally, by Definition 3.2, a state $\overline{x}^{gr}(\mu) \neq 0$ is uncontrollable if and only if

$$\left(\overline{x}^{gr}(\mu)\right)^{\mathrm{T}} x_u^{gr}(\Delta, 0, \mu) = 0 \text{ for all } \Delta > 0,\; \mu \in [0, 1],$$

and for all control $u(t)$. Here, we can see that

$$\begin{aligned}\left(\overline{x}^{gr}(\mu)\right)^{\mathrm{T}} x_u^{gr}(\Delta, 0, \mu) &= \left(\overline{x}^{gr}(\mu)\right)^{\mathrm{T}} \left(\int_0^{\Delta} e^{\tau A_{gr}}(\mu)B_{gr}(\mu)u^{gr}(\Delta - \tau, \mu)d\tau\right)\\ &= \int_0^{\Delta} \left(\overline{x}^{gr}(\mu)\right)^{\mathrm{T}} e^{\tau A_{gr}}(\mu)B_{gr}(\mu)u^{gr}(\Delta - \tau, \mu)\mathrm{d}\tau,\end{aligned}$$

for all $\Delta > 0$, $\mu \in [0, 1]$ and for all control $u(t)$. It follows that $\left(\overline{x}^{\mathrm{gr}}(\mu)\right)^{\mathrm{T}} e^{\tau A_{gr}}(\mu)$ $B_{gr}(\mu) = 0$ for all $\Delta > 0$, $\tau \in [0, \Delta]$ and $\mu \in [0, 1]$. Therefore, we have

$$B_{gr}^{\mathrm{T}}(\mu)e^{\tau A_{gr}^{\mathrm{T}}}(\mu)\overline{x}^{\mathrm{gr}}(\mu) = 0,$$

or equivalently,

$$\left(\overline{x}^{\mathrm{gr}}(\mu)\right)^{\mathrm{T}} \left(B_{\mathrm{gr}}(\mu) \;\; A_{\mathrm{gr}}(\mu)B_{gr}(\mu) \;\; A_{\mathrm{gr}}^2(\mu)B_{\mathrm{gr}}(\mu) \;\cdots\; A_{gr}^{n-1}(\mu)B_{gr}(\mu)\right) = 0,$$

for all $\mu \in [0, 1]$. Then, by using the transformation (4), we obtain

$$\overline{x}^{\mathrm{T}} \left(B \;\; AB \;\; A^2B \;\cdots\; A^{n-1}B\right) = 0.$$

Therefore, the proof is complete. □

Corollary 3.1 *We can conclude that (A, B) is controllable if and only if for all state $\overline{x} \in \mathcal{E}^n \backslash \{\overline{0}\}$, we have*

$$\left(\overline{x}^{\mathrm{gr}}(\mu)\right)^{\mathrm{T}} \left(B_{\mathrm{gr}}(\mu)\ A_{\mathrm{gr}}(\mu)B_{gr}(\mu)\ A_{gr}^2(\mu)B_{gr}(\mu)\ \cdots\ A_{gr}^{n-1}(\mu)B_{gr}(\mu)\right) \neq 0,$$

or equivalently,

$$rank\left(B_{gr}(\mu)\ A_{gr}(\mu)B_{gr}(\mu)\ A_{gr}^2(\mu)B_{gr}(\mu)\ \cdots\ A_{gr}^{n-1}(\mu)B_{gr}(\mu)\right) = n.$$

Thus, we define the controllability matrix by

$$\mathtt{K} := \left(B\ AB\ A^2B\ \cdots\ A^{n-1}B\right)$$

whose granular representation is given as

$$\mathtt{K}^{gr}(\mu) = \left(B_{gr}(\mu)\ A_{gr}(\mu)B_{gr}(\mu)\ A_{gr}^2(\mu)B_{gr}(\mu)\ \cdots\ A_{gr}^{n-1}(\mu)B_{gr}(\mu)\right), \quad \mu \in [0, 1].$$

Then, the system is controllable if and only if its granular controllability matrix is full row rank.

$$(A, B) \text{ is controllable } \Leftrightarrow rank\left(\mathtt{K}^{\mathrm{gr}}(\mu)\right) = n \tag{17}$$

3.2 The Stabilizability for Linear Time-Invariant Neutrosophic Systems

Consider a neutrosophic system

$$\frac{\mathrm{d}_{\mathrm{gr}}x(t)}{\mathrm{d}t} = A\left(x(t)\right), \tag{18}$$

where $x(t) \in \mathcal{E}^n$ is the state variables and $A : \mathcal{E}^n \to \mathcal{E}^n$ is an $\mathcal{E}$-valued function.

Definition 3.3 A state $x_* \in \mathcal{E}^n$ is said to be an equilibrium point of the system (18) if

$$A(x_*) = \hat{0},$$

that means $x_*^{\mathrm{gr}}(\mu)$ is a solution of the system $A_{gr}(x_*^{\mathrm{gr}}(\mu), \mu) = 0$ for all $\mu \in [0, 1]$.

Remark 3.3 Without loss of generality, we can assume that $x_* \equiv \hat{0}$ is an equilibrium of the neutrosophic system (18), that is $A(x_*) = \hat{0}$, or equivalently, $A_{\mathrm{gr}}(0, \mu) = 0$ for all $\mu \in [0, 1]$.

Definition 3.4 An equilibrium point $x_0 \equiv \hat{0}$ of the system (18) is said to be

(i) *stable* if for all $\epsilon > 0$, there exists $\delta(\epsilon) > 0$ such that for all solution $x(t)$ of the system (18) satisfying $\rho^{\mathrm{gr}}(x(0), \hat{0}) < \delta(\epsilon)$, it follows that $\rho^{\mathrm{gr}}(x(t), \hat{0}) < \epsilon$.

(ii) *asymptotically stable* if it is stable and there exists $c > 0$ such that if $x(t)$ is a solution of the system (18) satisfying $\rho^{gr}(x_0) < c$, $\lim_{t\to\infty} \rho^{gr}(x(t), \hat{0}) = 0$.

Example 3.1 Consider following Liénard neutrosophic differential equation

$$\frac{\mathrm{d}_{\mathrm{gr}}^2 x(t)}{\mathrm{d}t^2} = (-4+U)\frac{\mathrm{d}_{\mathrm{gr}} x(t)}{\mathrm{d}t} + (-3+U)x(t), \tag{19}$$

with the initial conditions $x(0) = 1$ and $\frac{\mathrm{d}_{\mathrm{gr}} x(0)}{\mathrm{d}t} = 0$.

Then, by denoting $x_1(t) = x(t)$ and $x_2(t) = \frac{\mathrm{d}_{\mathrm{gr}} x(t)}{\mathrm{d}t} + x(t)$, Eq. (19) can be transformed into following linear time-invariant neutrosophic system

$$\begin{pmatrix} \frac{\mathrm{d}_{\mathrm{gr}} x_1(t)}{\mathrm{d}t} \\ \frac{\mathrm{d}_{\mathrm{gr}} x_2(t)}{\mathrm{d}t} \end{pmatrix} = \begin{pmatrix} -1 & 1 \\ 0 & -3+U \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} \tag{20}$$

with initial condition $\begin{pmatrix} x_1(0) \\ x_2(0) \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ and the indeterminacy $U = [0, 1]$. For simplicity, we denote

$$X(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}, \qquad A = \begin{pmatrix} -1 & 1 \\ 0 & -3+U \end{pmatrix}.$$

Then, the given LTI neutrosophic system becomes $\frac{\mathrm{d}_{\mathrm{gr}} X(t)}{\mathrm{d}t} = AX(t)$ and it is easy to see that the state $X_e = \begin{pmatrix} 0 & 0 \end{pmatrix}^{\mathrm{T}}$ is an equilibrium point of (20). Now, we will prove that X_e is asymptotically stable. Indeed, the respective granular linear time-invariant system of the system (20)

$$\begin{pmatrix} \dfrac{\partial x_1^{\mathrm{gr}}(t,\mu)}{\partial t} \\ \dfrac{\partial x_2^{\mathrm{gr}}(t,\mu)}{\partial t} \end{pmatrix} = \begin{pmatrix} -1 & 1 \\ 0 & -3+\mu \end{pmatrix} \begin{pmatrix} x_1^{gr}(t,\mu) \\ x_2^{gr}(t,\mu) \end{pmatrix} \tag{21}$$

has the characteristic equation $(\lambda^{\mathrm{gr}}(\mu))^2 + (4-\mu)\lambda^{\mathrm{gr}}(\mu) + (3-\mu) = 0$ for each $\mu \in [0, 1]$. Then, the eigenvalues of the above system are

$$\lambda_1^{\mathrm{gr}}(\mu) = -1 \qquad \text{and} \qquad \lambda_2^{\mathrm{gr}}(\mu) = -3+\mu,$$

corresponding to the eigenvectors $v_1^{\mathrm{gr}}(\mu) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}^{\mathrm{T}}$ and $v_2^{\mathrm{gr}}(\mu) = \begin{pmatrix} 1 \\ \mu-2 \end{pmatrix}^{\mathrm{T}}$ for all $\mu \in [0, 1]$. Thus, we obtain the solution of granular LTI system (21) satisfying initial condition $\left(x_1^{gr}(0,\mu)\ x_2^{gr}(0,\mu)\right) = \begin{pmatrix} 0 & 1 \end{pmatrix}$ is

$$\begin{cases} x_1^{gr}(t) = \frac{1}{2-\mu}e^{-t} + \frac{1}{\mu-2}e^{(-3+\mu)t} \\ x_2^{gr}(t) = e^{(-3+\mu)t}. \end{cases}$$

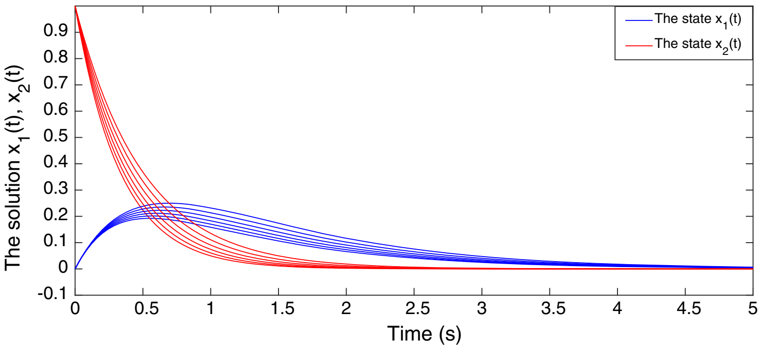

Fig. 7 The asymptotic behavior of the solution $[x_1(t)\ x_2(t)]^{\mathrm{T}}$ in Example 3.1

Hence, we can conclude that the solution X_e is asymptotically stable. The asymptotic behavior of the solution X_e is presented by Fig. 7.

Definition 3.5 The LTI neutrosophic system (14) is called stabilizable by a state-feedback control if there exists a neutrosophic matrix $P \in \mathrm{Mat}_{m\times n}(\mathcal{E})$ such that the corresponding closed-loop system

$$\begin{cases} \dfrac{\mathrm{d_{gr}}x(t)}{\mathrm{d}t} & = (A + BP)x(t), \quad t > 0, \\ x(0) & = x_0 \end{cases}$$

is asymptotically stable.

Remark 3.4 Stabilizability is related to both stability and controllability. It is well-known that if a system is stable then it is stabilizable, while if this system is controllable then we can use a state-feedback control to steer its eigenvalues to any locations in the s-plane. Hence, it follows that we can always use a state-feedback control to stabilize a controllable system, i.e., this state-feedback control will move the eigenvalues to the open left half of the s-plane. However, if the system is not controllable then we may not be always able to stabilizable it by using a state-feedback (Fig. 8).

Definition 3.6 Let $A \in \mathrm{Mat}_{n\times n}(\mathcal{E})$ and λ be an eigenvalue of the matrix A. Then,

(i) An eigenvalue λ is said to be unstable if $\mathrm{Re}\lambda \geq 0$.
(ii) An eigenvalue λ is not controllable if it cannot be moved by a state feedback.

Definition 3.7 The linear time-invariant neutrosophic system (14) is said to be stabilizable if all its unstable eigenvalues are controllable.

Here, the following theorem is used to discuss the stabilizability of LTI neutrosophic system (14). It is well-known that checking stabilizability is too complex, which we first need to find all eigenvalues of the system and then determine if they are controllable or not.

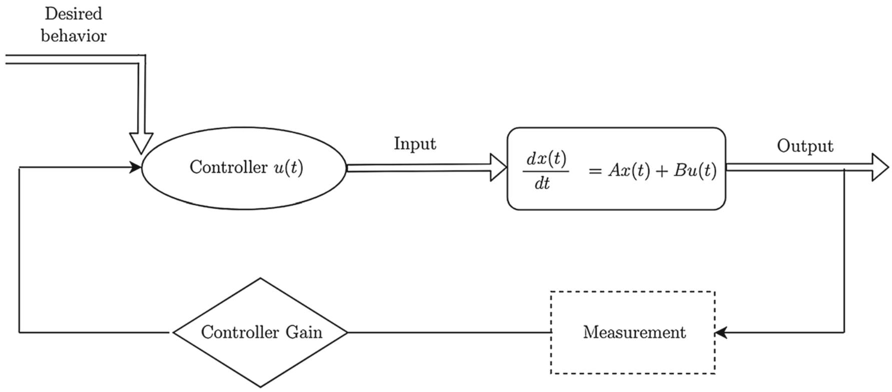

Fig. 8 Feedback control system with the gain controller P

Theorem 3.3 *Assume that λ_i is an eigenvalue of neutrosophic matrix A. Then, the eigenvalue $\lambda_i \in \lambda(A)$ is controllable if and only if its corresponding eigenvector v_i satisfies the condition $v_i^{\mathrm{T}} B \neq \hat{0}$.*

Proof Let $\lambda_i^{gr}(\mu_i)$ and $v_i^{gr}(\mu_i)$ be eigenvalue and eigenvector of the granular LTI system (15), respectively. We will prove that the controllability of the eigenvalue $\lambda_i^{gr}(\mu_i)$ is equivalent to $v_i^{\mathrm{gr}}(\mu_i) B_{\mathrm{gr}}(\mu) \neq 0$ for all $\mu, \mu_i \in [0, 1]$. Here, we only consider the case when all eigenvalues $\lambda_1^{\mathrm{gr}}(\mu_1), \lambda_2^{gr}(\mu_2), \ldots, \lambda_n^{gr}(\mu_n)$ are distinct real numbers. It is well-known that there exists a transformation matrix $\mathrm{S}^{\mathrm{gr}}(\mu)$ such that

$$\tilde{A}_{\mathrm{gr}}(\mu) = \left(\mathrm{S}^{gr}(\mu)\right)^{-1} A_{gr}(\mu) \mathrm{S}^{gr}(\mu) = \begin{pmatrix} \lambda_1^{gr}(\mu_1) & 0 & \cdots & 0 \\ 0 & \lambda_2^{gr}(\mu_2) & \cdots & 0 \\ \cdots & & \ddots & \cdots \\ 0 & \cdots & \cdots & \lambda_n^{gr}(\mu_n) \end{pmatrix}, \tag{22}$$

and

$$\tilde{B}_{\mathrm{gr}}(\mu) = \left(\mathrm{S}^{\mathrm{gr}}(\mu)\right)^{-1} B_{gr}(\mu), \quad \mu \in [0, 1].$$

On the other hand, it is well-known that the matrix $Q^{gr}(\mu) = \left(v_1^{\mathrm{gr}}(\mu_1)\ v_2^{\mathrm{gr}}(\mu_2) \cdots v_n^{\mathrm{gr}}(\mu_n)\right)$ is such that the matrix $(Q^{gr}(\mu))^{-1} \left(A_{gr}(\mu)\right)^{\mathrm{T}} Q^{gr}(\mu)$ has the diagonal form

$$\begin{pmatrix} \lambda_1^{\mathrm{gr}}(\mu_1) & 0 & \cdots & 0 \\ 0 & \lambda_2^{\mathrm{gr}}(\mu_2) & \cdots & 0 \\ \cdots & & \ddots & \cdots \\ 0 & \cdots & \cdots & \lambda_n^{\mathrm{gr}}(\mu_n) \end{pmatrix}.$$

Then, by using transpose transformation, we obtain

$$\left(Q^{\mathrm{gr}}(\mu)\right)^{\mathrm{T}} A_{gr}(\mu)\left[\left(Q^{gr}(\mu)\right)^{-1}\right]^{\mathrm{T}} = \begin{pmatrix} \lambda_1^{gr}(\mu_1) & 0 & \cdots & 0 \\ 0 & \lambda_2^{gr}(\mu_2) & \cdots & 0 \\ \cdots & & \ddots & \cdots \\ 0 & \cdots & \cdots & \lambda_n^{gr}(\mu_n) \end{pmatrix}.$$

Thus, by choosing $(\mathrm{S}^{gr}(\mu))^{-1} = \left[Q^{gr}(\mu)\right]^{\mathrm{T}}$, the matrix equality (22) is satisfied. In addition, we can see that

$$\begin{aligned}\tilde{B}_{\mathrm{gr}}(\mu) &= \left(\mathrm{S}^{gr}(\mu)\right)^{-1} B_{\mathrm{gr}}(\mu) = \left[Q^{gr}(\mu)\right]^{\mathrm{T}} B_{\mathrm{gr}}(\mu) \\ &= \left(v_1^{\mathrm{gr}}(\mu_1)B_{gr}(\mu)\ v_2^{gr}(\mu_2)B_{gr}(\mu)\cdots v_n^{gr}(\mu_n)B_{gr}(\mu)\right).\end{aligned}$$

Since the fact that the matrix $\tilde{A}_{gr}(\mu)$ is of the Jordan canonical form, it follows that the eigenvalue $\lambda_i^{gr}(\mu_i)$ is controllable if and only if $v_i^{gr}(\mu)B_{gr}(\mu) \neq 0$ for all $\mu \in [0, 1]$. □

4 Numerical Examples

To illustrate our obtained result, let us consider following examples.

Example 4.1 Consider a mechanical water pump consisting of a tank, two valves V_1, V_2 and an electrical circuit to control the system. The operation of this system is based on the following principle: The electrical circuit controls the amount of water flowing into the pump through the valve V_1 while the valve V_2 works mechanically as the outlet. The rate of the outlet flow depends on the height of the water in the mechanical pump. A higher level of water exerts more pressure on the mechanical valve V_2, creating a wider opening in the valve, thus releasing water at a faster rate. As the level of water drops, the opening of the valve that narrows the outlet flow of water is reduced. (see Fig. 9).
Next, we will construct the mathematical model for the mechanical pump. Firstly, assume that the rate of flow $f_{\mathrm{in}} := f_{\mathrm{in}}(t)$ at the input of the pump is a function of input control voltage $v(t)$, that is $f_{\mathrm{in}} = \kappa v(t)$, where $\kappa > 0$ is called the linearity constant.

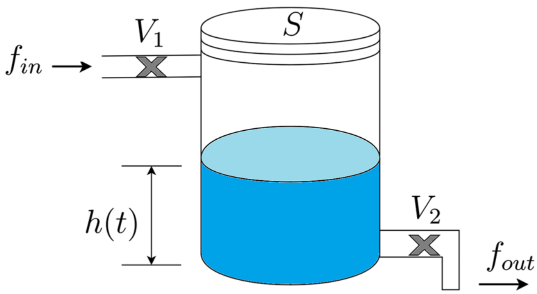

Fig. 9 The mechanical water pump

The valve V_2 is designed such that the outlet flow rate $f_{\text{out}} := f_{\text{out}}(t) = \lambda h(t)$ with $h(t)$ is the height of water level and $\lambda > 0$ denotes the outlet flow constant.

Let us denote $V(t)$ and S by the total volume of the water inside the tank and the cross-sectional area of the water tank, respectively, and assume that the initial height of water level is measured as $[h_0, h_0+\varepsilon]$, $\varepsilon > 0$, which means that we can consider the initial height of water level as a neutrosophic number $\tilde{h}_0 = h_0 + \varepsilon U$ with $U = [0, 1]$ is the indeterminacy. Then, we immediately obtain

$$\begin{cases} \dfrac{\mathrm{d}^{\mathrm{gr}} V(t)}{\mathrm{d}t} & = f_{\text{in}} - f_{\text{out}} = \kappa v(t) - \lambda h(t) \\ V(0) & = Ah(0). \end{cases}$$

Due to the fact that $V(t) = Sh(t)$, this system is equivalent to

$$\begin{cases} S\dfrac{\mathrm{d}^{\mathrm{gr}} h(t)}{\mathrm{d}t} & = \kappa v(t) - \lambda h(t) \\ h(0) & = \tilde{h}_0. \end{cases} \tag{23}$$

For simplicity, we denote $\alpha = \frac{\lambda}{S}$ and $\beta = \frac{\kappa}{S}$. Now, to solve the system (23), we multiply both sides of the differential equation of this system by $e^{\alpha t}$, and then by some fundamental computations, we get

$$\frac{\mathrm{d}^{\mathrm{gr}}}{\mathrm{d}t}\left(e^{\alpha t} h(t)\right) = \beta v(t).$$

By using Theorem 2.3, the solution of the system (23) is given as follows

$$h(t) = \tilde{h}_0 + \beta \int_0^t e^{-\alpha(t-\tau)} v(\tau)\mathrm{d}\tau, \quad t > 0 \tag{24}$$

In addition, by the formula (24), we can see that the mechanical water pump is controllable by the input control voltage $v(t)$ for all $t > 0$.

Remark 4.1 According to Theorem 3.3, it implies that for the stabilizability of the neutrosophic LTI differential system (14), we need to show that all of its unstable eigenvalues are controllable, that means the corresponding eigenvector v satisfies $v^{\mathrm{T}} B \neq 0$. Here, we will give an example to demonstrate this statement.

Example 4.2 Consider the following neutrosophic system

$$\frac{\mathrm{d}_{\mathrm{gr}} x(t)}{\mathrm{d}t} = \begin{pmatrix} -2 & 0 & 0 \\ 0 & 1+0.1U & 0 \\ 0 & 4 & -1 \end{pmatrix} x(t) + \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix} u(t), \tag{25}$$

where $x(t) = [x_1(t)\ x_2(t)\ x_3(t)]^{\mathrm{T}}$ is the state variables and the indeterminacy $U = [0, 1]$. By using the granular representation approach, we obtain following system

$$\frac{\partial x^{gr}(t,\mu)}{\partial t} = \begin{pmatrix} -2 & 0 & 0 \\ 0 & 1+0.1\mu & 0 \\ 0 & 4 & -1 \end{pmatrix} x^{\mathrm{gr}}(t,\mu) + \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix} u^{\mathrm{gr}}(t,\mu).$$

Here, we denote

$$A_{gr}(\mu) = \begin{pmatrix} -2 & 0 & 0 \\ 0 & 1+0.1\mu & 0 \\ 0 & 4 & -1 \end{pmatrix}, \qquad B_{\mathrm{gr}}(\mu) = \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix} \qquad (\mu \in [0, 1]).$$

Then, the system $\dfrac{\partial x^{\mathrm{gr}}(t,\mu)}{\partial t} = A_{gr}(\mu)x^{gr}(t,\mu) + A_{\mathrm{gr}}(\mu)u^{gr}(t,\mu)$ has the eigenvalues

$$\lambda_1^{\mathrm{gr}}(\mu) = -2, \qquad \lambda_2^{\mathrm{gr}}(\mu) = -1, \qquad \lambda_3^{\mathrm{gr}}(\mu) = 1 + 0.1\mu.$$

Moreover, the corresponding eigenvectors of $A_{gr}(\mu)$ are

$$v_1^{gr}(\mu) = \begin{pmatrix} 1\ 0\ 0 \end{pmatrix}^{\mathrm{T}}; \qquad v_2^{\mathrm{gr}}(\mu) = \begin{pmatrix} 0\ 0\ 1 \end{pmatrix}^{\mathrm{T}}; \qquad v_3^{\mathrm{gr}}(\mu) = \begin{pmatrix} 0\ 0.5 + 0.025\mu\ \ 1 \end{pmatrix}^{\mathrm{T}}.$$

By Definition 3.7 and Theorem 3.3, the system (25) is said to be stabilizable iff its unstable eigenvalue $\lambda_3^{gr}(\mu) = 1 + 0.1\mu$ is controllable, that means $\left(v_3^{gr}(\mu)\right)^{\mathrm{T}} B_{\mathrm{gr}}(\mu) \neq 0$ for all $\mu \in [0, 1]$. Indeed, we have

$$\left(v_3^{gr}(\mu)\right)^{\mathrm{T}} B_{gr}(\mu) = \begin{pmatrix} 0\ 0.5 + 0.025\mu\ \ 1 \end{pmatrix} \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix} = 2 + 0.05\mu \neq 0 \quad \text{for all } \mu \in [0, 1].$$

Hence, the system is stabilizable.

Remark 4.2 Next, in order to demonstrate the effectiveness of theoretical results, we will give a procedure to investigate controllable problems and stabilizable problems for neutrosophic LTI differential systems under granular computing:

Step 1. Transform the considered problem into mathematical model of the form (14);

Step 2. Convert the obtained neutrosophic LTI system (14) into the respective granular form (15);

Step 3. Employ the matrix criteria (17) to conclude the controllability of the obtained granular LTI systems. Moreover, if the considered system is controllable then it is also stabilizable by Remark 3.4.

Step 4. Moreover, we can determine the control $\hat{u}(\tau)$ of the form

$$\hat{u}(\tau) = -B^{\mathrm{T}} e^{(\Delta-\tau)A^{\mathrm{T}}} [\Pi(\Delta - \tau)]^{-1} \left(e^{\Delta A} x_0 \ominus^{gr} x_1\right)$$

that transfers the state x_0 into the state x_1 in time Δ.

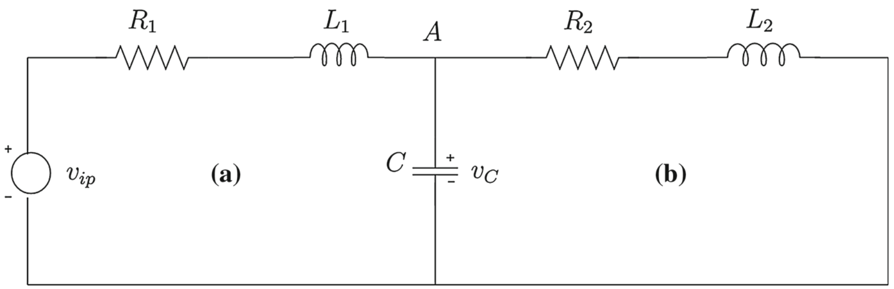

Fig. 10 The circuit diagram in Example 4.3

Table 2 Parameter values

L_1	0.01 H
L_2	0.02 H
C	$[2 \times 10^{-4}, 3 \times 10^{-4}]$ F
R_1	55 Ω
R_2	$[15, 16]\Omega$

Example 4.3 Consider a circuit diagram consisting of two resistors R_1, R_2, two inductors L_1, L_2, one capacitor C and one voltage source v_{ip} (see Fig. 10). The realistic values of circuit elements are in Table 2.

Here, the values of the resistor R_2 and the capacitor C are uncertain quantities given in form of neutrosophic numbers due to the errors in measurement and influence of environmental factors. In this case, neutrosophic number presentation has been considered as a better description in the formulation of this mathematical model. And, as a consequence, because of the uncertainty of R_2 and C, it follows that the matrix of state equations is neutrosophic matrix and thus, it also implies the uncertainty in the form of a solution.

In this example, we choose the currents i_1 and i_2 in inductors and the voltage v_C on capacitor be the state variables, respectively. Then, by applying Kirchhoff's voltage law to the loops (a) and (b), we obtain

$$
\begin{aligned}
v_{ip} &= R_1 i_1(t) + L_1 \frac{\mathrm{d}_{\mathrm{gr}} i_1(t)}{\mathrm{d}t} + v_C(t), \\
v_C(t) &= R_2 i_2(t) + L_2 \frac{\mathrm{d}_{\mathrm{gr}} i_2(t)}{\mathrm{d}t}.
\end{aligned}
$$

Moreover, by applying Kirchhoff's current law to node A, we immediately get that

$$
i_1(t) = i_2(t) + C \frac{\mathrm{d}_{\mathrm{gr}} v_C(t)}{\mathrm{d}t}.
$$

From three above equations, we have the following system of neutrosophic state equations

$$\begin{cases} \dfrac{\mathrm{d}_{\mathrm{gr}} i_1(t)}{\mathrm{d}t} & = \frac{1}{L_1}\left[v_{ip} \ominus^{gr} (R_1 i_1(t) + v_C(t))\right] \\ \dfrac{\mathrm{d}_{\mathrm{gr}} i_2(t)}{\mathrm{d}t} & = \frac{1}{L_2}\left(v_C(t) \ominus^{gr} R_2 i_2(t)\right) \\ \dfrac{\mathrm{d}_{\mathrm{gr}} v_C(t)}{\mathrm{d}t} & = \frac{1}{C}\left[i_1(t) \ominus^{gr} i_2(t)\right] \end{cases}$$

which has the granular representation as follows

$$\begin{cases} \dfrac{\partial i_1^{gr}(t,\mu)}{\partial t} = -\frac{R_1}{L_1} i_1^{gr}(t,\mu) - \frac{1}{L_1} v_C(t,\mu) + \frac{1}{L_1} v_{ip}(\mu) \\ \dfrac{\partial i_2^{gr}(t,\mu)}{\partial t} = -\frac{R_2}{L_2} i_2^{gr}(t,\mu) + \frac{1}{L_2} v_C(t,\mu) \\ \dfrac{\partial v_C^{gr}(t,\mu)}{\partial t} = \frac{1}{C} i_1^{gr}(t,\mu) - \frac{1}{C} i_2^{gr}(t,\mu) \end{cases} \tag{26}$$

The granular system (26) can be transformed into the following matrix form

$$\begin{pmatrix} \dfrac{\partial i_1^{gr}(t,\mu)}{\partial t} \\ \dfrac{\partial i_2^{gr}(t,\mu)}{\partial t} \\ \dfrac{\partial v_C^{gr}(t,\mu)}{\partial t} \end{pmatrix} = \begin{pmatrix} -\frac{R_1}{L_1} & 0 & -\frac{1}{L_1} \\ 0 & -\frac{R_2}{L_2} & \frac{1}{L_2} \\ \frac{1}{C} & -\frac{1}{C} & 0 \end{pmatrix} \begin{pmatrix} i_1^{gr}(t,\mu) \\ i_1^{gr}(t,\mu) \\ v_C^{gr}(t,\mu) \end{pmatrix} + \begin{pmatrix} \frac{1}{L_1} \\ 0 \\ 0 \end{pmatrix} v_{\mathrm{ip}}.$$

Assume that we can measure the value of the voltage v_{op} over the resistor R_2. Then, the output is

$$v_{\mathrm{op}} = \begin{pmatrix} 0 & R_2^{\mathrm{gr}}(\mu) & 0 \end{pmatrix} \begin{pmatrix} i_1^{gr}(t,\mu) \\ i_1^{gr}(t,\mu) \\ v_C^{gr}(t,\mu) \end{pmatrix}.$$

Next, by taking into account the system parameters, we obtain that

$$\begin{cases} \begin{pmatrix} \dfrac{\partial i_1^{gr}(t,\mu)}{\partial t} \\ \dfrac{\partial i_2^{gr}(t,\mu)}{\partial t} \\ \dfrac{\partial v_C^{gr}(t,\mu)}{\partial t} \end{pmatrix} = \begin{pmatrix} -5500 & 0 & -100 \\ 0 & -750 - 50\mu_1 & 50 \\ \frac{10{,}000}{2+\mu_2} & -\frac{10{,}000}{2+\mu_2} & 0 \end{pmatrix} \begin{pmatrix} i_1^{gr}(t,\mu) \\ i_1^{gr}(t,\mu) \\ v_C^{gr}(t,\mu) \end{pmatrix} + \begin{pmatrix} 100 \\ 0 \\ 0 \end{pmatrix} v_{\mathrm{ip}}, \\ v_{op} = \begin{pmatrix} 0 & 15 + \mu_1 & 0 \end{pmatrix} \begin{pmatrix} i_1^{gr}(t,\mu) \\ i_1^{gr}(t,\mu) \\ v_C^{gr}(t,\mu) \end{pmatrix}, \end{cases}$$

where $\mu \in [0, 1]$. Here, we denote

$$A_{gr}(\mu) = \begin{pmatrix} -5500 & 0 & -100 \\ 0 & -750 - 50\mu_1 & 50 \\ \dfrac{10{,}000}{2+\mu_2} & -\dfrac{10{,}000}{2+\mu_2} & 0 \end{pmatrix}; \quad B_{gr}(\mu) = \begin{pmatrix} 100 \\ 0 \\ 0 \end{pmatrix}; \quad C_{gr}(\mu) = \begin{pmatrix} 0 & 15+\mu_1 & 0 \end{pmatrix}.$$

Thus, we obtain the granular matrix $\mathtt{K}^{gr}(\mu)$, given by

$$\begin{aligned} \mathtt{K}^{gr}(\mu) &= \begin{pmatrix} B_{gr}(\mu) & A_{gr}(\mu)B_{gr}(\mu) & A_{gr}^2(\mu)B_{gr}(\mu) \end{pmatrix} \\ &= 100 \begin{pmatrix} 1 & -5500 & 10^4\left(3025 - \frac{10^2}{2+\mu_2}\right) \\ 0 & 0 & \frac{5\cdot 10^6}{2+\mu_2} \\ 0 & \frac{10^4}{2+\mu_2} & -\frac{55\cdot 10^6}{2+\mu_2} \end{pmatrix} \end{aligned}$$

has full row rank, that is $\mathrm{rank}(\mathtt{K}^{gr}(\mu)) = 3$ for all $\mu_1, \mu_2 \in [0, 1]$. Therefore, as a result of Theorem 3.2 and Corollary 3.1, we can conclude that the pair (A, B) is controllable. Moreover, due to Remark 3.4 and the controllability of (A, B), we also deduce that the considered system is stabilizable.

Example 4.4 An important actuator in control systems is DC motor system. The electric circuit of the armature and free-body diagram of the rotor is shown in Fig. 11. The physical parameters for our example are given in Table 3.

Here, we assume that the electromotive force constant is an uncertain quantity that can be represented as a neutrosophic number and the gear inertia and friction are

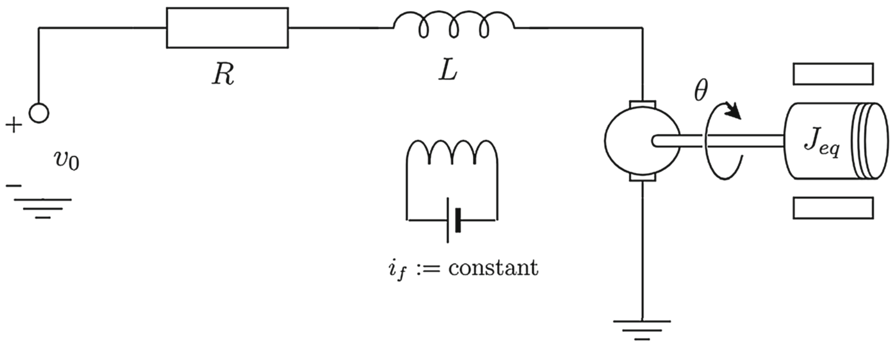

Fig. 11 The schematic of an armature-controlled DC motor system

Table 3 Parameter values

J_{eq}	Effective moment of inertia of the rotor	$0.1\,\mathrm{kg\,m^2}$
K_e	Electromotive force constant	$[5, 6]$ V/rad/s
K_t	Motor toque constant	5 N m/A
R	Electric resistance	1 Ω
L	Electric inductance	0 H

negligible. In addition, the electric inductance is not considered in the DC motor even though the current in the armature is AC in nature. The reasons for this phenomena are as follows:

- Since the load is fixed, the current magnitude through a coil-side remains fixed—though it changes the direction after 180° rotation. In this condition, the armature coils do not need to induce any emf in any other coil;
- In DC motor, the change in direction of current happens because of the physical change of terminals in the commutator. This change in direction creates opposite torque on the same conductor giving you motor rotation. The field is also supplied from DC having fixed current through it. Thus, the inductance does not play any role here.

Now, we will build the system of state equations of our model. First of all, by applying Kirchhoff's voltage law to the armature circuit, we obtain

$$L\frac{\mathrm{d}_{\mathrm{gr}}i(t)}{\mathrm{d}t}+i_aR+K_e\frac{\mathrm{d}_{gr}\theta(t)}{\mathrm{d}t}=v_0, \tag{27}$$

Moreover, the armature-controlled DC motor uses a constant field current, and thus, the motor toque is given as $\tau_m = K_t i_a$. In addition, since the motor torque equals the torque delivered to the load, we can see that

$$J_{\mathrm{eq}}\frac{\mathrm{d}^2_{gr}\theta(t)}{\mathrm{d}t^2}=K_t i_a. \tag{28}$$

Then, by substituting i_a in (27) into (28) and dividing both sides of this equation by J_{eq}, it yields

$$\frac{\mathrm{d}^2_{gr}\theta(t)}{\mathrm{d}t^2}=\frac{K_t}{J_{eq}}v_0 \ominus^{\mathrm{gr}} \frac{K_tK_e}{J_{eq}R}\cdot\frac{\mathrm{d}_{\mathrm{gr}}\theta(t)}{\mathrm{d}t} \tag{29}$$

By using horizontal membership function approach, the equation (29) becomes

$$\frac{\partial^2\theta^{gr}(t,\mu)}{\partial t^2}=\frac{K_t^{gr}(\mu)}{J_{eq}^{gr}(\mu)R^{gr}(\mu)}\left(v_0^{gr}(t,\mu)-K_e^{gr}(\mu)\frac{\partial\theta^{gr}(t,\mu)}{\partial t}\right).$$

Next, let $x_1^{gr}(\mu)=\theta^{gr}(t,\mu)$, $x_2^{gr}(\mu)=\dfrac{\partial\theta^{gr}(t,\mu)}{\partial t}$ and $u^{gr}(t,\mu)=v_0^{gr}(t,\mu)$ be the state variables and input control, respectively. Then, by taking into account the system's parameters, the above equation can be rewritten in following granular state-space form

$$\begin{pmatrix}\dfrac{\partial x_1^{gr}(\mu)}{\partial t}\\[2ex] \dfrac{\partial x_2^{gr}(\mu)}{\partial t}\end{pmatrix}=\begin{pmatrix}0 & 1\\ 0 & -50(5+\mu)\end{pmatrix}\begin{pmatrix}x_1^{gr}(\mu)\\ x_2^{gr}(\mu)\end{pmatrix}+\begin{pmatrix}0\\ 50\end{pmatrix}u^{gr}(t,\mu), \tag{30}$$

where $\mu \in [0, 1]$. For simplicity in representation, let us denote

$$A_{gr}(\mu) = \begin{pmatrix} 0 & 1 \\ 0 & -50(5+\mu) \end{pmatrix}, \qquad B_{gr}(\mu) = \begin{pmatrix} 0 \\ 50 \end{pmatrix}.$$

In this example, our aim is to investigate the stabilizability of the system (30) and find a control u that transfers the system from the initial state $x^{gr}(0, \mu) = \begin{pmatrix} 9 & 1 \end{pmatrix}^{\mathrm{T}}$ to the final state $x^{gr}(2, \mu) = \begin{pmatrix} 0 & 0 \end{pmatrix}^{\mathrm{T}}$. For the first aim, let us consider the granular matrix $\mathrm{K}^{gr}(\mu)$, given by

$$\mathrm{K}^{gr}(\mu) = \begin{pmatrix} B_{gr}(\mu) & A_{gr}(\mu)B_{gr}(\mu) \end{pmatrix} = 50 \begin{pmatrix} 0 & 1 \\ 1 & -50(5+\mu) \end{pmatrix}.$$

We can see that the above matrix has full row rank, that means the pair $(A_{gr}(\mu), B_{gr}(\mu))$ is controllable for all $\mu \in [0, 1]$. Thus, we imply the stabilizability of the system (30). Next, to find the control $u(t)$ as desired, let us recall from (16) the solution's form of the controlled system is as

$$x^{gr}(2, \mu) = e^{2A_{gr}}(\mu)x^{gr}(0, \mu) + \int_0^2 e^{(2-\tau)A_{gr}}(\mu)B_{gr}(\mu)u^{gr}(\tau, \mu)\mathrm{d}\tau.$$

Our goal is to find $u^{gr}(t, \mu)$ such that

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix} = e^{2A_{\mathrm{gr}}}(\mu) \begin{pmatrix} 9 \\ 1 \end{pmatrix} + \int_0^2 e^{(2-\tau)A_{\mathrm{gr}}}(\mu) \begin{pmatrix} 0 \\ 50 \end{pmatrix} u^{\mathrm{gr}}(\tau, \mu)\mathrm{d}\tau.$$

One can verify that one such $u^{\mathrm{gr}}(t, \mu)$ has following form

$$u^{\mathrm{gr}}(\tau, \mu) = -\left[e^{(2-\tau)A_{\mathrm{gr}}}(\mu)B_{gr}(\mu)\right]^{\mathrm{T}} [\Pi(2-\tau)]^{-1} e^{2A_{gr}}(\mu)x^{\mathrm{gr}}(0, \mu),$$

where

$$e^{tA_{\mathrm{gr}}}(\mu) = \begin{pmatrix} 1 & \frac{1-e^{-50(5+\mu)t}}{50(5+\mu)} \\ 0 & e^{-50(5+\mu)t} \end{pmatrix},$$

$$\Pi(2-\tau) = \int_0^2 e^{(2-\tau)A_{gr}}(\mu)B_{\mathrm{gr}}(\mu)\left[e^{(2-\tau)A_{gr}}(\mu)B_{\mathrm{gr}}(\mu)\right]^{\mathrm{T}} \mathrm{d}\tau.$$

Hence,

$$\begin{aligned} \Pi(2-\tau) &= \int_0^2 \begin{pmatrix} 1 & \frac{1-e^{-50(5+\mu)(2-\tau)}}{50(5+\mu)} \\ 0 & e^{-50(5+\mu)(2-\tau)} \end{pmatrix} \begin{pmatrix} 0 \\ 50 \end{pmatrix} \begin{pmatrix} 0 & 50 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \frac{1-e^{-50(5+\mu)(2-\tau)}}{50(5+\mu)} & e^{-50(5+\mu)(2-\tau)} \end{pmatrix} d\tau \\ &= \begin{pmatrix} \frac{50(1-e^{-100(5+\mu)})}{5+\mu} & 0 \\ \frac{e^{-200(5+\mu)}-2e^{-100(5+\mu)}+1}{2(5+\mu)^2} & \frac{25(1-e^{-200(5+\mu)})}{5+\mu} \end{pmatrix}. \end{aligned}$$

Table 4 The control strategy $u^{\mathrm{gr}}(t,\mu)$

μ	K_e	$u^{\mathrm{gr}}(t,\mu)$
0	5	$u^{\mathrm{gr}}(t,0)=0.3603e^{250(t-2)}-0.18015$
0.2	5.2	$u^{\mathrm{gr}}(t,0.2)=0.36027e^{260(t-2)}-0.18013$
0.4	5.4	$u^{\mathrm{gr}}(t,0.4)=0.36023e^{270(t-2)}-0.1801$
0.6	5.6	$u^{\mathrm{gr}}(t,0.6)=0.36019e^{280(t-2)}-0.18006$
0.8	5.8	$u^{\mathrm{gr}}(t,0.8)=0.36014e^{290(t-2)}-0.18002$
1	6	$u^{\mathrm{gr}}(t,1)=0.3601e^{300(t-2)}-0.17997$

The inverse of the above matrix is

$$\begin{pmatrix} \frac{5+\mu}{50(1-e^{-100(5+\mu)})} & 0 \\ -\frac{1}{2500(1+e^{-100(5+\mu)})} & \frac{5+\mu}{25(1-e^{-200(5+\mu)})} \end{pmatrix}.$$

Therefore, we obtain the control input $u^{gr}(t,\mu)$ is

$$\begin{aligned} u^{gr}(t,\mu) &= -\begin{pmatrix}0 & 50\end{pmatrix}\begin{pmatrix} 1 & 0 \\ \frac{1-e^{-50(5+\mu)(2-t)}}{50(5+\mu)} & e^{-50(5+\mu)(2-t)} \end{pmatrix} \\ &\quad\times \begin{pmatrix} \frac{5+\mu}{50(1-e^{-100(5+\mu)})} & 0 \\ -\frac{1}{2500(1+e^{-100(5+\mu)})} & \frac{5+\mu}{25(1-e^{-200(5+\mu)})} \end{pmatrix} \begin{pmatrix} 1 & \frac{1-e^{-100(5+\mu)}}{50(5+\mu)} \\ 0 & e^{-100(5+\mu)} \end{pmatrix} \begin{pmatrix} 9 \\ 1 \end{pmatrix} \\ &= \left[\frac{9-50e^{-100(5+\mu)}(5+\mu)}{25(1-e^{-200(5+\mu)})} + \frac{1}{1250(5+\mu)}\right] e^{50(5+\mu)(t-2)} \\ &\quad - \left[\frac{9}{50(1-e^{-100(5+\mu)})} + \frac{1}{2500(5+\mu)}\right]. \end{aligned}$$

In order to show the effect of different values of indeterminacy, Table 4 demonstrates the control input $u(t)$ that steers Eq. (29) from the initial state $x(0) = \begin{pmatrix}9 & 1\end{pmatrix}^{\mathrm{T}}$ to the state $x(2) = \begin{pmatrix}0 & 0\end{pmatrix}^{\mathrm{T}}$ in time $\Delta = 2$.

A plot of the control $u(t)$ versus time is given in Fig. 12.

5 Conclusions

We presented new results on granular computing with respect to neutrosophic-valued functions to form a new concept of derivatives and neutrosophic control systems. The essential concepts of neutrosophic analysis were introduced such as neutrosophic matrices, neutrosophic exponential matrix, neutrosophic differentiability, neutrosophic integral, neutrosophic complete metric space, etc, which allow us to consider LTI neutrosophic control systems in a new setting. Thanks to the horizontal membership functions approach, the LTI neutrosophic control systems can be converted into a class of LTI real control systems depending on parameters. Thus,

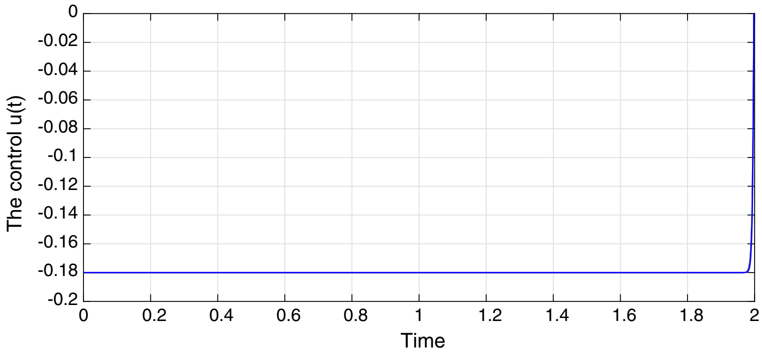

Fig. 12 A plot of the control action versus time of Example 4.4

the necessary and sufficient conditions for the controllability and stabilizability of neutrosophic LTI systems were investigated. Some real-life examples were examined to demonstrate the effectiveness of the theoretical results. The numerical examples showed that the proposed approach can prevent the shortcomings and disadvantages when operating in uncertain environments such as multiplicity solutions or unnatural behavior in modeling phenomena. For future research, we will study some open issues such as neutrosophic random control systems under granular differentiability.

Acknowledgements The authors are greatly indebted to Editor-in-Chief, Associate Editor, and the anonymous referees for their helpful comments and valuable suggestions, that greatly improve the quality and clarity of the paper. This research is funded by Vietnam National Foundation for Science and Technology Development (NAFOSTED) under Grant No. 101.02-2018.311.

References

1. M. Abdel-Basset, M. Mohamed, The role of single valued neutrosophic sets and rough sets in smart city: imperfect and incomplete information systems. Measurement **124**, 47–55 (2018)
2. M. Aslam, R.A.R. Bantan, N. Khan, Design of a new attribute control chart under neutrosophic statistics. Int. J. Fuzzy Syst. **21**(2), 433–440 (2019)
3. M. Aslam, A new failure-censored reliability test using neutrosophic statistical interval method. Int. J. Fuzzy Syst. **21**(4), 1214–1220 (2019)
4. B. Bede, S.G. Gal, Generalizations of the differential of fuzzy number-valued functions with applications to fuzzy differential equations. Fuzzy Sets Syst. **151**(3), 581–599 (2005)
5. S. Broumi, A. Bakali, M. Talea, F. Smarandache, Isolated single valued neutrosophic graphs. Neutrosophic Sets Syst. **11**, 74–78 (2016)
6. L.Q. Dat, N.T. Thong, L.H. Son, M. Ali, F. Smarandache, M. Abdel-Basset, H.V. Long, Linguistic approaches to interval complex neutrosophic sets in decision making. IEEE Access **7**, 38902–38917 (2019)
7. I. Deli, A novel defuzzificationmethod of SV-trapezoidal neutrosophic numbers andvmulti-attribute decision making: a comparative analysis. Soft Comput. (2019). https://doi.org/10.1007/s00500-019-03803-z
8. I. Deli, Some operators with IVGSVTrN-numbers and their applications to multiple criteria group decision making. Neutrosophic Sets Syst. **25**, 33–53 (2019)
9. A. Dey, L.H. Son, A. Pal, H.V. Long, Fuzzy minimum spanning tree with interval type 2 fuzzy arc length: formulation and a new genetic algorithm, Soft Comput., (2019). https://doi.org/10.1007/s00500-019-04166-1

10. X. Fu, Controllability of abstract neutral functional differential systems with unbounded delay. Appl. Math. Comput. **151**(2), 299–314 (2004)
11. S. Jha, L.H. Son, R. Kumar, I. Priyadarshini, F. Smarandache, H.V. Long, Neutrosophic image segmentation with Dice coefficients. Measurement **134**, 762–772 (2019)
12. L.W. Kong, Y.F. Wu, J. Ye, Misfire fault diagnosis method of gasoline engines using the cosine similarity measure of neutrosophic numbers. Neutrosophic Sets Syst. **8**, 42–45 (2015)
13. M. Landowski, Method with horizontal fuzzy numbers for solving real fuzzy linear systems. Soft Comput. **23**(12), 3921–3933 (2019)
14. H.V. Long, M. Ali, L.H. Son, M. Khan, D.N. Tu, A novel approach for fuzzy clustering based on neutrosophic association matrix. Comput. Ind. Eng. **127**, 687–697 (2019)
15. H.V. Long, N.P. Dong, An extension of Krasnoselskii's fixed point theorem and its application to nonlocal problems for implicit fractional differential systems with uncertainty. J. Fixed Point Theory Appl. **20**(1), 1–27 (2018)
16. M. Mazandarani, N. Pariz, A.V. Kamyad, Granular differentiability of fuzzy-number-valued functions. IEEE Trans. Fuzzy Syst. **26**(1), 310–323 (2018)
17. M. Mazandarani, Y. Zhao, Fuzzy Bang–Bang control problem under granular differentiability. J. Frankl. Inst. **355**(12), 4931–4951 (2018)
18. M. Mazandarani, N. Pariz, Sub-optimal control of fuzzy linear dynamical systems under granular differentiability concept. ISA Trans. **76**, 1–17 (2018)
19. P. Majumdar, Neutrosophic sets and its applications to decision making, in *Computational Intelligence for Big Data Analysis. Adaptation, Learning, and Optimization* ed. by D. Acharjya, S. Dehuri, S. Sanyal, vol. 19, (Springer, Cham, 2015)
20. R.E. Moore, R.B. Kearfott, J.M. Cloud, *Introduction to Interval Analysis* (SIAM, Philadelphia, 2009)
21. W. Pedrycz, S.M. Chen, *Granular Computing and Decision-Making: Interactive and Iterative Approaches* (Springer, Heidelberg, 2015)
22. X. Peng, G. Selvachandran, Pythagorean fuzzy set: state of the art and future directions. Artif. Intell. Rev. (2017). https://doi.org/10.1007/s10462-017-9596-9
23. C.I. Phillips, J.M. Parr, E.A. Riskin, Signals, Systems and Transforms, Prentice Hall (2007). ISBN 0-13-041207-4
24. A. Piegat, M. Landowski, Horizontal membership function and examples of its applications. Int. J. Fuzzy Syst. **17**(1), 22–30 (2015)
25. A. Piegat, M. Landowski, *Fuzzy arithmetic type-1 with HMFs, Uncertainty Modeling* (Springer, Berlin, 2017), pp. 233–250
26. A.A. Salama, S.A. Alblowi, Generalized neutrosophic set and generalized neutrousophic topological spaces. Comput. Sci. Eng. **2**(7), 129–132 (2012)
27. F. Smarandache, *Neutrosophy: Neutrosophic Probability, Set, and Logic* (American Research Press, Rehoboth, 1998)
28. F. Smarandache, *Introduction to Neutrosophic Measure, Neutrosophic Integral, and Neutrosophic Probability* (Sitech & Education Publisher, Craiova, 2013)
29. F. Smarandache, *Introduction to Neutrosophic Statistics* (Sitech & Education Publishing, Columbus, 2014)
30. F. Smarandache, *Neutrosophic Precalculus and Neutrosophic Calculus* (Europa-Nova, Brussels, 2015)
31. N.T.K. Son, N.P. Dong, L.H. Son, H.V. Long, Towards granular calculus of single-valued neutrosophic functions under granular computing. Multimed. Tools Appl. (2019). https://doi.org/10.1007/s11042-019-7388-8
32. N.T.K. Son, N.P. Dong, H.V. Long, Fuzzy delay differential equations under granular differentiability with applications. Comp. Appl. Math. **38**, 107 (2019)
33. M. Sarkar, T.K. Roy, Neutrosophic optimization technique and its application on structural design. J. Ultra Sci. Phys. Sci. **28**(6), 309–321 (2016)
34. M. Sarkar, S. Ghosh, T.K. Roy, Multi-objective welded beam optimization using neutrosophic optimization technique: a comparative study. J. Ultra Sci. Phys. Sci. **29**(6), 243–263 (2017)
35. I.R. Sumathi, V.M. Priya, A new perspective on neutrosophic differential equation. Int. J. Eng. Technol. **7**(10), 422–425 (2018)
36. F. Tas, S. Topal, F. Smarandache, Clustering neutrosophic data sets and neutrosophic valued metric spaces. Symmetry **10**(10), 430–442 (2018)
37. J. Ye, Neutrosophic number linear programming method and its application under neutrosophic number environments. Soft Comput. **22**(14), 4639–4646 (2017)

38. J. Ye, J. Chen, R. Yong, S. Du, Expression and analysis of joint roughness coefficient using neutrosophic number functions. Information **8**(2), 69 (2017)
39. J. Ye, W. Cui, Neutrosophic state feedback design method for SISO neutrosophic linear systems. Cogn. Syst. Res. **52**, 1056–1065 (2018)
40. S.H. Zăk, *Systems and Control, School of Electrical and Computer Engineering Purdue University* (Oxford University Press, Oxford, 2003)

Affiliations

Nguyen Thi Kim Son[1,2] · Nguyen Phuong Dong[3] · Le Hoang Son[4] · Mohamed Abdel-Basset[5] · Gunasekaran Manogaran[6] · Hoang Viet Long[7]

Nguyen Thi Kim Son
nguyenthikimson@tdtu.edu.vn

Nguyen Phuong Dong
nguyenphuongdong@hpu2.edu.vn

Le Hoang Son
sonlh@vnu.edu.vn

Mohamed Abdel-Basset
mohamed.abdlebasset@fci.zu.edu.eg

Gunasekaran Manogaran
gmanogaran@ucdavis.edu

1 Division of Computational Mathematics and Engineering, Institute for Computational Science, Ton Duc Thang University, Ho Chi Minh City, Vietnam

2 Faculty of Mathematics and Statistics, Ton Duc Thang University, Ho Chi Minh City, Vietnam

3 Department of Mathematics, Hanoi Pedagogical University 2, Phuc Yen City, Vietnam

4 VNU Information Technology Institute, Vietnam National University, Hanoi, Vietnam

5 Department of Operations Research and Decision Support, Faculty of Computers and Informatics, Zagazig University, Zagazig, Egypt

6 University of California, Davis, USA

7 Faculty of Information Technology, People's Police University of Technology and Logistics, Bac Ninh, Viet Nam