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Optimization of Financial Asset Neutrosophic Portfolios

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Abstract: The purpose of this paper was to model, with the help of neutrosophic fuzzy numbers, the optimal financial asset portfolios, offering additional information to those investing in the capital market. The optimal neutrosophic portfolios are those categories of portfolios consisting of two or more financial assets, modeled using neutrosophic triangular numbers, that allow for the determination of financial performance indicators, respectively the neutrosophic average, the neutrosophic risk, for each financial asset, and the neutrosophic covariance as well as the determination of the portfolio return, respectively of the portfolio risk. There are two essential conditions established by rational investors on the capital market to obtain an optimal financial assets portfolio, respectively by fixing the financial return at the estimated level as well as minimizing the risk of the financial assets neutrosophic portfolio. These conditions allowed us to compute the financial assets' share in the total value of the neutrosophic portfolios, for which the financial return reaches the level set by investors and the financial risk has the minimum value. In financial terms, the financial assets' share answers the legitimate question of rational investors in the capital market regarding the amount of money they must invest in compliance with the optimal conditions regarding the neutrosophic return and risk.

Keywords: financial assets; neutrosophic portfolio return; neutrosophic portfolios risk; optimal portfolios

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1. Introduction

The modern portfolio theory developed by H. Markowitz [1] paved the path for obtaining additional information on the capital market. These categories of information refer to the identification of correlations between the financial assets' return and risk, but also to the possibility of determining the financial asset portfolio's structure given an expected (desired) return level and focusing on minimizing the portfolio risk. However, modern portfolio theory does not provide sufficient information to capital market investors. Information such as the probability of accomplishment/non-accomplishment/uncertainty of some investment strategies are not the basis for financial decision-making. In the absence of this information, investors cannot make appropriate decisions on the capital market to ensure an effective correlation between the financial asset portfolios' return and risk.

The importance of the research problem identified above, namely the lack of information on the probability of carrying out investment strategies in the capital market, can be solved with the help of fuzzy neutrosophic numbers because they contain three categories of additional information that can help investors, namely, the degree of achievement of the performance indicators, the degree of non-achievement of the performance indicators encountered, especially in the situation of some scenarios of the investment strategies without perspective of realization as well as the degree of uncertainty that refers to those investment scenarios with uncertain probability. The performance indicators that will be modeled using neutrosophic fuzzy triangular numbers are the portfolio structure and the portfolio risk. Thus, the importance of this research resides in the fact that it offers additional information to those investing in the capital market, with the help of several portfolio performance indicators, which answers the legitimate question of rational investors in the capital market regarding the amount of money they must invest in compliance with the optimal conditions regarding the neutrosophic return and risk.

The innovation in the paper is ensured by the financial asset portfolio performance indicators, modeled with the use of neutrosophic fuzzy triangular numbers. In this context, specific neutrosophic performance indicators were obtained: the structure of the neutrosophic portfolio and its risk. The paper also lays the foundations for optimal financial asset neutrosophic portfolios such as those categories of financial assets that form a portfolio for which the portfolio return has a predetermined value (desired by investors) and the financial assets portfolio risk is minimal. The paper thus provides, through innovation, an additional information package for capital market investors generated by the specificity of neutrosophic fuzzy triangular numbers used in modeling.

The theoretical contribution of this research paper is given by the theoretical substantiation, with the help of neutrosophic fuzzy triangular numbers of the performance indicators specific to the optimal financial assets' portfolios, namely, portfolio structure, portfolio risk as well as portfolio return. These three performance indicators are specific to the optimal portfolios, respectively to those portfolio categories for which financial return reaches the level set by investors and the financial risk has the minimum value.

The fuzzy logic approach offers many advantages over the classic version of modeling the financial assets or financial asset portfolios' performance indicators, namely:

- Provides much more flexible calculation methodologies for modeling the financial assets performance to provide specialized information necessary for investors on the capital market.
- The fuzzy logic approach best describes the evolution of the financial assets value over a period, and as the membership degree specific to fuzzy numbers allows for the precise classification of the financial assets value by ranges of values assessed using linguistic terms: small, medium, or large.
- The fuzzy logic approach allows for the grouping of the financial assets values (for example, the shares value) that are the basis for determining the financial performance indicators (financial return, financial risk, etc.) in groups of values assessed using linguistic terms: small, medium, or large.
- The fuzzy logic approach allows for the determination of the financial performance indicators above-mentioned on value ranges also appreciated with the help of linguistic terms. Moreover, using the membership degree, it allows the establishment of their belonging at the value ranges.
- The fuzzy logic approach allows for additional information to be obtained if neutrosophic fuzzy numbers are used through the probabilities attached to them, which ensures the provision of additional information regarding the realization of the investment strategies on the capital market.

This research paper used neutrosophic fuzzy numbers because they best describe the investor's need for information on the likelihood of capital market investment success. They contain three categories of additional information for any investor, namely, the high probability of achieving an investment strategy conventionally denoted by (w), the low probability of achievement denoted by (u), and the uncertainty denoted by (y). All these three categories of probabilities are specific to neutrosophic fuzzy numbers only, are attached to, are modeled after rules specific to neutrosophic fuzzy numbers, and provide investors with additional information on how investment strategies can be achieved.

Neutrosophic triangular fuzzy numbers best describe the evolution of the values recorded by financial assets or financial asset portfolios. This is because the financial assets values, even if they follow an exponential trend, are often represented in the form of a "sawtooth" graph; in certain moments, they register maximum values and in other moments minimum values. Neutrosophic triangular fuzzy numbers thus best describe this type of behavior.

The remainder of the paper is organized as follows. Section 2 presents a literature review with a focus on the portfolio theory. Section 3 presents the prerequisites for the proposed approach. Section 4 discusses the optimization of a neutrosophic portfolio consisting of two financial assets, which is a basis for the discussion related to the optimal neutrosophic portfolios consisting of N-financial assets presented in Section 5. The paper ends with concluding remarks and references.

2. Literature Review

The financial assets optimizations were studied since the modern portfolio theory developed by H. Markowitz in 1952 [1], which opened the way to obtain supplementary information on the capital market. During the period 1975–2021, 1150 scientific articles were published on the Web of Science [2] database with this topic of interest. Since the 90s, an increase in interest in this field has been observed in terms of the number of studies published in ISI indexed journals (Figure 1).

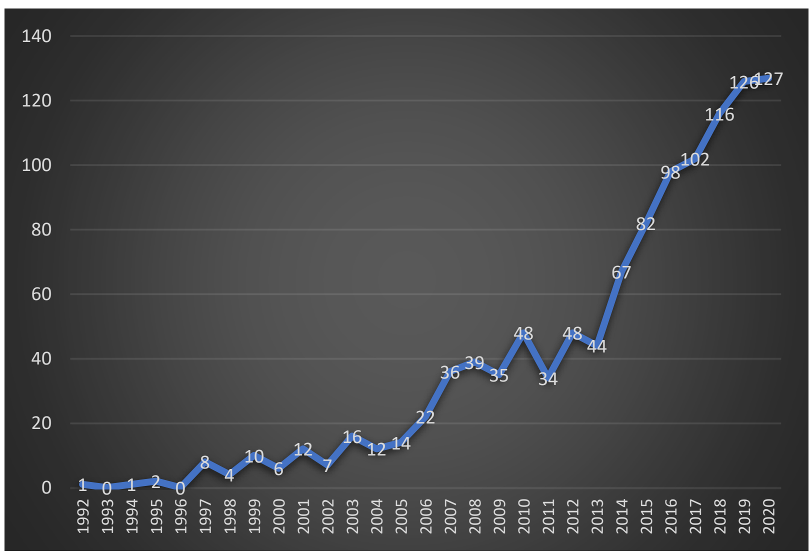

Figure 1. Number of studies published by year in the area of portfolio theory.

Regarding the Web of Science [2] categories, these studies targeted the following fields: operations research management science (23%), economics (20%), business finance (17%), mathematics interdisciplinary applications (13%), management (12.3%), computer science/AI (12%), social science mathematical methods (11%), engineering (11%), mathematics applied (8%), statistics probability (7.2%), etc. Only 1.6% of the total articles was aimed toward the mathematics field.

In terms of research areas, the top 10 was composed of business economics (41%), mathematics (25%), operation research (23%), computer science (21%), engineering (18%), mathematical methods (11%), automation control systems (4.2%), physics (3.8%), environmental science (3.2%), and science technology (3.1%). Most of them were published as articles (885), followed by proceedings papers (298), early access (24), book chapters (23), and reviews (18).

The financial assets optimizations captured the attention of researchers from all over the world during this period, but most of the published scientific studies came from China

(213), the USA (210), India (58), France (48), England (93), Italy (89), Austria (46), and Spain (33). These scored a h-index of 51 and an average citations per item of 11.21, being cited 12,897 times.

In the following, some of the most cited articles in this field will be discussed. Delage and Ye [3] proposed a model that describes uncertainty in the distribution form and moments, demonstrating in a practical example of portfolio selection that, for a wide range of cost functions, the associated min-max stochastic program can be solved efficiently.

Cavalcante et al. [4] analyzed the main challenges and open problems in financial markets by focusing on the scientific studies made on techniques for preprocessing and clustering of financial data, for forecasting future market movements, and for mining financial text information, thus contributing to the definition of a systematic procedure for guiding the task of building an intelligent trading system.

Onnela et al. [5] focused on taxonomy and portfolio analysis by analyzing the dynamics of market correlations using the minimum spanning tree description of correlations between stocks. The obtained results showed that the diversification aspect of portfolio optimization consists of the fact that the assets of the classic Markowitz portfolio are always located on the outer leaves of the tree. DeMiguel et al. [6] developed a general framework for finding portfolios that performed well out-of-sample in the presence of estimation error by using the traditional minimum-variance problem with the constraint that the norm of the portfolio-weight vector is smaller than a given threshold. As conclusions, the authors proposed several new portfolio strategies useful for investors who have a prior belief on portfolio weights rather than on moments of asset returns.

Bandi and Russell [7] developed a model that estimated the daily integrated variance of financial asset prices with the help of the microstructure noise, realized variance, and optimal sampling. Steinbach [8] used the mean-variance portfolio analysis to avoid the penalization of overperformance. The results were used in multiperiod models based on scenario trees to remove surplus money in future decisions, yielding approximate downside risk minimization. On the other hand, Tumminello et al. [9] studied several methods to quantitatively investigate the properties of correlation matrices in portfolio optimization and in asset price dynamics. Their paper focused on how to define and obtain hierarchical trees, correlation-based trees, and networks from a correlation matrix. Moreover, Karatzas [10] studied the existence of the numeraire portfolio under predictable convex constraints in a general semimartingale model of a financial market.

Varma et al. [11] analyzed the impact of enterprise-wide cross-functional coordination on enterprise performance, sustainability, and growth prospects. Within their research, they demonstrated the existence of cross-functional decision-making dependencies using an enterprise network model, targeting the interactions between enterprise planning decisions involving project financing, debt-equity balancing, R&D portfolio selection, risk hedging with real derivative instruments, supply chain asset creation, and marketing contracts. Papageorgio [12] addressed the decision-making problem in process industry supply chains, toward the development of optimal infrastructures and planning. His paper gathered the methodologies used in this field, offering a critical review from the financial and sustainability point of view.

As for the application of fuzzy logic in portfolio management, several studies in the literature have been dedicated to this area.

The advantages of using a fuzzy-logic-based asset allocation are underlined by North [13], while Karpenko et al. [14] showed that the fuzzy sets present some advantages that can support the assessment of corporate investment decisions in terms of risk and uncertainty.

Piasecki and Łyczkowska-Hanćkowiak [15] proposed some methods for approximating the ordered fuzzy numbers using trapezoidal ordered fuzzy numbers, simplifying in this manner the arithmetical operations with direct implications in the practical applications in the area of financial portfolio management.

The fuzzy portfolio optimization problem is discussed by Liu [16] in a research conducted on the Taiwan stock exchange. The author took advantage of the fuzzy data representation and fuzzy numbers and showed that the greater the amount of risk an investor is taking, the higher its potential return [16].

Dastkhan et al. [17] proposed models based on fuzzy mathematical programming and used the data provided by the New York Stock Exchange in order to illustrate their effectiveness. As a result, the authors state that the fuzzy mathematical approach used in portfolio management can be useful in providing a solution to the decision maker that will satisfy their preferences [17].

Using data from the Athens Stock Exchange, Chourmouziadis and Chatzoglou [18] proposed a short-term fuzzy system able to avoid high losses during bear markets. According to the authors, the proposed fuzzy model is superior to the returns of the buy and hold strategy [18].

Portfolio selection in public administration is addressed by Nassif et al. [19] through the use of fuzzy logic. Despite the usual approach in the portfolio management problem, focusing mostly on the financial return in the governmental sphere, the authors believe that the focus should also be put on the public benefits [19]. As a result, several experiments have been presented, simulated, and analyzed through the use of fuzzy logic.

Trying to overcome the drawback produced by the traditional assets allocation models, Hui et al. [20] used two fuzzy mathematical programming models in which the authors included expert adjustments not connected to the information found in the historical data. As a result, the authors underlined that the provided solution was able to produce a portfolio as efficient as the traditional allocation models, while minimizing the information vagueness [20].

Garcia-Crespo et al. [21] combined semantic technologies and fuzzy logic techniques and proposed a tool able to recommend investments based on both the psychological aspects of the investor and on the traditional financial indicators. Ferreira et al. [22] proposed a framework designated for portfolio optimization in private banking that took the advantages of the fuzzy approach while accounting for the personal features of the investor and the regulations imposed related to risk exposure.

In this context, the present paper combined the mathematical methods from portfolio theory with the principles of the neutrosophic theory, the neutrosophic fuzzy triangular numbers. Neutrosophic theory has been studied since 2000 and focuses on solving real life problems that involve uncertainty, impreciseness, vagueness, incompleteness, inconsistent, and indeterminacy [23–25]. The neutrosophic theory now has applications in various fields such as artificial intelligence, data mining, soft computing, decision making, information systems, image processing, computational modeling, robotics, medical diagnosis, biomedical engineering, investment problems, economic forecasting, social science, etc. [26–28].

The purpose of this research paper was to model, with the help of neutrosophic triangular fuzzy numbers, the financial performance indicators specific to the optimal portfolios. Thus, the object of modeling was the financial assets optimal portfolios characterized by the fact that the financial return is fixed at an estimated level and the financial asset neutrosophic portfolio's risk tends to a minimum. Although the global optimum can give relevant results in the research activity [29,30], however, for the proposed purpose of the elaborated research paper, the optimal portfolios best meet this objective.

3. Pre-Requisites

Financial asset portfolios have been the subject of numerous studies in the literature [31–34]. The conditions regarding the financial asset portfolios' return and risk are known as optimal portfolios and are based on setting a certain level for financial return that is conventionally denoted by (ρ) and minimizing the risk regarding the investments in the capital market, which is conventionally denoted by (σ_P^2). These two conditions are known in the literature as conditions for optimizing financial asset portfolios, which can be mathematically written as follows:

$$\begin{cases} R_P = \rho; \\ \sigma_P^2 \to min; \end{cases} \quad (1)$$

The results obtained in the studies carried out for the financial asset optimal conditions were spectacular. The defining elements of an optimal financial asset portfolio were established, among which we mention the portfolio structure of the form: $X_P = \frac{1}{D}[(\rho A - B)\Omega^{-1}R + (C - B\rho)\Omega^{-1}e]$, with A=e$\Omega^{-1}e^T$, $B = e\Omega^{-1}R^T = R\Omega^{-1}e^T$, $C = R\Omega^{-1}R^T$ and $D = AC - B^2$; and the portfolio risk of the form: $\sigma_P^2 = \frac{1}{D}(A\rho^2 - 2B\rho + C)$, known as the Markowitz's frontier. Within the present paper, the financial asset portfolios were modeled using fuzzy neutrosophic numbers. In these categories of financial asset portfolios are included those assets that cumulatively meet the following conditions [35]:

- contain financial assets in their structure, denoted by $(A_i); i = \overline{2, n}$, which have as KPIs: the neutrosophic return $\langle E_f(\widetilde{R_{A_i}}); w\widetilde{R_A}, u\widetilde{R_A}, y\widetilde{R_A}\rangle$, the neutrosophic risk $\langle \sigma f_{A_i}^2; w\tilde{\sigma}_A, u\tilde{\sigma}_A, y\tilde{\sigma}_A\rangle$, and the neutrosophic covariance that characterizes the intensity of the links between the neutrosophic returns of two financial assets $\langle cov(\widetilde{R_{A_1}}, \widetilde{R_{A_2}}); w\widetilde{R_{A_1}}, u\widetilde{R_{A_1}}, y\widetilde{R_{A_1}};\ w\widetilde{R_{A_2}}, u\widetilde{R_{A_2}}, y\widetilde{R_{A_2}}\rangle$; and
- allow the calculation of the neutrosophic portfolio return $\langle \tilde{R}_P; w\widetilde{Rp}, u\widetilde{Rp}, y\widetilde{Rp}\rangle$ and the neutrosophic portfolio risk $\langle \tilde{\sigma}_P^2; w\widetilde{\sigma p}, u\widetilde{\sigma p}, y\widetilde{\sigma p}\rangle$ as fundamental variables that define any neutrosophic portfolio $\langle \tilde{P}; w\tilde{P}, u\tilde{P}, y\tilde{P}\rangle$.

Thus, any financial asset (A_i) modeled using neutrosophic fuzzy numbers has its defining elements described by the average financial return, financial risk, and covariance, modeled using neutrosophic fuzzy numbers of the form [35,36]:

(a) The neutrosophic average return $\langle E_f(\widetilde{R_{A_i}}); w\widetilde{R_A}, u\widetilde{R_A}, y\widetilde{R_A}\rangle$ for the neutrosophic triangular fuzzy number $\widetilde{R_{A_i}} = \langle(\widetilde{R_{A_{ai}}}, \widetilde{R_{A_{bi}}}, \widetilde{R_{A_{ci}}}); w\widetilde{R_A}, u\widetilde{R_A}, y\widetilde{R_A}\rangle$, specific for the financial asset (A_i) and component part of the neutrosophic portfolio $(\tilde{P}; w\tilde{P}, u\tilde{P}, y\tilde{P},)$, any value of the financial asset return appreciated after the achievement degree by using the following coefficients: $w\widetilde{R_A} \in [0,1]$ for certain achievement degree, $u\widetilde{R_A} \in [0,1]$ for indeterminate achievement degree, and $y\widetilde{R_A} \in [0,1]$ for falsity achievement degree, determined as:

$$\langle E_f(\widetilde{R_A}); w\widetilde{R_A}, u\widetilde{R_A}, y\widetilde{R_A}\rangle = \langle\left(\frac{1}{6}\left(\widetilde{R_{A_{a1}}} + \widetilde{R_{A_{c1}}}\right) + \frac{2}{3}\widetilde{R_{A_{b1}}}\right); w\widetilde{R_A}, u\widetilde{R_A}, y\widetilde{R_A}\rangle \quad (2)$$

The membership function is modeled using neutrosophic fuzzy numbers as presented in Figure 2.

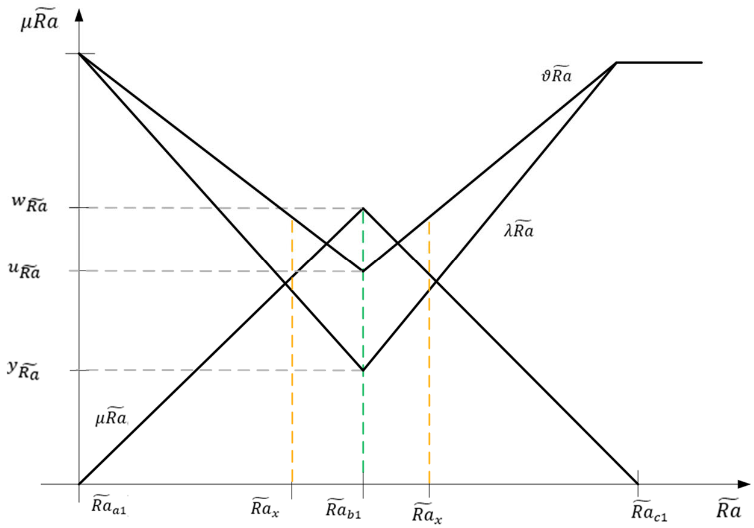

Figure 2. The financial assets' return modeled using neutrosophic fuzzy numbers.

(b) The neutrosophic risk $\langle \sigma f_{A_i}^2; w\tilde{\sigma}_A, u\tilde{\sigma}_A, y\tilde{\sigma}_A \rangle$ for the neutrosophic triangular fuzzy number $\tilde{\sigma}_{A_i} = \langle (\tilde{\sigma}_{A_{ai}}, \tilde{\sigma}_{A_{bi}}, \tilde{\sigma}_{A_{ci}}); w\tilde{\sigma}_A, u\tilde{\sigma}_A, y\tilde{\sigma}_A \rangle$ determined for the financial asset (A_i) and component part of the neutrosophic portfolio $\left(\widetilde{P}; w\widetilde{P}, u\widetilde{P}, y\widetilde{P},\right)$, any value of the financial asset risk appreciated after the achievement degree by using the following coefficients: $w\tilde{\sigma}_A \in [0,1]$ for certain achievement degree, $u\tilde{\sigma}_A \in [0,1]$ for indeterminate achievement degree, and $y\tilde{\sigma}_A \in [0,1]$ for falsity achievement degree, determined by the calculation formula:

$$\begin{aligned}
&\langle \sigma f_{A_i}^2; w\tilde{\sigma}_A, u\tilde{\sigma}_A, y\tilde{\sigma}_A \rangle \\
&\quad = \langle \frac{1}{4}\left[\left(\widetilde{R}_{A_{b1}} - \widetilde{R}_{A_{a1}}\right)^2 + \left(\widetilde{R}_{A_{c1}} - \widetilde{R}_{A_{b1}}\right)^2\right]; w\widetilde{R_A}, u\widetilde{R_A}, y\widetilde{R_A} \rangle \\
&\quad + \langle \frac{2}{3}\left[\widetilde{R}_{A_{a1}}\left(\widetilde{R}_{A_{b1}} - \widetilde{R}_{A_{a1}}\right) - \widetilde{R}_{A_{c1}}\left(\widetilde{R}_{A_{c1}} - \widetilde{R}_{A_{b1}}\right)\right]; w\widetilde{R_A}, u\widetilde{R_A}, y\widetilde{R_A} \rangle \\
&\quad + \langle \frac{1}{2}\left(\widetilde{R}_{A_{a1}}^{\,2} + \widetilde{R}_{A_{c1}}^{\,2}\right); w\widetilde{R_A}, u\widetilde{R_A}, y\widetilde{R_A} \rangle - \langle \frac{1}{2}E_f^2\left(\widetilde{Ra}_i\right); w\widetilde{R_A}, u\widetilde{R_A}, y\widetilde{R_A} \rangle
\end{aligned} \tag{3}$$

The membership function of the financial assets risk is modeled using neutrosophic fuzzy numbers, as presented in Figure 3.

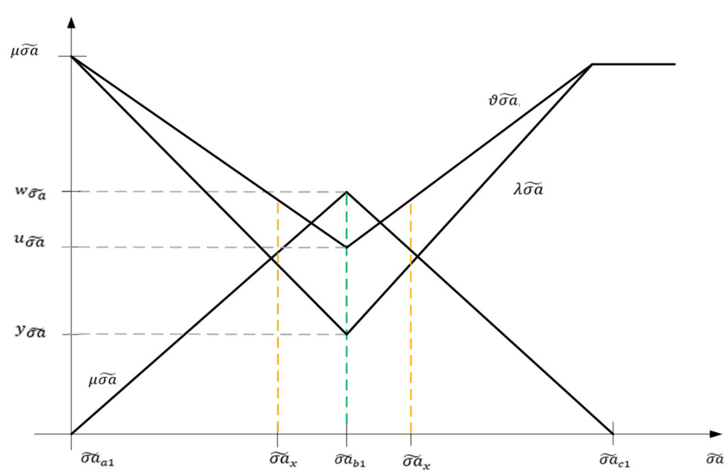

Figure 3. The financial assets risk modeled using neutrosophic fuzzy numbers.

(c) The neutrosophic covariance $\langle cov(\widetilde{R_{A_1}}, \widetilde{R_{A_2}}); w\widetilde{R_{A_1}}, u\widetilde{R_{A_1}}, y\widetilde{R_{A_1}};\ w\widetilde{R_{A_2}}, u\widetilde{R_{A_2}}, y\widetilde{R_{A_2}}\rangle$ for two neutrosophic triangular fuzzy numbers $\widetilde{R_{A_1}} = \langle(\widetilde{R_{A_{a1}}}, \widetilde{R_{A_{b1}}}, \widetilde{R_{A_{c1}}}); w\widetilde{R_{A_1}}, u\widetilde{R_{A_1}}, y\widetilde{R_{A_1}}\rangle$ and respectively $\widetilde{R_{A_2}} = \langle(\widetilde{R_{A_{a2}}}, \widetilde{R_{A_{b2}}}, \widetilde{R_{A_{c2}}}); w\widetilde{R_{A_2}}, u\widetilde{R_{A_2}}, y\widetilde{R_{A_2}}\rangle$ characterizing two financial assets (A_1, A_2) and component parts of the neutrosophic portfolio $(\widetilde{P}; w\widetilde{P}, u\widetilde{P}, y\widetilde{P},)$, any value of the financial asset covariance appreciated after the achievement degree by using the following coefficients: $w\widetilde{R_{A_1}}, w\widetilde{R_{A_2}} \in [0,1]$ for certain achievement degree, $u\widetilde{R_{A_1}}, u\widetilde{R_{A_2}} \in [0,1]$ for indeterminate achievement degree, and $y\widetilde{R_{A_1}}, y\widetilde{R_{A_2}} \in [0,1]$ for falsity achievement degree, determined by the calculation formula:

$$\langle cov(\widetilde{R_{A_1}}, \widetilde{R_{A_2}}); w\widetilde{R_{A_1}}, u\widetilde{R_{A_1}}, y\widetilde{R_{A_1}};\ w\widetilde{R_{A_2}}, u\widetilde{R_{A_2}}, y\widetilde{R_{A_2}}\rangle =$$

$$\begin{aligned}&\left\langle\left(\frac{1}{4}\left[(\widetilde{R_{A_{b11}}} - \widetilde{R_{A_{a11}}})(\widetilde{R_{A_{b21}}} - \widetilde{R_{A_{a21}}}) + (\widetilde{R_{A_{c11}}} - \widetilde{R_{A_{b11}}})(\widetilde{R_{A_{c21}}} - \widetilde{R_{A_{b21}}})\right]\right.\right.\\ &+ \frac{1}{3}\left\{\left[\widetilde{R_{A_{a21}}}(\widetilde{R_{A_{b11}}} - \widetilde{R_{A_{a11}}}) + \widetilde{R_{A_{a11}}}(\widetilde{R_{A_{b21}}} - \widetilde{R_{A_{a21}}})\right]\right.\\ &- \left.\left[\widetilde{R_{A_{c11}}}(\widetilde{R_{A_{c21}}} - \widetilde{R_{A_{b21}}}) + \widetilde{R_{A_{c21}}}(\widetilde{R_{A_{c11}}} - \widetilde{R_{A_{b11}}})\right]\right\} + \frac{1}{2}(\widetilde{R_{A_{a11}}}\widetilde{R_{A_{a21}}} + \widetilde{R_{A_{c11}}}\widetilde{R_{A_{c21}}})\\ &+ \left.\left.\frac{1}{2}E_f(\widetilde{R_{A_1}})E_f(\widetilde{R_{A_2}})\right); w\tilde{R}_{A_1} \wedge w\widetilde{R_{A_2}}, u\widetilde{R_{A_1}} \vee u\widetilde{R_{A_2}}, y\widetilde{R_{A_1}} \vee y\widetilde{R_{A_2}}\right\rangle\end{aligned} \quad (4)$$

These formulas have been demonstrated in [36] and we will not return upon these demonstrations in the present paper. The specific elements of a financial asset (A_i) modeled using neutrosophic fuzzy numbers were tested using practical applications, their major advantage being determined by the fact that they allow the characterization of any financial asset according to the perspective of achieving or not the values obtained for each category of indicators. Moreover, the research results were extended to a portfolio of financial assets $\langle\widetilde{P}; w\widetilde{P}, u\widetilde{P}, y\widetilde{P}\rangle$ for which the first fundamental element is the neutrosophic portfolio return denoted by $\langle\tilde{R}_P; w\widetilde{Rp}, u\widetilde{Rp}, y\widetilde{Rp}\rangle$, modeled using fuzzy triangular neutrosophic numbers of the form: $\widetilde{R_{A_i}} = \langle(\widetilde{R_{A_{ai}}}, \widetilde{R_{A_{bi}}}, \widetilde{R_{A_{ci}}}); w\widetilde{R_A}, u\widetilde{R_A}, y\widetilde{R_A}\rangle$, which is a variable that characterizes the neutrosophic portfolio and is determined by the formula:

$$\langle\tilde{R}_P; w\widetilde{Rp}, u\widetilde{Rp}, y\widetilde{Rp}\rangle = \sum_{i=1}^{n}\langle x_{A_i}\left(\frac{1}{6}(\tilde{R}_{A_{ai}} + \tilde{R}_{A_{ci}}) + \frac{2}{3}\tilde{R}_{A_{bi}}\right); w\tilde{R}_{A_i}, u\tilde{R}_{A_i}, y\tilde{R}_{A_i}\rangle \quad (5)$$

In addition, for the same portfolio $\langle\widetilde{P}; w\widetilde{P}, u\widetilde{P}, y\widetilde{P}\rangle$ modeled using neutrosophic triangular fuzzy numbers, the second fundamental element was obtained, which characterizes the same portfolio, the neutrosophic portfolio risk of the form: $\langle\tilde{\sigma}_P^2; w\widetilde{\sigma p}, u\widetilde{\sigma p}, y\widetilde{\sigma p}\rangle$, modeled with the help of neutrosophic fuzzy numbers: $\tilde{\sigma}_{A_i} = \langle(\tilde{\sigma}_{A_{ai}}, \tilde{\sigma}_{A_{bi}}, \tilde{\sigma}_{A_{ci}}); w\tilde{\sigma}_A, u\tilde{\sigma}_A, y\tilde{\sigma}_A\rangle$. This is also a fundamental element of the neutrosophic portfolio that is determined by the formula:

$$\begin{aligned}
&\langle\tilde{\sigma}_P^2; w\widetilde{\sigma p}, u\widetilde{\sigma p}, y\widetilde{\sigma p}\rangle \\
&\quad = \sum_{i=1}^{n} x_{A_i}^2 \langle \frac{1}{4}\left[\left(\widetilde{R_{A_{bi}}} - \widetilde{R_{A_{ai}}}\right)^2 + \left(\widetilde{R_{A_{ci}}} - \widetilde{R_{A_{bi}}}\right)^2\right]; w\widetilde{R_{A_i}}, u\widetilde{R_{A_i}}, y\widetilde{R_{A_i}}\rangle \\
&\quad + \langle \frac{2}{3}\left[\widetilde{R_{A_{ai}}}\left(\widetilde{R_{A_{bi}}} - \widetilde{R_{A_{ai}}}\right) - \widetilde{R_{A_{ci}}}\left(\widetilde{R_{A_{ci}}} - \widetilde{R_{A_{bi}}}\right)\right]; w\widetilde{R_{A_i}}, u\widetilde{R_{A_i}}, y\widetilde{R_{A_i}}\rangle \\
&\quad + \langle \frac{1}{2}\left(\widetilde{R_{A_{ai}}}^2 + \widetilde{R_{A_{ci}}}^2\right); w\widetilde{Ra}_i, u\widetilde{Ra}_i, y\widetilde{Ra}_i\rangle - \langle \frac{1}{2} E_f^2\left(\widetilde{R_{A_i}}\right); w\widetilde{R_{A_i}}, u\widetilde{R_{A_i}}, y\widetilde{R_{A_i}}\rangle \\
&\quad + 2\sum_{i=1}^{n}\sum_{j=1}^{n} x_{A_i}\, x_{A_j} \langle \left(\frac{1}{4}\left[\left(\widetilde{R_{A_{bi1}}} - \widetilde{R_{A_{ai1}}}\right)\left(\widetilde{R_{A_{bj1}}} - \widetilde{R_{A_{aj1}}}\right) + \left(\widetilde{R_{A_{ci1}}} - \widetilde{R_{A_{bi1}}}\right)\left(\widetilde{R_{A_{cj1}}} - \widetilde{R_{A_{bj1}}}\right)\right]\right. \\
&\quad + \frac{1}{3}\left\{\left[\widetilde{R_{A_{aj1}}}\left(\widetilde{R_{A_{bi1}}} - \widetilde{R_{A_{ai1}}}\right) + \widetilde{R_{A_{ai1}}}\left(\widetilde{R_{A_{bj1}}} - \widetilde{R\widetilde{R_A}a_{aj1}}\right)\right]\right. \\
&\quad \left. - \left[\widetilde{R_{A_{ci1}}}\left(\widetilde{R_{A_{cj1}}} - \widetilde{R_{A_{bj1}}}\right) + \widetilde{R_{A_{cj1}}}\left(\widetilde{R_{A_{ci1}}} - \widetilde{R_{A_{bi1}}}\right)\right]\right\} + \frac{1}{2}\left(\widetilde{R_{A_{ai1}}}\widetilde{R_{A_{aj1}}} + \widetilde{R_{A_{ci1}}}\widetilde{R_{A_{cj1}}}\right) \\
&\quad \left. + \frac{1}{2} E_f\left(\widetilde{R_{A_i}}\right) E_f\left(\widetilde{R_{A_j}}\right)\right); w\widetilde{R_{A_i}} \wedge w\widetilde{R_{A_j}}, u\widetilde{R_{A_i}} \vee u\widetilde{R_{A_j}}, y\widetilde{R_{A_i}} \vee y\widetilde{R_{A_j}}\rangle
\end{aligned} \tag{6}$$

As a result, it can be stated that any financial asset neutrosophic portfolio $\langle\widetilde{P}; w\widetilde{P}, u\widetilde{P}, y\widetilde{P}\rangle$ has two fundamental characteristic variables:

$$\langle\widetilde{P}; w\widetilde{P}, u\widetilde{P}, y\widetilde{P}\rangle = \begin{cases}\langle\tilde{R}_P; w\widetilde{Rp}, u\widetilde{Rp}, y\widetilde{Rp}\rangle \\ \langle\tilde{\sigma}_P^2; w\widetilde{\sigma p}, u\widetilde{\sigma p}, y\widetilde{\sigma p}\rangle\end{cases} \tag{7}$$

The results obtained in the previous studies [35,36] allow for the possibility of characterizing the financial asset portfolios both from the perspective of the average portfolio return and from the perspective of the financial portfolio risk, thus bringing an added value to the literature dealing with the problem of financial assets.

4. Optimization of a Neutrosophic Portfolio Consisting of Two Financial Assets

Theorem 1. *Let a neutrosophic portfolio of two financial assets denoted by* (A_1) *and* (A_2) *for which the neutrosophic average return and the neutrosophic average risk are determined:* $A_1: \langle\widetilde{R_{A_1}}, \tilde{\sigma}_{A_1}\rangle$*, respectively* $A_2: \langle\widetilde{R_{A_2}}, \tilde{\sigma}_{A_2}\rangle$.

This portfolio is an optimal neutrosophic portfolio, if two cumulative basic conditions are observed, namely:

- The two asset portfolio returns reached a level $\langle\tilde{R}_P; w\widetilde{Rp}, u\widetilde{Rp}, y\widetilde{Rp}\rangle = \langle\tilde{\rho}_{Rp}; w\widetilde{\rho_{Rp}}, u\widetilde{\rho_{Rp}}, y\widetilde{\rho_{Rp}}\rangle$; and
- The portfolio risk tends to a minimum: $\langle\tilde{\sigma}_P^2; w\widetilde{\sigma p}, u\widetilde{\sigma p}, y\widetilde{\sigma p}\rangle \to min$.

For such a portfolio, its structure is determined by the following relations:

$$\tilde{x}_{A_1} = \langle\frac{2\langle\tilde{R}_{A_2}\rangle\left[\langle\tilde{\rho}_{Rp}\rangle + \langle\tilde{R}_{A_1}\rangle\right] - \langle\tilde{R}_{A_1} + \tilde{R}_{A_2}\rangle\left[\langle\tilde{\rho}_{Rp}\rangle + \langle\tilde{R}_{A_2}\rangle\right]}{4\langle\tilde{R}_{A_1}\tilde{R}_{A_2}\rangle - \langle\tilde{R}_{A_1} + \tilde{R}_{A_2}\rangle^2} w\tilde{x}_{A_1} \wedge w\tilde{x}_{A_2}, u\tilde{x}_{A_1} \vee u\tilde{x}_{A_2}, y\tilde{x}_{A_1} \vee y\tilde{x}_{A_2}\rangle \tag{8}$$

and respectively:

$$\tilde{x}_{A_2} = \langle\frac{2\langle\tilde{R}_{A_1}\rangle\left[\langle\tilde{\rho}_{Rp}\rangle + \langle\tilde{R}_{A_2}\rangle\right] - \langle\tilde{R}_{A_1} + \tilde{R}_{A_2}\rangle\left[\langle\tilde{\rho}_{Rp}\rangle + \langle\tilde{R}_{A_1}\rangle\right]}{4\langle\tilde{R}_{A_1}\tilde{R}_{A_2}\rangle - \langle\tilde{R}_{A_1} + \tilde{R}_{A_2}\rangle^2} w\tilde{x}_{A_1} \wedge w\tilde{x}_{A_2}, u\tilde{x}_{A_1} \vee u\tilde{x}_{A_2}, y\tilde{x}_{A_1} \vee y\tilde{x}_{A_2}\rangle \tag{9}$$

Note 1: $\tilde{x}_{A_1}$ represents the asset share (A_1) in the total value of the portfolio, while $\tilde{x}_{A_2}$ represents the assets share (A_2) in the total value of the portfolio, with the meanings of the terms as they were presented within the theorem 1 demonstration.

Additionally, the portfolio risk is determined using the following formula:

$$\langle\tilde{\sigma}_P^2; w\widetilde{\sigma p}, u\widetilde{\sigma p}, y\widetilde{\sigma p}\rangle = \langle\left[\frac{2\langle\tilde{R}_{A_2}\rangle\left[\langle\tilde{\rho}_{Rp}\rangle + \langle\tilde{R}_{A_1}\rangle\right] - \langle\tilde{R}_{A_1} + \tilde{R}_{A_2}\rangle\left[\langle\tilde{\rho}_{Rp}\rangle + \langle\tilde{R}_{A_2}\rangle\right]}{4\langle\tilde{R}_{A_1}\tilde{R}_{A_2}\rangle - \langle\tilde{R}_{A_1} + \tilde{R}_{A_2}\rangle^2}\right]^2 \tilde{\sigma}_{A_1}^2; w\tilde{\sigma}_{A_1}, u\tilde{\sigma}_{A_1}, y\tilde{\sigma}_{A_1}\rangle + \tag{10}$$

$$+\langle\left[\frac{2\langle\tilde{R}_{A_1}\rangle[\langle\tilde{\rho}_{Rp}\rangle+\langle\tilde{R}_{A_2}\rangle]-\langle\tilde{R}_{A_1}+\tilde{R}_{A_2}\rangle[\langle\tilde{\rho}_{Rp}\rangle+\langle\tilde{R}_{A_1}\rangle]}{4\langle\tilde{R}_{A_1}\tilde{R}_{A_2}\rangle-\langle\tilde{R}_{A_1}+\tilde{R}_{A_2}\rangle^2}\right]^2\tilde{\sigma}_{A_2}^2; w\tilde{\sigma}_{A_2}, u\tilde{\sigma}_{A_2}, y\tilde{\sigma}_{A_2}\rangle+$$
$$+\langle 2\left(\frac{2\langle\tilde{R}_{A_2}\rangle[\langle\tilde{\rho}_{Rp}\rangle+\langle\tilde{R}_{A_1}\rangle]-\langle\tilde{R}_{A_1}+\tilde{R}_{A_2}\rangle[\langle\tilde{\rho}_{Rp}\rangle+\langle\tilde{R}_{A_2}\rangle]}{4\langle\tilde{R}_{A_1}\tilde{R}_{A_2}\rangle-\langle\tilde{R}_{A_1}+\tilde{R}_{A_2}\rangle^2}\right)\rangle\times$$
$$\times\langle\left(\frac{2\langle\tilde{R}_{A_1}\rangle[\langle\tilde{\rho}_{Rp}\rangle+\langle\tilde{R}_{A_2}\rangle]-\langle\tilde{R}_{A_1}+\tilde{R}_{A_2}\rangle[\langle\tilde{\rho}_{Rp}\rangle+\langle\tilde{R}_{A_1}\rangle]}{4\langle\tilde{R}_{A_1}\tilde{R}_{A_2}\rangle-\langle\tilde{R}_{A_1}+\tilde{R}_{A_2}\rangle^2}\right)\tilde{\sigma}_{A_1A_2}; w\tilde{\sigma}_{A_1}\wedge w\tilde{\sigma}_{A_2}, u\tilde{\sigma}_{A_1}\vee u\tilde{\sigma}_{A_2}, y\tilde{\sigma}_{A_1}\vee y\tilde{\sigma}_{A_2}\rangle$$

Note 2: $\langle\tilde{\sigma}_P^2; w\widetilde{\sigma p}, u\widetilde{\sigma p}, y\widetilde{\sigma p}\rangle$ represents the neutrosophic portfolio risk modeled using neutrosophic triangular fuzzy numbers whose relationship allows the determination of the portfolio risk consisting of two financial assets, for which the return registers a certain given value and the portfolio risk is minimal.

Demonstration: The financial asset portfolio consisting of two financial assets, marked with (A_1) and (A_2), which is characterized by neutrosophic return and risk, is modeled using neutrosophic triangular fuzzy numbers as follows:

$$A_1:\begin{cases}\widetilde{R_{A_1}}=\langle(\widetilde{R_{Aa1}},\widetilde{R_{Ab1}},\widetilde{R_{Ac1}}); w\widetilde{R_{A_1}}, u\widetilde{R_{A_1}}, y\widetilde{R_{A_1}}\rangle\\ \tilde{\sigma}_{A_1}=\langle(\tilde{\sigma}_{Aa1},\tilde{\sigma}_{Ab1},\tilde{\sigma}_{Ac1}); w\tilde{\sigma}_{A_1}, u\tilde{\sigma}_{A_1}, y\tilde{\sigma}_{A_1}\rangle\end{cases}\tag{11}$$

and

$$A_2:\begin{cases}\widetilde{R_{A_2}}=\langle(\widetilde{R_{Aa2}},\widetilde{R_{Ab2}},\widetilde{R_{Ac2}}); w\widetilde{R_{A_2}}, u\widetilde{R_{A_2}}, y\widetilde{R_{A_2}}\rangle\\ \tilde{\sigma}_{A_2}=\langle(\tilde{\sigma}_{Aa2},\tilde{\sigma}_{Ab2},\tilde{\sigma}_{Ac2}); w\tilde{\sigma}_{A_2}, u\tilde{\sigma}_{A_2}, y\tilde{\sigma}_{A_2}\rangle\end{cases}\tag{12}$$

The equations of the neutrosophic portfolio consisting of the two financial assets will be written as:

$$\begin{cases}\langle\tilde{R}_P; w\widetilde{Rp}, u\widetilde{Rp}, y\widetilde{Rp}\rangle=\langle x_{A_1}\tilde{R}_{A_1}; w\widetilde{R_{A_1}}, u\widetilde{R_{A_1}}, y\widetilde{R_{A_1}}\rangle+\langle x_{A_2}\tilde{R}_{A_2}; w\widetilde{R_{A_2}}, u\widetilde{R_{A_2}}, y\widetilde{R_{A_2}}\rangle\\ \langle\tilde{\sigma}_P^2; w\widetilde{\sigma p}, u\widetilde{\sigma p}, y\widetilde{\sigma p}\rangle=\langle x_{A_1}^2\tilde{\sigma}_{A_1}^2; w\tilde{\sigma}_{A_1}, u\tilde{\sigma}_{A_1}, y\tilde{\sigma}_{A_1}\rangle+\langle x_{A_2}^2\tilde{\sigma}_{A_2}^2; w\tilde{\sigma}_{A_2}, u\tilde{\sigma}_{A_2}, y\tilde{\sigma}_{A_2}\rangle+\\ \langle 2x_{A_1}x_{A_2}\tilde{\sigma}_{A_1A_2}; w\tilde{\sigma}_{A_1}\wedge w\tilde{\sigma}_{A_2}, u\tilde{\sigma}_{A_1}\vee u\tilde{\sigma}_{A_2}, y\tilde{\sigma}_{A_1}\vee y\tilde{\sigma}_{A_2}\rangle\\ x_{A_1}+x_{A_2}=1\end{cases}\tag{13}$$

The optimal conditions for a financial assets portfolio are determined by reaching a level $\langle\tilde{\rho}_{Rp}; w\widetilde{\rho_{Rp}}, u\widetilde{\rho_{Rp}}, y\widetilde{\rho_{Rp}}\rangle$ for the financial asset portfolio return $\langle\tilde{R}_P; w\widetilde{Rp}, u\widetilde{Rp}, y\widetilde{Rp}\rangle$ and minimizing the financial asset portfolio risk $\langle\tilde{\sigma}_P^2; w\widetilde{\sigma p}, u\widetilde{\sigma p}, y\widetilde{\sigma p}\rangle\to min$. Under these conditions are obtained the equations of a neutrosophic portfolio with optimization conditions, respectively:

$$\begin{cases}\langle\tilde{R}_P; w\widetilde{Rp}, u\widetilde{Rp}, y\widetilde{Rp}\rangle=\langle\tilde{\rho}_{Rp}; w\widetilde{\rho_{Rp}}, u\widetilde{\rho_{Rp}}, y\widetilde{\rho_{Rp}}\rangle\\ \langle\tilde{\sigma}_P^2; w\widetilde{\sigma p}, u\widetilde{\sigma p}, y\widetilde{\sigma p}\rangle\to min\\ x_{A_1}+x_{A_2}=1\end{cases}\tag{14}$$

The specific equations of a two financial asset neutrosophic portfolio with the optimization conditions above-mentioned become:

$$\begin{cases}\langle x_{A_1}\tilde{R}_{A_1}; w\widetilde{R_{A_1}}, u\widetilde{R_{A_1}}, y\widetilde{R_{A_1}}\rangle+\langle x_{A_2}\tilde{R}_{A_2}; w\widetilde{R_{A_2}}, u\widetilde{R_{A_2}}, y\widetilde{R_{A_2}}\rangle=\langle\tilde{\rho}_{Rp}; w\widetilde{\rho_{Rp}}, u\widetilde{\rho_{Rp}}, y\widetilde{\rho_{Rp}}\rangle\\ \langle x_{A_1}^2\tilde{\sigma}_{A_1}^2; w\tilde{\sigma}_{A_1}, u\tilde{\sigma}_{A_1}, y\tilde{\sigma}_{A_1}\rangle+\langle x_{A_2}^2\tilde{\sigma}_{A_2}^2; w\tilde{\sigma}_{A_2}, u\tilde{\sigma}_{A_2}, y\tilde{\sigma}_{A_2}\rangle+\\ \langle x_{A_1}x_{A_2}\tilde{\sigma}_{A_1A_2}; w\tilde{\sigma}_{A_1}\wedge w\tilde{\sigma}_{A_2}, u\tilde{\sigma}_{A_1}\vee u\tilde{\sigma}_{A_2}, y\tilde{\sigma}_{A_1}\vee y\tilde{\sigma}_{A_2}\rangle\to min\langle\tilde{\sigma}_P^2; w\widetilde{\sigma p}, u\widetilde{\sigma p}, y\widetilde{\sigma p}\rangle\\ x_{A_1}+x_{A_2}=1\end{cases}\tag{15}$$

or

$$\begin{cases}\langle x_{A_1}\tilde{R}_{A_1}; w\widetilde{R_{A_1}}, u\widetilde{R_{A_1}}, y\widetilde{R_{A_1}}\rangle+\langle x_{A_2}\tilde{R}_{A_2}; w\widetilde{R_{A_2}}, u\widetilde{R_{A_2}}, y\widetilde{R_{A_2}}\rangle=\langle\tilde{\rho}_{Rp}; w\widetilde{\rho_{Rp}}, u\widetilde{\rho_{Rp}}, y\widetilde{\rho_{Rp}}\rangle\\ min\langle x_{A_1}x_{A_2}\tilde{\sigma}_{A_1A_2}; w\tilde{\sigma}_{A_1}\wedge w\tilde{\sigma}_{A_2}, u\tilde{\sigma}_{A_1}\vee u\tilde{\sigma}_{A_2}, y\tilde{\sigma}_{A_1}\vee y\tilde{\sigma}_{A_2}\rangle\to min\langle\tilde{\sigma}_P^2; w\widetilde{\sigma p}, u\widetilde{\sigma p}, y\widetilde{\sigma p}\rangle\\ x_{A_1}+x_{A_2}=1\end{cases}\tag{16}$$

The optimal problem from above is solved with the help of the Lagrange function for which the objective function is to minimize the portfolio risk $\langle\tilde{\sigma}_P^2; w\widetilde{\sigma p}, u\widetilde{\sigma p}, y\widetilde{\sigma p}\rangle\to min$, under the conditions wherein the portfolio return $\langle\tilde{R}_P; w\widetilde{Rp}, u\widetilde{Rp}, y\widetilde{Rp}\rangle=$

$\langle\tilde{\rho}_{Rp}; w\widetilde{\rho_{Rp}}, u\widetilde{\rho_{Rp}}, y\widetilde{\rho_{Rp}}\rangle$ will reach a level of $\langle\tilde{\rho}_{Rp}; w\widetilde{\rho_{Rp}}, u\widetilde{\rho_{Rp}}, y\widetilde{\rho_{Rp}}\rangle$ and the sum of the weights of the two financial assets is equal to 1.

In these conditions, we will have:

$$L = f_{objective} - \lambda_1 C_1 - \lambda_2 C_2 \tag{17}$$

The Lagrange function for obtaining the optimal conditions can be written as follows:

$$\begin{aligned} L = min\ &\langle x_{A_1} x_{A_2} \tilde{\sigma}_{A_1A_2}; w\tilde{\sigma}_{A_1} \wedge w\tilde{\sigma}_{A_2}, u\tilde{\sigma}_{A_1} \vee u\tilde{\sigma}_{A_2}, y\tilde{\sigma}_{A_1} \vee y\tilde{\sigma}_{A_2}\rangle \\ &- \lambda_1\left(\langle x_{A_1}\tilde{R}_{A_1}; w\widetilde{R_{A_1}}, u\widetilde{R_{A_1}}, y\widetilde{R_{A_1}}\rangle + \langle x_{A_2}\tilde{R}_{A_2}; w\widetilde{R_{A_2}}, u\widetilde{R_{A_2}}, y\widetilde{R_{A_2}}\rangle - \rho_{\tilde{R}_P}\right) - \lambda_2\left(x_{A_1} + x_{A_2} - 1\right) \end{aligned} \tag{18}$$

The optimal conditions for the Lagrange function are set as:

$$\begin{cases} \dfrac{\partial L}{\partial x_{A1}} = 0 \\ \dfrac{\partial L}{\partial x_{A2}} = 0 \\ \dfrac{\partial L}{\partial \lambda_1} = 0 \\ \dfrac{\partial L}{\partial \lambda_2} = 0 \end{cases} \tag{19}$$

The optimal conditions for the neutrosophic portfolio formed by two financial assets will be rewritten as follows. After performing the calculations, we obtained:

$$\begin{cases} \langle x_{A_2}\tilde{\sigma}_{A_1A_2}; w\tilde{\sigma}_{A_1} \wedge w\tilde{\sigma}_{A_2}, u\tilde{\sigma}_{A_1} \vee u\tilde{\sigma}_{A_2}, y\tilde{\sigma}_{A_1} \vee y\tilde{\sigma}_{A_2}\rangle - \lambda_1\langle\tilde{R}_{A_1}; w\widetilde{R_{A_1}}, u\widetilde{R_{A_1}}, y\widetilde{R_{A_1}}\rangle - \lambda_2 = 0 \\ \langle x_{A_1}\tilde{\sigma}_{A_1A_2}; w\tilde{\sigma}_{A_1} \wedge w\tilde{\sigma}_{A_2}, u\tilde{\sigma}_{A_1} \vee u\tilde{\sigma}_{A_2}, y\tilde{\sigma}_{A_1} \vee y\tilde{\sigma}_{A_2}\rangle - \lambda_1\langle\tilde{R}_{A_2}; w\widetilde{R_{A_2}}, u\widetilde{R_{A_2}}, y\widetilde{R_{A_2}}\rangle - \lambda_2 = 0 \\ \langle x_{A_1}\tilde{R}_{A_1}; w\widetilde{R_{A_1}}, u\widetilde{R_{A_1}}, y\widetilde{R_{A_1}}\rangle + \langle x_{A_2}\tilde{R}_{A_2}; w\widetilde{R_{A_2}}, u\widetilde{R_{A_2}}, y\widetilde{R_{A_2}}\rangle - \langle\tilde{\rho}_{Rp}; w\widetilde{\rho_{Rp}}, u\widetilde{\rho_{Rp}}, y\widetilde{\rho_{Rp}}\rangle = 0 \\ x_{A_1} + x_{A_2} - 1 \end{cases} \tag{20}$$

The above system of linear equations can be rewritten as follows:

$$\begin{cases} \langle x_{A_2}\tilde{\sigma}_{A_1A_2}; w\tilde{\sigma}_{A_1} \wedge w\tilde{\sigma}_{A_2}, u\tilde{\sigma}_{A_1} \vee u\tilde{\sigma}_{A_2}, y\tilde{\sigma}_{A_1} \vee y\tilde{\sigma}_{A_2}\rangle = \lambda_1\langle\tilde{R}_{A_1}; w\widetilde{R_{A_1}}, u\widetilde{R_{A_1}}, y\widetilde{R_{A_1}}\rangle + \lambda_2 \\ \langle x_{A_1}\tilde{\sigma}_{A_1A_2}; w\tilde{\sigma}_{A_1} \wedge w\tilde{\sigma}_{A_2}, u\tilde{\sigma}_{A_1} \vee u\tilde{\sigma}_{A_2}, y\tilde{\sigma}_{A_1} \vee y\tilde{\sigma}_{A_2}\rangle = \lambda_1\langle\tilde{R}_{A_2}; w\widetilde{R_{A_2}}, u\widetilde{R_{A_2}}, y\widetilde{R_{A_2}}\rangle + \lambda_2 \\ \langle x_{A_1}\tilde{R}_{A_1}; w\widetilde{R_{A_1}}, u\widetilde{R_{A_1}}, y\widetilde{R_{A_1}}\rangle + \langle x_{A_2}\tilde{R}_{A_2}; w\widetilde{R_{A_2}}, u\widetilde{R_{A_2}}, y\widetilde{R_{A_2}}\rangle = \langle\tilde{\rho}_{Rp}; w\widetilde{\rho_{Rp}}, u\widetilde{\rho_{Rp}}, y\widetilde{\rho_{Rp}}\rangle \\ x_{A_1} + x_{A_2} = 1 \end{cases} \tag{21}$$

From the equations above, the values for (x_{A_1}) and (x_{A_2}) are obtained as functions of Lagrange parameters (λ_1) and (λ_2), as follows:

$$x_{A_1} = \frac{\lambda_1\langle\tilde{R}_{A_2}; w\widetilde{R_{A_2}}, u\widetilde{R_{A_2}}, y\widetilde{R_{A_2}}\rangle + \lambda_2}{\langle\tilde{\sigma}_{A_1A_2}; w\tilde{\sigma}_{A_1} \wedge w\tilde{\sigma}_{A_2}, u\tilde{\sigma}_{A_1} \vee u\tilde{\sigma}_{A_2}, y\tilde{\sigma}_{A_1} \vee y\tilde{\sigma}_{A_2}\rangle} \tag{22}$$

respectively:

$$x_{A_2} = \frac{\lambda_1\langle\tilde{R}_{A_1}; w\widetilde{R_{A_1}}, u\widetilde{R_{A_1}}, y\widetilde{R_{A_1}}\rangle + \lambda_2}{\langle\tilde{\sigma}_{A_1A_2}; w\tilde{\sigma}_{A_1} \wedge w\tilde{\sigma}_{A_2}, u\tilde{\sigma}_{A_1} \vee u\tilde{\sigma}_{A_2}, y\tilde{\sigma}_{A_1} \vee y\tilde{\sigma}_{A_2}\rangle} \tag{23}$$

By replacing the expressions for the asset weights (x_{A_1}) and (x_{A_2}) in the system equations for the optimal conditions, the following is obtained:

$$\begin{aligned} &\frac{\lambda_1\langle\tilde{R}_{A_2}; w\widetilde{R_{A_2}}, u\widetilde{R_{A_2}}, y\widetilde{R_{A_2}}\rangle + \lambda_2}{\langle\tilde{\sigma}_{A_1A_2}; w\tilde{\sigma}_{A_1} \wedge w\tilde{\sigma}_{A_2}, u\tilde{\sigma}_{A_1} \vee u\tilde{\sigma}_{A_2}, y\tilde{\sigma}_{A_1} \vee y\tilde{\sigma}_{A_2}\rangle}\langle\tilde{R}_{A_1}; w\widetilde{R_{A_1}}, u\widetilde{R_{A_1}}, y\widetilde{R_{A_1}}\rangle \\ &\quad + \frac{\lambda_1\langle\tilde{R}_{A_1}; w\widetilde{R_{A_1}}, u\widetilde{R_{A_1}}, y\widetilde{R_{A_1}}\rangle + \lambda_2}{\langle\tilde{\sigma}_{A_1A_2}; w\tilde{\sigma}_{A_1} \wedge w\tilde{\sigma}_{A_2}, u\tilde{\sigma}_{A_1} \vee u\tilde{\sigma}_{A_2}, y\tilde{\sigma}_{A_1} \vee y\tilde{\sigma}_{A_2}\rangle}\langle\tilde{R}_{A_2}; w\widetilde{R_{A_2}}, u\widetilde{R_{A_2}}, y\widetilde{R_{A_2}}\rangle \\ &\quad = \langle\tilde{\rho}_{Rp}; w\widetilde{\rho_{Rp}}, u\widetilde{\rho_{Rp}}, y\widetilde{\rho_{Rp}}\rangle \end{aligned} \tag{24}$$

respectively:

$$\frac{\lambda_1\langle\tilde{R}_{A_2}; w\widetilde{R_{A_2}}, u\widetilde{R_{A_2}}, y\widetilde{R_{A_2}}\rangle + \lambda_2}{\langle\tilde{\sigma}_{A_1A_2}; w\tilde{\sigma}_{A_1} \wedge w\tilde{\sigma}_{A_2}, u\tilde{\sigma}_{A_1} \vee u\tilde{\sigma}_{A_2}, y\tilde{\sigma}_{A_1} \vee y\tilde{\sigma}_{A_2}\rangle} + \frac{\lambda_1\langle\tilde{R}_{A_1}; w\widetilde{R_{A_1}}, u\widetilde{R_{A_1}}, y\widetilde{R_{A_1}}\rangle + \lambda_2}{\langle\tilde{\sigma}_{A_1A_2}; w\tilde{\sigma}_{A_1} \wedge w\tilde{\sigma}_{A_2}, u\tilde{\sigma}_{A_1} \vee u\tilde{\sigma}_{A_2}, y\tilde{\sigma}_{A_1} \vee y\tilde{\sigma}_{A_2}\rangle} = 1 \tag{25}$$

A system of equations with two unknowns of the form (λ_1) and (λ_2) is formed as follows:

$$\begin{cases} 2\lambda_1\langle\tilde{R}_{A_1}; w\widetilde{R_{A_1}}, u\widetilde{R_{A_1}}, y\widetilde{R_{A_1}}\rangle\langle\tilde{R}_{A_2}; w\widetilde{R_{A_2}}, u\widetilde{R_{A_2}}, y\widetilde{R_{A_2}}\rangle + \\ +\lambda_2\left(\langle\tilde{R}_{A_1}; w\widetilde{R_{A_1}}, u\widetilde{R_{A_1}}, y\widetilde{R_{A_1}}\rangle + \langle\tilde{R}_{A_2}; w\widetilde{R_{A_2}}, u\widetilde{R_{A_2}}, y\widetilde{R_{A_2}}\rangle\right) = \\ = \langle\tilde{\rho}_{Rp}; w\widetilde{\rho_{Rp}}, u\widetilde{\rho_{Rp}}, y\widetilde{\rho_{Rp}}\rangle\langle\tilde{\sigma}_{A_1A_2}; w\tilde{\sigma}_{A_1} \wedge w\tilde{\sigma}_{A_2}, u\tilde{\sigma}_{A_1} \vee u\tilde{\sigma}_{A_2}, y\tilde{\sigma}_{A_1} \vee y\tilde{\sigma}_{A_2}\rangle \\ \lambda_1\left(\langle\tilde{R}_{A_1}; w\widetilde{R_{A_1}}, u\widetilde{R_{A_1}}, y\widetilde{R_{A_1}}\rangle + \langle\tilde{R}_{A_2}; w\widetilde{R_{A_2}}, u\widetilde{R_{A_2}}, y\widetilde{R_{A_2}}\rangle\right) + 2\lambda_2 = \\ = \langle\tilde{\sigma}_{A_1A_2}; w\tilde{\sigma}_{A_1} \wedge w\tilde{\sigma}_{A_2}, u\tilde{\sigma}_{A_1} \vee u\tilde{\sigma}_{A_2}, y\tilde{\sigma}_{A_1} \vee y\tilde{\sigma}_{A_2}\rangle \end{cases} \tag{26}$$

In the above system, we operated the following notations to simplify the calculations:

$$\langle\tilde{R}_{A_1}\tilde{R}_{A_2}\rangle = \langle\tilde{R}_{A_1}; w\widetilde{R_{A_1}}, u\widetilde{R_{A_1}}, y\widetilde{R_{A_1}}\rangle\langle\tilde{R}_{A_2}; w\widetilde{R_{A_2}}, u\widetilde{R_{A_2}}, y\widetilde{R_{A_2}}\rangle \tag{27}$$

$$\langle\tilde{R}_{A_1} + \tilde{R}_{A_2}\rangle = \langle\tilde{R}_{A_1}; w\widetilde{R_{A_1}}, u\widetilde{R_{A_1}}, y\widetilde{R_{A_1}}\rangle + \langle\tilde{R}_{A_2}; w\widetilde{R_{A_2}}, u\widetilde{R_{A_2}}, y\widetilde{R_{A_2}}\rangle \tag{28}$$

$$\langle\tilde{\rho}_{Rp}\tilde{\sigma}_{A_1A_2}\rangle = \langle\tilde{\rho}_{Rp}; w\widetilde{\rho_{Rp}}, u\widetilde{\rho_{Rp}}, y\widetilde{\rho_{Rp}}\rangle\langle\tilde{\sigma}_{A_1A_2}; w\tilde{\sigma}_{A_1} \wedge w\tilde{\sigma}_{A_2}, u\tilde{\sigma}_{A_1} \vee u\tilde{\sigma}_{A_2}, y\tilde{\sigma}_{A_1} \vee y\tilde{\sigma}_{A_2}\rangle \tag{29}$$

$$\langle\tilde{\sigma}_{A_1A_2}\rangle = \langle\tilde{\sigma}_{A_1A_2}; w\tilde{\sigma}_{A_1} \wedge w\tilde{\sigma}_{A_2}, u\tilde{\sigma}_{A_1} \vee u\tilde{\sigma}_{A_2}, y\tilde{\sigma}_{A_1} \vee y\tilde{\sigma}_{A_2}\rangle \tag{30}$$

The solution of the above equation system results in the values for parameters (λ_1) and (λ_2) as follows:

$$\begin{cases} 2\langle\tilde{R}_{A_1}\tilde{R}_{A_2}\rangle\lambda_1 + \langle\tilde{R}_{A_1} + \tilde{R}_{A_2}\rangle\lambda_2 = \langle\tilde{\rho}_{Rp}\tilde{\sigma}_{A_1A_2}\rangle \\ \langle\tilde{R}_{A_1} + \tilde{R}_{A_2}\rangle\lambda_1 + 2\lambda_2 = \langle\tilde{\sigma}_{A_1A_2}\rangle \end{cases} \tag{31}$$

The solutions are obtained:

$$\lambda_1 = \frac{\begin{vmatrix} \langle\tilde{\rho}_{Rp}\tilde{\sigma}_{A_1A_2}\rangle & \langle\tilde{R}_{A_1} + \tilde{R}_{A_2}\rangle \\ \langle\tilde{\sigma}_{A_1A_2}\rangle & 2 \end{vmatrix}}{\begin{vmatrix} 2\langle\tilde{R}_{A_1}\tilde{R}_{A_2}\rangle & \langle\tilde{R}_{A_1} + \tilde{R}_{A_2}\rangle \\ \langle\tilde{R}_{A_1} + \tilde{R}_{A_2}\rangle & 2 \end{vmatrix}} = \frac{2\langle\tilde{\rho}_{Rp}\tilde{\sigma}_{A_1A_2}\rangle - \langle\tilde{\sigma}_{A_1A_2}\rangle\langle\tilde{R}_{A_1} + \tilde{R}_{A_2}\rangle}{4\langle\tilde{R}_{A_1}\tilde{R}_{A_2}\rangle - \langle\tilde{R}_{A_1} + \tilde{R}_{A_2}\rangle^2} \tag{32}$$

and

$$\lambda_2 = \frac{\begin{vmatrix} 2\langle\tilde{R}_{A_1}\tilde{R}_{A_2}\rangle & \langle\tilde{\rho}_{Rp}\tilde{\sigma}_{A_1A_2}\rangle \\ \langle\tilde{R}_{A_1} + \tilde{R}_{A_2}\rangle & \langle\tilde{\sigma}_{A_1A_2}\rangle \end{vmatrix}}{\begin{vmatrix} 2\langle\tilde{R}_{A_1}\tilde{R}_{A_2}\rangle & \langle\tilde{R}_{A_1} + \tilde{R}_{A_2}\rangle \\ \langle\tilde{R}_{A_1} + \tilde{R}_{A_2}\rangle & 2 \end{vmatrix}} = \frac{2\langle\tilde{R}_{A_1}\tilde{R}_{A_2}\rangle\langle\tilde{\sigma}_{A_1A_2}\rangle - \langle\tilde{\rho}_{Rp}\tilde{\sigma}_{A_1A_2}\rangle\langle\tilde{R}_{A_1} + \tilde{R}_{A_2}\rangle}{4\langle\tilde{R}_{A_1}\tilde{R}_{A_2}\rangle - \langle\tilde{R}_{A_1} + \tilde{R}_{A_2}\rangle^2} \tag{33}$$

Next, the following notations were operated to simplify the calculations, denoted by:

$$\langle\tilde{R}_{A_1}\rangle = \langle\tilde{R}_{A_1}; w\widetilde{R_{A_1}}, u\widetilde{R_{A_1}}, y\widetilde{R_{A_1}}\rangle \tag{34}$$

$$\langle\tilde{R}_{A_2}\rangle = \langle\tilde{R}_{A_2}; w\widetilde{R_{A_2}}, u\widetilde{R_{A_2}}, y\widetilde{R_{A_2}}\rangle \tag{35}$$

and the following is obtained:

$$x_{A_1} = \frac{\lambda_1\langle\tilde{R}_{A_2}; w\widetilde{R_{A_2}}, u\widetilde{R_{A_2}}, y\widetilde{R_{A_2}}\rangle + \lambda_2}{\langle\tilde{\sigma}_{A_1A_2}; w\tilde{\sigma}_{A_1} \wedge w\tilde{\sigma}_{A_2}, u\tilde{\sigma}_{A_1} \vee u\tilde{\sigma}_{A_2}, y\tilde{\sigma}_{A_1} \vee y\tilde{\sigma}_{A_2}\rangle} = \\ = \frac{\frac{2\langle\tilde{\rho}_{Rp}\tilde{\sigma}_{A_1A_2}\rangle - \langle\tilde{\sigma}_{A_1A_2}\rangle\langle\tilde{R}_{A_1} + \tilde{R}_{A_2}\rangle}{4\langle\tilde{R}_{A_1}\tilde{R}_{A_2}\rangle - \langle\tilde{R}_{A_1} + \tilde{R}_{A_2}\rangle^2}\langle\tilde{R}_{A_2}\rangle + \frac{2\langle\tilde{R}_{A_1}\tilde{R}_{A_2}\rangle\langle\tilde{\sigma}_{A_1A_2}\rangle - \langle\tilde{\rho}_{Rp}\tilde{\sigma}_{A_1A_2}\rangle\langle\tilde{R}_{A_1} + \tilde{R}_{A_2}\rangle}{4\langle\tilde{R}_{A_1}\tilde{R}_{A_2}\rangle - \langle\tilde{R}_{A_1} + \tilde{R}_{A_2}\rangle^2}}{\langle\tilde{\sigma}_{A_1A_2}\rangle} \tag{36}$$

After performing the calculations, the value for (x_{A_1}) is computed, being of the form:

$$x_{A_1} = \frac{\langle \tilde{R}_{A_2}\rangle\left[2\langle\tilde{\rho}_{Rp}\rangle - \langle\tilde{R}_{A_1} + \tilde{R}_{A_2}\rangle\right] + 2\langle\tilde{R}_{A_1}\tilde{R}_{A_2}\rangle - \langle\tilde{\rho}_{Rp}\rangle\langle\tilde{R}_{A_1} + \tilde{R}_{A_2}\rangle}{4\langle\tilde{R}_{A_1}\tilde{R}_{A_2}\rangle - \langle\tilde{R}_{A_1} + \tilde{R}_{A_2}\rangle^2} \tag{37}$$

Or, after rearranging the terms in the expression, $\left(x_{A_1}\right)$ is obtained:

$$\tilde{x}_{A_1} = \langle\frac{2\langle\tilde{R}_{A_2}\rangle\left[\langle\tilde{\rho}_{Rp}\rangle + \langle\tilde{R}_{A_1}\rangle\right] - \langle\tilde{R}_{A_1} + \tilde{R}_{A_2}\rangle\left[\langle\tilde{\rho}_{Rp}\rangle + \langle\tilde{R}_{A_2}\rangle\right]}{4\langle\tilde{R}_{A_1}\tilde{R}_{A_2}\rangle - \langle\tilde{R}_{A_1} + \tilde{R}_{A_2}\rangle^2} w\tilde{x}_{A_1} \wedge w\tilde{x}_{A_2}, u\tilde{x}_{A_1} \vee u\tilde{x}_{A_2}, y\tilde{x}_{A_1} \vee y\tilde{x}_{A_2}\rangle \tag{38}$$

To compute the asset weight (x_{A_2}), the above expression is replaced in:

$$x_{A_2} = \frac{\lambda_1\langle\tilde{R}_{A_1}; w\widetilde{R_{A_1}}, u\widetilde{R_{A_1}}, y\widetilde{R_{A_1}}\rangle + \lambda_2}{\langle\tilde{\sigma}_{A_1A_2}; w\tilde{\sigma}_{A_1} \wedge w\tilde{\sigma}_{A_2}, u\tilde{\sigma}_{A_1} \vee u\tilde{\sigma}_{A_2}, y\tilde{\sigma}_{A_1} \vee y\tilde{\sigma}_{A_2}\rangle} =$$

$$= \frac{\dfrac{2\langle\tilde{\rho}_{Rp}\tilde{\sigma}_{A_1A_2}\rangle - \langle\tilde{\sigma}_{A_1A_2}\rangle\langle\tilde{R}_{A_1} + \tilde{R}_{A_2}\rangle}{4\langle\tilde{R}_{A_1}\tilde{R}_{A_2}\rangle - \langle\tilde{R}_{A_1} + \tilde{R}_{A_2}\rangle^2}\langle\tilde{R}_{A_1}\rangle + \dfrac{2\langle\tilde{R}_{A_1}\tilde{R}_{A_2}\rangle\langle\tilde{\sigma}_{A_1A_2}\rangle - \langle\tilde{\rho}_{Rp}\tilde{\sigma}_{A_1A_2}\rangle\langle\tilde{R}_{A_1} + \tilde{R}_{A_2}\rangle}{4\langle\tilde{R}_{A_1}\tilde{R}_{A_2}\rangle - \langle\tilde{R}_{A_1} + \tilde{R}_{A_2}\rangle^2}}{\langle\tilde{\sigma}_{A_1A_2}\rangle} \tag{39}$$

After performing the calculations, the value for $\left(x_{A_2}\right)$ is obtained:

$$x_{A_2} = \frac{\langle \tilde{R}_{A_1}\rangle\left[2\langle\tilde{\rho}_{Rp}\rangle - \langle\tilde{R}_{A_1} + \tilde{R}_{A_2}\rangle\right] + 2\langle\tilde{R}_{A_1}\tilde{R}_{A_2}\rangle - \langle\tilde{\rho}_{Rp}\rangle\langle\tilde{R}_{A_1} + \tilde{R}_{A_2}\rangle}{4\langle\tilde{R}_{A_1}\tilde{R}_{A_2}\rangle - \langle\tilde{R}_{A_1} + \tilde{R}_{A_2}\rangle^2} \tag{40}$$

Or after rearranging the terms in the expression, (x_{A_1}) is obtained:

$$\tilde{x}_{A_2} = \langle\frac{2\langle\tilde{R}_{A_1}\rangle\left[\langle\tilde{\rho}_{Rp}\rangle + \langle\tilde{R}_{A_2}\rangle\right] - \langle\tilde{R}_{A_1} + \tilde{R}_{A_2}\rangle\left[\langle\tilde{\rho}_{Rp}\rangle + \langle\tilde{R}_{A_1}\rangle\right]}{4\langle\tilde{R}_{A_1}\tilde{R}_{A_2}\rangle - \langle\tilde{R}_{A_1} + \tilde{R}_{A_2}\rangle^2} w\tilde{x}_{A_1} \wedge w\tilde{x}_{A_2}, u\tilde{x}_{A_1} \vee u\tilde{x}_{A_2}, y\tilde{x}_{A_1} \vee y\tilde{x}_{A_2}\rangle \tag{41}$$

Under these conditions, the portfolio return is obtained:

$$\langle\tilde{R}_P; w\widetilde{Rp}, u\widetilde{Rp}, y\widetilde{Rp}\rangle = \langle x_{A_1}\tilde{R}_{A_1}; w\widetilde{R_{A_1}}, u\widetilde{R_{A_1}}, y\widetilde{R_{A_1}}\rangle + \langle x_{A_2}\tilde{R}_{A_2}; w\widetilde{R_{A_2}}, u\widetilde{R_{A_2}}, y\widetilde{R_{A_2}}\rangle \tag{42}$$

By replacing in the formula of $\langle\tilde{R}_P; w\widetilde{Rp}, u\widetilde{Rp}, y\widetilde{Rp}\rangle$, the calculation relation is obtained:

$$\begin{aligned}\langle\tilde{R}_P; w\widetilde{Rp}, u\widetilde{Rp}, y\widetilde{Rp}\rangle &= \langle\frac{2\langle\tilde{R}_{A_2}\rangle\left[\langle\tilde{\rho}_{Rp}\rangle + \langle\tilde{R}_{A_1}\rangle\right] - \langle\tilde{R}_{A_1} + \tilde{R}_{A_2}\rangle\left[\langle\tilde{\rho}_{Rp}\rangle + \langle\tilde{R}_{A_2}\rangle\right]}{4\langle\tilde{R}_{A_1}\tilde{R}_{A_2}\rangle - \langle\tilde{R}_{A_1} + \tilde{R}_{A_2}\rangle^2}\tilde{R}_{A_1}; w\widetilde{R_{A_1}}, u\widetilde{R_{A_1}}, y\widetilde{R_{A_1}}\rangle + \\ &+ \langle\frac{2\langle\tilde{R}_{A_1}\rangle\left[\langle\tilde{\rho}_{Rp}\rangle + \langle\tilde{R}_{A_2}\rangle\right] - \langle\tilde{R}_{A_1} + \tilde{R}_{A_2}\rangle\left[\langle\tilde{\rho}_{Rp}\rangle + \langle\tilde{R}_{A_1}\rangle\right]}{4\langle\tilde{R}_{A_1}\tilde{R}_{A_2}\rangle - \langle\tilde{R}_{A_1} + \tilde{R}_{A_2}\rangle^2}\tilde{R}_{A_2}; w\widetilde{R_{A_2}}, u\widetilde{R_{A_2}}, y\widetilde{R_{A_2}}\rangle\end{aligned} \tag{43}$$

The above relation represents the calculation formula for the portfolio return consisting of two financial assets for which the return $\langle\tilde{R}_P; w\widetilde{Rp}, u\widetilde{Rp}, y\widetilde{Rp}\rangle = \langle\tilde{\rho}_{Rp}; w\widetilde{\rho_{Rp}}, u\widetilde{\rho_{Rp}}, y\widetilde{\rho_{Rp}}\rangle$ and the portfolio risk $\langle\tilde{\sigma}_P^2; w\widetilde{\sigma p}, u\widetilde{\sigma p}, y\widetilde{\sigma p}\rangle \to min$. The calculation relationship for the portfolio risk is established with the formula:

$$\begin{aligned}\langle\tilde{\sigma}_P^2; w\widetilde{\sigma p}, u\widetilde{\sigma p}, y\widetilde{\sigma p}\rangle &= \langle x_{A_1}^2\tilde{\sigma}_{A_1}^2; w\tilde{\sigma}_{A_1}, u\tilde{\sigma}_{A_1}, y\tilde{\sigma}_{A_1}\rangle + \langle x_{A_2}^2\tilde{\sigma}_{A_2}^2; w\tilde{\sigma}_{A_2}, u\tilde{\sigma}_{A_2}, y\tilde{\sigma}_{A_2}\rangle + \\ &+\langle 2x_{A_1}x_{A_2}\tilde{\sigma}_{A_1A_2}; w\tilde{\sigma}_{A_1} \wedge w\tilde{\sigma}_{A_2}, u\tilde{\sigma}_{A_1} \vee u\tilde{\sigma}_{A_2}, y\tilde{\sigma}_{A_1} \vee y\tilde{\sigma}_{A_2}\rangle\end{aligned} \tag{44}$$

By replacing the expressions obtained for the weight of $\left(\tilde{x}_{A_1}\right)$ and $\left(\tilde{x}_{A_2}\right)$, the calculation relation for the portfolio risk is obtained:

$$\begin{aligned}\langle\tilde{\sigma}_P^2; w\widetilde{\sigma p}, u\widetilde{\sigma p}, y\widetilde{\sigma p}\rangle &= \langle\left[\frac{2\langle\tilde{R}_{A_2}\rangle\left[\langle\tilde{\rho}_{Rp}\rangle + \langle\tilde{R}_{A_1}\rangle\right] - \langle\tilde{R}_{A_1} + \tilde{R}_{A_2}\rangle\left[\langle\tilde{\rho}_{Rp}\rangle + \langle\tilde{R}_{A_2}\rangle\right]}{4\langle\tilde{R}_{A_1}\tilde{R}_{A_2}\rangle - \langle\tilde{R}_{A_1} + \tilde{R}_{A_2}\rangle^2}\right]^2\tilde{\sigma}_{A_1}^2; w\tilde{\sigma}_{A_1}, u\tilde{\sigma}_{A_1}, y\tilde{\sigma}_{A_1}\rangle + \\ &+ \langle\left[\frac{2\langle\tilde{R}_{A_1}\rangle\left[\langle\tilde{\rho}_{Rp}\rangle + \langle\tilde{R}_{A_2}\rangle\right] - \langle\tilde{R}_{A_1} + \tilde{R}_{A_2}\rangle\left[\langle\tilde{\rho}_{Rp}\rangle + \langle\tilde{R}_{A_1}\rangle\right]}{4\langle\tilde{R}_{A_1}\tilde{R}_{A_2}\rangle - \langle\tilde{R}_{A_1} + \tilde{R}_{A_2}\rangle^2}\right]^2\tilde{\sigma}_{A_2}^2; w\tilde{\sigma}_{A_2}, u\tilde{\sigma}_{A_2}, y\tilde{\sigma}_{A_2}\rangle + \\ &+ \langle 2\left(\frac{2\langle\tilde{R}_{A_2}\rangle\left[\langle\tilde{\rho}_{Rp}\rangle + \langle\tilde{R}_{A_1}\rangle\right] - \langle\tilde{R}_{A_1} + \tilde{R}_{A_2}\rangle\left[\langle\tilde{\rho}_{Rp}\rangle + \langle\tilde{R}_{A_2}\rangle\right]}{4\langle\tilde{R}_{A_1}\tilde{R}_{A_2}\rangle - \langle\tilde{R}_{A_1} + \tilde{R}_{A_2}\rangle^2}\right)\rangle \times \\ &\times \langle\left(\frac{2\langle\tilde{R}_{A_1}\rangle\left[\langle\tilde{\rho}_{Rp}\rangle + \langle\tilde{R}_{A_2}\rangle\right] - \langle\tilde{R}_{A_1} + \tilde{R}_{A_2}\rangle\left[\langle\tilde{\rho}_{Rp}\rangle + \langle\tilde{R}_{A_1}\rangle\right]}{4\langle\tilde{R}_{A_1}\tilde{R}_{A_2}\rangle - \langle\tilde{R}_{A_1} + \tilde{R}_{A_2}\rangle^2}\right)\tilde{\sigma}_{A_1A_2}; w\tilde{\sigma}_{A_1} \wedge w\tilde{\sigma}_{A_2}, u\tilde{\sigma}_{A_1} \vee u\tilde{\sigma}_{A_2}, y\tilde{\sigma}_{A_1} \vee y\tilde{\sigma}_{A_2}\rangle\end{aligned} \tag{45}$$

The above relation allows us to determine the portfolio risk consisting of two financial assets, modeled using triangular neutrosophic fuzzy numbers for which $\langle \tilde{R}_P; w\widetilde{Rp}, u\widetilde{Rp}, y\widetilde{Rp} \rangle = \langle \tilde{\rho}_{Rp}; w\widetilde{\rho_{Rp}}, u\widetilde{\rho_{Rp}}, y\widetilde{\rho_{Rp}} \rangle$ and portfolio risk is $\langle \tilde{\sigma}_P^2; w\widetilde{\sigma p}, u\widetilde{\sigma p}, y\widetilde{\sigma p} \rangle \to min$.

Example 1. *Let two financial assets (A_1, A_2) that have specific two neutrosophic triangular numbers for the return on financial assets of the form:*

$$\widetilde{R_{A_1}} = \langle (0.3\ 0.4\ 0.5); 0.5, 0.2, 0.3 \rangle \text{ for values of } \widetilde{R_A} \in [0.3; 0.5];$$

$$\widetilde{R_{A_2}} = \langle (0.2\ 0.3\ 0.4); 0.6, 0.3, 0.2 \rangle \text{ for values of } \widetilde{R_A} \in [0.2; 0.4];$$

Knowing that investors aim to achieve a portfolio return $\langle \tilde{R}_P; w\widetilde{Rp}, u\widetilde{Rp}, y\widetilde{Rp} \rangle = 0.3109$, it is required to establish the following:

(a) The neutrosophic average return for each asset: $E_f(\widetilde{Ra_1})$ and $E_f(\widetilde{Ra_2})$;
(b) The neutrosophic risk for each of the two assets: $\tilde{\sigma}_f a_1$ and $\tilde{\sigma}_f a_2$;
(c) The covariance between the two financial assets: $cov(\widetilde{Ra_1}, \widetilde{Ra_2})$;
(d) The portfolio return: $\langle \tilde{R}_P; w\widetilde{Rp}, u\widetilde{Rp}, y\widetilde{Rp} \rangle$ knowing that the weights of financial assets are $x_{A_1} = 0{,}4$ and $x_{A_2} = 0{,}6$;
(e) The portfolio risk: $\langle \tilde{\sigma}_P^2; w\widetilde{\sigma p}, u\widetilde{\sigma p}, y\widetilde{\sigma p} \rangle$; and
(f) The structure of the neutrosophic portfolio $(\tilde{x}_{A_1})$ and $(\tilde{x}_{A_2})$ for which $\langle \tilde{R}_P; w\widetilde{Rp}, u\widetilde{Rp}, y\widetilde{Rp} \rangle = 0{,}4000$ and the risk is minimal.

Solution:

Please note that all the computations required in order to solve the proposed example can be found in Appendix A.

(a) The determination of the neutrosophic average return for each financial asset, respectively for $\boldsymbol{E_f(\widetilde{Ra_1})}$ and $\boldsymbol{E_f(\widetilde{Ra_2})}$ is done using the calculation formula:

$$\langle \boldsymbol{E_f(\widetilde{R_A}); w\widetilde{R_A}, u\widetilde{R_A}, y\widetilde{R_A}} \rangle = \langle \left(\frac{1}{6}\left(\widetilde{R}_{A_{a1}} + \widetilde{R}_{A_{c1}}\right) + \frac{2}{3}\widetilde{R}_{A_{b1}} \right); \boldsymbol{w\widetilde{R_A}, u\widetilde{R_A}, y\widetilde{R_A}} \rangle$$

Thus, the following is obtained:

$$E_f(\widetilde{Ra_1}) = \langle 0{,}398; 0.5, 0.2, 0.3 \rangle$$

$$E_f(\widetilde{Ra_2}) = \langle 0{,}298; 0.5, 0.2, 0.3 \rangle;$$

The neutrosophic average return for the two financial assets is determined using the specific formula depending on its probability of realization, uncertainty, or failure of realization, according to the specific degrees of achievement of the neutrosophic numbers, namely: $\langle w\widetilde{R_A}, u\widetilde{R_A}, y\widetilde{R_A} \rangle$.

(b) The determination of the specific neutrosophic risk is made with the help of the formula:

$$\begin{aligned}
&\langle \boldsymbol{\sigma f_{A_i}^2; w\tilde{\sigma}_A, u\tilde{\sigma}_A, y\tilde{\sigma}_A} \rangle \\
&= \langle \frac{1}{4}\left[\left(\widetilde{R}_{A_{b1}} - \widetilde{R}_{A_{a1}}\right)^2 + \left(\widetilde{R}_{A_{c1}} - \widetilde{R}_{A_{b1}}\right)^2\right]; \boldsymbol{w\widetilde{R_A}, u\widetilde{R_A}, y\widetilde{R_A}} \rangle \\
&+ \langle \frac{2}{3}\left[\widetilde{R}_{A_{a1}}\left(\widetilde{R}_{A_{b1}} - \widetilde{R}_{A_{a1}}\right) - \widetilde{R}_{A_{c1}}\left(\widetilde{R}_{A_{c1}} - \widetilde{R}_{A_{b1}}\right)\right]; \boldsymbol{w\widetilde{R_A}, u\widetilde{R_A}, y\widetilde{R_A}} \rangle + \langle \frac{1}{2}\left(\widetilde{R}_{A_{a1}}^{\,2} + \widetilde{R}_{A_{c1}}^{\,2}\right); \boldsymbol{w\widetilde{R_A}, u\widetilde{R_A}, y\widetilde{R_A}} \rangle \\
&- \langle \frac{1}{2} \boldsymbol{E_f^2(\widetilde{Ra}_i); w\widetilde{R_A}, u\widetilde{R_A}, y\widetilde{R_A}} \rangle
\end{aligned}$$

Thus, the following are obtained:

$$\boldsymbol{\tilde{\sigma}_f^2 a_1 = \langle 0.083; 0.5, 0.2, 0.3 \rangle}$$

$$\boldsymbol{\widetilde{\sigma_f} a_1 = \langle 0,288; 0.5, 0.2, 0.3 \rangle}$$

$$\widetilde{\sigma}_f^2 a_2 = \langle \mathbf{0.048}; \mathbf{0.6}, \mathbf{0.3}, \mathbf{0.2} \rangle$$

$$\widetilde{\sigma_f} a_2 = \langle \mathbf{0,219}; \mathbf{0.5}, \mathbf{0.2}, \mathbf{0.3} \rangle$$

The risk of the two financial assets was determined using the specific formula, taking into account the degree of determination, namely the degree of certainty, uncertainty, and failure as follows: $\langle w\widetilde{R_A}, u\widetilde{R_A}, y\widetilde{R_A} \rangle$. The risk specific to these two assets is $\widetilde{\sigma_f} a_1 = \langle 0.288; 0.5, 0.2, 0.3 \rangle$ and $\widetilde{\sigma_f} a_2 = \langle 0.219; 0.6, 0.3, 0.2 \rangle$.

(c) The covariance between the two financial assets is also determined using the formula:

$$\begin{aligned} cov(\widetilde{Ra}_1, \widetilde{Ra}_2) = \langle \Bigg(& \frac{1}{4} [(\widetilde{Ra}_{b11} - \widetilde{Ra}_{a11})(\widetilde{Ra}_{b21} - \widetilde{Ra}_{a21}) + (\widetilde{Ra}_{c11} - \widetilde{Ra}_{b11})(\widetilde{Ra}_{c21} - \widetilde{Ra}_{b21})] \\ & + \frac{1}{3} \{ [\widetilde{Ra}_{a21}(\widetilde{Ra}_{b11} - \widetilde{Ra}_{a11}) + \widetilde{Ra}_{a11}(\widetilde{Ra}_{b21} - \widetilde{Ra}_{a21})] \\ & - [\widetilde{Ra}_{c11}(\widetilde{Ra}_{c21} - \widetilde{Ra}_{b21}) + \widetilde{Ra}_{c21}(\widetilde{Ra}_{c11} - \widetilde{Ra}_{b11})] \} + \frac{1}{2} (\widetilde{Ra}_{a11}\widetilde{Ra}_{a21} + \widetilde{Ra}_{c11}\widetilde{Ra}_{c21}) \\ & + \frac{1}{2} E_f(\widetilde{Ra}_1) E_f(\widetilde{Ra}_2) \Bigg); w\widetilde{Ra}_1 \wedge w\widetilde{Ra}_2, u\widetilde{Ra}_1 \vee u\widetilde{Ra}_2, y\widetilde{Ra}_1 \vee y\widetilde{Ra}_2 \rangle \end{aligned}$$

By replacing in the formula, the following is obtained:

$$cov(\widetilde{Ra}_1, \widetilde{Ra}_2) = \langle \mathbf{0.132}; \mathbf{0.6}, \mathbf{0.2}, \mathbf{0.2} \rangle$$

The variance-covariance matrix will be:

$$\Omega = \begin{pmatrix} \langle 0288; 0.5, 0.2, 0.3 \rangle & \langle 0.132; 0.6, 0.2, 0.2 \rangle \\ \langle 0.132; 0.6, 0.2, 0.2 \rangle & \langle 0.219; 0.6, 0.3, 0.2 \rangle \end{pmatrix}$$

The covariance between the two financial assets $cov(\widetilde{Ra}_1, \widetilde{Ra}_2) = \langle 0.132; 0.6, 0.2, 0.2 \rangle$ has a relatively small and positive value, which means that the connection between the two financial assets is relatively weak. When the value of one financial asset return increases, the value of the other financial asset return will also increase, while if the value of one of the financial asset return decreases, the value of the other financial asset will also decrease.

(d) The neutrosophic portfolio return consisting of two assets will be determined using the formula:

$$\langle \widetilde{R}_P; w\widetilde{Rp}, u\widetilde{Rp}, y\widetilde{Rp} \rangle = \sum_{i=1}^{n} \langle x_{A_i} \left(\frac{1}{6} (\widetilde{R}_{Aai} + \widetilde{R}_{Aci}) + \frac{2}{3} \widetilde{R}_{Abi} \right); w\widetilde{R}_{A_i}, u\widetilde{R}_{A_i}, y\widetilde{R}_{A_i} \rangle$$

By replacing in the above formula, the following is obtained:

$$\langle \widetilde{R}_P; w\widetilde{Rp}, u\widetilde{Rp}, y\widetilde{Rp} \rangle = \langle \mathbf{0,337}; \mathbf{0.5}, \mathbf{0.2}, \mathbf{0.3} \rangle$$

The portfolio risk is determined according to the weight of the two financial assets in the total portfolio and the neutrosophic average financial return. The average neutrosophic portfolio return has the value $\langle \widetilde{R}_P; w\widetilde{Rp}, u\widetilde{Rp}, y\widetilde{Rp} \rangle = \langle 0,337; 0.5, 0.2, 0.3 \rangle$, which indicates that the portfolio has a relatively low return value, leading to a relatively small neutrosophic portfolio risk.

(e) The neutrosophic portfolio risk is determined using the formula:

$$\begin{aligned} \langle \sigma_p^2; w\widetilde{\sigma p}, u\widetilde{\sigma p}, y\widetilde{\sigma p} \rangle & \\ = \langle x_{A_1}^2 \widetilde{\sigma}_{A_1}^2; w\widetilde{\sigma}_{A_1}, u\widetilde{\sigma}_{A_1}, y\widetilde{\sigma}_{A_1} \rangle & + \langle x_{A_2}^2 \widetilde{\sigma}_{A_2}^2; w\widetilde{\sigma}_{A_2}, u\widetilde{\sigma}_{A_2}, y\widetilde{\sigma}_{A_2} \rangle \\ + 2x_{A_1} x_{A_2} \widetilde{\sigma}_{A_1A_2}; w\widetilde{\sigma}_{A_1} \wedge w\widetilde{\sigma}_{A_2}, & u\widetilde{\sigma}_{A_1} \vee u\widetilde{\sigma}_{A_2}, y\widetilde{\sigma}_{A_1} \vee y\widetilde{\sigma}_{A_2} \end{aligned}$$

By replacing in the above formula, the following is obtained:

$$\langle \sigma_p^2; w\widetilde{\sigma p}, u\widetilde{\sigma p}, y\widetilde{\sigma p} \rangle = \langle \mathbf{0,036}; \mathbf{0.6}, \mathbf{0.2}, \mathbf{0.2} \rangle$$

$$\langle \sigma_p; w\widetilde{\sigma p}, u\widetilde{\sigma p}, y\widetilde{\sigma p} \rangle = \langle \mathbf{0.1897}; \mathbf{0.6}, \mathbf{0.2}, \mathbf{0.2} \rangle$$

The value of the neutrosophic portfolio risk $\langle\sigma_p; w\widetilde{\sigma p}, u\widetilde{\sigma p}, y\widetilde{\sigma p}\rangle = \langle 0.1897; 0.6, 0.2, 0.2\rangle$ obtained using the above formula is proportional to the value of the average neutrosophic return $\langle\tilde{R}_P; w\widetilde{Rp}, u\widetilde{Rp}, y\widetilde{Rp}\rangle = \langle 0{,}337; 0.5, 0.2, 0.3\rangle$, which validates the theory that a relatively low profitability value corresponds to a relatively low risk value.

(f) Determining the structure of the portfolio when the value of the return expected by investors is $\langle\tilde{R}_P; w\widetilde{Rp}, u\widetilde{Rp}, y\widetilde{Rp}\rangle = 0{,}400$, which can be done using formulas established according to theorem 1, as follows:

$$\tilde{\boldsymbol{x}}_{A_1} = \langle\frac{2\langle\tilde{\boldsymbol{R}}_{A_2}\rangle\left[\langle\tilde{\boldsymbol{\rho}}_{Rp}\rangle + \langle\tilde{\boldsymbol{R}}_{A_1}\rangle\right] - \langle\tilde{\boldsymbol{R}}_{A_1} + \tilde{\boldsymbol{R}}_{A_2}\rangle\left[\langle\tilde{\boldsymbol{\rho}}_{Rp}\rangle + \langle\tilde{\boldsymbol{R}}_{A_2}\rangle\right]}{4\langle\tilde{\boldsymbol{R}}_{A_1}\tilde{\boldsymbol{R}}_{A_2}\rangle - \langle\tilde{\boldsymbol{R}}_{A_1} + \tilde{\boldsymbol{R}}_{A_2}\rangle^2} \boldsymbol{w}\tilde{\boldsymbol{x}}_{A_1} \wedge \boldsymbol{w}\tilde{\boldsymbol{x}}_{A_2}, \boldsymbol{u}\tilde{\boldsymbol{x}}_{A_1} \vee \boldsymbol{u}\tilde{\boldsymbol{x}}_{A_2}, \boldsymbol{y}\tilde{\boldsymbol{x}}_{A_1} \vee \boldsymbol{y}\tilde{\boldsymbol{x}}_{A_2}\rangle$$

and

$$\tilde{\boldsymbol{x}}_{A_2} = \langle\frac{2\langle\tilde{\boldsymbol{R}}_{A_1}\rangle\left[\langle\tilde{\boldsymbol{\rho}}_{Rp}\rangle + \langle\tilde{\boldsymbol{R}}_{A_2}\rangle\right] - \langle\tilde{\boldsymbol{R}}_{A_1} + \tilde{\boldsymbol{R}}_{A_2}\rangle\left[\langle\tilde{\boldsymbol{\rho}}_{Rp}\rangle + \langle\tilde{\boldsymbol{R}}_{A_1}\rangle\right]}{4\langle\tilde{\boldsymbol{R}}_{A_1}\tilde{\boldsymbol{R}}_{A_2}\rangle - \langle\tilde{\boldsymbol{R}}_{A_1} + \tilde{\boldsymbol{R}}_{A_2}\rangle^2} \boldsymbol{w}\tilde{\boldsymbol{x}}_{A_1} \wedge \boldsymbol{w}\tilde{\boldsymbol{x}}_{A_2}, \boldsymbol{u}\tilde{\boldsymbol{x}}_{A_1} \vee \boldsymbol{u}\tilde{\boldsymbol{x}}_{A_2}, \boldsymbol{y}\tilde{\boldsymbol{x}}_{A_1} \vee \boldsymbol{y}\tilde{\boldsymbol{x}}_{A_2}\rangle$$

By replacing in the formulas, the following will be obtained:

$$\tilde{\boldsymbol{x}}_{A_1} = \langle \mathbf{1, 020; 0.5, 0.2, 0.3}\rangle$$

and

$$\tilde{\boldsymbol{x}}_{A_2} = \langle \mathbf{0.02; 0.5, 0.2, 0.3}\rangle$$

The first financial asset has a neutrosophic average return rate of $E_f(\widetilde{Ra}_1) = \langle 0{,}398; 0.5, 0.2, 0.3\rangle$ and therefore the share of investment in the first financial asset will be $\tilde{x}_{A_1} = \langle 1.020; 0.5, 0.2, 0.3\rangle$, while the second financial asset has a neutrosophic average return rate of $E_f(\widetilde{Ra}_2) = \langle 0{,}298; 0.5, 0.2, 0.3\rangle$ and a weight of $\tilde{x}_{A_2} = \langle 0.02; 0.5, 0.2, 0.3\rangle$. Thus, establishing the portfolio return formed with the structure of those two assets will be:

$$\langle\tilde{\boldsymbol{R}}_P; \boldsymbol{w}\widetilde{\boldsymbol{Rp}}, \boldsymbol{u}\widetilde{\boldsymbol{Rp}}, \boldsymbol{y}\widetilde{\boldsymbol{Rp}}\rangle = \langle \mathbf{0.4109; 0.5, 0.2, 0.3}\rangle$$

Compared to the proposed return $\langle\tilde{R}_P; w\widetilde{Rp}, u\widetilde{Rp}, y\widetilde{Rp}\rangle = \langle 0.4000;\ 0.5, 0.2, 0.3\rangle$, the portfolio return was obtained as $\langle\tilde{R}_P; w\widetilde{Rp}, u\widetilde{Rp}, y\widetilde{Rp}\rangle = \langle 0.4109; 0.5, 0.2, 0.3\rangle$, respectively a resulting error: $\Delta\varepsilon = \langle 0.4109; 0.5, 0.2, 0.3\rangle - \langle 0.4000;\ 0.5, 0.2, 0.3\rangle$, or $\Delta\varepsilon$ = 1.09% the acceptable error rate for the calculation accuracy. Regarding the portfolio risk, it will be determined as follows:

$$\langle\boldsymbol{\sigma}_p^2; \boldsymbol{w}\widetilde{\boldsymbol{\sigma p}}, \boldsymbol{u}\widetilde{\boldsymbol{\sigma p}}, \boldsymbol{y}\widetilde{\boldsymbol{\sigma p}}\rangle = \mathbf{0.0914; 0.6, 0.2, 0.2}$$

$$\langle\boldsymbol{\sigma}_p; \boldsymbol{w}\widetilde{\boldsymbol{\sigma p}}, \boldsymbol{u}\widetilde{\boldsymbol{\sigma p}}, \boldsymbol{y}\widetilde{\boldsymbol{\sigma p}}\rangle = \langle \mathbf{0.3023; 0.6, 0.2, 0.2}\rangle$$

The portfolio risk $\langle\sigma_p; w\widetilde{\sigma p}, u\widetilde{\sigma p}, y\widetilde{\sigma p}\rangle = \langle 0.3023; 0.6, 0.2, 0.2\rangle$ obtained by modeling using neutrosophic triangular fuzzy numbers is proportional to the value of the portfolio return $\langle\tilde{R}_P; w\widetilde{Rp}, u\widetilde{Rp}, y\widetilde{Rp}\rangle = \langle 0.4109; 0.5, 0.2, 0.3\rangle$, so the mathematical model obtained for determining the portfolio structure for $(\tilde{x}_{A_1})$ and $(\tilde{x}_{A_2})$ validates the calculations performed for the optimal neutrosophic portfolios consisting of two financial assets.

5. Optimal Neutrosophic Portfolios Consisting of N-Financial Assets

Theorem 2. *Let a neutrosophic portfolio consisting of N-financial assets* (A_i) *for which the neutrosophic average return and the neutrosophic average risk are determined:* $A_i: \langle\widetilde{R_{A_i}}, \tilde{\sigma}_{A_i}\rangle$.

This portfolio consisting of N-financial assets is an optimal neutrosophic portfolio if two basic cumulative conditions are respected, namely:

- The portfolio return reaches a level of $\langle\tilde{R}_P; w\widetilde{Rp}, u\widetilde{Rp}, y\widetilde{Rp}\rangle = \langle\tilde{\rho}_{Rp}; w\widetilde{\rho_{Rp}}, u\widetilde{\rho_{Rp}}, y\widetilde{\rho_{Rp}}\rangle$; and

- The portfolio risk tends to a minimum: $\langle \tilde{\sigma}_P^2; w\widetilde{\sigma p}, u\widetilde{\sigma p}, y\widetilde{\sigma p}\rangle \to min$.

For such a portfolio, its structure is determined by the calculation relations:

$$\tilde{X} = \langle \frac{1}{\langle\alpha\rangle\langle\gamma\rangle - \langle\beta\rangle^2}\left[\left(\langle\tilde{\rho}_{R_p}\rangle\langle\alpha\rangle - \langle\beta\rangle\right)\langle\tilde{\Omega}^{-1}\times\tilde{R}\rangle + \left(\langle\gamma\rangle - \langle\tilde{\rho}_{R_p}\rangle\langle\beta\rangle\right)\langle\tilde{\Omega}^{-1}\times\tilde{e}\rangle\right]; w\tilde{x}_{A_k}, u\tilde{x}_{A_k}, y\tilde{x}_{A_k}\rangle \quad (46)$$

with the following significations:

$$\begin{gathered}\langle\alpha\rangle = \langle\tilde{e}^T\times\tilde{\Omega}^{-1}\times\tilde{e}\rangle \\ \langle\beta\rangle = \langle\tilde{R}^T\times\tilde{\Omega}^{-1}\times\tilde{e}\rangle = \langle\tilde{e}^T\times\tilde{\Omega}^{-1}\times\tilde{R}\rangle \\ \langle\gamma\rangle = \langle\tilde{R}^T\times\tilde{\Omega}^{-1}\times\tilde{R}\rangle\end{gathered} \quad (47)$$

Note 3: $\langle\tilde{X}; w\tilde{x}_{A_k}, u\tilde{x}_{A_k}, y\tilde{x}_{A_k}\rangle$ represents the portfolio structure consisting of N-financial assets, where the portfolio meets the optimal portfolio conditions, namely the return is fixed to an estimated level and the risk tends to a minimum. The term meanings involved in the calculation formula are presented in the demonstration of theorem no. 2. The parameters $\langle\alpha\rangle, \langle\beta\rangle, \langle\gamma\rangle$ are used to simplify the calculation methodology for the portfolio structure.

For such an optimal neutrosophic portfolio, the portfolio risk is computed with the help of the formula:

$$\langle\tilde{\sigma}_P^2; w\widetilde{\sigma p}, u\widetilde{\sigma p}, y\widetilde{\sigma p}\rangle = \langle\frac{1}{\langle\alpha\rangle\langle\gamma\rangle - \langle\beta\rangle^2}\left[\langle\alpha\rangle\langle\tilde{\rho}_{R_p}\rangle^2 - 2\langle\beta\rangle\langle\tilde{\rho}_{R_p}\rangle + \langle\gamma\rangle\right]; w\widetilde{\sigma p}, u\widetilde{\sigma p}, y\widetilde{\sigma p}\rangle \quad (48)$$

Note 4: $\langle\tilde{\sigma}_P^2; w\widetilde{\sigma p}, u\widetilde{\sigma p}, y\widetilde{\sigma p}\rangle$ represents the calculation formula for the portfolio risk consisting of N-financial assets, with the known conditions of the optimal portfolio and with the meanings of the terms presented in demonstration of theorem no.2. The parameters $\langle\alpha\rangle, \langle\beta\rangle, \langle\gamma\rangle$ are parameters used to simplify the calculation methodology for the portfolio risk.

Note 5: The portfolio risk relationship is also known as the Markowitz frontier applied to this neutrosophic portfolio category, respectively those portfolios for which the degree of achievement, uncertainty, or non-realization is known, and represents the base for the investment decisions on capital market.

Demonstration: In order to establish the structure of an optimal neutrosophic portfolio consisting of N-financial assets, we started from the neutrosophic portfolio equations in analytical form, determined by the portfolio return $\langle\tilde{R}_P; w\widetilde{Rp}, u\widetilde{Rp}, y\widetilde{Rp}\rangle$, portfolio risk $\langle\sigma_p^2; w\widetilde{\sigma p}, u\widetilde{\sigma p}, y\widetilde{\sigma p}\rangle$ as well as the relationships between the financial asset weights in the total portfolio $\langle\tilde{x}_{A_i}\rangle$, according to the equations below:

$$\begin{cases}\langle\tilde{R}_P; w\widetilde{Rp}, u\widetilde{Rp}, y\widetilde{Rp}\rangle = \langle x_{A_1}\tilde{R}_{A_1}; w\tilde{R}_{A_1}, u\tilde{R}_{A_1}, y\tilde{R}_{A_1}\rangle + \langle x_{A_2}\tilde{R}_{A_2}; w\tilde{R}_{A_2}, u\tilde{R}_{A_2}, y\tilde{R}_{A_2}\rangle + \cdots + \\ \quad + \langle x_{A_n}\tilde{R}_{A_n}; w\tilde{R}_{A_n}, u\tilde{R}_{A_n}, y\tilde{R}_{A_n}\rangle \\ \langle\tilde{\sigma}_P^2; w\widetilde{\sigma p}, u\widetilde{\sigma p}, y\widetilde{\sigma p}\rangle = \langle x_{A_1}^2\tilde{\sigma}_{A_1}^2; w\tilde{\sigma}_{A_1}, u\tilde{\sigma}_{A_1}, y\tilde{\sigma}_{A_1}\rangle + \langle x_{A_2}^2\tilde{\sigma}_{A_2}^2; w\tilde{\sigma}_{A_2}, u\tilde{\sigma}_{A_2}, y\tilde{\sigma}_{A_2}\rangle + \\ \ldots + \langle x_{A_n}^2\tilde{\sigma}_{A_n}^2; w\tilde{\sigma}_{A_n}, u\tilde{\sigma}_{A_n}, y\tilde{\sigma}_{A_n}\rangle + \langle 2x_{A_1}x_{A_2}\tilde{\sigma}_{A_1A_2}; w\tilde{\sigma}_{A_1}\wedge w\tilde{\sigma}_{A_2}, u\tilde{\sigma}_{A_1}\vee u\tilde{\sigma}_{A_2}, y\tilde{\sigma}_{A_1}\vee y\tilde{\sigma}_{A_2}\rangle + \\ \langle 2x_{A_1}x_{A_3}\tilde{\sigma}_{A_1A_3}; w\tilde{\sigma}_{A_1}\wedge w\tilde{\sigma}_{A_3}, u\tilde{\sigma}_{A_1}\vee u\tilde{\sigma}_{A_3}, y\tilde{\sigma}_{A_1}\vee y\tilde{\sigma}_{A_3}\rangle + \cdots + \\ \langle 2x_{A_1}x_{A_n}\tilde{\sigma}_{A_1A_n}; w\tilde{\sigma}_{A_1}\wedge w\tilde{\sigma}_{A_n}, u\tilde{\sigma}_{A_1}\vee u\tilde{\sigma}_{A_n}, y\tilde{\sigma}_{A_1}\vee y\tilde{\sigma}_{A_n}\rangle + \\ \langle 2x_{A_2}x_{A_1}\tilde{\sigma}_{A_2A_1}; w\tilde{\sigma}_{A_2}\wedge w\tilde{\sigma}_{A_1}, u\tilde{\sigma}_{A_2}\vee u\tilde{\sigma}_{A_1}, y\tilde{\sigma}_{A_2}\vee y\tilde{\sigma}_{A_1}\rangle + \\ \langle 2x_{A_2}x_{A_3}\tilde{\sigma}_{A_2A_3}; w\tilde{\sigma}_{A_2}\wedge w\tilde{\sigma}_{A_3}, u\tilde{\sigma}_{A_2}\vee u\tilde{\sigma}_{A_3}, y\tilde{\sigma}_{A_2}\vee y\tilde{\sigma}_{A_3}\rangle + \cdots + \\ \langle 2x_{A_2}x_{A_n}\tilde{\sigma}_{A_2A_n}; w\tilde{\sigma}_{A_2}\wedge w\tilde{\sigma}_{A_n}, u\tilde{\sigma}_{A_2}\vee u\tilde{\sigma}_{A_n}, y\tilde{\sigma}_{A_2}\vee y\tilde{\sigma}_{A_n}\rangle + \\ \langle 2x_{A_n}x_{A_1}\tilde{\sigma}_{A_nA_1}; w\tilde{\sigma}_{A_n}\wedge w\tilde{\sigma}_{A_1}, u\tilde{\sigma}_{A_n}\vee u\tilde{\sigma}_{A_1}, y\tilde{\sigma}_{A_n}\vee y\tilde{\sigma}_{A_1}\rangle + \\ \langle 2x_{A_n}x_{A_2}\tilde{\sigma}_{A_nA_2}; w\tilde{\sigma}_{A_n}\wedge w\tilde{\sigma}_{A_2}, u\tilde{\sigma}_{A_n}\vee u\tilde{\sigma}_{A_2}, y\tilde{\sigma}_{A_n}\vee y\tilde{\sigma}_{A_2}\rangle + \cdots + \\ \langle 2x_{A_n}x_{A_{n-1}}\tilde{\sigma}_{A_nA_{n-1}}; w\tilde{\sigma}_{A_n}\wedge w\tilde{\sigma}_{A_{n-1}}, u\tilde{\sigma}_{A_n}\vee u\tilde{\sigma}_{A_{n-1}}, y\tilde{\sigma}_{A_n}\vee y\tilde{\sigma}_{A_{n-1}}\rangle \\ \langle x_{A_1}; w\tilde{R}_{A_1}, u\tilde{R}_{A_1}, y\tilde{R}_{A_1}\rangle + \langle x_{A_2}w\tilde{R}_{A_2}, u\tilde{R}_{A_2}, y\tilde{R}_{A_2}\rangle + \cdots + \langle x_{A_n}w\tilde{R}_{A_n}, u\tilde{R}_{A_n}, y\tilde{R}_{A_n}\rangle = 1\end{cases} \quad (49)$$

In a restricted form, the neutrosophic portfolio equations in analytical form can be written as follows:

$$\begin{cases} \langle \tilde{R}_P; w\widetilde{Rp}, u\widetilde{Rp}, y\widetilde{Rp}\rangle = \langle \sum_{k=1}^{n} \tilde{x}_{A_k}\tilde{R}_{A_k}; w\tilde{R}_{A_k}, u\tilde{R}_{A_k}, y\tilde{R}_{A_k}\rangle \\ \langle \tilde{\sigma}_P^2; w\widetilde{\sigma p}, u\widetilde{\sigma p}, y\widetilde{\sigma p}\rangle = \langle \sum_{k=1}^{n} \tilde{x}_{A_k}^2 \tilde{R}_{A_k}^2; w\tilde{R}_{A_k}, u\tilde{R}_{A_k}, y\tilde{R}_{A_k} + 2\sum_{k=1}^{n}\sum_{j=1}^{n} \tilde{x}_k\tilde{x}_j\tilde{\sigma}_{kj}; w\tilde{\sigma}_{kj}, u\tilde{\sigma}_{kj}, y\tilde{\sigma}_{kj}\rangle \\ \langle \sum_{i=1}^{n} \tilde{x}_{A_k}; w\tilde{R}_{A_k}, u\tilde{R}_{A_k}, y\tilde{R}_{A_k}\rangle = 1 \end{cases} \quad (50)$$

The equations of the financial asset neutrosophic portfolio written in a restricted form must respect the optimal conditions, respectively the neutrosophic portfolio return has to tend toward a certain value $\langle \tilde{R}_P; w\widetilde{Rp}, u\widetilde{Rp}, y\widetilde{Rp}\rangle = \langle \tilde{\rho}_{R_p}; w\tilde{R}_{R_P}, u\tilde{R}_{R_P}, y\tilde{R}_{R_P}\rangle$ and the value of the portfolio risk is minimal, respectively $\langle \tilde{\sigma}_P^2; w\widetilde{\sigma p}, u\widetilde{\sigma p}, y\widetilde{\sigma p}\rangle \to min$. Thus, the financial asset optimal neutrosophic portfolio equations become:

$$\begin{cases} \langle \sum_{k=1}^{n} \tilde{x}_{A_k}\tilde{R}_{A_k}; w\tilde{R}_{A_k}, u\tilde{R}_{A_k}, y\tilde{R}_{A_k}\rangle = \langle \tilde{\rho}_{R_p}; w\tilde{R}_{R_P}, u\tilde{R}_{R_P}, y\tilde{R}_{R_P}\rangle \\ \langle \frac{1}{2}\sum_{k=1}^{n}\sum_{j=1}^{n} \tilde{x}_k\tilde{x}_j\tilde{\sigma}_{kj}; w\tilde{\sigma}_{kj}, u\tilde{\sigma}_{kj}, y\tilde{\sigma}_{kj}\rangle \to min \\ \langle \sum_{i=1}^{n} \tilde{x}_{A_i}; w\tilde{R}_{A_i}, u\tilde{R}_{A_i}, y\tilde{R}_{A_i}\rangle = 1 \end{cases} \quad (51)$$

The optimal problem formulated with the help of the optimal neutrosophic portfolio equations is solved using the Lagrange function for which the objective function is to minimize the portfolio risk $\langle \tilde{\sigma}_P^2; w\widetilde{\sigma p}, u\widetilde{\sigma p}, y\widetilde{\sigma p}\rangle \to min$, given that the portfolio return $\langle \tilde{R}_P; w\widetilde{Rp}, u\widetilde{Rp}, y\widetilde{Rp}\rangle = \langle \tilde{\rho}_{Rp}; w\widetilde{\rho_{Rp}}, u\widetilde{\rho_{Rp}}, y\widetilde{\rho_{Rp}}\rangle$ will reach a level of $\langle \tilde{\rho}_{Rp}; w\widetilde{\rho_{Rp}}, u\widetilde{\rho_{Rp}}, y\widetilde{\rho_{Rp}}\rangle$ and the sum of the two financial assets weights is equal to 1. In these conditions, we will have:

$$L = fobjective - \lambda_1 C_1 - \lambda_2 C_2 \quad (52)$$

The Lagrange function to obtain the optimal conditions can be written as follows:

$$\begin{aligned} L = min\,\langle \frac{1}{2}\sum_{k=1}^{n}\sum_{j=1}^{n} \tilde{x}_{A_k}\tilde{x}_{A_j}\tilde{\sigma}_{A_{kj}}; w\tilde{\sigma}_{kj}, u\tilde{\sigma}_{kj}, y\tilde{\sigma}_{kj}\rangle - \lambda_1\left(\langle \sum_{k=1}^{n} \tilde{x}_{A_k}\tilde{R}_{A_k}; w\tilde{R}_{A_k}, u\tilde{R}_{A_k}, y\tilde{R}_{A_k}\rangle - \langle \tilde{\rho}_{R_p}; w\tilde{R}_{R_P}, u\tilde{R}_{R_P}, y\tilde{R}_{R_P}\rangle\right) \\ - \lambda_2\left(\langle \sum_{k=1}^{n} \tilde{x}_{A_k}; w\tilde{R}_{A_k}, u\tilde{R}_{A_k}, y\tilde{R}_{A_k}\rangle - 1\right) \end{aligned} \quad (53)$$

The optimal conditions for the Lagrange function are set as:

$$\begin{cases} \frac{\partial L}{\partial x_{A1}} = 0 \\ \frac{\partial L}{\partial x_{Ak}} = 0 \\ \frac{\partial L}{\partial \lambda_1} = 0 \\ \frac{\partial L}{\partial \lambda_2} = 0 \end{cases} \quad (54)$$

As a result of performing the calculations, the system of equations is obtained and can be written with the help of the mathematical operator of the form:

$$\begin{cases} \sum_{j=1}^{n} \langle \tilde{x}_{A_j} \tilde{\sigma}_{A_{kj}}; w\tilde{\sigma}_{kj}, u\tilde{\sigma}_{kj}, y\tilde{\sigma}_{kj} \rangle - \lambda_1 \langle \sum_{k=1}^{n} \tilde{R}_{A_k}; w\tilde{R}_{A_k}, u\tilde{R}_{A_k}, y\tilde{R}_{A_k} \rangle - \lambda_2 \langle 1; w\tilde{R}_{A_k}, u\tilde{R}_{A_k}, y\tilde{R}_{A_k} \rangle = 0 \\ \langle \sum_{k=1}^{n} \tilde{x}_{A_k} \tilde{R}_{A_k}; w\tilde{R}_{A_k}, u\tilde{R}_{A_k}, y\tilde{R}_{A_k} \rangle - \langle \tilde{\rho}_{R_p}; w\tilde{R}_{R_P}, u\tilde{R}_{R_P}, y\tilde{R}_{R_P} \rangle = 0 \\ \langle \sum_{k=1}^{n} \tilde{x}_{A_k}; w\tilde{R}_{A_k}, u\tilde{R}_{A_k}, y\tilde{R}_{A_k} \rangle - 1 = 0 \end{cases} \tag{55}$$

The matrix form of the above equations is written as follows:

$$\begin{cases} \langle \begin{pmatrix} \tilde{\sigma}_{A_{11}} & \tilde{\sigma}_{A_{12}} & \dots & \tilde{\sigma}_{A_{1n}} \\ \tilde{\sigma}_{A_{21}} & \tilde{\sigma}_{A_{22}} & \dots & \tilde{\sigma}_{A_{2n}} \\ \dots & \dots & \dots & \dots \\ \tilde{\sigma}_{A_{n1}} & \tilde{\sigma}_{A_{n2}} & \dots & \tilde{\sigma}_{A_{nn}} \end{pmatrix} \begin{pmatrix} \tilde{x}_1 \\ \tilde{x}_2 \\ \dots \\ \tilde{x}_n \end{pmatrix}; w\tilde{\sigma}_{kj}, u\tilde{\sigma}_{kj}, y\tilde{\sigma}_{kj} \rangle - \langle \lambda_1 \begin{pmatrix} \tilde{R}_{A_1} \\ \tilde{R}_{A_2} \\ \dots \\ \tilde{R}_{A_n} \end{pmatrix}; w\tilde{R}_{A_k}, u\tilde{R}_{A_k}, y\tilde{R}_{A_k} \rangle - \\ - \langle \lambda_2 \begin{pmatrix} 1 \\ 1 \\ \dots \\ 1 \end{pmatrix}; w\tilde{R}_{A_k}, u\tilde{R}_{A_k}, y\tilde{R}_{A_k} \rangle = 0 \\ \langle \left(\tilde{x}_{A_1} \tilde{x}_{A_2} \dots \tilde{x}_{A_n} \right) \begin{pmatrix} \tilde{R}_{A_1} \\ \tilde{R}_{A_2} \\ \dots \\ \tilde{R}_{A_n} \end{pmatrix}; w\tilde{R}_{A_k}, u\tilde{R}_{A_k}, y\tilde{R}_{A_k} \rangle = \langle \tilde{\rho}_{R_p}; w\tilde{R}_{R_P}, u\tilde{R}_{R_P}, y\tilde{R}_{R_P} \rangle \\ \langle \left(\tilde{x}_{A_1} \tilde{x}_{A_2} \dots \tilde{x}_{A_n} \right) \begin{pmatrix} 1 \\ 1 \\ \dots \\ 1 \end{pmatrix}; w\tilde{R}_{A_k}, u\tilde{R}_{A_k}, y\tilde{R}_{A_k} \rangle = 1 \end{cases} \tag{56}$$

In the system of equations above, we operated the following notations:

- The variance–covariance matrix is:

$$\tilde{\Omega} = \begin{pmatrix} \tilde{\sigma}_{A_{11}} & \tilde{\sigma}_{A_{12}} & \dots & \tilde{\sigma}_{A_{1n}} \\ \tilde{\sigma}_{A_{21}} & \tilde{\sigma}_{A_{22}} & \dots & \tilde{\sigma}_{A_{2n}} \\ \dots & \dots & \dots & \dots \\ \tilde{\sigma}_{A_{n1}} & \tilde{\sigma}_{A_{n2}} & \dots & \tilde{\sigma}_{A_{nn}} \end{pmatrix} \tag{57}$$

- The portfolio structure column vector is:

$$\tilde{X} = \begin{pmatrix} \tilde{x}_1 \\ \tilde{x}_2 \\ \dots \\ \tilde{x}_n \end{pmatrix} \tag{58}$$

- The portfolio return column vector is:

$$\tilde{R} = \begin{pmatrix} \tilde{R}_{A_1} \\ \tilde{R}_{A_2} \\ \dots \\ \tilde{R}_{A_n} \end{pmatrix} \tag{59}$$

- The portfolio unit column vector:

$$\tilde{e} = \begin{pmatrix} 1 \\ 1 \\ \dots \\ 1 \end{pmatrix} \tag{60}$$

With the above notations, the matrix system of equations for the optimal financial asset neutrosophic portfolio becomes:

$$\begin{cases} \langle \widetilde{\Omega} \times \tilde{X}; w\tilde{\sigma}_{kj}, u\tilde{\sigma}_{kj}, y\tilde{\sigma}_{kj} \rangle - \langle \lambda_1 \tilde{R}; w\tilde{R}_{A_k}, u\tilde{R}_{A_k}, y\tilde{R}_{A_k} \rangle - \langle \lambda_2 e; w\tilde{R}_{A_k}, u\tilde{R}_{A_k}, y\tilde{R}_{A_k} \rangle = 0 \\ \langle \tilde{X}^T \times \tilde{R}; w\tilde{R}_{A_k}, u\tilde{R}_{A_k}, y\tilde{R}_{A_k} \rangle = \langle \tilde{\rho}_{R_p}; w\tilde{R}_{R_P}, u\tilde{R}_{R_P}, y\tilde{R}_{R_P} \rangle \\ \langle \tilde{X}^T \times \tilde{e}; w\tilde{R}_{A_k}, u\tilde{R}_{A_k}, y\tilde{R}_{A_k} \rangle = 1 \end{cases} \tag{61}$$

In the system of equations above are operated the following notations:

$$\begin{gathered} \langle \widetilde{\Omega} \times \tilde{X} \rangle = \langle \widetilde{\Omega} \times \tilde{X}; w\tilde{\sigma}_{kj}, u\tilde{\sigma}_{kj}, y\tilde{\sigma}_{kj} \rangle \\ \langle \tilde{R} \rangle = \langle \tilde{R}; w\tilde{R}_{A_k}, u\tilde{R}_{A_k}, y\tilde{R}_{A_k} \rangle \\ \langle \tilde{e} \rangle = \langle e; w\tilde{R}_{A_k}, u\tilde{R}_{A_k}, y\tilde{R}_{A_k} \rangle \\ \langle \tilde{\rho}_{R_p} \rangle = \langle \tilde{\rho}_{R_p}; w\tilde{R}_{R_P}, u\tilde{R}_{R_P}, y\tilde{R}_{R_P} \rangle \\ \langle \tilde{X}^T \times \tilde{R} \rangle = \langle \tilde{X}^T \times R; w\tilde{R}_{A_k}, u\tilde{R}_{A_k}, y\tilde{R}_{A_k} \rangle \\ \langle \tilde{X}^T \times \tilde{e} \rangle = \langle \tilde{X}^T \times e; w\tilde{R}_{A_k}, u\tilde{R}_{A_k}, y\tilde{R}_{A_k} \rangle \end{gathered} \tag{62}$$

Thus, it will result in:

$$\begin{cases} \langle \widetilde{\Omega} \times \tilde{X} \rangle - \lambda_1 \langle \tilde{R} \rangle - \lambda_2 \langle e \rangle = 0 \\ \langle \tilde{X}^T \times \tilde{R} \rangle = \langle \tilde{\rho}_{R_p} \rangle \\ \langle \tilde{X}^T \times \tilde{e} \rangle = 1 \end{cases} \tag{63}$$

The structure vector of the optimal neutrosophic portfolio $\langle \tilde{X} \rangle$ is determined from the first equation of the above system and replaced in the other two equations, so that the following is obtained:

$$\tilde{X} = \lambda_1 \langle \widetilde{\Omega}^{-1} \times \tilde{R} \rangle + \lambda_2 \langle \widetilde{\Omega}^{-1} \times \tilde{e} \rangle \tag{64}$$

and respectively the system of equations of the form:

$$\begin{cases} \langle \tilde{R}^T \times \widetilde{\Omega}^{-1} \times \tilde{R} \rangle \lambda_1 + \langle \tilde{e}^T \times \widetilde{\Omega}^{-1} \times \tilde{R} \rangle \lambda_2 = \langle \tilde{\rho}_{R_p} \rangle \\ \langle \tilde{R}^T \times \widetilde{\Omega}^{-1} \times \tilde{e} \rangle \lambda_1 + \langle \tilde{e}^T \times \widetilde{\Omega}^{-1} \times \tilde{e} \rangle \lambda_2 = 1 \end{cases} \tag{65}$$

The solutions of the system are computed with the unknowns (λ_1) and (λ_2) with the help of Cramer's rules and results:

$$\lambda_1 = \frac{\begin{vmatrix} \langle \tilde{\rho}_{R_p} \rangle & \tilde{e}^T \times \widetilde{\Omega}^{-1} \times \tilde{R} \\ 1 & \langle \tilde{e}^T \times \widetilde{\Omega}^{-1} \times \tilde{e} \rangle \end{vmatrix}}{\begin{vmatrix} \langle \tilde{R}^T \times \widetilde{\Omega}^{-1} \times \tilde{R} \rangle & \langle \tilde{e}^T \times \widetilde{\Omega}^{-1} \times \tilde{R} \rangle \\ \langle \tilde{R}^T \times \widetilde{\Omega}^{-1} \times \tilde{e} \rangle & \langle \tilde{e}^T \times \widetilde{\Omega}^{-1} \times \tilde{e} \rangle \end{vmatrix}} = \frac{\langle \tilde{\rho}_{R_p} \rangle \langle \tilde{e}^T \times \widetilde{\Omega}^{-1} \times \tilde{e} \rangle - \tilde{e}^T \times \widetilde{\Omega}^{-1} \times \tilde{R}}{\langle \tilde{R}^T \times \widetilde{\Omega}^{-1} \times \tilde{R} \rangle \langle \tilde{e}^T \times \widetilde{\Omega}^{-1} \times \tilde{e} \rangle - \langle \tilde{R}^T \times \widetilde{\Omega}^{-1} \times \tilde{e} \rangle \langle \tilde{e}^T \times \widetilde{\Omega}^{-1} \times \tilde{R} \rangle} \tag{66}$$

Or after rearranging the terms of the equation results:

$$\lambda_1 = \frac{\langle \tilde{\rho}_{R_p} \rangle \langle \tilde{e}^T \times \widetilde{\Omega}^{-1} \times \tilde{e} \rangle - \tilde{e}^T \times \widetilde{\Omega}^{-1} \times \tilde{R}}{\langle \tilde{R}^T \times \widetilde{\Omega}^{-1} \times \tilde{R} \rangle \langle \tilde{e}^T \times \widetilde{\Omega}^{-1} \times \tilde{e} \rangle - \langle \tilde{R}^T \times \widetilde{\Omega}^{-1} \times \tilde{e} \rangle^2} \tag{67}$$

Under the same conditions, the following will be obtained:

$$\lambda_2 = \frac{\begin{vmatrix} \langle \tilde{R}^T \times \widetilde{\Omega}^{-1} \times \tilde{R} \rangle & \langle \tilde{\rho}_{R_p} \rangle \\ \langle \tilde{R}^T \times \widetilde{\Omega}^{-1} \times \tilde{e} \rangle & 1 \end{vmatrix}}{\begin{vmatrix} \langle \tilde{R}^T \times \widetilde{\Omega}^{-1} \times \tilde{R} \rangle & \langle \tilde{e}^T \times \widetilde{\Omega}^{-1} \times \tilde{R} \rangle \\ \langle \tilde{R}^T \times \widetilde{\Omega}^{-1} \times \tilde{e} \rangle & \langle \tilde{e}^T \times \widetilde{\Omega}^{-1} \times \tilde{e} \rangle \end{vmatrix}} = \frac{\langle \tilde{R}^T \times \widetilde{\Omega}^{-1} \times \tilde{R} \rangle - \langle \tilde{\rho}_{R_p} \rangle \langle \tilde{R}^T \times \widetilde{\Omega}^{-1} \times \tilde{e} \rangle}{\langle \tilde{R}^T \times \widetilde{\Omega}^{-1} \times \tilde{R} \rangle \langle \tilde{e}^T \times \widetilde{\Omega}^{-1} \times \tilde{e} \rangle - \langle \tilde{R}^T \times \widetilde{\Omega}^{-1} \times \tilde{e} \rangle \langle \tilde{e}^T \times \widetilde{\Omega}^{-1} \times \tilde{R} \rangle} \tag{68}$$

Or after rearranging the terms of the equation, we obtain:

$$\lambda_2 = \frac{\langle \tilde{R}^T \times \widetilde{\Omega}^{-1} \times \tilde{R} \rangle - \langle \tilde{\rho}_{R_p} \rangle \langle \tilde{R}^T \times \widetilde{\Omega}^{-1} \times \tilde{e} \rangle}{\langle \tilde{R}^T \times \widetilde{\Omega}^{-1} \times \tilde{R} \rangle \langle \tilde{e}^T \times \widetilde{\Omega}^{-1} \times \tilde{e} \rangle - \langle \tilde{R}^T \times \widetilde{\Omega}^{-1} \times \tilde{e} \rangle^2} \tag{69}$$

By replacing the expressions obtained for (λ_1) and (λ_2), the optimal neutrosophic portfolio structure will be obtained in the form:

$$\tilde{X} = \lambda_1 \langle \tilde{R} \times \widetilde{\Omega}^{-1} \rangle + \lambda_2 \langle \tilde{e} \times \widetilde{\Omega}^{-1} \rangle \tag{70}$$

$$\begin{aligned}\tilde{X} = \frac{1}{\langle \tilde{R}^T \times \widetilde{\Omega}^{-1} \times \tilde{R} \rangle \langle \tilde{e}^T \times \widetilde{\Omega}^{-1} \times \tilde{e} \rangle - \langle \tilde{R}^T \times \widetilde{\Omega}^{-1} \times \tilde{e} \rangle^2} \Big[\Big(\langle \tilde{\rho}_{R_p} \rangle \langle \tilde{e}^T \times \widetilde{\Omega}^{-1} \times \tilde{e} \rangle - \tilde{e}^T \times \widetilde{\Omega}^{-1} \times \tilde{R} \Big) \langle \widetilde{\Omega}^{-1} \times \tilde{R} \rangle \\ + \Big(\langle \tilde{R}^T \times \widetilde{\Omega}^{-1} \times \tilde{R} \rangle - \langle \tilde{\rho}_{R_p} \rangle \langle \tilde{R}^T \times \widetilde{\Omega}^{-1} \times \tilde{e} \rangle \langle \widetilde{\Omega}^{-1} \times \tilde{e} \rangle \Big) \Big]\end{aligned} \tag{71}$$

To simplify the calculations determined by the optimal neutrosophic portfolio structure, some additional notations can be used as follows:

$$\langle \alpha \rangle = \langle \tilde{e}^T \times \widetilde{\Omega}^{-1} \times \tilde{e} \rangle \tag{72}$$

$$\langle \beta \rangle = \langle \tilde{R}^T \times \widetilde{\Omega}^{-1} \times \tilde{e} \rangle = \langle \tilde{e}^T \times \widetilde{\Omega}^{-1} \times \tilde{R} \rangle \tag{73}$$

$$\langle \gamma \rangle = \langle \tilde{R}^T \times \widetilde{\Omega}^{-1} \times \tilde{R} \rangle \tag{74}$$

In these conditions, we obtain the structure of the optimal N-financial asset neutrosophic portfolio:

$$\tilde{X} = \langle \frac{1}{\langle \alpha \rangle \langle \gamma \rangle - \langle \beta \rangle^2} \Big[\Big(\langle \tilde{\rho}_{R_p} \rangle \langle \alpha \rangle - \langle \beta \rangle \Big) \langle \widetilde{\Omega}^{-1} \times \tilde{R} \rangle + \Big(\langle \gamma \rangle - \langle \tilde{\rho}_{R_p} \rangle \langle \beta \rangle \Big) \langle \widetilde{\Omega}^{-1} \times \tilde{e} \rangle \Big]; w\tilde{x}_{A_k}, u\tilde{x}_{A_k}, y\tilde{x}_{A_k} \rangle \tag{75}$$

To determine the optimal neutrosophic portfolio risk, we used the matrix writing of this indicator as follows:

$$\langle \tilde{\sigma}_P^2; w\widetilde{\sigma p}, u\widetilde{\sigma p}, y\widetilde{\sigma p} \rangle = \langle \left(\tilde{x}_{A_1} \tilde{x}_{A_2} \ldots \tilde{x}_{A_n} \right) \begin{pmatrix} \tilde{\sigma}_{A_{11}} & \tilde{\sigma}_{A_{12}} & \ldots & \tilde{\sigma}_{A_{1n}} \\ \tilde{\sigma}_{A_{21}} & \tilde{\sigma}_{A_{22}} & \ldots & \tilde{\sigma}_{A_{2n}} \\ \ldots & \ldots & \ldots & \ldots \\ \tilde{\sigma}_{A_{n1}} & \tilde{\sigma}_{A_{n2}} & \ldots & \tilde{\sigma}_{A_{nn}} \end{pmatrix} \begin{pmatrix} \tilde{x}_1 \\ \tilde{x}_2 \\ \ldots \\ \tilde{x}_n \end{pmatrix}; w\tilde{\sigma}_{kj}, u\tilde{\sigma}_{kj}, y\tilde{\sigma}_{kj} \rangle \tag{76}$$

Using the above notation results:

$$\langle \tilde{\sigma}_P^2; w\widetilde{\sigma p}, u\widetilde{\sigma p}, y\widetilde{\sigma p} \rangle = \langle \tilde{X}^T \times \widetilde{\Omega} \times \tilde{X}; w\tilde{\sigma}_{kj}, u\tilde{\sigma}_{kj}, y\tilde{\sigma}_{kj} \rangle \tag{77}$$

By replacing in the above formula, the expression of $\langle \tilde{X} \rangle$ is obtained:

$$\begin{aligned}\langle \tilde{\sigma}_P^2 \rangle = \langle \frac{1}{\langle \alpha \rangle \langle \gamma \rangle - \langle \beta \rangle^2} \Big[\Big(\langle \tilde{\rho}_{R_p} \rangle \langle \alpha \rangle - \langle \beta \rangle \Big) \langle \widetilde{\Omega}^{-1} \times \tilde{R}^T \rangle + \Big(\langle \gamma \rangle - \langle \tilde{\rho}_{R_p} \rangle \langle \beta \rangle \Big) \langle \widetilde{\Omega}^{-1} \times \tilde{e}^T \rangle \Big] \rangle \times \widetilde{\Omega} \\ \times \langle \frac{1}{\langle \alpha \rangle \langle \gamma \rangle - \langle \beta \rangle^2} \Big[\Big(\langle \tilde{\rho}_{R_p} \rangle \langle \alpha \rangle - \langle \beta \rangle \Big) \langle \widetilde{\Omega}^{-1} \times \tilde{R} \rangle + \Big(\langle \gamma \rangle - \langle \tilde{\rho}_{R_p} \rangle \langle \beta \rangle \Big) \langle \widetilde{\Omega}^{-1} \times \tilde{e} \rangle \Big] \rangle\end{aligned} \tag{78}$$

$$\begin{aligned}\langle \tilde{\sigma}_P^2 \rangle = \frac{1}{[\langle \alpha \rangle \langle \gamma \rangle - \langle \beta \rangle^2]^2} \Big[\Big(\langle \tilde{\rho}_{R_p} \rangle \langle \alpha \rangle - \langle \beta \rangle \Big) \langle \widetilde{\Omega}^{-1} \times \tilde{R}^T \rangle + \Big(\langle \gamma \rangle - \langle \tilde{\rho}_{R_p} \rangle \langle \beta \rangle \Big) \langle \widetilde{\Omega}^{-1} \times \tilde{e}^T \rangle \Big] \\ \times \Big[\Big(\langle \tilde{\rho}_{R_p} \rangle \langle \alpha \rangle - \langle \beta \rangle \Big) \langle \tilde{R} \rangle + \Big(\langle \gamma \rangle - \langle \tilde{\rho}_{R_p} \rangle \langle \beta \rangle \Big) \langle \tilde{e} \rangle \Big]\end{aligned} \tag{79}$$

$$\begin{aligned}\langle \tilde{\sigma}_P^2 \rangle = \frac{1}{[\langle \alpha \rangle \langle \gamma \rangle - \langle \beta \rangle^2]^2} \\ = \Big(\langle \tilde{\rho}_{R_p} \rangle \langle \alpha \rangle - \langle \beta \rangle \Big)^2 \langle \gamma \rangle + \Big(\langle \tilde{\rho}_{R_p} \rangle \langle \alpha \rangle - \langle \beta \rangle \Big) \Big(\langle \gamma \rangle - \langle \tilde{\rho}_{R_p} \rangle \langle \beta \rangle \Big) \langle \beta \rangle \\ + \Big(\langle \gamma \rangle - \langle \tilde{\rho}_{R_p} \rangle \langle \beta \rangle \Big) \Big(\langle \tilde{\rho}_{R_p} \rangle \langle \alpha \rangle - \langle \beta \rangle \Big) \langle \beta \rangle + \Big(\langle \gamma \rangle - \langle \tilde{\rho}_{R_p} \rangle \langle \beta \rangle \Big)^2 \langle \alpha \rangle\end{aligned} \tag{80}$$

$$\langle\tilde{\sigma}_P^2\rangle = \frac{1}{[\langle\alpha\rangle\langle\gamma\rangle - \langle\beta\rangle^2]^2}\Big[\Big(\langle\tilde{\rho}_{R_p}\rangle^2\,\langle\alpha\rangle^2 - 2\langle\beta\rangle\langle\alpha\rangle\,\langle\tilde{\rho}_{R_p}\rangle + \langle\beta\rangle^2\Big)\langle\gamma\rangle + \Big(\langle\alpha\rangle\langle\gamma\rangle\,\langle\tilde{\rho}_{R_p}\rangle - \langle\alpha\rangle\langle\beta\rangle\,\langle\tilde{\rho}_{R_p}\rangle^2 - \langle\beta\rangle\langle\gamma\rangle + \langle\beta\rangle^2\,\langle\tilde{\rho}_{R_p}\rangle\Big)\langle\beta\rangle\Big] + \Big(\langle\alpha\rangle\langle\gamma\rangle\,\langle\tilde{\rho}_{R_p}\rangle - \langle\beta\rangle\langle\gamma\rangle - \langle\alpha\rangle\langle\beta\rangle\,\langle\tilde{\rho}_{R_p}\rangle^2 + \langle\tilde{\rho}_{R_p}\rangle\,\langle\beta\rangle^2\Big)\langle\beta\rangle + \Big(\langle\gamma\rangle^2 - 2\langle\beta\rangle\langle\gamma\rangle\,\langle\tilde{\rho}_{R_p}\rangle + \langle\tilde{\rho}_{R_p}\rangle^2\,\langle\beta\rangle^2\Big)\langle\alpha\rangle \quad (81)$$

$$\langle\tilde{\sigma}_P^2\rangle = \frac{1}{[\langle\alpha\rangle\langle\gamma\rangle - \langle\beta\rangle^2]^2}\Big(\langle\alpha\rangle^2\langle\gamma\rangle\,\langle\tilde{\rho}_{R_p}\rangle^2 - 2\langle\alpha\rangle\langle\beta\rangle\langle\gamma\rangle\,\langle\tilde{\rho}_{R_p}\rangle + \langle\beta\rangle^2\langle\gamma\rangle + \langle\alpha\rangle\langle\beta\rangle\langle\gamma\rangle\,\langle\tilde{\rho}_{R_p}\rangle - \langle\alpha\rangle\langle\beta\rangle^2\,\langle\tilde{\rho}_{R_p}\rangle^2 - \langle\beta\rangle^2\langle\gamma\rangle + \langle\beta\rangle^3\,\langle\tilde{\rho}_{R_p}\rangle + \langle\alpha\rangle\langle\beta\rangle\langle\gamma\rangle\,\langle\tilde{\rho}_{R_p}\rangle - \langle\beta\rangle^2\langle\gamma\rangle - \langle\alpha\rangle\langle\beta\rangle^2\,\langle\tilde{\rho}_{R_p}\rangle^2 + \langle\tilde{\rho}_{R_p}\rangle\,\langle\beta\rangle^3 + \langle\alpha\rangle\langle\gamma\rangle^2 - 2\langle\alpha\rangle\langle\beta\rangle\langle\gamma\rangle\,\langle\tilde{\rho}_{R_p}\rangle + \langle\alpha\rangle\langle\beta\rangle^2\,\langle\tilde{\rho}_{R_p}\rangle^2\Big) \quad (82)$$

$$\langle\tilde{\sigma}_P^2\rangle = \frac{1}{[\langle\alpha\rangle\langle\gamma\rangle - \langle\beta\rangle^2]^2}\Big[\langle\tilde{\rho}_{R_p}\rangle^2\,(\langle\alpha\rangle^2\langle\gamma\rangle - \langle\alpha\rangle\langle\beta\rangle^2) + \langle\tilde{\rho}_{R_p}\rangle\,(2\langle\beta\rangle^3 - 2\langle\alpha\rangle\langle\beta\rangle\langle\gamma\rangle) + (\langle\beta\rangle^2\langle\gamma\rangle - 2\langle\beta\rangle^2\langle\gamma\rangle + \langle\alpha\rangle\langle\gamma\rangle^2)\Big] \quad (83)$$

$$\langle\tilde{\sigma}_P^2\rangle = \frac{1}{[\langle\alpha\rangle\langle\gamma\rangle - \langle\beta\rangle^2]^2}\langle\tilde{\rho}_{R_p}\rangle^2\,\langle\alpha\rangle(\langle\alpha\rangle\langle\gamma\rangle - \langle\beta\rangle^2) - 2\,\langle\tilde{\rho}_{R_p}\rangle\,\langle\beta\rangle(\langle\alpha\rangle\langle\gamma\rangle - \langle\beta\rangle^2) + \langle\gamma\rangle(\langle\alpha\rangle\langle\gamma\rangle - \langle\beta\rangle^2) \quad (84)$$

After solving the calculations in the above formula, the final form is obtained to determine the risk of the optimal neutrosophic portfolio of the form:

$$\langle\tilde{\sigma}_P^2; w\widetilde{\sigma p}, u\widetilde{\sigma p}, y\widetilde{\sigma p}\rangle = \langle\frac{1}{\langle\alpha\rangle\langle\gamma\rangle - \langle\beta\rangle^2}\Big[\langle\alpha\rangle\,\langle\tilde{\rho}_{R_p}\rangle^2 - 2\langle\beta\rangle\,\langle\tilde{\rho}_{R_p}\rangle + \langle\gamma\rangle\Big]; w\widetilde{\sigma p}, u\widetilde{\sigma p}, y\widetilde{\sigma p}\rangle \quad (85)$$

The optimal neutrosophic portfolio risk equation is also known as the Markowitz's frontier and measures the N-assets portfolio risk for which the portfolio return reaches a value of $\langle\tilde{R}_P; w\widetilde{Rp}, u\widetilde{Rp}, y\widetilde{Rp}\rangle = \langle\tilde{\rho}_{R_p}; w\tilde{R}_{R_P}, u\tilde{R}_{R_P}, y\tilde{R}_{R_P}\rangle$ and the portfolio risk tends to a minimum $\langle\tilde{\sigma}_P^2; w\widetilde{\sigma p}, u\widetilde{\sigma p}, y\widetilde{\sigma p}\rangle \to min$.

Example 2. *Let there be three financial assets* (A_1, A_2, A_3) *to which three neutrosophic triangular numbers are specific for the financial assets' return, of the form:*

$$\widetilde{R_{A_1}} = \langle(0.3\ 0.4\ 0.5); 0.5, 0.2, 0.3\rangle \text{ for } \widetilde{R_A} \in [0.3; 0.5];$$

$$\widetilde{R_{A_2}} = \langle(0.2\ 0.3\ 0.4); 0.6, 0.3, 0.2\rangle \text{ for } \widetilde{R_A} \in [0,2; 0,4];$$

$$\widetilde{R_{A_3}} = \langle(0.25\ 0.35\ 0.4); 0.4, 0.3, 0.3\rangle \text{ for } \widetilde{R_A} \in [0,3; 0,6];$$

It is required to establish:

(a) The neutrosophic average returns for each of these three financial assets;
(b) The neutrosophic risk for each of these three financial assets;
(c) The covariance between these financial assets, the variance–covariance matrix $(\tilde{\Omega})$ *and its inverse* $(\tilde{\Omega}^{-1})$*;*
(d) The portfolio structure taking into account that the neutrosophic average return pursued is $\langle\tilde{R}_P; w\widetilde{Rp}, u\widetilde{Rp}, y\widetilde{Rp}\rangle = \langle 0.3109;\ 0.5, 0.2, 0.3\rangle$ *and the risk tends to be minimum* $\langle\tilde{\sigma}_P^2; w\widetilde{\sigma p}, u\widetilde{\sigma p}, y\widetilde{\sigma p}\rangle \to min$*;* and
(e) The optimal portfolio risk for which its structure was determined.

Solution:

Please note that all the computation needed in order to solve this example is provided in Appendix B.

(a) The determination of the average neutrosophic return is done with the help of the formula:

$$\langle E_f\big(\widetilde{R_A}\big); w\widetilde{R_A}, u\widetilde{R_A}, y\widetilde{R_A}\rangle = \langle\Big(\frac{1}{6}\big(\widetilde{R_{A_{a1}}} + \widetilde{R_{A_{c1}}}\big) + \frac{2}{3}\widetilde{R_{A_{b1}}}\Big); w\widetilde{R_A}, u\widetilde{R_A}, y\widetilde{R_A}\rangle$$

Thus, we will have:

$$E_f\left(\widetilde{Ra}_1\right) = \langle 0{,}398; 0.5, 0.2, 0.3\rangle$$

$$E_f\left(\widetilde{Ra}_2\right) = \langle 0{,}298; 0.5, 0.2, 0.3\rangle;$$

$$E_f\left(\widetilde{Ra}_3\right) = \langle 0{,}340; 0.5, 0.2, 0.3\rangle;$$

(b) The determination of the neutrosophic risk for each of the financial assets are made with the formula:

$$\begin{aligned}\langle \sigma f_{A_i}^2; w\tilde{\sigma}_A, u\tilde{\sigma}_A, y\tilde{\sigma}_A\rangle &= \langle \frac{1}{4}\left[\left(\widetilde{R}_{A_{b1}} - \widetilde{R}_{A_{a1}}\right)^2 + \left(\widetilde{R}_{A_{c1}} - \widetilde{R}_{A_{b1}}\right)^2\right]; w\widetilde{R_A}, u\widetilde{R_A}, y\widetilde{R_A}\rangle \\ &+ \langle \frac{2}{3}\left[\widetilde{R}_{A_{a1}}\left(\widetilde{R}_{A_{b1}} - \widetilde{R}_{A_{a1}}\right) - \widetilde{R}_{A_{c1}}\left(\widetilde{R}_{A_{c1}} - \widetilde{R}_{A_{b1}}\right)\right]; w\widetilde{R_A}, u\widetilde{R_A}, y\widetilde{R_A}\rangle + \langle \frac{1}{2}\left(\widetilde{R}_{A_{a1}}^{\,2} + \widetilde{R}_{A_{c1}}^{\,2}\right); w\widetilde{R_A}, u\widetilde{R_A}, y\widetilde{R_A}\rangle \\ &- \langle \frac{1}{2} E_f^2\left(\widetilde{Ra}_i\right); w\widetilde{R_A}, u\widetilde{R_A}, y\widetilde{R_A}\rangle\end{aligned}$$

Thus, we will have:

$$\tilde{\sigma}_f^2 a_1 = \langle 0.083; 0.5, 0.2, 0.3\rangle$$

$$\tilde{\sigma}_f a_1 = \langle 0{,}288; 0.5, 0.2, 0.3\rangle$$

$$\tilde{\sigma}_f^2 a_2 = \langle 0.048; 0.6, 0.3, 0.2\rangle$$

$$\tilde{\sigma}_f a_2 = \langle 0{,}219; 0.5, 0.2, 0.3\rangle$$

$$\tilde{\sigma}_f^2 a_3 = \langle 0.0655; 0.6, 0.3, 0.2\rangle$$

$$\tilde{\sigma}_f a_3 = \langle 0{,}255; 0.5, 0.2, 0.3\rangle$$

(c) To determine the variance-covariance matrix ($\tilde{\Omega}$), the covariance for all three financial assets is computed as follows:

$$\begin{aligned}cov\left(\widetilde{Ra}_1, \widetilde{Ra}_2\right) = \langle\Bigg(&\frac{1}{4}\left[\left(\widetilde{Ra}_{b11} - \widetilde{Ra}_{a11}\right)\left(\widetilde{Ra}_{b21} - \widetilde{Ra}_{a21}\right) + \left(\widetilde{Ra}_{c11} - \widetilde{Ra}_{b11}\right)\left(\widetilde{Ra}_{c21} - \widetilde{Ra}_{b21}\right)\right] \\ &+ \frac{1}{3}\left\{\left[\widetilde{Ra}_{a21}\left(\widetilde{Ra}_{b11} - \widetilde{Ra}_{a11}\right) + \widetilde{Ra}_{a11}\left(\widetilde{Ra}_{b21} - \widetilde{Ra}_{a21}\right)\right]\right. \\ &- \left.\left[\widetilde{Ra}_{c11}\left(\widetilde{Ra}_{c21} - \widetilde{Ra}_{b21}\right) + \widetilde{Ra}_{c21}\left(\widetilde{Ra}_{c11} - \widetilde{Ra}_{b11}\right)\right]\right\} + \frac{1}{2}\left(\widetilde{Ra}_{a11}\widetilde{Ra}_{a21} + \widetilde{Ra}_{c11}\widetilde{Ra}_{c21}\right) \\ &+ \frac{1}{2} E_f\left(\widetilde{Ra}_1\right) E_f\left(\widetilde{Ra}_2\right)\Bigg); w\widetilde{Ra}_1 \wedge w\widetilde{Ra}_2, u\widetilde{Ra}_1 \vee u\widetilde{Ra}_2, y\widetilde{Ra}_1 \vee y\widetilde{Ra}_2\rangle\end{aligned}$$

By replacing the values in the formula, the following is obtained:

$$cov\left(\widetilde{Ra}_1, \widetilde{Ra}_2\right) = \langle 0.132; 0.6, 0.2, 0.2\rangle$$

$$cov\left(\widetilde{Ra}_1, \widetilde{Ra}_3\right) = \langle 0.2007; 0.5, 0.2, 0.3\rangle$$

$$cov\left(\widetilde{Ra}_2, \widetilde{Ra}_3\right) = \langle 0.1542;\ 0.6, 0.3, 0.2\rangle$$

The variance-covariance matrix will have the following form:

$$\left(\tilde{\Omega}\right) = \begin{pmatrix} \langle 0.288; 0.5, 0.2, 0.3\rangle & \langle 0.132; 0.6, 0.2, 0.2\rangle & \langle 0.200; 0.5, 0.2, 0.3\rangle \\ \langle 0.132; 0.6, 0.2, 0.2\rangle & \langle 0.219; 0.5, 0.2, 0.3\rangle & \langle 0.154;\ 0.6, 0.3, 0.2\rangle \\ \langle 0.200; 0.5, 0.2, 0.3\rangle & \langle 0.154;\ 0.6, 0.3, 0.2\rangle & \langle 0.255; 0.5, 0.2, 0.3\rangle \end{pmatrix}$$

The inverse of the variance-covariance matrix will be established as:

$$det\,\widetilde{\Omega} = \begin{vmatrix} \langle 0.288; 0.5, 0.2, 0.3\rangle & \langle 0.132; 0.6, 0.2, 0.2\rangle & \langle 0.200; 0.5, 0.2, 0.3\rangle \\ \langle 0.132; 0.6, 0.2, 0.2\rangle & \langle 0.219; 0.5, 0.2, 0.3\rangle & \langle 0.154;\ 0.6, 0.3, 0.2\rangle \\ \langle 0.200; 0.5, 0.2, 0.3\rangle & \langle 0.154;\ 0.6, 0.3, 0.2\rangle & \langle 0.255; 0.5, 0.2, 0.3\rangle \end{vmatrix} = $$
$$= 0.00093 \neq 0$$

It can be observed that the variance-covariance matrix is invertible because $det\,\widetilde{\Omega} \neq 0$. The $\widetilde{\Omega}^T$ is set as:

$$\widetilde{\Omega}^T = \begin{pmatrix} \langle 0.288; 0.5, 0.2, 0.3\rangle & \langle 0.132; 0.6, 0.2, 0.2\rangle & \langle 0.200; 0.5, 0.2, 0.3\rangle \\ \langle 0.132; 0.6, 0.2, 0.2\rangle & \langle 0.219; 0.5, 0.2, 0.3\rangle & \langle 0.154;\ 0.6, 0.3, 0.2\rangle \\ \langle 0.200; 0.5, 0.2, 0.3\rangle & \langle 0.154;\ 0.6, 0.3, 0.2\rangle & \langle 0.255; 0.5, 0.2, 0.3\rangle \end{pmatrix}$$

The disjoint matrix is set as:

$$\widetilde{\Omega}^* = \begin{pmatrix} \langle -0.0048; 0.6, 0.3, 0.2\rangle & \langle 0.0088; 0.6, 0.2, 0.2\rangle & \langle 0.0021;\ 0.6, 0.2, 0.2\rangle \\ \langle 0.0088;\ 0.6, 0.2, 0.2\rangle & \langle -0.0346;\ 0.5, 0.2, 0.3\rangle & \langle 0.0116;\ 0.6, 0.2, 0.2\rangle \\ \langle 0.0021;\ 0.6, 0.2, 0.2\rangle & \langle 0.0116;\ 0.6, 0.2, 0.2\rangle & \langle -0.0045;\ 0.6, 0.2, 0.2\rangle \end{pmatrix}$$

Based on the disjoint matrix, the variance-covariance matrix is established according to the formula:

$$\widetilde{\Omega}^{-1} = \frac{1}{0.00093}\begin{pmatrix} \langle -0.0048; 0.6, 0.3, 0.2\rangle & \langle 0.0088; 0.6, 0.2, 0.2\rangle & \langle 0.0021;\ 0.6, 0.2, 0.2\rangle \\ \langle 0.0088;\ 0.6, 0.2, 0.2\rangle & \langle -0.0346;\ 0.5, 0.2, 0.3\rangle & \langle 0.0116;\ 0.6, 0.2, 0.2\rangle \\ \langle 0.0021;\ 0.6, 0.2, 0.2\rangle & \langle 0.0116;\ 0.6, 0.2, 0.2\rangle & \langle -0.0045;\ 0.6, 0.2, 0.2\rangle \end{pmatrix}$$

The final form of the variance-covariance matrix will be:

$$\widetilde{\Omega}^{-1} = \begin{pmatrix} \langle -5.16;\ 0.6, 0.3, 0.2\rangle & \langle 9.46;\ 0.6, 0.2, 0.2\rangle & \langle 2.25;\ 0.6, 0.2, 0.2\rangle \\ \langle 9.46;\ 0.6, 0.2, 0.2\rangle & \langle -37.20;\ 0.5, 0.2, 0.3\rangle & \langle 12.47; 0.6, 0.2, 0.2\rangle \\ \langle 2.25;\ 0.6, 0.2, 0.2\rangle & 12.47;\ 0.6, 0.2, 0.2 & \langle -4.83; 0.6, 0.2, 0.2\rangle \end{pmatrix}$$

(d) To obtain the portfolio structure in terms of return $\langle \tilde{R}_P; w\widetilde{Rp}, u\widetilde{Rp}, y\widetilde{Rp}\rangle = \langle 0.3109;\ 0.5, 0.2, 0.3\rangle$ and minimize the portfolio risk $\langle \tilde{\sigma}_P^2; w\widetilde{\sigma p}, u\widetilde{\sigma p}, y\widetilde{\sigma p}\rangle \to min,$ the auxiliary calculations for determining the parameters $\langle \alpha\rangle$, $\langle \beta\rangle$, and $\langle \gamma\rangle$ will be performed as follows:

$$\langle \alpha\rangle = \langle \tilde{e}^T \times \widetilde{\Omega}^{-1} \times \tilde{e}\rangle$$

$$\langle \beta\rangle = \langle \tilde{R}^T \times \widetilde{\Omega}^{-1} \times \tilde{e}\rangle = \langle \tilde{e}^T \times \widetilde{\Omega}^{-1} \times \tilde{R}\rangle$$

$$\langle \gamma\rangle = \langle \tilde{R}^T \times \widetilde{\Omega}^{-1} \times \tilde{R}\rangle$$

$$\langle \alpha\rangle = \langle 1.17; 0.6, 0.2, 0.2\rangle$$

$$\langle \beta\rangle = \langle 3.14; 0.6, 0.2, 0.2\rangle$$

$$\langle \gamma\rangle = \langle 1.01; 0.6, 0.2, 0.2\rangle$$

The other variables that intervene in the calculation formula of the portfolio structure are determined, namely:

$$\langle \widetilde{\Omega}^{-1} \times \tilde{R}\rangle = \begin{pmatrix} \langle 1.18;\ 0.6, 0.2, 0.2\rangle \\ \langle 0.76;\ 0.6, 0.2, 0.2\rangle \\ \langle 1.19;\ 0.6, 0.2, 0.2\rangle \end{pmatrix}$$

$$\langle \widetilde{\Omega}^{-1} \times \tilde{e}\rangle = \begin{pmatrix} \langle 6.55; 0.6, 0.2, 0.2\ \rangle \\ \langle -15.27; 0.6, 0.2, 0.2\rangle \\ \langle 9.89;\ \ 0.6, 0.2, 0.2\rangle \end{pmatrix}$$

Under these conditions, the structure of the optimal portfolio will be established according to the calculation formula:

$$\tilde{X} = \langle \frac{1}{\langle\alpha\rangle\langle\gamma\rangle - \langle\beta\rangle^2}\left[\left(\langle\tilde{\rho}_{R_p}\rangle\langle\alpha\rangle - \langle\beta\rangle\right)\langle\widetilde{\Omega}^{-1} \times \tilde{R}\rangle + \left(\langle\gamma\rangle - \langle\tilde{\rho}_{R_p}\rangle\langle\beta\rangle\right)\langle\widetilde{\Omega}^{-1} \times \tilde{e}\rangle\right]; w\tilde{x}_{A_k}, u\tilde{x}_{A_k}, y\tilde{x}_{A_k}\rangle$$

$$\tilde{X} = \begin{pmatrix} \langle 0.3795;\ 0.6, 0.2, 0.2\rangle \\ \langle 0.1815;\ 0.6, 0.2, 0.2\rangle \\ \langle 0.3938;\ 0.6, 0.2, 0.2\rangle \end{pmatrix}$$

The interpretation of the results is that in order to achieve a level of return for the proposed neutrosophic portfolio, respectively a value of $\langle\tilde{R}_P; w\widetilde{Rp}, u\widetilde{Rp}, y\widetilde{Rp}\rangle = \langle 0.3109;\ 0.5, 0.2, 0.3\rangle$, the investors will make an investment of 37.95% in the first financial asset, 18.15% in the second financial asset, and 39.38% in the third financial asset. These results can be verified by determining:

$$\langle\tilde{R}_P; w\widetilde{Rp}, u\widetilde{Rp}, y\widetilde{Rp}\rangle = \langle 0.3198;\ 0.6, 0.2, 0.2\rangle$$

As a conclusion, the verifications show that by creating a portfolio with the above structure: $\tilde{x}_{A_1} = 37.95\%$, $\tilde{x}_{A_2} = 18.15\%$, and $\tilde{x}_{A_3} = 39.38\%$, a return of $\langle\tilde{R}_P; w\widetilde{Rp}, u\widetilde{Rp}, y\widetilde{Rp}\rangle = \langle 0.3198;\ 0.6, 0.2, 0.2\rangle$ will be obtained compared to the expected return of $\langle\tilde{R}_P; w\widetilde{Rp}, u\widetilde{Rp}, y\widetilde{Rp}\rangle = \langle 0.3109;\ 0.5, 0.2, 0.3\rangle$. Thus, this verification validates the research results by a neutrosophic return error rate, calculated using the formula: $\Delta\varepsilon = \langle 0.3109;\ 0.5, 0.2, 0.3\rangle - \langle 0{,}3198;\ 0.6, 0.2, 0.2\rangle = 0.8\%$, an error rate compared to the proposed neutrosophic return.

(e) The portfolio risk is determined with the specific formula resulted from theorem no. 2, according to which we will have:

$$\langle\tilde{\sigma}_P^2; w\widetilde{\sigma p}, u\widetilde{\sigma p}, y\widetilde{\sigma p}\rangle = \langle \frac{1}{\langle\alpha\rangle\langle\gamma\rangle - \langle\beta\rangle^2}\left[\langle\alpha\rangle\langle\tilde{\rho}_{R_p}\rangle^2 - 2\langle\beta\rangle\langle\tilde{\rho}_{R_p}\rangle + \langle\gamma\rangle\right]; w\widetilde{\sigma p}, u\widetilde{\sigma p}, y\widetilde{\sigma p}\rangle$$

$$\langle\tilde{\sigma}_P^2; w\widetilde{\sigma p}, u\widetilde{\sigma p}, y\widetilde{\sigma p}\rangle = \langle 0.0909;\ 0.6, 0.2, 0.2\rangle$$

$$\langle\sigma_p; w\widetilde{\sigma p}, u\widetilde{\sigma p}, y\widetilde{\sigma p}\rangle = \langle 0.3016; 0.6, 0.2, 0.2\rangle$$

The value of the optimal neutrosophic portfolio risk determined using theorem no.2 $\langle\sigma_p; w\widetilde{\sigma p}, u\widetilde{\sigma p}, y\widetilde{\sigma p}\rangle = \langle 0.3016; 0.6, 0.2, 0.2\rangle$ indicates the proportionality relation that exists between the desired neutrosophic portfolio return and the portfolio risk. There is a relationship between return and risk. Moreover, the elaborated theorems were validated by the practical applications presented in the paper, which allows, in the future, that the neutrosophic financial asset portfolio structure can be built so that between return and risk is a correlation required by investment prudence in the capital market.

6. Conclusions

The optimal neutrosophic portfolios are those categories of portfolios consisting of two or more financial assets, modeled using neutrosophic triangular numbers that allow for the determination of financial performance indicators, respectively the neutrosophic average return of the neutrosophic risk for each financial asset and the neutrosophic covariance as well as the determination of the portfolio return of the portfolio risk.

Moreover, to be considered optimal neutrosophic portfolios, they must meet two additional conditions, namely: fixing the financial return at an estimated level as well as minimizing the financial asset neutrosophic portfolio's risk. For this category of neutrosophic portfolios within this study, we determined two KPIs, namely: the portfolio structure that allows for the determination of the financial asset weight in the total value of the portfolio under optimal portfolio conditions under the conditions of a given value for return and under the conditions of minimizing the portfolio risk.

Both the structure of the optimal neutrosophic portfolio and the optimal portfolio risk were tested, validated, and verified on financial assets portfolios, modeled using neutrosophic fuzzy triangular numbers consisting of two and N-financial assets. The results were verified in the paper to certify the calculation formulas related to the portfolio structure and to the portfolio risk. The use of neutrosophic fuzzy triangular numbers to model financial asset portfolios has the advantage of introducing a category of additional information for investors on the capital market, namely the degree of realization/uncertainty/failure of the KPIs studied in this paper.

The theoretical contribution of this research paper is given by the theoretical substantiation with the help of neutrosophic fuzzy triangular numbers of the performance indicators specific to the optimal financial asset portfolios, namely, the portfolio structure, portfolio risk as well as portfolio return. These three performance indicators are specific to the optimal portfolios, respectively to those portfolio categories for which financial return reaches the level set by investors and the financial risk has the minimum value.

The practical implications of the research paper consist of substantiating the investment decisions on the capital market by using neutrosophic performance indicators, namely the neutrosophic portfolio structure, the neutrosophic portfolio risk, and the neutrosophic portfolio return. All the above performance indicators apply to optimal financial asset portfolios. To test the calculation of these three performance indicators using neutrosophic triangular fuzzy numbers, two practical calculation examples were presented that aimed to apply financial performance indicators for a portfolio of two financial assets as well as for a portfolio of N financial assets.

The limitations of modeling the optimal financial asset portfolios using fuzzy triangular numbers can be represented by the computation complexity performed, but nevertheless, the practical applications can be summarized in applying the formulas that have been demonstrated using the theorems.

The future research directions are the following: the use of other artificial intelligence techniques to model the optimal financial asset portfolios such as "swarm particles" and genetic algorithms, but also in extending, with the help of neutrosophic fuzzy numbers, the modeling of financial asset portfolio performance indicators such as conjugated neutrosophic portfolios, or the financial assets assessment when the financial asset portfolio also contains a zero risk financial asset like treasury bonds.

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Appendix A. Numerical Calculation for Example 1

(a)

$$\langle E_f(\widetilde{R_A}); w\widetilde{R_A}, u\widetilde{R_A}, y\widetilde{R_A}\rangle = \langle\left(\frac{1}{6}\left(\widetilde{R_{A_{a1}}} + \widetilde{R_{A_{c1}}}\right) + \frac{2}{3}\widetilde{R_{A_{b1}}}\right); w\widetilde{R_A}, u\widetilde{R_A}, y\widetilde{R_A}\rangle$$

$$E_f(\widetilde{Ra_1}) = \langle\left(\frac{1}{6}(0.3 + 0.5) + \frac{2}{3}0.4\right); 0.5, 0.2, 0.3\rangle$$

$$E_f\left(\widetilde{Ra}_1\right) = \langle\left(\frac{1}{6}0.8 + \frac{2}{3}0.4\right); 0.5, 0.2, 0.3\rangle$$

$$E_f\left(\widetilde{Ra}_1\right) = \langle 0{,}398; 0.5, 0.2, 0.3\rangle$$

$$E_f\left(\widetilde{Ra}_2\right) = \langle\left(\frac{1}{6}(0.2 + 0.4) + \frac{2}{3} \times 0.3\right); 0.6, 0.3, 0.2\rangle$$

$$E_f\left(\widetilde{Ra}_2\right) = \langle\left(\frac{1}{6}0.6 + \frac{2}{3}0.3\right); 0.6, 0.3, 0.2\rangle$$

$$E_f\left(\widetilde{Ra}_2\right) = \langle 0.298; 0.5, 0.2, 0.3\rangle;$$

(b)

$$\begin{aligned}\langle \sigma f_{A_i}^2; w\tilde{\sigma}_A, u\tilde{\sigma}_A, y\tilde{\sigma}_A\rangle &= \langle\frac{1}{4}\left[\left(\widetilde{R_{A_{b1}}} - \widetilde{R_{A_{a1}}}\right)^2 + \left(\widetilde{R_{A_{c1}}} - \widetilde{R_{A_{b1}}}\right)^2\right]; w\widetilde{R_A}, u\widetilde{R_A}, y\widetilde{R_A}\rangle \\ &+ \langle\frac{2}{3}\left[\widetilde{R_{A_{a1}}}\left(\widetilde{R_{A_{b1}}} - \widetilde{R_{A_{a1}}}\right) - \widetilde{R_{A_{c1}}}\left(\widetilde{R_{A_{c1}}} - \widetilde{R_{A_{b1}}}\right)\right]; w\widetilde{R_A}, u\widetilde{R_A}, y\widetilde{R_A}\rangle + \langle\frac{1}{2}\left(\widetilde{R_{A_{a1}}}^2 + \widetilde{R_{A_{c1}}}^2\right); w\widetilde{R_A}, u\widetilde{R_A}, y\widetilde{R_A}\rangle \\ &- \langle\frac{1}{2}E_f^2\left(\widetilde{Ra}_i\right); w\widetilde{R_A}, u\widetilde{R_A}, y\widetilde{R_A}\rangle\end{aligned}$$

$$\begin{aligned}\tilde{\sigma}_f^2 a_1 = \langle\frac{1}{4}[(0.4 - 0.3)^2 + (0.5 - 0.4)^2]; 0.5, 0.2, 0.3\rangle + \langle\frac{2}{3}\left(0.3(0.4 - 0.3) - 0.5(0.5 - 0.4)\right); 0.5, 0.2, 0.3\rangle \\ + \langle\frac{1}{2}(0.3^2 + 0.5^2); 0.5, 0.2, 0.3\rangle - \langle\frac{1}{2}(0.398)^2; 0.5, 0.2, 0.3\rangle;\end{aligned}$$

$$\tilde{\sigma}_f^2 a_1 = \langle\frac{1}{4}(0.01 + 0.01); 0.5, 0.2, 0.3\rangle + \langle\frac{2}{3}(0.03 - 0{,}05); 0.5, 0.2, 0.3\rangle + \langle\frac{1}{2}0.34; 0.5, 0.2, 0.3\rangle - \langle\frac{1}{2}0.158; 0.5, 0.2, 0.3\rangle;$$

$$\tilde{\sigma}_f^2 a_1 = \langle(0.005 - 0.013 + 0{,}17 - 0.079; 0.5, 0.2, 0.3)\rangle;$$

$$\tilde{\sigma}_f^2 a_1 = \langle 0.083; 0.5, 0.2, 0.3\rangle$$

$$\tilde{\sigma}_f a_1 = \langle 0.288; 0.5, 0.2, 0.3\rangle$$

$$\begin{aligned}\tilde{\sigma}_f^2 a_2 = \langle\frac{1}{4}[(0.3 - 0.2)^2 + (0.4 - 0.3)^2]; 0.6, 0.3, 0.2\rangle + \langle\frac{2}{3}\left(0.2(0.3 - 0.2) - 0.4(0.4 - 0.3)\right); 0.6, 0.3, 0.2\rangle \\ + \langle\frac{1}{2}(0.2^2 + 0.4^2); 0.6, 0.3, 0.2\rangle - \langle\frac{1}{2}(0.298)^2; 0.6, 0.3, 0.2\rangle\end{aligned}$$

$$\tilde{\sigma}_f^2 a_2 = \langle\frac{1}{4}(0.01 + 0.01); 0.6, 0.3, 0.2\rangle + \langle\frac{2}{3}(0.02 - 0.04); 0.6, 0.3, 0.2\rangle + \langle\frac{1}{2}0.20; 0.6, 0.3, 0.2\rangle - \langle\frac{1}{2}0.088; 0.6, 0.3, 0.2\rangle;$$

$$\tilde{\sigma}_f^2 a_2 = \langle(0.005 - 0.013 + 0{,}10 - 0.044; 0.6, 0.3, 0.2)\rangle$$

$$\tilde{\sigma}_f^2 a_2 = \langle 0.048; 0.6, 0.3, 0.2\rangle$$

$$\tilde{\sigma}_f a_2 = \langle 0.219; 0.5, 0.2, 0.3\rangle$$

(c)

$$cov\left(\widetilde{Ra}_1, \widetilde{Ra}_2\right) = \langle\left(\frac{1}{4}\left[\left(\widetilde{Ra}_{b11} - \widetilde{Ra}_{a11}\right)\left(\widetilde{Ra}_{b21} - \widetilde{Ra}_{a21}\right) + \left(\widetilde{Ra}_{c11} - \widetilde{Ra}_{b11}\right)\left(\widetilde{Ra}_{c21} - \widetilde{Ra}_{b21}\right)\right] + \frac{1}{3}\left\{\left[\widetilde{Ra}_{a21}\left(\widetilde{Ra}_{b11} - \widetilde{Ra}_{a11}\right) + \widetilde{Ra}_{a11}\left(\widetilde{Ra}_{b21} - \widetilde{Ra}_{a21}\right)\right] - \left[\widetilde{Ra}_{c11}\left(\widetilde{Ra}_{c21} - \widetilde{Ra}_{b21}\right) + \widetilde{Ra}_{c21}\left(\widetilde{Ra}_{c11} - \widetilde{Ra}_{b11}\right)\right]\right\} + \frac{1}{2}\left(\widetilde{Ra}_{a11}\widetilde{Ra}_{a21} + \widetilde{Ra}_{c11}\widetilde{Ra}_{c21}\right) + \frac{1}{2}E_f\left(\widetilde{Ra}_1\right)E_f\left(\widetilde{Ra}_2\right)\right); w\widetilde{Ra}_1 \wedge w\widetilde{Ra}_2, u\widetilde{Ra}_1 \vee u\widetilde{Ra}_2, y\widetilde{Ra}_1 \vee y\widetilde{Ra}_2\rangle$$

$$cov\left(\widetilde{Ra}_1, \widetilde{Ra}_2\right) = \langle\frac{1}{4}[(0.4-0.3)(0.3-0.2) + (0.5-0.4)(0.4-0.3)] + \frac{1}{3}[0.2(0.4-0.3) + 0.3(0.3-0.2)] - [0.5(0.4-0.3) + 0.4(0.5-0.4)] + \frac{1}{2}(0.3 * 0.2 + 0.5 * 0.4) + \frac{1}{2}0{,}398 * 0{,}298; 0.5 \wedge 0.6, 0.2 \vee 0.3, 0.3 \vee 0.2\rangle$$

$$cov\left(\widetilde{Ra}_1, \widetilde{Ra}_2\right) = \langle\frac{1}{4}[0.01 + 0.01] + \frac{1}{3}[0.02 + 0.03] - [0.05 + 0.04] + \frac{1}{2}(0.06 + 0.20) + \frac{1}{2}0.118; 0.6, 0.2, 0.2\rangle$$

$$cov\left(\widetilde{Ra}_1, \widetilde{Ra}_2\right) = \langle 0.021 - 0.013 + 0.065 + 0.059; 0.6, 0.2, 0.2\rangle$$

$$cov\left(\widetilde{Ra}_1, \widetilde{Ra}_2\right) = \langle 0.132; 0.6, 0.2, 0.2\rangle$$

$$\Omega = \begin{pmatrix} \langle 0288; 0.5, 0.2, 0.3\rangle & \langle 0.132; 0.6, 0.2, 0.2\rangle \\ \langle 0.132; 0.6, 0.2, 0.2\rangle & \langle 0.219; 0.6, 0.3, 0.2\rangle \end{pmatrix}$$

(d)

$$\langle\tilde{R}_P; w\widetilde{Rp}, u\widetilde{Rp}, y\widetilde{Rp}\rangle = \sum_{i=1}^{n} \langle x_{A_i}\left(\frac{1}{6}\left(\tilde{R}_{A_{ai}} + \tilde{R}_{A_{ci}}\right) + \frac{2}{3}\tilde{R}_{A_{bi}}\right); w\tilde{R}_{A_i}, u\tilde{R}_{A_i}, y\tilde{R}_{A_i}\rangle$$

$$\langle\tilde{R}_P; w\widetilde{Rp}, u\widetilde{Rp}, y\widetilde{Rp}\rangle = \langle 0.4\left(\frac{1}{6}(0.3 + 0.5) + \frac{2}{3} \times 0.4\right); 0.5, 0.2, 0.3\rangle + \langle 0.6\left(\frac{1}{6}(0.2 + 0.4) + \frac{2}{3} \times 0.3\right); 0.6, 0.3, 0.2\rangle$$

$$\langle\tilde{R}_P; w\widetilde{Rp}, u\widetilde{Rp}, y\widetilde{Rp}\rangle = \langle 0.4\left(\frac{1}{6}0.8 + \frac{2}{3}0.4\right); 0.5, 0.2, 0.3\rangle + \langle 0.6\left(\frac{1}{6}0.6 + \frac{2}{3}0.3\right); 0.6, 0.3, 0.2\rangle$$

$$\langle\tilde{R}_P; w\widetilde{Rp}, u\widetilde{Rp}, y\widetilde{Rp}\rangle = \langle 0.4 \times 0.398; 0.5, 0.2, 0.3\rangle + \langle 0.6 \times 0.298; 0.5, 0.2, 0.3\rangle$$

$$\langle\tilde{R}_P; w\widetilde{Rp}, u\widetilde{Rp}, y\widetilde{Rp}\rangle = \langle 0.159; 0.5, 0.2, 0.3\rangle + \langle 0.178;\ 0.5, 0.2, 0.3\rangle\ \langle\tilde{R}_P; w\widetilde{Rp}, u\widetilde{Rp}, y\widetilde{Rp}\rangle = \langle 0.337; 0.5, 0.2, 0.3\rangle$$

(e)

$$\langle\sigma_p^2; w\widetilde{\sigma p}, u\widetilde{\sigma p}, y\widetilde{\sigma p}\rangle = \langle x_{A_1}^2\tilde{\sigma}_{A_1}^2; w\tilde{\sigma}_{A_1}, u\tilde{\sigma}_{A_1}, y\tilde{\sigma}_{A_1}\rangle + \langle x_{A_2}^2\tilde{\sigma}_{A_2}^2; w\tilde{\sigma}_{A_2}, u\tilde{\sigma}_{A_2}, y\tilde{\sigma}_{A_2}\rangle + 2x_{A_1}x_{A_2}\tilde{\sigma}_{A_1A_2}; w\tilde{\sigma}_{A_1} \wedge w\tilde{\sigma}_{A_2}, u\tilde{\sigma}_{A_1} \vee u\tilde{\sigma}_{A_2}, y\tilde{\sigma}_{A_1} \vee y\tilde{\sigma}_{A_2}$$

$$\langle\sigma_p^2; w\widetilde{\sigma p}, u\widetilde{\sigma p}, y\widetilde{\sigma p}\rangle = \langle 0.16 \times \langle 0.083; 0.5, 0.2, 0.3\rangle\rangle + \langle 0.36 \times \langle 0.048; 0.6, 0.3, 0.2\rangle\rangle + \langle 2 \times 0.4 \times 0.6 \times \langle 0.132; 0.6, 0.2, 0.2\rangle\rangle$$

$$\langle\sigma_p^2; w\widetilde{\sigma p}, u\widetilde{\sigma p}, y\widetilde{\sigma p}\rangle = \langle 0.013; 0.5, 0.2, 0.3\rangle + \langle 0.017; 0.6, 0.3, 0.2\rangle + \langle 0.006; 0.6, 0.2, 0.2\rangle$$

$$\langle\sigma_p^2; w\widetilde{\sigma p}, u\widetilde{\sigma p}, y\widetilde{\sigma p}\rangle = \langle 0.036; 0.6, 0.2, 0.2\rangle$$

$$\langle\sigma_p; w\widetilde{\sigma p}, u\widetilde{\sigma p}, y\widetilde{\sigma p}\rangle = \sqrt{\langle 0.036; 0.6, 0.2, 0.2\rangle}$$

$$\langle\sigma_p; w\widetilde{\sigma p}, u\widetilde{\sigma p}, y\widetilde{\sigma p}\rangle = \langle 0.1897; 0.6, 0.2, 0.2\rangle$$

(f)

$$\tilde{x}_{A_1} = \langle \frac{2\langle \tilde{R}_{A_2} \rangle [\langle \tilde{\rho}_{Rp} \rangle + \langle \tilde{R}_{A_1} \rangle] - \langle \tilde{R}_{A_1} + \tilde{R}_{A_2} \rangle [\langle \tilde{\rho}_{Rp} \rangle + \langle \tilde{R}_{A_2} \rangle]}{4\langle \tilde{R}_{A_1} \tilde{R}_{A_2} \rangle - \langle \tilde{R}_{A_1} + \tilde{R}_{A_2} \rangle^2} w\tilde{x}_{A_1} \wedge w\tilde{x}_{A_2}, u\tilde{x}_{A_1} \vee u\tilde{x}_{A_2}, y\tilde{x}_{A_1} \vee y\tilde{x}_{A_2} \rangle$$

$$\tilde{x}_{A_2} = \langle \frac{2\langle \tilde{R}_{A_1} \rangle [\langle \tilde{\rho}_{Rp} \rangle + \langle \tilde{R}_{A_2} \rangle] - \langle \tilde{R}_{A_1} + \tilde{R}_{A_2} \rangle [\langle \tilde{\rho}_{Rp} \rangle + \langle \tilde{R}_{A_1} \rangle]}{4\langle \tilde{R}_{A_1} \tilde{R}_{A_2} \rangle - \langle \tilde{R}_{A_1} + \tilde{R}_{A_2} \rangle^2} w\tilde{x}_{A_1} \wedge w\tilde{x}_{A_2}, u\tilde{x}_{A_1} \vee u\tilde{x}_{A_2}, y\tilde{x}_{A_1} \vee y\tilde{x}_{A_2} \rangle$$

By replacing in the formulas will be obtained:

$$\tilde{x}_{A_1} = \langle \frac{2 \times \langle 0.298 \rangle [\langle 0.4000 + 0.398 \rangle] - \langle 0.398 + 0.298 \rangle [0.4000 + 0.298]}{4\langle 0.398 \times 0.298 \rangle - \langle 0.398 + 0.298 \rangle^2}; 0.5, 0.2, 0.3 \rangle$$

$$\tilde{x}_{A_1} = \langle \frac{\langle 0.596 \rangle \langle 0.7980 \rangle - \langle 0.696 \rangle \langle 0.6980 \rangle}{\langle 0.4744 \rangle - \langle 0.4844 \rangle}; 0.5, 0.2, 0.3 \rangle$$

$$\tilde{x}_{A_1} = \langle \frac{\langle 0.4756 \rangle - \langle 0.4858 \rangle}{\langle 0.4744 \rangle - \langle 0.4844 \rangle}; 0.5, 0.2, 0.3 \rangle$$

$$\tilde{x}_{A_1} = \langle \frac{-\langle 0.0102 \rangle}{-0.0100}; 0.5, 0.2, 0.3 \rangle$$

$$\tilde{x}_{A_1} = \langle 1{,}020; 0.5, 0.2, 0.3 \rangle$$

$$\tilde{x}_{A_2} = \langle \frac{2 \times \langle 0.398 \rangle [\langle 0.400 + 0.298 \rangle] - \langle 0.398 + 0.298 \rangle [0.400 + 0.398]}{4\langle 0.398 \times 0.298 \rangle - \langle 0.398 + 0.298 \rangle^2}; 0.5, 0.2, 0.3 \rangle$$

$$\tilde{x}_{A_2} = \langle \frac{\langle 0.796 \rangle [\langle 0.698 \rangle] - \langle 0.696 \rangle [0.7980]}{\langle 0.4744 \rangle - \langle 0.4844 \rangle}; 0.5, 0.2, 0.3 \rangle$$

$$\tilde{x}_{A_2} = \langle \frac{\langle 0.5556 \rangle - \langle 0.5554 \rangle}{\langle 0.4744 \rangle - \langle 0.4844 \rangle}; 0.5, 0.2, 0.3 \rangle$$

$$\tilde{x}_{A_2} = \langle \frac{\langle 0.0002 \rangle}{\langle -0.0100 \rangle}; 0.5, 0.2, 0.3 \rangle$$

$$\tilde{x}_{A_2} = \langle 0.02; 0.5, 0.2, 0.3 \rangle$$

$$\langle \tilde{R}_P; w\widetilde{Rp}, u\widetilde{Rp}, y\widetilde{Rp} \rangle = \langle 1.020 \times 0{,}398; 0.5, 0.2, 0.3 \rangle + \langle 0.02 \times 0.298; 0.5, 0.2, 0.3 \rangle$$

$$\langle \tilde{R}_P; w\widetilde{Rp}, u\widetilde{Rp}, y\widetilde{Rp} \rangle = \langle 0.405;\ 0.5, 0.2, 0.3 \rangle + \langle 0.0059; 0.5, 0.2, 0.3 \rangle$$

$$\langle \tilde{R}_P; w\widetilde{Rp}, u\widetilde{Rp}, y\widetilde{Rp} \rangle = \langle 0.4109; 0.5, 0.2, 0.3 \rangle$$

$$\langle \sigma_p^2; w\widetilde{\sigma p}, u\widetilde{\sigma p}, y\widetilde{\sigma p} \rangle = \langle 1.02^2 \times 0.083; 0.5, 0.2, 0.3 \rangle + \langle 0.02^2 \times 0.048; 0.6, 0.3, 0.2 \rangle + \langle 2 \times 1.02 \times 0.02 \times 0.13$$

$$\langle \sigma_p^2; w\widetilde{\sigma p}, u\widetilde{\sigma p}, y\widetilde{\sigma p} \rangle = \langle 0.086; 0.5, 0.2, 0.3 \rangle + \langle 0.0001; 0.6, 0.3, 0.2 \rangle + \langle 0.0053; 0.6, 0.2, 0.2 \rangle$$

$$\langle \sigma_p^2; w\widetilde{\sigma p}, u\widetilde{\sigma p}, y\widetilde{\sigma p} \rangle = 0.0914; 0.6, 0.2, 0.2$$

$$\langle \sigma_p; w\widetilde{\sigma p}, u\widetilde{\sigma p}, y\widetilde{\sigma p} \rangle = \sqrt{\langle 0.0914; 0.6, 0.2, 0.2 \rangle}$$

$$\langle \sigma_p; w\widetilde{\sigma p}, u\widetilde{\sigma p}, y\widetilde{\sigma p} \rangle = \langle 0.3023; 0.6, 0.2, 0.2 \rangle$$

Appendix B. Numerical Calculation for Example 2

(a)

$$\langle E_f(\widetilde{R_A}); w\widetilde{R_A}, u\widetilde{R_A}, y\widetilde{R_A}\rangle = \langle\left(\frac{1}{6}\left(\widetilde{R_{A_{a1}}} + \widetilde{R_{A_{c1}}}\right) + \frac{2}{3}\widetilde{R_{A_{b1}}}\right); w\widetilde{R_A}, u\widetilde{R_A}, y\widetilde{R_A}\rangle$$

$$E_f(\widetilde{Ra}_1) = \langle\left(\frac{1}{6}(0.3 + 0.5) + \frac{2}{3}0.4\right); 0.5, 0.2, 0.3\rangle$$

$$E_f(\widetilde{Ra}_1) = \langle\left(\frac{1}{6}0.8 + \frac{2}{3}0.4\right); 0.5, 0.2, 0.3\rangle$$

$$E_f(\widetilde{Ra}_1) = \langle 0.398; 0.5, 0.2, 0.3\rangle$$

$$E_f(\widetilde{Ra}_2) = \langle\left(\frac{1}{6}(0.2 + 0.4) + \frac{2}{3} \times 0.3\right); 0.6, 0.3, 0.2\rangle$$

$$E_f(\widetilde{Ra}_2) = \langle\left(\frac{1}{6}0.6 + \frac{2}{3}0.3\right); 0.6, 0.3, 0.2\rangle$$

$$E_f(\widetilde{Ra}_2) = \langle 0.298; 0.5, 0.2, 0.3\rangle;$$

$$E_f(\widetilde{Ra}_3) = \langle\left(\frac{1}{6}(0.25 + 0.4) + \frac{2}{3} \times 0.35\right); 0.6, 0.3, 0.2\rangle$$

$$E_f(\widetilde{Ra}_3) = \langle\left(\frac{1}{6}0.65 + \frac{2}{3}0.35\right); 0.6, 0.3, 0.2\rangle$$

$$E_f(\widetilde{Ra}_3) = \langle 0.340; 0.5, 0.2, 0.3\rangle;$$

(b)

$$\begin{aligned}\langle \sigma f_{A_i}^2; w\tilde{\sigma}_A, u\tilde{\sigma}_A, y\tilde{\sigma}_A\rangle &= \langle\frac{1}{4}\left[\left(\widetilde{R_{A_{b1}}} - \widetilde{R_{A_{a1}}}\right)^2 + \left(\widetilde{R_{A_{c1}}} - \widetilde{R_{A_{b1}}}\right)^2\right]; w\widetilde{R_A}, u\widetilde{R_A}, y\widetilde{R_A}\rangle \\ &+ \langle\frac{2}{3}\left[\widetilde{R_{A_{a1}}}\left(\widetilde{R_{A_{b1}}} - \widetilde{R_{A_{a1}}}\right) - \widetilde{R_{A_{c1}}}\left(\widetilde{R_{A_{c1}}} - \widetilde{R_{A_{b1}}}\right)\right]; w\widetilde{R_A}, u\widetilde{R_A}, y\widetilde{R_A}\rangle + \langle\frac{1}{2}\left(\widetilde{R_{A_{a1}}}^2 + \widetilde{R_{A_{c1}}}^2\right); w\widetilde{R_A}, u\widetilde{R_A}, y\widetilde{R_A}\rangle \\ &- \langle\frac{1}{2}E_f^2(\widetilde{Ra}_i); w\widetilde{R_A}, u\widetilde{R_A}, y\widetilde{R_A}\rangle\end{aligned}$$

$$\begin{aligned}\tilde{\sigma}_f^2 a_1 = \langle\frac{1}{4}[(0.4 - 0.3)^2 + (0.5 - 0.4)^2]; 0.5, 0.2, 0.3\rangle + \langle\frac{2}{3}\left(0.3(0.4 - 0.3) - 0.5(0.5 - 0.4)\right); 0.5, 0.2, 0.3\rangle \\ + \langle\frac{1}{2}(0.3^2 + 0.5^2); 0.5, 0.2, 0.3\rangle - \langle\frac{1}{2}(0.398)^2; 0.5, 0.2, 0.3\rangle\end{aligned}$$

$$\tilde{\sigma}_f^2 a_1 = \langle\frac{1}{4}(0.01 + 0.01); 0.5, 0.2, 0.3\rangle + \langle\frac{2}{3}(0.03 - 0{,}05); 0.5, 0.2, 0.3\rangle + \langle\frac{1}{2}0.34; 0.5, 0.2, 0.3\rangle - \langle\frac{1}{2}0.158; 0.5, 0.2, 0.3\rangle$$

$$\tilde{\sigma}_f^2 a_1 = \langle(0.005 - 0.013 + 0{,}17 - 0.079; 0.5, 0.2, 0.3)\rangle$$

$$\tilde{\sigma}_f^2 a_1 = \langle 0.083; 0.5, 0.2, 0.3\rangle$$

$$\tilde{\sigma}_f a_1 = \langle 0.288; 0.5, 0.2, 0.3\rangle$$

$$\begin{aligned}\tilde{\sigma}_f^2 a_2 = \langle\frac{1}{4}[(0.3 - 0.2)^2 + (0.4 - 0.3)^2]; 0.6, 0.3, 0.2\rangle + \langle\frac{2}{3}\left(0.2(0.3 - 0.2) - 0.4(0.4 - 0.3)\right); 0.6, 0.3, 0.2\rangle \\ + \langle\frac{1}{2}(0.2^2 + 0.4^2); 0.6, 0.3, 0.2\rangle - \langle\frac{1}{2}(0.298)^2; 0.6, 0.3, 0.2\rangle\end{aligned}$$

$$\tilde{\sigma}_f^2 a_2 = \langle \frac{1}{4}(0.01 + 0.01); 0.6, 0.3, 0.2\rangle + \langle \frac{2}{3}(0.02 - 0.04); 0.6, 0.3, 0.2\rangle + \langle \frac{1}{2}0.20; 0.6, 0.3, 0.2\rangle - \langle \frac{1}{2}0.088; 0.6, 0.3, 0.2\rangle$$

$$\tilde{\sigma}_f^2 a_2 = \langle (0.005 - 0.013 + 0{,}10 - 0.044; 0.6, 0.3, 0.2)\rangle$$

$$\tilde{\sigma}_f^2 a_2 = \langle 0.048; 0.6, 0.3, 0.2\rangle$$

$$\tilde{\sigma}_f a_2 = \langle 0.219; 0.5, 0.2, 0.3\rangle$$

$$\tilde{\sigma}_f^2 a_3 = \langle \frac{1}{4}[(0.35 - 0.25)^2 + (0.4 - 0.35)^2]; 0.6, 0.3, 0.2\rangle + \langle \frac{2}{3}\big(0.25(0.35 - 0.25) - 0.4(0.4 - 0.35)\big); 0.6, 0.3, 0.2\rangle + \langle \frac{1}{2}(0.25^2 + 0.4^2); 0.6, 0.3, 0.2\rangle - \langle \frac{1}{2}(0.340)^2; 0.6, 0.3, 0.2\rangle$$

$$\tilde{\sigma}_f^2 a_3 = \langle \frac{1}{4}(0.01 + 0.02); 0.6, 0.3, 0.2\rangle + \langle \frac{2}{3}(0.025 - 0.02); 0.6, 0.3, 0.2\rangle + \langle \frac{1}{2}0.225; 0.6, 0.3, 0.2\rangle - \langle \frac{1}{2}0.115; 0.6, 0.3, 0.2\rangle$$

$$\tilde{\sigma}_f^2 a_3 = \langle (0.0075 + 0.003 + 0{,}112 - 0.057; 0.6, 0.3, 0.2)\rangle$$

$$\tilde{\sigma}_f^2 a_3 = \langle 0.0655; 0.6, 0.3, 0.2\rangle$$

$$\tilde{\sigma}_f a_3 = \langle 0.255; 0.5, 0.2, 0.3\rangle$$

(c)

$$cov\big(\widetilde{Ra}_1, \widetilde{Ra}_2\big) = \langle \Bigg(\frac{1}{4}\big[\big(\widetilde{Ra}_{b11} - \widetilde{Ra}_{a11}\big)\big(\widetilde{Ra}_{b21} - \widetilde{Ra}_{a21}\big) + \big(\widetilde{Ra}_{c11} - \widetilde{Ra}_{b11}\big)\big(\widetilde{Ra}_{c21} - \widetilde{Ra}_{b21}\big)\big] + \frac{1}{3}\big\{\big[\widetilde{Ra}_{a21}\big(\widetilde{Ra}_{b11} - \widetilde{Ra}_{a11}\big) + \widetilde{Ra}_{a11}\big(\widetilde{Ra}_{b21} - \widetilde{Ra}_{a21}\big)\big] - \big[\widetilde{Ra}_{c11}\big(\widetilde{Ra}_{c21} - \widetilde{Ra}_{b21}\big) + \widetilde{Ra}_{c21}\big(\widetilde{Ra}_{c11} - \widetilde{Ra}_{b11}\big)\big]\big\} + \frac{1}{2}\big(\widetilde{Ra}_{a11}\widetilde{Ra}_{a21} + \widetilde{Ra}_{c11}\widetilde{Ra}_{c21}\big) + \frac{1}{2}E_f\big(\widetilde{Ra}_1\big)E_f\big(\widetilde{Ra}_2\big)\Bigg); w\widetilde{Ra}_1 \wedge w\widetilde{Ra}_2, u\widetilde{Ra}_1 \vee u\widetilde{Ra}_2, y\widetilde{Ra}_1 \vee y\widetilde{Ra}_2\rangle$$

$$cov\big(\widetilde{Ra}_1, \widetilde{Ra}_2\big) = \langle \frac{1}{4}[(0.4 - 0.3)(0.3 - 0.2) + (0.5 - 0.4)(0.4 - 0.3)] + \frac{1}{3}[0.2(0.4 - 0.3) + 0.3(0.3 - 0.2)] - [0.5(0.4 - 0.3) + 0.4(0.5 - 0.4)] + \frac{1}{2}(0.3 * 0.2 + 0.5 * 0.4) + \frac{1}{2}0.398 * 0.298; 0.5 \wedge 0.6, 0.2 \vee 0.3, 0.3 \vee 0.2\rangle$$

$$cov\big(\widetilde{Ra}_1, \widetilde{Ra}_2\big) = \langle \frac{1}{4}[0.01 + 0.01] + \frac{1}{3}[0.02 + 0.03] - [0.05 + 0.04] + \frac{1}{2}(0.06 + 0.20) + \frac{1}{2}0.118; 0.6, 0.2, 0.2\rangle$$

$$cov\big(\widetilde{Ra}_1, \widetilde{Ra}_2\big) = \langle 0.021 - 0.013 + 0.065 + 0.059; 0.6, 0.2, 0.2\rangle$$

$$cov\big(\widetilde{Ra}_1, \widetilde{Ra}_2\big) = \langle 0.132; 0.6, 0.2, 0.2\rangle$$

$$cov\big(\widetilde{Ra}_1, \widetilde{Ra}_3\big) = \langle \frac{1}{4}[(0.4 - 0.3)(0.35 - 0.25) + (0.5 - 0.4)(0.4 - 0.35)] + \frac{1}{3}[0.25(0.4 - 0.35) + 0.3(0.35 - 0.25)] - [0.5(0.4 - 0.35) + 0.4(0.5 - 0.4)] + \frac{1}{2}(0.3 * 0.25 + 0.5 * 0.4) + \frac{1}{2}0.398 * 0.340; 0.5 \wedge 0.4, 0.2 \vee 0.3, 0.3 \vee 0.3\rangle$$

$$cov\big(\widetilde{Ra}_1, \widetilde{Ra}_3\big) = \langle \frac{1}{4}[0.01 + 0.005] + \frac{1}{3}[0.0125 + 0.03] - [0.025 + 0.04] + \frac{1}{2}(0.075 + 0.2) + \frac{1}{2}0.135; 0.5, 0.2, 0.3\rangle$$

$$cov\big(\widetilde{Ra}_1, \widetilde{Ra}_3\big) = \langle 0.00375 - 0.0075 + 0{,}137 + 0.0675; 0.5, 0.2, 0.3\rangle$$

$$cov\left(\widetilde{Ra}_1, \widetilde{Ra}_3\right) = \langle 0.2007; 0.5, 0.2, 0.3\rangle$$

$$cov\left(\widetilde{Ra}_2, \widetilde{Ra}_3\right) = \langle \frac{1}{4}[(0.3-0.2)(0.35-0.25)+(0.4-0.3)(0.4-0.35)] + \frac{1}{3}[0.25(0.3-0.2)+0.2(0.35-0.25)] - [0.4(0.4-0.35)+0.4(0.4-0.3)] + \frac{1}{2}(0.2*0.25+0.4*0.4) + \frac{1}{2}0.298 * 0.340; 0.6\wedge 0.4, 0.3\vee 0.3, 0.2\vee 0.3\rangle$$

$$cov\left(\widetilde{Ra}_2, \widetilde{Ra}_3\right) = \langle \frac{1}{4}[0.01+0.005] + \frac{1}{3}[0.045+0.01] - [0.02+0.04] + \frac{1}{2}(0.05+0.16) + \frac{1}{2}0.101; 0.6, 0.3, 0.2\rangle$$

$$cov\left(\widetilde{Ra}_2, \widetilde{Ra}_3\right) = \langle 0.00375 - 0.005 + 0.105 + 0.0505;\ 0.6, 0.3, 0.2\rangle$$

$$cov\left(\widetilde{Ra}_2, \widetilde{Ra}_3\right) = \langle 0.1542;\ 0.6, 0.3, 0.2\rangle$$

$$(\widetilde{\Omega}) = \begin{pmatrix} \langle 0.288; 0.5, 0.2, 0.3\rangle & \langle 0.132; 0.6, 0.2, 0.2\rangle & \langle 0.200; 0.5, 0.2, 0.3\rangle \\ \langle 0.132; 0.6, 0.2, 0.2\rangle & \langle 0.219; 0.5, 0.2, 0.3\rangle & \langle 0.154;\ 0.6, 0.3, 0.2\rangle \\ \langle 0.200; 0.5, 0.2, 0.3\rangle & \langle 0.154;\ 0.6, 0.3, 0.2\rangle & \langle 0.255; 0.5, 0.2, 0.3\rangle \end{pmatrix}$$

$$det\,\widetilde{\Omega} = \begin{vmatrix} \langle 0.288; 0.5, 0.2, 0.3\rangle & \langle 0.132; 0.6, 0.2, 0.2\rangle & \langle 0.200; 0.5, 0.2, 0.3\rangle \\ \langle 0.132; 0.6, 0.2, 0.2\rangle & \langle 0{,}219; 0.5, 0.2, 0.3\rangle & \langle 0.154;\ 0.6, 0.3, 0.2\rangle \\ \langle 0.200; 0.5, 0.2, 0.3\rangle & \langle 0.154;\ 0.6, 0.3, 0.2\rangle & \langle 0.255; 0.5, 0.2, 0.3\rangle \end{vmatrix} =$$

$$= 0.00003 + 0.0010 + 0.0010 - 0.0006 - 0.0001 - 0.0004 = 0.00093 \neq 0$$

$$\widetilde{\Omega}^T = \begin{pmatrix} \langle 0.288; 0.5, 0.2, 0.3\rangle & \langle 0.132; 0.6, 0.2, 0.2\rangle & \langle 0.200; 0.5, 0.2, 0.3\rangle \\ \langle 0.132; 0.6, 0.2, 0.2\rangle & \langle 0.219; 0.5, 0.2, 0.3\rangle & \langle 0.154;\ 0.6, 0.3, 0.2\rangle \\ \langle 0.200; 0.5, 0.2, 0.3\rangle & \langle 0.154;\ 0.6, 0.3, 0.2\rangle & \langle 0.255; 0.5, 0.2, 0.3\rangle \end{pmatrix}$$

$$\widetilde{\Omega}^* = \begin{pmatrix} \langle -0.0048; 0.6, 0.3, 0.2\rangle & \langle 0.0088; 0.6, 0.2, 0.2\rangle & \langle 0.0021;\ 0.6, 0.2, 0.2\rangle \\ \langle 0.0088;\ 0.6, 0.2, 0.2\rangle & \langle -0.0346;\ 0.5, 0.2, 0.3\rangle & \langle 0.0116;\ 0.6, 0.2, 0.2\rangle \\ \langle 0.0021;\ 0.6, 0.2, 0.2\rangle & \langle 0.0116;\ 0.6, 0.2, 0.2\rangle & \langle -0.0045;\ 0.6, 0.2, 0.2\rangle \end{pmatrix}$$

$$\widetilde{\Omega}^{-1} = \frac{1}{0.00093}\begin{pmatrix} \langle -0.0048; 0.6, 0.3, 0.2\rangle & \langle 0.0088; 0.6, 0.2, 0.2\rangle & \langle 0.0021;\ 0.6, 0.2, 0.2\rangle \\ \langle 0.0088;\ 0.6, 0.2, 0.2\rangle & \langle -0.0346;\ 0.5, 0.2, 0.3\rangle & \langle 0.0116;\ 0.6, 0.2, 0.2\rangle \\ \langle 0.0021;\ 0.6, 0.2, 0.2\rangle & \langle 0.0116;\ 0.6, 0.2, 0.2\rangle & \langle -0.0045;\ 0.6, 0.2, 0.2\rangle \end{pmatrix}$$

$$\widetilde{\Omega}^{-1} = \begin{pmatrix} \langle -5{,}16;\ 0.6, 0.3, 0.2\rangle & \langle 9.46;\ 0.6, 0.2, 0.2\rangle & \langle 2.25;\ 0.6, 0.2, 0.2\rangle \\ \langle 9.46;\ 0.6, 0.2, 0.2\rangle & \langle -37.20;\ 0.5, 0.2, 0.3\rangle & \langle 12.47; 0.6, 0.2, 0.2\rangle \\ \langle 2.25;\ 0.6, 0.2, 0.2\rangle & 12.47;\ 0.6, 0.2, 0.2 & \langle -4.83; 0.6, 0.2, 0.2\rangle \end{pmatrix}$$

(d)

$$\langle\alpha\rangle = \langle \tilde{e}^T \times \widetilde{\Omega}^{-1} \times \tilde{e}\rangle$$

$$\langle\beta\rangle = \langle \tilde{R}^T \times \widetilde{\Omega}^{-1} \times \tilde{e}\rangle = \langle \tilde{e}^T \times \widetilde{\Omega}^{-1} \times \tilde{R}\rangle$$

$$\langle\gamma\rangle = \langle \tilde{R}^T \times \widetilde{\Omega}^{-1} \times \tilde{R}\rangle$$

$$\langle\alpha\rangle = (1\ 1\ 1)\begin{pmatrix} \langle -5.16;\ 0.6, 0.3, 0.2\rangle & \langle 9.46;\ 0.6, 0.2, 0.2\rangle & \langle 2.25;\ 0.6, 0.2, 0.2\rangle \\ \langle 9.46;\ 0.6, 0.2, 0.2\rangle & \langle -37.20;\ 0.5, 0.2, 0.3\rangle & \langle 12.47; 0.6, 0.2, 0.2\rangle \\ \langle 2.25;\ 0.6, 0.2, 0.2\rangle & \langle 12.47;\ 0.6, 0.2, 0.2\rangle & \langle -4.83; 0.6, 0.2, 0.2\rangle \end{pmatrix}\begin{pmatrix}1\\1\\1\end{pmatrix}$$

$$\langle\alpha\rangle = (\langle 6.55;\ 0.6, 0.2, 0.2\rangle\ \langle -15.27;\ 0.6, 0.2, 0.2\rangle\ \langle 9.89; 0.6, 0.2, 0.2\rangle)\begin{pmatrix}1\\1\\1\end{pmatrix}$$

$$\langle\alpha\rangle = \langle 1.17; 0.6, 0.2, 0.2\rangle$$

$$\langle\beta\rangle = (1\ 1\ 1)\begin{pmatrix} \langle -5.16;\ 0.6, 0.3, 0.2\rangle & \langle 9.46;\ 0.6, 0.2, 0.2\rangle & \langle 2.25;\ 0.6, 0.2, 0.2\rangle \\ \langle 9.46;\ 0.6, 0.2, 0.2\rangle & \langle -37.20;\ 0.5, 0.2, 0.3\rangle & \langle 12.47; 0.6, 0.2, 0.2\rangle \\ \langle 2.25;\ 0.6, 0.2, 0.2\rangle & \langle 12.47;\ 0.6, 0.2, 0.2\rangle & \langle -4.83; 0.6, 0.2, 0.2\rangle \end{pmatrix}\begin{pmatrix} \langle 0.316; 0.5, 0.2 \\ \langle 0.199; 0.5, 0.2 \\ \langle 0.416; 0.4, 0.3 \end{pmatrix}$$

$$\langle\beta\rangle = (\langle 6.55;\ 0.6, 0.2, 0.2\rangle\ \langle -15.27;\ 0.6, 0.2, 0.2\rangle\ \langle 9.89; 0.6, 0.2, 0.2\rangle)\begin{pmatrix} \langle 0.316; 0.5, 0.2, 0.3\rangle \\ \langle 0.199; 0.5, 0.2, 0.3\rangle \\ \langle 0.416; 0.4, 0.3, 0.3\rangle \end{pmatrix}$$

$$\langle\beta\rangle = \langle 2.06; 0.6, 0.2, 0.2\rangle - \langle 3.03; 0.6, 0.2, 0.2\rangle + \langle 4.11;\ 0.6, 0.2, 0.2\rangle$$

$$\langle\beta\rangle = \langle 3.14; 0.6, 0.2, 0.2\rangle$$

$$\langle\gamma\rangle = (\langle 0.316; 0.5, 0.2, 0.3\rangle\langle 0.199; 0.5, 0.2, 0.3\rangle\langle 0.416; 0.4, 0.3, 0.3\rangle) \times$$
$$\times\begin{pmatrix} \langle -5.16;\ 0.6, 0.3, 0.2\rangle & \langle 9.46;\ 0.6, 0.2, 0.2\rangle & \langle 2.25;\ 0.6, 0.2, 0.2\rangle \\ \langle 9.46;\ 0.6, 0.2, 0.2\rangle & \langle -37.20;\ 0.5, 0.2, 0.3\rangle & \langle 12.47; 0.6, 0.2, 0.2\rangle \\ \langle 2.25;\ 0.6, 0.2, 0.2\rangle & \langle 12.47;\ 0.6, 0.2, 0.2\rangle & \langle -4.83; 0.6, 0.2, 0.2\rangle \end{pmatrix}\begin{pmatrix} \langle 0.316; 0.5, 0.2, \\ \langle 0.199; 0.5, 0.2, \\ \langle 0.416; 0.4, 0.3, \end{pmatrix}$$

$$\langle\gamma\rangle = (\langle 1.18;\ 0.6, 0.2, 0.2\rangle\langle 0.76;\ 0.6, 0.2, 0.2\rangle)\langle 1.19;\ 0.6, 0.2, 0.2\rangle\begin{pmatrix} \langle 0.316; 0.5, 0.2, 0.3\rangle \\ \langle 0.199; 0.5, 0.2, 0.3\rangle \\ \langle 0.416; 0.4, 0.3, 0.3\rangle \end{pmatrix}$$

$$\langle\gamma\rangle = \langle 1.01; 0.6, 0.2, 0.2\rangle$$

$$\langle\widetilde{\Omega}^{-1} \times \tilde{R}\rangle = \begin{pmatrix} \langle -5.16;\ 0.6, 0.3, 0.2\rangle & \langle 9.46;\ 0.6, 0.2, 0.2\rangle & \langle 2.25;\ 0.6, 0.2, 0.2\rangle \\ \langle 9.46;\ 0.6, 0.2, 0.2\rangle & \langle -37.20;\ 0.5, 0.2, 0.3\rangle & \langle 12.47; 0.6, 0.2, 0.2\rangle \\ \langle 2.25;\ 0.6, 0.2, 0.2\rangle & \langle 12.47;\ 0.6, 0.2, 0.2\rangle & \langle -4.83; 0.6, 0.2, 0.2\rangle \end{pmatrix}\begin{pmatrix} \langle 0.316; 0.5, 0.2 \\ \langle 0.199; 0.5, 0.2 \\ \langle 0.416; 0.4, 0.3 \end{pmatrix}$$

$$\langle\widetilde{\Omega}^{-1} \times \tilde{R}\rangle = \begin{pmatrix} \langle 1.18;\ 0.6, 0.2, 0.2\rangle \\ \langle 0.76;\ 0.6, 0.2, 0.2\rangle \\ \langle 1.19;\ 0.6, 0.2, 0.2\rangle \end{pmatrix}$$

$$\langle\widetilde{\Omega}^{-1} \times \tilde{e}\rangle = \begin{pmatrix} \langle -5.16;\ 0.6, 0.3, 0.2\rangle & \langle 9.46;\ 0.6, 0.2, 0.2\rangle & \langle 2.25;\ 0.6, 0.2, 0.2\rangle \\ \langle 9.46;\ 0.6, 0.2, 0.2\rangle & \langle -37.20;\ 0.5, 0.2, 0.3\rangle & \langle 12.47; 0.6, 0.2, 0.2\rangle \\ \langle 2.25;\ 0.6, 0.2, 0.2\rangle & \langle 12.47;\ 0.6, 0.2, 0.2\rangle & \langle -4.83; 0.6, 0.2, 0.2\rangle \end{pmatrix}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$$\langle\widetilde{\Omega}^{-1} \times \tilde{e}\rangle = \begin{pmatrix} \langle 6.55; 0.6, 0.2, 0.2\ \rangle \\ \langle -15.27; 0.6, 0.2, 0.2\rangle \\ \langle 9.89;\ \ 0.6, 0.2, 0.2\rangle \end{pmatrix}$$

$$\tilde{X} = \langle\frac{1}{\langle\alpha\rangle\langle\gamma\rangle - \langle\beta\rangle^2}\Big[\Big(\langle\tilde{\rho}_{R_p}\rangle\langle\alpha\rangle - \langle\beta\rangle\Big)\langle\widetilde{\Omega}^{-1} \times \tilde{R}\rangle + \Big(\langle\gamma\rangle - \langle\tilde{\rho}_{R_p}\rangle\langle\beta\rangle\Big)\langle\widetilde{\Omega}^{-1} \times \tilde{e}\rangle\Big]; w\tilde{x}_{A_k}, u\tilde{x}_{A_k}, y\tilde{x}_{A}$$

$$\tilde{X} = \frac{1}{\langle 1.17\rangle\langle 1.01\rangle - 3.14^2}\left[(\langle 0.3109\rangle\langle 1.17\rangle - \langle 3.14\rangle)\begin{pmatrix} \langle 1.18;\ 0.6, 0.2, 0.2\rangle \\ \langle 0.76;\ 0.6, 0.2, 0.2\rangle \\ \langle 1.19;\ 0.6, 0.2, 0.2\rangle \end{pmatrix}\right]$$
$$+ (\langle 1{,}01\rangle - \langle 3.14\rangle\langle 0.3109\rangle)\begin{pmatrix} \langle 6.55; 0.6, 0.2, 0.2\ \rangle \\ \langle -15.27; 0.6, 0.2, 0.2\rangle \\ \langle 9.89;\ \ 0.6, 0.2, 0.2\rangle \end{pmatrix}$$

$$\tilde{X} = \langle -0.11\rangle\left[\langle -2.77\rangle\begin{pmatrix} \langle 1.18;\ 0.6, 0.2, 0.2\rangle \\ \langle 0.76;\ 0.6, 0.2, 0.2\rangle \\ \langle 1.19;\ 0.6, 0.2, 0.2\rangle \end{pmatrix} - \langle 0.03\rangle\begin{pmatrix} \langle 6.55; 0.6, 0.2, 0.2\ \rangle \\ \langle -15.27; 0.6, 0.2, 0.2\rangle \\ \langle 9.89;\ \ 0.6, 0.2, 0.2\rangle \end{pmatrix}\right]$$

$$\tilde{X} = \langle -0.11 \rangle \left[\begin{pmatrix} \langle -3.26;\ 0.6, 0.2, 0.2 \rangle \\ \langle -2.10;\ 0.6, 0.2, 0.2 \rangle \\ \langle -3.29;\ 0.6, 0.2, 0.2 \rangle \end{pmatrix} - \begin{pmatrix} \langle 0.19; 0.6, 0.2, 0.2 \rangle \\ \langle -0.45; 0.6, 0.2, 0.2 \rangle \\ \langle 0.29;\ 0.6, 0.2, 0.2 \rangle \end{pmatrix} \right]$$

$$\tilde{X} = \langle -0.11 \rangle \begin{pmatrix} \langle -3.45;\ 0.6, 0.2, 0.2 \rangle \\ \langle -1.65;\ 0.6, 0.2, 0.2 \rangle \\ \langle -3.58;\ 0.6, 0.2, 0.2 \rangle \end{pmatrix}$$

$$\tilde{X} = \begin{pmatrix} \langle 0.3795;\ 0.6, 0.2, 0.2 \rangle \\ \langle 0.1815;\ 0.6, 0.2, 0.2 \rangle \\ \langle 0.3938;\ 0.6, 0.2, 0.2 \rangle \end{pmatrix}$$

$$\begin{aligned} \langle \tilde{R}_P; w\widetilde{Rp}, u\widetilde{Rp}, y\widetilde{Rp} \rangle \\ &= \langle 0.316; 0.5, 0.2, 0.3 \rangle \langle 0.3795;\ 0.6, 0.2, 0.2 \rangle + \langle 0.199; 0.5, 0.2, 0.3 \rangle \langle 0.1815;\ 0.6, 0. \\ &+ \langle 0.416; 0.4, 0.3, 0.3 \rangle \langle 0.3938;\ 0.6, 0.2, 0.2 \rangle \end{aligned}$$

$$\langle \tilde{R}_P; w\widetilde{Rp}, u\widetilde{Rp}, y\widetilde{Rp} \rangle = \langle 0.1199; 0.6, 0.2, 0.2 \rangle + \langle 0.0361; 0.6, 0.2, 0.2 \rangle + \langle 0.1638;\ 0.6, 0.2, 0.2 \rangle$$

$$\langle \tilde{R}_P; w\widetilde{Rp}, u\widetilde{Rp}, y\widetilde{Rp} \rangle = \langle 0.3198;\ 0.6, 0.2, 0.2 \rangle$$

(e)

$$\langle \tilde{\sigma}_P^2; w\widetilde{\sigma p}, u\widetilde{\sigma p}, y\widetilde{\sigma p} \rangle = \langle \frac{1}{\langle \alpha \rangle \langle \gamma \rangle - \langle \beta \rangle^2} \left[\langle \alpha \rangle \langle \tilde{\rho}_{R_p} \rangle^2 - 2 \langle \beta \rangle \langle \tilde{\rho}_{R_p} \rangle + \langle \gamma \rangle \right]; w\widetilde{\sigma p}, u\widetilde{\sigma p}, y\widetilde{\sigma p} \rangle$$

$$\langle \tilde{\sigma}_P^2; w\widetilde{\sigma p}, u\widetilde{\sigma p}, y\widetilde{\sigma p} \rangle = \frac{1}{\langle 1{,}17 \rangle \langle 1{,}01 \rangle - 3{,}14^2} [\langle 1.17 \rangle \langle 0.3109 \rangle^2 - 2 \langle 3.14 \rangle \langle 0.3109 \rangle + \langle 1.01 \rangle]; 0.6, 0.2$$

$$\langle \tilde{\sigma}_P^2; w\widetilde{\sigma p}, u\widetilde{\sigma p}, y\widetilde{\sigma p} \rangle = \langle -0.11 \rangle [\langle 0{,}1130 \rangle - \langle 1.95 \rangle + \langle 1.01 \rangle]; 0.6, 0.2, 0.2$$

$$\langle \tilde{\sigma}_P^2; w\widetilde{\sigma p}, u\widetilde{\sigma p}, y\widetilde{\sigma p} \rangle = \langle -0.11 \rangle \langle -0.827 \rangle; 0.6, 0.2, 0.2$$

$$\langle \tilde{\sigma}_P^2; w\widetilde{\sigma p}, u\widetilde{\sigma p}, y\widetilde{\sigma p} \rangle = \langle 0.0909;\ 0.6, 0.2, 0.2 \rangle$$

$$\langle \sigma_p; w\widetilde{\sigma p}, u\widetilde{\sigma p}, y\widetilde{\sigma p} \rangle = \sqrt{\langle 0.0909;\ 0.6, 0.2, 0.2 \rangle}$$

$$\langle \sigma_p; w\widetilde{\sigma p}, u\widetilde{\sigma p}, y\widetilde{\sigma p} \rangle = \langle 0.3016; 0.6, 0.2, 0.2 \rangle$$

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