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Probabilistic Single-Valued (Interval) Neutrosophic Hesitant Fuzzy Set and Its Application in Multi-Attribute Decision Making

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Abstract: The uncertainty and concurrence of randomness are considered when many practical problems are dealt with. To describe the aleatory uncertainty and imprecision in a neutrosophic environment and prevent the obliteration of more data, the concept of the probabilistic single-valued (interval) neutrosophic hesitant fuzzy set is introduced. By definition, we know that the probabilistic single-valued neutrosophic hesitant fuzzy set (PSVNHFS) is a special case of the probabilistic interval neutrosophic hesitant fuzzy set (PINHFS). PSVNHFSs can satisfy all the properties of PINHFSs. An example is given to illustrate that PINHFS compared to PSVNHFS is more general. Then, PINHFS is the main research object. The basic operational relations of PINHFS are studied, and the comparison method of probabilistic interval neutrosophic hesitant fuzzy numbers (PINHFNs) is proposed. Then, the probabilistic interval neutrosophic hesitant fuzzy weighted averaging (PINHFWA) and the probability interval neutrosophic hesitant fuzzy weighted geometric (PINHFWG) operators are presented. Some basic properties are investigated. Next, based on the PINHFWA and PINHFWG operators, a decision-making method under a probabilistic interval neutrosophic hesitant fuzzy circumstance is established. Finally, we apply this method to the issue of investment options. The validity and application of the new approach is demonstrated.

Keywords: probabilistic single-valued (interval) neutrosophic hesitant fuzzy set; multi-attribute decision making; aggregation operator

1. Introduction

In real life, uncertainty widely exists, like an expert system, information fusion, intelligent computations and medical diagnoses. When some decision problems need to be solved, establishing mathematical models of uncertainty plays an important role. Especially when dealing with big data problems, the uncertainty must be considered. Therefore, to describe the uncertainty of the problems, Zadeh [1] presented the fuzzy set theory. Next, many new types of fuzzy set theory have been developed, including the intuitionistic fuzzy set [2], hesitant fuzzy set (HFS) [3], dual hesitant fuzzy set (DHFS) [4], interval-valued intuitionistic fuzzy set (IVIFS) [5,6], necessary and possible hesitant fuzzy sets [7] and dual hesitant fuzzy probability [8]. The fuzzy set theory is a useful tool to figure out uncertain information [9]. In addition, Fuzzy set theory has also been applied to algebraic systems [10–13].

Simultaneously, in actual productions, statistical uncertainty needs to be considered. The probabilistic method is not always effective when we deal with epistemic uncertain problems [14]. Thus, those problems makes researchers attempt to combine fuzzy set theory with probability theory as a new fuzzy concept. For example, (1) probability theory as a method of knowledge representation [15–18]; (2) increase the probability value when processing fuzzy decision making problems [19–21]; (3) through the combination of stochastic simulation with nonlinear programming, the fuzzy values can be generated [22,23]. In [24], Hao et al. lists a detailed summary. In the probabilistic fuzzy circumstances, probabilistic data will be lost easily. Thus, under the fuzzy linguistic environments [25–27], Pang et al. [28] established a new type of probabilistic fuzzy linguistic term set and successfully solved these issues. In some practical issues, it is necessary to fully consider the ambiguity and probability. In 2016, Xu and Zhou [29] produced the hesitant probabilistic fuzzy set (HPFS). Then, Hao et al. [24] researched a new probabilistic dual hesitant fuzzy set (PDHFS) and applied it to the uncertain risk evaluation issues.

In [30], Smarandache introduced the neutrosophic set (NS) as a new type of fuzzy set. The NS A includes three independent members: truth membership $T_A(x) \in [0,1]$, indeterminacy membership $I_A(x) \in [0,1]$ and falsity membership $F_A(x) \in [0,1]$. NS theory has been widely used in algebraic systems [31–36]. Next, some new types of NS were introduced, like single-valued NS (SVNS) [37] and interval NS (INS) [38]. Ye utilized SVNS theory applied to different types of decision making (DM) issues [39–41]. In [42], Ye presented a simplified neutrosophic set (SNS). Xu and Xia utilized HFS theory for actual life productions [43–46]. Next, in a hesitant fuzzy environment, a group DM method was introduced by Xu et al. [47]. However, there are some types of questions that are difficult to solve by HFS. Thus, Zhu [4] introduced a DHFS theory. Then, Ye [48] established a correlation coefficient of DHFS. When decision makers are making decisions, DHFS theory cannot express the doubts of decision makers, completely. Next, in 2005, a single-valued neutrosophic hesitant fuzzy set (SVNHFS) was established by Ye [49], and interval neutrosophic hesitant fuzzy set (INHFS) was introduced by Liu [50]. Recently, neutrosophic fuzzy set theory has been widely researched and applied [51–55].

The aleatory uncertainty needs to be considered under the probabilistic neutrosophic hesitant fuzzy environments. Recently, fuzzy random variables have been used to describe probability information in uncertainty. However, in the above NS theories, the probabilities is not considered. Thus, if a neutrosophic multi-attribute decision making (MADM) problem under the probabilistic surroundings needs to be solved, the probabilities as a part of a fuzzy system will be lost. Until now, this problem has not given an effective solution. Peng et al. [56] proposed a new method: the probability multi-valued neutrosophic set (PMVNS). The PMVNS theory successfully solves multi-criteria group decision-making problems without loss of information. Then, we offer the notion of probabilistic SVNHFS (the probabilistic interval neutrosophic hesitant fuzzy set (PINHFS)) based on fuzzy set, HFS, PDHFS, NS and IVNHFS. To solve the MADM problems under the probabilistic interval neutrosophic hesitant fuzzy circumstance, the concept of PINHFS is used. By comparison, we find that the application of PINHFS is wider than that of the probabilistic single-valued neutrosophic hesitant fuzzy set (PSVNHFS), and it is closer to real life. Thus, we can study the case of the interval.

The rest of the paper is organized as follows: Section 2 briefly describes some basic definitions. In Section 3, the concepts of PSVNHFS and PINHFS are introduced, respectively. Next, PINHFS is the main research object. The comparison method of probabilistic interval neutrosophic hesitant fuzzy numbers (PINHFNs) is proposed. In Section 4, the basic operation laws of PINHFN are investigated. The probabilistic interval neutrosophic hesitant fuzzy weighted averaging (PINHFWA) and the probability interval neutrosophic hesitant fuzzy weighted geometric (PINHFWG) operators are established, and some basic properties are studied in Section 5. In Section 6, a MAMD method based on the PINHFWA and PINHFWG operators is proposed. Section 7 gives an illustrative example according to our method. To explain that PINHFS comparedto PSVNHFS is more extensive, in Section 8, the PSVNHFS being a special case of PINHFS, the probabilistic single-valued neutrosophic hesitant fuzzy weighted averaging (PSVNHFWA) and probabilistic single-valued neutrosophic hesitant fuzzy weighted geometric (PSVNHFWG) operators are introduced and a numerical example given to illustrate. Last, we summarize the conclusion and further research work.

2. Preliminaries

Let us review some fundamental definitions of HFS, SVNHFS and INHFS in this section.

Definition 1. *([3]) Let X be a non-empty finite set; an HFS A on X is defined in terms of a function $h_A(x)$ that when applied to X returns a finite subset of [0, 1], and we can express HFSs by:*

$$A = \{\langle x, h_A(x)\rangle | x \in X\},$$

where $h_A(x)$ is a set of some different values in [0, 1], representing the possible membership degrees of the element $x \in X$ to A. We call $h_A(x)$ a hesitant fuzzy element (HFE), denoted by h, which reads $h = \{\lambda | \lambda \in h\}$.

Definition 2. *([49]) Let X be a fixed set; an SVNHFS on X is defined as:*

$$N = \{\langle x, \tilde{t}(x), \tilde{i}(x), \tilde{f}(x)\rangle | x \in X\}$$

in which $\tilde{t}(x)$, $\tilde{i}(x)$ and $\tilde{f}(x)$ are three sets of some values in [0, 1], denoting the possible truth-membership hesitant degrees, indeterminacy-membership hesitant degrees and falsity-membership hesitant degrees of the element $x \in X$ to the set N, respectively, with the conditions $0 \leq \delta, \gamma, \eta \leq 1$ and $0 \leq \delta^+ + \gamma^+ + \eta^+ + \leq 3$, where $\delta \in \tilde{t}(x), \gamma \in \tilde{i}(x), \eta \in \tilde{f}(x)$, $\delta^+ \in \tilde{t}(x) = \bigcup_{\delta \in \tilde{t}(x)} max\delta$, $\gamma^+ \in \tilde{i}(x) = \bigcup_{\gamma \in \tilde{i}(x)} max\gamma$, $\eta^+ \in \tilde{f}(x) = \bigcup_{\eta \in \tilde{f}(x)} max\eta$ for $x \in X$.

Definition 3. *([50]) Let X be a non-empty finite set; an interval neutrosophic hesitant fuzzy set (INHFS) on X is represented by:*

$$A = \{(x, T_A(x), I_A(x), F_A(x)) | x \in X\},$$

where $T_A(x) = \{\tilde{\alpha} | \tilde{\alpha} \in T_A(x)\}$, $I_A(x) = \{\tilde{\beta} | \tilde{\beta} \in I_A(x)\}$ and $T_A(x) = \{\tilde{\gamma} | \tilde{\gamma} \in F_A(x)\}$ are three sets of some interval values in real unit interval [0, 1], which denotes the possible truth-membership hesitant degrees, indeterminacy-membership hesitant degrees and falsity-membership hesitant fuzzy degrees of element $x \in X$ to the set A and satisfies these limits: $\tilde{\alpha} = [\alpha^L, \alpha^U] \subseteq [0,1]$, $\tilde{\beta} = [\beta^L, \beta^U] \subseteq [0,1]$, $\tilde{\gamma} = [\gamma^L, \gamma^U] \subseteq [0,1]$ and $0 \leq sup\tilde{\alpha}^+ + sup\tilde{\beta}^+ + sup\tilde{\gamma}^+ \leq 3$, where $\tilde{\alpha}^+ = \bigcup_{\tilde{\alpha} \in T_A(x)} max\{\tilde{\alpha}\}$, $\tilde{\beta}^+ = \bigcup_{\tilde{\beta} \in I_A(x)} max\{\tilde{\beta}\}$ and $\tilde{\gamma}^+ = \bigcup_{\tilde{\gamma} \in F_A(x)} max\{\tilde{\gamma}\}$ for $x \in X$.

3. The Probabilistic Single-Valued (Interval) Neutrosophic Hesitant Fuzzy Set

In this section, the concepts of PSVNHFS and PINHFS are introduced. Since PINHFS is more general than PSVNHFS, the situation of PINHFS is mainly discussed.

Definition 4. *Let X be a fixed set. A probabilistic single-valued neutrosophic hesitant fuzzy set (PSVNHFS) on X is defined by the following mathematical symbol:*

$$NP = \{\langle x, \tilde{t}(x)|P^{\tilde{t}}(x), \tilde{i}(x)|P^{\tilde{i}}(x), \tilde{f}(x)|P^{\tilde{f}}(x)\rangle | x \in X\}. \tag{1}$$

The components $\tilde{t}(x)|P^{\tilde{t}}(x)$, $\tilde{i}(x)|P^{\tilde{i}}(x)$ and $\tilde{f}(x)|P^{\tilde{f}}(x)$ are three sets of some possible elements where $\tilde{t}(x)$, $\tilde{i}(x)$ and $\tilde{f}(x)$ represent the possible truth-membership hesitant degrees, indeterminacy-membership *hesitant degrees and falsity-membership hesitant degrees to the set X of x, respectively.* $P^{\tilde{t}}(x)$, $P^{\tilde{i}}(x)$ *and* $P^{\tilde{f}}(x)$ *are the corresponding probabilistic information for these three types of degrees. There is:*

$$0 \leq \alpha, \beta, \gamma \leq 1, 0 \leq \delta^+ + \gamma^+ + \eta^+ + \leq 3; P_a^{\tilde{t}} \in [0,1], P_b^{\tilde{i}} \in [0,1], P_c^{\tilde{f}} \in [0,1]; \sum_{a=1}^{\#\tilde{t}} P_a^{\tilde{t}} = 1, \sum_{b=1}^{\#\tilde{i}} P_b^{\tilde{i}} = 1, \sum_{c=1}^{\#\tilde{f}} P_c^{\tilde{f}} = 1.$$

where $\alpha \in \tilde{t}(x)$, $\beta \in \tilde{i}(x)$, $\gamma \in \tilde{f}(x)$. $\alpha^+ \in \tilde{t}^+(x) == \bigcup_{\alpha \in \tilde{t}(x)} max\alpha$, $\beta^+ \in \tilde{i}^+(x) = \bigcup_{\beta \in \tilde{i}(x)} max\beta$, $\gamma^+ \in \tilde{f}^+(x) = \bigcup_{\gamma \in \tilde{f}(x)} max\gamma$, $P_a^{\tilde{t}} \in P^{\tilde{t}}$, $P_b^{\tilde{i}} \in P^{\tilde{i}}$, $P_c^{\tilde{f}} \in P^{\tilde{f}}$. *The symbols* $\#\tilde{t}$, $\#\tilde{i}$ *and* $\#\tilde{f}$ *are the total numbers of elements in the components* $\tilde{t}(x)|P^{\tilde{t}}(x)$, $\tilde{i}(x)|P^{\tilde{i}}(x)$ *and* $\tilde{f}(x)|P^{\tilde{f}}(x)$, *respectively.*

For convenience, we call $\tilde{n}p = \langle \tilde{t}(x)|P^{\tilde{t}}(x), \tilde{i}(x)|P^{\tilde{i}}(x), \tilde{f}(x)|P^{\tilde{f}}(x)\rangle$ a probabilistic single-valued neutrosophic hesitant fuzzy number (PSVNHFN). It is defined by the mathematical symbol: $\tilde{n} = \{\tilde{t}|P^{\tilde{t}}, \tilde{i}|P^{\tilde{i}}, \tilde{f}|P^{\tilde{f}}\}$.

Next, a numerical example about investment options is used to explain the PSVNHFS.

Example 1. *OF four investment selections* A_h, *select the only investment option of an investment company. The investment corporation wants to have an effective evaluation and to choose the best investment opportunity; thus, the decision maker needs to use the PSVNHFS theory. According to the practical situation, there are three main attributes: (1)* C_1 *is the hazard of investment; (2)* C_2 *is the future outlook; (3)* C_3 *is the environment index. Thus, the data on these four options are represented by SVNHFS, as illustrated in Tables 1–4. Every table is called a probabilistic single-valued neutrosophic hesitant fuzzy decision matrix (PSVNHFDM).*

Table 1. A probabilistic single-valued neutrosophic hesitant fuzzy decision matrix (PSVNHFDM) D_1 with respect to A_1.

Attributes	Investment Selection A_1
C_1	$\{\{0.3|0.2, 0.4|0.3, 0.5|0.5\}, \{0.1|1\}, \{0.3|0.6, 0.4|0.4\}\}$
C_2	$\{\{0.5|0.5, 0.6|0.5\}, \{0.2|0.2, 0.3|0.8\}, \{0.3|0.4, 0.4|0.6\}\}$
C_3	$\{\{0.2|0.1, 0.3|0.9\}, \{0.1|0.3, 0.2|0.7\}, \{0.5|0.2, 0.6|0.8\}\}$

Table 2. PSVNHFDM D_2 with respect to A_2.

Attributes	Investment Selection A_2
C_1	$\{\{0.6|0.1, 0.7|0.9\}, \{0.1|0.4, 0.2|0.6\}, \{0.2|0.5, 0.3|0.5\}\}$
C_2	$\{\{0.6|0.2, 0.7|0.8\}, \{0.1|1\}, \{0.3|1\}\}$
C_3	$\{\{0.6|0.3, 0.7|0.7\}, \{0.1|0.6, 0.2|0.4\}, \{0.1|0.7, 0.2|0.3\}\}$

Table 3. PSVNHFDM D_3 with respect to A_3.

Attributes	Investment Selection A_3
C_1	$\{\{0.5|0.5, 0.6|0.5\}, \{0.4|1\}, \{0.2|0.2, 0.3|0.8\}\}$
C_2	$\{\{0.6|1\}, \{0.3|1\}, \{0.4|1\}\}$
C_3	$\{\{0.5|0.6, 0.6|0.4\}, \{0.1|1\}, \{0.3|1\}\}$

Table 4. PSVNHFDM D_4 with respect to A_4.

Attributes	Investment Selection A_4
C_1	$\{\{0.7|0.4, 0.8|0.6\}, \{0.1|1\}, \{0.1|0.1, 0.2|0.9\}\}$
C_2	$\{\{0.6|0.6, 0.7|0.4\}, \{0.1|1\}, \{0.2|1\}\}$
C_3	$\{\{0.3|0.9, 0.5|0.1\}, \{0.2|1\}, \{0.1|0.1, 0.2|0.8, 0.3|0.1\}\}$

In general, in the real world, if the three types of hesitant degrees of the PSVNHFS are interval values, this is a special case of INHFS. This kind of interval is more able to express the problems that people encounter when making choices in real life. However, the PSVNHFS is not an effective tool to solve this problem. Thus, we need to propose a new method to solve this problem. Then, the SVNHFS can be used as a special case of the probabilistic interval neutrosophic hesitant fuzzy circumstance. Thus, the probabilistic interval neutrosophic hesitant fuzzy set (PINHFS) is proposed and studied. The advantages of this are: SVNHFS can be studied in a wider range; the scope of application is also broader and closer to real life. Hence, we will give the concept of PINHFS. Simultaneously, in the rest of this paper, we take PINHFS as an example to conduct research.

Definition 5. *Let X be a fixed set, a probabilistic interval neutrosophic hesitant fuzzy set (PINHFS) on X is defined by the following mathematical symbol:*

$$N = \{\langle x, T(x)|P^T(x), I(x)|P^I(x), F(x)|P^F(x)\rangle | x \in X\}.$$

The components $T(x)|P^T(x)$, $I(x)|P^I(x)$ and $F(x)|P^F(x)$ are three sets of possible elements where $T(x)$, $I(x)$ and $F(x)$ are three sets of some interval values in the real unit interval [0, 1], which denotes the possible truth-membership hesitant degrees, indeterminacy-membership hesitant degrees and falsity-membership hesitant fuzzy degrees of element $x \in X$ to the set N, respectively. $P^T(x)$, $P^I(x)$ and $P^F(x)$ are the corresponding probabilistic information for these three types of degrees. There is:

$$\tilde{\alpha} = [\alpha^L, \alpha^U] \subseteq [0,1], \tilde{\beta} = [\beta^L, \beta^U] \subseteq [0,1], \tilde{\gamma} = [\gamma^L, \gamma^U] \subseteq [0,1]; 0 \leq sup\tilde{\alpha}^+ + sup\tilde{\beta}^+ + sup\tilde{\gamma}^+ \leq 3;$$
$$P_a^T \in [0,1], P_b^I \in [0,1], P_c^F \in [0,1], \sum_{a=1}^{\#T} P_a^T = 1, \sum_{b=1}^{\#I} P_b^I = 1, \sum_{c=1}^{\#F} P_c^F = 1;$$

where $\tilde{\alpha} \in T(x)$, $\tilde{\beta} \in I(x)$ and $\tilde{\gamma} \in F(x)$. $\tilde{\alpha}^+ = \bigcup_{\tilde{\alpha} \in T_A(x)} max\{\tilde{\alpha}\}$, $\tilde{\beta}^+ = \bigcup_{\tilde{\beta} \in I_A(x)} max\{\tilde{\beta}\}$, and $\tilde{\gamma}^+ = \bigcup_{\tilde{\gamma} \in F_A(x)} max\{\tilde{\gamma}\}$. $P_a^T \in P^T$, $P_b^I \in P^I$, $P_c^F \in P^F$. The symbols #T, #I and $\#\tilde{f}$ are the total numbers of elements in the components $T(x)|P^T(x)$, $I(x)|P^I(x)$ and $F(x)|P^F(x)$, respectively.

For convenience, we call $n = \langle T(x)|P^T(x), I(x)|P^I(x), F(x)|P^F(x)\rangle$ a probabilistic interval neutrosophic hesitant fuzzy number (PINHFN). It is defined by the mathematical symbol: $n = \{T|P^T, I|P^I, F|P^F\}$

If $\alpha^L = \alpha^U$, $\beta^L = \beta^U$, $\gamma^L = \gamma^U$, the PINHFS is transformed into the PSVNHFS.

Therefore, we know PINHFS is more general than PSVNHFS. PSVNHFS can satisfy all the properties of PINHFS. Thus, this paper mainly studies PINHFS.

Definition 6. *For a PINHFN n, where $a = 1, 2, ..., \#T$, $b = 1, 2, ..., \#I$, $c = 1, 2, ..., \#F$, the score function $s(n)$ is defined as:*

$$s(n) = \frac{\sum_{a=1}^{\#T}(\alpha_a^L + \alpha_a^U)P_a^T + \sum_{b=1}^{\#I}(2 - (\beta_b^L + \beta_b^U))P_b^I + \sum_{c=1}^{\#F}(2 - (\gamma_c^L + \beta_c^U))P_c^F}{6}, \quad (2)$$

where #T, #I and $\#\tilde{f}$ are the total numbers of elements in the components $T(x)|P^T(x)$, $I(x)|P^I(x)$ and $F(x)|P^F(x)$, respectively.

Definition 7. *For a PINHFN n, where $a = 1, 2, ..., \#T$, $b = 1, 2, ..., \#I$, $c = 1, 2, ..., \#F$, the deviation function $d(n)$ is defined as:*

$$d(n) = \frac{\sum_{a=1}^{\#T}(\alpha_a^L + \alpha_a^U - 2s(n))^2 \cdot P_a^T + \sum_{b=1}^{\#I}(2 - \beta_b^L - \beta_b^U - 2s(n))^2 \cdot P_b^I + \sum_{c=1}^{\#F}(2 - \gamma_c^L - \beta_c^U - 2s(n))^2 \cdot P_c^F}{4} \quad (3)$$

where $\#T$, $\#I$ and $\#\tilde{f}$ are the total numbers of elements in the components $T(x)|P^T(x)$, $I(x)|P^I(x)$ and $F(x)|P^F(x)$, respectively.

Definition 8. *Let n_1 and n_2 be two PINHFNs, the comparison of the method for n_1 and n_2 is as follows:*

(1) If $s(n_1) > s(n_2)$, then $n_1 > n_2$;
(2) If $s(n_1) = s(n_2), d(n_1) > d(n_2)$, then $n_1 > n_2$;
(3) If $s(n_1) = s(n_2), d(n_1) = d(n_2)$, then $n_1 = n_2$.

4. Some Basic Operations of PINHFNs

Definition 9. *Let $n_1 = \{T_1|P^{T_1}, I_1|P^{I_1}, F_1|P^{F_1}\}$ and $n_2 = \{T_2|P^{T_2}, I_2|P^{I_2}, F_2|P^{F_2}\}$ be two PINHFNs, then:*

(1) $$(n_1)^c = \bigcup_{\tilde{\alpha}_1\in T_1,\tilde{\beta}_1\in I_1,\tilde{\gamma}_1\in F_1} \{\tilde{\gamma}_1|P_1^{F_1}, [1-\beta_1^U, 1-\beta_1^L]|P_1^{I_1}, \tilde{\alpha}_1|P_1^{T_1}\},$$

(2) $$n_1 \cap n_2 = \bigcap_{\substack{\tilde{\alpha}_1\in T_1,\tilde{\beta}\in I_1,\tilde{\gamma}_1\in F_1,\\ \tilde{\eta}_2\in T_2,\tilde{\theta}\in I_2,\tilde{\mu}_2\in F_2}} \{\{\tilde{\alpha}_1 \cap \tilde{\eta}_2|\frac{P_2^{T_1}P_2^{T_2}}{\sum P_1^{T_1}P_2^{T_2}}\}, \{\tilde{\beta}_1 \cup \tilde{\theta}_2|\frac{P_1^{I_1}P_2^{I_2}}{\sum P_1^{I_1}P_2^{I_2}}\},$$
$$\{\tilde{\gamma}_1 \cup \tilde{\mu}_2|\frac{P_1^{F_1}P_2^{F_2}}{\sum P_1^{F_1}P_2^{F_2}}\}\},$$

(3) $$n_1 \cup n_2 = \bigcup_{\tilde{\alpha}_1\in T_1,\tilde{\beta}\in I_1,\tilde{\gamma}_2\in F_1,\tilde{\eta}_2\in T_2,\tilde{\theta}\in I_2,\tilde{\mu}_2\in F_2} \{\{\tilde{\alpha}_1 \cup \tilde{\eta}_2|\frac{P_1^{T_1}P_2^{T_2}}{\sum P_1^{T_1}P_2^{T_2}}\}, \{\tilde{\beta}_1 \cap \tilde{\theta}_2|\frac{P_1^{I_1}P_2^{I_2}}{\sum P_{b'}^{I_1}P_2^{I_2}}\},$$
$$\{\tilde{\gamma}_1 \cap \tilde{\mu}_2|\frac{P_1^{F_1}P_2^{F_2}}{\sum P_1^{F_1}P_2^{F_2}}\}\},$$

(4) $$(n_1)^\lambda = \bigcup_{\tilde{\alpha}_1\in T_1,\tilde{\beta}_1\in I_1,\tilde{\gamma}_1\in F_1} \{\{[(\alpha_1^L)^\lambda, (\alpha_1^U)^\lambda]|P_1^{T_1}\}, \{[1-(1-\beta_1^L)^\lambda, 1-(1-\beta_1^U)^\lambda]|P_1^{I_1}\},$$
$$\{[1-(1-\gamma_1^L), 1-(1-\gamma_1^U)]^\lambda|P_1^{F_1}\}\},$$

(5) $$\lambda(n_1) = \bigcup_{\tilde{\alpha}_1\in T_1,\tilde{\beta}_1\in I_1,\tilde{\gamma}_1\in F_1} \{\{[1-(1-\lambda_1^L)^\lambda, 1-(1-\lambda_1^L)^\lambda]|P_1^{T_1}\}, \{[(\beta_1^L)^\lambda, (\beta_1^U)^\lambda]|P_1^{I_1}\}, \{[(\gamma_1^L)^\lambda, (\gamma_1^U)^\lambda]|P_1^{F_1}\}\},$$

(6) $$n_1 \oplus n_2 = \bigcup_{\substack{\tilde{\alpha}_1\in T_1,\tilde{\beta}_1\in I_1,\tilde{\gamma}_1\in F_1,\\ \tilde{\eta}_2\in T_2,\tilde{\theta}_2\in I_2,\tilde{\mu}_2\in F_2}} \{\{[\alpha_1^L + \eta_2^L - \alpha_2^L\eta_2^L,\ \alpha_1^U + \eta_2^U - \alpha_2^U\eta_2^U]|P_1^{T_1}P_2^{T_2}\},$$
$$\{[\beta_1^L\theta_2^L,\ \beta_1^U\theta_2^U]|P_1^{I_1}P_2^{I_2}\}, \{[\gamma_1^L\mu_2^L, \gamma_1^U\mu_2^U]|P_1^{F_1}P_{c^2}^{F_2}\}\},$$

(7) $$n_1 \otimes n_2 = \bigcup_{\substack{\tilde{\alpha}_1\in T_1,\tilde{\beta}_1\in I_1,\tilde{\gamma}_1\in F_1,\\ \tilde{\eta}_2\in T_2,\tilde{\theta}_2\in I_2,\tilde{\mu}_2\in F_2}} \{\{[\alpha_1^L\eta_2^L, \alpha_1^U\eta_2^U]|P_1^{T_1}P_2^{T_2}\},$$
$$\{[\beta_1^L + \theta_2^L - \beta_1^L\theta_2^L,\ \beta_1^U + \theta_2^U - \beta_1^U\theta_2^U|P_1^{I_1}P_2^{I_2}\},$$
$$\{[\gamma_1^L + \mu_2^L - \gamma_1^L\mu_2^L,\ \gamma_1^U + \mu_2^U - \gamma_1^U\mu_2^U]|P_1^{F_1}P_{c^2}^{F_2}\}\},$$

where $P_1^{T_1}$; $P_1^{I_1}$ and $P_1^{F_1}$ are hesitant probabilities of $\tilde{\alpha}_1 \in T_1$, $\tilde{\beta}_1 \in I_1$ and $\tilde{\gamma}_1 \in F_1$, respectively. $P_2^{T_2}$; $P_2^{I_2}$ and $P_1^{F_2}$ are corresponding hesitant probabilities of $\tilde{\eta}_2 \in T_2$, $\tilde{\theta}_2 \in I_2$ and $\tilde{\mu}_2 \in F_2$.

Theorem 1. *Let n_1 and n_2 be two PINHFNs, then $(n_1)^c$, $n_1 \cap n_2$, $n_1 \cup n_2$, $(n_1)^\lambda$, $\lambda(n_1)$, $n_1 \oplus n_2$ and $n_1 \otimes n_2$ are PINHFNs.*

Proof. By Definition 5, Definition 9, it is easy to prove the result. □

Theorem 2. *Let* $n_1 = (T_1|P^{T_1}, I_1|P^{I_1}, F_1|P^{F_1})$, $n_2 = (T_2|P^{T_2}, I_2|P^{I_2}, F_2|P^{F_2})$ *and* $n_3 = (T_3|P^{T_3}, I_3|P^{I_3}, F_3|P^{F_3})$ *be three PINHFNs,* $\lambda, \lambda_1, \lambda_2 \geq 0$, *then:*

(1) $n_1 \oplus n_2 = n_2 \oplus n_1$; $n_1 \otimes n_2 = n_2 \otimes n_1$,
(2) $(n_1 \oplus n_2) \oplus n_3 = n_1 \oplus (n_2 \oplus n_3)$; $(n_1 \otimes n_2) \otimes n_3 = n_1 \otimes (n_2 \otimes n_3)$,
(3) $\lambda(n_1 \oplus n_2) = \lambda(n_1) \oplus \lambda(n_2)$,
(4) $(n_1 \otimes n_2)^\lambda = (n_1)^\lambda \otimes (n_2)^\lambda$,
(5) $(n_1)^{\lambda_1+\lambda_2} = (n_1)^{\lambda_1} \otimes (n_1)^{\lambda_2}$; $(\lambda_1 + \lambda_2)n_1 = \lambda_1(n_1) \oplus \lambda_2(n_1)$.

Proof. If $P_1^{T_1}$; $P_1^{I_1}$ and $P_1^{F_1}$ are probabilities of $\tilde{\alpha}_1 \in T_1$, $\tilde{\beta}_1 \in I_1$ and $\tilde{\gamma}_1 \in F_1$, respectively. $P_2^{T_2}$, $P_2^{I_2}$ and $P_2^{F_2}$ are corresponding probabilities of $\tilde{\eta}_2 \in T_2$, $\tilde{\theta}_2 \in I_2$ and $\tilde{\mu}_2 \in F_2$. $P_3^{T_3}$, $P_3^{I_3}$ and $P_3^{F_3}$ are corresponding probabilities of $\tilde{\xi}_3 \in T_3$, $\tilde{\sigma}_3 \in I_3$ and $\tilde{\phi}_3 \in F_3$, then we have:

(1) By Definition 9, we can get that (1) is true.

(2)

$$\begin{aligned}(n_1 \oplus n_2) \oplus n_3 = &\bigcup_{\substack{\tilde{\alpha}_1 \in T_1, \tilde{\beta}_1 \in I_1, \tilde{\gamma}_1 \in F_1, \\ \tilde{\eta}_2 \in T_2, \tilde{\theta}_2 \in I_2, \tilde{\mu}_2 \in F_2 \\ \tilde{\xi}_3 \in T_3, \tilde{\sigma}_3 \in I_3, \tilde{\phi}_3 \in F_3}} \{\{[\alpha_1^L + (\eta_2^L + \xi_3^L - \eta_2^L \xi_3^L) - \alpha_1^L(\eta_2^L + \xi_3^L - \eta_2^L \xi_3^L), \\ &\alpha_1^U + (\eta_2^U + \xi_3^U - \eta_2^U \xi_3^U) - \alpha_1^U(\eta_2^U + \xi_3^U - \eta_2^U \xi_3^U)]|P_1^{T_1}(P_2^{T_2}P_3^{T_3})\}, \\ &\{[\beta_1^L(\theta_2^L \sigma_3^L), \beta_1^U(\theta_2^U \sigma_3^U)]|P_1^{I_1}(P_2^{I_2}P_3^{I_3})\}, \\ &\{[\lambda_1^L(\mu_2^L \phi_3^L), \lambda_1^U(\mu_2^U \phi_3^U)]|P_1^{F_1}(P_2^{F_2}P_3^{F_3})\}\} \\ = &\, n_1 \oplus (n_2 \oplus n_3).\end{aligned}$$

Similarly, we can obtain $(n_1 \otimes n_2) \otimes n_3 = n_1 \otimes (n_2 \otimes n_3)$.

(3)

$$\begin{aligned}\lambda(n_1 \oplus n_2) = &\bigcup_{\substack{\tilde{\alpha}_1 \in T_1, \tilde{\beta}_1 \in I_1, \tilde{\gamma}_1 \in F_1, \\ \tilde{\eta}_2 \in T_2, \tilde{\theta}_2 \in I_2, \tilde{\mu}_2 \in F_2}} \{\{[1 - (1 - (\alpha_1^L + \eta_2^L - \alpha_1^L \eta_2^L))^\lambda, 1 - (1 - (\alpha_1^U + \eta_2^U - \alpha_1^U \eta_2^U))^\lambda]|P_1^{T_1} PT_{22}\} \\ &\{[(\beta_1^L)^\lambda (\theta_2^L)^\lambda, (\beta_1^U)^\lambda (\theta_2^U)^\lambda]|P_1^{I_1} P_2^{I_2}\}, \{[(\gamma_1^L)^\lambda (\mu_2^L)^\lambda, (\gamma_1^U)^\lambda (\mu_2^U)^\lambda]|P_1^{F_1} P_2^{F_2}\}\} \\ = &\bigcup_{\tilde{\alpha}_1 \in T_1, \tilde{\beta}_1 \in I_1, \tilde{\gamma}_1 \in F_1} \{\{[1 - (1 - \alpha_1^L)^\lambda, 1 - (1 - \alpha_1^U)^\lambda]|P_1^{T_1}\}, \{[(\beta_1^L)^\lambda, (\beta_1^U)^\lambda|P_1^{I_1}\}, \{[(\gamma_1^L)^\lambda, (\gamma_1^U)^\lambda]|P_1^{F_1}\}\} \\ &\oplus \bigcup_{\tilde{\eta}_2 \in T_2, \tilde{\theta}_2 \in I_2, \tilde{\mu}_2 \in F_2} \{\{[1 - (1 - \eta_2^L)^\lambda, 1 - (1 - \eta_2^U)^\lambda]|P_2^{T_2}\}, \{[(\theta_2^L)^\lambda, (\theta_2^U)^\lambda]|P_2^{I_2}\}, \{[(\mu_2^L)^\lambda, (\mu_2^U)^\lambda]|P_2^{F_2}\}\} \\ = &\, \lambda(n_1) \oplus \lambda(n_2).\end{aligned}$$

(4)

$$(n_1 \otimes n_2)^{\lambda} = \bigcup_{\substack{\tilde{\alpha}_1 \in T_1, \tilde{\beta}_1 \in I_1, \tilde{\gamma}_1 \in F_1, \\ \tilde{\eta}_2 \in T_2, \tilde{\theta}_2 \in I_2, \tilde{\mu}_2 \in F_2}} \{\{[(\alpha_1^L \eta_2^L)^{\lambda}, (\alpha_1^U \eta_2^U)^{\lambda}] | P_1^{T_1} P_2^{T_2}\}, \\ \{[1-(1-(\beta_1^L+\theta_2^L-\beta_1^L\theta_2^L))^{\lambda}, 1-(1-(\beta_1^U+\theta_2^U-\beta_1^U\theta_2^U))^{\lambda}] | P_1^{I_1} P_2^{I_2}\}, \\ \{[1-(1-(\gamma_1^L+\mu_2^L-\gamma_1^L\mu_2^L))^{\lambda}, 1-(1-(\gamma_1^U+\mu_2^U-\gamma_1^U\mu_2^U))^{\lambda}] | P_1^{F_1} P_2^{F_2}\}\}$$

$$= \bigcup_{\tilde{\alpha}_1 \in T_1, \tilde{\beta}_1 \in I_1, \tilde{\gamma}_1 \in F_1} \{\{[(\alpha_1^L)^{\lambda}, (\alpha_1^U)^{\lambda}] | P_1^{T_1}\}, \{[1-(1-\beta_1^L)^{\lambda}, 1-(1-\beta_1^U)^{\lambda}] | P_1^{I_1}\}, \\ \{[1-(1-\gamma_1^L)^{\lambda}, 1-(1-\gamma_1^U)^{\lambda}] | P_1^{F_1}\}\}$$

$$\otimes \bigcup_{\tilde{\eta}_2 \in T_2, \tilde{\theta}_2 \in I_2, \tilde{\mu}_2 \in F_2} \{\{[(\eta_2^L)^{\lambda}, (\eta_2^U)^{\lambda}] | P_2^{T_2}\}, \{[1-(1-\theta_2^L)^{\lambda}, 1-(1-\theta_2^U)^{\lambda}] | P_2^{I_2}\}, \\ \{[1-(1-\mu_2^L)^{\lambda}, 1-(1-\mu_2^U)^{\lambda}] | P_2^{F_2}\}\}$$

$$= (n_1)^{\lambda} \otimes (n_2)^{\lambda}.$$

(5)

$$(n_1)^{\lambda_1+\lambda_2} = \bigcup_{\tilde{\alpha}_1 \in T_1, \tilde{\beta}_1 \in I_1, \tilde{\gamma}_1 \in F_1} \{\{[(\alpha_1^L)^{\lambda_1+\lambda_2}, (\alpha_1^U)^{\lambda_1+\lambda_2}] | P_1^{T_1}\}, \{[1-(1-\beta_1^L)^{\lambda_1+\lambda_2}, 1-(1-\beta_1^U)^{\lambda_1+\lambda_2}] | P_1^{I_1}\}, \\ \{[1-(1-\gamma_1^L)^{\lambda_1+\lambda_2}, 1-(1-\gamma_1^U)^{\lambda_1+\lambda_2}] | P_1^{F_1}\}\}$$

$$= \bigcup_{\tilde{\alpha}_1 \in T_1, \tilde{\beta}_1 \in I_1, \tilde{\gamma}_1 \in F_1} \{\{\alpha_{a'}^{\lambda_1} | P_{a'}^{\tilde{t}_1}\}, \{(1-(1-\beta_{b'})^{\lambda_1}) | P_{b'}^{\tilde{i}_1}\}, \{(1-(1-\gamma_{c'})^{+\lambda_1}) | P_{c'}^{\tilde{f}_1}\}\} \\ \otimes \bigcup_{\tilde{\alpha}_1 \in T_1, \tilde{\beta}_1 \in I_1, \tilde{\gamma}_1 \in F_1} \{\{[(\alpha_1^L)^{\lambda_2}, (\alpha_1^U)^{\lambda_2}] | P_1^{T_1}\}, \{[1-(1-\beta_1^L)^{\lambda_2}, 1-(1-\beta_1^U)^{\lambda_2}] | P_1^{I_1}\}, \\ \{[1-(1-\gamma_1^L)^{\lambda_2}, 1-(1-\gamma_1^U)^{\lambda_2}] | P_1^{F_1}\}\}$$

$$= (n_1)^{\lambda_1} \otimes (n_1)^{\lambda_2}.$$

Similarly, we have $(\lambda_1 + \lambda_2)n_1 = \lambda_1(n_1) \oplus \lambda_2(n_1)$. □

Theorem 3. *Let n_1 and n_2 be two PINHFNs, $\lambda \geq 0$, then:*

(1) $((n_1)^c)^{\lambda} = (\lambda(n_1))^c$,
(2) $\lambda(n_1)^c = ((n_1)^{\lambda})^c$,
(3) $(n_1)^c \oplus {n_2}^c = (n_1 \otimes n_2)^c$,
(4) $(n_1)^c \otimes (n_2)^c = (n_1 \oplus n_2)^c$.

Proof. $P_1^{T_1}$; $P_1^{I_1}$ and $P_1^{F_1}$ are hesitant probabilities of $\tilde{\alpha}_1 \in T_1$, $\tilde{\beta}_1 \in I_1$ and $\tilde{\gamma}_1 \in F_1$, respectively. $P_2^{T_2}$, $P_2^{I_2}$ and $P_2^{F_2}$ are corresponding hesitant probabilities of $\tilde{\eta}_2 \in T_2$, $\tilde{\theta}_2 \in I_2$ and $\tilde{\mu}_2 \in F_2$. Then:

(1)

$$
\begin{aligned}
((n_1)^c)^\lambda &= (\bigcup_{\tilde{\alpha}_1\in T_1,\tilde{\beta}_1\in I_1,\tilde{\gamma}_1\in F_1}\{\{[\gamma_1^L,\gamma_1^U]|P_1^{F_1}\},\{[1-\beta_1^U,1-\beta_1^U]|P_1^{I_1}\},\{\alpha_1^L,\alpha_1^U]|P_1^{T_1}\}\})^\lambda \\
&= \bigcup_{\tilde{\alpha}_1\in T_1,\tilde{\beta}_1\in I_1,\tilde{\gamma}_1\in F_1}\{\{[(\gamma_1^L)^\lambda,(\gamma_1^U)^\lambda]|P_1^{F_1}\},\{[1-(\beta_1^U)^\lambda,1-(\beta_1^U)^\lambda]|P_1^{I_1}\}, \\
&\qquad [1-(1-\alpha_1^L)^\lambda,1-(1-\alpha_1^U)^\lambda]|P_1^{\tilde{I}_1}\} \\
&= (\lambda(\bigcup_{\tilde{\alpha}_1\in T_1,\tilde{\beta}_1\in I_1,\tilde{\gamma}_1\in F_1}\{\{[\alpha_1^L,\alpha_1^U]|P_1^{T_1}\},[\beta_1^L,\beta_1^U]|P_1^{I_1}\},[\gamma_1^L,\gamma_1^U]|P_1^{F_1}\}\}))^c \\
&= (\lambda(n_1))^c.
\end{aligned}
$$

(2)

$$
\begin{aligned}
\lambda(n_1)^c &= \lambda(\bigcup_{\tilde{\alpha}_1\in T_1,\tilde{\beta}_1\in I_1,\tilde{\gamma}_1\in F_1}\{\{[\gamma_1^L,\gamma_1^U]|P_1^{F_1}\},[1-\beta_1^U,1-\beta_1^U]|P_1^{I_1}\},\{[\alpha_1^L,\alpha_1^L]|P_1^{T_1}\}\}) \\
&= \bigcup_{\alpha_{a'}\in\tilde{t}_1,\beta_{b'}\in\tilde{i}_1,\gamma_{c'}\in\tilde{f}_1}\{\{1-(1-\gamma_1^L)^\lambda,1-(1-\gamma_1^U)^\lambda]|P_1^{F_1}\},\{[(1-\beta_1^U)^\lambda,(1-\beta_1^L)^\lambda]|P_1^{I_1}\}, \\
&\qquad \{[(\alpha_1^L)^\lambda,(\alpha_1^U)^\lambda]|P_1^{T_1}\}\} \\
&= (\bigcup_{\alpha_{a'}\in\tilde{t}_1,\beta_{b'}\in\tilde{i}_1,\gamma_{c'}\in\tilde{f}_1}\{\{[(\alpha_1^L)^\lambda,(\alpha_1^U)^\lambda]|P_1^{T_1}\},\{[1-(1-\beta_1^L)^\lambda,1-(1-\beta_1^U)^\lambda]|P_1^{I_1}\}, \\
&\qquad \{[1-(1-\gamma_1^L)^\lambda,1-(1-\gamma_1^U)^\lambda]|P_1^{F_1}\}\})^c \\
&= ((n_1)^\lambda)^c.
\end{aligned}
$$

(3)

$$
\begin{aligned}
(n_1)^c\oplus(n_2)^c &= (\bigcup_{\tilde{\alpha}_1\in T_1,\tilde{\beta}_1\in I_1,\tilde{\gamma}_1\in F_1}\{\{[\gamma_1^L,\gamma_1^U]|P_1^{F_1}\},\{[1-\beta_1^U,1-\beta_1^L]|P_1^{I_1}\},\{[\alpha_1^L,\alpha_1^U]|P_1^{T_1}\}\}) \\
&\quad \oplus(\bigcup_{\tilde{\eta}_2\in T_2,\tilde{\theta}_2\in I_2,\tilde{\mu}_2\in F_2}\{[\mu_2^L,\mu_2^U]|P_2^{F_2}\},\{[1-\theta_2^U,1-\theta_2^L]|P_2^{I_2}\},\{[\eta_2^L,\eta_2^U]|P_2^{T_2}\}\}) \\
&= \bigcup_{\substack{\tilde{\alpha}_1\in T_1,\tilde{\beta}_1\in I_1,\tilde{\gamma}_1\in F_1,\\ \tilde{\eta}_2\in T_2,\tilde{\theta}_2\in I_2,\tilde{\mu}_2\in F_2}}\{\{[\gamma_1^L+\mu_2^L-\gamma_1^L\mu_2^L,\gamma_1^U+\mu_2^U-\gamma_1^U\mu_2^U]|P_1^{F_1}P_2^{F_2}\}, \\
&\qquad \{[(1-\beta_2^L)(1-\theta_2^L),(1-\beta_2^U)(1-\theta_2^U)]|P_1^{I_1}P_2^{I_2}\},\{[\alpha_1^L\eta_2^L,\alpha_1^U\eta_2^U]|P_1^{T_1}P_2^{T_2}\}\} \\
&= (\bigcup_{\substack{\tilde{\alpha}_1\in T_1,\tilde{\beta}_1\in I_1,\tilde{\gamma}_1\in F_1,\\ \tilde{\eta}_2\in T_2,\tilde{\theta}_2\in I_2,\tilde{\mu}_2\in F_2}}\{\{[\alpha_1^L\eta_2^L,\alpha_1^U\eta_2^U]|P_1^{T_1}P_2^{T_2}\},\{[\beta_1^L+\theta_2^L-\beta_1^L\theta_2^L,\beta_1^U+\theta_2^U-\beta_1^U\theta_2^U]|P_1^{I_1}P_2^{I_2}\}, \\
&\qquad \{[\gamma_1^L+\mu_2^L-\gamma_1^L\mu_2^L,\gamma_1^U+\mu_2^U-\gamma_1^U\mu_2^U]|P_1^{F_1}P_2^{F_2}\}\})^c \\
&= (n_1\otimes n_2)^c.
\end{aligned}
$$

(4)

$$
\begin{aligned}
(n_1)^c \otimes (n_2)^c &= (\bigcup_{\tilde{\alpha}_1 \in T_1, \tilde{\beta}_1 \in I_1, \tilde{\gamma}_1 \in F_1} \{\{[\gamma_1^L, \gamma_1^U] | P_1^{F_1}\}, \{[1-\beta_1^U, 1-\beta_1^L] | P_1^{I_1}\}, \{[\alpha_1^L, \alpha_1^U] | P_1^{T_1}\}\}) \\
&\quad \otimes (\bigcup_{\tilde{\eta}_2 \in T_2, \tilde{\theta}_2 \in I_2, \tilde{\mu}_2 \in F_2} \{\{[\mu_2^L, \mu_2^U] | P_2^{F_2}\}, \{[1-\theta_2^U, 1-\theta_2^L] | P_2^{I_2}\}, \{[\eta_2^L, \eta_2^U] | P_2^{T_2}\}\}) \\
&= \bigcup_{\substack{\tilde{\alpha}_1 \in T_1, \tilde{\beta}_1 \in I_1, \tilde{\gamma}_1 \in F_1, \\ \tilde{\eta}_2 \in T_2, \tilde{\theta}_2 \in I_2, \tilde{\mu}_2 \in F_2}} \{\{[\gamma_1^L \mu_2^L, \gamma_1^U \mu_2^U] | P_1^{F_1} P_2^{F_2}\}, \{[1-\beta_1^U \theta_2^U, 1-\beta_1^L \theta_2^L] | P_1^{I_1} P_2^{I_2}\}, \\
&\qquad \{[\alpha_1^L + \eta_2^L - \alpha_1^L \eta_2^L, \alpha_1^U + \eta_2^U - \alpha_1^U \eta_2^U] | P_1^{T_1} P_2^{T_2}\}\} \\
&= (\bigcup_{\substack{\tilde{\alpha}_1 \in T_1, \tilde{\beta}_1 \in I_1, \tilde{\gamma}_1 \in F_1, \\ \tilde{\eta}_2 \in T_2, \tilde{\theta}_2 \in I_2, \tilde{\mu}_2 \in F_2}} \{\{[\alpha_1^L + \eta_2^L - \alpha_1^L \eta_2^L, \alpha_1^U + \eta_2^U - \alpha_1^U \eta_2^U] | P_1^{T_1} P_2^{T_2}\}, \{[\beta_1^L \theta_2^L, \beta_1^U \theta_2^U] | P_1^{I_1} P_2^{I_2}\}, \\
&\qquad \{[\gamma_1^L \mu_2^L, \gamma_1^U \mu_2^U] | P_1^{F_1} P_2^{F_2}\}\})^c \\
&= (n_1 \oplus n_2)^c.
\end{aligned}
$$

□

The PSVNHFS also satisfies the above properties, and the process of the proof is omitted.

5. The Basic Aggregation Operators for PINHFSs

Definition 10. *Let n_j $(x = 1, 2, \cdots, X)$ be a non-empty collection of PINHFNs, then a probabilistic interval neutrosophic hesitant fuzzy weighted averaging (PINHFWA) operator can be indicated as:*

$$
\begin{aligned}
&PINHFWA(n_1, n_2, \cdots, n_X) = \bigoplus_{j=1}^{X} w_j(n_j) \\
&= \bigcup \{\{[1 - \prod_{j=1}^{X}(1-\alpha_j^L)^{w_j}, 1 - \prod_{j=1}^{X}(1-\alpha_j^U)^{w_j}] | \prod_{j=1}^{X} P_j^{T_j}\}, \\
&\qquad \{[\prod_{j=1}^{X}(\beta_j^L)^{w_j}, \prod_{j=1}^{X}(\beta_j^U)^{w_j}] | \prod_{j=1}^{X} P_j^{I_j}\}, \{[\prod_{j=1}^{X}(\gamma_j^L)^{w_j}, \prod_{j=1}^{X}(\gamma_j^L)^{w_j}] | \prod_{j=1}^{X} P_j^{F_j}\}\},
\end{aligned}
\tag{4}
$$

where $[\alpha_j^L, \alpha_j^U] = \tilde{\alpha}_j \in T_j$, $[\beta_j^L, \beta_j^U] = \tilde{\beta}_j \in I_j$, $[\gamma_j^L, \gamma_j^U] = \tilde{\gamma} \in F_j$, $P_j^{T_j}$, $P_j^{I_j}$ and $P_j^{F_j}$ are corresponding hesitant probabilities of $\tilde{\alpha}_j \in T_j$, $\tilde{\beta}_j \in I_j$ and $\tilde{\gamma}_j \in F_j$. $j = 1, 2, \cdots, X$, w_j is the weight of n_j and $\sum_{j=1}^{X} w_j = 1$. If all wights are $\frac{1}{X}$, then the PINHFWA operator reduces to the probabilistic interval neutrosophic hesitant fuzzy averaging (PINHFA) operator:

$$
\begin{aligned}
&PINHFA(n_1, n_2, \cdots, n_X) = \bigoplus_{j=1}^{X} \frac{1}{X}(n_j) \\
&= \bigcup \{\{[1 - \prod_{j=1}^{X}(1-\alpha_j^L)^{\frac{1}{X}}, 1 - \prod_{j=1}^{X}(1-\alpha_j^L)^{\frac{1}{X}}] | \prod_{j=1}^{X} P_j^{T_j}\}, \\
&\qquad \{[\prod_{j=1}^{X}(\beta_j^L)^{\frac{1}{X}}, \prod_{j=1}^{X}(\beta_j^U)^{\frac{1}{X}}] | \prod_{j=1}^{X} P_j^{I_j}\}, \{[\prod_{j=1}^{X}(\gamma_j^L)^{\frac{1}{X}}, \prod_{j=1}^{X}(\gamma_j^U)^{\frac{1}{X}}] | \prod_{j=1}^{X} P_j^{F_j}\}\}.
\end{aligned}
\tag{5}
$$

Theorem 4. *(Monotonicity) Let* $n_j = \{\{\tilde{\alpha}_j|P_J^{T_j}\},\{\tilde{\beta}_j|P_j^{I_j}\},\{\tilde{\gamma}_j|P_j^{F_j}\}\}$ *and* $m_j = \{\{\tilde{\eta}_j|P_j^{T_j^*}\},\{\tilde{\theta}_2|P_j^{I_j^*}\},\{\tilde{\mu}_j|P_j^{F_j^*}\}\}$ *be two collections of PINHFNs;* $w_j (j = 1,2,\cdots,X)$ *is weight, and* $\sum_{j=1}^{n} w_j = 1$*. If* $P_j^{T_j} = P_j^{T_j^*}$, $P_j^{I_j} = P_j^{I_j^*}$, $P_j^{F_j} = P_j^{F_j^*}$ *and* $\alpha_j^L \le \eta_j^L$, $\alpha_j^U \le \eta_j^U$, $\beta_j^L \ge \theta_j^L$, $\beta_j^U \ge \theta_j^U$, $\gamma_j^L \ge \mu_j^L$, $\gamma_j^U \ge \mu_j^U$*, then:*

$$PINHFWA(n_1,n_2,\cdots,n_X) \le PINHFWA(m_1,m_2,\cdots,m_X). \tag{6}$$

Proof. Since $\alpha_j^L \le \eta_j^L$, $\alpha_j^U \le \eta_j^U$, $\beta_j^L \ge \theta_j^L$, $\beta_j^U \ge \theta_j^U$, $\gamma_j^L \ge \mu_j^L$, $\gamma_j^U \ge \mu_j^U$ for all j, we have:

$$1-\prod(1-\alpha_j^L)^{w_j} \le 1-\prod(1-\eta_j^L)^{w_j}, 1-\prod(1-\alpha_j^U)^{w_j} \le 1-\prod(1-\eta_j^U)^{w_j};$$
$$\prod(\beta_j^L)^{w_j} \ge \prod(\theta_j^L)^{w_j}, \prod(\beta_j^U)^{w_j} \ge \prod(\theta_j^U)^{w_j};$$
$$\prod(\gamma_j^L)^{w_j} \ge \prod(\mu_j^L)^{w_j}, \prod(\gamma_j^U)^{w_j} \ge \prod(\mu_j^U)^{w_j}.$$

Simultaneously, we have $P_j^{T_j} = P_j^{T_j^*}$, $P_j^{I_j} = P_j^{I_j^*}$, $P_j^{F_j} = P_j^{F_j^*}$, so we can obtain:

$$(1-\prod(1-\alpha_j^L)^{w_j})\prod P_j^{T_j} - \prod(\beta_j^L)^{w_j}\prod P_j^{I_j} - \prod(\gamma_j^L)^{w_j}\prod P_j^{F_j}\} \le$$
$$(1-\prod(1-\eta_j^L)^{w_j})\prod P_j^{T_j^*} - \prod(\theta_j^L)^{w_j}\prod P_j^{I_j^*}\prod(\mu_j^L)^{w_j}\prod P_j^{F_j^*},$$
$$(1-\prod(1-\alpha_j^U)^{w_j})\prod P_j^{T_j} - \prod(\beta_j^U)^{w_j}\prod P_j^{I_j} - \prod(\gamma_j^U)^{w_j}\prod P_j^{F_j}\} \le$$
$$(1-\prod(1-\eta_j^U)^{w_j})\prod P_j^{T_j^*} - \prod(\theta_j^U)^{w_j}\prod P_j^{I_j^*}\prod(\mu_j^U)^{w_j}\prod P_j^{F_j^*},$$

then by the score function 6 and Definition 8, we have $PINHFWA(n_1,n_2,\cdots,n_X) \le PINHFWA(m_1,m_2,\cdots,m_X)$. □

Theorem 5. *(Boundedness) Let* $n_j = \{\{\tilde{\alpha}_j|P_j^{T_j}\},\{\tilde{\beta}_j|P^{I_j}\},\{\tilde{\gamma}_j|P^{F_j}\}\}$ *be a PINHFN* $(j = 1,2,\cdots,X)$, $\tilde{\alpha}_j \in T_j$, $\tilde{\beta}_b \in I_j$, $\tilde{\gamma}_j \in F_j$, $P_j^{T_j}$; $P_j^{I_j}$ *and* $P_j^{F_j}$ *are hesitant probabilities of* $\tilde{\alpha}_j$, $\tilde{\beta}_j$ *and* $\tilde{\gamma}_j$*, respectively.* w_j $(j = 1,2,\cdots,X)$ *is a weight, and* $\sum_{j=1}^{X} w_j = 1$*. If:*

$$N^- = \{\{[min\{\alpha_j^L\}, min\{\alpha_j^U\}]|min\{P_j^{T_j}\}\},\{[max\{\beta_j^L\}, max\{\beta_j^U\}]|max\{P_j^{I_j}\}\},\{[max\{\gamma_j^L\}, max\{\gamma_j^L\}]|max\{P_j^{F_j}\}\}\},$$
$$N^+ = \{\{[max\{\alpha_j^L\}, max\{\alpha_j^U\}]|max\{P_j^{T_j}\}\},\{[min\{\beta_j^L\}, min\{\beta_j^U\}]|min\{P_j^{I_j}\}\},\{[min\{\gamma_j^L\}, min\{\gamma_j^L\}]|min\{P_j^{F_j}\}\}\}.$$

Then:

$$PINHFWA(N^-) \le PINHFWA(n_1,n_2,\cdots,n_X) \le PINHFWA(N^+) \tag{7}$$

Proof. For all PINHFNs n_l, we have:

$$min\{\alpha_j^L\} \le \alpha_j^L \le max\{\alpha_j^L\}, min\{\alpha_j^U\} \le \alpha_j^U \le max\{\alpha_j^U\};$$
$$min\{\beta_j^L\} \le \beta_j^L \le max\{\beta_j^L\}, min\{\beta_j^U\} \le \beta_j^U \le max\{\beta_j^U\};$$
$$min\{\gamma_j^L\} \le \gamma_j^L \le max\{\gamma_j^L\}, min\{\gamma_j^U\} \le \gamma_j^U \le max\{\gamma_j^U\};$$
$$min\{P_j^{T_j}\} \le P_j^{T_j} \le max\{P_j^{T_j}\}, min\{P_j^{I_j}\} \le P_j^{I_l} \le max\{P_j^{I_j}\},$$
$$min\{P_j^{F_j}\} \le P_j^{F_j} \le max\{P_j^{F_j}\}.$$

Thus,

$$\begin{aligned}
1-\prod(1-\alpha_j^L)^{w_j} &\geq 1-\prod(1-min\{\alpha_j^L\})^{w_j}=1-(1-min\{\alpha_l^L\})^{\Sigma w_j}=min\{\alpha_j^L\},\\
1-\prod(1-\alpha_j^U)^{w_j} &\geq 1-\prod(1-min\{\alpha_j^U\})^{w_j}=1-(1-min\{\alpha_l^U\})^{\Sigma w_j}=min\{\alpha_j^U\},\\
\prod(\beta_j^L)^{w_j} &\leq \prod(max\{\beta_j^L\})^{w_j}=(max\{\beta_j^L\})^{\Sigma w_j}=max\{\beta_j^L\},\\
\prod(\beta_j^U)^{w_j} &\leq \prod(max\{\beta_j^U\})^{w_j}=(max\{\beta_j^U\})^{\Sigma w_j}=max\{\beta_j^U\},\\
\prod(\gamma_j^L)^{w_j} &\leq \prod(max\{\gamma_j^L\})^{w_j}=(max\{\gamma_j^L\})^{\Sigma w_j}=max\{\gamma_j^L\},\\
\prod(\gamma_j^U)^{w_j} &\leq \prod(max\{\gamma_j^U\})^{w_j}=(max\{\gamma_j^U\})^{\Sigma w_j}=max\{\gamma_j^U\}.
\end{aligned}$$

Next, by Definition 10, we have:

$$\begin{aligned}
NHPFWA(N^-)=&\bigcup\{\{[min\{\alpha_j^L\},min\{\alpha_j^U\}]|\prod min\{P_j^{T_j}\},\{[max\{\beta_j^L\},max\{\beta_j^U\}]|\prod max\{P_j^{I_j}\}\},\\
&\{[max\{\gamma_j^L\},[max\{\gamma_j^U\}]|\prod max\{P_j^{F_j}\}\}\}.
\end{aligned}$$

By score function 6 and Definition 8, we can obtain $PINHFWA(N^-) \leq PINHFWA(n_1,n_2,\cdots,n_X)$. Similarly, we have $PINHFWA(n_1,n_2,\cdots,n_X) \leq PINHFWA(N^+)$. □

Theorem 6. *(Idempotency) If* $n_j=\{\{[\alpha^L,\alpha^U]|P_1\},\{[\beta^L,\beta^U]|P_2\},\{[\gamma^L,\gamma^U]|P_3\}\}$, $j=1,2,\cdots,X$, w_j *is the weight of* n_j, $\sum_{j=1}^{X} w_j=1$, *then:*

$$PINHFWA(n_1,n_2,\cdots,n_X)=\{\{[\alpha^L,\alpha^U]|P_1\},\{[\beta^L,\beta^U]|P_2\},\{[\gamma^L,\gamma^U]|P_3\}\}. \tag{8}$$

Proof. Since $n_j=\{\{[\alpha^L,\alpha^U]|P_1\},\{[\beta^L,\beta^U]|P_2\},\{[\gamma^L,\gamma^U]|P_3\}\}$, thus we have:

$$\begin{aligned}
&1-\prod(1-\alpha^L)^{w_j}=1-(1-\alpha^L)^{\Sigma w_j}=\alpha^L, 1-\prod(1-\alpha^U)^{w_j}=1-(1-\alpha^U)^{\Sigma w_j}=\alpha^U;\\
&\prod(\beta^L)^{w_j}=(\beta^L)^{\Sigma w_j}=\beta^L, \prod(\beta^U)^{w_j}=(\beta^U)^{\Sigma w_j}=\beta^U,\\
&\prod(\gamma^L)^{w_j}=(\gamma^L)^{\Sigma w_j}=\gamma^L, \prod(\gamma^U)^{w_j}=(\gamma^U)^{\Sigma w_j}=\gamma^U,\\
&\prod(P_1)^{w_j}=(P_1)^{\Sigma w_j}=P_1, \prod(P_2)^{w_j}=(P_2)^{\Sigma w_j}=P_2, \prod(P_3)^{w_j}=(P_3)^{\Sigma w_j}=P_3.
\end{aligned}$$

It is easy to get:

$$PINHFWA(\tilde{np}_1,\tilde{np}_2,\cdots,\tilde{np}_X)=\{\{[\alpha^L,\alpha^U]|P_1\},\{[\beta^L,\beta^U]|P_2\},\{[\gamma^L,\gamma^U]|P_3\}\}.$$

□

Theorem 7. *(Commutativity) If* $A=\{n_1,n_2,\cdots,n_X\}$ *is a collection and* $B=\{m_1,m_2,\cdots,m_X\}$ *is a new permutation of A, then:*

$$PINHFWA(n_1,n_2,\cdots,n_X)=PINHFWA(m_1,m_2,\cdots,m_X).$$

Proof. By Definition 10, it is easy to prove it. □

Definition 11. *Let n_j ($j = 1, 2, \cdots, X$) be a non-empty collection of PINHFNs; a probability interval neutrosophic hesitant fuzzy weighted geometric (PINHFWG) operator can be indicated as:*

$$PINHFWG(n_1, n_2, \cdots, n_X) = \bigotimes_{j=1}^{X} w_j(n_j)$$
$$= \bigcup \{\{[\prod_{j=1}^{X}(\alpha_j^L)^{w_j}, \prod_{j=1}^{X}(\alpha_j^U)^{w_j}] | \prod_{j=1}^{X} P_j^{T_j}\}, \{[1 - \prod_{j=1}^{X}(1-\beta_j^L)^{w_j}, 1 - \prod_{j=1}^{X}(1-\beta_j^U)^{w_j}] | \prod_{j=1}^{X} P_j^{I_j}\},$$
$$\{[1 - \prod_{j=1}^{X}(1-\gamma_j^L)^{w_j}, 1 - \prod_{j=1}^{X}(1-\gamma_j^U)^{w_j}] | \prod_{j=1}^{X} P_j^{F_j}\}\}, \quad (9)$$

where $[\alpha_j^L, \alpha_j^U] = \tilde{\alpha}_j \in T_j$, $[\beta_j^L, \beta_j^U] = \tilde{\beta}_j \in I_j$, $[\gamma_j^L, \gamma_j^U] = \tilde{\gamma} \in F_j$, $P_j^{T_j}$, $P_j^{I_j}$ and $P_j^{F_j}$ are corresponding hesitant probabilities of $\tilde{\alpha}_j$, $\tilde{\beta}_j$ and $\tilde{\gamma}_j$. $j = 1, 2, \cdots, X$, w_j is the weight of n_j and $\sum_{j=1}^{X} w_j = 1$. If all wights are $\frac{1}{X}$, then the PINHFWG operator converts to the probabilistic interval neutrosophic hesitant fuzzy geometric (PINHFG) operator:

$$PINHFG(n_1, n_2, \cdots, n_X) = \bigotimes_{j=1}^{X} \frac{1}{X}(n_j)$$
$$= \bigcup \{\{[\prod_{j=1}^{X}(\alpha_j^L)^{\frac{1}{X}}, \prod_{j=1}^{X}(\alpha_j^U)^{\frac{1}{X}}] | \prod_{j=1}^{X} P_j^{T_j}\}, \{[1 - \prod_{j=1}^{X}(1-\beta_j^L)^{\frac{1}{X}}, 1 - \prod_{j=1}^{X}(1-\beta_j^U)^{\frac{1}{X}}] | \prod_{j=1}^{X} P_j^{I_j}\},$$
$$\{[1 - \prod_{j=1}^{X}(1-\gamma_j^L)^{\frac{1}{X}}, 1 - \prod_{j=1}^{X}(1-\gamma_j^U)^{\frac{1}{X}}] | \prod_{j=1}^{X} P_j^{F_j}\}\}. \quad (10)$$

Theorem 8. *(Monotonicity) Let $n_j = \{\{\tilde{\alpha}_j | P_J^{T_j}\}, \{\tilde{\beta}_j | P_j^{I_j}\}, \{\tilde{\gamma}_j | P_j^{F_j}\}\}$ and $m_j = \{\{\tilde{\eta}_j | P_j^{T_j^*}\}, \{\tilde{\theta}_2 | P_j^{I_j^*}\}, \{\tilde{\mu}_j | P_j^{F_j^*}\}\}$ be two collections of PINHFNs; $w_j (j = 1, 2, \cdots, X)$ is weight, and $\sum_{j=1}^{n} w_j = 1$. If $P_j^{T_j} = P_j^{T_j^*}$, $P_j^{I_j} = P_j^{I_j^*}$, $P_j^{F_j} = P_j^{F_j^*}$ and $\alpha_j^L \le \eta_j^L$, $\alpha_j^U \le \eta_j^U$, $\beta_j^L \ge \theta_j^L$, $\beta_j^U \ge \theta_j^U$, $\gamma_j^L \ge \mu_j^L$, $\gamma_j^U \ge \mu_j^U$, then:*

$$PINHFWG(n_1, n_2, \cdots, n_X) \le PINHFWG(m_1, m_2, \cdots, m_X). \quad (11)$$

Proof. This is similar to Theorem 4. □

Theorem 9. *(Boundedness) Let $n_j = \{\{\tilde{\alpha}_j | P_j^{T_j}\}, \{\tilde{\beta}_j | P^{I_j}\}, \{\tilde{\gamma}_j | P^{F_j}\}\}$ be a PINHFN ($j = 1, 2, \cdots, X$), $\tilde{\alpha}_j \in T_j$, $\tilde{\beta}_b \in I_j$, $\tilde{\gamma}_j \in F_j$, $P_j^{T_j}$; $P_j^{I_j}$ and $P_j^{F_j}$ are hesitant probabilities of $\tilde{\alpha}_j$, $\tilde{\beta}_j$ and $\tilde{\gamma}_j$, respectively. w_j ($j = 1, 2, \cdots, X$) is a weight, and $\sum_{j=1}^{X} w_j = 1$. If:*

$$P^- = \{\{[min\{\alpha_j^L\}, min\{\alpha_j^U\}] | min\{P_j^{T_j}\}\}, \{[max\{\beta_j^L\}, max\{\beta_j^U\}] | max\{P_j^{I_j}\}\}, \{[max\{\gamma_j^L\}, max\{\gamma_j^L\}] | max\{P_j^{F_j}\}\}\},$$
$$P^+ = \{\{[max\{\alpha_j^L\}, max\{\alpha_j^U\}] | max\{P_j^{T_j}\}\}, \{[min\{\beta_j^L\}, min\{\beta_j^U\}] | min\{P_j^{I_j}\}\}, \{[min\{\gamma_j^L\}, min\{\gamma_j^L\}] | min\{P_j^{F_j}\}\}\},$$

then:

$$PINHFWG(P^-) \le PINHFWG(n_1, n_2, \cdots, n_X) \le PINHFWG(P^+) \quad (12)$$

Proof. This is similar to Theorem 5. □

Theorem 10. *(Idempotency) If* $n_j = \{\{[\alpha^L, \alpha^U]|P_1\}, \{[\beta^L, \beta^U]|P_2\}, \{[\gamma^L, \gamma^U]|P_3\}\}$, $j = 1, 2, \cdots, X$, w_j *is the weight of* n_j, $\sum_{j=1}^{X} w_j = 1$, *then:*

$$PINHFWG(n_1, n_2, \cdots, n_X) = \{\{[\alpha^L, \alpha^U]|P_1\}, \{[\beta^L, \beta^U]|P_2\}, \{[\gamma^L, \gamma^U]|P_3\}\}. \tag{13}$$

Proof. This is similar to Theorem 6. □

Theorem 11. *(Commutativity) If* $A = \{n_1, n_2, \cdots, n_X\}$ *is a collection and* $B = \{m_1, m_2, \cdots, m_X\}$ *is a new permutation of A, then:*

$$PINHFWG(n_1, n_2, \cdots, n_X) = PINHFWG(m_1, m_2, \cdots, m_X).$$

Proof. We can obtain it by Definition 13. □

Lemma 1. *[3] Let* $x_i \geq 0$, $w_i \geq 0$, $i = 1, 2, \cdots, n$ *and* $\sum_{i=1}^{n} w_i = 1$, *then:*

$$\prod_{i=1}^{n} (x_i)^{w_i} \leq \sum_{i=1}^{n} x_i w_i,$$

Theorem 12. *If* $n_j = \{\{\tilde{\alpha}_j|P_j^{T_j}\}, \{\tilde{\beta}_j|P^{I_j}\}, \{\tilde{\gamma}_j|P^{F_j}\}\}$ *is a collection of PINHFNs and* $j = 1, 2, \cdots, X$, w_j *is the weight of* n_j, $w_j \geq 0$ *and* $\sum_{j=1}^{X} w_j = 1$, *then:*

$$PINHFWG(n_1, n_2, \cdots, n_X) \leq PINHFWA(n_1, n_2, \cdots, n_X),$$
$$PINHFG(n_1, n_2, \cdots, n_X) \leq PINHFA(n_1, n_2, \cdots, n_X).$$

Proof. Since $\tilde{\alpha}_j = [\alpha_j^L, \alpha_j^U]$, $\tilde{\beta}_j = [\beta_j^L, \beta_j^U]$, $\tilde{\gamma}_j = [\gamma_j^L, \gamma_j^U]$, $\alpha_j^L, \alpha_j^U \in [0, 1]$. Thus, By Lemma 1, we have:

$$\prod (\alpha_j^L)^{w_j} \leq \sum w_j \alpha_j^L = 1 - \sum w_j (1 - \alpha_j^L) \leq 1 - \prod (1 - \alpha_j^L)^{w_j},$$
$$\prod (\alpha_j^U)^{w_j} \leq \sum w_j \alpha_j^U = 1 - \sum w_j (1 - \alpha_j^U) \leq 1 - \prod (1 - \alpha_j^U)^{w_j}.$$

Thus, we can obtain:

$$\prod (\alpha_j^L)^{w_j} \prod P_j^{T_j} \leq (1 - \prod (1 - \alpha_j^L)^{w_j}) \prod P_j^{T_j},$$
$$\prod (\alpha_j^U)^{w_j} \prod P_j^{T_j} \leq (1 - \prod (1 - \alpha_j^U)^{w_j}) \prod P_j^{T_j}.$$

Similarly, we can also get:

$$\prod (\beta_j^L)^{w_j} \prod P_j^{I_j} \leq (1 - \prod (1 - \beta_j^L)^{w_j}) \prod P_j^{I_j}, \prod (\beta_j^U)^{w_j} \prod P_j^{I_j} \leq (1 - \prod (1 - \beta_j^U)^{w_j}) \prod P_j^{I_j},$$
$$\prod (\gamma_j^L)^{w_j} \prod P_j^{F_j} \leq (1 - \prod (1 - \gamma_j^L)^{w_j}) \prod P_j^{F_j}, \prod (\gamma_j^U)^{w_j} \prod P_j^{F_j} \leq (1 - \prod (1 - \gamma_j^U)^{w_j}) \prod P_j^{F_j}.$$

Next, by the score function 6, we know:

$$PINHFWG(n_1, n_2, \cdots, n_X) \leq PINHFWA(n_1, n_2, \cdots, n_X).$$

Similar to the above process of the proof, we know inequality $PINHFG(n_1, n_2, \cdots, n_X) \leq PINHFA(n_1, n_2, \cdots, n_X)$ is right. □

6. MADM Based on the PINHFWA and PINHFWG Operators

In this section, the PINHFWA and PINHFWG operators are used to solve MADM problems with probabilistic interval neutrosophic hesitant fuzzy circumstances.

Let $A = \{A_1, A_2, \cdots, A_M\}$ be a collection of options and $C = \{C_1, C_2, \cdots, C_N\}$ be a set of attributes. In order to assess A_h $(h = 1, 2, \cdots, M)$ with the attribute C_k $(k = 1, 2, \cdots, N)$ represented by the PINHFN $n_{hk} = \{T_{hk}|P^{T_{hk}}, I_{hk}|P^{I_{hk}}, F_{hk}|P^{F_{hk}}\}$, next, we can construct a probabilistic interval neutrosophic hesitant fuzzy decision matrix (PINHFDM) $D = (n_{hk})_{M\times N}$ $(h = 1, 2, \cdots, M; k = 1, 2, \cdots, N)$. The weight vector of C is $w = (w_1, w_2, \cdots, w_N)$. Then, the evaluation steps can select an optimal option:

- Step 1. Use the PINHFWA or PINHFWG operator to aggregate N PINHFNs for an alternative A_h, $h = 1, 2, \cdots, M$.
- Step 2. Calculate the score values of all PINHFNs; if we get the same for $s(n)$, then we need to compare the deviation values.
- Step 3. Rank and select the optimal option A_h.

7. Illustrative Example

The background of the numerical case comes from Example 1. Therefore, this section is not covered in detail. The weight vector of C is $w = (0.35, 0.25, 0.4)$. Thus, four PINHFDMs are established, illustrated in Tables 5–8.

Table 5. A probabilistic interval neutrosophic hesitant fuzzy decision matrix (PINHFDM) D_1 with respect to A_1.

Attributes	Investment Selection A_1
C_1	$\{\{[0.3,0.4]|0.1,[0.4,0.4]|0.1,[0.4,0.5]|0.8\},\{[0.1,0.2]|1\},\{0.3,0.4]|1\}\}$
C_2	$\{\{[0.4,0.5]|0.5,[0.5,0.6]]|0.5\},\{[0.2,0.3]|1\},\{[0.3,0.3]|0.7,[0.3,0.4]|0.3\}\}$
C_3	$\{\{[0.2,0.3]|1\},\{[0.1,0.2]|1\},\{[0.4,0.5]|0.7,[0.5,0.6]|0.3\}\}$

Table 6. PINHFDM D_2 with respect to A_2.

Attributes	Investment Selection A_2
C_1	$\{\{[0.6,0.7]|1\},\{[0.1,0.2]|1\},\{[0.1,0.2]|0.2,[0.2,0.3]|0.8\}\}$
C_2	$\{\{[0.6,0.7]|1\},\{[0.1,0.1]|1\},\{[0.2,0.3]|1\}\}$
C_3	$\{\{[0.6,0.7]|1\},\{[0.1,0.2]|1\},\{[0.1,0.2]|1\}\}$

Table 7. PINHFDM D_3 with respect to A_3.

Attributes	Investment Selection A_3
C_1	$\{\{[0.3,0.4]|0.3,[0.5,0.6]|0.7\},\{[0.2,0.4]|1\},\{[0.2,0.3]|1\}\}$
C_2	$\{\{[0.5,0.6]|1\},\{[0.2,0.3]|1\},\{[0.3,0.4]|1\}\}$
C_3	$\{\{[0.5,0.6]|1\},\{[0.1,0.2]|0.4,[0.2,0.3]|0.6\},\{[0.2,0.3]|1\}\}$

Table 8. PINHFDM D_4 with respect to A_4.

Attributes	Investment Selection A_4
C_1	$\{\{[0.7,0.8]|1\},\{[0,0.1]|1\},\{[0.1,0.2]|1\}\}$
C_2	$\{\{[0.6,0.7]|1\},\{[0,0.1]|1\},\{[0.2,0.2]|1\}\}$
C_3	$\{\{[0.3,0.5]|1\},\{[0.2,0.3]|1\},\{[0.1,0.2]|0.2,[0.3,0.3]|0.8\}\}$

- Step 1. Select the PINHFWA operator to aggregate all PINHFNs of n_{hk} $(h = 1, 2, 3, 4; k = 1, 2, 3)$ to obtain the collective PINHFN n_h $(h = 1, 2, 3, 4)$ for the alternative A_h $(h = 1, 2, 3, 4)$.

$$
\begin{aligned}
n_1 &= PINHFWA(n_{11}, n_{12}, n_{13}) \\
&= \{\{[0.2895, 0.3903]|0.05, [0.3212, 0.4234]|0.05, [0.3268, 0.3903]|0.05, [0.3568, 0.4234]|0.05, \\
&\quad [0.3268, 0.4280]|0.4, [0.3568, 0.4590]|0.4\}, \\
&\quad \{[0.1189, 0.2213]|1\}, \\
&\quad \{[0.3366, 0.407]|0.49, [0.368, 0.4378]|0.21, [0.3366, 0.4373]|0.21, [0.368, 0.4704]|0.09\}; \\
n_2 &= PINHFWA(n_{21}, n_{22}, n_{23}) \\
&= \{\{[0.6, 0.7]|1\}, \{[0.1, 0.1682]|1\}, \{[0.1189, 0.2213]|0.2, [0.1516, 0.2551]|0.8\}\}; \\
n_3 &= PINHFWA(n_{31}, n_{32}, n_{33}) \\
&= \{\{[0.4375, 0.5390]|0.3, [0.5, 0.6]|0.7\}, \{[0.1516, 0.2821]|0.4, [0.2, 0.3318]|0.6\}, \{[0.2213, 0.3224]|1\}\}; \\
n_4 &= PINHFWA(n_{41}, n_{42}, n_{43}) \\
&= \{\{0.5476, 0.6807]|1\}, \{[0, 0.1552]|1\}, \{[0.1189, 0.2]|0.2, [0.1845, 0.2352]|0.8\}\}.
\end{aligned}
$$

- Step 2. By (2), count the score values of all PINHFNs n_h $(h = 1, 2, 3, 4)$,

$$n_1 = 0.6104, n_2 = 0.7731, n_3 = 0.6711, \tilde{np}_4 = 0.7789.$$

- Step 3. Rank the PINHFNs by Definition 8; we have:

$$A_4 > A_2 > A_3 > A_1.$$

Thus, we know that A_4 is the best choice.

Next, we will make use of the PINHFWG operator to solve the MADM problem.

- Step 1′. Aggregate PINHFNs n_{hk} $(h = 1, 2, 3, 4; k = 1, 2, 3)$ by taking advantage of the PINHFWG operator to get the collective PINHFN n_h for A_h.

$$
\begin{aligned}
n_1 &= PINHFWG(n_{11}, n_{12}, n_{13}) \\
&= \{\{[0.2741, 0.377]|0.05, [0.2898, 0.3946]|0.05, [0.3031, 0.377]|0.05, [0.3205, 0.3946]|0.05, \\
&\quad [0.3031, 0.4076]|0.4, [0.3205, 0.4266]|0.4\} \\
&\quad \{[0.1261, 0.2263]|1\}, \\
&\quad \{[0.3419, 0.4203]|0.49, [0.3881, 0.4698]|0.21, [0.3419, 0.4422]|0.21, [0.3881, 0.4898]|0.09\}\}; \\
n_2 &= PINHFWG(n_{21}, n_{22}, n_{23}) \\
&= \{\{[0.6, 0.7]|1\}, \{[0.1, 0.1761]|1\}, \{[0.1261, 0.2263]|0.2, [0.1614, 0.2616]|0.8\}\}; \\
n_3 &= PINHFWG(n_{31}, n_{32}, n_{33}) \\
&= \{\{[0.4181, 0.5206]|0.3, [0.5, 0.6]|0.7\}, \{[0.1614, 0.3004]|0.4, [0.2000, 0.3368]|0.6\}, \\
&\quad \{[0.2263, 0.3265]|1\}\}; \\
n_4 &= PINHFWG(n_{41}, n_{42}, n_{43}) \\
&= \{\{[0.4799, 0.6411]|1\}, \{[0.0854, 0.1861]|1\}, \{[0.1261, 0.2000]|0.2, [0.2097, 0.2416]|0.8\}\}.
\end{aligned}
$$

- Step 2′. By Definition 6, we have:

$$n_1 = 0.595, n_2 = 0.7692, n_3 = 0.6653, n_4 = 0.7372.$$

- Step 3′. Rank A_h $(h = 1, 2, 3, 4)$ on the basis of Step 2′,

$$A_2 > A_4 > A_3 > A_1.$$

Thus, A_2 is the best choice.

8. The Basic Aggregation Operator for PSVNHFS

In this subsection, we construct the PSVNHFWA operator and the PSVNHFWG operator. The comparison method of PIVNHFNs is proposed.

Definition 12. *Let $\tilde{np}_x$ $(x = 1, 2, \cdots, X)$ be a non-empty collection of PSVNHFNs , then a PSVNHFWA operator can be indicated as:*

$$PSVNHFWA(\tilde{np}_1, \tilde{np}_2, \cdots, \tilde{np}_X) = \bigoplus_{x=1}^{X} w_x(\tilde{np}_x)$$
$$= \bigcup\{\{(1 - \prod_{j=1}^{X}(1-\alpha_j)^{w_j}) | \prod_{j=1}^{X} P^{\tilde{t}_j}\}, \{\prod_{j=1}^{X} \beta_j^{w_j} | \prod_{k=1}^{X} P^{\tilde{i}_j}\}, \{\prod_{j=1}^{X} \gamma_j^{w_j} | \prod_{j=1}^{X} P^{\tilde{f}_j}\}\}, \tag{14}$$

where $\alpha_j \in \tilde{t}_j, \beta_j \in \tilde{i}_j, \gamma_j \in \tilde{f}_j$, $j = 1, 2, \cdots, X$, w_j is the weight of $\tilde{np}_j$ and $\sum_{j=1}^{X} w_j = 1$.

Definition 13. *Let $\tilde{np}_x$ $(x = 1, 2, \cdots, X)$ be a non-empty collection of PSVNHFNs, then the PSVNHFWG operator can be indicated as:*

$$PSVNHFWG(\tilde{np}_1, \tilde{np}_2, \cdots, \tilde{np}_X) = \bigotimes_{j=1}^{X} w_j(\tilde{np}_j)$$
$$= \bigcup\{\{\prod_{j=1}^{X}(\alpha_j)^{w_j}) | \prod_{j=1}^{X} P^{\tilde{t}_j}\}, \{(1 - \prod_{j=1}^{X}(1-\beta_j)^{w_j}) | \prod_{j=1}^{X} P^{\tilde{i}_j}\}, \{(1 - \prod_{j=1}^{X}(1-\gamma_j)^{w_j}) | \prod_{j=1}^{X} P^{\tilde{f}_j}\}\}, \tag{15}$$

where $\alpha_j \in \tilde{t}_j, \beta_j \in \tilde{i}_j, \gamma_j \in \tilde{f}_j$, $j = 1, 2, \cdots, X$, w_j is the weight of $\tilde{np}_j$ and $\sum_{j=1}^{X} w_j = 1$.

Since the PSVNHFN is a special case of PINHFN, thus the score function $s(\tilde{np})$, deviation function $d(\tilde{np})$ and sorting method can utilize Definition 6, Definition 7 and Definition 8, respectively. In order to solve the MADM problem of the probabilistic single-valued neutrosophic hesitant fuzzy circumstance, the algorithm can use the same method described in Section 6. Next, The application can use Example 1.

- Step 1. Select the PSVNHFWA operator to aggregate all PSVNHFNs of $(\tilde{np})_{hk}$ $(h = 1, 2, 3, 4; k = 1, 2, 3)$ to obtain the PSVNHFN $\tilde{np}_h$ $(h = 1, 2, 3, 4)$ for the option A_h $(h = 1, 2, 3, 4)$.

$$\tilde{np}_1 = \{\{0.3212|0.01, 0.3568|0.015, 0.3966|0.025, 0.3580|0.01, 0.3917|0.015, 0.4293|0.025, 0.3565|0.09, 0.3903|0.1350, 0.4280|0.2250, 0.3914|0.09, 0.4234|0.1350, 0.4590|0.2250\}, \{0.1189|0.06, 0.1569|0.14, 0.1316|0.24, 0.1737|0.56\}, \{0.368|0.048, 0.407|0.032, 0.3955|0.072, 0.4373|0.048, 0.3959|0.192, 0.4378|0.128, 0.4254|0.288, 0.4704|0.192\}\}$$

$$\tilde{np}_2 = \{\{0.6|0.006, 0.6435|0.014, 0.6383|0.054, 0.6776|0.126, 0.6278|0.024, 0.6682|0.056, 0.6634|0.216, 0.7|0.504\}, \{0.1|0.24, 0.132|0.16, 0.1275|0.36, 0.1682|0.24\}, \{0.1677|0.35, 0.2213|0.15, 0.1933|0.35, 0.2551|0.15\}\};$$

$$\tilde{np}_3 = \{\{0.5271|0.3, 0.5675|0.2, 0.5627|0.3, 0.6|0.2\}, \{0.2138|1\}, \{0.2797|0.2, 0.3224|0.8\}\};$$

$$\tilde{np}_4 = \{\{0.5476|0.216, 0.6045|0.024, 0.579|0.144, 0.632|0.016, 0.6074|0.324, 0.6569|0.036, 0.6347|0.216, 0.6807|0.024\}, \{0.132|1\}, \{0.1189|0.01, 0.1569|0.08, 0.1846|0.01, 0.1516|0.09, 0.2|0.72, 0.2352|0.09\}\}.$$

- Step 2. By (2), count the score values of all $\tilde{np}_h$ $(h = 1, 2, 3, 4)$,

$$s(\tilde{np}_1) = 0.6108, s(\tilde{np}_2) = 0.7839, s(\tilde{np}_3) = 0.6776, s(\tilde{np}_4) = 0.7579.$$

- Step 3. Rank the PSVNHFNs by Definition 8; we have.

$$A_2 > A_4 > A_3 > A_1.$$

Thus, we know that A_2 is the best choice.

Next, we will make use of the PSVNHFWG operator to solve Example 1.

- Step 1′. Aggregate PSVNHFNs $\tilde{np}_h k$ $(h = 1,2,3,4; k = 1,2,3)$ by taking advantage of the PSVNHFWG operator to get the $\tilde{np}_h$ for A_h.

$$\tilde{np}_1 = \{\{0.2898|0.01, 0.3409|0.09, 0.3033|0.01, 0.3568|0.09, 0.3205|0.015, 0.377|0.135, 0.3355|0.015, 0.3946|0.135, 0.3466|0.025, 0.4076|0.225, 0, 3627|0.025, 0.4266|0.225\}, \{0.1261|0.06, 0.1663|0.14, 0.1548|0.24, 0.1937|0.56\}, \{0.3881|0.048, 0.4404|0.192, 0.4113|0.072, 0.4615|0.288, 0.4203|0.032, 0.4698|0.128, 0.4422|0.048, 0.4898|0.192\}\},$$

$$\tilde{np}_2 = \{\{0.6|0.006, 0.6382|0.014, 0.6236|0.024, 0.6632|0.056, 0.6333|0.054, 0.6735|0.126, 0.6581|0.216, 0.7|0.504\}, \{0.1|0.24, 0.1414|0.16, 0.1363|0.36, 0.1761|0.24\}, \{0.1889|0.35, 0.2263|0.15, 0.226|0.35, 0.2616|0.15\}\}.$$

$$\tilde{np}_3 = \{\{0.5233|0.3, 0.5629|0.2, 0.5578|0.3, 0.6|0.2\}, \{0.2666|1\}, \{0.2942|0.2, 0.3265|0.8\}\}.$$

$$\tilde{np}_4 = \{\{0.4799|0.216, 0.5887|0.024, 0.4988|0.144, 0.6119|0.016, 0.5029|0.324, 0.6169|0.036, 0.5226|0.216, 0.6411|0.024\}, \{0.1414|1\}, \{0.1261|0.01, 0.1663|0.08, 0.2097|0.01, 0.1614|0.09, 0.2|0.72, 0.2416|0.09\}\}.$$

- Step 2′. By Formula (2), we have:

$$s(\tilde{np}_1) = 0.5507, s(\tilde{np}_2) = 0.7741, s(\tilde{np}_3) = 0.6568, s(\tilde{np}_4) = 0.7248.$$

- Step 3′. Rank A_h $(h = 1,2,3,4)$ by Definition 8,

$$A_2 > A_4 > A_3 > A_1.$$

Thus, A_2 is the best choice.

In order to demonstrated the effectiveness of our approaches, a comparison was established with other methods. They are shown in Tables 9 and 10.

Table 9. Comparison of the results obtained by different methods under the single-valued neutrosophic hesitant fuzzy circumstance.

Method	Sort of Results	Best Alternative	Worst Alternative
$SVNHFWA$ $operator$ [49]	$A_4 > A_2 > A_3 > A_1$	A_3	A_4
$SVNHFWG$ $operator$ [49]	$A_2 > A_4 > A_3 > A_1$	A_2	A_1
$PSVNHFWA$ $operator$	$A_2 > A_4 > A_3 > A_1$	A_2	A_1
$PSVNHFWG$ $operator$	$A_2 > A_4 > A_3 > A_1$	A_2	A_1

Table 10. Comparison of the results obtained by different methods under the interval neutrosophic hesitant fuzzy circumstance.

Method	Sort of Results	Best Alternative	Worst Alternative
GWA $operator(1 \leq \lambda \leq 39)$ [50]	$A_3 > A_1 > A_2 > A_4$	A_3	A_4
$PINHFWA$ $operator$	$A_4 > A_2 > A_3 > A_1$	A_4	A_1
$PINHFWG$ $operator$	$A_2 > A_4 > A_3 > A_1$	A_2	A_1

In [49], Ye introduced the single-valued neutrosophic hesitant fuzzy weighted averaging (SVNHFWA) and single-valued neutrosophic hesitant fuzzy weighted geometric (SVNHFWG) operators and applied them to the single-valued neutrosophic hesitant fuzzy circumstance. In [50], Liu proposed the generalized weighted aggregation (GWA) operator and established the MADM method under the interval neutrosophic hesitant fuzzy circumstance. However, probability is not considered in [49,50]. The ranking results are presented in Table 9 and Table 10. According to the Table 9, A_2 is always the best choice, A_1 is always the worst option. According to the Table 10, the best option is A_4 under the group's major points, whereas the best selection is A_2 under the individual major points. A_1 is always the worst choice. Apparently, the SVNHFS, IVHFS and PSVNHFS are special cases of PINHFS. Thus, the PINHFS is is wider than other methods.

9. Conclusions

In this paper, as a generation of fuzzy set theory, a new concept of PSVNHFS (PINHFS) is proposed based on the NHS and INS. The score function and the deviation function are defined. A comparison method is proposed. PSVNHFS is a special case of PINHFS; thus, PINHFS has a wider range of applications. Therefore, this paper mainly discusses the situation of the interval. Then, some basic operation laws of PINHFNs are introduced and investigated. Next, the PINHFWA and PINHFWG operators are presented, and some properties are studied. PSVNHFSs also satisfies the properties mentioned above. We can determine the optimal alternative by utilizing the PINHFWA (PINHFWG) operator. Finally, a numerical example was given. It is proven that the new approach is more flexible and suitable for practical issues. In addition, an example raised in this paper is to explain that PINHFS is more general than PSVNHFS. In the future, others aggregation operators of PINHFNs can be researched, and more practical applications in other areas can be solved, like medical diagnoses.

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