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Relations between the Complex Neutrosophic Sets with Their Applications in Decision Making

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Received: 23 July 2018; Accepted: 24 August 2018; Published: 1 September 2018

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Abstract: The basic aim of soft computing is to trade precision for a tractableness and reduction in solution cost by pushing the limits of tolerance for imprecision and uncertainty. This paper introduces a novel soft computing technique called complex neutrosophic relation (CNR) to evaluate the degree of interaction between two complex neutrosophic sets (CNSs). CNSs are used to represent two-dimensional information that are imprecise, uncertain, incomplete and indeterminate. The Cartesian product of CNSs and subsequently the complex neutrosophic relation is formally defined. This relation is generalised from a conventional single valued neutrosophic relation (SVNR), based on CNSs, where the ranges of values of CNR are extended to the unit circle in complex plane for its membership functions instead of [0, 1] as in the conventional SVNR. A new algorithm is created using a comparison matrix of the SVNR after mapping the complex membership functions from complex space to the real space. This algorithm is then applied to scrutinise the impact of some teaching strategies on the student performance and the time frame(phase) of the interaction between these two variables. The notion of inverse, complement and composition of CNRs along with some related theorems and properties are introduced. The performance and utility of the composition concept in real-life situations is also demonstrated. Then, we define the concepts of projection and cylindric extension for CNRs along with illustrative examples. Some interesting properties are also obtained. Finally, a comparison between different existing relations and CNR to show the ascendancy of our proposed CNR is provided.

Keywords: neutrosophic set; complex neutrosophic; complex neutrosophic relation

1. Introduction

In recent years, a substantial amount of growth has been noticed on the application of soft computing techniques in science, engineering and other disciplines. Soft computing is an emerging collection of methodologies and techniques that aim to deal with imprecision, uncertainty, partial truths and approximation to solve everyday and advanced problems with durability, tractability, stability and low solution cost. The principal constituents of soft computing are: fuzzy systems; evolutionary computation, including genetic algorithms; neural networks, including neural computing; machine learning and probabilistic reasoning. Fuzzy systems, including fuzzy logic and fuzzy set theory, play a leading role in soft computing as they bring basic ideas to other soft computing methodologies.

Fuzzy and intuitionistic fuzzy sets [1–4] were introduced to handle uncertain and incomplete data, which is ubiquitous in real situations. Because decision-makers often have limited time and knowledge, the vagueness of their opinions must also be taken into consideration; for this reason, the integration of fuzzy and intuitionistic fuzzy sets in the multi criteria decision-making (MCDM) methods has garnered the attention of many researchers [5–10]. Subsequently, these fuzzy sets have been actively applied in various MCDM problems using a myriad of different approaches of operational research such as TOPSIS [11], AHP [12], PROMETHEE [13] and others. However, the aforementioned uncertainty sets do not handle the indeterminate and inconsistent information. To overcome this issue, Smarandache [14] originally gave a concept of a neutrosophic set from a philosophical point of view, which is a part of neutrosophy. Neutrosophic set is characterised independently by a truth-membership function, indeterminacy-membership function and falsity-membership function, denoted by T, I, F, respectively, where the indeterminacy is quantified explicitly. The ranges of the functions T, I and F are subsets of the real standard or nonstandard interval $]^{-}0, 1^{+}[$. To constrain them in the real standard interval $[0, 1]$ for convenient science and engineering applications, a single-valued neutrosophic set (SVNS) and its operators were introduced by Wang et al. [15] as the subclasses of the neutrosophic sets. Recent years have seen the rapid utilization and usage of SVNS in many real life problems [16–19]. On the other hand, Ye [20] put forward a concept of a simplified neutrosophic set (SNS), which is also a subclass of the neutrosophic set and includes a SVNS, and defined basic operational laws of SNSs. Then, he applied this concept to solve many MCDM problems [20–22].

In order to incorporate the advantages of complex numbers to the notion of fuzzy sets and its generalizations, the introduction of fuzzy sets was followed by their extension to the complex fuzzy set [23]. In particular, complex fuzzy sets are not simply a linear extension of conventional fuzzy sets. Rather, complex fuzzy sets allow a natural extension of fuzzy set theory to problems that are either very difficult or impossible to address with one-dimensional grades of membership. Complex fuzzy set progressed rapidly to complex fuzzy logic [24] and complex intuitionistic fuzzy sets [25]. All of the former structures cannot handle imprecise, indeterminate, inconsistent, and incomplete information that has periodic nature. Therefore, Ali and Smarandache [26] initialised the CNS with three complex valued membership functions to handle all the types of uncertainty, including indeterminacy with periodicity. CNS is actually an extension of the SVNS to the complex space, which makes it superior to all of the aforementioned uncertainty sets.

Like a set, a relation is of fundamental importance in all engineering, science, and mathematically based fields. Relations are intimately involved in logic, approximate reasoning, rule-based systems, nonlinear simulation, synthetic evaluation, classification, pattern recognition, and control. The relations between fuzzy sets and intutionistic fuzzy sets is a topic that has been extensively studied [27–31]. In the neutrosophic environment, Yang et al. [32] proposed single valued neutrosophic relations (SVNRs) and study their properties, while Abu Qamar and Hassan [33] derived Q-neutrosophic soft relations as an extension of the neutrosophic soft relation [34]. Among other notable research in this relatively unexplored area are the complex fuzzy relations [23], complex Atanassov's intuitionistic fuzzy (CAIF) relations [35] and the complex neutrosophic soft expert sets and relations [36,37].

We will extend the studies on CAIF relation [29] and SVNRs [27] by establishing a novel notion called complex neutrosophic relation (CNR), which is constructed to hold the advantages of SVNRs while maintaining the features of complex numbers in CAIF relation as follows: on the one hand, SVNR has the ability to handle imprecise, indeterminate, inconsistent, and incomplete information embedded in the relation body. On the other hand, CAIF relations have the complexity feature, which has the ability to capture information pertaining to the time frame of the interaction between the parameters. All of these features together will be contained in our proposed CNR.

The remainder of this paper is organized as follows. In Section 2, we recapitulate some of the fundamental concepts related to neutrosophic sets and CNSs. Throughout Section 3, we define the concept of the Cartesian product of two CNSs as a forerunner of the concept of CNR. Subsequently, the concept of CNR will be defined. A decision-making algorithm is also generated in this section. This algorithm examines the affectivity of number of teaching techniques using the characteristics of the CNR. In Section 4, we define some basic operations on CNR such as the complement, inverse and composition of CNRs. This section also gives the definitions of projection and cylindric extension for CNRs. Section 5 provides a comprehensive comparison among CNR and other recent approaches to manifest the dominance of our proposed method. Section 6 outlines the conclusion of this paper and suggests directions for future research.

2. Preliminaries

In this section, a summary of the literature on neutrosophic set and CNS relevant to this paper is presented.

We begin by recalling the definition of neutrosophic set, followed by the definition of SVNS.

Definition 1. *[14] Let U be a universe of discourse, and a neutrosophic set N in U is defined as: $A = \{< u; T_N(u); I_N(u); F_N(u) >; u \in U\}$, where $T_N(u), I_N(u)$ and $F_N(u)$ are the truth membership function, the indeterminacy membership function and the falsity membership function, respectively, such that T; I; $F : X \rightarrow]^{-}0; 1^{+}[$ and $^{-}0 \leq T_N(u) + I_N(u) + F_N(u) \leq 3^{+}$.*

In order to apply neutrosophic set on the scientific fields, its parameters should have to be specified. Hence, Wang et al. [15] provided the following definition.

Definition 2. *[15] Let U be a universe of discourse. A single valued neutrosophic set (SVNS) S in U is defined as:*

$$S = \int_U \langle T(U), I(U), F(U) \rangle / u,\ u \in U,$$

when U is continuous and

$$S = \sum_{i=1}^{n} \langle T(U_i), I(U_i), F(U_i) \rangle / u_i,\ u_i \in U,$$

when U is discrete, where $T, I, F : U \rightarrow [0,1]$ are the truth-membership function, the indeterminacy-membership function and the falsity-membership function, respectively.

Ali and Smarandache [26] conceptualized complex neutrosophic set and gave the basic operations in the following three definitions.

Definition 3. *[26] Let a universe of discourse X, a complex neutrosophic set S in X is characterized by a truth membership function $T_S(x)$, an indeterminacy membership function $I_S(x)$, and a falsity membership function $F_S(x)$ that assign an element $x \in X$ a complex-valued grade of $T_S(x)$, $I_S(x)$, and $F_S(x)$ in S. By definition, the values $T_S(x)$, $I_S(x)$, $F_S(x)$ and their sum may all within the unit circle in the complex plane and are of the form, $T_S(x) = p_S(x) \cdot e^{j\mu_S(x)}$, $I_S(x) = q_S(x) \cdot e^{j\nu_S(x)}$ and $F_S(x) = r_S(x) \cdot e^{j\omega_S(x)}$, each of $p_S(x), q_S(x), r_S(x)$ and $\mu_S(x), \nu_S(x), \omega_S(x)$ are, respectively, real valued and $p_S(x), q_S(x), r_S(x) \in [0,1]$ such that $0^{-} \leq P_S(x) + q_S(x) + r_S(x) \leq 3^{+}$.*

Definition 4. *[26] Let A and B be two CNSs, where A is characterized by a truth membership function $T_A(x) = p_A(x) \cdot e^{j\mu_A(x)}$, an indeterminacy membership function $I_A(x) = q_A(x) \cdot e^{j\nu_A(x)}$ and a falsity membership function $F_A(x) = r_A(x) \cdot e^{j\omega_A(x)}$ and B is characterized by a truth membership function $T_B(x) = p_B(x) \cdot e^{j\mu_B(x)}$, an indeterminacy membership function $I_B(x) = q_B(x) \cdot e^{j\nu_B(x)}$ and a falsity membership function $F_B(x) = r_B(x) \cdot e^{j\omega_B(x)}$.*

We define complement of A, union and intersection of A and B as follows.

1. *The complement of A , denoted as $c(A)$, is specified by functions:*
   $T_{c(A)}(x) = p_{c(A)}(x) \cdot e^{j\mu_{c(A)}(x)} = r_A(x) \cdot e^{j(2\pi - \mu_A(x))}$,
   $I_{c(A)}(x) = q_{c(A)}(x) \cdot e^{j\nu_{c(A)}(x)} = (1 - q_A(x)) \cdot e^{j(2\pi - \nu_A(x))}$, *and*
   $F_{c(A)}(x) = r_{c(A)}(x) \cdot e^{j\omega_{c(A)}(x)} = p_A(x) \cdot e^{j(2\pi - \omega_A(x))}$.
2. *A is said to be complex neutrosophic subset of B ($A \subseteq B$) if and only if the following conditions are satisfied:*

- $T_A(u) \leq T_B(u)$ *such that* $p_A(u) \leq p_B(u)$ *and* $\mu_A(u) \leq \mu_B(u)$.
- $I_A(u) \geq I_B(u)$ *such that* $q_A(u) \geq q_B(u)$ *and* $\nu_A(u) \geq \nu_B(u)$.
- $F_A(u) \geq F_B(u)$ *such that* $r_A(u) \geq r_B(u)$ *and* $\omega_A(u) \geq \omega_B(u)$.

3. *The union(intersection) of A and B, denoted as* $A \cup_N (\cap_N) B$ *and the truth membership function* $T_{A\cup(\cap)B}(x)$ *, the indeterminacy membership function* $I_{A\cup(\cap)B}(x)$ *, and the falsity membership function* $F_{A\cup(\cap)B}(x)$ *are defined as:*

$$T_{A\cup(\cap)B}(x) = [(p_A(x) \vee (\wedge) p_B(x))] \cdot e^{j(\mu_A(x)\vee(\wedge)\mu_B(x))},$$

$$I_{A\cup(\cap)B}(x) = [(q_A(x) \wedge (\vee) q_B(x))] \cdot e^{j(\nu_A(x)\wedge(\vee)\nu_B(x))} \text{ and}$$

$$F_{A\cup(\cap)B}(x) = [(r_A(x) \wedge (\vee) r_B(x))] \cdot e^{j(\omega_A(x)\wedge(\vee)\omega_B(x))},$$

where $\vee$ = *max and* $\wedge$ = *min* .

Definition 5. *[26] Let* A_n *be N CNSs on X,* $(n = 1, 2, ..., N)$*, and* $T_{A_n}(x) = p_{A_n}(x) \cdot e^{j\mu_{A_n}(x)}$ *be a complex-valued membership function,* $I_{A_n}(x) = q_{A_n}(x) \cdot e^{j\nu_{A_n}(x)}$ *be a complex-valued indeterminacy membership function and* $F_{A_n}(x) = r_{A_n}(x) \cdot e^{j\omega_{A_n}(x)}$ *be a complex-valued nonmembership function. The Cartesian products of* A_n*, denoted as* $A_1 \times A_2 \times ... \times A_N$*, are specified by the functions:*

$$\begin{aligned} T_{A_1\times A_2\times...\times A_N}(x) &= p_{A_1\times A_2\times...\times A_N}(x) \cdot e^{j\cdot\mu_{A_1\times A_2\times...\times A_N}(x)} \\ &= min(p_{A_1}(x_1), p_{A_2}(x_1), ..., p_{A_N}(x_N)) \cdot e^{jmin(\mu_{A_1}(x_1),\mu_{A_2}(x_2),...,\mu_{A_N}(x_N))}, \end{aligned}$$

$$\begin{aligned} I_{A_1\times A_2\times...\times A_N}(x) &= q_{A_1\times A_2\times...\times A_N}(x) \cdot e^{j\cdot\nu_{A_1\times A_2\times...\times A_N}(x)} \\ &= max(q_{A_1}(x_1), q_{A_2}(x_1), ..., q_{A_N}(x_N)) \cdot e^{jmax(\nu_{A_1}(x_1),\nu_{A_2}(x_2),...,\nu_{A_N}(x_N))}, \end{aligned}$$

$$\begin{aligned} F_{A_1\times A_2\times...\times A_N}(x) &= r_{A_1\times A_2\times...\times A_N}(x) \cdot e^{j.\omega_{A_1\times A_2\times...\times A_N}(x)} \\ &= max(r_{A_1}(x_1), r_{A_2}(x_1), ..., r_{A_N}(x_N)) \cdot e^{jmax(\omega_{A_1}(x_1),\omega_{A_2}(x_2),...,\omega_{A_N}(x_N))}. \end{aligned}$$

3. Complex Neutrosophic Relations

In this section, we propose the definition of Cartesian product between two CNSs and subsequently, the formal definition of the CNR. We also apply the CNR method to solve a MCDM problem.

The following is the definition of the Cartesian product between two CNSs:

Definition 6. *Let X and Y be two CNSs over the universal sets U and V, respectively. The Cartesian product of X and Y, denoted by* $X \times Y$ *is a CNS defined as* $X \times Y = \Big\{\langle (u,v), T_{X\times Y}(u,v), I_{X\times Y}(u,v), F_{X\times Y}(u,v)\rangle : (u,v) \in U \times V\Big\}$*, where* $T_{X\times Y}(u,v)$ *is a complex- valued truth membership function,* $I_{X\times Y}(u,v)$ *is a complex-valued indeterminacy membership function and* $F_{X\times Y}(u,v)$ *is a complex- valued falsity membership function and* $\forall (u,v) \in U \times V$*,*

$$T_{X\times Y}(u,v) = min(p_X(u), p_Y(v)) \cdot e^{jmin(\mu_X(u),\mu_Y(v))},$$

$$I_{X\times Y}(u,v) = max(q_X(u), q_Y(v)) \cdot e^{jmax(\nu_X(u),\nu_Y(v))}, \text{ and}$$

$$F_{X\times Y}(u,V) = max(r_X(u), r_Y(v)) \cdot e^{jmax(\omega_x(u),\omega_Y(v))}.$$

We will now define the concept of CNR as follows:

Definition 7. *Let X and Y be two CNSs over the universes U and V, respectively. A complex neutrosophic relation from X to Y is a complex neutrosophic subset of* $X \times Y$*. Thus, a CNR from X to Y, denoted by* $R(X,Y)$*, where* $R(X,Y) \subseteq X \times Y$*. As always,* $R(X,Y)$ *may be represented as the set of ordered sequence*

$R(X,Y) = \Big\{\langle (u,v), T_R(u,v), I_R(u,v), F_R(u,v)\rangle : (u,v) \in U \times V\Big\}$, *where* $\forall u \in U, v \in V$, $T_R(u,v) = p_R(u,v) \cdot e^{j\mu_R(u,v)}$, $I_R(u,v) = q_R(u,v) \cdot e^{j\nu_R(u,v)}$ *and* $F_R(u,v) = r_R(u,v) \cdot e^{j\omega_R(u,v)}$. *The values* $T_R(u,v), I_R(u,v), F_R(u,v)$ *are within the unit circle in the complex plane and both the amplitude terms* $p_R(u,v), q_R(u,v), r_R(u,v)$ *and the phase terms* $\mu_R(u,v), \nu_R(u,v), \omega_R(u,v)$ *are real valued such that* $p_R(u,v), q_R(u,v), r_R(u,v) \in [0,1]$ *and* $0 \leq p_R(u,v) + q_R(u,v) + r_R(u,v) \leq 3$.

Now, we put forward a real-life application of CNR to reveal its ability to describe and analyze real events.

3.1. Complex Neutrosophic Relation in Education

CNR can be effectively used to measure the interaction between several educational variables where the factor of time plays a key role and the indeterminacy is unavoidable.

Now, we give an example of a relation between two CNSs.

Example 1. *Suppose a study is conducted to determine the most influential teaching strategy contributing to student achievement. Let* U *be the set of teaching techniques in education applied on a certain group of students, where* $U = \{u_1 =$ *cooperative learning,* $u_2 =$ *self-learning,* $u_3 =$ *traditional teaching* $\}$. *Let* V *be a set of indicators that measure the student's achievement and his interaction, where* $V = \{v_1 =$ *academic achievement,* $v_2 =$ *emotional interaction,* $v_3 =$ *social interaction* $\}$. *Let* X *and* Y *be two CNSs over* U *and* V, *respectively, defined as follows:*

$$X = \left\{ \frac{u_1}{\langle 0.9e^{i2\pi(11/12)}, 0.2e^{i2\pi(1/12)}, 0.1e^{i2\pi(1/48)}\rangle}, \frac{u_2}{\langle 0.3e^{i2\pi(4/12)}, 0.6e^{i2\pi(4/12)}, 0.1e^{i2\pi(4/12)}\rangle}, \frac{u_3}{\langle 0.2e^{i2\pi(3/12)}, 0.6e^{i2\pi(4/12)}, 0.9e^{i2\pi(7/12)}\rangle} \right\},$$

$$Y = \left\{ \frac{v_1}{\langle 0.9e^{i2\pi(12/12)}, 0.3e^{i2\pi(3/12)}, 0.1e^{i2\pi(1/24)}\rangle}, \frac{v_2}{\langle 0.4e^{i2\pi(5/12)}, 0.5e^{i2\pi(4/12)}, 0.3e^{i2\pi(8/12)}\rangle}, \frac{v_3}{\langle 0.8e^{i2\pi(12/12)}, 0.5e^{i2\pi(9/12)}, 0.2e^{i2\pi(2/12)}\rangle} \right\}.$$

Now, we compute the relation between the two CNSs X *and* Y *to investigate the effect of modern methods in education on the student's performance. Our CNR denoted by* $R(X,Y)$ *such that* $R(X,Y) \subseteq X \times Y$, *is*

$$\begin{aligned} R(X,Y) = \Bigg\{ & \frac{(u_1,v_1)}{\langle 0.9e^{i2\pi(11/12)}, 0.3e^{i2\pi(3/12)}, 0.1e^{i2\pi(1/24)}\rangle}, \frac{(u_1,v_2)}{\langle 0.4e^{i2\pi(5/12)}, 0.5e^{i2\pi(4/12)}, 0.3e^{i2\pi(8/12)}\rangle}, \\ & \frac{(u_1,v_3)}{\langle 0.8e^{i2\pi(11/12)}, 0.5e^{i2\pi(9/12)}, 0.2e^{i2\pi(2/12)}\rangle}, \frac{(u_2,v_1)}{\langle 0.3e^{i2\pi(4/12)}, 0.6e^{i2\pi(4/12)}, 0.1e^{i2\pi(4/12)}\rangle}, \\ & \frac{(u_2,v_2)}{\langle 0.3e^{i2\pi(4/12)}, 0.6e^{i2\pi(4/12)}, 0.3e^{i2\pi(8/12)}\rangle}, \frac{(u_2,v_3)}{\langle 0.3e^{i2\pi(4/12)}, 0.6e^{i2\pi(9/12)}, 0.2e^{i2\pi(4/12)}\rangle} \\ & \frac{(u_3,v_1)}{\langle 0.2e^{i2\pi(3/12)}, 0.6e^{i2\pi(4/12)}, 0.9e^{i2\pi(7/12)}\rangle}, \frac{(u_3,v_2)}{\langle 0.2e^{i2\pi(3/12)}, 0.6e^{i2\pi(4/12)}, 0.9e^{i2\pi(8/12)}\rangle}, \\ & \frac{(u_3,v_3)}{\langle 0.2e^{i2\pi(3/12)}, 0.6e^{i2\pi(9/12)}, 0.9e^{i2\pi(8/12)}\rangle} \Bigg\}. \end{aligned}$$

Suppose that the relation between X *and* Y *are measured within a time frame of 12 months. In our example, the terms of truth amplitude, indeterminate amplitude and false amplitude of* $R(X,Y)$ *measure the membership degree of the impact of the modern methods in education on the student's performance, the indeterminacy membership degree of the impact of the modern methods in education on the student's performance and the non-membership degree of the impact of the modern methods in education on the student's performance, respectively, whereas the truth phase term, the indeterminate phase term and the false phase term of* $R(X,Y)$ *represent the period of time in which the modern methods influence the student's performance, the period of time in which we are unable to determine if the modern methods influence the student's performance or does not influence the student's performance and the period of time in which the modern methods does not influence the* *student's performance, respectively. Since the phase terms in $R(X,Y)$ represent periods of time and $R(X,Y)$ represents the relation between the modern methods in education and the student's performance within the time frame of 12 months, then, in each complex neutrosophic value, the range value of each of the truth, indeterminate and false phase terms should be between* 0 *and* 1.

In the following discussion, we will study the influence of the cooperative learning on the academic achievement of the student.

The term $\frac{(u_1,v_1)}{\langle 0.9e^{i(11/12)\pi},\, 0.3e^{i(3/12)\pi},\, 0.1e^{i(1/24)\pi}\rangle}$ implies that the cooperative learning strongly influences the student's academic achievement. The complex-valued truth membership function $0.9e^{i(11/12)\pi}$ indicates that there is a strong influence with degree 0.9 and this influence span of 11 months is considered a very long time of influence, the complex-valued indeterminacy membership function $0.3e^{i(3/12)\pi}$ can be interpreted as we are unable to determine if there is an influence of the cooperative learning on the academic achievement or not with a degree of 0.3, and this influence is not evident for three months. For the complex-valued falsity membership function $0.1e^{i(1/24)\pi}$, we presume with a degree of 0.1 that there is no influence and the time with no influence is one month.

Next, the CNR R is used together with a generalized algorithm to solve the decision-making problem stated at the beginning of this section. This algorithm is employed to rank the educational techniques that affect the student performance corresponding to their influence strength. The algorithm steps are given as follows:

Step 1: Input the CNSs X and Y.

Step 2: Calculate The CNR R of $X \times Y$.

Step 3: Convert the CNR R, which is actually a CNS to the SVNS $\widehat{R}$ by obtaining the weighted aggregation values of $T_{\widehat{R}}(u,v), I_{\widehat{R}}(u,v)$ and $F_{\widehat{R}}(u,v)$, $\forall(u,v) \in X \times Y$ as in the following formulas:

$$T_{\widehat{R}}(u,v) = w_1 p_R(u,v) + w_2(1/2\pi)\mu_R(u,v),$$

$$I_{\widehat{R}}(u,v) = w_1 q_R(u,v) + w_2(1/2\pi)\nu_R(u,v),$$

$$F_{\widehat{R}}(u,v) = w_1 r_R(u,v) + w_2(1/2\pi)\omega_R(u,v),$$

where $p_R(u,v), q_R(u,v), r_R(u,v)$ and $\mu_R(u,v), \nu_R(u,v), \omega_R(u,v)$ are the amplitude and phase terms in the CNR R, respectively. $T_{\widehat{R}}(u,v), I_{\widehat{R}}(u,v)$ and $F_{\widehat{R}}(u,v)$ are the truth membership function, indeterminacy membership function and falsity membership function in the SVNS $\widehat{R}$, respectively, and w_1, w_2 are the weights for the amplitude terms (degrees of influence) and the phase terms (times of influence), respectively, where w_1 and $w_2 \in [0,1]$ and $w_1 + w_2 = 1$.

Step 4: Compute the comparison matrix of the SVNR $(\widehat{R})$. Comparison matrix of $(\widehat{R})$ is a matrix whose rows comprise the experimental(predictor)variables u_1, u_2 and u_3 and the columns represent the response or dependent variables v_1, v_2 and v_3. The entries c_{ij} are calculated as $c_{ij} = a - (b + c)$, where a is the integer calculated as how many times $T_{\widehat{R}}(u_i, v_j)$ exceeds or is equal to $T_{\widehat{R}}(u_k, v_j)$, for $u_i \neq u_k$, $\forall u_k \in U$), b is the integer calculated as how many times $I_{\widehat{R}}(u_i, v_j)$ exceeds or is equal to $I_{\widehat{R}}(u_k, v_j)$, for $u_i \neq u_k$, $\forall u_k \in U$) and c is the integer calculated as how many times $F_{\widehat{R}}(u_i, v_j)$ exceeds or is equal to $F_{\widehat{R}}(u_k, v_j)$, for $u_i \neq u_k$, $\forall u_k \in U$).

Step 5: Find the values of the score M_i of u_i, $\forall i$.

Step 6: Determine the value of the highest score $M = max M_i$, $\forall i$.

Now, to change the form of the CNS R to the real state $(\widehat{R})$, we assume that the weight vectors are $w_1 = 0.8$ and $w_2 = 0.2$. To illustrate this step, we calculate $T_{\widehat{R}}(u,v), I_{\widehat{R}}(u,v)$ and $F_{\widehat{R}}(u,v)$ for u_1 and v_1 as shown below:

$$\begin{aligned}
T_{\widehat{R}}(u_1,v_1) &= w_1 p_R(u_1,v_1) + w_2(1/2\pi)\mu_R(u_1,v_1) \\
&= 0.8(0.9) + 0.2(1/2\pi)(2\pi)(11/12) \\
&= 0.9, \\
I_{\widehat{R}}(u_1,v_1) &= w_1 q_R(u_1,v_1) + w_2(1/2\pi)\nu_R(u_1,v_1) \\
&= 0.8(0.3) + 0.2(1/2\pi)(2\pi)(3/12) \\
&= 0.29, \\
F_{\widehat{R}}(u_1,v_1) &= w_1 r_R(u_1,v_1) + w_2(1/2\pi)\omega_R(u_1,v_1) \\
&= 0.8(0.1) + 0.2(1/2\pi)(2\pi)(1/24) \\
&= 0.09.
\end{aligned}$$

Then, for u_1 and v_1, the single-valued neutrosophic value is $(T_{\widehat{R}}(u_1,v_1), I_{\widehat{R}}(u_1,v_1), F_{\widehat{R}}(u_1,v_1)) = (0.9, 0.29, 0.09)$.

In the same manner, we calculate the other single-valued neutrosophic value, $\forall u \in U$ and $v \in V$. The SVNR $(\widehat{R})$ can be presented by the following relational matrix:

$$\widehat{R}(u,v) = \begin{bmatrix} & u_1 & u_2 & u_3 \\ v_1 & (0.9,0.29,0.09) & (0.31,0.55,0.15) & (0.21,0.55,0.84) \\ v_2 & (0.40,0.47,0.37) & (0.31,0.55,0.37) & (0.21,0.55,0.85) \\ v_3 & (0.82,0.55,0.19) & (0.31,0.66,0.23) & (0.21,0.63,0.85) \end{bmatrix}.$$

The comparison matrix of the SVNR $(\widehat{R})$ is as shown in Table 1.

Table 1. Comparison matrix of $(\widehat{R})$.

$\widehat{R}$	v_1	v_2	v_3
u_1	2	1	2
u_2	−2	−2	−2
u_3	−4	−4	−3

Computing the score for each of the u_i, we have the scores as in Table 2.

Table 2. Score of (u_i) .

u_i	M_i
u_1	5
u_2	−6
u_3	−11

From Table 2, $maxM_i = 5$. Thus, the decision is to choose the element u_1 as the appropriate solution. Therefore, we conclude that the cooperative learning is the most influential technique improving student performance, followed by self-learning and traditional teaching.

4. Operations on Complex Neutrosophic Relation

We will now introduce some basic operations on CNR such as the complement, inverse and composition of CNRs. We will begin by first proposing the definition of the complement of CNR.

Definition 8. *Let R be a CNR on $U \times V$, where: $R = \left\{\langle (u,v), T_R(u,v), I_R(u,v), F_R(u,v)\rangle : (u,v) \in U \times V\right\}$. Then, the complex neutrosophic complement relation of R, denoted by R^c, is defined as*

$$R^c = \left\{\langle (u,v), T_{R^c}(u,v), I_{R^c}(u,v), F_{R^c}(u,v)\rangle : (u,v) \in U \times V\right\}, \text{ where}$$

$$T_{R^c}(u,v) = p_{R^c}(u,v) \cdot e^{j\mu_{R^c}(u,v)} = r_R(u,v) \cdot e^{j(2\pi - \mu_R(u,v))},$$

$$I_{R^c}(u,v) = q_{R^c}(u,v) \cdot e^{j\nu_{R^c}(u,v)} = (1 - q_R(u,v)) \cdot e^{j(2\pi - \nu_R(u,v))}, \text{ and}$$

$$F_{R^c}(u,v) = r_{R^c}(u,v) \cdot e^{j\omega_{R^c}(u,v)} = p_R(u,v) \cdot e^{j(2\pi - \omega_R(u,v))},.$$

Next, we will give the definition of the inverse of a CNR and give a proposition on the inverse of a CNR.

Definition 9. *Let R be a CNR from X to Y. The inverse of R is denoted as R^{-1} and is a CNR from Y to X defined as $R^{-1} = \left\{ \langle (v,u), T_{R^{-1}}(u,v), I_{R^{-1}}(u,v), F_{R^{-1}}(u,v) \rangle : (v,u) \in V \times U \right\}$, where $\forall u \in U$ and $v \in V$, $T_{R^{-1}}(u,v) = T_R(v,u)$, $I_{R^{-1}}(u,v) = I_R(v,u)$ and $F_{R^{-1}}(u,v) = F_R(v,u)$.*

Proposition 1. *Let X and Y be two CNSs over U and V, respectively. Suppose that R and S are two CNRs from X to Y. Then, the following results hold true:*

1. $(R^{-1})^{-1} = R$,
2. *If $R \subseteq S$ then $R^{-1} \subseteq S^{-1}$.*

Proving $\forall u \in U$ *and* $v \in V$, *we have*

1. $(R^{-1})^{-1} = \left\{ \langle (u,v), T_{(R^{-1})^{-1}}(u,v), I_{(R^{-1})^{-1}}(u,v), F_{(R^{-1})^{-1}}(u,v) \rangle : (u,v) \in U \times V \right\}$, *where* $T_{(R^{-1})^{-1}}(u,v) = T_{R^{-1}}(v,u) = T_R(u,v)$, $I_{(R^{-1})^{-1}}(u,v) = I_{R^{-1}}(v,u) = I_R(u,v)$ *and* $F_{(R^{-1})^{-1}}(u,v) = F_{R^{-1}}(v,u) = F_R(u,v)$. *This implies that* $(R^{-1})^{-1} = R$.
2. *If $R \subseteq S$, then,*
   $T_R(u,v) \le T_S(u,v)$
   $\Rightarrow T_R(u,v) = T_{R^{-1}}(v,u) \le T_S(u,v) = T_{S^{-1}}(v,u)$
   $\Rightarrow T_{R^{-1}}(v,u) \le T_{S^{-1}}(v,u)$. *Similarly, we can show that* $I_{R^{-1}}(v,u) \ge I_{S^{-1}}(v,u)$ *and* $F_{R^{-1}}(v,u) \ge F_{S^{-1}}(v,u)$. *This completes the proof.*

In this part, we propose the axiomatic definition of the composition of CNRs followed by an example illustrates application of this concept in real life. We then give two theorems on the composition concept.

Definition 10. *Let X, Y and Z be three CNSs over the universes U, V and W, respectively. Let R be a CNR from X to Y and S a CNR from Y to Z. The composition of the CNRs R and S is an CNR from X to Z, defined as: $R \circ S = \left\{ \langle (u,w), T_{R \circ S}(u,w), I_{R \circ S}(u,w), F_{R \circ S}(u,w) \rangle : (u,w) \in U \times W \right\}$, where $\forall (u,w) \in U \times W$ and $v \in V$,*

$$T_{R \circ S}(u,w) = p_{R \circ S}(u,w) \cdot e^{j\mu_{R \circ S}(u,w)},$$

where $p_{R \circ S}(u,w) = max[p_R(u,v), p_S(v,w)] = max[min(p_X(u), p_Y(v)), min(p_Y(v), p_Z(w))]$ *and* $\mu_{R \circ S}(u,w) = max[\mu_R(u,w), \mu_S(u,w)] = max[min(\mu_X(u), \mu_Y(v)), min(\mu_Y(v), \mu_Z(w))]$,

$$I_{R \circ S}(u,w) = q_{R \circ S}(u,w) \cdot e^{j\nu_{R \circ S}(u,w)},$$

where $q_{R \circ S}(u,w) = min[q_R(u,v), q_S(v,w)] = min[max(q_X(u), q_Y(v)), max(q_Y(v), q_Z(w))]$ *and* $\nu_{R \circ S}(u,w) = min[\nu_R(u,w), \nu_S(u,w)] = min[max(\nu_X(u), \nu_Y(v)), max(\nu_Y(v), \nu_Z(w))]$,

$$F_{R \circ S}(u,w) = r_{R \circ S}(u,w) \cdot e^{j\omega_{R \circ S}(u,w)},$$

where $r_{R\circ S}(u,w) = min[r_R(u,v), r_S(v,w)] = min[max(r_X(u), r_Y(v)), max(r_Y(v), r_Z(w))]$ *and* $\omega_{R\circ S}(u,w) = min[\omega_R(u,w), \omega_S(u,w)] = min[max(\omega_X(u), \omega_Y(v)), max(\omega_Y(v), \omega_Z(w))]$.

This relationship can be written as $R \circ S(u,w) = R(u,v)\widetilde{\cap}_N S(v,w)$.

The following example demonstrates the utilization of the composition of the CNRs in real life.

Example 2. *Suppose X, Y and Z are three CNSs that represent the sets of Chinese financial indicators, Malaysian financial indicators and Malaysian public opinion indicators, respectively. Suppose the interactions among these sets are measured over the limited time frame of 12 months using the CNRs R and S, where* $R(X,Y)$ *represents the effect of the Chinese financial indicators to Malaysian financial indicators, and* $S(Y,Z)$ *represents the effect of the Japanese financial indicators to Malaysian public opinion indicators.*

The composition of CNRs $R(X,Y)$ *and* $S(Y,Z)$ *obtains a new CNR* $T(X,Z)$*, which represents the effect of the Chinese financial indicators to the Malaysian public opinion indicators.*

For the sake of illustration, it will suffice in this example to consider the composition of the following two approximations in the CNRs $R(X,Y)$ *and* $S(Y,Z)$.

1- $\langle T_R(u,v), I_R(u,v), F_R(u,v)\rangle = \left\langle 0.9e^{i(10/12)\pi},\ 0.3e^{i(3/12)\pi},\ 0.1e^{i(1/24)\pi}\right\rangle$*, where* $u \in X$ *and* $v \in Y$ *represent, respectively, the exchange rate of the Chinese Yuan and the inflation rate in Malaysia. This approximation measures the truth, indeterminacy and falsity for both degree and phase (period) of the influence of Chinese Yuan's exchange rate on the inflation rate in Malaysia.*

2- $\langle T_S(v,w), I_S(v,w), F_S(v,w)\rangle = \left\langle 0.8e^{i(11/12)\pi},\ 0.7e^{i(1/12)\pi},\ 0.05e^{i(1/48)\pi}\right\rangle$*, where* $v \in Y$ *and* $w \in Z$ *represent, respectively, the inflation rate in Malaysia and confidence in the Malaysian economy. This approximation measures the truth, indeterminacy and falsity for both degree and phase (period) of the influence of the exchange rate of Chinese Yuan on the confidence in the Malaysian economy.*

The result of this composition is:

$\langle T_{R\circ S}(u,w), I_{R\circ S}(u,w), F_{R\circ S}(u,w)\rangle = \left\langle 0.9e^{i(11/12)\pi},\ 0.3e^{i(1/12)\pi},\ 0.05e^{i(1/48)\pi}\right\rangle$*. The components* $T_{R\circ S}(u,w), I_{R\circ S}(u,w), F_{R\circ S}(u,w)$ *measure, respectively, the truth, indeterminacy and falsity for both degree and phase (period) of the influence of the inflation rate in Malaysia on the confidence in the Malysian economy.*

In view of Definition 10, we prove the following results.

Theorem 1. *Let* X, Y *and Z be three complex neutrosophic sets over the universes U, V and W, respectively. Let R be a CNR from X to Y and S a CNR from Y to Z. Then,* $(R \circ S)^{-1} = S^{-1} \circ R^{-1}$.

Proof. For all $(u,v) \in U \times V$ and $(v,w) \in V \times W$, let
$(R \circ S)^{-1} = \left\{\langle (u,w), T_{(R\circ S)^{-1}}(u,w), I_{(R\circ S)^{-1}}(u,w), F_{(R\circ S)^{-1}}(u,w)\rangle : (u,w) \in U \times W\right\}$, and
$S^{-1} \circ R^{-1} = \left\{\langle (u,w), T_{S^{-1}\circ R^{-1}}(u,w), I_{S^{-1}\circ R^{-1}}(u,w), F_{S^{-1}\circ R^{-1}}(u,w)\rangle : (u,w) \in U \times W\right\}$.

To prove the equality, we have to show that
$T_{(R\circ S)^{-1}}(u,w) = T_{S^{-1}\circ R^{-1}}(u,w)$, $I_{(R\circ S)^{-1}}(u,w) = I_{S^{-1}\circ R^{-1}}(u,w)$, and $F_{(R\circ S)^{-1}}(u,w) = F_{S^{-1}\circ R^{-1}}(u,w)$.

Therefore,

$$\begin{aligned}
p_{(R\circ S)^{-1}}(u,w) &= p_{(R\circ S)}(w,u) \\
&= max[p_R(w,v), p_S(v,u)] = max[p_S(v,u), p_R(w,v)] \\
&= max[min(p_Y(v), p_X(u)), min(p_Z(w), p_Y(v))] \\
&= max[min(p_X(u), p_Y(v)), min(p_Y(v), p_Z(w))] \\
&= max[p_{S^{-1}(u,v)}, p_{R^{-1}(v,w)}] \\
&= p_{S^{-1}\circ R^{-1}}(u,w),
\end{aligned}$$

which implies $p_{(R\circ S)^{-1}}(u,w) = p_{S^{-1}\circ R^{-1}}(u,w)$. Similarly, we can show that $\mu_{(R\circ S)^{-1}}(u,w) = \mu_{S^{-1}\circ R^{-1}}(u,w)$, proving that $T_{(R\circ S)^{-1}}(u,w) = T_{S^{-1}\circ R^{-1}}(u,w)$. Similarly, we can show it also holds for the identity and falsity terms and this completes the proof. □

Theorem 2. *Let X, Y, Z and W be CNSs over the universes U, V, L and M, respectively. Let R be a CNR from X to Y, S a CNR from Y to Z and T a CNR from Z to W. Then, $R\circ(S\circ T) = (R\circ S)\circ T$.*

Proof. For all $(u,v)\in U\times V, (v,l)\in V\times L, (l,m)\in L\times M$, let
$R\circ(S\circ T) = \left\{\left\langle (u,m), T_{R\circ(S\circ T)}(u,m), I_{R\circ(S\circ T)}(u,m), F_{R\circ(S\circ T)}(u,m)\right\rangle : (u,m)\in U\times M\right\}$,
$(R\circ S)\circ T = \left\{\left\langle (u,m), T_{(R\circ S)\circ T}(u,m), I_{(R\circ S)\circ T}(u,m), F_{(R\circ S)\circ T}(u,m)\right\rangle : (u,m)\in U\times M\right\}$.

To prove the equality, we have to show that $T_{R\circ(S\circ T)}(u,m) = T_{(R\circ S)\circ T}(u,m), I_{R\circ(S\circ T)}(u,m) = I_{(R\circ S)\circ T}(u,m)$ and $F_{R\circ(S\circ T)}(u,m) = F_{(R\circ S)\circ T}(u,m)$.

Therefore,

$$\begin{aligned}
p_{R\circ(S\circ T)}(u,m) &= max(p_R(u,v), p_{S\circ T}(v,m)) = max[p_R(u,v), max(p_S(v,l), p_T(l,m))],\\
&= max[min(p_X(u), p_Y(v)), max[min(p_Y(v), p_Z(l)), min(p_Z(l), p_W(m))]],\\
&= max[min(p_X(u), p_Y(v)), min(p_Y(v), p_Z(l)), min(p_Z(l), p_W(m))],\\
&= max[max[min(p_X(u), p_Y(v)), min(p_Y(v), p_Z(l))], min(p_Z(l), p_W(m))],\\
&= max[max[p_R(u,v), p_S(v,l)], p_T(l,m)],\\
&= max[p_{(R\circ S)}(u,l), p_T(l,m)],\\
&= p_{(R\circ S)\circ T}(u,m),
\end{aligned}$$

which implies that $p_{R\circ(S\circ T)}(u,m) = p_{(R\circ S)\circ T}(u,m)$. Similarly, we can show that

$\mu_{R\circ(S\circ T)}(u,m) = \mu_{(R\circ S)\circ T}(u,m)$, proving that $T_{R\circ(S\circ T)}(u,m) = T_{(R\circ S)\circ T}(u,m)$.

The proofs for the identity and falsity terms can be similarly proven and this completes the proof. □

In order to give a deeper insight into this issue, we introduce the definitions of projection and cylindric extension for CNRs.

Definition 11. *Let U and V be two universes and R be a CNR on $U\times V$. Then, for all $u\in U$ and $v\in V$,*
1- The projection of R on U is a CNS R_{p_U}, defined respectively, by the complex valued truth, indeterminacy and falsity membership functions:

$$T_{R_{p_U}}(u) = max_v P_R(u,v)\cdot e^{i max_v \mu_R(u,v)},$$

$$I_{R_{p_U}}(u) = min_v q_R(u,v)\cdot e^{i min_v \nu_R(u,v)},$$

$$F_{R_{p_U}}(u) = min_v r_R(u,v)\cdot e^{i min_v \omega_R(u,v)}.$$

2- The projection of R on V is a CNS R_{p_V}, defined respectively, by the complex valued truth, indeterminacy and falsity membership functions:

$$T_{R_{p_V}}(v) = max_u P_R(u,v)\cdot e^{i max_u \mu_R(u,v)},$$

$$I_{R_{p_V}}(v) = min_u q_R(u,v)\cdot e^{i min_u \nu_R(u,v)},$$

$$F_{R_{p_V}}(v) = min_u r_R(u,v)\cdot e^{i min_u \omega_R(u,v)}.$$

Example 3. *Let $U = \{u_1, u_2, u_3\}$ and $V = \{v_1, v_2\}$ be two universes. Let R be a CNR on $U \times V$ defined as follows:*

$$R(U,V) = \left\{ \frac{(u_1,v_1)}{\langle 0.1e^{i2\pi(7/12)}, 0.5e^{i2\pi(8/12)}, 0.7e^{i2\pi(5/24)} \rangle}, \frac{(u_1,v_2)}{\langle 0.4e^{i2\pi(5/12)}, 0.5e^{i2\pi(4/12)}, 0.3e^{i2\pi(8/12)} \rangle}, \frac{(u_2,v_1)}{\langle 0.5e^{i2\pi(4/12)}, 0.5e^{i2\pi(4/12)}, 0.1e^{i2\pi(4/12)} \rangle}, \frac{(u_2,v_2)}{\langle 0.4e^{i2\pi(4/12)}, 0.8e^{i2\pi(4/12)}, 0.3e^{i2\pi(8/12)} \rangle}, \frac{(u_3,v_1)}{\langle 0.1e^{i2\pi(3/12)}, 0.3e^{i2\pi(4/12)}, 0.9e^{i2\pi(7/12)} \rangle}, \frac{(u_3,v_2)}{\langle 0.2e^{i2\pi(3/12)}, 0.6e^{i2\pi(4/12)}, 0.2e^{i2\pi(8/12)} \rangle} \right\}.$$

The projection of the CNR R on U is given by:

$$R_{p_U}(u)$$

$$= \left\{ \frac{u_1}{\langle max(0.1,0.4)e^{i2\pi max(7/12,5/12)}, min(0.5,0.5)e^{i2\pi min(8/12,4/12)}, min(0.7,0.3)e^{i2\pi min(5/24,8/12)} \rangle} \frac{u_2}{\langle max(0.5,0.4)e^{i2\pi max(4/12,4/12)}, min(0.5,0.8)e^{i2\pi min(4/12,4/12)}, min(0.1,0.3)e^{i2\pi min(4/12,8/12)} \rangle}, \frac{u_3}{\langle max(0.1,0.2)e^{i2\pi max(3/12,3/12)}, min(0.3,0.6)e^{i2\pi min(4/12,4/12)}, min(0.9,0.2)e^{i2\pi min(7/12,8/12)} \rangle} \right\}.$$

$$= \left\{ \frac{u_1}{\langle 0.4e^{i2\pi(7/12)}, 0.5e^{i2\pi(4/12)}, 0.3e^{i2\pi(5/24)} \rangle} \frac{u_2}{\langle 0.5e^{i2\pi(4/12)}, 0.5e^{i2\pi(4/12)}, 0.1e^{i2\pi(4/12)} \rangle}, \frac{u_3}{\langle 0.2e^{i2\pi(3/12)}, 0.3e^{i2\pi(4/12)}, 0.2e^{i2\pi(7/12)} \rangle} \right\}.$$

The projection of the CNR R on V is given by:

$$R_{p_V}(v)$$

$$= \left\{ \frac{v_1}{\langle max(0.1,0.5,0.1)e^{i2\pi max(7/12,4/12,3/12)}, min(0.5,0.5,0.3)e^{i2\pi min(8/12,4/12,4/12)}, min(0.7,0.1,0.9)e^{i2\pi min(5/24,4/12,7/12)} \rangle} \frac{v_2}{\langle max(0.4,0.4,0.2)e^{i2\pi max(5/12,4/12,3/12)}, min(0.5,0.8,0.6)e^{i2\pi min(4/12,4/12,4/12)}, min(0.3,0.3,0.2)e^{i2\pi min(8/12,8/12,8/12)} \rangle} \right\}.$$

$$= \left\{ \frac{v_1}{\langle 0.5e^{i2\pi(7/12)}, 0.3e^{i2\pi(4/12)}, 0.1e^{i2\pi(4/12)} \rangle} \frac{v_2}{\langle 0.4e^{i2\pi(5/12)}, 0.5e^{i2\pi(4/12)}, 0.2e^{i2\pi(8/12)} \rangle} \right\}.$$

Definition 12. *Let U and V be two universes and R be a CNR on $U \times V$. Then, for all $u \in U$ and $v \in V$,*

1- The cylindric extension of R_{p_U} to $U \times V$, is a CNR $R_{p_{UE}}$ given by:

$$T_{R_{p_{UE}}}(u,v) = T_{R_{p_U}}(u),$$
$$I_{R_{p_{UE}}}(u,v) = I_{R_{p_U}}(u),$$
$$F_{R_{p_{UE}}}(u,v) = F_{R_{p_U}}(u).$$

2- The cylindric extension of R_{p_V} to $U \times V$, is a CNR $R_{p_{VE}}$ given by:

$$T_{R_{p_{VE}}}(u,v) = T_{R_{p_V}}(v),$$
$$I_{R_{p_{VE}}}(u,v) = I_{R_{p_V}}(v),$$
$$F_{R_{p_{VE}}}(u,v) = F_{R_{p_V}}(v).$$

Example 4. *Consider Example 3 to find the cylindric extension of R_{p_U} and R_{p_V}. Then,*

$$
\begin{aligned}
& R_{p_{UE}} \\
& = \left\{ \frac{(u_1,v_1)}{\langle 0.4e^{i2\pi(7/12)},\, 0.5e^{i2\pi(4/12)},\, 0.3e^{i2\pi(5/24)} \rangle},\, \frac{(u_1,v_2)}{\langle 0.4e^{i2\pi(7/12)},\, 0.5e^{i2\pi(4/12)},\, 0.3e^{i2\pi(5/24)} \rangle}, \right. \\
& \frac{(u_2,v_1)}{\langle 0.5e^{i2\pi(4/12)},\, 0.5e^{i2\pi(4/12)},\, 0.1e^{i2\pi(4/12)} \rangle},\, \frac{(u_2,v_2)}{\langle 0.5e^{i2\pi(4/12)},\, 0.5e^{i2\pi(4/12)},\, 0.1e^{i2\pi(4/12)} \rangle}, \\
& \left. \frac{(u_3,v_1)}{\langle 0.2e^{i2\pi(3/12)},\, 0.3e^{i2\pi(4/12)},\, 0.2e^{i2\pi(7/12)} \rangle},\, \frac{(u_3,v_2)}{\langle 0.2e^{i2\pi(3/12)},\, 0.3e^{i2\pi(4/12)},\, 0.2e^{i2\pi(7/12)} \rangle} \right\},
\end{aligned}
$$

and

$$
\begin{aligned}
& R_{p_{VE}} \\
& = \left\{ \frac{(u_1,v_1)}{\langle 0.5e^{i2\pi(7/12)},\, 0.3e^{i2\pi(4/12)},\, 0.1e^{i2\pi(4/12)} \rangle},\, \frac{(u_2,v_1)}{\langle 0.5e^{i2\pi(7/12)},\, 0.3e^{i2\pi(4/12)},\, 0.1e^{i2\pi(4/12)} \rangle}, \right. \\
& \frac{(u_3,v_1)}{\langle 0.5e^{i2\pi(7/12)},\, 0.3e^{i2\pi(4/12)},\, 0.1e^{i2\pi(4/12)} \rangle},\, \frac{(u_1,v_2)}{\langle 0.4e^{i2\pi(5/12)},\, 0.5e^{i2\pi(4/12)},\, 0.2e^{i2\pi(8/12)} \rangle}, \\
& \left. \frac{(u_2,v_2)}{\langle 0.4e^{i2\pi(5/12)},\, 0.5e^{i2\pi(4/12)},\, 0.2e^{i2\pi(8/12)} \rangle},\, \frac{(u_3,v_2)}{\langle 0.4e^{i2\pi(5/12)},\, 0.5e^{i2\pi(4/12)},\, 0.2e^{i2\pi(8/12)} \rangle} \right\}.
\end{aligned}
$$

5. Comparison and Discussion

We have discussed the influence of the modern methods in education to the student performance, and the time of the influence of these modern methods on the student performance. In this section, we will compare our proposed CNR model to three other existing models, the CAIF relation [35], SVNR [32] and CNSER [37].

As an extension of fuzzy and intuitionistic fuzzy relations, SVNR [32] was developed to measure the degree of the interaction between two SVNSs, with the ability to handle all types of uncertainties including indeterminacy, which is beyond the scope of fuzzy and intuitionistic fuzzy relations. However, SVNR fails to deal with problems that involve two-dimensional information/date i.e., two different types of information/data pertaining to the problem parameters.

From Example 1, it can be seen that SVNR is not able to solve the decision making problem presented, which involves two types of neutrosophic information (the degree of the influence and the total time of the influence) since SVNR lacks the phase terms which represent the time frame of this problem.

CAIF relation [35] can overcome the problems inherent in using SVNSs by virtue of the phase terms which have the ability to represent the time frame of the interaction between the variables. However, the CAIF relation cannot be used to describe the space $U \times V$ which is explained by three complex-valued membership functions, since the CAIF relation is characterized by complex-valued truth membership and complex-valued falsity membership functions which handle only incomplete and uncertainty information.

In a bid to address these shortcomings in the CAIF relation and SVNR , Al-Quran and Hassan [37] introduced the CNSER which can handle all types of uncertainty including indeterminacy with complex-valued truth membership function, complex-valued indeterminacy membership function and complex-valued falsity membership function.

Although this model managed to overcome all of the drawbacks of the previous models, it cannot be applied to our decision-making problem, since it is constructed to solve decision-making problems by incorporating the idea of the expert as well as the parametric factors, whereas CNR, can be successfully applied to the various disciplines without considering the parameterization factor during the analysis.

On the other hand, the decision-making algorithm based on the CNSER, presented in [37], shows a discernible bias in one of its common steps which is choosing the highest grade among all the u's (alternatives or elements in the universal set) that attached with the same parameter. This might lead to drop some of the u's who had not obtained the highest grade, but with the best overall performance.

This inaccurate step can clearly affect the final decision, especially when the universal set contains more than three elements with extremely small differences in the values of the membership functions. Unlike the CNSER method and the other methods that were discussed, our proposed model and its accompanying decision-making algorithms can be used in any type of situation including ones in which the values of the membership functions of the alternatives are extremely close to another, thus making our proposed method superior compared to other recent approaches and also existing methods in the literature. The comparative analysis provided above explains the importance and necessity of our proposed CNR model.

6. Conclusions

This paper derived and investigated the relations among the CNSs and used these relations to describe and handle a real decision-making problem. The Cartesian product between two CNSs is defined as a prerequisite to define the CNR. Subsequently, the formal definition of the CNR is presented based on the definition of the CNSs. A new algorithm has been proposed and its detailed decision steps constructed. It was shown to be workable and successful in producing a desired result as illustrated by the application in education. We then presented some fundamental operators on the CNR such as complement and inverse of CNR. The axiomatic definition of the composition of CNRs is also defined along with an example to illustrate the applicability of this concept to derive useful information in reality by combining two CNRs. We provided some theorems on the previous operations and derived some properties with illustrative examples. The concepts of projection and cylindric extension for CNRs is also defined and illustrated by examples. We further demonstrated the superiority of the proposed CNR model through conducting a comparison between our model and three other models, which are SVNR, CAIF relation and CNSER. This novel concept provides a significant contribution to the sets and relations by adding a crucial aspect, which is the time factor to measure not only the degree of interaction between two objects but also the time of this interaction. This idea can also be conveyed to the area of the cognitive maps to measure the intersection time of the neutrosophic cognitive map parameters. The structure of the CNR is also rehabilitated to describe the relations between periodic phenomena that have uncertain data where the amplitude terms represent the uncertainty and the phase terms represent the periodicity semantic. Thus, the CNR may provide a powerful framework to represent problems with uncertainty and periodicity simultaneously in the field of physics, signal processing, stock marketing, and so on. The phase term of the CNR may also represent distance, direction, temperature, pressure or any variable that affects and interacts with its corresponding amplitude term in the decision process. This new interpretation of the phase term may open avenues for many applications in the field of physics and other natural sciences; for example, the CNR may be used as a new descriptor for the relation between vector quantities, where the amplitude and phase terms of the CNR, respectively, represent the magnitudes and the directions of the vector quantities. The development of CNR is important as it forms a basis for the derivation of complex neutrosophic logic.

Author Contributions: Conceptualization, A.A.-Q. and S.A.; Formal analysis, A.A.-Q. and S.A.; Investigation, A.A.-Q. and S.A.; Methodology, A.A.-Q.; Writing- original draft, A. A.-Q.; Writing-review and editing, S.A.

Funding: This research received no external funding.

Conflicts of Interest: The authors declare no conflict of interest.

Abbreviations

The following abbreviations are used in this manuscript:

CNR	Complex neutrosophic relation
CNSs	Complex neutrosophic sets
SVNR	Single valued neutrosophic relation
CAIF	Complex Atanassov's intuitionistic fuzzy
MCDM	Multi criteria decision-makig

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