Refined Neutrosophic Rings I

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Abstract

The study of refined neutrosophic rings is the objective of this paper. Substructures of refined neutrosophic rings and their elementary properties are presented. It is shown that every refined neutrosophic ring is a ring.

Keywords: Neutrosophy, refined neutrosophic set, refined neutrosophic group, refined neutrosophic ring.

1 Introduction

The notion of neutrosophic ring $R(I)$ generated by the ring R and the indeterminacy component I was introduced for the first time in the literature by Vasantha Kandasamy and Smarandache in.[12] Since then, further studies have been carried out on neutrosophic ring, neutrosophic nearring and neutrosophic hyperring see.[1,3,4,6–8] Recently, Smarandache[10] introduced the notion of refined neutrosophic logic and neutrosophic set with the splitting of the neutrosophic components $< T, I, F >$ into the form $< T_1, T_2, \ldots, T_p; I_1, I_2, \ldots, I_r; F_1, F_2, \ldots, F_s >$ where T_i, I_i, F_i can be made to represent different logical notions and concepts. In,[11] Smarandache introduced refined neutrosophic numbers in the form $(a, b_1I_1, b_2I_2, \ldots, b_nI_n)$ where $a, b_1, b_2, \ldots, b_n \in \mathbb{R}$ or $\mathbb{C}$. The concept of refined neutrosophic algebraic structures was introduced by Agboola in[5] and in particular, refined neutrosophic groups and their substructures were studied. The present paper is devoted to the study of refined neutrosophic rings and their substructures. It is shown that every refined neutrosophic ring is a ring.

For the purposes of this paper, it will be assumed that I splits into two indeterminacies I_1 [**contradiction** (true (T) and false (F))] and I_2 [**ignorance** (true (T) or false (F))]. It then follows logically that:

$$I_1I_1 = I_1^2 = I_1, \quad (1)$$
$$I_2I_2 = I_2^2 = I_2, \text{ and} \quad (2)$$
$$I_1I_2 = I_2I_1 = I_1. \quad (3)$$

If X is any nonempty set, then the set

$$X(I_1, I_2) =< X, I_1, I_2 >= \{(x, yI_1, zI_2) : x, y, z \in X\} \quad (4)$$

is called a refined neutrosophic set generated by X, I_1 and I_2. For $x, y, z \in X$, any element of $X(I_1, I_2)$ is of the form (x, yI_1, zI_2) and it is called a refined neutrosophic element.

If $+$ and . are the usual addition and multiplication of numbers, then I_k with $k = 1, 2$ have the following properties:

(1) $I_k + I_k + \cdots + I_k = nI_k$.

(2) $I_k + (-I_k) = 0$.

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(3) $I_k.I_k.\cdots.I_k = I_k^n = I_k$ for all positive integer $n > 1$.

(4) $0.I_k = 0$.

(5) I_k^{-1} is undefined with respect to multiplication and therefore does not exist.

For any two elements $(a, bI_1, cI_2), (d, eI_1, fI_2) \in X(I_1, I_2)$, we define

$$(a, bI_1, cI_2) + (d, eI_1, fI_2) = (a + d, (b + e)I_1, (c + f)I_2), \quad (5)$$

$$(a, bI_1, cI_2).(d, eI_1, fI_2) = (ad, (ae + bd + be + bf + ce)I_1, (af + cd + cf)I_2). \quad (6)$$

For any algebraic structure $(X, *)$, the couple $(X(I_1, I_2), *)$ is called a refined neutrosophic algebraic structure and it is named according to the laws (axioms) satisfied by $*$. For instance, if $(X, *)$ is a group, then $(X(I_1, I_2), *)$ is called a refined neutrosophic group generated by X, I_1, I_2.

Given any two refined neutrosophic algebraic structures $(X(I_1, I_2), *)$ and $(Y(I_1, I_2), *')$, the mapping $\phi : (X(I_1, I_2), *) \to (Y(I_1, I_2), *')$ is called a neutrosophic homomorphism if the following conditions hold:

(1) $\phi((a, bI_1, cI_2) * (d, eI_1, fI_2)) = \phi((a, bI_1, cI_2)) *' \phi((d, eI_1, fI_2)) \quad \forall (a, bI_1, cI_2), (d, eI_1, fI_2) \in X(I_1, I_2)$.

(2) $\phi(I_k) = I_k$ for $k = 1, 2$.

Example 1.1. [5] Let $\mathbb{Z}_2(I_1, I_2) = \{(0, 0, 0), (1, 0, 0), (0, I_1, 0), (0, 0, I_2), (0, I_1, I_2), (1, I_1, 0), (1, 0, I_2), (1, I_1, I_2)\}$. Then $(\mathbb{Z}_2(I_1, I_2), +)$ is a commutative refined neutrosophic group of integers modulo 2. Generally for a positive integer $n \geq 2$, $(\mathbb{Z}_n(I_1, I_2), +)$ is a finite commutative refined neutrosophic group of integers modulo n.

Example 1.2. [5] Let $(G(I_1, I_2), *)$ and and $(H(I_1, I_2), *')$ be two refined neutrosophic groups. Let $\phi : G(I_1, I_2) \times H(I_1, I_2) \to G(I_1, I_2)$ be a mapping defined by $\phi(x, y) = x$ and let $\psi : G(I_1, I_2) \times H(I_1, I_2) \to H(I_1, I_2)$ be a mapping defined by $\psi(x, y) = y$. Then ϕ and ψ are refined neutrosophic group homomorphisms.

For more details about refined neutrosophic sets, refined neutrosophic numbers and refined neutrosophic groups, we refer to.[5,10,11]

2 Main Results

Definition 2.1. Let $(R, +, .)$ be any ring. The abstract system $(R(I_1, I_2), +, .)$ is called a refined neutrosophic ring generated by R, I_1, I_2.

The abstract system $(R(I_1, I_2), +, .)$ is called a commutative refined neutrosophic ring if for all $x, y \in R(I_1, I_2)$, we have $xy = yx$. If there exists an element $e = (1, 0, 0) \in R(I_1, I_2)$ such that $ex = xe = x$ for all $x \in R(I_1, I_2)$, then we say that $(R(I_1, I_2), +, .)$ is a refined neutrosophic ring with unity.

Definition 2.2. Let $(R(I_1, I_2), +, .)$ be a refined neutrosophic ring and let $n \in \mathbb{Z}^+$.

(i) If for the least positive integer n such that $nx = 0$ for all $x \in R(I_1, I_2)$, then we call $(R(I_1, I_2), +, .)$ a refined neutrosophic ring of characteristic n and n is called the characteristic of $(R(I_1, I_2), +, .)$.

(ii) $(R(I_1, I_2), +, .)$ is called a refined neutrosophic ring of characteristic zero if for all $x \in R(I_1, I_2)$, $nx = 0$ is possible only if $n = 0$.

Example 2.3. (i) $\mathbb{Z}(I_1, I_2), \mathbb{Q}(I_1, I_2), \mathbb{R}(I_1, I_2), \mathbb{C}(I_1, I_2)$ are commutative refined neutrosophic rings with unity of characteristics zero.

(ii) Let $\mathbb{Z}_2(I_1, I_2) = \{(0, 0, 0), (1, 0, 0), (0, I_1, 0), (0, 0, I_2), (0, I_1, I_2), (1, I_1, 0), (1, 0, I_2), (1, I_1, I_2)\}$. Then $(\mathbb{Z}_2(I_1, I_2), +, .)$ is a commutative refined neutrosophic ring of integers modulo 2 of characteristic 2. Generally for a positive integer $n \geq 2$, $(\mathbb{Z}_n(I_1, I_2), +, .)$ is a finite commutative refined neutrosophic ring of integers modulo n of characteristic n.

Example 2.4. Let $M_{n\times n}^{\mathbb{R}}(I_1,I_2) = \left\{ \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} : a_{ij} \in \mathbb{R}(I_1,I_2) \right\}$ be a refined neutrosophic set of all $n \times n$ matrix. Then $(M_{n\times n}^{\mathbb{R}}(I_1,I_2),+,.)$ is a non-commutative refined neutrosophic ring under matrix multiplication.

Theorem 2.5. *Let $(R(I_1,I_2),+,.)$ be any refined neutrosophic ring. Then $(R(I_1,I_2),+,.)$ is a ring.*

Proof. It is clear that $(R(I_1,I_2),+)$ is an abelian group and and that $(R(I_1,I_2),.)$ is a semigroup. It remains to show that the distributive laws hold. To this end, let $x = (a_1,a_2I_1,a_3I_2), y = (b_1,b_2I_1,b_3I_2), z = (c_1,c_2I_1,c_3I_2)$ be any arbitrary elements of $R(I_1,I_2)$. Then

$$\begin{aligned} x(y+z) &= (a_1,a_2I_1,a_3I_2)((b_1,b_2I_1,b_3I_2)+(c_1,c_2I_1,c_3I_2)) \\ &= (a_1,a_2I_1,a_3I_2)(b_1+c_1,(b_2+c_2)I_1,b_3+c_3)I_2) \\ &= (a_1(b_1+c_1),a_1(b_2+c_2)+a_2(b_1+c_1)+a_2(b_2+c_2)+a_2(b_3+c_3)+a_3(b_2+c_2))I_1, \\ &\quad (a_1(b_3+c_3)+a_3(b_1+c_1)+a_3(b_3+c_3))I_2) \\ &= (a_1b_1+a_1c_1,(a_1b_2+a_1c_2+a_2b_1+a_2c_1+a_2b_2+a_2c_2+a_2b_3+a_2c_3+a_3b_2+a_3c_2)I_1, \\ &\quad (a_1b_3+a_1c_3+a_3b_1+a_3c_1+a_3b_3+a_3c_3)I_2). \end{aligned}$$

Also,

$$\begin{aligned} xy+xz &= ((a_1,a_2I_1,a_3I_2))((b_1,b_2I_1,b_3I_2))+((a_1,a_2I_1,a_3I_2))((c_1,c_2I_1,c_3I_2)) \\ &= (a_1b_1,(a_1b_2+a_2b_1+a_2b_2+a_2b_3+a_3b_2)I_1, \\ &\quad (a_1b_3+a_3b_1+a_3b_3)I_2)+(a_1c_1,(a_1c_2+a_2c_1+a_2c_2+a_2c_3+a_3c_2)I_1, \\ &\quad (a_1c_3+a_3c_1+a_3c_3)I_2) \\ &= (a_1b_1+a_1c_1,(a_1b_2+a_2b_1+a_2b_2+a_2b_3+a_3b_2+a_1c_2+a_2c_1+a_2c_2+a_2c_3+a_3c_2)I_1, \\ &\quad (a_1b_3+a_3b_1+a_3b_1+a_3b_3+a_1c_3+a_3c_1+a_3c_3)I_2). \end{aligned}$$

These show that $x(y+z) = xy+xz$. Similarly, it can be shown that $(y+z)x = yx+zx$. Hence $(R(I_1,I_2),+,.)$ is a ring. □

Definition 2.6. Let $(R(I_1,I_2),+,.)$ be a refined neutrosophic ring and let $J(I_1,I_2)$ be a nonempty subset of $R(I_1,I_2)$. $J(I_1,I_2)$ is called a refined neutrosophic subring of $R(I_1,I_2)$ if $(J(I_1,I_2),+,.)$ is itself a refined neutrosophic ring.

It is essential that $J(I_1,I_2)$ contains a proper subset which is a ring. Otherwise, $J(I_1,I_2)$ will be called a pseudo refined neutrosophic subring of $R(I_1,I_2)$.

Example 2.7. Let $(R(I_1,I_2),+,.) = (\mathbb{Z}(I_1,I_2),+)$ be the refined neutrosophic ring of integers. The set $J(I_1,I_2) = n\mathbb{Z}(I_1,I_2))$ for all positive integer n is a refined neutrosophic subring of $R(I_1,I_2)$.

Example 2.8. Let $(R(I_1,I_2),+,.) = (\mathbb{Z}_6(I_1,I_2),+)$ be the refined neutrosophic ring of integers modulo 6. The set

$$\begin{aligned} J(I_1,I_2) &= \{(0,0,0),(0,I_1,0),(0,0,I_2),(0,I_1,I_2), \\ &\quad (0,2I_1,0),(0,0,2I_2),(0,2I_1,2I_2), \\ &\quad (0,3I_1,0),(0,0,3I_2),(0,3I_1,3I_2), \\ &\quad (0,4I_1,0),(0,0,4I_2),(0,4I_1,4I_2), \\ &\quad (0,5I_1,0),(0,0,5I_2),(0,5I_1,5I_2)\}. \end{aligned}$$

is a refined neutrosophic subring of $R(I_1,I_2)$.

Theorem 2.9. *Let $\{J_k(I_1,I_2)\}_1^n$ be a family of all refined neutrosophic subrings (pseudo refined neutrosophic subrings) of a refined neutrosophic ring $(R(I_1,I_2),+,.)$. Then $\bigcap_1^n J_k(I_1,I_2)\}$ is a refined neutrosophic subring (pseudo refined neutrosophic subring) of $R(I_1,I_2)$.*

Definition 2.10. Let $A(I_1, I_2)$ and $B(I_1, I_2)$ be any two refined neutrosophic subrings (pseudo refined neutrosophic subrings) of a refined neutrosophic ring $(R(I_1, I_2), +)$. We define the sum $A(I_1, I_2) \oplus B(I_1, I_2)$ by the set

$$A(I_1, I_2) \oplus B(I_1, I_2) = \{a + b : a \in A(I_1, I_2), b \in B(I_1, I_2)\} \quad (7)$$

which is a refined neutrosophic subring (pseudo refined neutrosophic subring) of $R(I_1, I_2)$

Theorem 2.11. *Let $A(I_1, I_2)$ be any refined neutrosophic subring of a refined neutrosophic ring $(R(I_1, I_2), +)$ and let $B(I_1, I_2)$ be any pseudo refined neutrosophic subring of $(R(I_1, I_2), +)$. Then:*

(i) $A(I_1, I_2) \oplus A(I_1, I_2) = A(I_1, I_2)$.

(ii) $B(I_1, I_2) \oplus B(I_1, I_2) = B(I_1, I_2)$.

(iii) $A(I_1, I_2) \oplus B(I_1, I_2)$ *is a refined neutrosophic subring of* $R(I_1, I_2)$.

Definition 2.12. Let R be a non-empty set and let $+$ and . be two binary operations on R such that:

(i) $(R, +)$ is an abelian group.

(ii) $(R, .)$ is a semigroup.

(iii) There exists $x, y, z \in R$ such that

$$x(y + z) = xy + xz, (y + z)x = yx + zx.$$

(iv) R contains elements of the form (x, yI_1, zI_2) with $x, y, z \in R$ such that $y, z \neq 0$ for at least one value.

Then $(R, +, .)$ is called a pseudo refined neutrosophic ring.

Example 2.13. Let R be a set given by

$$R = \{(0, 0, 0), (0, 2I_1, 0), (0, 0, 2I_2), (0, 4I_1, 0), (0, 0, 4I_2), (0, 6I_1, 0), (0, 0, 6I_2)\}.$$

Then $(R, +, .)$ is a pseudo refined neutrosophic ring which is also a refined neutrosophic ring where $+$ and . are addition and multiplication modulo 8.

Example 2.14. Let $R(I_1, I_2) = \mathbb{Z}_{12}(I_1, I_2)$ be a refined neutrosophic ring of integers modulo 12 and let T be a subset of $\mathbb{Z}_{12}(I_1, I_2)$ given by

$$\begin{aligned} T = \{&(0, 0, 0), (0, 2I_1, 0), (0, 0, 2I_2), (0, 4I_1, 0), (0, 0, 4I_2), (0, 4I_1, 0), (0, 0, 4I_2), \\ &(0, 6I_1, 0), (0, 0, 6I_2)(0, 8I_1, 0), (0, 0, 8I_2), (0, 10I_1, 0), (0, 0, 10I_2)\}. \end{aligned}$$

It is clear that $(T, +, .)$ is a pseudo refined neutrosophic ring.

Since $T \subset R(I_1, I_2)$, it follows that $T \cup R(I_1, I_2) \subseteq R(I_1, I_2)$ and consequently, $(T \cup R(I_1, I_2), +, .)$ is a refined neutrosophic ring.

Theorem 2.15. *Let $(R(I_1, I_2), +, .)$ be any refined neutrosophic ring and let $(T, +, .)$ be any pseudo refined neutrosophic ring. Then $(T \cup R(I_1, I_2), +, .)$ is a refined neutrosophic ring if and only if $T \subset R(I_1, I_2)$.*

Theorem 2.16. *Let $(R(I_1, I_2), +, .)$ be any refined neutrosophic ring and let $(T, +, .)$ be any pseudo refined neutrosophic ring. Then $(T \oplus R(I_1, I_2), +, .)$ is a refined neutrosophic ring.*

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