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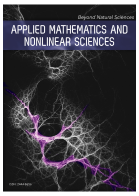


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# Some Characterizations of Neutrosophic Submodules of an $R$-module

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**Abstract**

In this paper, we investigate some new characterizations in the algebraic nature of neutrosophic submodule defined over a classical module using single valued neutrosophic set. We additionally characterized the neutrosophic point and determined the algebraic properties of neutrosophic point using the operations defined on neutrosophic submodule. Finley we examined a neutrosophic set as a generator of a neutrosophic submodule and derived some related concepts.

**Keywords:** Neutrosophic set, Neutrosophic point, Neutrosophic submodule, Neutrosophic ideal, Neutrosophic submodule generated by neutrosophic set.

## 1 Introduction

A fuzzy set represents vague concepts and contexts expressed in natural language by means of graded membership of elements in $[0,1]$ which is introduced by Lotfi A. Zadeh in 1965 [14, 25]. In 1986, Attanassov put forward intuitionistic fuzzy set hypothesis as a delineation of a set in which every segment is corresponding with a participation grades and non enrollment [3]. In 1995, Smarandache outlined neutrosophic set in which each element of a set is represented by three differing types of membership values and objective is to narrow the gap between the vague, ambiguous and inexact real world situations [5–7, 23]. Neutrosophic set theory gives a thorough scientific stage in which wispy and uncertain hypothetical phenomena can be managed by hierarchal membership of components.

The algebraic structure in pure mathematics cloning with uncertainty has been studied by some authors. In 1971, Azriel Rosenfield bestowed a seminal paper on fuzzy subgroup and W.J. Liu developed the idea of fuzzy normal subgroup and fuzzy subring. Consolidating neutrosophic set hypothesis with algebraic structures is a rising pattern in the region of mathematical research. In 2011, Isaac.P, P.P.John [12] recognized some algebraic nature of intuitionistic fuzzy submodule of a classical module. Neutrosophic algebraical structures

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and its properties provide us a solid mathematical foundation to clarify connected scientific ideas in designing, information mining and economics. In this paper we discuss about the generators of a neutrosophic submodule and some related results.

## 2 Preliminaries

**Definition 2.1.** [2] Let $R$ be a commutative ring with unity. A module $M$ over $R$, denoted as $M_R$ is an abelian group with a law of composition written '+'and a scalar multiplication $R \times M \to M$, written $(r,v) \rightsquigarrow rv$, that satisfy these axioms

1. $1v = v$
2. $(rs)v = r(sv)$
3. $(r+s)v = rv + sv$
4. $r(v+v') = rv + rv' \;\; \forall\, r,s \in R$ and $v, v` \in M$.

**Definition 2.2.** [2] A submodule $N$ of $M_R$ is a nonempty subset of $M_R$ that is closed under addition and scalar multiplication.

**Definition 2.3.** [21,24] A neutrosophic set P of the universal set X ($NS(X)$) is defined as

$$P = \{(\eta, t_P(\eta), i_P(\eta), f_P(\eta)) : \eta \in X\}$$

where $t_P, i_P, f_P : X \to (^-0, 1^+)$. The three components $t_P, i_P$ *and* $f_P$ represent membership value (Percentage of truth), indeterminacy (Percentage of indeterminacy) and non membership value (Percentage of falsity) respectively. These components are functions of non standard unit interval $(^-0, 1^+)$ [18].

**Remark 2.1.** [10,21] If the components of a neutrosophic set $P$, $t_P, i_P, f_P : X \to [0,1]$, then P is known as single valued neutrosophic set(SVNS).

**Remark 2.2.** In this paper, we discuss about the algebraic structure $M_R$-module with underlying set as SVNS. For simplicity SVNS will be called neutrosophic set.

**Remark 2.3.** $U^X$ denotes the set of all neutrosophic subset of $X$ or neutrosophic power set of $X$.

**Definition 2.4.** [17,21,22] Let $P,\ Q \in U^X$. Then P is contained in Q, denoted as $P \subseteq Q$ if and only if $P(\eta) \leqslant Q(\eta)\ \forall\ \eta \in X$, this means that

$$t_P(\eta) \leq t_Q(\eta),\ i_P(\eta) \leq i_Q(\eta),\ f_P(\eta) \geq f_Q(\eta),\ \forall\ \eta \in X$$

**Definition 2.5.** [13,19,21] The complement of $P = \{(x, t_P(x), i_P(x), f_P(x) : x \in X\} \in U^X$ is denoted by $P^C$ and defined as $P^C = \{x, f_P(x), 1 - i_P(x), t_P(x) : x \in X\}$.

**Remark 2.4.** $(P^C)^C = P$ where $P \in U^X$

**Definition 2.6.** [8,13,21] Let $P,\ Q \in U^X$

1. The union $C = \{\eta, t_C(\eta), i_C(\eta), f_C(\eta) : \eta \in X\}$ of P and Q [17] is denoted by $C = P \cup Q$ where

$$t_C(\eta) = t_P(\eta) \vee t_Q(\eta)$$

$$i_C(\eta) = i_P(\eta) \vee i_Q(\eta)$$

$$f_C(\eta) = f_P(\eta) \wedge f_Q(\eta)$$

2. The intersection $C = \{\eta, t_C(\eta), i_C(\eta), f_C(\eta) : \eta \in X\}$ of P and Q [17] is denoted by $C = P \cap Q$ where

$$t_C(\eta) = t_P(\eta) \wedge t_Q(\eta)$$
$$i_C(\eta) = i_P(\eta) \wedge i_Q(\eta)$$
$$f_C(\eta) = f_P(\eta) \vee f_Q(\eta)$$

**Definition 2.7.** [17,22]For any $P = \{(\eta, t_P(\eta), i_P(\eta), f_P(\eta)) : \eta \in X\} \in U^X$, the support $P^*$ of $P$ can be defined as

$$P^* = \{\eta \in X, t_P(\eta) > 0, i_P(\eta) > 0, f_P(\eta) < 1\}$$

.

**Definition 2.8.** [1,16] Let $P = \{(\eta, t_P(\eta), i_P(\eta), f_P(\eta)) : \eta \in R\}$ be an $NS(R)$. Then $P$ is called a neutrosophic ideal of $R$ if it satisfies the following conditions $\forall\ \eta, \theta \in R$

1. $t_P(\eta - \theta) \geq t_P(\eta) \wedge t_P(\theta)$
2. $i_P(\eta - \theta) \geq i_P(\eta) \wedge i_P(\theta)$
3. $f_P(\eta - \theta) \leq f_P(\eta) \vee f_P(\theta)$
4. $t_P(\eta\theta) \geq t_P(\eta) \vee t_P(\theta)$
5. $i_P(\eta\theta) \geq i_P(\eta) \vee i_P(\theta)$
6. $f_P(\eta\theta) \leq f_P(\eta) \wedge f_P(\theta)$

**Remark 2.4.** We denote the set of all neutrosophic ideals of $R$ by $U(R)$

## 3 Neutrosophic submodule

**Definition 3.1.** [8,9] A neutrosophic subset $P \in U^{M_R}$ is called a neutrosophic submodule of $M_R$ if

1. $t_P(0) = 1, i_P(0) = 1, f_P(0) = 0$
2. $t_P(\eta + \theta) \geq t_P(\eta) \wedge t_P(\theta)$
   $i_P(\eta + \theta) \geq i_P(\eta) \wedge i_P(\theta)$
   $f_P(\eta + \theta) \leq f_P(\eta) \vee f_P(\theta),\ for\ all \eta, \theta\ in\ M_R$
3. $t_P(\gamma\eta) \geq t_P(\eta)$
   $i_P(\gamma\eta) \geq i_P(\eta)$
   $f_P(\gamma\eta) \leq f_P(\eta), for\ all\ \eta\ in\ M_R,\ for\ all\ \gamma\ in\ R$

**Remark 3.1.** We denote neutrosophic submodules over $M_R$ using single valued neutrosophic set by $U(M)$.

**Remark 3.2.** If $P \in U(M)$, then the neutrosophic components of $P$ can be denoted as $(t_P(\eta), i_P(\eta), f_P(\eta))\ \forall\ \eta \in M_R$.

**Definition 3.2.** [8] A neutrosophic subset $\gamma P = \{\eta, t_{\gamma P}(\eta), i_{\gamma P}(\eta), f_{\gamma P}(\eta) : \eta \in M_R, \gamma \in R\}$ of $M_R$ where $P \in U^M$ defined as follows

$$t_{\gamma P}(\eta) = \vee\{t_P(\theta) : \theta \in M_R, \eta = \gamma\theta\}$$
$$i_{\gamma P}(\eta) = \vee\{i_P(\theta) : \theta \in M_R, \eta = \gamma\theta\}$$
$$f_{\gamma P}(\eta) = \wedge\{f_P(\theta) : \theta \in M_R, \eta = \gamma\theta\}$$

**Proposition 3.1.** Let $P = \{\eta, t_P(\eta), i_P(\eta), f_P(\eta); \eta \in M_R\} \in U^{M_R}$, then $t_{\gamma P}(\gamma\eta) \geq t_P(\eta)$, $i_{\gamma P}(\gamma\eta) \geq i_P(\eta)$ and $f_{\gamma P}(\gamma\eta) \leq f_P(\eta)$.

*Proof.* We have

$$t_{\gamma P}(\gamma\eta) = \vee\{t_P(\theta) : \theta \in M_R, \gamma\eta = \gamma\theta\} \geq t_P(\eta), \forall\ \eta \in M_R$$

Similarly $i_{\gamma P}(\gamma\eta) \geq i_P(\eta)$ . Also

$$f_{\gamma P}(\gamma\eta) = \wedge\{f_P(\theta) : \theta \in M_R, \gamma\eta = \gamma\theta\} \leq f_P(\eta), \forall\ \eta \in M_R$$

□

**Definition 3.3.** [8] Let $P = \{\eta, t_P(\eta), i_P(\eta), f_P(\eta); \eta \in M_R\} \in U^{M_R}$, then

$$-P = \{\eta, t_{-P}(\eta), i_{-P}(\eta), f_{-P}(\eta); \eta \in M_R\} \in U^{M_R}$$

where

$$t_{-P}(\eta) = t_P(-\eta),\ i_{-P}(\eta) = i_P(-\eta),\ f_{-P}(\eta) = f_P(-\eta),\ \forall\ \eta \in M_R$$

**Proposition 3.2.** [8] If $P = \{\eta, t_P(\eta), i_P(\eta), f_P(\eta); \eta \in M_R\} \in U^{M_R}$, then $1.P = P$ and $(-1)P = -P$

**Theorem 3.1.** *[8] Let $P \in U^{M_R}$, then $P \in U(M)$ if and only if the following properties are satisfied $\forall\ \eta, \theta \in M_R,\ \gamma, \beta \in R$*

$$i)\ \ t_P(0) = 1, i_P(0) = 1, f_P(0) = 0$$

$$ii)\ \ t_P(\gamma\eta + \beta\theta) \geq t_P(\eta) \wedge t_P(\theta),\ i_P(\gamma\eta + \beta\theta) \geq i_P(\eta) \wedge i_P(\theta),\ f_P(\gamma\eta + \beta\theta) \leq f_P(\eta) \vee f_P(\theta)$$

**Theorem 3.2.** *Let $P \in U(M)$. Then $P^*$ is a neutrosophic submodule of $M_R$.*

*Proof.* Given $P \in U(M)$ and $P^* = \{\eta \in M_R, t_P(\eta) > 0, i_P(\eta) > 0, f_P(\eta) < 1\}$. Let $\eta, \theta \in P^*$. Then

$$t_P(\eta) > 0, i_P(\eta) > 0, f_P(\eta) < 1$$

$$t_P(\theta) > 0, i_P(\theta) > 0, f_P(\theta) < 1$$

To prove that $\gamma\eta + \beta\theta \in P^*$ *where* $\gamma, \beta \in R$
$\Rightarrow$ to prove that $t_P(\gamma\eta + \beta\theta) > 0,\ i_P(\gamma\eta + \beta\theta) > 0,\ f_P(\gamma\eta + \beta\theta) < 1$
Now

$$\begin{aligned} t_P(\gamma\eta + \beta\theta) &\geq t_P(\gamma\eta) \wedge t_P(\beta\theta) \\ &\geq t_P(\eta) \wedge t_P(\theta) \\ &> 0 \end{aligned}$$

In the same way, we can prove the other two inequalities. Hence the proof.

□

**Definition 3.4.** Let $P_i,\ i \in J$ be an arbitrary non empty family of $U^{M_R}$, then

1. $\bigcap_{i \in J} P_i = \{\eta, t_{\bigcap_{i \in J} P_i}(\eta), i_{\bigcap_{i \in J} P_i}(\eta), f_{\bigcap_{i \in J} P_i}(\eta) : \eta \in M_R\}$ where

$$t_{\bigcap_{i \in J} P_i}(\eta) = \bigwedge_{i \in J} t_{P_i}(\eta)$$

$$i_{\bigcap_{i \in J} P_i}(\eta) = \bigwedge_{i \in J} i_{P_i}(\eta)$$

$$f_{\bigcap_{i \in J} P_i}(\eta) = \bigvee_{i \in J} f_{P_i}(\eta)$$

2. $\bigcup_{i\in J} P_i = \{\eta, t_{\bigcup_{i\in J} P_i}(\eta), i_{\bigcup_{i\in J} P_i}(\eta), f_{\bigcup_{i\in J} P_i}(\eta) : \eta \in M_R\}$ where

$$t_{\bigcup_{i\in J} P_i}(\eta) = \bigvee_{i\in J} t_{P_i}(\eta)$$

$$i_{\bigcup_{i\in J} P_i}(\eta) = \bigvee_{i\in J} i_{P_i}(\eta)$$

$$f_{\bigcup_{i\in J} P_i}(\eta) = \bigwedge_{i\in J} f_{P_i}(\eta)$$

**Proposition 3.3.** Let $P_i,\ i \in J$ be an arbitrary non empty family of $U^{M_R}$, then $\gamma(\bigcup_{i\in J} P_i) = \bigcup_{i\in J}(\gamma P_i)\ for\ \gamma \in R$

*Proof.* Consider $\gamma \bigcup_{i\in J} P_i = \{\eta, t_{\gamma\bigcup_{i\in J} P_i}(\eta), i_{\gamma\bigcup_{i\in J} P_i}(\eta), f_{\gamma\bigcup_{i\in J} P_i}(\eta) : \eta \in M_R,\ \gamma \in R\}$
Now

$$\begin{aligned} t_{\gamma\bigcup_{i\in J} P_i}(\eta) &= \vee\{t_{\bigcup_{i\in J} P_i}(\theta) : \theta \in M_R, \eta = \gamma\theta\} \\ &= \vee\{\bigvee_{i\in J} t_{P_i}(\theta) : \theta \in M_R, \eta = \gamma\theta\} \\ &= \bigvee_{i\in J} t_{\gamma P_i}(\eta) \\ &= t_{\bigcup_{i\in J} \gamma P_i}(\eta) \end{aligned}$$

Similarly $i_{\gamma\bigcup_{i\in J} P_i}(\eta) = i_{\bigcup_{i\in J} \gamma P_i}(\eta)$
Now

$$\begin{aligned} f_{\gamma\bigcup_{i\in J} P_i}(\eta) &= \wedge\{f_{\bigcup_{i\in J} P_i}(\theta) : \theta \in M_R, \eta = \gamma\theta\} \\ &= \wedge\{\bigwedge_{i\in J} f_{P_i}(\theta) : \theta \in M_R, \eta = \gamma\theta\} \\ &= \bigwedge_{i\in J} f_{\gamma P_i}(\eta) \\ &= f_{\bigcup_{i\in J} \gamma P_i}(\eta) \end{aligned}$$

Hence $\gamma(\bigcup_{i\in J} P_i) = \bigcup_{i\in J}(\gamma P_i)\ for\ \gamma \in R$ □

**Theorem 3.3.** *Let $P_i,\ i \in J$ be an arbitrary non empty family of $U(M)$, then $\bigcap_{i\in J} P_i \in U(M)$*

*Proof.* We have $\bigcap_{i\in J} P_i = \{\eta, t_{\bigcap_{i\in J} P_i}(\eta), i_{\bigcap_{i\in J} P_i}(\eta), f_{\bigcap_{i\in J} P_i}(\eta) : \eta \in M_R\}$ and

$$t_{\bigcap_{i\in J} P_i}(0) = \bigwedge_{i\in J} t_{P_i}(0) = 1$$

$$i_{\bigcap_{i\in J} P_i}(0) = \bigwedge_{i\in J} i_{P_i}(0) = 1$$

$$f_{\bigcap_{i\in J} P_i}(0) = \bigvee_{i\in J} f_{P_i}(0) = 0$$

Now

$$\begin{aligned} t_{\bigcap_{i\in J} P_i}(\gamma\eta + \beta\theta) &= \bigwedge_{i\in J} t_{P_i}(\gamma\eta + \beta\theta) \\ &\geq \bigwedge_{i\in J} (t_{P_i}(\eta) \wedge t_{P_i}(\theta)) \\ &= [\bigwedge_{i\in J} t_{P_i}(\eta)] \wedge [\bigwedge_{i\in J} t_{P_i}(\theta)] \\ &= t_{\bigcap_{i\in J} P_i}(\eta) \wedge t_{\bigcap_{i\in J} P_i}(\theta) \end{aligned}$$

in the same way we can derive

$$i_{\bigcap_{i\in J} P_i}(\eta + \theta) \geq i_{\bigcap_{i\in J} P_i}(\eta) \wedge i_{\bigcap_{i\in J} P_i}(\theta)$$
$$f_{\bigcap_{i\in J} P_i}(\eta + \theta) \leq f_{\bigcap_{i\in J} P_i}(\eta) \vee f_{\bigcap_{i\in J} P_i}(\theta)$$

Hence $\bigcap_{i\in J} P_i \in U(M)$ □

**Definition 3.5.** [20] Let $P,Q \in U^{M_R}$, then the sum

$$P+Q = \{\eta, t_{P+Q}(\eta), t_{P+Q}(\eta), t_{P+Q}(\eta) : \eta \in M_R\} \in U^{M_R}$$

defined as follows

$$t_{P+Q}(\eta) = \vee\{t_P(\theta) \wedge t_Q(\vartheta) | \eta = \theta + \vartheta, \theta, \vartheta \in M_R\}$$
$$i_{P+Q}(\eta) = \vee\{i_P(\theta) \wedge i_Q(\vartheta) | \eta = \theta + \vartheta, \theta, \vartheta \in M_R\}$$
$$f_{P+Q}(\eta) = \wedge\{f_P(\theta) \vee f_B(\vartheta) | \eta = \theta + \vartheta, \theta, \vartheta \in M_R\}$$

**Definition 3.6.** Let $P_i,\ i \in J$ be an arbitrary family of $U(M)$ where $P_i = \{\eta, t_{P_i}(\eta), i_{P_i}(\eta), f_{P_i}(\eta) : \eta \in M\}$ for each $i \in J$. Then

$$\sum_{i\in J} P_i = \{\eta, t_{\sum\limits_{i\in J} P_i}(\eta), i_{\sum\limits_{i\in J} P_i}(\eta), f_{\sum\limits_{i\in J} P_i}(\eta) : \eta \in M_R\}$$

where

$$t_{\sum\limits_{i\in J} P_i}(\eta) = \vee\{\bigwedge_{i\in J} t_{P_i}(\eta_i) : \eta_i \in M_R, \sum_{i\in J} \eta_i = \eta\}\ \forall \eta \in M_R$$
$$i_{\sum\limits_{i\in J} P_i}(x) = \vee\{\bigwedge_{i\in J} i_{P_i}(M_i) : \eta_i \in M_R, \sum_{i\in J} \eta_i = \eta\}\ \forall \eta \in M_R$$
$$f_{\sum\limits_{i\in J} P_i}(x) = \wedge\{\bigvee_{i\in J} f_{P_i}(\eta_i) : \eta_i \in M_R, \sum_{i\in J} \eta_i = \eta\}\ \forall \eta \in M_R$$

where, in $\sum\limits_{i\in J} \eta_i$, at most finitely $\eta_i \neq 0$.

**Theorem 3.4.** *If $P,Q \in U(M)$, then $P+Q \in U(M)$*

*Proof.* It is enough to prove $P+Q$ satisfies the properties listed below $\forall \eta, \theta \in M_R, \gamma, \beta \in R$

1. $t_{P+Q}(0) = 1, i_{P+Q}(0) = 1, f_{P+Q}(0) = 0.$
2. $t_{A+B}(\gamma\eta + \beta\theta) \geq t_{P+Q}(\eta) \wedge t_{P+Q}(\theta),\ i_{P+Q}(\gamma\eta + \beta\theta) \geq i_{P+Q}(\eta) \wedge i_{P+Q}(\theta),$
   $f_{A+B}(\gamma\eta + \beta\theta) \leq f_{P+Q}(\eta) \vee f_{P+Q}(\theta)$

From the definition 3.5, property 1 is obvious because $P,Q \in U(M)$.
Consider

$$\begin{aligned}
t_{P+Q}(\eta) \wedge t_{P+Q}(\theta) &= \bigvee\{t_P(\eta_1) \wedge t_Q(\eta_2) : \eta = \eta_1 + \eta_2\} \wedge \bigvee\{t_P(\theta_1) \wedge t_Q(\theta_2) : \theta = \theta_1 + \theta_2\} \\
&\leq \bigvee\{t_P(\gamma\eta_1) \wedge t_Q(\gamma\eta_2) : \gamma\eta = \gamma\eta_1 + \gamma\eta_2\} \wedge \bigvee\{t_P(\beta\theta_1) \wedge t_Q(\beta\theta_2) : \beta\theta = \beta\theta_1 + \beta\theta_2\} \\
&= \bigvee\{[t_P(\gamma\eta_1) \wedge t_P(\beta\theta_1)] \wedge [t_Q(\gamma\eta_2) \wedge t_Q(\beta\theta_2)] : \gamma\eta = \gamma\eta_1 + \gamma\eta_2, \beta\theta = \beta\theta_1 + \beta\theta_2\} \\
&\leq \bigvee\{t_P(\gamma\eta_1 + \beta\theta_1) \wedge t_Q(\gamma\eta_2 + \beta\theta_2) : \gamma\eta + \beta\theta = \gamma\eta_1 + \beta\theta_1 + \gamma\eta_2 + \beta\theta_2\} \\
&\leq t_{P+Q}(\gamma\eta + \beta\theta)\ \textit{where}\ \gamma\eta + \beta\theta = \gamma(\eta_1 + \eta_2) + \beta(\theta_1 + \theta_2)\ \forall\ \eta_1, \eta_2, \theta_1, \theta_2 \in M_R
\end{aligned}$$

Similarly, $i_{P+Q}(\gamma\eta + \beta\theta) \geq i_{P+Q}(\eta) \wedge i_{P+Q}(\theta)$, $f_{P+Q}(\gamma\eta + \beta\theta) \leq f_{P+Q}(\eta) \wedge f_{P+Q}(\theta)$
$\Rightarrow P+ \in U(M)$. □

**Corollary 3.4.1.** Let $P_i,\ i \in J$ be a family of neutrosophic submodules of an $M_R$. Then $\sum\limits_{i\in J} P_i \in U(M)$.

**Definition 3.7.** For any $\eta \in X$, the neutrosophic point $\hat{N}_{\{\eta\}}$ is defined as

$$\hat{N}_{\{\eta\}}(s) = \{(s, t_{\hat{N}_{\{\eta\}}}, i_{\hat{N}_{\{\eta\}}}, f_{\hat{N}_{\{\eta\}}}) : s \in X\}$$

where

$$\hat{N}_{\{\eta\}}(s) = \begin{cases} (1,1,0) & \eta = s \\ (0,0,1) & \eta \neq s \end{cases}$$

**Remark 3.3.** Let $X$ be a non empty set. The neutrosophic point $\hat{N}_{\{0\}}$ in $X$ is defined as $\hat{N}_{\{0\}}(x) = \{(x, t_{\hat{N}_{\{0\}}}, i_{\hat{N}_{\{0\}}}, f_{\hat{N}_{\{0\}}}) : x \in X\}$ where

$$\hat{N}_{\{0\}}(x) = \begin{cases} (1,1,0) & x = 0 \\ (0,0,1) & x \neq 0 \end{cases}$$

**Theorem 3.5.** *Let $P \in U(M)$. $P = \hat{N}_{\{0\}} \Leftrightarrow P^* = \{0\}$*

*Proof.* If $P = \hat{N}_{\{0\}}$, and $P^* = \{\eta \in M_R, t_P(\eta) > 0, i_P(\eta) > 0, f_P(\eta) < 1\} = \{0\}$.
Conversely, if $P^* = \{0\} \Rightarrow t_P(0) > 0, i_P(0) > 0, f_P(\eta) < 1$ and $t_P(\eta) = 0,\ i_P(\eta) = 0$ *and* $f_P(\eta) = 1\ \forall\ \eta \neq 0$.
Therefore

$$P(\eta) = \begin{cases} (1,1,0) & \eta = 0 \\ (0,0,1) & \eta \neq 0 \end{cases} = \hat{N}_{\{0\}}$$

. □

## 4 Neutrosophic Submodule Generated by Neutrosophic Set

In this section we study about the $U(M)$ of $M_R$ generated by single valued neutrosophic set defined over a classical module .

**Definition 4.1.** Let $p = \{\eta, t_P(\eta), i_P(\eta), f_P(\eta) : \eta \in M_R\} \in U^M$. Then the $U(M)$ of $M_R$ generated by neutrosophic set $P$ can be denoted and defined as

$$\langle P \rangle = \cap \{Q | P \subseteq Q : Q \in U(M)\}$$

**Remark 4.1.** If $Q = \langle P \rangle$, then $P$ is called generator of $Q$.

**Theorem 4.1.** *Let $P_i = \{(\eta, t_{P_i}(\eta), i_{P_i}(\eta), f_{P_i}(\eta) : i \in J,\ \eta \in M_R\}$ be an arbitrary non empty family of $NS(M_R)$. Then $\langle \cup_{i \in J} Pi \rangle = \sum_{i \in J} P_i$*

*Proof.* By a corollary 3.4.1,we can write

$$\sum_{i \in J} P_i = \{\eta, t_{\sum\limits_{i \in J} P_i}(\eta), i_{\sum\limits_{i \in J} P_i}(\eta), f_{\sum\limits_{i \in J} P_i}(\eta) : \eta \in M_R\} \in U(M)$$

where, *for all* $\eta$ *in* $M_R$

$$t_{\sum\limits_{i \in J} P_i}(\eta) = \vee \{\bigwedge_{i \in J} t_{P_i}(\eta_i) : \eta_i \in M_R, \sum_{i \in J} \eta_i = \eta\}$$

$$i_{\sum\limits_{i \in J} P_i}(\eta) = \vee \{\bigwedge_{i \in J} i_{P_i}(\eta_i) : \eta_i \in M_R, \sum_{i \in J} \eta_i = \eta\}$$

$$f_{\sum\limits_{i \in J} P_i}(\eta) = \wedge \{\bigvee_{i \in J} f_{P_i}(\eta_i) : \eta_i \in M_R, \sum_{i \in J} \eta_i = \eta\}$$

where, in $\sum_{i \in J} \eta_i$ finitely $\eta_i$'$s \neq 0$
So we can conclude *for all* $\eta$ *in* $M_R$

1. $t_{P_i}(\eta) \leq t_{\sum\limits_{i \in J} P_i}(\eta), \forall \eta \in M_R$

2. $i_{P_i}(\eta) \leq i_{\sum\limits_{i \in J} P_i}(\eta), \forall \eta \in M_R$

3. $f_{P_i}(\eta) \geq f_{\sum_{i\in J} P_i}(\eta), \forall \eta \in M_R$

Hence $P_i \subseteq \sum_{i\in J} P_i$, $\forall i \in J$.

Now to prove that $\sum_{i\in J} P_i$ is the least neutrosophic submodule and $\sum_{i\in J} P_i$ contains all $P_i^{'}s$. Let $Q = \{\eta, t_Q(\eta), i_Q(\eta), f_Q(\eta) : \eta \in M_R\} \in U(M)$ and $P_i \subseteq Q, \forall i \in J$, which means that

$$t_{P_i}(\eta) \leq t_Q(\eta),\ i_{P_i}(\eta) \leq i_Q(\eta),\ f_{P_i}(\eta) \geq f_Q(\eta)\ \forall\ i \in J$$

.

Let $\eta \in M_R$ where $\sum_{i\in J} \eta_i = \eta$ and only finitely $\eta_i' s \neq 0$, then

$$\begin{aligned} t_{\sum_{i\in J} P_i}(\eta) &= \vee\{\wedge_{i\in J} t_{P_i}(\eta_i) : \eta_i \in M_R, \sum_{i\in J} \eta_i = \eta\} \\ &\leq \vee\{\wedge_{i\in J} t_Q(\eta_i) : \eta_i \in M, \sum_{i\in J} \eta_i = \eta\} \\ &\leq \vee\{t_Q(\sum_{i\in J} \eta_i) : \eta_i \in M_R, \sum_{i\in J} \eta_i = \eta\} \\ &= t_Q(\eta) \end{aligned}$$

In the same way, $i_{\sum_{i\in J} P_i}(\eta) \leq i_Q(\eta), f_{\sum_{i\in J} P_i}(\eta) \geq f_Q(\eta)$.

$\Rightarrow \sum_{i\in J} P_i \subseteq Q$ . Hence $\sum_{i\in J} P_i \in U(M)$ is the smallest one and contains all $P_i' s$. Therefore $\sum_{i\in J} P_i$ is the smallest $U(M)$ which contains $\cup_{i\in J} P_i \subseteq \sum_{i\in J} P_i$. Hence $\langle \cup_{i\in J} P_i \rangle = \sum_{i\in J} P_i$ □

**Definition 4.2.** Let $C \in U(R)$ and $P \in NS(M_R)$ . Define the operations $C \circledcirc P$ and $C \circledast P$ as $NS(M_R)$ as follows

1. $C \circledcirc P\ (\eta) = (\eta, t_{C\circledcirc P}(\eta), i_{C\circledcirc P}(\eta), f_{C\circledcirc P}(\eta))\ \forall\ \eta \in M$ where

$$\begin{aligned} t_{C\circledcirc P}(\eta) &= \vee\{t_C(\gamma) \wedge t_P(\theta) : \gamma \in R, \theta \in M, \gamma\theta = \eta\} \\ i_{C\circledcirc P}(\eta) &= \vee\{i_C(\gamma) \wedge i_P(\theta) : \gamma \in R, \theta \in M, \gamma\theta = \eta\} \\ f_{C\circledcirc P}(\eta) &= \wedge\{f_C(\gamma) \vee f_P(\theta) : \gamma \in R, \theta \in M, \gamma\theta = \eta\} \end{aligned}$$

2. $C \circledast P\ (\eta) = (\eta, t_{C\circledast P}(\eta), i_{C\circledast P}(\eta), f_{C\circledast P}(\eta))\ \forall\ \eta \in M$ where

$$t_{C\circledast P}(\eta) = \vee\{\wedge_{i=1}^{n}(t_C(\gamma_i) \wedge t_P(\eta_i) : \gamma_i \in R, \eta_i \in M, \sum_{i=1}^{n} \gamma_i\eta_i = \eta, 1 \leq i \leq n, n \in N\}$$

$$i_{C\circledast A}(\eta) = \vee\{\wedge_{i=1}^{n}(i_C(\gamma_i) \wedge i_A(\eta_i) : \gamma_i \in \gamma, \eta_i \in M, \sum_{i=1}^{n} \gamma_i\eta_i = \eta, 1 \leq i \leq n, n \in N\}$$

$$f_{C\circledast A}(\eta) = \wedge\{\vee_{i=1}^{n}(f_C(\gamma_i) \vee f_A(\eta_i) : \gamma_i \in \gamma, \eta_i \in M, \sum_{i=1}^{n} \gamma_i\eta_i = \eta, 1 \leq i \leq n, n \in N\}$$

**Theorem 4.2.** *Let $P \in U^{M_R}$, then*

1. $\forall\ \gamma \in R,\ \hat{N}_{\{\gamma\}} \circledcirc P = \gamma P$

2. $\forall\ \gamma \in R,\ \eta \in M, \hat{N}_{\{\gamma\}} \circledast P(\eta) = \{\eta, t_{\hat{N}_{\{\gamma\}}\circledast P}(\eta), i_{\hat{N}_{\{\gamma\}}\circledast P}(\eta), f_{\hat{N}_{\{\gamma\}}\circledast P}(\eta)\}$ *where* $1 \leq i \leq n,\ n \in \mathbf{N}$

$$t_{\hat{N}_{\{\gamma\}}\circledast P}(\eta) = \vee\{\wedge_{i=1}^{n} t_P(\eta_i) : \eta_i \in M, \gamma\sum_{i=1}^{n} \eta_i = \eta\}$$

$$i_{\hat{N}_{\{\gamma\}}\circledast P}(\eta) = \vee\{\wedge_{i=1}^{n} i_P(\eta_i) : \eta_i \in M, \gamma\sum_{i=1}^{n} \eta_i = \eta\}$$

$$f_{\hat{N}_{\{\gamma\}}\circledast P}(\eta) = \wedge\{\vee_{i=1}^{n} f_P(\eta_i) : \eta_i \in M_R, \gamma\sum_{i=1}^{n} \eta_i = \eta\}$$

*Proof.* (1) The neutrosophic point $\hat{N}_{\{\gamma\}}$, for any $\gamma \in R$, is defined as $\hat{N}_{\{\gamma\}}(\varsigma) = \{(\varsigma, t_{\hat{N}_{\{\gamma\}}}, i_{\hat{N}_{\{\gamma\}}}, f_{\hat{N}_{\{\gamma\}}}) : \varsigma \in R\}$ where

$$\hat{N}_{\{\gamma\}}(\varsigma) = \begin{cases} (1,1,0) & \gamma = \varsigma \\ (0,0,1) & \gamma \neq \varsigma \end{cases}$$

Consider $\hat{N}_{\{\gamma\}} \circledcirc P(\eta) = \{(\eta, t_{\hat{N}_{\{\gamma\}} \circledcirc P}(\eta), i_{\hat{N}_{\{\gamma\}} \circledcirc P}(\eta), f_{\hat{N}_{\{\gamma\}} \circledcirc P}(\eta))\}\ \forall\ \eta \in M_R,\ \gamma \in R$, we have

$$\begin{aligned} t_{\hat{N}_{\{\gamma\}} \circledcirc P}(\eta) &= \vee\{t_{\hat{N}_{\{\gamma\}}}(\varsigma) \wedge t_P(\theta) : \varsigma \in R, \theta \in M_R, \varsigma\theta = \eta\} \\ &= \vee\{t_P(\theta) : \theta \in M, \gamma = \eta\} \\ &= t_{\gamma P}(\eta) \end{aligned}$$

Similarly we get, $i_{\hat{N}_{\{\gamma\}} \circledcirc P}(\eta) = i_{\gamma P}(\eta), f_{\hat{N}_{\{\gamma\}} \circledcirc P}(\eta) = f_{\gamma P}(\eta)$
(2) Now consider, for any $\gamma \in R, \eta \in M_R,\ 1 \leq i \leq n,\ n \in \mathbf{N}$

$$\begin{aligned} t_{\hat{N}_{\{\gamma\}} \circledast P}(\eta) &= \vee\{\wedge_{i=1}^{n}(t_{\hat{N}_{\{\gamma\}}}(\gamma_i) \wedge t_P(\eta_i) : r_i \in R, \eta_i \in M, \sum_{i=1}^{n} \gamma_i \eta_i = \eta\} \\ &= \vee\{\wedge_{i=1}^{n} t_P(\eta_i) : \eta_i \in M_R, \gamma \sum_{i=1}^{n} \eta_i = \eta\} \end{aligned}$$

Similarly we get

$$i_{\hat{N}_{\{\gamma\}} \circledast P}(\eta) = \vee\{\wedge_{i=1}^{n} i_P(\eta_i) : \eta_i \in M_R, \gamma \sum_{i=1}^{n} \eta_i = \eta\}$$

$$f_{\hat{N}_{\{\gamma\}} \circledast P}(\eta) = \wedge\{\vee_{i=1}^{n} f_P(\eta_i) : \eta_i \in M_R, \gamma \sum_{i=1}^{n} \eta_i = \eta\}$$

□

**Theorem 4.3.** *If $P \in U(R)$ and $Q \in U(M)$, then $P \circledast Q \in U(M)$*

*Proof.* From the definition of $P \circledast Q$, we can write, for all $1 \leq i \leq n,\ n \in \mathbf{N}$

$$\begin{aligned} t_{P \circledast Q}(0) &= \vee\{\wedge_{i=1}^{n}(t_P(\gamma_i) \wedge t_Q(\eta_i) : \gamma_i \in R, \eta_i \in M_R, \sum_{i=1}^{n} \gamma_i \eta_i = 0,\} \\ &= 1 \ when\ \gamma_i = \eta_i = 0\ \forall\ i,\ since\ t_P(0) \geq t_P(\gamma)\ \forall\ \gamma \in R \end{aligned}$$

Similarly $i_{P \circledast Q}(0) = 1$ *and* $f_{P \circledast Q}(0) = 0$.

Then prove that $t_{P\circledast Q}(\eta+\theta)\geq t_{P\circledast Q}(\eta)\wedge t_{P\circledast Q}(\theta)$ for $\eta,\theta\in M_R, 1\leq i\leq n, n\in\mathbf{N}$

$$\begin{aligned}
t_{P\circledast Q}(\eta+\theta) &= \vee\{\wedge_{i=1}^{n}(t_P(\gamma_i)\wedge t_Q(z_i):\gamma_i\in R, z_i\in M, \sum_{i=1}^{n}\gamma_i z_i=\eta+\theta\}\\
&\geq \vee\{\wedge_{i=1}^{n}(t_P(\varsigma_i)\wedge t_Q(\eta_i+\theta_i):\varsigma_i\in R,\ \eta_i,\theta_i\in M, \sum_{i=1}^{n}\varsigma_i(\eta_i+\theta_i)=\eta+\theta,\forall\ i\}\\
&\geq \vee\{\wedge_{i=1}^{n}(t_P(\varsigma_i)\wedge(t_Q(\eta_i)\wedge t_Q(\theta_i))):\varsigma_i\in R,\ \eta_i,\theta_i\in M, \sum_{i=1}^{n}\varsigma_i(\eta_i+\theta_i)=\eta+\theta,\forall\ i\}\\
&= \vee\{\wedge_{i=1}^{n}(t_P(\varsigma_i)\wedge(t_Q(\eta_i))\wedge(t_P(\varsigma_i))\wedge t_Q(\theta_i)):\\
&\quad \varsigma_i\in R,\ \eta_i,\ \theta_i\in M, \sum_{i=1}^{n}\varsigma_i(\eta_i+\theta_i)=\eta+\theta\}\\
&\geq \vee\{\wedge_{i=1}^{n}(t_P(\varsigma_i)\wedge t_Q(\eta_i)):\varsigma_i\in R,\eta_i\in M, \sum_{i=1}^{n}\varsigma_i\eta_i=\eta\}\wedge\\
&\quad \vee\{\wedge_{i=1}^{n}(t_P(\varsigma_i)\wedge t_Q(\theta_i)):\varsigma_i\in R,\theta_i\in M, \sum_{i=1}^{n}\varsigma_i\theta_i=\theta\}\\
&= t_{P\circledast Q}(\eta)\wedge t_{P\circledast Q}(\theta)
\end{aligned}$$

Similarly we can prove that $i_{P\circledast Q}(\eta+\theta)\geq i_{P\circledast Q}(\eta)\wedge i_{P\circledast Q}(\theta)\ \eta,\theta\in M$ and
$f_{P\circledast Q}(\eta+\theta)\leq f_{P\circledast Q}(\eta)\vee f_{P\odot Q}(\theta)\ \eta,\theta\in M_R$.
Now for all $1\leq i\leq n, n\in\mathbf{N}$

$$\begin{aligned}
t_{P\circledast Q}(\gamma\eta) &= \vee\{\wedge_{i=1}^{n}(t_P(\gamma_i)\wedge t_Q(\eta_i)):\gamma_i\in R,\eta_i\in M_R, \sum_{i=1}^{n}\gamma_i\eta_i=\gamma\eta\}\\
&\geq \vee\{\wedge_{i=1}^{n}(t_P(\gamma\varsigma_i)\wedge t_Q(\theta_i)):\varsigma_i\in R,\theta_i\in M_R, \gamma\sum_{i=1}^{n}\varsigma_i\theta_i=\gamma\eta\}[\,when\ \gamma=1]\\
&\geq \vee\{\wedge_{i=1}^{n}(t_P(\varsigma_i)\wedge t_Q(\theta_i)):\varsigma_i\in R,\theta_i\in M_R, \sum_{i=1}^{n}\varsigma_i\theta_i=\eta, 1\leq i\leq n, n\in N\}\\
&\quad [\,since\ P\in U(R)\Rightarrow t_P(\gamma\varsigma_i)\geq t_P(\gamma)\vee t_P(\varsigma_i)\geq t_P(\varsigma_i)\,]\\
&= t_{P\circledast Q}(\eta)
\end{aligned}$$

Similarly, $i_{P\circledast Q}(r\eta)\geq i_{P\circledast Q}(\eta)$ *and* $f_{P\circledast Q}(r\eta)\leq f_{P\circledast Q}(\eta)$.
Hence $P\circledast Q\in U(M)$. □

**Theorem 4.4.** *Let $P\in U^{M_R}$ and corresponding to P, define $Q\in U^{M_R}$ such that $Q=\{\eta, t_Q(\eta), i_Q(\eta), f_Q(\eta):\eta\in M_R\}$ where*

$$t_Q(\eta)=\begin{cases}1 & \eta=0\\ \vee\{\wedge_{i=1}^{n}t_P(\eta_i):\sum_{i=1}^{n}\gamma_i\eta_i=\eta,\eta_i\in M_R,\gamma_i\in R\} & otherwise\end{cases}$$

$$i_Q(\eta)=\begin{cases}1 & \eta=0\\ \vee\{\wedge_{i=1}^{n}i_P(\eta_i):\sum_{i=1}^{n}\gamma_i\eta_i=\eta,\eta_i\in M,\gamma_i\in R\} & otherwise\end{cases}$$

$$f_Q(\eta)=\begin{cases}0 & \eta=0\\ \wedge\{\vee_{i=1}^{n}f_P(\eta_i):\sum_{i=1}^{n}\gamma_i\eta_i=\eta,\eta_i\in M_R,\gamma_i\in R\} & otherwise\end{cases}$$

*where $1\leq i\leq n,\ n\in N$. Then $Q\in U(M)$ and $\langle P\rangle=Q$.*

*Proof.* From the definition of $Q$, $t_P(\eta) \leq t_Q(\eta),\ i_P(\eta) \leq i_Q(\eta)\ and\ f_P(\eta) \geq f_Q(\eta)\ \forall \eta \in M_R,\ then\ P \subseteq Q$ .
We know $t_Q(0) = 1, i_Q(0) = 1\ and\ f_Q(0) = 0$. Let $\gamma \in R, \eta \in M_R$.
If $\gamma\eta = 0$ then

$$t_Q(\gamma\eta) = 1 \geq t_Q(\eta),\ i_Q(\gamma\eta) = 1 \geq i_Q(\eta)\ and\ f_Q(\gamma\eta) = 0 \leq f_Q(\eta)$$

Suppose $\gamma\eta \neq 0$ then $\eta \neq 0\ and\ \forall\ 1 \leq i \leq n, n \in \mathbf{N}$

$$\begin{aligned}
t_Q(\gamma\eta) &= \vee\{\wedge_{i=1}^{n} t_P(\eta_i) : \sum_{i=1}^{n} \gamma_i \eta_i = \gamma\eta, \eta_i \in M_R, \gamma_i \in R,\ \} \\
&\geq \vee\{\wedge_{i=1}^{n} t_P(\eta_i) : \sum_{i=1}^{n} \gamma\varsigma_i \eta_i = \gamma\eta, \eta_i \in M_R, \varsigma_i \in R\} \\
&\geq \vee\{\wedge_{i=1}^{n} t_P(\eta_i) : \gamma\sum_{i=1}^{n} \varsigma_i \eta_i = \gamma\eta, \eta_i \in M_R, \varsigma_i \in R\} \\
&\geq \vee\{\wedge_{i=1}^{n} t_P(\eta_i) : \sum_{i=1}^{n} \varsigma_i \eta_i = \eta, \eta_i \in M_R, \varsigma_i \in R\} \\
&\quad (when \gamma = 1) \\
&= t_Q(\eta)
\end{aligned}$$

In the same way we can show that $i_Q(\gamma\eta) \geq i_Q(\eta)$ and $f_Q(\gamma\eta) \leq f_Q(\eta)$
Suppose $\eta, \theta\ and\ \eta + \theta \neq 0, \forall\ 1 \leq i \leq n, n \in \mathbf{N}$, then

$$\begin{aligned}
t_Q(\eta+\theta) &= \vee\{\wedge_{i=1}^{n} t_A(z_i) : \sum_{i=1}^{n} \gamma_i z_i = \eta + \theta, z_i \in M_R, \gamma_i \in R\} \\
&\geq \vee\{\wedge_{i=1}^{n} t_P(z_i) : z_i = \eta_i + \theta_i, \sum_{i=1}^{n} \gamma_i(\eta_i + \theta_i) = \eta + \theta, \eta_i, \theta_i \in M, \gamma_i \in R\} \\
&\geq \vee\{(\wedge_{i=1}^{n} t_P(\eta_i)) \wedge (\wedge_{i=1}^{n} t_P(\theta_i)) : z_i = \eta_i + \theta_i, \sum_{i=1}^{n} \gamma_i \eta_i + \sum_{i=1}^{n} \gamma_i \theta_i = \eta + \theta, \\
&\quad \eta_i, \theta_i \in M_R, \gamma_i \in R\} \\
&\geq \vee\{(\wedge_{i=1}^{n} t_P(\eta_i) : \sum_{i=1}^{n} \gamma_i \eta_i = \eta, \eta_i \in M_R, \gamma_i \in R\} \wedge \\
&\quad \vee\{(\wedge_{i=1}^{n} t_P(\theta_i) : \sum_{i=1}^{n} \gamma_i \theta_i = \theta, \theta_i \in M, \gamma_i \in R\} \\
&= t_Q(\eta) \wedge t_Q(\theta)
\end{aligned}$$

Similarly, $i_Q(\eta+\theta) \geq i_Q(\eta) \wedge i_Q(\theta)$ and $f_Q(\eta+\theta) \leq f_Q(\eta) \vee f_Q(\theta)$.
$\Rightarrow\ Q \in U(M)$ and $P \subseteq Q$.
Now consider $S = \{\eta, t_S(\eta), i_S(\eta), f_S(\eta) : \eta \in M_R\} \in U(M)$ and which contains $P$. Now to prove that $Q \subseteq S$.
From the assumption, $P \subseteq S$,
$\Rightarrow$

$$t_P(\eta) \leq t_S(\eta),\ i_P(\eta) \leq i_S(\eta)\ Pnd\ f_P(\eta) \geq f_S(\eta).$$

Now for all $1 \leq i \leq n,\ n \in \mathbf{N}$

$$
\begin{aligned}
t_Q(\eta) &= \vee\{\wedge_{i=1}^{n} t_P(\eta_i) : \sum_{i=1}^{n} \gamma_i \eta_i = \eta, \eta_i \in M_R, \gamma_i \in R\} \\
&\leq \vee\{\wedge_{i=1}^{n} t_S(\eta_i) : \sum_{i=1}^{n} \gamma_i \eta_i = \eta, \eta_i \in M_R, \gamma_i \in R\} \\
&\quad [\text{we know } t_S(\eta) = t_S(\sum_{i=1}^{n} \gamma_i \eta_i) \geq \wedge_{i=1}^{n} t_S(\gamma_i \eta_i) \geq \wedge_{i=1}^{n} t_S(\eta_i) \Rightarrow t_S(\eta) \leq \vee(\wedge_{i=1}^{n} t_S(\eta_i))] \\
&\leq t_S(\eta)
\end{aligned}
$$

We can derive in the same pattern, $i_Q(\eta) \leq i_S(\eta)$ and $f_Q(\eta) \geq f_S(\eta)$. $\Rightarrow Q \subseteq C$. Thus we can conclude $\langle P \rangle = Q$. □

## 5 Conclusion

Neutrosophic submodule is one of the generalizations of a classical algebraic structure, module. The study of neutrosophic submodule give extra promptitude to the classic algebraic structures rather than fuzzy or intuitionistic fuzzy sets because of the investigation of three different level graded functions of each element in $[0,1]$. This paper has developed a method to identify generator of $U(M)$ and derived algebraic results with the help of some algebraic operators as neutrosophic sets. This work are often extended to the generators of arbitrary nonempty family of neutrosophic submodules and structure preserving properties like isomorphism of neutrosophic submodules. Neutrosophic submodules provide us a solid mathematical foundation to clarify connected scientific ideas in image processing, control theory and economic science.

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