Some interval-valued neutrosophic hesitant fuzzy uncertain linguistic Bonferroni mean aggregation operators and their application in multiple attribute decision making

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Abstract: Interval-valued neutrosophic hesitant fuzzy uncertain linguistic set (IVNHFULS) has the advantages of both neutrosophic numbers and interval-valued hesitant fuzzy uncertain linguistic variable. In this paper, we firstly introduce the definition, the operational laws, and the score function of IVNHFULS. Then, we combine interval-valued neutrosophic hesitant fuzzy uncertain linguistic set with Bonferroni mean operator and propose some new aggregation operators, such as the interval-valued neutrosophic hesitant fuzzy uncertain linguistic Bonferroni mean (IVNHFULBM) operator, the interval-valued neutrosophic hesitant fuzzy uncertain linguistic weighted Bonferroni mean (IVNHFULWBM) operator, the interval-valued neutrosophic hesitant fuzzy uncertain linguistic geometric Bonferroni mean (IVNHFULGBM) operator, the interval-valued neutrosophic hesitant fuzzy uncertain linguistic weighted geometric Bonferroni mean (IVNHFULWGBM) operator. At the same time, the related properties of these operators are discussed. Furthermore, we propose two multiple attribute decision making methods based on IVNHFULWBM operator and IVNHFULWGBM operator. Finally, we give an illustrative example to demonstrate the practicality and effectiveness of the proposed methods.

Keywords: Multiple attribute decision making; Interval-valued neutrosophic hesitant fuzzy uncertain linguistic set; Bonferroni mean; Interval-valued neutrosophic hesitant fuzzy uncertain linguistic Bonferroni mean aggregation operator.

1. Introduction

Multiple attribute decision making (MADM) has been a hot research topic. In the real decision making, because of the complexity and the uncertainty of decision making problems, how to realistically express the fuzzy and uncertain information is very important. Zadeh [1] firstly proposed the Fuzzy set (FS), which has been successfully applied in various fields. Due to the FS only consider the membership degree, Atanassov [2] further developed the intuitionistic fuzzy set (IFS) based on the FS. In recent years, Smarandache [3] proposed the neutrosophic set (NS) which consisted of membership degree, non-membership degree and indeterminacy -membership degree. In addition, some subsets of neutrosophic set have been proposed. For example, Wang et al. [4][5] further proposed the interval-valued neutrosophic set (IVNS) and the single-valued neutrosophic set (SVNS). Due to the membership degree of FS, the membership degree and non-membership degree of IFS, the membership degree and non-membership degree and indeterminacy-membership degree of NS are a single value ranged from 0 to 1. As we all known, the single values are asymmetric and not comprehensive so that they inappropriately describe the incomplete information. Based on this, Torra and Narukawa [6-7] introduced the hesitant fuzzy sets (HFS). In HFS, the membership degree is a set of several possible values between 0 and 1. The obvious difference between FS and HFS is that the membership degree is a single value or a set of possible values ranged from 0 to 1. When the decision makers evaluate the alternatives, HFS is a useful tool to describe the evaluation values with hesitance. Chen et al. [8] extended HFS to interval-valued hesitant fuzzy sets (IVHFS), in which the membership degree is a set of possible interval values between 0 and 1. Ye [9] presented a neutrosophic hesitant fuzzy set (NHFS), which is the combination HFS with SVNS. In real decision making, especially for qualitative information, the linguistic variables are suitable to express the evaluation values and it can improve the flexibility and reliability of the decision making information. So far, the linguistic variables have been widely studied [10-12], which could be expressed as single values [13] or interval values [14]. The interval linguistic terms are also known as uncertain linguistic variables. Rodriguez et al. [15-16] originally gave the hesitant fuzzy linguistic term sets (HFTLS) which is used to describe the linguistic variables. Lin et al. [17] developed the hesitant fuzzy linguistic sets (HFLS) and the hesitant fuzzy uncertain linguistic set (HFULS) on the basis of HFS and linguistic variables or uncertain linguistic variables. The characteristics of the HFLS and HFULS are that the membership degree can be expressed as a set of hesitant fuzzy linguistic variables or hesitant fuzzy uncertain linguistic variables. Similar to the HFS, HFLS doesn't take the non-membership degree, indeterminacy-membership degree into account so that HFLS has the weaknesses in avoiding the information distortion and losing effectively in handling the indeterminate, incomplete and inconsistent problems. In order to overcome the deficiencies, in this paper, we extend NHFS to interval-valued neutrosophic hesitant fuzzy set (IVNHFS) and cobmine IVNHFS with uncertain linguistic variables. Consequently, we finally give the concept of interval-valued neutrosophic hesitant fuzzy uncertain linguistic set (IVNHFULS), which is composed of two main parts: an uncertain linguistic term and an interval-valued neutrosophic hesitant fuzzy element. The two parts respectively express an evaluation value and the hesitancy for the given evaluation value. Certainly, the second part also represents the interval-valued membership degree, interval-valued non-membership degree and interval-valued indeterminacy-membership degree associated with the specific uncertain linguistic term.

Up to now the information aggregation operators are studied deeply, and they have been widely used in various field. All sorts of operators have been proposed to deal with many different situations. For example, Xu [18-19] developed the uncertain linguistic weighted averaging (ULWA) operator, the uncertain linguistic ordered weighted averaging (ULOWA) operator, the uncertain linguistic hybrid averaging (ULHA) operator, the uncertain linguistic geometric averaging (ULGA) operator, uncertain linguistic weighted geometric averaging (ULWGA) operator, uncertain linguistic ordered weighted geometric averaging (ULOWGA) operator, and induced uncertain linguistic ordered weighted geometric averaging (IULOWGA) operator under uncertain linguistic environment. Wei [20] proposed an uncertain linguistic hybrid geometric averaging (ULHGA) operator. In order to more accurately and flexibly deal with the hesitant fuzzy information, Lin et al. [17] developed some hesitant fuzzy uncertain linguistic arithmetic aggregation operators, such as hesitant fuzzy uncertain linguistic weighted averaging (HFULWA) operator, hesitant fuzzy uncertain linguistic ordered weighted averaging (HFULOWA) operator, hesitant fuzzy uncertain linguistic hybrid averaging (HFULHA) operator. Ju and Liu [21] gave the interval-valued hesitant uncertain linguistic weighted averaging (IVHULWA) operator, the interval-valued hesitant uncertain linguistic weighted geometric averaging (IVHULWGA) operator. Liu et al. [22] proposed some interval-valued hesitant uncertain linguistic generalized aggregation operators, including the generalized interval-valued hesitant uncertain linguistic weighted averaging (GIVHULWA) operator, the generalized interval-valued hesitant uncertain linguistic ordered weighted averaging (GIVHULOWA) operator and the generalized interval-valued hesitant uncertain linguistic hybrid averaging (GIVHULHA) operator. In order to better handle the interrelationship between input arguments, Bonferroni [23] originally introduced the Bonferroni Mean (BM) operator. The extensions of the BM operator have been constantly proposed since it was initially introduced. Yager [24][25] and Beliakov et al. [26] proposed some generalized BM operators. Zhu et al.[27] combined BM operator with geometric operator and finally proposed the geometric Bonferroni mean (GBM) operator. In this paper, we will extend the BM operator to the interval-valued neutrosophic hesitant fuzzy uncertain linguistic environment for the sake of possessing the advantages of both BM operator and IVNHFULS, and flexibly handling the indeterminate, incomplete, inconsistent information and effectively considering the interrelationship of the interval-valued neutrosophic hesitant fuzzy uncertain linguistic information.

The remainder of this paper is shown as follows. In section 2, we mainly introduce the basic concepts of the interval-valued neutrosophic hesitant fuzzy uncertain linguistic set (IVNHFULS), and the operational rules and the characteristics of IVNHFULS. In section 3, we propose some interval-valued neutrosophic hesitant fuzzy uncertain linguistic BM aggregation operators, such as the interval-valued neutrosophic hesitant fuzzy uncertain linguistic Bonferroni mean (IVNHFULBM) operator, the interval-valued neutrosophic hesitant fuzzy uncertain linguistic weighted Bonferroni mean (IVNHFULWBM) operator, the interval-valued neutrosophic hesitant fuzzy uncertain linguistic geometric Bonferroni mean (IVNHFULGBM) operator, the interval-valued neutrosophic hesitant fuzzy uncertain linguistic weighted geometric Bonferroni mean (IVNHFULWGBM) operator. In section 4, two multiple attribute decision making methods based on IVNHFULWBM operator and IVNHFULWGBM operator are proposed. In section 5, we give a numerical example to prove the effectiveness of the proposed methods.

2. Preliminaries

2.1 The uncertain linguistic variables

Suppose that $S=(s_0,s_1,\cdots,s_{l-1})$ consists of the finite and complete ordered elements, where l is an odd value. In general, l is equal to 3,5,7,9, etc. when $l=9$, $S=(s_0,s_1,s_2,s_3,s_4,s_5,s_6,s_7,s_8)$ =(extremely poor, very poor, poor, slightly poor, fair, slightly good, good, very good, extremely good). Here, $s_\alpha\,(\alpha=1,2,...,l-1)$ can be defined as a linguistic variable.

Let s_i and s_j are any two elements in linguistic set S, they must meet the following conditions [28][29]:

(1) If $i>j$, then $s_i>s_j$ (i.e., s_i is better than s_j);

(2) Negative operator: $neg(s_i)=s_{l-1-i}$;

(3) If $s_i\ge s_j$, then $\max(s_i,s_j)=s_i$;

(4) If $s_i\le s_j$, then $\min(s_i,s_j)=s_i$.

For any linguistic set $S=(s_0,s_1,\cdots,s_{l-1})$, the element s_α is a strictly monotonically increasing function of its subscript α [30]. In order to minimize the loss of linguistic information, the discrete linguistic set $S=(s_0,s_1,\cdots,s_{l-1})$ is extended to a continuous linguistic set $\bar{S}=\{s_\theta\mid\theta\in[0,q]\}$, q is sufficiently large number. The operational laws are defined as follows:[28][29]

(1) $\beta s_i = s_{\beta \times i} \quad \beta \geq 0$ (1)

(2) $s_i \oplus s_j = s_{i+j}$ (2)

(3) $s_i / s_j = s_{i/j}$ (3)

(4) $(s_i)^n = s_{i^n} \quad n \geq 0$ (4)

Definition 1 [31]**.** Let $\tilde{s} = [s_a, s_b]$, $s_a, s_b \in S$ and $b \geq a \geq 0$, s_a, s_b are the lower limit and upper limit of $\tilde{s}$, respectively, then, $\tilde{s}$ is called an uncertain linguistic variable.

Let $\tilde{s}_1 = [s_{a1}, s_{b1}]$, $\tilde{s}_2 = [s_{a2}, s_{b2}]$ be any two uncertain linguistic variables, the operation rules are defined as follows[32][33]:

(1) $\tilde{s}_1 \oplus \tilde{s}_2 = [s_{a1}, s_{b1}] \oplus [s_{a2}, s_{b2}] = [s_{a1+a2}, s_{b1+b2}]$ (5)

(2) $\tilde{s}_1 \otimes \tilde{s}_2 = [s_{a1}, s_{b1}] \otimes [s_{a2}, s_{b2}] = [s_{a1\times a2}, s_{b1\times b2}]$ (6)

(3) $\tilde{s}_1 / \tilde{s}_2 = [s_{a1}, s_{b1}] / [s_{a2}, s_{b2}] = [s_{a1/b2}, s_{b1/a2}] \quad if \quad a2 \neq 0, b2 \neq 0$ (7)

(4) $\lambda \tilde{s}_1 = \lambda [s_{a1}, s_{b1}] = [s_{\lambda * a1}, s_{\lambda * b1}] \quad \lambda > 0$ (8)

2.2 The neutrosophic set

Definition 2 [34]**.** Let X be a universe of discourse, with a generic element in X denoted by x. A neutrosophic number A in X is

$$A(x) = < x \mid (T_A(x), I_A(x), F_A(x)) > \tag{9}$$

where, $T_A(x)$ is the truth-membership function, $I_A(x)$ is the indeterminacy-membership function, and $F_A(x)$ is the falsity-membership function. $T_A(x), I_A(x)$ and $F_A(x)$ are real standard or nonstandard subsets of $]0^-, 1^+[$.

There is no restriction on the sum of $T_A(x)$, $I_A(x)$ and $F_A(x)$, so $0^- \leq T_A(x) + I_A(x) + F_A(x) \leq 3^+$.

Definition 3 [35]**.** Let X be a universe of discourse, with a generic element in X denoted by x. A single valued neutrosophic number A in X is

$$A(x) = < x \mid (T_A(x), I_A(x), F_A(x)) > \tag{10}$$

where $T_A(x)$ is the truth-membership function, $I_A(x)$ is the indeterminacy-membership function, and $F_A(x)$ is the falsity-membership function. For each point x in X, we have $T_A(x), I_A(x), F_A(x) \in [0,1]$, and $0 \leq T_A(x) + I_A(x) + F_A(x) \leq 3$.

Definition 4 [35]. Let X be a universe of discourse, with a generic element in X denoted by x. A interval neutrosophic set A in X is

$$A(x) = < x \mid (T_A(x), I_A(x), F_A(x)) > \tag{11}$$

where, $T_A(x)$ is the truth-membership function, $I_A(x)$ is the indeterminacy-membership function, and $F_A(x)$ is the falsity-membership function. For each point x in X, we have

$T_A(x), I_A(x), F_A(x) \subseteq [0,1]$, and $0 \le \sup(T_A(x)) + \sup(I_A(x)) + \sup(F_A(x)) \le 3$.

For convenience, we can use $x = ([T^L, T^U], [I^L, I^U], [F^L, F^U])$ to represent an element in INS, and can call an interval neutrosophic number (INN).

2.3 Interval-valued hesitant fuzzy set

Definition 5 [36][37]**.** A hesitant fuzzy set (HFS) E on X is defined in terms of a function that when applied to X, which returns a finite subset of [0, 1], to be easily understood, the HFS can be expressed by mathematical symbol as follows:

$$E = \{< x, h(x) > | \, x \in X\} \tag{12}$$

where $h(x)$ is a set of some different values in [0, 1], representing the possible membership degrees of the element $x \in X$ to E .From now on, it will be convenient to call $h(x)$ a hesitant fuzzy element (HFE) and E the set of all hesitant fuzzy elements (HFEs).

For any three HFEs h, h_1 and h_2, Torra [22] defined some basic operations shown as follow:

(1) $h^c = \bigcup_{\gamma \in h} \{1 - \gamma\}$, (13)

(2) $h_1 \bigcup h_2 = \bigcup_{\gamma_1 \in h_1, \gamma_2 \in h_2} \max\{\gamma_1, \gamma_2\}$, (14)

(3) $h_1 \bigcap h_2 = \bigcup_{\gamma_1 \in h_1, \gamma_2 \in h_2} \min\{\gamma_1, \gamma_2\}$. (15)

After that, Xia and Xu [23] defined four operations about the HFEs h, h_1, h_2 with a positive scale n:

(1) $h^n = \bigcup_{\gamma \in h} \{\gamma^n\}$, (16)

(2) $nh = \bigcup_{\gamma \in h} \{1 - (1 - \gamma)^n\}$, (17)

(3) $h_1 \oplus h_2 = \bigcup_{\gamma_1 \in h_1, \gamma_2 \in h_2} \{\gamma_1 + \gamma_2 - \gamma_1 \gamma_2\}$, (18)

(4) $h_1 \otimes h_2 = \bigcup_{\gamma_1 \in h_1, \gamma_2 \in h_2} \{\gamma_1 \gamma_2\}$. (19)

Definition 6 [38]**.** An interval-valued hesitant fuzzy set (IVHFS) $\tilde{E}$ on X is defined in terms of a function that when applied to X, which returns a set of finite closed sub-intervals in[0,1], the IVHFS can be expressed by mathematical symbol as follows:

$$\tilde{E} = \{< x, \tilde{h}(x) > | \, x \in X\} \tag{20}$$

where $\tilde{h}(x)$ is a set of some different interval values in [0,1], representing the possible membership degrees of the element $x \in X$ to $\tilde{E}$. From now on, it will be convenient to call $\tilde{h}(x)$ an interval-valued hesitant fuzzy element (IVHFE) and $\tilde{E}$ the set of all hesitant fuzzy elements

(IVHFEs).

2.4 interval-valued neutrosophic hesitant fuzzy uncertain linguistic set

Definition 7 [39]**.** Let X is a fixed set and $\tilde{S}$ is an uncertain linguistic set, then the hesitant fuzzy uncertain linguistic set (HFULS) on X can be defined as follows:

$$A = \{< x, [s_{\theta(x)}, s_{\tau(x)}], h(x)) >| x \in X\} \tag{21}$$

where $[s_{\theta(x)}, s_{\tau(x)}] \in \tilde{S}$ and $h(x)$ is a set of some values in [0,1], denoting the possible membership degree of the element $x \in X$ to the uncertain linguistic variable $[s_{\theta(x)}, s_{\tau(x)}]$.From now on, it will be convenient to call $a =< [s_{\theta(a)}, s_{\tau(a)}], h(a) >$ a hesitant fuzzy uncertain linguistic element (HFULE) and A the set of all HFULEs.

Definition 8. Let X be a fixed set and $\tilde{S}$ be an uncertain linguistic set, then an interval-valued neutrosophic hesitant fuzzy uncertain linguistic set (IVNHFULS) on X can be expressed by:

$$A = \{< x, [s_{\theta(x)}, s_{\tau(x)}], (\tilde{t}(x), \tilde{i}(x), \tilde{f}(x)) >| x \in X\} \tag{22}$$

where $[s_{\theta(x)}, s_{\tau(x)}] \in \tilde{S}$, $\tilde{t}(x) = \{\tilde{\gamma} | \tilde{\gamma} \in \tilde{t}(x)\}$, $\tilde{i}(x) = \{\tilde{\delta} | \tilde{\delta} \in \tilde{i}(x)\}$ and $\tilde{f}(x) = \{\tilde{\eta} | \tilde{\eta} \in \tilde{f}(x)\}$ are three sets of some values in [0,1], which represents the possible truth-membership degrees, indeterminacy-membership degrees, and falsity-membership degrees of the element $x \in X$ to the uncertain linguistic variable $[s_{\theta(x)}, s_{\tau(x)}]$, and satisfies these limits :

$\tilde{\gamma} \in [0,1]$, $\tilde{\delta} \in [0,1]$, $\tilde{\eta} \in [0,1]$ and $0 \le \sup \tilde{\gamma}^+ + \sup \tilde{\delta}^+ + \sup \tilde{\eta}^+ \le 3$,

where $\tilde{\gamma}^+ = \cup_{\tilde{\gamma} \in \tilde{t}(x)} \max\{\tilde{\gamma}\}$, $\tilde{\delta}^+ = \cup_{\tilde{\delta} \in \tilde{i}(x)} \max\{\tilde{\delta}\}$, and $\tilde{\eta}^+ = \cup_{\tilde{\eta} \in \tilde{f}(x)} \max\{\tilde{\eta}\}$ for $x \in X$.

Hence, $a =< [s_{\theta(a)}, s_{\tau(a)}], \{\tilde{t}(a), \tilde{i}(a), \tilde{f}(a)\} >$ can be described as an interval-valued neutrosophic hesitant fuzzy uncertain linguistic element (IVNHFULE). So, A is the set of all IVNHFULEs.

Let $\tilde{a} =< [s_{\theta(\tilde{a})}, s_{\tau(\tilde{a})}], (\tilde{t}(\tilde{a}), \tilde{i}(\tilde{a}), \tilde{f}(\tilde{a})) >$ and $\tilde{b} =< [s_{\theta(\tilde{b})}, s_{\tau(\tilde{b})}], (\tilde{t}(\tilde{b}), \tilde{i}(\tilde{b}), \tilde{f}(\tilde{b})) >$ be any two IVNHFULEs, the operation rules are defined as follows:

(1)
$$\tilde{a} \oplus \tilde{b} =< [s_{\theta(\tilde{a})+\theta(\tilde{b})}, s_{\tau(\tilde{a})+\tau(\tilde{b})}], \bigcup_{\substack{\tilde{r}(\tilde{a}) \in \tilde{t}(\tilde{a}), \tilde{r}(\tilde{b}) \in \tilde{t}(\tilde{b}) \\ \tilde{\delta}(\tilde{a}) \in \tilde{i}(\tilde{a}), \tilde{\delta}(\tilde{b}) \in \tilde{i}(\tilde{b}) \\ \tilde{\eta}(\tilde{a}) \in \tilde{f}(\tilde{a}), \tilde{\eta}(\tilde{b}) \in \tilde{f}(\tilde{b})}} ([\tilde{r}_{\tilde{a}}^{L} + \tilde{r}_{\tilde{b}}^{L} - \tilde{r}_{\tilde{a}}^{L} \tilde{r}_{\tilde{b}}^{L}, \tilde{r}_{\tilde{a}}^{U} + \tilde{r}_{\tilde{b}}^{U} - \tilde{r}_{\tilde{a}}^{U} \tilde{r}_{\tilde{b}}^{U}], [\tilde{\delta}_{\tilde{a}}^{L} \tilde{\delta}_{\tilde{b}}^{L}, \tilde{\delta}_{\tilde{a}}^{U} \tilde{\delta}_{\tilde{b}}^{U}], [\tilde{\eta}_{\tilde{a}}^{L} \tilde{\eta}_{\tilde{b}}^{L}, \tilde{\eta}_{\tilde{a}}^{U} \tilde{\eta}_{\tilde{b}}^{U}]) > \tag{23}$$

(2)
$$\tilde{a} \otimes \tilde{b} =< [s_{\theta(\tilde{a}) \times \theta(\tilde{b})}, s_{\tau(\tilde{a}) \times \tau(\tilde{b})}], \bigcup_{\substack{\tilde{r}(\tilde{a}) \in \tilde{t}(\tilde{a}), \tilde{r}(\tilde{b}) \in \tilde{t}(\tilde{b}) \\ \tilde{\delta}(\tilde{a}) \in \tilde{i}(\tilde{a}), \tilde{\delta}(\tilde{b}) \in \tilde{i}(\tilde{b}) \\ \tilde{\eta}(\tilde{a}) \in \tilde{f}(\tilde{a}), \tilde{\eta}(\tilde{b}) \in \tilde{f}(\tilde{b})}} ([\tilde{r}_{\tilde{a}}^{L} \tilde{r}_{\tilde{b}}^{L}, \tilde{r}_{\tilde{a}}^{U} \tilde{r}_{\tilde{b}}^{U}], [\tilde{\delta}_{\tilde{a}}^{L} + \tilde{\delta}_{\tilde{b}}^{L} - \tilde{\delta}_{\tilde{a}}^{L} \tilde{\delta}_{\tilde{b}}^{L}, \tilde{\delta}_{\tilde{a}}^{U} + \tilde{\delta}_{\tilde{b}}^{U} - \tilde{\delta}_{\tilde{a}}^{U} \tilde{\delta}_{\tilde{b}}^{U}] [\tilde{\eta}_{\tilde{a}}^{L} + \tilde{\eta}_{\tilde{b}}^{L} - \tilde{\eta}_{\tilde{a}}^{L} \tilde{\eta}_{\tilde{b}}^{L}, \tilde{\eta}_{\tilde{a}}^{U} + \tilde{\eta}_{\tilde{b}}^{U} - \tilde{\eta}_{\tilde{a}}^{U} \tilde{\eta}_{\tilde{b}}^{U}]) > \tag{24}$$

(3) $\lambda\tilde{a} = \langle [s_{\lambda\times\theta(\tilde{a})}, s_{\lambda\times\tau(\tilde{a})}], \bigcup_{\tilde{r}(\tilde{a})\in\tilde{t}(\tilde{a}),\tilde{\delta}(\tilde{a})\in\tilde{i}(\tilde{a}),\tilde{\eta}(\tilde{a})\in\tilde{f}(\tilde{a})} ([1-(1-\tilde{r}_{\tilde{a}}^{L})^{\lambda}, 1-(1-\tilde{r}_{\tilde{a}}^{U})^{\lambda}],$

$$[(\tilde{\delta}_{\tilde{a}}^{L})^{\lambda}, (\tilde{\delta}_{\tilde{a}}^{U})^{\lambda}], [(\tilde{\eta}_{\tilde{a}}^{L})^{\lambda}, (\tilde{\eta}_{\tilde{a}}^{U})^{\lambda}]) \rangle \quad (25)$$

(4) $\tilde{a}^{\lambda} = \langle [s_{\theta(\tilde{a})^{\lambda}}, s_{\tau(\tilde{a})^{\lambda}}], \bigcup_{\tilde{r}(\tilde{a})\in\tilde{t}(\tilde{a}),\tilde{\delta}(\tilde{a})\in\tilde{i}(\tilde{a}),\tilde{\eta}(\tilde{a})\in\tilde{f}(\tilde{a})} ([(\tilde{r}_{\tilde{a}}^{L})^{\lambda}, (\tilde{r}_{\tilde{a}}^{U})^{\lambda}], [1-(1-\tilde{\delta}_{\tilde{a}}^{L})^{\lambda}, 1-(1-\tilde{\delta}_{\tilde{a}}^{U})^{\lambda}],$

$$[1-(1-\tilde{\eta}_{\tilde{a}}^{L})^{\lambda}, 1-(1-\tilde{\eta}_{\tilde{a}}^{U})^{\lambda}]) \rangle \quad (26)$$

Let $\tilde{a} = \langle [s_{\theta(\tilde{a})}, s_{\tau(\tilde{a})}], (\tilde{t}(\tilde{a}), \tilde{i}(\tilde{a}), \tilde{f}(\tilde{a})) \rangle$, $\tilde{b} = \langle [s_{\theta(\tilde{b})}, s_{\tau(\tilde{b})}], (\tilde{t}(\tilde{b}), \tilde{i}(\tilde{b}), \tilde{f}(\tilde{b})) \rangle$ are any two IVNHFULEs, and $\lambda, \lambda_1, \lambda_2 \geq 0$, then we have:

(1) $\tilde{a} \oplus \tilde{b} = \tilde{b} \oplus \tilde{a}$ (27)

(2) $\tilde{a} \otimes \tilde{b} = \tilde{b} \otimes \tilde{a}$ (28)

(3) $\lambda(\tilde{a} \oplus \tilde{b}) = \lambda\tilde{a} \oplus \lambda\tilde{b}$ (29)

(4) $\lambda_1\tilde{a} \oplus \lambda_2\tilde{a} = (\lambda_1 + \lambda_2)\tilde{a}$ (30)

(5) $\tilde{a}^{\lambda} \otimes \tilde{b}^{\lambda} = (\tilde{a} \otimes \tilde{b})^{\lambda}$ (31)

(6) $\tilde{a}^{\lambda_1} \otimes \tilde{a}^{\lambda_2} = \tilde{a}^{\lambda_1+\lambda_2}$ (32)

Proof.

(1) The formula (27) is obviously right according to the operational rule (1) expressed by (23)

(2) The formula (28) is obviously right according to the operational rule (2) expressed by (24)

(3) For the left of the formula (29), we have

$$\lambda(\tilde{a} \oplus \tilde{b}) = \lambda(\langle [s_{\theta(\tilde{a})+\theta(\tilde{b})}, s_{\tau(\tilde{a})+\tau(\tilde{b})}], \bigcup_{\substack{\tilde{r}(\tilde{a})\in\tilde{t}(\tilde{a}),\tilde{r}(\tilde{b})\in\tilde{t}(\tilde{b}) \\ \tilde{\delta}(\tilde{a})\in\tilde{i}(\tilde{a}),\tilde{\delta}(\tilde{b})\in\tilde{i}(\tilde{b}) \\ \tilde{\eta}(\tilde{a})\in\tilde{f}(\tilde{a}),\tilde{\eta}(\tilde{b})\in\tilde{f}(\tilde{b})}} ([\tilde{r}_{\tilde{a}}^{L} + \tilde{r}_{\tilde{b}}^{L} - \tilde{r}_{\tilde{a}}^{L}\tilde{r}_{\tilde{b}}^{L}, \tilde{r}_{\tilde{a}}^{U} + \tilde{r}_{\tilde{b}}^{U} - \tilde{r}_{\tilde{a}}^{U}\tilde{r}_{\tilde{b}}^{U}],$$
$$[\tilde{\delta}_{\tilde{a}}^{L}\tilde{\delta}_{\tilde{b}}^{L}, \tilde{\delta}_{\tilde{a}}^{U}\tilde{\delta}_{\tilde{b}}^{U}], [\tilde{\eta}_{\tilde{a}}^{L}\tilde{\eta}_{\tilde{b}}^{L}, \tilde{\eta}_{\tilde{a}}^{U}\tilde{\eta}_{\tilde{b}}^{U}]) \rangle)$$

$$= \langle [s_{\lambda(\theta(\tilde{a})+\theta(\tilde{b}))}, s_{\lambda(\tau(\tilde{a})+\tau(\tilde{b}))}], \bigcup_{\substack{\tilde{r}(\tilde{a})\in\tilde{t}(\tilde{a}),\tilde{r}(\tilde{b})\in\tilde{t}(\tilde{b}) \\ \tilde{\delta}(\tilde{a})\in\tilde{i}(\tilde{a}),\tilde{\delta}(\tilde{b})\in\tilde{i}(\tilde{b}) \\ \tilde{\eta}(\tilde{a})\in\tilde{f}(\tilde{a}),\tilde{\eta}(\tilde{b})\in\tilde{f}(\tilde{b})}} ([1-(1-\tilde{r}_{\tilde{a}}^{L})^{\lambda}(1-\tilde{r}_{\tilde{b}}^{L})^{\lambda}, 1-(1-\tilde{r}_{\tilde{a}}^{U})^{\lambda}(1-\tilde{r}_{\tilde{b}}^{U})^{\lambda}],$$
$$[(\tilde{\delta}_{\tilde{a}}^{L}\tilde{\delta}_{\tilde{b}}^{L})^{\lambda}, (\tilde{\delta}_{\tilde{a}}^{U}\tilde{\delta}_{\tilde{b}}^{U})^{\lambda}], [(\tilde{\eta}_{\tilde{a}}^{L}\tilde{\eta}_{\tilde{b}}^{L})^{\lambda}, (\tilde{\eta}_{\tilde{a}}^{U}\tilde{\eta}_{\tilde{b}}^{U})^{\lambda}]) \rangle$$

and for the right of the formula (29), we have

$$= \langle [s_{\lambda\times\theta(\tilde{a})}, s_{\lambda\times\tau(\tilde{a})}], \bigcup_{\tilde{r}(\tilde{a})\in\tilde{t}(\tilde{a}),\tilde{\delta}(\tilde{a})\in\tilde{i}(\tilde{a}),\tilde{\eta}(\tilde{a})\in\tilde{f}(\tilde{a})} ([1-(1-\tilde{r}_{\tilde{a}}^{L})^{\lambda}, 1-(1-\tilde{r}_{\tilde{a}}^{U})^{\lambda}]$$
$$[(\tilde{\delta}_{\tilde{a}}^{L})^{\lambda}, (\tilde{\delta}_{\tilde{a}}^{U})^{\lambda}], [(\tilde{\eta}_{\tilde{a}}^{L})^{\lambda}, (\tilde{\eta}_{\tilde{a}}^{U})^{\lambda}]) \rangle$$

$$\oplus \langle [s_{\lambda\times\theta(\tilde{b})}, s_{\lambda\times\tau(\tilde{b})}], \bigcup_{\tilde{r}(\tilde{b})\in\tilde{t}(\tilde{b}),\tilde{\delta}(\tilde{b})\in\tilde{i}(\tilde{b}),\tilde{\eta}(\tilde{b})\in\tilde{f}(\tilde{b})} ([1-(1-\tilde{r}_{\tilde{b}}^{L})^{\lambda}, 1-(1-\tilde{r}_{\tilde{b}}^{U})^{\lambda}]$$
$$[(\tilde{\delta}_{\tilde{b}}^{L})^{\lambda}, (\tilde{\delta}_{\tilde{b}}^{U})^{\lambda}], [(\tilde{\eta}_{\tilde{b}}^{L})^{\lambda}, (\tilde{\eta}_{\tilde{b}}^{U})^{\lambda}]) \rangle$$

$$= \langle [s_{\lambda(\theta(\tilde{a})+\theta(\tilde{b}))}, s_{\lambda(\tau(\tilde{a})+\tau(\tilde{b}))}], \bigcup_{\substack{\tilde{r}(\tilde{a})\in\tilde{t}(\tilde{a}),\tilde{r}(\tilde{b})\in\tilde{t}(\tilde{b}) \\ \tilde{\delta}(\tilde{a})\in\tilde{i}(\tilde{a}),\tilde{\delta}(\tilde{b})\in\tilde{i}(\tilde{b}) \\ \tilde{\eta}(\tilde{a})\in\tilde{f}(\tilde{a}),\tilde{\eta}(\tilde{b})\in\tilde{f}(\tilde{b})}} [(1-(1-\tilde{r}_{\tilde{a}}^{L})^{\lambda}(1-\tilde{r}_{\tilde{b}}^{L})^{\lambda}, 1-(1-\tilde{r}_{\tilde{a}}^{U})^{\lambda}(1-\tilde{r}_{\tilde{b}}^{U})^{\lambda}],$$
$$[(\tilde{\delta}_{\tilde{a}}^{L}\tilde{\delta}_{\tilde{b}}^{L})^{\lambda}, (\tilde{\delta}_{\tilde{a}}^{U}\tilde{\delta}_{\tilde{b}}^{U})^{\lambda}], [(\tilde{\eta}_{\tilde{a}}^{L}\tilde{\eta}_{\tilde{b}}^{L})^{\lambda}, (\tilde{\eta}_{\tilde{a}}^{U}\tilde{\eta}_{\tilde{b}}^{U})^{\lambda}]) \rangle$$

So, we can get $\lambda(\tilde{a} \oplus \tilde{b}) = \lambda\tilde{a} \oplus \lambda\tilde{b}$, which completes the proof of formula (29).

(4) For formula (30), we have

$$\lambda_1\tilde{a} \oplus \lambda_2\tilde{a} = <[s_{\lambda_1\times\theta(\tilde{a})}, s_{\lambda_1\times\tau(\tilde{a})}], \bigcup_{\substack{\tilde{r}(\tilde{a})\in\tilde{t}(\tilde{a}),\\ \tilde{\delta}(\tilde{a})\in\tilde{i}(\tilde{a}),\\ \tilde{\eta}(\tilde{a})\in\tilde{f}(\tilde{a})}} ([1-(1-\tilde{r}_{\tilde{a}}^{L})^{\lambda_1}, 1-(1-\tilde{r}_{\tilde{a}}^{U})^{\lambda_1}], [(\tilde{\delta}_{\tilde{a}}^{L})^{\lambda_1}, (\tilde{\delta}_{\tilde{a}}^{U})^{\lambda_1}], [(\tilde{\eta}_{\tilde{a}}^{L})^{\lambda_1}, (\tilde{\eta}_{\tilde{a}}^{U})^{\lambda_1}]) >$$

$$\oplus <[s_{\lambda_2\times\theta(\tilde{a})}, s_{\lambda_2\times\tau(\tilde{a})}], \bigcup_{\substack{\tilde{r}(\tilde{a})\in\tilde{t}(\tilde{a}),\\ \tilde{\delta}(\tilde{a})\in\tilde{i}(\tilde{a}),\\ \tilde{\eta}(\tilde{a})\in\tilde{f}(\tilde{a})}} ([1-(1-\tilde{r}_{\tilde{a}}^{L})^{\lambda_2}, 1-(1-\tilde{r}_{\tilde{a}}^{U})^{\lambda_2}], [(\tilde{\delta}_{\tilde{a}}^{L})^{\lambda_2}, (\tilde{\delta}_{\tilde{a}}^{U})^{\lambda_2}], [(\tilde{\eta}_{\tilde{a}}^{L})^{\lambda_2}, (\tilde{\eta}_{\tilde{a}}^{U})^{\lambda_2}]) >$$

$$= <[s_{(\lambda_1+\lambda_2)\times\theta(\tilde{a})}, s_{(\lambda_1+\lambda_2)\times\tau(\tilde{a})}], \bigcup_{\substack{\tilde{r}(\tilde{a})\in\tilde{t}(\tilde{a}),\\ \tilde{\delta}(\tilde{a})\in\tilde{i}(\tilde{a}),\\ \tilde{\eta}(\tilde{a})\in\tilde{f}(\tilde{a})}} ([1-(1-\tilde{r}_{\tilde{a}}^{L})^{\lambda_1}(1-\tilde{r}_{\tilde{a}}^{L})^{\lambda_2}, 1-(1-\tilde{r}_{\tilde{a}}^{U})^{\lambda_1}(1-\tilde{r}_{\tilde{a}}^{U})^{\lambda_2}], [(\tilde{\delta}_{\tilde{a}}^{L})^{\lambda_1}(\tilde{\delta}_{\tilde{a}}^{L})^{\lambda_2}, (\tilde{\delta}_{\tilde{a}}^{U})^{\lambda_1}(\tilde{\delta}_{\tilde{a}}^{U})^{\lambda_2}], [(\tilde{\eta}_{\tilde{a}}^{L})^{\lambda_1}(\tilde{\eta}_{\tilde{a}}^{L})^{\lambda_2}, (\tilde{\eta}_{\tilde{a}}^{U})^{\lambda_1}(\tilde{\eta}_{\tilde{a}}^{U})^{\lambda_2}]) >$$

$$= <[s_{(\lambda_1+\lambda_2)\times\theta(\tilde{a})}, s_{(\lambda_1+\lambda_2)\times\tau(\tilde{a})}], \bigcup_{\substack{\tilde{r}(\tilde{a})\in\tilde{t}(\tilde{a}),\\ \tilde{\delta}(\tilde{a})\in\tilde{i}(\tilde{a}),\\ \tilde{\eta}(\tilde{a})\in\tilde{f}(\tilde{a})}} ([1-(1-\tilde{r}_{\tilde{a}}^{L})^{\lambda_1+\lambda_2}, 1-(1-\tilde{r}_{\tilde{a}}^{U})^{\lambda_1+\lambda_2}], [(\tilde{\delta}_{\tilde{a}}^{L})^{\lambda_1+\lambda_2}, (\tilde{\delta}_{\tilde{a}}^{U})^{\lambda_1+\lambda_2}], [(\tilde{\eta}_{\tilde{a}}^{L})^{\lambda_1+\lambda_2}, (\tilde{\eta}_{\tilde{a}}^{U})^{\lambda_1+\lambda_2}]) >$$

$$= (\lambda_1 + \lambda_2)\tilde{a}$$

So, we can get the formula (30) is right.

(5) For the left of the formula (31), we have

$$\tilde{a}^{\lambda} \otimes \tilde{b}^{\lambda} = <[s_{\theta(\tilde{a})^{\lambda}}, s_{\tau(\tilde{a})^{\lambda}}], \bigcup_{\substack{\tilde{r}(\tilde{a})\in\tilde{t}(\tilde{a}),\\ \tilde{\delta}(\tilde{a})\in\tilde{i}(\tilde{a}),\\ \tilde{\eta}(\tilde{a})\in\tilde{f}(\tilde{a})}} ([(\tilde{r}_{\tilde{a}}^{L})^{\lambda}, (\tilde{r}_{\tilde{a}}^{U})^{\lambda}], [1-(1-\tilde{\delta}_{\tilde{a}}^{L})^{\lambda}, 1-(1-\tilde{\delta}_{\tilde{a}}^{U})^{\lambda}], [1-(1-\tilde{\eta}_{\tilde{a}}^{L})^{\lambda}, 1-(1-\tilde{\eta}_{\tilde{a}}^{U})^{\lambda}]) >$$

$$\otimes <[s_{\theta(\tilde{b})^{\lambda}}, s_{\tau(\tilde{b})^{\lambda}}], \bigcup_{\substack{\tilde{r}(\tilde{b})\in\tilde{t}(\tilde{b}),\\ \tilde{\delta}(\tilde{b})\in\tilde{i}(\tilde{b}),\\ \tilde{\eta}(\tilde{b})\in\tilde{f}(\tilde{b})}} ([(\tilde{r}_{\tilde{b}}^{L})^{\lambda}, (\tilde{r}_{\tilde{b}}^{U})^{\lambda}], [1-(1-\tilde{\delta}_{\tilde{b}}^{L})^{\lambda}, 1-(1-\tilde{\delta}_{\tilde{b}}^{U})^{\lambda}], [1-(1-\tilde{\eta}_{\tilde{b}}^{L})^{\lambda}, 1-(1-\tilde{\eta}_{\tilde{b}}^{U})^{\lambda}]) >$$

$$= <[s_{\theta(\tilde{a})^{\lambda}\times\theta(\tilde{b})^{\lambda}}, s_{\tau(\tilde{a})^{\lambda}\times\tau(\tilde{b})^{\lambda}}], \bigcup_{\substack{\tilde{r}(\tilde{a})\in\tilde{t}(\tilde{a}),\tilde{r}(\tilde{b})\in\tilde{t}(\tilde{b}),\\ \tilde{\delta}(\tilde{a})\in\tilde{i}(\tilde{a}),\tilde{\delta}(\tilde{b})\in\tilde{i}(\tilde{b}),\\ \tilde{\eta}(\tilde{a})\in\tilde{f}(\tilde{a}),\tilde{\eta}(\tilde{b})\in\tilde{f}(\tilde{b})}} ([(\tilde{r}_{\tilde{a}}^{L})^{\lambda}(\tilde{r}_{\tilde{b}}^{L})^{\lambda}, (\tilde{r}_{\tilde{a}}^{U})^{\lambda}(\tilde{r}_{\tilde{b}}^{U})^{\lambda}], [1-(1-\tilde{\delta}_{\tilde{a}}^{L})^{\lambda}(1-\tilde{\delta}_{\tilde{b}}^{L})^{\lambda}, 1-(1-\tilde{\delta}_{\tilde{a}}^{U})^{\lambda}(1-\tilde{\delta}_{\tilde{b}}^{U})^{\lambda}], [1-(1-\tilde{\eta}_{\tilde{a}}^{L})^{\lambda}(1-\tilde{\eta}_{\tilde{b}}^{L})^{\lambda}, 1-(1-\tilde{\eta}_{\tilde{a}}^{U})^{\lambda}(1-\tilde{\eta}_{\tilde{b}}^{U})^{\lambda}]) >$$

$$= <[s_{(\theta(\tilde{a})\times\theta(\tilde{b}))^{\lambda}}, s_{(\tau(\tilde{a})\times\tau(\tilde{b}))^{\lambda}}], \bigcup_{\substack{\tilde{r}(\tilde{a})\in\tilde{t}(\tilde{a}),\tilde{r}(\tilde{b})\in\tilde{t}(\tilde{b}),\\ \tilde{\delta}(\tilde{a})\in\tilde{i}(\tilde{a}),\tilde{\delta}(\tilde{b})\in\tilde{i}(\tilde{b}),\\ \tilde{\eta}(\tilde{a})\in\tilde{f}(\tilde{a}),\tilde{\eta}(\tilde{b})\in\tilde{f}(\tilde{b})}} ([(\tilde{r}_{\tilde{a}}^{L}\tilde{r}_{\tilde{b}}^{L})^{\lambda}, (\tilde{r}_{\tilde{a}}^{U}\tilde{r}_{\tilde{b}}^{U})^{\lambda}], [1-((1-\tilde{\delta}_{\tilde{a}}^{L})(1-\tilde{\delta}_{\tilde{b}}^{L}))^{\lambda}, 1-((1-\tilde{\delta}_{\tilde{a}}^{U})(1-\tilde{\delta}_{\tilde{b}}^{U}))^{\lambda}], [1-((1-\tilde{\eta}_{\tilde{a}}^{L})(1-\tilde{\eta}_{\tilde{b}}^{L}))^{\lambda}, 1-((1-\tilde{\eta}_{\tilde{a}}^{U})(1-\tilde{\eta}_{\tilde{b}}^{U}))^{\lambda}]) >$$

and for the right of the formula (31), we have

$$(\tilde{a} \otimes \tilde{b})^{\lambda} = (<[s_{\theta(\tilde{a})\times\theta(\tilde{b})}, s_{\tau(\tilde{a})\times\tau(\tilde{b})}], \bigcup_{\substack{\tilde{r}(\tilde{a})\in\tilde{t}(\tilde{a}),\tilde{r}(\tilde{b})\in\tilde{t}(\tilde{b})\\ \tilde{\delta}(\tilde{a})\in\tilde{i}(\tilde{a}),\tilde{\delta}(\tilde{b})\in\tilde{i}(\tilde{b})\\ \tilde{\eta}(\tilde{a})\in\tilde{f}(\tilde{a}),\tilde{\eta}(\tilde{b})\in\tilde{f}(\tilde{b})}} ([\tilde{r}_{\tilde{a}}^{L}\tilde{r}_{\tilde{b}}^{L}, \tilde{r}_{\tilde{a}}^{U}\tilde{r}_{\tilde{b}}^{U}], [\tilde{\delta}_{\tilde{a}}^{L} + \tilde{\delta}_{\tilde{b}}^{L} - \tilde{\delta}_{\tilde{a}}^{L}\tilde{\delta}_{\tilde{b}}^{L}, \tilde{\delta}_{\tilde{a}}^{U} + \tilde{\delta}_{\tilde{b}}^{U} - \tilde{\delta}_{\tilde{a}}^{U}\tilde{\delta}_{\tilde{b}}^{U}], [\tilde{\eta}_{\tilde{a}}^{L} + \tilde{\eta}_{\tilde{b}}^{L} - \tilde{\eta}_{\tilde{a}}^{L}\tilde{\eta}_{\tilde{b}}^{L}, \tilde{\eta}_{\tilde{a}}^{U} + \tilde{\eta}_{\tilde{b}}^{U} - \tilde{\eta}_{\tilde{a}}^{U}\tilde{\eta}_{\tilde{b}}^{U}]) >)^{\lambda}$$

$$=<[s_{(\theta(\tilde{a})\times\theta(\tilde{b}))^{\lambda}}, s_{(\tau(\tilde{a})\times\tau(\tilde{b}))^{\lambda}}], \bigcup_{\substack{\tilde{r}(\tilde{a})\in\tilde{t}(\tilde{a}),\tilde{r}(\tilde{b})\in\tilde{t}(\tilde{b}),\\ \tilde{\delta}(\tilde{a})\in\tilde{i}(\tilde{a}),\tilde{\delta}(\tilde{b})\in\tilde{i}(\tilde{b}),\\ \tilde{\eta}(\tilde{a})\in\tilde{f}(\tilde{a}),\tilde{\eta}(\tilde{b})\in\tilde{f}(\tilde{b})}} ([(\tilde{r}_{\tilde{a}}^{L}\tilde{r}_{\tilde{b}}^{L})^{\lambda}, (\tilde{r}_{\tilde{a}}^{U}\tilde{r}_{\tilde{b}}^{U})^{\lambda}],$$

$$[1-((1-\tilde{\delta}_{\tilde{a}}^{L})(1-\tilde{\delta}_{\tilde{b}}^{L}))^{\lambda}, 1-((1-\tilde{\delta}_{\tilde{a}}^{U})(1-\tilde{\delta}_{\tilde{b}}^{U}))^{\lambda}], [1-((1-\tilde{\eta}_{\tilde{a}}^{L})(1-\tilde{\eta}_{\tilde{b}}^{L}))^{\lambda}, 1-((1-\tilde{\eta}_{\tilde{a}}^{U})(1-\tilde{\eta}_{\tilde{b}}^{U}))^{\lambda}]) >$$

So, we can get the formula (31) is right.

(6) For formula (32), we have

$$\tilde{a}^{\lambda_1} \otimes \tilde{a}^{\lambda_2} =<[s_{\theta(\tilde{a})^{\lambda_1}}, s_{\tau(\tilde{a})^{\lambda_1}}], \bigcup_{\substack{\tilde{r}(\tilde{a})\in\tilde{t}(\tilde{a}),\\ \tilde{\delta}(\tilde{a})\in\tilde{i}(\tilde{a}),\\ \tilde{\eta}(\tilde{a})\in\tilde{f}(\tilde{a})}} ([(\tilde{r}_{\tilde{a}}^{L})^{\lambda_1}, (\tilde{r}_{\tilde{a}}^{U})^{\lambda_1}], [1-(1-\tilde{\delta}_{\tilde{a}}^{L})^{\lambda_1}, 1-(1-\tilde{\delta}_{\tilde{a}}^{U})^{\lambda_1}],$$

$$[1-(1-\tilde{\eta}_{\tilde{a}}^{L})^{\lambda_1}, 1-(1-\tilde{\eta}_{\tilde{a}}^{U})^{\lambda_1}]) >$$

$$\otimes <[s_{\theta(\tilde{a})^{\lambda_2}}, s_{\tau(\tilde{a})^{\lambda_2}}], \bigcup_{\substack{\tilde{r}(\tilde{a})\in\tilde{t}(\tilde{a}),\\ \tilde{\delta}(\tilde{a})\in\tilde{i}(\tilde{a}),\\ \tilde{\eta}(\tilde{a})\in\tilde{f}(\tilde{a})}} ([(\tilde{r}_{\tilde{a}}^{L})^{\lambda_2}, (\tilde{r}_{\tilde{a}}^{U})^{\lambda_2}], [1-(1-\tilde{\delta}_{\tilde{a}}^{L})^{\lambda_2}, 1-(1-\tilde{\delta}_{\tilde{a}}^{U})^{\lambda_2}],$$

$$[1-(1-\tilde{\eta}_{\tilde{a}}^{L})^{\lambda_2}, 1-(1-\tilde{\eta}_{\tilde{a}}^{U})^{\lambda_2}]) >$$

$$=<[s_{\theta(\tilde{a})^{\lambda_1}\times\theta(\tilde{a})^{\lambda_2}}, s_{\tau(\tilde{a})^{\lambda_1}\times\tau(\tilde{a})^{\lambda_2}}], \bigcup_{\substack{\tilde{r}(\tilde{a})\in\tilde{t}(\tilde{a}),\\ \tilde{\delta}(\tilde{a})\in\tilde{i}(\tilde{a}),\\ \tilde{\eta}(\tilde{a})\in\tilde{f}(\tilde{a})}} ([(\tilde{r}_{\tilde{a}}^{L})^{\lambda_1}(\tilde{r}_{\tilde{a}}^{L})^{\lambda_2}, (\tilde{r}_{\tilde{a}}^{U})^{\lambda_1}(\tilde{r}_{\tilde{a}}^{U})^{\lambda_2}],$$

$$[1-(1-\tilde{\delta}_{\tilde{a}}^{L})^{\lambda_1}(1-\tilde{\delta}_{\tilde{a}}^{L})^{\lambda_2}, 1-(1-\tilde{\delta}_{\tilde{a}}^{U})^{\lambda_1}(1-\tilde{\delta}_{\tilde{a}}^{U})^{\lambda_2}], [1-(1-\tilde{\eta}_{\tilde{a}}^{L})^{\lambda_1}(1-\tilde{\eta}_{\tilde{a}}^{L})^{\lambda_2}, 1-(1-\tilde{\eta}_{\tilde{a}}^{U})^{\lambda_1}(1-\tilde{\eta}_{\tilde{a}}^{U})^{\lambda_2}]) >$$

$$=<[s_{\theta(\tilde{a})^{\lambda_1+\lambda_2}}, s_{\tau(\tilde{a})^{\lambda_1+\lambda_2}}], \bigcup_{\substack{\tilde{r}(\tilde{a})\in\tilde{t}(\tilde{a}),\\ \tilde{\delta}(\tilde{a})\in\tilde{i}(\tilde{a}),\\ \tilde{\eta}(\tilde{a})\in\tilde{f}(\tilde{a})}} ([(\tilde{r}_{\tilde{a}}^{L})^{\lambda_1+\lambda_2}, (\tilde{r}_{\tilde{a}}^{U})^{\lambda_1+\lambda_2}], [1-(1-\tilde{\delta}_{\tilde{a}}^{L})^{\lambda_1+\lambda_2}, 1-(1-\tilde{\delta}_{\tilde{a}}^{U})^{\lambda_1+\lambda_2}],$$

$$[1-(1-\tilde{\eta}_{\tilde{a}}^{L})^{\lambda_1+\lambda_2}, 1-(1-\tilde{\eta}_{\tilde{a}}^{U})^{\lambda_1+\lambda_2}]) >$$

$$= \tilde{a}^{\lambda_1+\lambda_2}$$

So, we can get the formula (32) is right.

Definition 9 [9,15]**.** Let $\tilde{a} =<[s_{\theta(\tilde{a})}, s_{\tau(\tilde{a})}], (\tilde{t}(\tilde{a}), \tilde{i}(\tilde{a}), \tilde{f}(\tilde{a})) >$ be an interval-valued neutrosophic hesitant fuzzy uncertain linguistic element (IVNHFULE) , then the score function $S(\tilde{a})$ of $\tilde{a}$ can be defined as follows:

$$S(\tilde{a}) = \frac{[\frac{1}{\#\tilde{t}}\sum_{\tilde{r}(\tilde{a})\in\tilde{t}(\tilde{a})}(\frac{\tilde{r}_{\tilde{a}}^{L}+\tilde{r}_{\tilde{a}}^{U}}{2}) + \frac{1}{\#\tilde{i}}\sum_{\tilde{\delta}(\tilde{a})\in\tilde{i}(\tilde{a})}(\frac{(1-\tilde{\delta}_{\tilde{a}}^{L})+(1-\tilde{\delta}_{\tilde{a}}^{U})}{2}) + \frac{1}{\#\tilde{f}}\sum_{\eta(\tilde{a})\in\tilde{f}(\tilde{a})}(\frac{(1-\tilde{\eta}_{\tilde{a}}^{L})+(1-\tilde{\eta}_{\tilde{a}}^{U})}{2})]}{3} \times s_{\frac{\theta(\tilde{a})+\tau(\tilde{a})}{2}}$$

$$= s_{\frac{(\theta(\tilde{a})+\tau(\tilde{a}))*[\frac{1}{\#\tilde{t}}\sum_{\tilde{r}(\tilde{a})\in\tilde{t}(\tilde{a})}(\frac{\tilde{r}_{\tilde{a}}^{L}+\tilde{r}_{\tilde{a}}^{U}}{2})+\frac{1}{\#\tilde{i}}\sum_{\tilde{\delta}(\tilde{a})\in\tilde{i}(\tilde{a})}(\frac{(1-\tilde{\delta}_{\tilde{a}}^{L})+(1-\tilde{\delta}_{\tilde{a}}^{U})}{2})+\frac{1}{\#\tilde{f}}\sum_{\eta(\tilde{a})\in\tilde{f}(\tilde{a})}(\frac{(1-\tilde{\eta}_{\tilde{a}}^{L})+(1-\tilde{\eta}_{\tilde{a}}^{U})}{2})]}{6}} \quad (33)$$

where $\#\tilde{t}$, $\#\tilde{i}$, $\#\tilde{f}$ are the number of the interval numbers in $\tilde{t}(\tilde{a})$, $\tilde{i}(\tilde{a})$, $\tilde{f}(\tilde{a})$.

Let $\tilde{a} =<[s_{\theta(\tilde{a})}, s_{\tau(\tilde{a})}], (\tilde{t}(\tilde{a}), \tilde{i}(\tilde{a}), \tilde{f}(\tilde{a})) >$ and $\tilde{b} =<[s_{\theta(\tilde{b})}, s_{\tau(\tilde{b})}], (\tilde{t}(\tilde{b}), \tilde{i}(\tilde{b}), \tilde{f}(\tilde{b})) >$ are any two IVNHFULEs, the comparison method of IVNHFULEs is expressed as follows:

(1) If $S(\tilde{a}) > S(\tilde{b})$,then $\tilde{a} > \tilde{b}$

(2) If $S(\tilde{a}) < S(\tilde{b})$,then $\tilde{a} < \tilde{b}$

(3) If $S(\tilde{a}) = S(\tilde{b})$,then $\tilde{a} = \tilde{b}$

2.4 The Bonferroni mean (BM) operator

Definition 10 [40]**.** Let $a_i (i = 1,2,..., n)$ be a set of non-negative numbers, and $p, q \geq 0$, the Bonferroni mean operator is defined as follows:

$$BM^{p,q}(a_1, a_2,..., a_n) = \left(\frac{1}{n(n-1)} \sum_{\substack{i,j=1 \\ j \neq i}}^{n} a_i{}^p a_j{}^q \right)^{\frac{1}{p+q}} \tag{34}$$

In real decision making, the weight of input arguments plays an important part in decision process. But the BM operator doesn't consider the importance of input argument itself. Hence, Zhou [41] developed a weighted Bonferroni mean (WBM) operator.

Definition 11 [41]**.** Let $a_i (i = 1,2,..., n)$ be a set of non-negative numbers, and p , $q \geq 0$. The weighted Bonferroni mean operator is defined as follows:

$$WBM^{p,q}(a_1, a_2,..., a_n) = \left(\sum_{\substack{i,j=1 \\ i \neq j}}^{n} \frac{w_i w_j}{1 - w_i} a_i{}^p a_j{}^q \right)^{1/p+q} \tag{35}$$

where $W = (w_1, w_2,..., w_n)^T$ is the weight vector of $a_i (i = 1,2,..., n)$, satisfying $0 \leq w_i \leq 1$ and

$\sum_{i=1}^{n} w_i = 1$.

The WBM operator has four desirable properties as follows:

Theorem 1 [41] (Reducibility).

Let $W = (\frac{1}{n}, \frac{1}{n},..., \frac{1}{n})^T$ be the weight vector of $a_i (i = 1,2,..., n)$,then

$$WBM^{p,q}(a_1, a_2,..., a_n) = \left(\frac{1}{n(n-1)} \sum_{\substack{i,j=1 \\ i \neq j}}^{n} a_i{}^p a_j{}^q \right)^{\frac{1}{p+q}} = BM^{p,q}(a_1, a_2,..., a_n).$$

Theorem 2 [41] (Idempotency).

Let $a_j = a (j = 1,2,..., n)$, then $WBM^{p,q}(a_1, a_2,..., a_n) = a$.

Theorem 3 [41] (Permutation).

Let $(a_1, a_2,..., a_n)$ be any permutation of $(a_1', a_2',..., a_n')$, then

$$WBM^{p,q}(a_1', a_2',..., a_n') = WBM^{p,q}(a_1, a_2,..., a_n).$$

Theorem 4 [41] (Monotonicity).

If $a_j \geq b_j (j = 1,2,..., n)$, then $WBM^{p,q}(a_1, a_2,..., a_n) \geq WBM^{p,q}(b_1, b_2,..., b_n)$.

Theorem 5 [41] (Boundedness).

The $WBM^{p,q}$ operator lies between the max and min operators, i.e.,

$$\min(a_1, a_2, ..., a_n) \le WBM^{p,q}(a_1, a_2, ..., a_n) \le \max(a_1, a_2, ..., a_n).$$

When the WBM operator select different values to the parameters p and q, some special cases of WBM can be obtained as follows:

(1) If $q = 0$, then $WBM^{p,0}(a_1, a_2, ..., a_n) = \left(\sum_{i=1}^{n} w_i a_i^{\,p}\right)^{\frac{1}{p}}$.

(2) If $p = 1$ and $q = 0$, then $WBM^{1,0}(a_1, a_2, ..., a_n) = \sum_{i=1}^{n} w_i a_i$.

(3) If $p = \frac{1}{2}$ and $q = \frac{1}{2}$, then $WBM^{\frac{1}{2},\frac{1}{2}}(a_1, a_2, ..., a_n) = \sum_{\substack{i,j=1 \\ j \ne i}}^{n} \frac{w_i w_j}{1 - w_i} (a_i a_j)^{\frac{1}{2}}$.

(4) If $p = 1$ and $q = 1$, then $WBM^{1,1}(a_1, a_2, ..., a_n) = \left(\sum_{\substack{i,j=1 \\ j \ne i}}^{n} \frac{w_i w_j}{1 - w_i} a_i a_j\right)^{\frac{1}{2}}$.

2.5 The geometric Bonferroni mean (GBM) operator

Definition 12 [42]**.** Let $a_i (i = 1,2,...,n)$ be a set of non-negative numbers, and $p, q \ge 0$, the geometric Bonferroni mean operator is defined as follows:

$$GBM^{p,q}(a_1, a_2, ..., a_n) = \frac{1}{p+q} \prod_{\substack{i,j=1 \\ j \ne i}}^{n} (pa_i + qa_j)^{\frac{1}{n(n-1)}} \tag{36}$$

Similar to the BM operator, the GBN operator also doesn't consider the weights of the input arguments. The weighted geometric Bonferroni mean (WGBM) operator was proposed by Sun and Liu [43].

Definition 13 [43]**.** Let $a_i (i = 1,2,...,n)$ be a set of non-negative numbers, and p , q $\ge$ 0 . The weighted geometric Bonferroni mean operator is defined as follows:

$$WGBM^{p,q}(a_1, a_2, ..., a_n) = \frac{1}{p+q} \prod_{\substack{i,j=1 \\ j \ne i}}^{n} (pa_i + qa_j)^{\frac{w_i w_j}{1 - w_i}} \tag{37}$$

where $W = (w_1, w_2, ..., w_n)^T$ is the weight vector of $a_i (i = 1,2,...,n)$,satisfying $0 \le w_i \le 1$ and

$\sum_{i=1}^{n} w_i = 1$.

It is easy to demonstrate the WGBM operator has some properties as follows:

Theorem 6 [43] **(Reducibility).**

Let $W = (\frac{1}{n}, \frac{1}{n}, ..., \frac{1}{n})^T$ be the weight vector of $a_i (i = 1,2,...,n)$,then

$$WGBM^{p,q}(a_1, a_2, ..., a_n) = GBM^{p,q}(a_1, a_2, ..., a_n).$$

Theorem 7 [43] **(Idempotency).**

Let $a_j = a (j = 1,2,...,n)$, then $WGBM^{p,q}(a_1, a_2,..., a_n) = a$.

Theorem 8 [43] **(Permutation).**

Let $(a_1, a_2,..., a_n)$ be any permutation of $(a_1', a_2',..., a_n')$, then

$$WGBM^{p,q}(a_1', a_2',..., a_n') = WGBM^{p,q}(a_1, a_2,..., a_n).$$

Theorem 9 [43] **(Monotonicity).**

If $a_j \geq b_j (j = 1,2,...,n)$, then $WGBM^{p,q}(a_1, a_2,..., a_n) \geq WGBM^{p,q}(b_1, b_2,..., b_n)$.

Theorem 10 [43] **(Boundedness).**

The $WGBM^{p,q}$ operator lies between the max and min operators, i.e. ,

$$\min(a_1, a_2,..., a_n) \leq WGBM^{p,q}(a_1, a_2,..., a_n) \leq \max(a_1, a_2,..., a_n)$$

When the WGBM operator select different parameters p and q, some special cases of WGBM operator can be obtained as follows:

(1) If $q = 0$, then $WGBM^{p,q}(a_1, a_2,..., a_n) = \frac{1}{p}\prod_{i=1}^{n}(pa_i)^{w_i}$

(2) If $q = 0, p = 1$ then $WGBM^{p,q}(a_1, a_2,..., a_n) = \prod_{i=1}^{n} a_i^{w_i}$

(3) If $p = \frac{1}{2}$ and $q = \frac{1}{2}$,then $WGBM^{\frac{1}{2},\frac{1}{2}}(a_1, a_2,..., a_n) = \prod_{\substack{i,j=1 \\ j \neq i}}^{n}(\frac{1}{2}a_i + \frac{1}{2}a_j)^{\frac{w_i w_j}{1-w_i}}$

(4) If $p = 1$ and $q = 1$,then $WGBM^{1,1}(a_1, a_2,..., a_n) = \frac{1}{2}\prod_{\substack{i,j=1 \\ j \neq i}}^{n}(a_i + a_j)^{\frac{w_i w_j}{1-w_i}}$

3. Interval-valued neutrosophic hesitant fuzzy uncertain linguistic Bonferroni mean aggregation operators

Interval-valued neutrosophic hesitant fuzzy uncertain linguistic set (IVNHFULS) is an important tool to describe the decision maker's preference and to handle the hesitant, uncertain, incomplete information. Bonferroni mean (BM) can replace the simple averaging of other mean type operators because of the excellent modeling capability.

In this section, we combine IVNHFULS with BM operator and propose some interval-valued neutrosophic hesitant fuzzy uncertain linguistic Bonferroni mean aggregation operators.

3.1 The interval-valued neutrosophic hesitant fuzzy uncertain linguistic Bonferroni mean operator

Definition 14. Let $\tilde{a}_i = \langle [s_{\theta(\tilde{a}_i)}, s_{\tau(\tilde{a}_i)}], (\tilde{t}(\tilde{a}_i), \tilde{i}(\tilde{a}_i), \tilde{f}(\tilde{a}_i)) \rangle$ $(i = 1,2,...,n)$ be the set of all IVNHFULEs, and $p, q \geq 0$, then the interval-valued neutrosophic hesitant fuzzy uncertain linguistic Bonferroni mean operator can be defined as follows:

$$IVNHFULBM^{p,q}(\tilde{a}_1,\tilde{a}_2,...,\tilde{a}_n)=\left(\frac{1}{n(n-1)}\underset{\substack{i,j=1\\ j\neq i}}{\overset{n}{\oplus}}\tilde{a}_i^{\,p}\otimes\tilde{a}_j^{\,q}\right)^{\frac{1}{p+q}} \tag{38}$$

Theorem 11. Let $\tilde{a}_i=<[s_{\theta(\tilde{a}_i)},s_{\tau(\tilde{a}_i)}],(\tilde{t}(\tilde{a}_i),\tilde{i}(\tilde{a}_i),\tilde{f}(\tilde{a}_i))>$ $(i=1,2,...,n)$ be the set of IVNHFULEs, and $p\,,\,q\;\geq\;0$, then, the result aggregated from Definition 14 is still an IVNHFULE, and even

$$IVNHFULBM^{p,q}(\tilde{a}_1,\tilde{a}_2,...,\tilde{a}_n)=\left(\frac{1}{n(n-1)}\overset{n}{\underset{i=1}{\oplus}}\overset{n}{\underset{j=1,j\neq i}{\oplus}}\tilde{a}_i^{\,p}\otimes\tilde{a}_j^{\,q}\right)^{\frac{1}{p+q}}$$

$$=\left\langle\left[s_{\left(\sum_{i=1}^{n}\sum_{j=1,j\neq i}^{n}\frac{1}{n(n-1)}.\theta(\tilde{a}_i)^p.\theta(\tilde{a}_j)^q\right)^{\frac{1}{p+q}}},s_{\left(\sum_{i=1}^{n}\sum_{j=1,j\neq i}^{n}\frac{1}{n(n-1)}.\tau(\tilde{a}_i)^p.\tau(\tilde{a}_j)^q\right)^{\frac{1}{p+q}}}\right],\right.$$

$$\bigcup_{\tilde{r}(\tilde{a}_1)\in\tilde{t}(\tilde{a}_1),\tilde{r}(\tilde{a}_2)\in\tilde{t}(\tilde{a}_2),...,\tilde{r}(\tilde{a}_n)\in\tilde{t}(\tilde{a}_n)}\left(\left(1-\prod_{i=1}^{n}\prod_{j=1,j\neq i}^{n}(1-(r_{\tilde{a}_i}{}^{L})^p.(r_{\tilde{a}_j}{}^{L})^q)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{p+q}},\left(1-\prod_{i=1}^{n}\prod_{j=1,j\neq i}^{n}(1-(r_{\tilde{a}_i}{}^{U})^p.(r_{\tilde{a}_j}{}^{U})^q)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{p+q}}\right), \tag{39}$$

$$\bigcup_{\tilde{\delta}(\tilde{a}_1)\in\tilde{i}(\tilde{a}_1),\tilde{\delta}(\tilde{a}_2)\in\tilde{i}(\tilde{a}_2)...,\tilde{\delta}(\tilde{a}_n)\in\tilde{i}(\tilde{a}_n)}\left(1-\left(1-\left(\prod_{i=1}^{n}\prod_{j=1,j\neq i}^{n}\left(1-(1-\delta_{\tilde{a}_i}{}^{L})^p(1-\delta_{\tilde{a}_j}{}^{L})^q\right)\right)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{p+q}},1-\left(1-\left(\prod_{i=1}^{n}\prod_{j=1,j\neq i}^{n}\left(1-(1-\delta_{\tilde{a}_i}{}^{u})^p(1-\delta_{\tilde{a}_j}{}^{u})^q\right)\right)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{p+q}}\right),$$

$$\left.\bigcup_{\tilde{\eta}(\tilde{a}_1)\in\tilde{f}(\tilde{a}_1),\tilde{\eta}(\tilde{a}_2)\in\tilde{f}(\tilde{a}_2),...,\tilde{\eta}(\tilde{a}_n)\in\tilde{f}(\tilde{a}_n)}\left(1-\left(1-\left(\prod_{i=1}^{n}\prod_{j=1,j\neq i}^{n}\left(1-(1-\eta_{\tilde{a}_i}{}^{L})^p(1-\eta_{\tilde{a}_j}{}^{L})^q\right)\right)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{p+q}},1-\left(1-\left(\prod_{i=1}^{n}\prod_{j=1,j\neq i}^{n}\left(1-(1-\eta_{\tilde{a}_i}{}^{u})^p(1-\eta_{\tilde{a}_j}{}^{u})^q\right)\right)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{p+q}}\right)\right\rangle$$

Proof .

By the operational rules of the interval-valued neutrosopic hesitant fuzzy uncertain linguistic variables, we have

$$\tilde{a}_i^{\,p}=<[s_{\theta(\tilde{a}_i)^{\,p}},s_{\tau(\tilde{a}_i)^{\,p}}],\bigcup_{\tilde{r}(\tilde{a}_i)\in\tilde{t}(\tilde{a}_i)}\{(r_{\tilde{a}_i}{}^{L})^p,(r_{\tilde{a}_i}{}^{U})^p\}>$$
$$\tilde{a}_j^{\,q}=<[s_{\theta(\tilde{a}_j)^{\,q}},s_{\tau(\tilde{a}_j)^{\,q}}],\bigcup_{\tilde{r}(\tilde{a}_j)\in\tilde{t}(\tilde{a}_j)}\{(r_{\tilde{a}_j}{}^{L})^q,(r_{\tilde{a}_j}{}^{U})^q\}>$$

$$\begin{aligned}\tilde{a}_i^{\,p}\otimes\tilde{a}_j^{\,q}=<[&s_{\theta(\tilde{a}_i)^{\,p}\cdot\theta(\tilde{a}_j)^{\,q}},s_{\tau(\tilde{a}_i)^{\,p}\cdot\tau(\tilde{a}_j)^{\,q}}],\\ &\bigcup_{\tilde{r}(\tilde{a}_i)\in\tilde{t}(\tilde{a}_i),\tilde{r}(\tilde{a}_j)\in\tilde{t}(\tilde{a}_j)}((r_{\tilde{a}_i}{}^{L})^p.(r_{\tilde{a}_j}{}^{L})^q,(r_{\tilde{a}_i}{}^{U})^p.(r_{\tilde{a}_j}{}^{U})^q),\\ &\bigcup_{\tilde{\delta}(\tilde{a}_i)\in\tilde{i}(\tilde{a}_i),\tilde{\delta}(\tilde{a}_j)\in\tilde{i}(\tilde{a}_j)}(1-(1-\tilde{\delta}_{\tilde{a}_i}{}^{L})^P(1-\tilde{\delta}_{\tilde{a}_j}{}^{L})^P,1-(1-\tilde{\delta}_{\tilde{a}_i}{}^{U})^P(1-\tilde{\delta}_{\tilde{a}_j}{}^{U})^P)\\ &\bigcup_{\tilde{\eta}(\tilde{a}_i)\in\tilde{f}(\tilde{a}_i),\tilde{\eta}(\tilde{a}_j)\in\tilde{f}(\tilde{a}_j)}(1-(1-\tilde{\eta}_{\tilde{a}_i}{}^{L})^P(1-\tilde{\eta}_{\tilde{a}_j}{}^{L})^P,1-(1-\tilde{\eta}_{\tilde{a}_i}{}^{U})^P(1-\tilde{\eta}_{\tilde{a}_j}{}^{U})^P)>\end{aligned}$$

then

$$\frac{1}{n(n-1)}\tilde{a}_i^{\,p}\otimes\tilde{a}_j^{\,q}=\left\langle\left[s_{\frac{1}{n(n-1)}.\theta(\tilde{a}_i)^p.\theta(\tilde{a}_j)^q},s_{\frac{1}{n(n-1)}.\tau(\tilde{a}_i)^p.\tau(\tilde{a}_j)^q}\right],\right.$$

$$\left(\bigcup_{\tilde{r}(\tilde{a}_i)\in\tilde{t}(\tilde{a}_i),\tilde{r}(\tilde{a}_j)\in\tilde{t}(\tilde{a}_j)}\left(1-(1-r(_{\tilde{a}_i}{}^{L})^p.(r_{\tilde{a}_j}{}^{L})^q)^{\frac{1}{n(n-1)}},1-(1-(r_{\tilde{a}_i}{}^{U})^p.(r_{\tilde{a}_j}{}^{U})^q)^{\frac{1}{n(n-1)}}\right),\right.$$

$$\bigcup_{\tilde{\delta}(\tilde{a}_i)\in\tilde{i}(\tilde{a}_i),\tilde{\delta}(\tilde{a}_j)\in\tilde{i}(\tilde{a}_j)}\left((1-(1-\delta_{\tilde{a}_i}{}^{L})^p(1-\delta_{\tilde{a}_j}{}^{L})^q)^{\frac{1}{n(n-1)}},(1-(1-\delta_{\tilde{a}_i}{}^{U})^p(1-\delta_{\tilde{a}_j}{}^{U})^q)^{\frac{1}{n(n-1)}}\right),$$

$$\left.\left.\bigcup_{\tilde{\eta}(\tilde{a}_i)\in\tilde{f}(\tilde{a}_i),\tilde{\eta}(\tilde{a}_j)\in\tilde{f}(\tilde{a}_j)}\left((1-(1-\eta_{\tilde{a}_i}{}^{L})^p(1-\eta_{\tilde{a}_j}{}^{L})^q)^{\frac{1}{n(n-1)}},(1-(1-\eta_{\tilde{a}_i}{}^{U})^p(1-\eta_{\tilde{a}_j}{}^{U})^q)^{\frac{1}{n(n-1)}}\right)\right)\right\rangle$$

and

$$\frac{1}{n(n-1)}\mathop{\oplus}_{i=1}^{n}\mathop{\oplus}_{j=i}^{n}\tilde{a}_i^{\,p}\otimes\tilde{a}_j^{\,q}=\mathop{\oplus}_{i=1}^{n}\mathop{\oplus}_{j=i}^{n}\frac{1}{n(n-1)}\tilde{a}_i^{\,p}\otimes\tilde{a}_j^{\,q}$$

$$=\left\langle\left[s_{\sum_{i=1}^{n}\sum_{j=1,j\neq i}^{n}\frac{1}{n(n-1)}\theta(\tilde{a}_i)^p\theta(\tilde{a}_j)^q},s_{\sum_{i=1}^{n}\sum_{j=1,j\neq i}^{n}\frac{1}{n(n-1)}\tau(\tilde{a}_i)^p\tau(\tilde{a}_j)^q}\right],\right.$$

$$\bigcup\tilde{r}(\tilde{a}_1)\in\tilde{t}(\tilde{a}_1),\tilde{r}(\tilde{a}_2)\in\tilde{t}(\tilde{a}_2),\ldots,\tilde{r}(\tilde{a}_n)\in\tilde{t}(\tilde{a}_n)$$
$$\left(1-\prod_{i=1}^{n}\prod_{j=1,j\neq i}^{n}(1-(r_{\tilde{a}_i}{}^{L})^p.(r_{\tilde{a}_j}{}^{L})^q)^{\frac{1}{n(n-1)}},1-\prod_{i=1}^{n}\prod_{j=1,j\neq i}^{n}(1-(r_{\tilde{a}_i}{}^{U})^p.(r_{\tilde{a}_j}{}^{U})^q)^{\frac{1}{n(n-1)}}\right),$$

$$\bigcup\tilde{\delta}(\tilde{a}_1)\in\tilde{i}(\tilde{a}_1),\tilde{\delta}(\tilde{a}_2)\in\tilde{i}(\tilde{a}_2)\ldots,\tilde{\delta}(\tilde{a}_n)\in\tilde{i}(\tilde{a}_n)$$
$$\left(\prod_{i=1}^{n}\prod_{j=1,j\neq i}^{n}\left(1-(1-\delta_{\tilde{a}_i}{}^{L})^p(1-\delta_{\tilde{a}_j}{}^{L})^q\right)^{\frac{1}{n(n-1)}},\prod_{i=1}^{n}\prod_{j=1,j\neq i}^{n}\left(1-(1-\delta_{\tilde{a}_i}{}^{U})^p(1-\delta_{\tilde{a}_j}{}^{U})^q\right)^{\frac{1}{n(n-1)}}\right),$$

$$\bigcup\tilde{\eta}(\tilde{a}_1)\in\tilde{f}(\tilde{a}_1),\tilde{\eta}(\tilde{a}_2)\in\tilde{f}(\tilde{a}_2),\ldots,\tilde{\eta}(\tilde{a}_n)\in\tilde{f}(\tilde{a}_n)$$
$$\left.\left(\prod_{i=1}^{n}\prod_{j=1,j\neq i}^{n}\left(1-(1-\eta_{\tilde{a}_i}{}^{L})^p(1-\eta_{\tilde{a}_j}{}^{L})^q\right)^{\frac{1}{n(n-1)}},\prod_{i=1}^{n}\prod_{j=1,j\neq i}^{n}\left(1-(1-\eta_{\tilde{a}_i}{}^{U})^p(1-\eta_{\tilde{a}_j}{}^{U})^q\right)^{\frac{1}{n(n-1)}}\right)\right\rangle$$

Then

$$\left(\frac{1}{n(n-1)}\mathop{\oplus}_{i=1}^{n}\mathop{\oplus}_{j=i}^{n}\tilde{a}_i^{\,p}\otimes\tilde{a}_j^{\,q}\right)^{\frac{1}{p+q}}=\left\langle\left[s_{\left(\sum_{i=1}^{n}\sum_{j=1,j\neq i}^{n}\frac{1}{n(n-1)}.\theta(\tilde{a}_i)^p.\theta(\tilde{a}_j)^q\right)^{\frac{1}{p+q}}},s_{\left(\sum_{i=1}^{n}\sum_{j=1,j\neq i}^{n}\frac{1}{n(n-1)}.\tau(\tilde{a}_i)^p.\tau(\tilde{a}_j)^q\right)^{\frac{1}{p+q}}}\right],\right.$$

$$\bigcup\tilde{r}(\tilde{a}_1)\in\tilde{t}(\tilde{a}_1),\tilde{r}(\tilde{a}_2)\in\tilde{t}(\tilde{a}_2),\ldots,\tilde{r}(\tilde{a}_n)\in\tilde{t}(\tilde{a}_n)$$
$$\left(\left(1-\prod_{i=1}^{n}\prod_{j=1,j\neq i}^{n}(1-(r_{\tilde{a}_i}{}^{L})^p.(r_{\tilde{a}_j}{}^{L})^q)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{p+q}},\left(1-\prod_{i=1}^{n}\prod_{j=1,j\neq i}^{n}(1-(r_{\tilde{a}_i}{}^{U})^p.(r_{\tilde{a}_j}{}^{U})^q)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{p+q}}\right),$$

$$\bigcup\tilde{\delta}(\tilde{a}_1)\in\tilde{i}(\tilde{a}_1),\tilde{\delta}(\tilde{a}_2)\in\tilde{i}(\tilde{a}_2)\ldots,\tilde{\delta}(\tilde{a}_n)\in\tilde{i}(\tilde{a}_n)$$
$$\left(1-\left(1-\prod_{i=1}^{n}\prod_{j=1,j\neq i}^{n}\left(1-(1-\delta_{\tilde{a}_i}{}^{L})^p(1-\delta_{\tilde{a}_j}{}^{L})^q\right)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{p+q}},1-\left(1-\prod_{i=1}^{n}\prod_{j=1,j\neq i}^{n}\left(1-(1-\delta_{\tilde{a}_i}{}^{U})^p(1-\delta_{\tilde{a}_j}{}^{U})^q\right)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{p+q}}\right),$$

$$\cup \tilde{\eta}(\tilde{a}_1) \in \tilde{f}(\tilde{a}_1), \tilde{\eta}(\tilde{a}_2) \in \tilde{f}(\tilde{a}_2), \ldots, \tilde{\eta}(\tilde{a}_n) \in \tilde{f}(\tilde{a}_n)$$

$$\left(1 - \left(1 - \prod_{i=1}^{n} \prod_{j=1, j\neq i}^{n} \left(1-(1-\eta_{\tilde{a}_i}{}^{L})^{p}(1-\eta_{\tilde{a}_j}{}^{L})^{q}\right)^{\frac{1}{n(n-1)}} \right)^{\frac{1}{p+q}}, 1 - \left(1 - \prod_{i=1}^{n} \prod_{j=1, j\neq i}^{n} \left(1-(1-\eta_{\tilde{a}_i}{}^{u})^{p}(1-\eta_{\tilde{a}_j}{}^{u})^{q}\right)^{\frac{1}{n(n-1)}} \right)^{\frac{1}{p+q}} \right) \Bigg\rangle$$

which completes the proof of theorem 11.

Moreover, the IVNHFULBM operator also has the following properties:

Theorem 12 (Idempotency).

Let $\tilde{a}_j = <[S_{\theta(a)}, s_{\tau(a)}], ([r_a{}^L, r_a{}^U], [\delta_a{}^L, \delta_a{}^U], [\eta_a{}^L, \eta_a{}^U]) >= \tilde{a} \quad (j = 1,2,\ldots,n)$, then

$$IVNHFULBM^{p,q}(\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_n) = \tilde{a}$$

Proof .

Since $\tilde{a}_j = <[s_{\theta(a)}, s_{\tau(a)}], \{[r_a{}^L, r_a{}^U]\} >= \tilde{a}\,(j = 1,2,\ldots, n)$, then according to the theorem 11, we can get

$$IVNHFULBM^{p,q}(\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_n) = \left(\frac{1}{n(n-1)} \mathop{\oplus}_{i=1}^{n} \mathop{\oplus}_{j=1, j\neq i}^{n} \tilde{a}_i{}^{p} \otimes \tilde{a}_j{}^{q} \right)^{\frac{1}{p+q}}$$

$$= \Bigg\langle \left[s_{\left(\sum_{i=1}^{n}\sum_{j=1, j\neq i}^{n} \frac{1}{n(n-1)} .\theta(\tilde{a})^{p} .\theta(\tilde{a})^{q}\right)^{\frac{1}{p+q}}}, s_{\left(\sum_{i=1}^{n}\sum_{j=1, j\neq i}^{n} \frac{1}{n(n-1)} .\tau(\tilde{a})^{p} .\tau(\tilde{a})^{q}\right)^{\frac{1}{p+q}}} \right],$$

$$\cup \tilde{r}(\tilde{a}_1) \in \tilde{t}(\tilde{a}_1), \tilde{r}(\tilde{a}_2) \in \tilde{t}(\tilde{a}_2), \ldots, \tilde{r}(\tilde{a}_n) \in \tilde{t}(\tilde{a}_n)$$

$$\left(\left(1 - \prod_{i=1}^{n} \prod_{j=1, j\neq i}^{n} (1-(r_{\tilde{a}}{}^{L})^{p} .(r_{\tilde{a}}{}^{L})^{q})^{\frac{1}{n(n-1)}} \right)^{\frac{1}{p+q}}, \left(1 - \prod_{i=1}^{n} \prod_{j=1, j\neq i}^{n} (1-(r_{\tilde{a}}{}^{U})^{p} .(r_{\tilde{a}}{}^{U})^{q})^{\frac{1}{n(n-1)}} \right)^{\frac{1}{p+q}} \right),$$

$$\cup \tilde{\delta}(\tilde{a}_1) \in \tilde{i}(\tilde{a}_1), \tilde{\delta}(\tilde{a}_2) \in \tilde{i}(\tilde{a}_2) \ldots, \tilde{\delta}(\tilde{a}_n) \in \tilde{i}(\tilde{a}_n)$$

$$\left(1 - \left(1 - \prod_{i=1}^{n} \prod_{j=1, j\neq i}^{n} \left(1-(1-\delta_{\tilde{a}}{}^{L})^{p}(1-\delta_{\tilde{a}}{}^{L})^{q}\right)^{\frac{1}{n(n-1)}} \right)^{\frac{1}{p+q}}, 1 - \left(1 - \prod_{i=1}^{n} \prod_{j=1, j\neq i}^{n} \left(1-(1-\delta_{\tilde{a}}{}^{U})^{p}(1-\delta_{\tilde{a}}{}^{U})^{q}\right)^{\frac{1}{n(n-1)}} \right)^{\frac{1}{p+q}} \right),$$

$$\cup \tilde{\eta}(\tilde{a}_1) \in \tilde{f}(\tilde{a}_1), \tilde{\eta}(\tilde{a}_2) \in \tilde{f}(\tilde{a}_2), \ldots, \tilde{\eta}(\tilde{a}_n) \in \tilde{f}(\tilde{a}_n)$$

$$\left(1 - \left(1 - \prod_{i=1}^{n} \prod_{j=1, j\neq i}^{n} \left(1-(1-\eta_{\tilde{a}}{}^{L})^{p}(1-\eta_{\tilde{a}}{}^{L})^{q}\right)^{\frac{1}{n(n-1)}} \right)^{\frac{1}{p+q}}, 1 - \left(1 - \prod_{i=1}^{n} \prod_{j=1, j\neq i}^{n} \left(1-(1-\eta_{\tilde{a}}{}^{U})^{p}(1-\eta_{\tilde{a}}{}^{U})^{q}\right)^{\frac{1}{n(n-1)}} \right)^{\frac{1}{p+q}} \right) \Bigg\rangle$$

$$= <[s_{\theta(a)}, s_{\tau(a)}], ([r_a{}^L, r_a{}^U], [\delta_a{}^L, \delta_a{}^U], [\eta_a{}^L, \eta_a{}^U]) >= \tilde{a}$$

So, we complete the proof of the theorem 12.

Theorem 13 (Commutativity).

Let $(\tilde{a}_1', \tilde{a}_2', \ldots, \tilde{a}_n')$ be any permutation of $(\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_n)$, then we can get

$$IVNHFULBM^{p,q}(\tilde{a}_1', \tilde{a}_2', \ldots, \tilde{a}_n') = IVNHFULBM^{p,q}(\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_n).$$

Proof.

$$IVNHFULBM^{p,q}(\tilde{a}_1{}', \tilde{a}_2{}', ..., \tilde{a}_n{}') = \left(\frac{1}{n(n-1)} \underset{i=1}{\overset{n}{\oplus}} \underset{j=1, j\neq i}{\overset{n}{\oplus}} \tilde{a}_i{}'^p \otimes \tilde{a}_j{}'^q \right)^{\frac{1}{p+q}}$$

$$= \left\langle \left[s_{\left(\sum_{i=1}^{n}\sum_{j=1, j\neq i}^{n} \frac{1}{n(n-1)} \cdot \theta(\tilde{a}_i{}')^p \cdot \theta(\tilde{a}_j{}')^q\right)^{\frac{1}{p+q}}}, s_{\left(\sum_{i=1}^{n}\sum_{j=1, j\neq i}^{n} \frac{1}{n(n-1)} \cdot \tau(\tilde{a}_i{}')^p \cdot \tau(\tilde{a}_j{}')^q\right)^{\frac{1}{p+q}}} \right], \right.$$

$$\bigcup_{\tilde{r}(\tilde{a}'_1) \in \tilde{t}(\tilde{a}_1{}'), \tilde{r}(\tilde{a}_2{}') \in \tilde{t}(\tilde{a}_2{}'), ..., \tilde{r}(\tilde{a}_n{}') \in \tilde{t}(\tilde{a}_n{}')} \left(\left(1 - \prod_{i=1}^{n} \prod_{j=1, j\neq i}^{n} (1 - (r_{\tilde{a}_i{}'}{}^L)^p \cdot (r_{\tilde{a}_j{}'}{}^L)^q)^{\frac{1}{n(n-1)}} \right)^{\frac{1}{p+q}}, \left(1 - \prod_{i=1}^{n} \prod_{j=1, j\neq i}^{n} (1 - (r_{\tilde{a}_i{}'}{}^U)^p \cdot (r_{\tilde{a}_j{}'}{}^U)^q)^{\frac{1}{n(n-1)}} \right)^{\frac{1}{p+q}} \right),$$

$$\bigcup_{\tilde{\delta}(\tilde{a}_1{}') \in \tilde{i}(\tilde{a}_1{}'), \tilde{\delta}(\tilde{a}_2{}') \in \tilde{i}(\tilde{a}_2{}')..., \tilde{\delta}(\tilde{a}_n{}') \in \tilde{i}(\tilde{a}_n{}')} \left(1 - \left(1 - \prod_{i=1}^{n} \prod_{j=1, j\neq i}^{n} \left(1 - (1 - \delta_{\tilde{a}_i{}'}{}^L)^p (1 - \delta_{\tilde{a}_j{}'}{}^L)^q \right)^{\frac{1}{n(n-1)}} \right)^{\frac{1}{p+q}}, 1 - \left(1 - \prod_{i=1}^{n} \prod_{j=1, j\neq i}^{n} \left(1 - (1 - \delta_{\tilde{a}_i{}'}{}^U)^p (1 - \delta_{\tilde{a}_j{}'}{}^U)^q \right)^{\frac{1}{n(n-1)}} \right)^{\frac{1}{p+q}} \right),$$

$$\left. \bigcup_{\tilde{\eta}(\tilde{a}_1{}') \in \tilde{f}(\tilde{a}_1{}'), \tilde{\eta}(\tilde{a}_2{}') \in \tilde{f}(\tilde{a}_2{}'), ..., \tilde{\eta}(\tilde{a}_n{}') \in \tilde{f}(\tilde{a}_n{}')} \left(1 - \left(1 - \prod_{i=1}^{n} \prod_{j=1, j\neq i}^{n} \left(1 - (1 - \eta_{\tilde{a}_i{}'}{}^L)^p (1 - \eta_{\tilde{a}_j{}'}{}^L)^q \right)^{\frac{1}{n(n-1)}} \right)^{\frac{1}{p+q}}, 1 - \left(1 - \prod_{i=1}^{n} \prod_{j=1, j\neq i}^{n} \left(1 - (1 - \eta_{\tilde{a}_i{}'}{}^U)^p (1 - \eta_{\tilde{a}_j{}'}{}^U)^q \right)^{\frac{1}{n(n-1)}} \right)^{\frac{1}{p+q}} \right) \right\rangle$$

Since $\{\tilde{a}_1', \tilde{a}_2', ..., \tilde{a}_n'\}$ is any permutation of $\{\tilde{a}_1, \tilde{a}_2, ..., \tilde{a}_n\}$, then we have

$$\sum_{i=1}^{n} \theta(\tilde{a}_i{}')^p = \sum_{i=1}^{n} \theta(\tilde{a}_i)^p, \sum_{j=1, j\neq i}^{n} \theta(\tilde{a}_j{}')^q = \sum_{j=1, j\neq i}^{n} \theta(\tilde{a}_j)^q,$$
$$\sum_{i=1}^{n} \tau(\tilde{a}_i)^p = \sum_{i=1}^{n} \tau(\tilde{a}_i{}')^p, \sum_{j=1, j\neq i}^{n} \tau(\tilde{a}_j{}')^q = \sum_{j=1, j\neq i}^{n} \tau(\tilde{a}_j)^q.$$

so, we can get

$$\left(\sum_{i=1}^{n} \sum_{j=1, j\neq i}^{n} \frac{1}{n(n-1)} \theta(\tilde{a}_i{}')^p \theta(\tilde{a}_j{}')^q \right)^{\frac{1}{p+q}} = \left(\frac{1}{n(n-1)} \sum_{i=1}^{n} \theta(\tilde{a}_i{}')^p \sum_{j=1, j\neq i}^{n} \theta(\tilde{a}_j{}')^q \right)^{\frac{1}{p+q}}$$
$$= \left(\frac{1}{n(n-1)} \sum_{i=1}^{n} \theta(\tilde{a}_i)^p \sum_{j=1, j\neq i}^{n} \theta(\tilde{a}_j)^q \right)^{\frac{1}{p+q}}$$

$$\left(\sum_{i=1}^{n} \sum_{j=1, j\neq i}^{n} \frac{1}{n(n-1)} \tau(\tilde{a}_i{}')^p \tau(\tilde{a}_j{}')^q \right)^{\frac{1}{p+q}} = \left(\frac{1}{n(n-1)} \sum_{i=1}^{n} \tau(\tilde{a}_i{}')^p \sum_{j=1, j\neq i}^{n} \tau(\tilde{a}_j{}')^q \right)^{\frac{1}{p+q}}$$
$$= \left(\frac{1}{n(n-1)} \sum_{i=1}^{n} \tau(\tilde{a}_i)^p \sum_{j=1, j\neq i}^{n} \tau(\tilde{a}_j)^q \right)^{\frac{1}{p+q}}$$

and

$$\prod_{i=1}^{n}\prod_{j=1,j\neq i}^{n}(1-(r_{\tilde{a}_i'}{}^{L})^{p}.(r_{\tilde{a}_j'}{}^{L})^{q})^{\frac{1}{n(n-1)}}=\left(\prod_{i=1}^{n}\prod_{j=1,j\neq i}^{n}(1-(r_{\tilde{a}_i'}{}^{L})^{p}.(r_{\tilde{a}_j'}{}^{L})^{q})\right)^{\frac{1}{n(n-1)}}$$

$$=\left(\prod_{i=1}^{n}\prod_{j=1,j\neq i}^{n}(1-(r_{\tilde{a}_i}{}^{L})^{p}.(r_{\tilde{a}_j}{}^{L})^{q})\right)^{\frac{1}{n(n-1)}}=\prod_{i=1}^{n}\prod_{j=1,j\neq i}^{n}(1-(r_{\tilde{a}_i}{}^{L})^{p}.(r_{\tilde{a}_j}{}^{L})^{q})^{\frac{1}{n(n-1)}}$$

Similarly,

$$\prod_{i=1}^{n}\prod_{j=1,j\neq i}^{n}(1-(r_{\tilde{a}_i'}{}^{U})^{p}(r_{\tilde{a}_j'}{}^{U})^{q})^{\frac{1}{n(n-1)}}=\prod_{i=1}^{n}\prod_{j=1,j\neq i}^{n}(1-(r_{\tilde{a}_i}{}^{U})^{p}(r_{\tilde{a}_j}{}^{U})^{q})^{\frac{1}{n(n-1)}}$$

$$\prod_{i=1}^{n}\prod_{j=1,j\neq i}^{n}\left(1-(1-\delta_{\tilde{a}_i'}{}^{L})^{p}(1-\delta_{\tilde{a}_j'}{}^{L})^{q}\right)=\prod_{i=1}^{n}\prod_{j=1,j\neq i}^{n}\left(1-(1-\delta_{\tilde{a}_i}{}^{L})^{p}(1-\delta_{\tilde{a}_j}{}^{L})^{q}\right)$$

$$\prod_{i=1}^{n}\prod_{j=1,j\neq i}^{n}\left(1-(1-\delta_{\tilde{a}_i'}{}^{U})^{p}(1-\delta_{\tilde{a}_j'}{}^{U})^{q}\right)=\prod_{i=1}^{n}\prod_{j=1,j\neq i}^{n}\left(1-(1-\delta_{\tilde{a}_i}{}^{U})^{p}(1-\delta_{\tilde{a}_j}{}^{U})^{q}\right)$$

$$\prod_{i=1}^{n}\prod_{j=1,j\neq i}^{n}\left(1-(1-\eta_{\tilde{a}_i'}{}^{L})^{p}(1-\eta_{\tilde{a}_j'}{}^{L})^{q}\right)=\prod_{i=1}^{n}\prod_{j=1,j\neq i}^{n}\left(1-(1-\eta_{\tilde{a}_i}{}^{L})^{p}(1-\eta_{\tilde{a}_j}{}^{L})^{q}\right)$$

$$\prod_{i=1}^{n}\prod_{j=1,j\neq i}^{n}\left(1-(1-\eta_{\tilde{a}_i'}{}^{U})^{p}(1-\eta_{\tilde{a}_j'}{}^{U})^{q}\right)=\prod_{i=1}^{n}\prod_{j=1,j\neq i}^{n}\left(1-(1-\eta_{\tilde{a}_i}{}^{U})^{p}(1-\eta_{\tilde{a}_j}{}^{U})^{q}\right)$$

So, according to the theorem 13, we can get

$IVNHFULBM^{p,q}(\tilde{a}_1,\tilde{a}_2,...,\tilde{a}_n)=IVNHFULBM^{p,q}(\tilde{a}_1',\tilde{a}_2',...,\tilde{a}_n')$.

Theorem 14 (Monotonicity).

Suppose $(\tilde{a}_1,\tilde{a}_2,...,\tilde{a}_n)$ and $(\tilde{b}_1,\tilde{b}_2,...,\tilde{b}_n)$ are any two sets of IVNHFULEs. If $s_{\theta(\tilde{a}_i)}\leq s_{\theta(\tilde{b}_i)}$, $s_{\tau(\tilde{a}_i)}\leq s_{\tau(\tilde{b}_i)}$, $r_{\tilde{a}_i}{}^{L}\leq r_{\tilde{b}_i}{}^{L}$, $r_{\tilde{a}_i}{}^{U}\leq r_{\tilde{b}_i}{}^{U}$, $\delta_{\tilde{a}_i}{}^{L}\geq\delta_{\tilde{b}_i}{}^{L}$, $\delta_{\tilde{a}_i}{}^{U}\geq\delta_{\tilde{b}_i}{}^{U}$ and $\eta_{\tilde{a}_i}{}^{L}\geq\eta_{\tilde{b}_i}{}^{L}$, $\eta_{\tilde{a}_i}{}^{U}\geq\eta_{\tilde{b}_i}{}^{U}$ for all $i=1,2,...,n$,then

$$IVNHFULBM^{p,q}(\tilde{a}_1,\tilde{a}_2,...,\tilde{a}_n)\leq IVNHFULBM^{p,q}(\tilde{b}_1,\tilde{b}_2,...,\tilde{b}_n)$$

Proof.

(1) Since $s_{\theta(\tilde{a}_i)}\leq s_{\theta(\tilde{b}_i)}$, then $\theta(\tilde{a}_i)\leq\theta(\tilde{b}_i)$, according to theorem 11 , we have

$$\left(\sum_{i=1}^{n}\sum_{j=1,j\neq i}^{n}\frac{1}{n(n-1)}\theta(\tilde{a}_i)^{p}\times\theta(\tilde{a}_j)^{q}\right)^{\frac{1}{p+q}}\leq\left(\sum_{i=1}^{n}\sum_{j=1,j\neq i}^{n}\frac{1}{n(n-1)}\theta(\tilde{b}_i)^{p}\times\theta(\tilde{b}_j)^{q}\right)^{\frac{1}{p+q}},$$

(2) Similar to (1),we have

$$\left(\sum_{i=1}^{n}\sum_{j=1,j\neq i}^{n}\frac{1}{n(n-1)}\tau(\tilde{a}_i)^{p}\times\tau(\tilde{a}_j)^{q}\right)^{\frac{1}{p+q}}\leq\left(\sum_{i=1}^{n}\sum_{j=1,j\neq i}^{n}\frac{1}{n(n-1)}\tau(\tilde{b}_i)^{p}\times\tau(\tilde{b}_j)^{q}\right)^{\frac{1}{p+q}}$$

(3) Since $r_{\tilde{a}_i}{}^{L}\leq r_{\tilde{b}_i}{}^{L}$ for all i and $p,q\geq 0$, then $(r_{\tilde{a}_i}{}^{L})^{p}\leq(r_{\tilde{b}_i}{}^{L})^{p}$,

$$(r_{\tilde{a}_i}{}^L)^p (r_{\tilde{a}_j}{}^L)^q \le (r_{\tilde{b}_i}{}^L)^p (r_{\tilde{b}_j}{}^L)^q, 1-(r_{\tilde{a}_i}{}^L)^p (r_{\tilde{a}_j}{}^L)^q \ge 1-(r_{\tilde{b}_i}{}^L)^p (r_{\tilde{b}_j}{}^L)^q,$$

$$\prod_{i=1}^{n}\prod_{j=1, j\ne i}^{n}\left(1-(r_{\tilde{a}_i}{}^L)^p (r_{\tilde{a}_j}{}^L)^q\right)^{\frac{1}{n(n-1)}} \ge \prod_{i=1}^{n}\prod_{j=1, j\ne i}^{n}\left(1-(r_{\tilde{b}_i}{}^L)^p (r_{\tilde{b}_j}{}^L)^q\right)^{\frac{1}{n(n-1)}}$$

and

$$1-\prod_{i=1}^{n}\prod_{j=1, j\ne i}^{n}\left(1-(r_{\tilde{a}_i}{}^L)^p (r_{\tilde{a}_j}{}^L)^q\right)^{\frac{1}{n(n-1)}} \le 1-\prod_{i=1}^{n}\prod_{j=1, j\ne i}^{n}\left(1-(r_{\tilde{b}_i}{}^L)^p (r_{\tilde{b}_j}{}^L)^q\right)^{\frac{1}{n(n-1)}}$$

So

$$\left(1-\prod_{i=1}^{n}\prod_{j=1, j\ne i}^{n}\left(1-(r_{\tilde{a}_i}{}^L)^p (r_{\tilde{a}_j}{}^L)^q\right)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{p+q}} \le \left(1-\prod_{i=1}^{n}\prod_{j=1, j\ne i}^{n}\left(1-(r_{\tilde{b}_i}{}^L)^p (r_{\tilde{b}_j}{}^L)^q\right)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{p+q}}$$

Similar to (3), we have

$$\left(1-\prod_{i=1}^{n}\prod_{j=1, j\ne i}^{n}\left(1-(r_{\tilde{a}_i}{}^U)^p (r_{\tilde{a}_j}{}^U)^q\right)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{p+q}} \le \left(1-\prod_{i=1}^{n}\prod_{j=1, j\ne i}^{n}\left(1-(r_{\tilde{b}_i}{}^U)^p (r_{\tilde{b}_j}{}^U)^q\right)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{p+q}}$$

(4) since $\delta_{\tilde{a}_i}{}^L \ge \delta_{\tilde{b}_i}{}^L$ for all i and $p,q \ge 0$, then $(1-\delta_{\tilde{a}_i}{}^L)^p \le (1-\delta_{\tilde{b}_i}{}^L)^p$,

$$(1-\delta_{\tilde{a}_i}{}^L)^p (1-\delta_{\tilde{a}_j}{}^L)^q \le (1-\delta_{\tilde{b}_i}{}^L)^p (1-\delta_{\tilde{b}_j}{}^L)^q, 1-(1-\delta_{\tilde{a}_i}{}^L)^p (1-\delta_{\tilde{a}_j}{}^L)^q \ge 1-(1-\delta_{\tilde{b}_i}{}^L)^p (1-\delta_{\tilde{b}_j}{}^L)^q$$

$$\left(1-(1-\delta_{\tilde{a}_i}{}^L)^p (1-\delta_{\tilde{a}_j}{}^L)^q\right)^{\frac{1}{n(n+1)}} \ge \left(1-(1-\delta_{\tilde{b}_i}{}^L)^p (1-\delta_{\tilde{b}_j}{}^L)^q\right)^{\frac{1}{n(n+1)}}$$

And

$$\prod_{i=1}^{n}\prod_{j=1, j\ne i}^{n}\left(1-(1-\delta_{\tilde{a}_i}{}^L)^p (1-\delta_{\tilde{a}_j}{}^L)^q\right)^{\frac{1}{n(n-1)}} \ge \prod_{i=1}^{n}\prod_{j=1, j\ne i}^{n}\left(1-(1-\delta_{\tilde{b}_i}{}^L)^p (1-\delta_{\tilde{b}_j}{}^L)^q\right)^{\frac{1}{n(n-1)}}$$

Then

$$\left(1-\prod_{i=1}^{n}\prod_{j=1, j\ne i}^{n}\left(1-(1-\delta_{\tilde{a}_i}{}^L)^p (1-\delta_{\tilde{a}_j}{}^L)^q\right)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{p+q}}$$
$$\le \left(1-\prod_{i=1}^{n}\prod_{j=1, j\ne i}^{n}\left(1-(1-\delta_{\tilde{b}_i}{}^L)^p (1-\delta_{\tilde{b}_j}{}^L)^q\right)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{p+q}}$$

So

$$1-\left(1-\prod_{i=1}^{n}\prod_{j=1, j\ne i}^{n}\left(1-(1-\delta_{\tilde{a}_i}{}^L)^p (1-\delta_{\tilde{a}_j}{}^L)^q\right)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{p+q}}$$
$$\ge 1-\left(1-\prod_{i=1}^{n}\prod_{j=1, j\ne i}^{n}\left(1-(1-\delta_{\tilde{b}_i}{}^L)^p (1-\delta_{\tilde{b}_j}{}^L)^q\right)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{p+q}}$$

Similarly,

$$1-\left(1-\prod_{i=1}^{n}\prod_{j=1,j\neq i}^{n}\left(1-(1-\delta_{\tilde{a}_i}{}^{U})^{p}(1-\delta_{\tilde{a}_j}{}^{U})^{q}\right)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{p+q}}$$
$$\geq 1-\left(1-\prod_{i=1}^{n}\prod_{j=1,j\neq i}^{n}\left(1-(1-\delta_{\tilde{b}_i}{}^{U})^{p}(1-\delta_{\tilde{b}_j}{}^{U})^{q}\right)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{p+q}}$$

(5) Similar to (4), we can know

$$1-\left(1-\prod_{i=1}^{n}\prod_{j=1,j\neq i}^{n}\left(1-(1-\eta_{\tilde{a}_i}{}^{L})^{p}(1-\eta_{\tilde{a}_j}{}^{L})^{q}\right)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{p+q}}$$
$$\geq 1-\left(1-\prod_{i=1}^{n}\prod_{j=1,j\neq i}^{n}\left(1-(1-\eta_{\tilde{b}_i}{}^{L})^{p}(1-\eta_{\tilde{b}_j}{}^{L})^{q}\right)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{p+q}}$$

$$1-\left(1-\prod_{i=1}^{n}\prod_{j=1,j\neq i}^{n}\left(1-(1-\eta_{\tilde{a}_i}{}^{U})^{p}(1-\eta_{\tilde{a}_j}{}^{U})^{q}\right)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{p+q}}$$
$$\geq 1-\left(1-\prod_{i=1}^{n}\prod_{j=1,j\neq i}^{n}\left(1-(1-\eta_{\tilde{b}_i}{}^{U})^{p}(1-\eta_{\tilde{b}_j}{}^{U})^{q}\right)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{p+q}}$$

So, we can get $IVNHFULBM^{p,q}(\tilde{a}_1,\tilde{a}_2,...,\tilde{a}_n)\leq IVNHFULBM^{p,q}(\tilde{b}_1,\tilde{b}_2,...,\tilde{b}_n)$,

which completes the proof of the theorem 14.

Theorem 15 (Boundedness).

Let $\tilde{a}_i\,(i=1,2,...,n)$ be a set of IVNHFULEs. If $s_{\theta-}=\min_{1\leq i\leq n}\{s_{\theta(\tilde{a}_i)}\}$, $s_{\tau-}=\min_{1\leq i\leq n}\{s_{\tau(\tilde{a}_i)}\}$,

$s_{\theta+}=\max_{1\leq i\leq n}\{s_{\theta(\tilde{a}_i)}\}$, $s_{\tau+}=\max_{1\leq i\leq n}\{s_{\tau(\tilde{a}_i)}\}$, $r^{L-}=\min_{1\leq i\leq n}\{r_{\tilde{a}_i}{}^{L}\mid[r_{\tilde{a}_i}{}^{L},r_{\tilde{a}_i}{}^{U}]\in t(\tilde{a}_i)\}$,

$r^{U-}=\min_{1\leq i\leq n}\{r_{\tilde{a}_i}{}^{U}\mid[r_{\tilde{a}_i}{}^{L},r_{\tilde{a}_i}{}^{U}]\in t(\tilde{a}_i)\}$, $r^{L+}=\max_{1\leq i\leq n}\{r_{\tilde{a}_i}{}^{L}\mid[r_{\tilde{a}_i}{}^{L},r_{\tilde{a}_i}{}^{U}]\in t(\tilde{a}_i)\}$

$r^{U+}=\max_{1\leq i\leq n}\{r_{\tilde{a}_i}{}^{U}\mid[r_{\tilde{a}_i}{}^{L},r_{\tilde{a}_i}{}^{U}]\in t(\tilde{a}_i)\}$, $\delta^{L-}=\min_{1\leq i\leq n}\{\delta_{\tilde{a}_i}{}^{L}\mid[\delta_{\tilde{a}_i}{}^{L},\delta_{\tilde{a}_i}{}^{U}]\in i(\tilde{a}_i)\}$,

$\delta^{U-}=\min_{1\leq i\leq n}\{\delta_{\tilde{a}_i}{}^{U}\mid[\delta_{\tilde{a}_i}{}^{L},\delta_{\tilde{a}_i}{}^{U}]\in i(\tilde{a}_i)\}$, $\delta^{L+}=\max_{1\leq i\leq n}\{\delta_{\tilde{a}_i}{}^{L}\mid[\delta_{\tilde{a}_i}{}^{L},\delta_{\tilde{a}_i}{}^{U}]\in i(\tilde{a}_i)\}$

$\delta^{U+}=\max_{1\leq i\leq n}\{\delta_{\tilde{a}_i}{}^{U}\mid[\delta_{\tilde{a}_i}{}^{L},\delta_{\tilde{a}_i}{}^{U}]\in i(\tilde{a}_i)\}$, $\eta^{L-}=\min_{1\leq i\leq n}\{\eta_{\tilde{a}_i}{}^{L}\mid[\eta_{\tilde{a}_i}{}^{L},\eta_{\tilde{a}_i}{}^{U}]\in f(\tilde{a}_i)\}$,

$\eta^{U-}=\min_{1\leq i\leq n}\{\eta_{\tilde{a}_i}{}^{U}\mid[\eta_{\tilde{a}_i}{}^{L},\eta_{\tilde{a}_i}{}^{U}]\in f(\tilde{a}_i)\}$, $\eta^{L+}=\max_{1\leq i\leq n}\{\eta_{\tilde{a}_i}{}^{L}\mid[\eta_{\tilde{a}_i}{}^{L},\eta_{\tilde{a}_i}{}^{U}]\in f(\tilde{a}_i)\}$

$\eta^{U+}=\max_{1\leq i\leq n}\{\eta_{\tilde{a}_i}{}^{U}\mid[\eta_{\tilde{a}_i}{}^{L},\eta_{\tilde{a}_i}{}^{U}]\in f(\tilde{a}_i)\}$, for all $i=1,2,...,n$,

$\tilde{a}^{-}=<[s_{\theta-},s_{\tau-}],([r^{L-},r^{U-}],[\delta^{L+},\delta^{U+}],[\eta^{L+},\eta^{L+}])>$ and

$\tilde{a}^{+}=<[s_{\theta+},s_{\tau+}],([r^{L+},r^{U+}],[\delta^{L-},\delta^{U-}],[\eta^{L-},\eta^{U-}])>$ then

$$\tilde{a}^- \le IVNHFULBM^{p,q}(\tilde{a}_1, \tilde{a}_2, ..., \tilde{a}_n) \le \tilde{a}^+$$

Proof.

Based on above terms, we can get

(1) $\theta_-^{\,p+q} \le \theta(\tilde{a}_i)^p \theta(\tilde{a}_j)^q \le \theta_+^{\,p+q}$

$$\left(\sum_{i=1}^{n}\sum_{j=1, j\neq i}^{n}\frac{1}{n(n-1)}\theta_-^{\,p+q}\right)^{\frac{1}{p+q}} \le \left(\sum_{i=1}^{n}\sum_{j=1, j\neq i}^{n}\frac{1}{n(n-1)}.\theta(\tilde{a}_i)^p.\theta(\tilde{a}_j)^q\right)^{\frac{1}{p+q}} \le \left(\sum_{i=1}^{n}\sum_{j=1, j\neq i}^{n}\frac{1}{n(n-1)}\theta_+^{\,p+q}\right)^{\frac{1}{p+q}}$$

$$\theta_- \le \left(\sum_{i=1}^{n}\sum_{j=1, j\neq i}^{n}\frac{1}{n(n-1)}.\theta(\tilde{a}_i)^p.\theta(\tilde{a}_j)^q\right)^{\frac{1}{p+q}} \le \theta_+$$

Similarly, we can know

$$\tau_- \le \left(\sum_{i=1}^{n}\sum_{j=1, j\neq i}^{n}\frac{1}{n(n-1)}\tau(\tilde{a}_i)^p \tau(\tilde{a}_j)^q\right)^{\frac{1}{p+q}} \le \tau_+$$

(2) $(r^{L-})^{p+q} \le (r_{\tilde{a}_i}{}^{L})^p.(r_{\tilde{a}_j}{}^{L})^q \le (r^{L+})^{p+q}$

$$(1-(r^{L-})^{p+q})^{\frac{1}{n(n-1)}} \le (1-(r_{\tilde{a}_i}{}^{L})^p.(r_{\tilde{a}_j}{}^{L})^q)^{\frac{1}{n(n-1)}} \le (1-(r^{L+})^{p+q})^{\frac{1}{n(n-1)}}$$

$$\prod_{i=1}^{n}\prod_{j=1, j\neq i}^{n}(1-(r^{L-})^{p+q})^{\frac{1}{n(n-1)}} \le \prod_{i=1}^{n}\prod_{j=1, j\neq i}^{n}(1-(r_{\tilde{a}_i}{}^{L})^p.(r_{\tilde{a}_j}{}^{L})^q)^{\frac{1}{n(n-1)}} \le \prod_{i=1}^{n}\prod_{j=1, j\neq i}^{n}(1-(r^{L+})^{p+q})^{\frac{1}{n(n-1)}}$$

$$\left(1-\prod_{i=1}^{n}\prod_{j=1, j\neq i}^{n}(1-(r^{L+})^{p+q})^{\frac{1}{n(n-1)}}\right)^{\frac{1}{p+q}} \le \left(1-\prod_{i=1}^{n}\prod_{j=1, j\neq i}^{n}(1-(r_{\tilde{a}_i}{}^{L})^p.(r_{\tilde{a}_j}{}^{L})^q)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{p+q}}$$
$$\le \left(1-\prod_{i=1}^{n}\prod_{j=1, j\neq i}^{n}(1-(r^{L-})^{p+q})^{\frac{1}{n(n-1)}}\right)^{\frac{1}{p+q}}$$

$$r^{L+} \le \left(1-\prod_{i=1}^{n}\prod_{j=1, j\neq i}^{n}(1-(r_{\tilde{a}_i}{}^{L})^p.(r_{\tilde{a}_j}{}^{L})^q)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{p+q}} \le r^{L-}$$

Likewise, we also get

$$r^{U+} \le \left(1-\prod_{i=1}^{n}\prod_{j=1, j\neq i}^{n}(1-(r_{\tilde{a}_i}{}^{U})^p.(r_{\tilde{a}_j}{}^{U})^q)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{p+q}} \le r^{U-}$$

(3) $(1-\delta^{L+})^{p+q} \le (1-\delta_{\tilde{a}_i}{}^{L})^p(1-\delta_{\tilde{a}_j}{}^{L})^q \le (1-\delta^{L-})^{p+q}$

$$\prod_{i=1}^{n}\prod_{j=1, j\neq i}^{n}\left(1-(1-\delta^{L-})^{p+q}\right) \le \prod_{i=1}^{n}\prod_{j=1, j\neq i}^{n}\left(1-(1-\delta_{\tilde{a}_i}{}^{L})^p(1-\delta_{\tilde{a}_j}{}^{L})^q\right) \le \prod_{i=1}^{n}\prod_{j=1, j\neq i}^{n}\left(1-(1-\delta^{L+})^{p+q}\right)$$

$$1-\left(1-\prod_{i=1}^{n}\prod_{j=1,j\neq i}^{n}\left(1-(1-\delta^{L+})^{p+q}\right)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{p+q}} \leq 1-\left(1-\prod_{i=1}^{n}\prod_{j=1,j\neq i}^{n}\left(1-(1-\delta_{\tilde{a}_i}{}^{L})^{p}(1-\delta_{\tilde{a}_j}{}^{L})^{q}\right)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{p+q}}$$

$$\leq \quad 1-\left(1-\prod_{i=1}^{n}\prod_{j=1,j\neq i}^{n}\left(1-(1-\delta^{L-})^{p+q}\right)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{p+q}}$$

$$\delta^{L+} \leq 1-\left(1-\prod_{i=1}^{n}\prod_{j=1,j\neq i}^{n}\left(1-(1-\delta_{\tilde{a}_i}{}^{L})^{p}(1-\delta_{\tilde{a}_j}{}^{L})^{q}\right)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{p+q}} \leq \delta^{L-}$$

Similarly,

$$\delta^{U+} \leq 1-\left(1-\prod_{i=1}^{n}\prod_{j=1,j\neq i}^{n}\left(1-(1-\delta_{\tilde{a}_i}{}^{U})^{p}(1-\delta_{\tilde{a}_j}{}^{U})^{q}\right)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{p+q}} \leq \delta^{U-}$$

(4)Similar to the proof of the (3), the results can be got.

$$\eta^{L+} \leq 1-\left(1-\prod_{i=1}^{n}\prod_{j=1,j\neq i}^{n}\left(1-(1-\eta_{\tilde{a}_i}{}^{L})^{p}(1-\eta_{\tilde{a}_j}{}^{L})^{q}\right)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{p+q}} \leq \eta^{L-}$$

$$\eta^{U+} \leq 1-\left(1-\prod_{i=1}^{n}\prod_{j=1,j\neq i}^{n}\left(1-(1-\eta_{\tilde{a}_i}{}^{U})^{p}(1-\eta_{\tilde{a}_j}{}^{U})^{q}\right)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{p+q}} \leq \eta^{U-}$$

So, we can know $\tilde{a}^{-} \leq IVNHFULBM^{p,q}(\tilde{a}_1,\tilde{a}_2,...,\tilde{a}_n) \leq \tilde{a}^{+}$

which completes the proof of theorem 15.

In the following, we will discuss some special cases of the IVNHFULBM operator with respect to the different parameters p and q.

(1) If $q=0$, then

$$IVNHFULBM^{p,0}(\tilde{a}_1,\tilde{a}_2,...,\tilde{a}_n)=\left(\frac{1}{n(n-1)}\mathop{\oplus}_{i=1}^{n}\mathop{\oplus}_{j=1,j\neq i}^{n}\tilde{a}_i{}^{p}\right)^{\frac{1}{p}}=\left(\frac{1}{n}\mathop{\oplus}_{i=1}^{n}\tilde{a}_i{}^{p}\right)^{\frac{1}{p}}$$

$$=\left\langle\left[s_{\left(\sum_{i=1}^{n}\frac{1}{n}\theta(\tilde{a}_i)^{p}\right)^{\frac{1}{p}}},s_{\left(\sum_{i=1}^{n}\frac{1}{n}\tau(\tilde{a}_i)^{p}\right)^{\frac{1}{p}}}\right],\bigcup_{\tilde{r}(\tilde{a}_1)\in\tilde{t}(\tilde{a}_1),\tilde{r}(\tilde{a}_2)\in\tilde{t}(\tilde{a}_2),...,\tilde{r}(\tilde{a}_n)\in\tilde{t}(\tilde{a}_n)}\left(\left(1-\prod_{i=1}^{n}(1-(r_{\tilde{a}_i}{}^{L})^{p})^{\frac{1}{n}}\right)^{\frac{1}{p}},\left(1-\prod_{i=1}^{n}(1-(r_{\tilde{a}_i}{}^{U})^{p})^{\frac{1}{n}}\right)^{\frac{1}{p}}\right),\right.$$

$$\bigcup_{\tilde{\delta}(\tilde{a}_1)\in\tilde{i}(\tilde{a}_1),\tilde{\delta}(\tilde{a}_2)\in\tilde{i}(\tilde{a}_2)...,\tilde{\delta}(\tilde{a}_n)\in\tilde{i}(\tilde{a}_n)}\left(1-\left(1-\left(\prod_{i=1}^{n}\left(1-(1-\delta_{\tilde{a}_i}{}^{L})^{p}\right)\right)^{\frac{1}{n}}\right)^{\frac{1}{p}},1-\left(1-\left(\prod_{i=1}^{n}\left(1-(1-\delta_{\tilde{a}_i}{}^{U})^{p}\right)\right)^{\frac{1}{n}}\right)^{\frac{1}{p}}\right),$$

$$\bigcup \tilde{\eta}(\tilde{a}_1)\in \tilde{f}(\tilde{a}_1),\tilde{\eta}(\tilde{a}_2)\in \tilde{f}(\tilde{a}_2),...,\tilde{\eta}(\tilde{a}_n)\in \tilde{f}(\tilde{a}_n)$$

$$\left\{\left(1-\left(1-\left(\prod_{i=1}^{n}\left(1-(1-\eta_{\tilde{a}_i}{}^{L})^{p}\right)\right)^{\frac{1}{n}}\right)^{\frac{1}{p}},1-\left(1-\left(\prod_{i=1}^{n}\left(1-(1-\eta_{\tilde{a}_i}{}^{U})^{p}\right)\right)^{\frac{1}{n}}\right)^{\frac{1}{p}}\right)\right\}\right\rangle$$

(2) If $p=1$ and $q=0$, then

$$IVNHFULBM^{1,0}(\tilde{a}_1,\tilde{a}_2,...,\tilde{a}_n)=\left(\frac{1}{n(n-1)}\mathop{\oplus}_{i=1}^{n}\mathop{\oplus}_{j=1,j\neq i}^{n}\tilde{a}_i\right)=\left(\frac{1}{n}\mathop{\oplus}_{i=1}^{n}\tilde{a}_i\right)$$

$$\left\langle\left[s_{\left(\sum_{i=1}^{n}\sum_{j=1,j\neq i}^{n}\frac{1}{n}\theta(\tilde{a}_i)\right)},s_{\left(\sum_{i=1}^{n}\sum_{j=1,j\neq i}^{n}\frac{1}{n}\tau(\tilde{a}_i)\right)}\right],\bigcup_{\tilde{r}(\tilde{a}_i)\in\tilde{t}(\tilde{a}_i)}\left\{\left(1-\prod_{i=1}^{n}(1-r_{\tilde{a}_i}{}^{L})^{\frac{1}{n}}\right),\left(1-\prod_{i=1}^{n}(1-r_{\tilde{a}_i}{}^{U})^{\frac{1}{n}}\right)\right\},\right.$$

$$\left.\bigcup_{\tilde{\delta}(\tilde{a}_i)\in\tilde{i}(\tilde{a}_i)}\left\{\left(\prod_{i=1}^{n}\delta_{\tilde{a}_i}{}^{L}\right)^{\frac{1}{n}},\left(\prod_{i=1}^{n}\delta_{\tilde{a}_i}{}^{U}\right)^{\frac{1}{n}}\right\},\bigcup_{\tilde{\eta}(\tilde{a}_i)\in\tilde{f}(\tilde{a}_i)}\left\{\left(\prod_{i=1}^{n}\eta_{\tilde{a}_i}{}^{L}\right)^{\frac{1}{n}},\left(\prod_{i=1}^{n}\eta_{\tilde{a}_i}{}^{U}\right)^{\frac{1}{n}}\right\}\right\rangle$$

(3) If $p=\frac{1}{2}$ and $q=\frac{1}{2}$, then

$$IVNHFULBM^{\frac{1}{2},\frac{1}{2}}(\tilde{a}_1,\tilde{a}_2,...,\tilde{a}_n)=\left(\frac{1}{n(n-1)}\mathop{\oplus}_{i=1}^{n}\mathop{\oplus}_{j=1,j\neq i}^{n}\tilde{a}_i^{\frac{1}{2}}\otimes\tilde{a}_j^{\frac{1}{2}}\right)$$

$$\left\langle\left[s_{\left(\sum_{i=1}^{n}\sum_{j=1,j\neq i}^{n}\frac{1}{n(n-1)}\theta(\tilde{a}_i)^{\frac{1}{2}}\theta(\tilde{a}_j)^{\frac{1}{2}}\right)},s_{\left(\sum_{i=1}^{n}\sum_{j=1,j\neq i}^{n}\frac{1}{n(n-1)}\tau(\tilde{a}_i)^{\frac{1}{2}}\tau(\tilde{a}_j)^{\frac{1}{2}}\right)}\right],\right.$$

$$\bigcup_{\tilde{r}(\tilde{a}_i)\in\tilde{t}(\tilde{a}_i),\tilde{r}(\tilde{a}_j)\in\tilde{t}(\tilde{a}_j)}\left\{\left(1-\prod_{i=1}^{n}\prod_{j=1,j\neq i}^{n}(1-(r_{\tilde{a}_i}{}^{L}r_{\tilde{a}_j}{}^{L})^{\frac{1}{2}})^{\frac{1}{n(n-1)}}\right),\left(1-\prod_{i=1}^{n}\prod_{j=1,j\neq i}^{n}(1-(r_{\tilde{a}_i}{}^{U}r_{\tilde{a}_j}{}^{U})^{\frac{1}{2}})^{\frac{1}{n(n-1)}}\right)\right\},$$

$$\bigcup \tilde{\delta}(\tilde{a}_i)\in\tilde{i}(\tilde{a}_i),\tilde{\delta}(\tilde{a}_j)\in\tilde{i}(\tilde{a}_j)$$

$$\left\{\left(\prod_{i=1}^{n}\prod_{j=1,j\neq i}^{n}\left(1-(1-\delta_{\tilde{a}_i}{}^{L})^{\frac{1}{2}}(1-\delta_{\tilde{a}_j}{}^{L})^{\frac{1}{2}}\right)\right)^{\frac{1}{n(n-1)}},\left(\prod_{i=1}^{n}\prod_{j=1,j\neq i}^{n}\left(1-(1-\delta_{\tilde{a}_i}{}^{U})^{\frac{1}{2}}(1-\delta_{\tilde{a}_j}{}^{U})^{\frac{1}{2}}\right)\right)^{\frac{1}{n(n-1)}}\right\},$$

$$\bigcup \tilde{\eta}(\tilde{a}_i)\in\tilde{f}(\tilde{a}_i),\tilde{\eta}(\tilde{a}_j)\in\tilde{f}(\tilde{a}_j)$$

$$\left.\left\{\left(\prod_{i=1}^{n}\prod_{j=1,j\neq i}^{n}\left(1-(1-\eta_{\tilde{a}_i}{}^{L})^{\frac{1}{2}}(1-\eta_{\tilde{a}_j}{}^{L})^{\frac{1}{2}}\right)\right)^{\frac{1}{n(n-1)}},\left(\prod_{i=1}^{n}\prod_{j=1,j\neq i}^{n}\left(1-(1-\eta_{\tilde{a}_i}{}^{U})^{\frac{1}{2}}(1-\eta_{\tilde{a}_j}{}^{U})^{\frac{1}{2}}\right)\right)^{\frac{1}{n(n-1)}}\right\}\right\rangle$$

(4) If $p=1$ and $q=1$, then

$$IVNHFULBM^{1,1}(\tilde{a}_1,\tilde{a}_2,...,\tilde{a}_n)=\left(\frac{1}{n(n-1)}\mathop{\oplus}_{i=1}^{n}\mathop{\oplus}_{j=1,j\neq i}^{n}\tilde{a}_i\otimes\tilde{a}_j\right)^{\frac{1}{2}}$$

$$= \left\langle \left[s_{\left(\sum_{i=1}^{n} \sum_{j=1, j\neq i}^{n} \frac{1}{n(n-1)} \theta(\tilde{a}_i)\theta(\tilde{a}_j) \right)^{\frac{1}{2}}}, s_{\left(\sum_{i=1}^{n} \sum_{j=1, j\neq i}^{n} \frac{1}{n(n-1)} \tau(\tilde{a}_i)\tau(\tilde{a}_j) \right)^{\frac{1}{2}}} \right], \right.$$

$$\bigcup_{\tilde{r}(\tilde{a}_i) \in \tilde{t}(\tilde{a}_i), \tilde{r}(\tilde{a}_j) \in \tilde{t}(\tilde{a}_j)} \left(\left(1 - \prod_{i=1}^{n} \prod_{j=1, j\neq i}^{n} (1 - r_{\tilde{a}_i}^{L} r_{\tilde{a}_j}^{L})^{\frac{1}{n(n-1)}} \right)^{\frac{1}{2}}, \left(1 - \prod_{i=1}^{n} \prod_{j=1, j\neq i}^{n} (1 - r_{\tilde{a}_i}^{U} r_{\tilde{a}_j}^{U})^{\frac{1}{n(n-1)}} \right)^{\frac{1}{2}} \right),$$

$$\bigcup_{\tilde{\delta}(\tilde{a}_i) \in \tilde{i}(\tilde{a}_i), \tilde{\delta}(\tilde{a}_j) \in \tilde{i}(\tilde{a}_j)} \left(1 - \left(1 - \left(\prod_{i=1}^{n} \prod_{j=1, j\neq i}^{n} \left(1 - (1-\delta_{\tilde{a}_i}^{L})(1-\delta_{\tilde{a}_j}^{L}) \right) \right)^{\frac{1}{n(n-1)}} \right)^{\frac{1}{2}}, 1 - \left(1 - \left(\prod_{i=1}^{n} \prod_{j=1, j\neq i}^{n} \left(1 - (1-\delta_{\tilde{a}_i}^{U})(1-\delta_{\tilde{a}_j}^{U}) \right) \right)^{\frac{1}{n(n-1)}} \right)^{\frac{1}{2}} \right),$$

$$\left. \bigcup_{\tilde{\eta}(\tilde{a}_i) \in \tilde{f}(\tilde{a}_i), \tilde{\eta}(\tilde{a}_j) \in \tilde{f}(\tilde{a}_j)} \left(1 - \left(1 - \left(\prod_{i=1}^{n} \prod_{j=1, j\neq i}^{n} \left(1 - (1-\eta_{\tilde{a}_i}^{L})(1-\eta_{\tilde{a}_j}^{L}) \right) \right)^{\frac{1}{n(n-1)}} \right)^{\frac{1}{2}}, 1 - \left(1 - \left(\prod_{i=1}^{n} \prod_{j=1, j\neq i}^{n} \left(1 - (1-\eta_{\tilde{a}_i}^{U})(1-\eta_{\tilde{a}_j}^{U}) \right) \right)^{\frac{1}{n(n-1)}} \right)^{\frac{1}{2}} \right) \right\rangle$$

3.2 The interval-valued neutrosophic hesitant fuzzy uncertain linguistic weighted Bonferroni mean operator

Definition 15. Let $\tilde{a}_i = <[s_{\theta(\tilde{a}_i)}, s_{\tau(\tilde{a}_i)}], (\tilde{t}(\tilde{a}_i), \tilde{i}(\tilde{a}_i), \tilde{f}(\tilde{a}_i))>$ $(i = 1,2,...,n)$ be the set of all IVNHFULEs, and p, $q \geq 0$, then an interval-valued neutrosophic hesitant fuzzy uncertain linguistic weighted Bonferroni mean operator can be expressed as follows:

$$IVNHFULWBM^{p,q}(\tilde{a}_1, \tilde{a}_2, ..., \tilde{a}_n) = \left(\mathop{\oplus}_{i=1}^{n} \mathop{\oplus}_{j=1, j\neq i}^{n} \frac{w_i w_j}{1-w_i} \tilde{a}_i^{\,p} \otimes \tilde{a}_j^{\,q} \right)^{\frac{1}{p+q}} \tag{40}$$

Theorem 16. Let $\tilde{a}_i = <[s_{\theta(\tilde{a}_i)}, s_{\tau(\tilde{a}_i)}], (\tilde{t}(\tilde{a}_i), \tilde{i}(\tilde{a}_i), \tilde{f}(\tilde{a}_i))>$ $(i = 1,2,...,n)$ be the set of all IVNHFULEs, and p, $q \geq 0$, then, the result aggregated from Definition 15 is still an IVNHFULE, and even

$$IVNHFULWBM^{p,q}(\tilde{a}_1, \tilde{a}_2, ..., \tilde{a}_n) = \left(\mathop{\oplus}_{i=1}^{n} \mathop{\oplus}_{j=1, j\neq i}^{n} \frac{w_i w_j}{1-w_i} \tilde{a}_i^{\,p} \otimes \tilde{a}_j^{\,q} \right)^{\frac{1}{p+q}}$$

$$= \left\langle \left[s_{\left(\sum_{i=1}^{n} \sum_{j=1, j\neq i}^{n} \frac{w_i w_j}{1-w_i} \theta(\tilde{a}_i)^{p} \theta(\tilde{a}_j)^{q} \right)^{\frac{1}{p+q}}}, s_{\left(\sum_{i=1}^{n} \sum_{j=1, j\neq i}^{n} \frac{w_i w_j}{1-w_i} \tau(\tilde{a}_i)^{p} \tau(\tilde{a}_j)^{q} \right)^{\frac{1}{p+q}}} \right], \right.$$

$$\bigcup_{\tilde r(\tilde a_i)\in \tilde t(\tilde a_i),\tilde r(\tilde a_j)\in \tilde t(\tilde a_j)} \left(\left(1-\prod_{i=1}^{n}\prod_{j=1,j\ne i}^{n}\left(1-(r_{\tilde a_i}^{L})^{p}(r_{\tilde a_j}^{L})^{q}\right)^{\frac{w_iw_j}{1-w_i}}\right)^{\frac{1}{p+q}}, \left(1-\prod_{i=1}^{n}\prod_{j=1,j\ne i}^{n}\left(1-(r_{\tilde a_i}^{U})^{p}(r_{\tilde a_j}^{U})^{q}\right)^{\frac{w_iw_j}{1-w_i}}\right)^{\frac{1}{p+q}} \right) \tag{41}$$

$$\bigcup_{\tilde \delta(\tilde a_i)\in \tilde i(\tilde a_i),\tilde \delta(\tilde a_j)\in \tilde i(\tilde a_j)} \left(1-\left(1-\prod_{i=1}^{n}\prod_{j=1,j\ne i}^{n}\left(1-(1-\delta_{\tilde a_i}^{L})^{p}(1-\delta_{\tilde a_j}^{L})^{q}\right)^{\frac{w_iw_j}{1-w_i}}\right)^{\frac{1}{p+q}}, 1-\left(1-\prod_{i=1}^{n}\prod_{j=1,j\ne i}^{n}\left(1-(1-\delta_{\tilde a_i}^{U})^{p}(1-\delta_{\tilde a_j}^{U})^{q}\right)^{\frac{w_iw_j}{1-w_i}}\right)^{\frac{1}{p+q}} \right)$$

$$\bigcup_{\tilde \eta(\tilde a_i)\in \tilde f(\tilde a_i),\tilde \eta(\tilde a_j)\in \tilde f(\tilde a_j)} \left(1-\left(1-\prod_{i=1}^{n}\prod_{j=1,j\ne i}^{n}\left(1-(1-\eta_{\tilde a_i}^{L})^{p}(1-\eta_{\tilde a_j}^{L})^{q}\right)^{\frac{w_iw_j}{1-w_i}}\right)^{\frac{1}{p+q}}, 1-\left(1-\prod_{i=1}^{n}\prod_{j=1,j\ne i}^{n}\left(1-(1-\eta_{\tilde a_i}^{U})^{p}(1-\eta_{\tilde a_j}^{U})^{q}\right)^{\frac{w_iw_j}{1-w_i}}\right)^{\frac{1}{p+q}} \right)$$

The proof of Theorem 16 can be easily completed similar to Theorem 11.

The IVNHFULWBM operator satisfies the following properties:

Theorem 17 (Reducibility).

Let $W = (\frac{1}{n},\frac{1}{n},...,\frac{1}{n})^{T}$ be the weight vector of $\tilde a_i\, (i=1,2,...,n)$, then

$$IVNHFULWBM^{p,q}(\tilde a_1,\tilde a_2,...,\tilde a_n) = IVNHFULBM^{p,q}(\tilde a_1,\tilde a_2,...,\tilde a_n)$$

Proof.

Since $W = (\frac{1}{n},\frac{1}{n},...,\frac{1}{n})^{T}$ is the weight vector of $\tilde a_i\, (i=1,2,...,n)$,then according to definition 15,we have

$$IVNHFULWBM^{p,q}(\tilde a_1,\tilde a_2,...,\tilde a_n) = \left(\mathop{\oplus}_{i=1}^{n}\mathop{\oplus}_{j=1,j\ne i}^{n}\frac{w_iw_j}{1-w_i}\tilde a_i^{\,p}\otimes \tilde a_j^{\,q}\right)^{\frac{1}{p+q}}$$

$$= \left(\mathop{\oplus}_{i=1}^{n}\mathop{\oplus}_{j=1,j\ne i}^{n}\frac{\frac{1}{n}\times\frac{1}{n}}{1-\frac{1}{n}}\tilde a_i^{\,p}\otimes \tilde a_j^{\,q}\right)^{\frac{1}{p+q}} = \left(\frac{1}{n(n-1)}\mathop{\oplus}_{i=1}^{n}\mathop{\oplus}_{j=1,j\ne i}^{n}\tilde a_i^{\,p}\otimes \tilde a_j^{\,q}\right)^{\frac{1}{p+q}}$$

$$= IVNHFULBM^{p,q}(\tilde a_1,\tilde a_2,...,\tilde a_n)$$

which completes the proof of theorem 17.

Theorem 18 (Idempotency).

Let $\tilde a_j = <[S_{\theta(a)}, s_{\tau(a)}], ([r_a^{L}, r_a^{U}], [\delta_a^{L}, \delta_a^{U}], [\eta_a^{L}, \eta_a^{U}])> = \tilde a \quad (j=1,2,...,n)$,then

$$IVNHFULWBM^{p,q}(\tilde a_1,\tilde a_2,...,\tilde a_n) = \tilde a\,.$$

The proof of Theorem 18 can be easily completed similar to Theorem 12.

Theorem 19 (Commutativity).

If $(\tilde a_1',\tilde a_2',...,\tilde a_n')$ be any permutation of $(\tilde a_1,\tilde a_2,...,\tilde a_n)$, then we can get

$$IVNHFULWBM^{p,q}(\tilde{a}_1', \tilde{a}_2', ..., \tilde{a}_n') = IVNHFULWBM^{p,q}(\tilde{a}_1, \tilde{a}_2, ..., \tilde{a}_n).$$

The proof of Theorem 19 can be easily completed similar to Theorem 13.

Theorem 20 (Monotonicity).

Suppose $(\tilde{a}_1, \tilde{a}_2, ..., \tilde{a}_n)$ and $(\tilde{b}_1, \tilde{b}_2, ..., \tilde{b}_n)$ are two sets of IVNHFULEs. If $s_{\theta(\tilde{a}_i)} \le s_{\theta(\tilde{b}_i)}$, $s_{\tau(\tilde{a}_i)} \le s_{\tau(\tilde{b}_i)}$, $r_{\tilde{a}_i}{}^L \le r_{\tilde{b}_i}{}^L$, $r_{\tilde{a}_i}{}^U \le r_{\tilde{b}_i}{}^U$, $\delta_{\tilde{a}_i}{}^L \ge \delta_{\tilde{b}_i}{}^L$, $\delta_{\tilde{a}_i}{}^U \ge \delta_{\tilde{b}_i}{}^U$ and $\eta_{\tilde{a}_i}{}^L \ge \eta_{\tilde{b}_i}{}^L$, $\eta_{\tilde{a}_i}{}^U \ge \eta_{\tilde{b}_i}{}^U$ for all $i = 1,2,...,n$, then $IVNHFULWBM^{p,q}(\tilde{a}_1, \tilde{a}_2, ..., \tilde{a}_n) \le IVNHFULWBM^{p,q}(\tilde{b}_1, \tilde{b}_2, ..., \tilde{b}_n)$

The proof of Theorem 20 can be easily completed similar to Theorem 14.

Theorem 21 (Boundedness).

Let $\tilde{a}_i (i = 1,2,...,n)$ be a set of IVNHFULEs. If $s_{\theta-} = \min_{1\le i\le n}\{s_{\theta(\tilde{a}_i)}\}$, $s_{\tau-} = \min_{1\le i\le n}\{s_{\tau(\tilde{a}_i)}\}$,

$s_{\theta+} = \max_{1\le i\le n}\{s_{\theta(\tilde{a}_i)}\}$, $s_{\tau+} = \max_{1\le i\le n}\{s_{\tau(\tilde{a}_i)}\}$, $r^{L-} = \min_{1\le i\le n}\{r_{\tilde{a}_i}{}^L \mid [r_{\tilde{a}_i}{}^L, r_{\tilde{a}_i}{}^U] \in t(\tilde{a}_i)\}$,

$r^{U-} = \min_{1\le i\le n}\{r_{\tilde{a}_i}{}^U \mid [r_{\tilde{a}_i}{}^L, r_{\tilde{a}_i}{}^U] \in t(\tilde{a}_i)\}$, $r^{L+} = \max_{1\le i\le n}\{r_{\tilde{a}_i}{}^L \mid [r_{\tilde{a}_i}{}^L, r_{\tilde{a}_i}{}^U] \in t(\tilde{a}_i)\}$

$r^{U+} = \max_{1\le i\le n}\{r_{\tilde{a}_i}{}^U \mid [r_{\tilde{a}_i}{}^L, r_{\tilde{a}_i}{}^U] \in t(\tilde{a}_i)\}$, $\delta^{L-} = \min_{1\le i\le n}\{\delta_{\tilde{a}_i}{}^L \mid [\delta_{\tilde{a}_i}{}^L, \delta_{\tilde{a}_i}{}^U] \in i(\tilde{a}_i)\}$,

$\delta^{U-} = \min_{1\le i\le n}\{\delta_{\tilde{a}_i}{}^U \mid [\delta_{\tilde{a}_i}{}^L, \delta_{\tilde{a}_i}{}^U] \in i(\tilde{a}_i)\}$, $\delta^{L+} = \max_{1\le i\le n}\{\delta_{\tilde{a}_i}{}^L \mid [\delta_{\tilde{a}_i}{}^L, \delta_{\tilde{a}_i}{}^U] \in i(\tilde{a}_i)\}$

$\delta^{U+} = \max_{1\le i\le n}\{\delta_{\tilde{a}_i}{}^U \mid [\delta_{\tilde{a}_i}{}^L, \delta_{\tilde{a}_i}{}^U] \in i(\tilde{a}_i)\}$, $\eta^{L-} = \min_{1\le i\le n}\{\eta_{\tilde{a}_i}{}^L \mid [\eta_{\tilde{a}_i}{}^L, \eta_{\tilde{a}_i}{}^U] \in f(\tilde{a}_i)\}$,

$\eta^{U-} = \min_{1\le i\le n}\{\eta_{\tilde{a}_i}{}^U \mid [\eta_{\tilde{a}_i}{}^L, \eta_{\tilde{a}_i}{}^U] \in f(\tilde{a}_i)\}$, $\eta^{L+} = \max_{1\le i\le n}\{\eta_{\tilde{a}_i}{}^L \mid [\eta_{\tilde{a}_i}{}^L, \eta_{\tilde{a}_i}{}^U] \in f(\tilde{a}_i)\}$

$\eta^{U+} = \max_{1\le i\le n}\{\eta_{\tilde{a}_i}{}^U \mid [\eta_{\tilde{a}_i}{}^L, \eta_{\tilde{a}_i}{}^U] \in f(\tilde{a}_i)\}$, for all $i = 1,2,..., n$,

$\tilde{a}^- = <[s_{\theta-}, s_{\tau-}], ([r^{L-}, r^{U-}], [\delta^{L+}, \delta^{U+}], [\eta^{L+}, \eta^{L+}])>$ and

$\tilde{a}^+ = <[s_{\theta+}, s_{\tau+}], ([r^{L+}, r^{U+}], [\delta^{L-}, \delta^{U-}], [\eta^{L-}, \eta^{U-}])>$ then

$$\tilde{a}^- \le IVNHFULWBM^{p,q}(\tilde{a}_1, \tilde{a}_2, ..., \tilde{a}_n) \le \tilde{a}^+.$$

The proof of Theorem 21 can be easily completed similar to Theorem 15.

In the following, we will discuss some specials of the IVNHFULWBM operator with respect to the different parameters p and q.

(1) If $q = 0$, then

$$IVNHFULWBM^{p,0}(\tilde{a}_1, \tilde{a}_2, ..., \tilde{a}_n) = \left(\frac{w_i w_i}{1-w_i} \mathop{\oplus}_{i=1}^{n} \mathop{\oplus}_{j=1, j\ne i}^{n} \tilde{a}_i{}^p\right)^{\frac{1}{p}} = \left(\mathop{\oplus}_{i=1}^{n} w_i \tilde{a}_i{}^p\right)^{\frac{1}{p}}$$

$$= \left\langle \left[s_{\left(\sum_{i=1}^{n} w_i \theta(\tilde{a}_i)^p\right)^{\frac{1}{p}}}, s_{\left(\sum_{i=1}^{n} w_i \tau(\tilde{a}_i)^p\right)^{\frac{1}{p}}} \right], \right.$$

$$\bigcup \tilde{r}(\tilde{a}_1) \in \tilde{t}(\tilde{a}_1), \tilde{r}(\tilde{a}_2) \in \tilde{t}(\tilde{a}_2), ..., \tilde{r}(\tilde{a}_n) \in \tilde{t}(\tilde{a}_n)$$

$$\left(\left(1 - \prod_{i=1}^{n} (1 - (r_{\tilde{a}_i}{}^{L})^{p})^{w_i} \right)^{\frac{1}{p}}, \left(1 - \prod_{i=1}^{n} (1 - (r_{\tilde{a}_i}{}^{U})^{p})^{w_i} \right)^{\frac{1}{p}} \right),$$

$$\bigcup \tilde{\delta}(\tilde{a}_1) \in \tilde{i}(\tilde{a}_1), \tilde{\delta}(\tilde{a}_2) \in \tilde{i}(\tilde{a}_2) ..., \tilde{\delta}(\tilde{a}_n) \in \tilde{i}(\tilde{a}_n)$$

$$\left(1 - \left(1 - \prod_{i=1}^{n} \left(1 - (1 - \delta_{\tilde{a}_i}{}^{L})^{p}\right)^{w_i} \right)^{\frac{1}{p}}, 1 - \left(1 - \prod_{i=1}^{n} \left(1 - (1 - \delta_{\tilde{a}_i}{}^{U})^{p}\right)^{w_i} \right)^{\frac{1}{p}} \right),$$

$$\bigcup \tilde{\eta}(\tilde{a}_1) \in \tilde{f}(\tilde{a}_1), \tilde{\eta}(\tilde{a}_2) \in \tilde{f}(\tilde{a}_2), ..., \tilde{\eta}(\tilde{a}_n) \in \tilde{f}(\tilde{a}_n)$$

$$\left(1 - \left(1 - \prod_{i=1}^{n} \left(1 - (1 - \eta_{\tilde{a}_i}{}^{L})^{p}\right)^{w_i} \right)^{\frac{1}{p}}, 1 - \left(1 - \prod_{i=1}^{n} \left(1 - (1 - \eta_{\tilde{a}_i}{}^{U})^{p}\right)^{w_i} \right)^{\frac{1}{p}} \right) \Bigg\rangle$$

(2) If $p = 1$ and $q = 0$,then

$$IVNHFULWBM^{1,0}(\tilde{a}_1, \tilde{a}_2, ..., \tilde{a}_n) = \left(\bigoplus_{i=1}^{n} w_i \tilde{a}_i \right)$$

$$\Bigg\langle \left[s_{\left(\sum_{i=1}^{n} w_i \theta(\tilde{a}_i) \right)}, s_{\left(\sum_{i=1}^{n} w_i \tau(\tilde{a}_i) \right)} \right], \bigcup_{\tilde{r}(\tilde{a}_i) \in \tilde{t}(\tilde{a}_i)} \left(\left(1 - \prod_{i=1}^{n} (1 - r_{\tilde{a}_i}{}^{L})^{w_i} \right), \left(1 - \prod_{i=1}^{n} (1 - r_{\tilde{a}_i}{}^{U})^{w_i} \right) \right),$$

$$\bigcup_{\tilde{\delta}(\tilde{a}_i) \in \tilde{i}(\tilde{a}_i)} \left(\prod_{i=1}^{n} (\delta_{\tilde{a}_i}{}^{L})^{w_i}, \prod_{i=1}^{n} (\delta_{\tilde{a}_i}{}^{U})^{w_i} \right), \bigcup_{\tilde{\eta}(\tilde{a}_i) \in \tilde{f}(\tilde{a}_i)} \left(\prod_{i=1}^{n} (\eta_{\tilde{a}_i}{}^{L})^{w_i}, \prod_{i=1}^{n} (\eta_{\tilde{a}_i}{}^{U})^{w_i} \right) \Bigg\rangle$$

(3) If $p = \frac{1}{2}$ and $q = \frac{1}{2}$, then

$$IVNHFULWBM^{\frac{1}{2},\frac{1}{2}}(\tilde{a}_1, \tilde{a}_2, ..., \tilde{a}_n) = \left(\bigoplus_{i=1}^{n} \bigoplus_{j=1, j \neq i}^{n} \frac{w_i w_j}{1 - w_i} \tilde{a}_i^{\frac{1}{2}} \otimes \tilde{a}_j^{\frac{1}{2}} \right)$$

$$= \Bigg\langle \left[s_{\left(\sum_{i=1}^{n} \sum_{j=1, j \neq i}^{n} \frac{w_i w_j}{1 - w_i} \theta(\tilde{a}_i)^{\frac{1}{2}} \theta(\tilde{a}_j)^{\frac{1}{2}} \right)}, s_{\left(\sum_{i=1}^{n} \sum_{j=1, j \neq i}^{n} \frac{w_i w_j}{1 - w_i} \tau(\tilde{a}_i)^{\frac{1}{2}} \tau(\tilde{a}_j)^{\frac{1}{2}} \right)} \right],$$

$$\bigcup \tilde{r}(\tilde{a}_i) \in \tilde{t}(\tilde{a}_i), \tilde{r}(\tilde{a}_j) \in \tilde{t}(\tilde{a}_j)$$

$$\left(\left(1 - \prod_{i=1}^{n} \prod_{j=1, j \neq i}^{n} (1 - (r_{\tilde{a}_i}{}^{L} r_{\tilde{a}_j}{}^{L})^{\frac{1}{2}})^{\frac{w_i w_j}{1 - w_i}} \right), \left(1 - \prod_{i=1}^{n} \prod_{j=1, j \neq i}^{n} (1 - (r_{\tilde{a}_i}{}^{U} r_{\tilde{a}_j}{}^{U})^{\frac{1}{2}})^{\frac{w_i w_j}{1 - w_i}} \right) \right),$$

$$\bigcup_{\tilde{\delta}(\tilde{a}_i)\in\tilde{i}(\tilde{a}_i),\tilde{\delta}(\tilde{a}_j)\in\tilde{i}(\tilde{a}_j)}\left(\left(\prod_{i=1}^{n}\prod_{j=1,j\neq i}^{n}\left(1-(1-\delta_{\tilde{a}_i}{}^{L})^{\frac{1}{2}}(1-\delta_{\tilde{a}_j}{}^{L})^{\frac{1}{2}}\right)^{\frac{w_iw_j}{1-w_i}}\right),\left(\prod_{i=1}^{n}\prod_{j=1,j\neq i}^{n}\left(1-(1-\delta_{\tilde{a}_i}{}^{U})^{\frac{1}{2}}(1-\delta_{\tilde{a}_j}{}^{U})^{\frac{1}{2}}\right)^{\frac{w_iw_j}{1-w_i}}\right)\right),$$

$$\left.\bigcup_{\tilde{\eta}(\tilde{a}_i)\in\tilde{f}(\tilde{a}_i),\tilde{\eta}(\tilde{a}_j)\in\tilde{f}(\tilde{a}_j)}\left(\left(\prod_{i=1}^{n}\prod_{j=1,j\neq i}^{n}\left(1-(1-\eta_{\tilde{a}_i}{}^{L})^{\frac{1}{2}}(1-\eta_{\tilde{a}_j}{}^{L})^{\frac{1}{2}}\right)^{\frac{w_iw_j}{1-w_i}}\right),\left(\prod_{i=1}^{n}\prod_{j=1,j\neq i}^{n}\left(1-(1-\eta_{\tilde{a}_i}{}^{U})^{\frac{1}{2}}(1-\eta_{\tilde{a}_j}{}^{U})^{\frac{1}{2}}\right)^{\frac{w_iw_j}{1-w_i}}\right)\right)\right\rangle$$

(4) If $p=1$ and $q=1$, then

$$IVNHFULWBM^{1,1}(\tilde{a}_1,\tilde{a}_2,...,\tilde{a}_n)=\left(\mathop{\oplus}_{i=1}^{n}\mathop{\oplus}_{j=1,j\neq i}^{n}\frac{w_iw_j}{1-w_i}\tilde{a}_i\otimes\tilde{a}_j\right)^{\frac{1}{2}}$$

$$=\left\langle\left[s_{\left(\sum_{i=1}^{n}\sum_{j=1,j\neq i}^{n}\frac{w_iw_j}{1-w_i}\theta(\tilde{a}_i)\theta(\tilde{a}_j)\right)^{\frac{1}{2}}},s_{\left(\sum_{i=1}^{n}\sum_{j=1,j\neq i}^{n}\frac{w_iw_j}{1-w_i}\tau(\tilde{a}_i)\tau(\tilde{a}_j)\right)^{\frac{1}{2}}}\right],\right.$$

$$\bigcup_{\tilde{r}(\tilde{a}_i)\in\tilde{t}(\tilde{a}_i),\tilde{r}(\tilde{a}_j)\in\tilde{t}(\tilde{a}_j)}\left(\left(1-\prod_{i=1}^{n}\prod_{j=1,j\neq i}^{n}(1-r_{\tilde{a}_i}{}^{L}r_{\tilde{a}_j}{}^{L})^{\frac{w_iw_j}{1-w_i}}\right)^{\frac{1}{2}},\left(1-\prod_{i=1}^{n}\prod_{j=1,j\neq i}^{n}(1-r_{\tilde{a}_i}{}^{U}r_{\tilde{a}_j}{}^{U})^{\frac{w_iw_j}{1-w_i}}\right)^{\frac{1}{2}}\right),$$

$$\bigcup_{\tilde{\delta}(\tilde{a}_i)\in\tilde{i}(\tilde{a}_i),\tilde{\delta}(\tilde{a}_j)\in\tilde{i}(\tilde{a}_j)}\left(1-\left(1-\prod_{i=1}^{n}\prod_{j=1,j\neq i}^{n}\left(1-(1-\delta_{\tilde{a}_i}{}^{L})(1-\delta_{\tilde{a}_j}{}^{L})\right)^{\frac{w_iw_j}{1-w_i}}\right)^{\frac{1}{2}},1-\left(1-\prod_{i=1}^{n}\prod_{j=1,j\neq i}^{n}\left(1-(1-\delta_{\tilde{a}_i}{}^{U})(1-\delta_{\tilde{a}_j}{}^{U})\right)^{\frac{w_iw_j}{1-w_i}}\right)^{\frac{1}{2}}\right),$$

$$\left.\bigcup_{\tilde{\eta}(\tilde{a}_i)\in\tilde{f}(\tilde{a}_i),\tilde{\eta}(\tilde{a}_j)\in\tilde{f}(\tilde{a}_j)}\left(1-\left(1-\prod_{i=1}^{n}\prod_{j=1,j\neq i}^{n}\left(1-(1-\eta_{\tilde{a}_i}{}^{L})(1-\eta_{\tilde{a}_j}{}^{L})\right)^{\frac{w_iw_j}{1-w_i}}\right)^{\frac{1}{2}},1-\left(1-\prod_{i=1}^{n}\prod_{j=1,j\neq i}^{n}\left(1-(1-\eta_{\tilde{a}_i}{}^{U})(1-\eta_{\tilde{a}_j}{}^{U})\right)^{\frac{w_iw_j}{1-w_i}}\right)^{\frac{1}{2}}\right)\right\rangle$$

3.3 The interval-valued neutrosophic hesitant fuzzy uncertain linguistic geometric Bonferroni mean operator

Definition 16. Let $\tilde{a}_i=<[s_{\theta(\tilde{a}_i)},s_{\tau(\tilde{a}_i)}],(\tilde{t}(\tilde{a}_i),\tilde{i}(\tilde{a}_i),\tilde{f}(\tilde{a}_i))>$ $(i=1,2,...,n)$ be the set of all IVNHFULEs, and $p,q\geq 0$, then an interval-valued neutrosophic hesitant fuzzy uncertain linguistic geometric Bonferroni mean operator can be defined as follows:

$$IVNHFULGBM^{p,q}(\tilde{a}_1,\tilde{a}_2,...,\tilde{a}_n)=\frac{1}{p+q}\mathop{\otimes}_{i=1}^{n}\mathop{\otimes}_{j=1,j\neq i}^{n}(p\tilde{a}_i\oplus q\tilde{a}_j)^{\frac{1}{n(n-1)}} \tag{42}$$

Theorem 22. Let $\tilde{a}_i=<[s_{\theta(\tilde{a}_i)},s_{\tau(\tilde{a}_i)}],(\tilde{t}(\tilde{a}_i),\tilde{i}(\tilde{a}_i),\tilde{f}(\tilde{a}_i))>$ $(i=1,2,...,n)$ be the set of all IVNHFULEs, and $p, q \geq 0$, then, the result aggregated from Definition 16 is still an IVNHFULE, and even

$$IVNHFULGBM^{p,q}(\tilde{a}_1, \tilde{a}_2, ..., \tilde{a}_n) = \frac{1}{p+q} \underset{i=1}{\overset{n}{\otimes}} \underset{j=1, j\neq i}{\overset{n}{\otimes}} (p\tilde{a}_i \oplus q\tilde{a}_j)^{\frac{1}{n(n-1)}}$$

$$= \left\langle \left[s_{\frac{1}{p+q}\prod_{i=1}^{n}\prod_{j=1, j\neq i}^{n}\left(p\theta(\tilde{a}_i)+q\theta(\tilde{a}_j)\right)^{\frac{1}{n(n+1)}}}, s_{\frac{1}{p+q}\prod_{i=1}^{n}\prod_{j=1, j\neq i}^{n}\left(p\tau(\tilde{a}_i)+q\tau(\tilde{a}_j)\right)^{\frac{1}{n(n+1)}}} \right], \right.$$

$$\bigcup_{\substack{\tilde{r}(\tilde{a}_i)\in\tilde{t}(\tilde{a}_i) \\ \tilde{r}(\tilde{a}_j)\in\tilde{t}(\tilde{a}_j)}} \left(1-\left(1-\prod_{i=1}^{n}\prod_{j=1, j\neq i}^{n}\left(1-(1-r_{\tilde{a}_i}{}^{L})^{p}(1-r_{\tilde{a}_j}{}^{L})^{q}\right)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{p+q}}, \right.$$

$$\left. 1-\left(1-\prod_{i=1}^{n}\prod_{j=1, j\neq i}^{n}\left(1-(1-r_{\tilde{a}_i}{}^{U})^{p}(1-r_{\tilde{a}_j}{}^{U})^{q}\right)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{p+q}} \right), \tag{43}$$

$$\bigcup_{\substack{\tilde{\delta}(\tilde{a}_i)\in\tilde{i}(\tilde{a}_i) \\ \tilde{\delta}(\tilde{a}_j)\in\tilde{i}(\tilde{a}_j)}} \left(\left(1-\prod_{i=1}^{n}\prod_{j=1, j\neq i}^{n}\left(1-(\delta_{\tilde{a}_i}{}^{L})^{p}(\delta_{\tilde{a}_j}{}^{L})^{q}\right)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{p+q}}, \left(1-\prod_{i=1}^{n}\prod_{j=1, j\neq i}^{n}\left(1-(\delta_{\tilde{a}_i}{}^{U})^{p}(\delta_{\tilde{a}_j}{}^{U})^{q}\right)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{p+q}} \right),$$

$$\left. \bigcup_{\substack{\tilde{\eta}(\tilde{a}_i)\in\tilde{f}(\tilde{a}_i) \\ \tilde{\eta}(\tilde{a}_j)\in\tilde{f}(\tilde{a}_j)}} \left(\left(1-\prod_{i=1}^{n}\prod_{j=1, j\neq i}^{n}\left(1-(\eta_{\tilde{a}_i}{}^{L})^{p}(\eta_{\tilde{a}_j}{}^{L})^{q}\right)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{p+q}}, \left(1-\prod_{i=1}^{n}\prod_{j=1, j\neq i}^{n}\left(1-(\eta_{\tilde{a}_i}{}^{U})^{p}(\eta_{\tilde{a}_j}{}^{U})^{q}\right)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{p+q}} \right) \right\rangle$$

The proof of Theorem 22 can be easily completed similar to Theorem 11.

The IVNHFULGBM operator satisfies the following properties:

Theorem 23 (Idempotency).

Let $\tilde{a}_j =< [S_{\theta(a)}, s_{\tau(a)}], ([r_a{}^L, r_a{}^U], [\delta_a{}^L, \delta_a{}^U], [\eta_a{}^L, \eta_a{}^U]) >= \tilde{a}$ $(j = 1,2,...,n)$,then

$$IVNHFULGBM^{p,q}(\tilde{a}_1, \tilde{a}_2, ..., \tilde{a}_n) = \tilde{a} \ .$$

The proof of Theorem 23 can be easily completed similar to Theorem 12.

Theorem 24 (Commutativity).

If $(\tilde{a}_1', \tilde{a}_2', ..., \tilde{a}_n')$ be any permutation of $(\tilde{a}_1, \tilde{a}_2, ..., \tilde{a}_n)$, then we can get

$$IVNHFULGBM^{p,q}(\tilde{a}_1', \tilde{a}_2', ..., \tilde{a}_n') = IVNHFULGBM^{p,q}(\tilde{a}_1, \tilde{a}_2, ..., \tilde{a}_n) \ .$$

The proof of Theorem 24 can be easily completed similar to Theorem 13.

Theorem 25 (Monotonicity).

Suppose $(\tilde{a}_1, \tilde{a}_2, ..., \tilde{a}_n)$ and $(\tilde{b}_1, \tilde{b}_2, ..., \tilde{b}_n)$ are two sets of IVNHFULEs. If $s_{\theta(\tilde{a}_i)} \leq s_{\theta(\tilde{b}_i)}$, $s_{\tau(\tilde{a}_i)} \leq s_{\tau(\tilde{b}_i)}$, $r_{\tilde{a}_i}{}^L \leq r_{\tilde{b}_i}{}^L$, $r_{\tilde{a}_i}{}^U \leq r_{\tilde{b}_i}{}^U$, $\delta_{\tilde{a}_i}{}^L \geq \delta_{\tilde{b}_i}{}^L$, $\delta_{\tilde{a}_i}{}^U \geq \delta_{\tilde{b}_i}{}^U$ and $\eta_{\tilde{a}_i}{}^L \geq \eta_{\tilde{b}_i}{}^L$, $\eta_{\tilde{a}_i}{}^U \geq \eta_{\tilde{b}_i}{}^U$ for all $i = 1,2,...,n$,then

$$IVNHFULGBM^{p,q}(\tilde{a}_1, \tilde{a}_2, ..., \tilde{a}_n) \leq IVNHFULGBM^{p,q}(\tilde{b}_1, \tilde{b}_2, ..., \tilde{b}_n)$$

The proof of Theorem 25 can be easily completed similar to Theorem 14.

Theorem 26 (Boundedness).

Let $\tilde{a}_i (i = 1,2,..., n)$ be a set of IVNHFULEs. If $s_{\theta-} = \min_{1\le i\le n}\{s_{\theta(\tilde{a}_i)}\}$, $s_{\tau-} = \min_{1\le i\le n}\{s_{\tau(\tilde{a}_i)}\}$,

$s_{\theta+} = \max_{1\le i\le n}\{s_{\theta(\tilde{a}_i)}\}$, $s_{\tau+} = \max_{1\le i\le n}\{s_{\tau(\tilde{a}_i)}\}$, $r^{L-} = \min_{1\le i\le n}\{r_{\tilde{a}_i}{}^L \mid [r_{\tilde{a}_i}{}^L, r_{\tilde{a}_i}{}^U] \in t(\tilde{a}_i)\}$,

$r^{U-} = \min_{1\le i\le n}\{r_{\tilde{a}_i}{}^U \mid [r_{\tilde{a}_i}{}^L, r_{\tilde{a}_i}{}^U] \in t(\tilde{a}_i)\}$, $r^{L+} = \max_{1\le i\le n}\{r_{\tilde{a}_i}{}^L \mid [r_{\tilde{a}_i}{}^L, r_{\tilde{a}_i}{}^U] \in t(\tilde{a}_i)\}$

$r^{U+} = \max_{1\le i\le n}\{r_{\tilde{a}_i}{}^U \mid [r_{\tilde{a}_i}{}^L, r_{\tilde{a}_i}{}^U] \in t(\tilde{a}_i)\}$, $\delta^{L-} = \min_{1\le i\le n}\{\delta_{\tilde{a}_i}{}^L \mid [\delta_{\tilde{a}_i}{}^L, \delta_{\tilde{a}_i}{}^U] \in i(\tilde{a}_i)\}$,

$\delta^{U-} = \min_{1\le i\le n}\{\delta_{\tilde{a}_i}{}^U \mid [\delta_{\tilde{a}_i}{}^L, \delta_{\tilde{a}_i}{}^U] \in i(\tilde{a}_i)\}$, $\delta^{L+} = \max_{1\le i\le n}\{\delta_{\tilde{a}_i}{}^L \mid [\delta_{\tilde{a}_i}{}^L, \delta_{\tilde{a}_i}{}^U] \in i(\tilde{a}_i)\}$

$\delta^{U+} = \max_{1\le i\le n}\{\delta_{\tilde{a}_i}{}^U \mid [\delta_{\tilde{a}_i}{}^L, \delta_{\tilde{a}_i}{}^U] \in i(\tilde{a}_i)\}$, $\eta^{L-} = \min_{1\le i\le n}\{\eta_{\tilde{a}_i}{}^L \mid [\eta_{\tilde{a}_i}{}^L, \eta_{\tilde{a}_i}{}^U] \in f(\tilde{a}_i)\}$,

$\eta^{U-} = \min_{1\le i\le n}\{\eta_{\tilde{a}_i}{}^U \mid [\eta_{\tilde{a}_i}{}^L, \eta_{\tilde{a}_i}{}^U] \in f(\tilde{a}_i)\}$, $\eta^{L+} = \max_{1\le i\le n}\{\eta_{\tilde{a}_i}{}^L \mid [\eta_{\tilde{a}_i}{}^L, \eta_{\tilde{a}_i}{}^U] \in f(\tilde{a}_i)\}$

$\eta^{U+} = \max_{1\le i\le n}\{\eta_{\tilde{a}_i}{}^U \mid [\eta_{\tilde{a}_i}{}^L, \eta_{\tilde{a}_i}{}^U] \in f(\tilde{a}_i)\}$, for all $i = 1,2,..., n$,

$\tilde{a}^- = \langle [s_{\theta-}, s_{\tau-}], ([r^{L-}, r^{U-}], [\delta^{L+}, \delta^{U+}], [\eta^{L+}, \eta^{L+}]) \rangle$ and

$\tilde{a}^+ = \langle [s_{\theta+}, s_{\tau+}], ([r^{L+}, r^{U+}], [\delta^{L-}, \delta^{U-}], [\eta^{L-}, \eta^{U-}]) \rangle$ then

$$\tilde{a}^- \le IVNHFULGBM^{p,q}(\tilde{a}_1, \tilde{a}_2, ..., \tilde{a}_n) \le \tilde{a}^+ .$$

The proof of Theorem 26 can be easily completed similar to Theorem 15.

In the following, we will discuss some specials of the IVNHFULGBM operator with respect to the different parameters p and q.

(1) If $q = 0$, then

$$IVNHFULGBM^{p,0}(\tilde{a}_1, \tilde{a}_2, ..., \tilde{a}_n) = \frac{1}{p} \overset{n}{\underset{i=1}{\otimes}} (p\tilde{a}_i)^{\frac{1}{n}}$$

$$= \left\langle \left[s_{\prod_{i=1}^{n}\theta(\tilde{a}_i)^{\frac{1}{n}}}, s_{\prod_{i=1}^{n}\tau(\tilde{a}_i)^{\frac{1}{n}}} \right], \bigcup_{\substack{\tilde{r}(\tilde{a}_i)\in \tilde{t}(\tilde{a}_i)\\ \tilde{r}(\tilde{a}_j)\in \tilde{t}(\tilde{a}_j)}} \left(1-\left(1-\prod_{i=1}^{n}\left(1-(1-r_{\tilde{a}_i}{}^L)^p\right)^{\frac{1}{n}}\right)^{\frac{1}{p}}, 1-\left(1-\prod_{i=1}^{n}\left(1-(1-r_{\tilde{a}_i}{}^U)^p\right)^{\frac{1}{n}}\right)^{\frac{1}{p}} \right), \right.$$

$$\bigcup_{\substack{\tilde{\delta}(\tilde{a}_i)\in \tilde{i}(\tilde{a}_i)\\ \tilde{\delta}(\tilde{a}_j)\in \tilde{i}(\tilde{a}_j)}} \left(\left(1-\prod_{i=1}^{n}\left(1-(\delta_{\tilde{a}_i}{}^L)^p\right)^{\frac{1}{n}}\right)^{\frac{1}{p}}, \left(1-\prod_{i=1}^{n}\left(1-(\delta_{\tilde{a}_i}{}^U)^p\right)^{\frac{1}{n}}\right)^{\frac{1}{p}} \right),$$

$$\left. \bigcup_{\substack{\tilde{\eta}(\tilde{a}_i)\in \tilde{f}(\tilde{a}_i)\\ \tilde{\eta}(\tilde{a}_j)\in \tilde{f}(\tilde{a}_j)}} \left(\left(1-\prod_{i=1}^{n}\left(1-(\eta_{\tilde{a}_i}{}^L)^p\right)^{\frac{1}{n}}\right)^{\frac{1}{p}}, \left(1-\prod_{i=1}^{n}\left(1-(\eta_{\tilde{a}_i}{}^U)^p\right)^{\frac{1}{n}}\right)^{\frac{1}{p}} \right) \right\rangle$$

(2) If $p = 1$ and $q = 0$, then

$$IVNHFULGBM^{1,0}(\tilde{a}_1,\tilde{a}_2,...,\tilde{a}_n)=\underset{i=1}{\overset{n}{\otimes}}\tilde{a}_i^{\frac{1}{n}}$$

$$=\left\langle\left[s_{\prod_{i=1}^{n}\theta(\tilde{a}_i)^{\frac{1}{n}}},s_{\prod_{i=1}^{n}\tau(\tilde{a}_i)^{\frac{1}{n}}}\right],\bigcup_{\substack{\tilde{r}(\tilde{a}_i)\in\tilde{t}(\tilde{a}_i)\\ \tilde{r}(\tilde{a}_j)\in\tilde{t}(\tilde{a}_j)}}\left(\prod_{i=1}^{n}\left(r_{\tilde{a}_i}{}^{L}\right)^{\frac{1}{n}},\prod_{i=1}^{n}\left(r_{\tilde{a}_i}{}^{U}\right)^{\frac{1}{n}}\right),\right.$$

$$\left.\bigcup_{\substack{\tilde{\delta}(\tilde{a}_i)\in\tilde{i}(\tilde{a}_i)\\ \tilde{\delta}(\tilde{a}_j)\in\tilde{i}(\tilde{a}_j)}}\left(1-\prod_{i=1}^{n}\left(1-\delta_{\tilde{a}_i}{}^{L}\right)^{\frac{1}{n}},1-\prod_{i=1}^{n}\left(1-\delta_{\tilde{a}_i}{}^{U}\right)^{\frac{1}{n}}\right),\bigcup_{\substack{\tilde{\eta}(\tilde{a}_i)\in\tilde{f}(\tilde{a}_i)\\ \tilde{\eta}(\tilde{a}_j)\in\tilde{f}(\tilde{a}_j)}}\left(1-\prod_{i=1}^{n}\left(1-\eta_{\tilde{a}_i}{}^{L}\right)^{\frac{1}{n}},1-\prod_{i=1}^{n}\left(1-\eta_{\tilde{a}_i}{}^{U}\right)^{\frac{1}{n}}\right)\right\rangle$$

(3) If $p=\frac{1}{2}$ and $q=\frac{1}{2}$, then

$$IVNHFULGBM^{\frac{1}{2},\frac{1}{2}}(\tilde{a}_1,\tilde{a}_2,...,\tilde{a}_n)=\underset{i=1}{\overset{n}{\otimes}}\underset{j=1,j\neq i}{\overset{n}{\otimes}}\left(\frac{1}{2}\tilde{a}_i\oplus\frac{1}{2}\tilde{a}_j\right)^{\frac{1}{n(n-1)}}$$

$$=\left\langle\left[s_{\prod_{i=1}^{n}\prod_{j=1,j\neq i}^{n}\left(\frac{1}{2}\theta(\tilde{a}_i)+\frac{1}{2}\theta(\tilde{a}_j)\right)^{\frac{1}{n(n+1)}}},s_{\prod_{i=1}^{n}\prod_{j=1,j\neq i}^{n}\left(\frac{1}{2}\tau(\tilde{a}_i)+\frac{1}{2}\tau(\tilde{a}_j)\right)^{\frac{1}{n(n+1)}}}\right],\right.$$

$$\bigcup_{\substack{\tilde{r}(\tilde{a}_i)\in\tilde{t}(\tilde{a}_i)\\ \tilde{r}(\tilde{a}_j)\in\tilde{t}(\tilde{a}_j)}}\left(\prod_{i=1}^{n}\prod_{j=1,j\neq i}^{n}\left(1-(1-r_{\tilde{a}_i}{}^{L})^{\frac{1}{2}}(1-r_{\tilde{a}_j}{}^{L})^{\frac{1}{2}}\right)^{\frac{1}{n(n-1)}},\prod_{i=1}^{n}\prod_{j=1,j\neq i}^{n}\left(1-(1-r_{\tilde{a}_i}{}^{U})^{\frac{1}{2}}(1-r_{\tilde{a}_j}{}^{U})^{\frac{1}{2}}\right)^{\frac{1}{n(n-1)}}\right),$$

$$\bigcup_{\substack{\tilde{\delta}(\tilde{a}_i)\in\tilde{i}(\tilde{a}_i)\\ \tilde{\delta}(\tilde{a}_j)\in\tilde{i}(\tilde{a}_j)}}\left(1-\prod_{i=1}^{n}\prod_{j=1,j\neq i}^{n}\left(1-(\delta_{\tilde{a}_i}{}^{L}\delta_{\tilde{a}_j}{}^{L})^{\frac{1}{2}}\right)^{\frac{1}{n(n-1)}},1-\prod_{i=1}^{n}\prod_{j=1,j\neq i}^{n}\left(1-(\delta_{\tilde{a}_i}{}^{U}\delta_{\tilde{a}_j}{}^{U})^{\frac{1}{2}}\right)^{\frac{1}{n(n-1)}}\right),$$

$$\left.\bigcup_{\substack{\tilde{\eta}(\tilde{a}_i)\in\tilde{f}(\tilde{a}_i)\\ \tilde{\eta}(\tilde{a}_j)\in\tilde{f}(\tilde{a}_j)}}\left(1-\prod_{i=1}^{n}\prod_{j=1,j\neq i}^{n}\left(1-(\eta_{\tilde{a}_i}{}^{L}\eta_{\tilde{a}_j}{}^{L})^{\frac{1}{2}}\right)^{\frac{1}{n(n-1)}},1-\prod_{i=1}^{n}\prod_{j=1,j\neq i}^{n}\left(1-(\eta_{\tilde{a}_i}{}^{U}\eta_{\tilde{a}_j}{}^{U})^{\frac{1}{2}}\right)^{\frac{1}{n(n-1)}}\right)\right\rangle$$

(4) If $p=1$ and $q=1$, then

$$IVNHFULGBM^{1,1}(\tilde{a}_1,\tilde{a}_2,...,\tilde{a}_n)=\frac{1}{2}\underset{i=1}{\overset{n}{\otimes}}\underset{j=1,j\neq i}{\overset{n}{\otimes}}(\tilde{a}_i\oplus\tilde{a}_j)^{\frac{1}{n(n-1)}}$$

$$=\left\langle\left[s_{\frac{1}{2}\prod_{i=1}^{n}\prod_{j=1,j\neq i}^{n}\left(\theta(\tilde{a}_i)+\theta(\tilde{a}_j)\right)^{\frac{1}{n(n+1)}}},s_{\frac{1}{2}\prod_{i=1}^{n}\prod_{j=1,j\neq i}^{n}\left(\tau(\tilde{a}_i)+\tau(\tilde{a}_j)\right)^{\frac{1}{n(n+1)}}}\right],\right.$$

$$\bigcup_{\substack{\tilde{r}(\tilde{a}_i)\in\tilde{t}(\tilde{a}_i)\\ \tilde{r}(\tilde{a}_j)\in\tilde{t}(\tilde{a}_j)}}\left(1-\left(1-\prod_{i=1}^{n}\prod_{j=1,j\neq i}^{n}\left(1-(1-r_{\tilde{a}_i}{}^{L})(1-r_{\tilde{a}_j}{}^{L})\right)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{2}},1-\left(1-\prod_{i=1}^{n}\prod_{j=1,j\neq i}^{n}\left(1-(1-r_{\tilde{a}_i}{}^{U})(1-r_{\tilde{a}_j}{}^{U})\right)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{2}}\right),$$

$$\bigcup_{\substack{\tilde{\delta}(\tilde{a}_i)\in\tilde{i}(\tilde{a}_i)\\ \tilde{\delta}(\tilde{a}_j)\in\tilde{i}(\tilde{a}_j)}}\left(\left(1-\prod_{i=1}^{n}\prod_{j=1,j\neq i}^{n}\left(1-\delta_{\tilde{a}_i}{}^{L}\delta_{\tilde{a}_j}{}^{L}\right)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{2}},\left(1-\prod_{i=1}^{n}\prod_{j=1,j\neq i}^{n}\left(1-\delta_{\tilde{a}_i}{}^{U}\delta_{\tilde{a}_j}{}^{U}\right)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{2}}\right),$$

$$\bigcup_{\substack{\tilde{\eta}(\tilde{a}_i)\in\tilde{f}(\tilde{a}_i)\\ \tilde{\eta}(\tilde{a}_j)\in\tilde{f}(\tilde{a}_j)}}\left(\left(1-\prod_{i=1}^{n}\prod_{j=1,j\neq i}^{n}\left(1-\eta_{\tilde{a}_i}{}^{L}\eta_{\tilde{a}_j}{}^{L}\right)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{2}},\left(1-\prod_{i=1}^{n}\prod_{j=1,j\neq i}^{n}\left(1-\eta_{\tilde{a}_i}{}^{U}\eta_{\tilde{a}_j}{}^{U}\right)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{2}}\right)\right\rangle$$

3.4 The interval-valued neutrosophic hesitant fuzzy uncertain linguistic weighted geometric Bonferroni mean operator

Definition 16. Let $\tilde{a}_i = <[s_{\theta(\tilde{a}_i)}, s_{\tau(\tilde{a}_i)}], (\tilde{t}(\tilde{a}_i), \tilde{i}(\tilde{a}_i), \tilde{f}(\tilde{a}_i))>$ $(i = 1,2,...,n)$ be the set of all IVNHFULEs, and p , q $\geq$ 0 , then the interval-valued neutrosophic hesitant fuzzy uncertain linguistic weighted geometric Bonferroni mean operator can be defined as follows:

$$IVNHFULWGBM^{\,p,q}(\tilde{a}_1,\tilde{a}_2,...,\tilde{a}_n)=\frac{1}{p+q}\underset{i=1}{\overset{n}{\otimes}}\underset{j=1,j\neq i}{\overset{n}{\otimes}}(p\tilde{a}_i\oplus q\tilde{a}_j)^{\frac{w_iw_j}{1-w_i}} \tag{44}$$

Theorem 27. Let $\tilde{a}_i = <[s_{\theta(\tilde{a}_i)}, s_{\tau(\tilde{a}_i)}], (\tilde{t}(\tilde{a}_i), \tilde{i}(\tilde{a}_i), \tilde{f}(\tilde{a}_i))>$ $(i = 1,2,...,n)$ be the set of all IVNHFULEs, and p , q $\geq$ 0 , then, the result aggregated from Definition 16 is still an IVNHFULE, and even

$$IVNHFULWGBM^{\,p,q}(\tilde{a}_1,\tilde{a}_2,...,\tilde{a}_n)=\frac{1}{p+q}\underset{i=1}{\overset{n}{\otimes}}\underset{j=1,j\neq i}{\overset{n}{\otimes}}(p\tilde{a}_i\oplus q\tilde{a}_j)^{\frac{w_iw_j}{1-w_i}}$$

$$=\left\langle\left[s_{\frac{1}{p+q}\prod_{i=1}^{n}\prod_{j=1,j\neq i}^{n}\left(p\theta(\tilde{a}_i)+q\theta(\tilde{a}_j)\right)^{\frac{w_iw_j}{1-w_i}}},s_{\frac{1}{p+q}\prod_{i=1}^{n}\prod_{j=1,j\neq i}^{n}\left(p\tau(\tilde{a}_i)+q\tau(\tilde{a}_j)\right)^{\frac{w_iw_j}{1-w_i}}}\right],\right.$$

$$\bigcup_{\substack{\tilde{r}(\tilde{a}_i)\in\tilde{t}(\tilde{a}_i)\\ \tilde{r}(\tilde{a}_j)\in\tilde{t}(\tilde{a}_j)}}\left(1-\left(1-\prod_{i=1}^{n}\prod_{j=1,j\neq i}^{n}\left(1-(1-r_{\tilde{a}_i}{}^{L})^{p}(1-r_{\tilde{a}_j}{}^{L})^{q}\right)^{\frac{w_iw_j}{1-w_i}}\right)^{\frac{1}{p+q}},\right.$$

$$\left.1-\left(1-\prod_{i=1}^{n}\prod_{j=1,j\neq i}^{n}\left(1-(1-r_{\tilde{a}_i}{}^{U})^{p}(1-r_{\tilde{a}_j}{}^{U})^{q}\right)^{\frac{w_iw_j}{1-w_i}}\right)^{\frac{1}{p+q}}\right), \tag{45}$$

$$\bigcup_{\substack{\tilde{\delta}(\tilde{a}_i)\in\tilde{i}(\tilde{a}_i)\\ \tilde{\delta}(\tilde{a}_j)\in\tilde{i}(\tilde{a}_j)}}\left(\left(1-\prod_{i=1}^{n}\prod_{j=1,j\neq i}^{n}\left(1-(\delta_{\tilde{a}_i}{}^{L})^{p}(\delta_{\tilde{a}_j}{}^{L})^{q}\right)^{\frac{w_iw_j}{1-w_i}}\right)^{\frac{1}{p+q}},\left(1-\prod_{i=1}^{n}\prod_{j=1,j\neq i}^{n}\left(1-(\delta_{\tilde{a}_i}{}^{U})^{p}(\delta_{\tilde{a}_j}{}^{U})^{q}\right)^{\frac{w_iw_j}{1-w_i}}\right)^{\frac{1}{p+q}}\right),$$

$$\bigcup_{\substack{\tilde{\eta}(\tilde{a}_i)\in\tilde{f}(\tilde{a}_i)\\ \tilde{\eta}(\tilde{a}_j)\in\tilde{f}(\tilde{a}_j)}}\left(\left(1-\prod_{i=1}^{n}\prod_{j=1,j\neq i}^{n}\left(1-(\eta_{\tilde{a}_i}{}^{L})^{p}(\eta_{\tilde{a}_j}{}^{L})^{q}\right)^{\frac{w_iw_j}{1-w_i}}\right)^{\frac{1}{p+q}},\left(1-\prod_{i=1}^{n}\prod_{j=1,j\neq i}^{n}\left(1-(\eta_{\tilde{a}_i}{}^{U})^{p}(\eta_{\tilde{a}_j}{}^{U})^{q}\right)^{\frac{w_iw_j}{1-w_i}}\right)^{\frac{1}{p+q}}\right)\right\rangle$$

The IVNHFULWGBM operator satisfies the following properties:

Theorem 28 (Reducibility).

Let $W = (\frac{1}{n}, \frac{1}{n}, ..., \frac{1}{n})^T$ be the weight vector of $\tilde{a}_i (i = 1,2,..., n)$, then

$$IVNHFULWGBM^{p,q}(\tilde{a}_1, \tilde{a}_2, ..., \tilde{a}_n) = IVNHFULGBM^{p,q}(\tilde{a}_1, \tilde{a}_2, ..., \tilde{a}_n)$$

The proof of Theorem 28 can be easily completed similar to Theorem 17.

Theorem 29 (Idempotency).

Let $\tilde{a}_j =< [S_{\theta(a)}, s_{\tau(a)}], ([r_a^L, r_a^U], [\delta_a^L, \delta_a^U], [\eta_a^L, \eta_a^U]) >= \tilde{a} \quad (j = 1,2,..., n)$,

then

$$IVNHFULWGBM^{p,q}(\tilde{a}_1, \tilde{a}_2, ..., \tilde{a}_n) = \tilde{a}\ .$$

The proof of Theorem 29 can be easily completed similar to Theorem 12.

Theorem 30 (Commutativity).

If $(\tilde{a}_1', \tilde{a}_2', ..., \tilde{a}_n')$ be any permutation of $(\tilde{a}_1, \tilde{a}_2, ..., \tilde{a}_n)$, then we can get

$$IVNHFULWGBM^{p,q}(\tilde{a}_1', \tilde{a}_2', ..., \tilde{a}_n') = IVNHFULWGBM^{p,q}(\tilde{a}_1, \tilde{a}_2, ..., \tilde{a}_n)\ .$$

The proof of Theorem 30 can be easily completed similar to Theorem 13.

Theorem 31 (Monotonicity).

Suppose $(\tilde{a}_1, \tilde{a}_2, ..., \tilde{a}_n)$ and $(\tilde{b}_1, \tilde{b}_2, ..., \tilde{b}_n)$ are two sets of IVNHFULEs. If $s_{\theta(\tilde{a}_i)} \le s_{\theta(\tilde{b}_i)}$, $s_{\tau(\tilde{a}_i)} \le s_{\tau(\tilde{b}_i)}$, $r_{\tilde{a}_i}^L \le r_{\tilde{b}_i}^L$, $r_{\tilde{a}_i}^U \le r_{\tilde{b}_i}^U$, $\delta_{\tilde{a}_i}^L \ge \delta_{\tilde{b}_i}^L$, $\delta_{\tilde{a}_i}^U \ge \delta_{\tilde{b}_i}^U$ and $\eta_{\tilde{a}_i}^L \ge \eta_{\tilde{b}_i}^L$, $\eta_{\tilde{a}_i}^U \ge \eta_{\tilde{b}_i}^U$ for all $i = 1,2,..., n$,then

$$IVNHFULWGBM^{p,q}(\tilde{a}_1, \tilde{a}_2, ..., \tilde{a}_n) \le IVNHFULWGBM^{p,q}(\tilde{b}_1, \tilde{b}_2, ..., \tilde{b}_n)$$

The proof of Theorem 31 can be easily completed similar to Theorem 14.

Theorem 32 (Boundedness).

Let $\tilde{a}_i (i = 1,2,..., n)$ be a collection of IVNHFULEs. If $s_{\theta-} = \min_{1 \le i \le n}\{s_{\theta(\tilde{a}_i)}\}$, $s_{\tau-} = \min_{1 \le i \le n}\{s_{\tau(\tilde{a}_i)}\}$,

$s_{\theta+} = \max_{1 \le i \le n}\{s_{\theta(\tilde{a}_i)}\}$, $s_{\tau+} = \max_{1 \le i \le n}\{s_{\tau(\tilde{a}_i)}\}$, $r^{L-} = \min_{1 \le i \le n}\{r_{\tilde{a}_i}^L \mid [r_{\tilde{a}_i}^L, r_{\tilde{a}_i}^U] \in t(\tilde{a}_i)\}$,

$r^{U-} = \min_{1 \le i \le n}\{r_{\tilde{a}_i}^U \mid [r_{\tilde{a}_i}^L, r_{\tilde{a}_i}^U] \in t(\tilde{a}_i)\}$, $r^{L+} = \max_{1 \le i \le n}\{r_{\tilde{a}_i}^L \mid [r_{\tilde{a}_i}^L, r_{\tilde{a}_i}^U] \in t(\tilde{a}_i)\}$

$r^{U+} = \max_{1 \le i \le n}\{r_{\tilde{a}_i}^U \mid [r_{\tilde{a}_i}^L, r_{\tilde{a}_i}^U] \in t(\tilde{a}_i)\}$, $\delta^{L-} = \min_{1 \le i \le n}\{\delta_{\tilde{a}_i}^L \mid [\delta_{\tilde{a}_i}^L, \delta_{\tilde{a}_i}^U] \in i(\tilde{a}_i)\}$,

$\delta^{U-} = \min_{1 \le i \le n}\{\delta_{\tilde{a}_i}^U \mid [\delta_{\tilde{a}_i}^L, \delta_{\tilde{a}_i}^U] \in i(\tilde{a}_i)\}$, $\delta^{L+} = \max_{1 \le i \le n}\{\delta_{\tilde{a}_i}^L \mid [\delta_{\tilde{a}_i}^L, \delta_{\tilde{a}_i}^U] \in i(\tilde{a}_i)\}$

$\delta^{U+} = \max_{1 \le i \le n}\{\delta_{\tilde{a}_i}^U \mid [\delta_{\tilde{a}_i}^L, \delta_{\tilde{a}_i}^U] \in i(\tilde{a}_i)\}$, $\eta^{L-} = \min_{1 \le i \le n}\{\eta_{\tilde{a}_i}^L \mid [\eta_{\tilde{a}_i}^L, \eta_{\tilde{a}_i}^U] \in f(\tilde{a}_i)\}$,

$\eta^{U-} = \min_{1 \le i \le n}\{\eta_{\tilde{a}_i}^U \mid [\eta_{\tilde{a}_i}^L, \eta_{\tilde{a}_i}^U] \in f(\tilde{a}_i)\}$, $\eta^{L+} = \max_{1 \le i \le n}\{\eta_{\tilde{a}_i}^L \mid [\eta_{\tilde{a}_i}^L, \eta_{\tilde{a}_i}^U] \in f(\tilde{a}_i)\}$

$\eta^{U+} = \max_{1 \le i \le n}\{\eta_{\tilde{a}_i}^U \mid [\eta_{\tilde{a}_i}^L, \eta_{\tilde{a}_i}^U] \in f(\tilde{a}_i)\}$, for all $i = 1,2,..., n$,

$\tilde{a}^- =< [s_{\theta-}, s_{\tau-}], ([r^{L-}, r^{U-}], [\delta^{L+}, \delta^{U+}], [\eta^{L+}, \eta^{L+}]) >$ and

$\tilde{a}^{+} = <[s_{\theta+}, s_{\tau+}],([r^{L+}, r^{U+}],[\delta^{L-}, \delta^{U-}],[\eta^{L-}, \eta^{U-}])>$ then

$$\tilde{a}^{-} \leq IVNHFULWGBM^{p,q}(\tilde{a}_1, \tilde{a}_2, ..., \tilde{a}_n) \leq \tilde{a}^{+}.$$

The proof of Theorem 32 can be easily completed similar to Theorem 15.

In the following, we will discuss some specials of the IVNHFULWGBM operator with respect to the different parameters p and q.

(1) If $q = 0$, then

$$IVNHFULWGBM^{p,0}(\tilde{a}_1, \tilde{a}_2, ..., \tilde{a}_n) = \frac{1}{p} \otimes_{i=1}^{n} (p\tilde{a}_i)^{w_i}$$

$$= \left\langle \left[s_{\prod_{i=1}^{n} \theta(\tilde{a}_i)^{w_i}}, s_{\prod_{i=1}^{n} \tau(\tilde{a}_i)^{w_i}} \right], \bigcup_{\substack{\tilde{r}(\tilde{a}_i) \in \tilde{t}(\tilde{a}_i) \\ \tilde{r}(\tilde{a}_j) \in \tilde{t}(\tilde{a}_j)}} \left(1 - \left(1 - \prod_{i=1}^{n} \left(1 - (1 - r_{\tilde{a}_i}{}^{L})^{p} \right)^{w_i} \right)^{\frac{1}{p}}, 1 - \left(1 - \prod_{i=1}^{n} \left(1 - (1 - r_{\tilde{a}_i}{}^{U})^{p} \right)^{w_i} \right)^{\frac{1}{p}} \right), \right.$$

$$\bigcup_{\substack{\tilde{\delta}(\tilde{a}_i) \in \tilde{i}(\tilde{a}_i) \\ \tilde{\delta}(\tilde{a}_j) \in \tilde{i}(\tilde{a}_j)}} \left(\left(1 - \prod_{i=1}^{n} \left(1 - (\delta_{\tilde{a}_i}{}^{L})^{p} \right)^{w_i} \right)^{\frac{1}{p}}, \left(1 - \prod_{i=1}^{n} \left(1 - (\delta_{\tilde{a}_i}{}^{U})^{p} \right)^{w_i} \right)^{\frac{1}{p}} \right),$$

$$\left. \bigcup_{\substack{\tilde{\eta}(\tilde{a}_i) \in \tilde{f}(\tilde{a}_i) \\ \tilde{\eta}(\tilde{a}_j) \in \tilde{f}(\tilde{a}_j)}} \left(\left(1 - \prod_{i=1}^{n} \left(1 - (\eta_{\tilde{a}_i}{}^{L})^{p} \right)^{w_i} \right)^{\frac{1}{p}}, \left(1 - \prod_{i=1}^{n} \left(1 - (\eta_{\tilde{a}_i}{}^{U})^{p} \right)^{w_i} \right)^{\frac{1}{p}} \right) \right\rangle$$

(2) If $p = 1$ and $q = 0$, then

$$IVNHFULWGBM^{1,0}(\tilde{a}_1, \tilde{a}_2, ..., \tilde{a}_n) = \otimes_{i=1}^{n} \tilde{a}_i{}^{w_i}$$

$$= \left\langle \left[s_{\prod_{i=1}^{n} \theta(\tilde{a}_i)^{w_i}}, s_{\prod_{i=1}^{n} \tau(\tilde{a}_i)^{w_i}} \right], \bigcup_{\substack{\tilde{r}(\tilde{a}_i) \in \tilde{t}(\tilde{a}_i) \\ \tilde{r}(\tilde{a}_j) \in \tilde{t}(\tilde{a}_j)}} \left(\prod_{i=1}^{n} \left(r_{\tilde{a}_i}{}^{L} \right)^{w_i}, \prod_{i=1}^{n} \left(r_{\tilde{a}_i}{}^{U} \right)^{w_i} \right), \right.$$

$$\left. \bigcup_{\substack{\tilde{\delta}(\tilde{a}_i) \in \tilde{i}(\tilde{a}_i) \\ \tilde{\delta}(\tilde{a}_j) \in \tilde{i}(\tilde{a}_j)}} \left(1 - \prod_{i=1}^{n} \left(1 - \delta_{\tilde{a}_i}{}^{L} \right)^{w_i}, 1 - \prod_{i=1}^{n} \left(1 - \delta_{\tilde{a}_i}{}^{U} \right)^{w_i} \right), \bigcup_{\substack{\tilde{\eta}(\tilde{a}_i) \in \tilde{f}(\tilde{a}_i) \\ \tilde{\eta}(\tilde{a}_j) \in \tilde{f}(\tilde{a}_j)}} \left(1 - \prod_{i=1}^{n} \left(1 - \eta_{\tilde{a}_i}{}^{L} \right)^{w_i}, 1 - \prod_{i=1}^{n} \left(1 - \eta_{\tilde{a}_i}{}^{U} \right)^{w_i} \right) \right\rangle$$

(3) If $p = \frac{1}{2}$ and $q = \frac{1}{2}$, then

$$IVNHFULWGBM^{\frac{1}{2},\frac{1}{2}}(\tilde{a}_1, \tilde{a}_2, ..., \tilde{a}_n) = \otimes_{i=1}^{n} \otimes_{j=1, j \neq i}^{n} \left(\frac{1}{2}\tilde{a}_i \oplus \frac{1}{2}\tilde{a}_j \right)^{\frac{w_i w_j}{1-w_i}}$$

$$= \left\langle \left[s_{\prod_{i=1}^{n} \prod_{j=1, j \neq i}^{n} \left(\frac{1}{2}\theta(\tilde{a}_i) + \frac{1}{2}\theta(\tilde{a}_j) \right)^{\frac{w_i w_j}{1-w_i}}}, s_{\prod_{i=1}^{n} \prod_{j=1, j \neq i}^{n} \left(\frac{1}{2}\tau(\tilde{a}_i) + \frac{1}{2}\tau(\tilde{a}_j) \right)^{\frac{w_i w_j}{1-w_i}}} \right], \right.$$

$$\bigcup_{\substack{\tilde{r}(\tilde{a}_i)\in\tilde{t}(\tilde{a}_i)\\ \tilde{r}(\tilde{a}_j)\in\tilde{t}(\tilde{a}_j)}}\left(\prod_{i=1}^{n}\prod_{j=1,j\neq i}^{n}\left(1-(1-r_{\tilde{a}_i}{}^{L})^{\frac{1}{2}}(1-r_{\tilde{a}_j}{}^{L})^{\frac{1}{2}}\right)^{\frac{w_i w_j}{1-w_i}},\prod_{i=1}^{n}\prod_{j=1,j\neq i}^{n}\left(1-(1-r_{\tilde{a}_i}{}^{U})^{\frac{1}{2}}(1-r_{\tilde{a}_j}{}^{U})^{\frac{1}{2}}\right)^{\frac{w_i w_j}{1-w_i}}\right),$$

$$\bigcup_{\substack{\tilde{\delta}(\tilde{a}_i)\in\tilde{i}(\tilde{a}_i)\\ \tilde{\delta}(\tilde{a}_j)\in\tilde{i}(\tilde{a}_j)}}\left(1-\prod_{i=1}^{n}\prod_{j=1,j\neq i}^{n}\left(1-(\delta_{\tilde{a}_i}{}^{L}\delta_{\tilde{a}_j}{}^{L})^{\frac{1}{2}}\right)^{\frac{w_i w_j}{1-w_i}},1-\prod_{i=1}^{n}\prod_{j=1,j\neq i}^{n}\left(1-(\delta_{\tilde{a}_i}{}^{U}\delta_{\tilde{a}_j}{}^{U})^{\frac{1}{2}}\right)^{\frac{w_i w_j}{1-w_i}}\right),$$

$$\bigcup_{\substack{\tilde{\eta}(\tilde{a}_i)\in\tilde{f}(\tilde{a}_i)\\ \tilde{\eta}(\tilde{a}_j)\in\tilde{f}(\tilde{a}_j)}}\left(1-\prod_{i=1}^{n}\prod_{j=1,j\neq i}^{n}\left(1-(\eta_{\tilde{a}_i}{}^{L}\eta_{\tilde{a}_j}{}^{L})^{\frac{1}{2}}\right)^{\frac{w_i w_j}{1-w_i}},1-\prod_{i=1}^{n}\prod_{j=1,j\neq i}^{n}\left(1-(\eta_{\tilde{a}_i}{}^{U}\eta_{\tilde{a}_j}{}^{U})^{\frac{1}{2}}\right)^{\frac{w_i w_j}{1-w_i}}\right)\Bigg\rangle$$

(4) If $p=1$ and $q=1$, then

$$IVNHFULWGBM^{1,1}(\tilde{a}_1,\tilde{a}_2,...,\tilde{a}_n)=\frac{1}{2}\mathop{\otimes}_{i=1}^{n}\mathop{\otimes}_{j=1,j\neq i}^{n}(\tilde{a}_i\oplus\tilde{a}_j)^{\frac{w_i w_j}{1-w_i}}$$

$$=\Bigg\langle\left[s_{\frac{1}{2}\prod_{i=1}^{n}\prod_{j=1,j\neq i}^{n}(\theta(\tilde{a}_i)+\theta(\tilde{a}_j))^{\frac{w_i w_j}{1-w_i}}},s_{\frac{1}{2}\prod_{i=1}^{n}\prod_{j=1,j\neq i}^{n}(\tau(\tilde{a}_i)+\tau(\tilde{a}_j))^{\frac{w_i w_j}{1-w_i}}}\right],$$

$$\bigcup_{\substack{\tilde{r}(\tilde{a}_i)\in\tilde{t}(\tilde{a}_i)\\ \tilde{r}(\tilde{a}_j)\in\tilde{t}(\tilde{a}_j)}}\left(1-\left(1-\prod_{i=1}^{n}\prod_{j=1,j\neq i}^{n}\left(1-(1-r_{\tilde{a}_i}{}^{L})(1-r_{\tilde{a}_j}{}^{L})\right)^{\frac{w_i w_j}{1-w_i}}\right)^{\frac{1}{2}},1-\left(1-\prod_{i=1}^{n}\prod_{j=1,j\neq i}^{n}\left(1-(1-r_{\tilde{a}_i}{}^{U})(1-r_{\tilde{a}_j}{}^{U})\right)^{\frac{w_i w_j}{1-w_i}}\right)^{\frac{1}{2}}\right),$$

$$\bigcup_{\substack{\tilde{\delta}(\tilde{a}_i)\in\tilde{i}(\tilde{a}_i)\\ \tilde{\delta}(\tilde{a}_j)\in\tilde{i}(\tilde{a}_j)}}\left(\left(1-\prod_{i=1}^{n}\prod_{j=1,j\neq i}^{n}\left(1-\delta_{\tilde{a}_i}{}^{L}\delta_{\tilde{a}_j}{}^{L}\right)^{\frac{w_i w_j}{1-w_i}}\right)^{\frac{1}{2}},\left(1-\prod_{i=1}^{n}\prod_{j=1,j\neq i}^{n}\left(1-\delta_{\tilde{a}_i}{}^{U}\delta_{\tilde{a}_j}{}^{U}\right)^{\frac{w_i w_j}{1-w_i}}\right)^{\frac{1}{2}}\right),$$

$$\bigcup_{\substack{\tilde{\eta}(\tilde{a}_i)\in\tilde{f}(\tilde{a}_i)\\ \tilde{\eta}(\tilde{a}_j)\in\tilde{f}(\tilde{a}_j)}}\left(\left(1-\prod_{i=1}^{n}\prod_{j=1,j\neq i}^{n}\left(1-\eta_{\tilde{a}_i}{}^{L}\eta_{\tilde{a}_j}{}^{L}\right)^{\frac{w_i w_j}{1-w_i}}\right)^{\frac{1}{2}},\left(1-\prod_{i=1}^{n}\prod_{j=1,j\neq i}^{n}\left(1-\eta_{\tilde{a}_i}{}^{U}\eta_{\tilde{a}_j}{}^{U}\right)^{\frac{w_i w_j}{1-w_i}}\right)^{\frac{1}{2}}\right)\Bigg\rangle$$

4. Two multiple attribute decision making methods based on IVNHFULWBM operator and IVNHFULWGBM operator

In a multiple attribute decision making problem with interval-valued neutrosophic hesitant uncertain linguistic information, let $A=\{A_1,A_2,...,A_m\}$ be a discrete set of alternatives, $C=\{C_1,C_2,...,C_n\}$ be a set of attributes. The evaluation values can be represented by the form of interval-valued neutrosophic hesitant uncertain linguistic elements $n_{ij}=\left\langle\left[s_{\theta(n_{ij})},s_{\tau(n_{ij})}\right],\left(\tilde{t}(n_{ij}),\tilde{i}(n_{ij}),\tilde{f}(n_{ij})\right)\right\rangle$, $\tilde{t}(n_{ij})=\{\tilde{r}(n_{ij})\mid\tilde{r}(n_{ij})\in\tilde{t}(n_{ij})\}$, $\tilde{i}(n_{ij})=\{\tilde{\delta}(n_{ij})\mid\tilde{\delta}(n_{ij})\in\tilde{i}(n_{ij})\}$, $\tilde{f}(n_{ij})=\{\tilde{\eta}(n_{ij})\mid\tilde{\eta}(n_{ij})\in\tilde{f}(n_{ij})\}$. Therefore, we can get the decision matrix $N=(n_{ij})_{m\times n}$. The weighting vector of attributes is given by $W=(w_1,w_2,...,w_n)^T$ with $w_j\geq 0$,

$\sum_{j=1}^{n} w_j = 1$. Then, the steps of the decision making method are described as follows:

Step1: Utilized the IVNHFULWBM operator and IVNHFULWGBM operator to derive the overall assessment values $N_i (i = 1,2,...,n)$ of the alternatives $A_i (i = 1,2,...,n)$.

Step2: Calculate the score values of the overall assessment values $N_i (i = 1,2,...,m)$ by the Eq.(33).

Step3: Rank all the alternatives $\{A_1, A_2,..., A_n\}$ by score values $S(N_i)$, and select the best alternative.

Step4: End.

5. A numerical example

In this section, we give a numerical example to demonstrate the MADM methods based on IVNHFULWBM and IVNHFULWGBM operators.

Suppose that there is a company, which wants to invest a sum of money to an industry. There are four companies as alternatives to be chosen, including: (1) A_1 is a car company; (2) A_2 is a food company; (3) A_3 is a computer company; (4) A_4 is an arms company. There are three evaluation attributes, including: (1) C_1 is the risk; (2) C_2 is the growth; (3) C_3 is the environment. We can know the attributes C_1 and C_2 are benefit criteria, and the type of C3 is cost criteria. The weight vector of the attribute is $w = \{0.25, 0.35, 0.4\}$ .The final evaluation values are expressed by the IVNHFULEs, and shown in Table 1.

Table 1 Interval-valued neutrosophic hesitant fuzzy uncertain linguistic decision matrix

	C1	C2	C3
A1	<[S_4,S_5],([0.3,0.4],[0.4,0.4]),([0.1,0.2]),([0.3,0.4])>	<[S_3,S_5],([0.4,0.5],[0.5,0.6]),([0.2,0.3],[0.3,0.3]),([0.3,0.4])>	<[S_4,S_5],([0.2,0.3]),([0.1,0.2],[0.4,0.5]),([0.5,0.6])>
A2	<[S_3,S_5],([0.6,0.7]),([0.1,0.2]),([0.1,0.2],[0.2,0.3])>	<[S_2,S_4],([0.6,0.7]),([0.1,0.1]),([0.2,0.3])>	<[S_3,S_5],([0.6,0.7]),([0.1,0.2]),([0.1,0.2])>
A3	<[S_2,S_3],([0.3,0.4],[0.5,0.6]),([0.2,0.4]),([0.2,0.3])>	<[S_2,S_4],([0.5,0.6]),([0.2,0.3]),([0.3,0.4])>	<[S_3,S_5],([0.5,0.6]),([0.1,0.2],[0.2,0.3]),([0.2,0.3])>
A4	<[S_1,S_3],([0.7,0.8]),([0.2,0.3]),([0.1,0.2])>	<[S_2,S_5],([0.6,0.7]),([0.3,0.3]),([0.2,0.2])>	<[S_2,S_5],([0.3,0.5],[0.2,0.3]),([0.1,0.2]),([0.3,0.3])>

5.1 Procedure of decision making method based on the IVNHFULWBM operator

Step1: Utilized the IVNHFULWBM operator to derive the overall assessment values $N_i (i = 1,2,...,n)$ of the alternatives $A_i (i = 1,2,...,n)$. We can get

N_1=<[$S_{2.90}$,$S_{4.61}$],([0.2853,0.3869],[0.3088,0.4118],[0.3175,0.3869],[0.3429,0.4118]),([0.1255,0.2268],[0.2393,0.3431],[0.1482,0.2268],[0.2689,0.3431]),([0.3768,0.4783])>

N_2=<[$S_{2.71}$,$S_{4.72}$],([0.6000,0.7000]),([0.1000,0.1714]),([0.1255,0.2268],[0.1623,0.2629])>

N_3=<[$S_{2.34}$,$S_{3.97}$],([0.4258,0.5274],[0.5000,0.6000]),([0.1623,0.2988],[0.2000,0.3352]),([0.2268,0.3273])>

N_4=<[$S_{1.61}$,$S_{4.24}$],([0.5107,0.6569],[0.4644,0.5743]),([0.1882,0.2692]),([0.2021,0.2373])>

Step2: Calculate the score values of the overall assessment values $N_i(i=1,2,...,m)$, which are shown as follows (for convenience, we can use the subscript):

S(N_1)=2.1135, S(N_2)=2.8716, S(N_3)=2.0908, S(N_4)= 2.0554

Step3: Rank all the alternatives $\{A_1, A_2,..., A_n\}$ by score values $S(N_i)$, and select the best alternative.

So, $A_2 \succ A_1 \succ A_3 \succ A_4$, and the best alternative is A_2.

Step4: End.

5.2 Procedure of decision making method based on the IVNHFULWGBM operator

Step1: Utilized the IVNHFULWGBM operator to derive the overall evaluation values $N_i(i=1,2,...,n)$ of the alternatives $A_i(i=1,2,...,n)$. We can get

N_1=<[$S_{2.95}$,$S_{4.62}$],([0.2900,0.3911],[0.3164,0.4189],[0.3276,0.3911],[0.3557,0.4189]),([0.1246,0.2260],[0.2174,0.3249],[0.1452,0.2260],[0.2496,0.3249]),([0.3678,0.4697])>

N_2=<[$S_{2.71}$,$S_{4.72}$],([0.6000,0.7000]),([0.1000,0.1704]),([0.1246,0.2260],[0.1583,0.2601])>

N_3=<[$S_{2.36}$,$S_{4.01}$],([0.4330,0.5343],[0.5000,0.6000]),([0.1583,0.2904],[0.2000,0.3333]),([0.2260,0.3266])>

N_4=<[$S_{1.64}$,$S_{4.28}$],([0.5411,0.6751],[0.5128,0.6223]),([0.1822,0.2601]),([0.1914,0.2346])>

Step2: Calculate the score values of the overall assessment values $N_i(i=1,2,...,m)$, which are shown as follows (for convenience, we can use the subscript):

S(N_1)=2.1648, S(N_2)=2.8975, S(N_3)=2.1183, S(N_4)=2.1222

Step3: Rank all the alternatives $\{A_1, A_2,..., A_n\}$ by score values $S(N_i)$,and select the best alternative.

So, $A_2 \succ A_1 \succ A_4 \succ A_3$, and the best alternative is A_2.

Step4: End.

In this section, we propose two methods based on IVNHFULWBM operator and IVNHFULWGBM operator to solve the multiple attribute decision making problems in interval-valued neutrosophic hesitant fuzzy uncertain linguistic environment. Finally, we get the same best alternative. At the same time, we can find that the main advantages of the developed methods in this paper are that not only accommodating the interval-valued hesitant uncertain linguistic environment but also considering the interrelationship of the input arguments.

5.3 Analysis of the influence of the parameters p and q

In order to demonstrate the influence of the parameters p and q on decision making results, we use the different values p and q in IVNHFULWBM operator and IVNHFULWGBM operator to rank the alternatives. The ranking results are shown in Table 2 and Table 3.

Table 2 Ordering of the alternatives by utilizing the different p, q in IVNHFULWBM operator

p, q	Score function $S(N_i)$	Ranking
$p = q = \frac{1}{2}$	$S(N_1)=2.0983, S(N_2)=2.8630, S(N_3)=2.0805, S(N_4)=2.0286$	$A_2 \succ A_1 \succ A_3 \succ A_4$
$p = q = 1$	$S(N_1)=2.1135$, $S(N_2)=2.8716$, $S(N_3)=2.0908$, $S(N_4)=2.0554$	$A_2 \succ A_1 \succ A_3 \succ A_4$
$p = 1, q = 0$	$S(N_1)=2.1706$, $S(N_2)=2.9115$, $S(N_3)=2.1564$, $S(N_4)=2.1570$	$A_2 \succ A_1 \succ A_4 \succ A_3$
$p = 0, q = 1$	$S(N_1)=2.2300$, $S(N_2)=2.8692$, $S(N_3)=2.1141$, $S(N_4)=2.1670$	$A_2 \succ A_1 \succ A_4 \succ A_3$
$p = 1, q = 2$	$S(N_1)=2.1601$, $S(N_2)=2.8801$, $S(N_3)=2.1084$, $S(N_4)=2.1095$	$A_2 \succ A_1 \succ A_4 \succ A_3$
$p = 2, q = 1$	$S(N_1)=2.1435$, $S(N_2)=2.8922$, $S(N_3)=2.1239$, $S(N_4)=2.1081$	$A_2 \succ A_1 \succ A_3 \succ A_4$
$p = 2, q = 2$	$S(N_1)=2.1473$, $S(N_2)=2.8886$, $S(N_3)=2.1115$, $S(N_4)=2.1149$	$A_2 \succ A_1 \succ A_4 \succ A_3$
$p = 5, q = 5$	$S(N_1)=2.2531$, $S(N_2)=2.9360$, $S(N_3)=2.1694$, $S(N_4)=2.2846$	$A_2 \succ A_4 \succ A_1 \succ A_3$
$p = 10, q = 0$	$S(N_1)=2.6175$, $S(N_2)=3.0393$, $S(N_3)=2.4994$, $S(N_4)=2.5667$	$A_2 \succ A_1 \succ A_4 \succ A_3$
$p = 0, q = 10$	$S(N_1)=2.6442$, $S(N_2)=3.0184$, $S(N_3)=2.4619$, $S(N_4)=2.5649$	$A_2 \succ A_1 \succ A_4 \succ A_3$
$p = 15, q = 15$	$S(N_1)=2.4632$, $S(N_2)=3.0276$, $S(N_3)=2.2789$, $S(N_4)=2.5187$	$A_2 \succ A_4 \succ A_1 \succ A_3$

Table 3 Ordering of the alternatives by utilizing the different p, q in IVNHFULWGBM operator

p, q	Score function $S(N_i)$	Ranking
$p = q = \frac{1}{2}$	$S(N_1)=2.1727, S(N_2)=2.9003, S(N_3)=2.1208, S(N_4)=2.1310$	$A_2 \succ A_1 \succ A_4 \succ A_3$
$p = q = 1$	$S(N_1)=2.1648$, $S(N_2)=2.8975$, $S(N_3)=2.1183$, $S(N_4)=2.1222$	$A_2 \succ A_1 \succ A_4 \succ A_3$
$p = 1, q = 0$	$S(N_1)=2.0379$, $S(N_2)=2.8962$, $S(N_3)=2.0850$, $S(N_4)=1.1963$	$A_2 \succ A_3 \succ A_1 \succ A_4$
$p = 0, q = 1$	$S(N_1)=2.0995$, $S(N_2)=2.8501$, $S(N_3)=2.0462$, $S(N_4)=1.9921$	$A_2 \succ A_1 \succ A_3 \succ A_4$
$p = 1, q = 2$	$S(N_1)=2.1492$, $S(N_2)=2.8819$, $S(N_3)=2.1009$, $S(N_4)=2.0921$	$A_2 \succ A_1 \succ A_3 \succ A_4$
$p = 2, q = 1$	$S(N_1)=2.1283$, $S(N_2)=2.8979$, $S(N_3)=2.1139$, $S(N_4)=2.0855$	$A_2 \succ A_1 \succ A_3 \succ A_4$
$p = 2, q = 2$	$S(N_1)=2.1477$, $S(N_2)=2.8915$, $S(N_3)=2.1125$, $S(N_4)=2.1028$	$A_2 \succ A_1 \succ A_3 \succ A_4$

$p=5, q=5$	S(N$_1$)=2.1072, S(N$_2$)=2.8783, S(N$_3$)=2.0961, S(N$_4$)=2.0574	$A_2 \succ A_1 \succ A_3 \succ A_4$
$p=10, q=0$	S(N$_1$)=1.8177, S(N$_2$)=2.8326, S(N$_3$)=1.9907, S(N$_4$)=1.8845	$A_2 \succ A_3 \succ A_4 \succ A_1$
$p=0, q=10$	S(N$_1$)=1.8639, S(N$_2$)=2.7869, S(N$_3$)=1.9555, S(N$_4$)=1.8948	$A_2 \succ A_3 \succ A_4 \succ A_1$
$p=15, q=15$	S(N$_1$)=2.2248, S(N$_2$)=3.4678, S(N$_3$)=2.3038, S(N$_4$)=2.4967	$A_2 \succ A_4 \succ A_3 \succ A_1$

As we can see from Table 2 and Table 3, the ranking of the alternatives may be different for the different values p, q in IVNHFULWBM operator and IVNHFULWGBM operator. But the best alternative is the same although the different values p, q. Thus, the organization can properly select the desirable alternative according to his interest and the actual needs. In general, we can get $p = q = 1$ or $p = q = \frac{1}{2}$.

6. Conclusions

Interval-valued neutrosophic hesitant fuzzy uncertain linguistic set (IVNHFULS) can be as an important tool of describing the decision maker’s preference and handling the hesitant, uncertain, incomplete information, which plays a vital role in decision making. Bonferroni mean (BM) can replace the simple averaging of other mean type operators because of the excellent modeling capability. In real decision making, how to flexibly and rationally deal with input arguments is a vital problem. In order to have the advantages of both IVNHFULS and BM, we extend BM operator to interval-valued neutrosophic hesitant fuzzy uncertain linguistic environment. Hence, we finally give some interval-valued neutrosophic hesitant fuzzy uncertain linguistic BM aggregation operators, such as interval-valued neutrosophic hesitant fuzzy uncertain linguistic Bonferroni mean (IVNHFULBM) operator, interval-valued neutrosophic hesitant fuzzy uncertain linguistic weighted Bonferroni mean (IVNHFULWBM) operator, interval-valued neutrosophic hesitant fuzzy uncertain linguistic geometric Bonferroni mean (IVNHFULGBM) operator, interval-valued neutrosophic hesitant fuzzy uncertain linguistic weighted geometric Bonferroni mean (IVNHFULWGBM) operator. It is essential to verify the effectiveness of IVNHFULWBM operator and IVNHFULWGBM operator, so we apply the two operators to deal with a numerical example. To get the desirable alternative decision maker wanted, we change the values of p, q according to their interest and actual need.

In the further research, we should dug deeper into the applications of the proposed methods and constantly improve the multiple attribute decision making method.

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Compliance with Ethical Standards

(1) Disclosure of potential conflicts of interest

We declare that we do have no commercial or associative interests that represent a conflict of interests in connection with this manuscript. There are no professional or other personal interests that can inappropriately influence our submitted work.

(2) Research involving human participants and/or animals

This article does not contain any studies with human participants or animals performed by any of the authors.

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