Some New Operations of (α, β, γ) Interval Cut Set of Interval Valued Neutrosophic Sets

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Abstract. In this paper, we define the disjunctive sum, difference and Cartesian product of two interval valued neutrosophic sets and study their basic properties. The notions of the (α, β, γ) interval cut set of interval valued neutrosophic sets and the (α, β, γ) strong interval cut set of interval valued neutrosophic sets are put forward. Some related properties have been established with proof, examples and counter examples.

Keywords: (α, β, γ) *interval cut set of interval valued neutrosophic sets; cut set; disjunctive sum; interval valued neutrosophic sets; neutrosophic set.*

1 Introduction

Many theories have been developed for uncertainties, including the theory of probability, the theory of fuzzy sets, the theory of intuitionistic fuzzy sets, the theory of rough sets, and so on. Although many new techniques have been developed as a result of these theories there are still some difficulties. Major difficulties arise due to inadequacy of parameters. In 1999, Smarandache [1,2] proposed the concept of the neutrosophic set (NS) by adding an independent indeterminacy-membership function, which is a generalization of a classic set, a fuzzy set [3], an intuitionistic fuzzy set [4], and so on. In an NS, the indeterminacy is quantified explicitly and truth-membership function (T), indeterminacy-membership function (I), and false-membership function (F) are completely independent and from a scientific or engineering point of view, the NS operators need to be specified. Therefore, Wang, *et al.* [5] defined the single valued neutrosophic set (SVNS) and then provided the set-theoretic operations and various properties of single valued neutrosophic sets. Wang, *et al.* [6] proposed the set-theoretic operations on an instance of the neutrosophic set, called the interval valued neutrosophic set (IVNS), which is more flexible and practical than the NS. Works on single valued neutrosophic sets (SVNS) and interval valued neutrosophic sets (IVNS) and their hybrid structure in theory and applications have progressed rapidly (e.g., [7-23]). Also, neutrosophic sets have been extended to neutrosophic models both in theory [24-28] and in applications [29-32].

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The organization of this paper is as follows. In Section 2, some basic properties of neutrosophic sets are redefined. In Section 3, the (α, β, γ) interval cut sets of interval valued neutrosophic sets are defined and some properties of these (α, β, γ) interval cut sets are given. In Section 4, the operations of addition, subtraction, multiplication and division over neutrosophic multi-sets are defined based on the interval valued neutrosophic set. Finally, the conclusion is given.

2 Preliminaries

In this section, we recall some concepts of neutrosophic set theory and interval valued neutrosophic set theory. See the refences for more detailed explanations related to this subsection and its background.

Definition 2.1 [4]**:** An intuitionistic fuzzy set (IFS) A in X is:

$$A = \{\langle x, \mu_A(x), \nu_A(x) \ \rangle : x \in X\},$$

where the functions $\mu_A(x),\ \nu_A(x) : X \to [0,1]$ with the condition $0 \leq \mu_A(x) + \nu_A(x) \leq 1$. The numbers $\mu_A(x)$ and $\nu_A(x)$ represent, respectively, the degree of membership and degree of non-membership of the element $x \in X$.

Definition 2.2 [33]**:** Let $A = \{\langle x, \mu_A(x), \nu_A(x) \ \rangle : x \in X\}$ be any intuitionistic fuzzy set on the universal set X. For any ordered pair $\langle \alpha, \beta \rangle$, where $\alpha \in [0,1]$, $\beta \in [0,1]$, and $0 \leq \alpha + \beta \leq 1$, let:

$$A_{\langle \alpha, \beta \rangle} = \{x : \mu_A(x) \geq \alpha, \nu_A(x) \leq \beta : x \in X\},$$

which is called the $\langle \alpha, \beta \rangle$-cut set (or level set) of intuitionistic fuzzy set A.

Definition 2.3 [34]**:** Let X be an universe of discourse with a generic element in X, denoted by x. Then, a neutrosophic (NS) set A is an object with the following form:

$$A = \{\langle x : T_A(x), I_A(x), F_A(x) \rangle, x \in X\}$$

A neutrosophic set A in X is characterized by truth-membership function T_A, indeterminacy membership function I_A, and falsity-membership function F_A. $T_A(x)$, $I_A(x)$ and $F_A(x)$ are real standard or non-standard subsets of $]^-0, 1^+[$. That is:

$$T_A(x) : X \to]^-0, 1^+[$$

$$I_A(x) : X \to]^-0, 1^+[$$

$$F_A(x) : X \to]^-0, 1^+[$$

There is no restriction on the sum of $T_A(x)$, $I_A(x)$ and $F_A(x)$, so:

$$^{-}0 \leq supT_A(x) + supI_A(x) + supF_A(x) \leq 3^{+}.$$

Instead of $]^{-}0, 1^{+}[$, we need to take the interval [0, 1] for technical applications, because $]^{-}0, 1^{+}[$ will be difficult to apply in real applications, for example in scientific and engineering problems.

For two NS:

$$A_{NS} = \{\langle x\,, \mathrm{T_A(x)},\ \mathrm{I_A(x)},\ \mathrm{F_A(x)}\rangle\ |\mathrm{x} \in \mathrm{X}\}$$

and

$$B_{NS} = \{\langle x\,, \mathrm{T_B(x)},\ \mathrm{I_B(x)},\ \mathrm{F_B(x)}\rangle\ |\mathrm{x} \in \mathrm{X}\}$$

the two relations are defined as follows:

(1) $A_{NS} \subseteq B_{NS}$ if and only if

$$\mathrm{T_A(x)} \leq \mathrm{T_B(x)},\ \ \mathrm{I_A(x)} \geq \mathrm{I_B(x)},\ \ \mathrm{F_A(x)} \geq \mathrm{F_B(x)}$$

(2) $A_{NS} = B_{NS}$ if and only if

$$\mathrm{T_A(x)} = \mathrm{T_B(x)},\ \ \mathrm{I_A(x)} = \mathrm{I_B(x)},\ \ \mathrm{F_A(x)} = \mathrm{F_B(x)}.$$

Definition 2.4: [5,34] Let X be a space of points (objects) with a generic element in X denoted by x. An interval valued neutrosophic set (IVNS) A in X is characterized by truth-membership function T_A, indeterminacy-membership function I_A, and falsity-membership function F_A. For each point $\mathrm{x} \in \mathrm{X}$, $T_A(x)$, $I_A(x)$, $F_A(x) \subseteq [0, 1]$.

This can be written as:

$$A_{\mathrm{IVNS}} = \left\{\left\langle \begin{matrix} x, [\inf T_A(x)\,, \sup T_A(x)], [\inf I_A(x)\,, \sup I_A(x)], \\ [\inf F_A(x)\,, \sup F_A(x)] \end{matrix} \right\rangle : \mathrm{x} \in \mathrm{X}\right\}$$

Definition 2.5 [5]**:** For two IVNS,

$$A_{\mathrm{IVNS}} = \left\{\left\langle \begin{matrix} x, [\inf T_A(x)\,, \sup T_A(x)], [\inf I_A(x)\,, \sup I_A(x)], \\ [\inf F_A(x)\,, \sup F_A(x)] \end{matrix} \right\rangle : \mathrm{x} \in \mathrm{X}\right\}$$

and

$$B_{\mathrm{IVNS}} = \left\{\left\langle \begin{matrix} x, [\inf T_B(x)\,, \sup T_B(x)], [\inf I_B(x)\,, \sup I_B(x)], \\ [\inf F_B(x)\,, \sup F_B(x)] \end{matrix} \right\rangle : \mathrm{x} \in \mathrm{X}\right\},$$

we now present the set-theoretic operators on the interval neutrosophic set.

1. An interval valued neutrosophic set A is contained in another interval valued neutrosophic set B, $A_{\mathrm{IVNS}} \subseteq B_{\mathrm{IVNS}}$, if and only if

$\inf T_A(x) \leq \inf T_B(x), \sup T_A(x) \leq \sup T_B(x),$

$\inf I_A(x) \geq \inf I_B(x), \sup I_A(x) \geq \sup I_B(x),$

$\inf F_A(x) \geq \inf F_B(x), \sup F_A(x) \geq \sup F_B(x),$

for all $x \in X$.

2. Two interval valued neutrosophic sets A and B are equal, written as $A_{\text{IVNS}} = B_{\text{IVNS}}$, if and only if A ⊆ B and B ⊆ A, i.e.

   $\inf T_A(x) = \inf T_B(x), \sup T_A(x) = \sup T_B(x),$

   $\inf I_A(x) = \inf I_B(x), \sup I_A(x) = \sup I_B(x),$

   $\inf F_A(x) = \inf F_B(x), \sup F_A(x) = \sup F_B(x),$

   for any x ∈ X.

3. An interval neutrosophic set A is empty if and only if

   $inf\ T_A(x) = sup\ T_A(x) = 0,$

   $inf\ I_A(x) = supI_A(x) = 1$ and

   $inf\ F_A(x) = sup\ F_A(x) = 0,$

   for all $x \in X$.

4. The complement of an interval neutrosophic set A is denoted by A^c and is defined by

$$A_{\text{IVNS}^C} = \left\{ \begin{matrix} x, [\inf F_A(x), \sup F_A(x)], \\ [1 - \sup I_A(x), 1 - inf I_A(x)], : \text{x} \in \text{X} \\ [\inf T_A(x), \sup T_A(x)] \end{matrix} \right\},$$

   for all x in X.

5. The intersection of two neutrosophic sets A and B is a neutrosophic set $A \cap B$, whose truth-membership, indeterminacy-membership and falsity-membership functions are related to those of A and B by

$$A_{IVNS} \cap B_{IVNS} = \left\{ \begin{matrix} x, [\inf T_A(x) \wedge \inf T_B(x), \sup T_A(x) \wedge \sup T_B(x), \\ \langle\ [\inf I_A(x) \vee \inf I_B(x), \sup I_A(x) \vee \sup I_B(x)],\ \rangle : x \in \text{X} \\ [\inf F_A(x) \vee \inf F_B(x), \sup F_A(x) \vee \sup F_B(x) \end{matrix} \right\},$$

   for all x in X.

6. The union of two interval neutrosophic sets A and B is an interval neutrosophic set $A_{\text{IVNS}} \cup B_{\text{IVNS}}$, whose truth-membership, indeterminacy-membership and false-membership are related to those of A and B by

$$A_{\text{IVNS}} \cup B_{\text{IVNS}} = \left\{ \left\langle \begin{array}{c} x, [\inf T_A(x) \vee \inf T_B(x), \sup T_A(x) \vee \sup T_B(x)] \\ [\inf I_A(x) \wedge \inf I_B(x), \sup I_A(x) \wedge \sup I_B(x)], \\ [\inf F_A(x) \wedge \inf F_B(x), \sup F_A(x) \wedge \sup F_B(x)] \end{array} \right\rangle : x \in X \right\}$$

for all $x \in X$.

7. The difference of two interval neutrosophic sets A and B is an interval neutrosophic set $A_{\text{IVNS}} \backslash B_{\text{IVNS}}$, whose truth-membership, indeterminacy-membership and falsity-membership functions are related to those of A and B by

   $A_{\text{IVNS}} \backslash B_{\text{IVNS}} =$

   $$\left\{ \left\langle \begin{array}{c} x, [\min\{infT_A(x), infF_B(x)\}, \min\{supT_A(x), supF_B(x)\}], \\ [\max(\inf I_A(x), 1 - supI_B(x)), \max(\sup I_A(x), 1 - \inf I_B(x))], \\ [max(infF_A(x), inf\ T_B(x)), max(sup\ F_A(x), sup\ T_B(x))] \end{array} \right\rangle x \in X \right\}$$

8. The addition of two interval neutrosophic sets A and B is an interval neutrosophic set $A_{\text{IVNS}} + B_{\text{IVNS}}$, whose truth-membership, indeterminacy-membership and falsity-membership functions are related to those of A and B by

   $A_{\text{IVNS}} + B_{\text{IVNS}} =$

   $$\left\{ \left\langle \begin{array}{c} \mathrm{x}, [\min(\inf \mathrm{T_A}(\mathrm{x}) + \inf \mathrm{T_B}(\mathrm{x}), 1), \min(\sup \mathrm{T_A}(\mathrm{x}) + \sup \mathrm{T_B}(\mathrm{x}), 1)], \\ [\min(\inf \mathrm{I_A}(\mathrm{x}) + \inf \mathrm{I_B}(\mathrm{x}), 1), \min(\sup \mathrm{I_A}(\mathrm{x}) + \sup \mathrm{I_B}(\mathrm{x}), 1)], \\ [\min(\inf \mathrm{F_A}(\mathrm{x}) + \inf \mathrm{F_B}(\mathrm{x}), 1), \min(\sup \mathrm{F_A}(\mathrm{x}) + \sup \mathrm{F_B}(\mathrm{x}), 1)] \end{array} \right\rangle \mathrm{x} \in \mathrm{X} \right\}$$

   for all $x \in X$.

9. The scalar multiplication of interval neutrosophic set A is $A_{\text{IVNS}}.a$, whose truth-membership, indeterminacy-membership and falsity-membership functions are related to those of A by

   $$A_{\text{IVNS}}.a = \left\{ \left\langle \begin{array}{c} x, [\min(\inf T_A(x).a, 1), \min(\sup T_A(x).a, 1)], \\ [\min(\inf I_A(x).a, 1), \min(\sup I_A(x).a, 1)], \\ [\min(\inf F_A(x).a, 1), \min(\sup F_A(x).a, 1)] \end{array} \right\rangle : x \in X \right\}$$

   for all $x \in X$, $a \in \mathrm{R}^+$.

10. The scalar division of interval neutrosophic set A is A_{IVNS}/a, whose truth-membership, indeterminacy-membership and falsity-membership functions are related to those of A by

    $$A_{\text{IVNS}}/a = \left\{ \left\langle \begin{array}{c} x, [\min(\inf T_A(x)/a, 1), \min(\sup T_A(x)/a, 1)], \\ [\min(\inf I_A(x)/a, 1), \min(\sup I_A(x)/a, 1)], \\ [\min(\inf F_A(x)/a, 1), \min(\sup F_A(x).a, 1)] \end{array} \right\rangle : x \in X \right\}$$

    for all $x \in X$, $a \in R^+$.

3 (α, β, γ) Interval Cut Set of Interval Valued Neutrosophic Sets

Definition 3.1 Let

$$A_{\text{IVNS}} = \left\{ \left\langle \begin{array}{l} x, [\inf T_A(x), \sup T_A(x)], \\ [\inf I_A(x), \sup I_A(x)], \\ [\inf F_A(x), \sup F_A(x)] \end{array} \right\rangle : x \in X \right\}$$

be any interval valued neutrosophic set on X. For any ordered $\alpha = [\alpha_1, \alpha_2]$, $\beta = [\beta_1, \beta_2]$ and $\gamma = [\gamma_1, \gamma_2] \in [0,1]$, such that $0 \le \alpha + \beta + \gamma \le 3$. We define the (α, β, γ) cut of interval valued neutrosophic set A, denoted by $A_{([\alpha_1,\alpha_2],[\beta_1,\beta_2],[\gamma_1,\gamma_2])}$:

$$A_{([\alpha_1,\alpha_2],[\beta_1,\beta_2],[\gamma_1,\gamma_2])} = \left\{ \begin{array}{l} x: inf\ T_A(x) \ge \alpha_1,\ sup\ T_A(x) \ge \alpha_2, \\ inf\ I_A(x) \le \beta_1, sup\ I_A(x) \le \beta_2,\ \ x \in X \\ inf\ F_A(x) \le \gamma_1, sup\ F_A(x) \le \gamma_2; \end{array} \right\}$$

That is,

$$A_{([\alpha_1,\alpha_2],[\beta_1,\beta_2],[\gamma_1,\gamma_2])} = \left\{ \left\langle \begin{array}{l} x: \inf T_A(x) \wedge \alpha_1 = \alpha_1, \sup T_A(x) \wedge \alpha_2 = \alpha_2, \\ \inf I_A(x) \vee \beta_1 = \beta_1, \sup I_A(x) \vee \beta_2 = \beta_2\ , \\ \inf F_A(x) \vee \gamma_1 = \gamma_1, \sup F_A(x) \vee \gamma_2 = \gamma_2 \end{array} \right\rangle x \in X \right\}.$$

We define

$A_{([\alpha_1,\alpha_2])} = \{x: \inf T_A(x) \ge \alpha_1, \sup T_A(x) \ge \alpha_2; x \in X\}$ the α-cut of the truth membership function generated by A.

$A_{([\beta_1,\beta_2])} = \{x:\ inf\ I_A(x) \le \beta_1,\ sup\ I_A(x) \le\ \beta_2\ ; x \in X\}$ the β-cut of the indeterminacy membership function generated by A and

$A_{([\gamma_1,\gamma_2])} = \{x:\ inf\ F_A(x) \le \gamma_1, sup\ F_A(x) \le \gamma_2; x \in X\}$ the γ-cut of the falsity membership function generated by A.

Example 3.2 Let $= \{x_1, x_2, x_3\}$. Let us consider an interval valued neutrosophic set:

$A = \{\ x_1$: ([0.5, 0.6]; [0.3, 0.4]; [0.2,0.3]);

x_2: ([0.5, 0.8]; [0.4, 0.6]; [0.4, 0.5]);

x_3: ([0.1, 0.2]; [0.4, 0.5]; [0.7, 0.8])}

Let $\alpha_1 = 0.3$; $\beta_1 = 0.4$; $\gamma_1\ = 0.5 \in [0,1]$ and $\alpha_2 = 0.5$; $\beta_2 = 0.6$; $\gamma_2 =\ 0.7\ \in$ [0,1].

Then,

$A_{([\alpha_1,\alpha_2],[\beta_1,\beta_2],[\gamma_1,\gamma_2])} = \{x_1, x_2\}$. Here, x_3 is not an element of this cut set. Moreover,

$$A_{([\alpha_1,\alpha_2])} = \{x_1, x_2\}$$

$$A_{([\beta_1,\beta_2])} = \{x_1, x_2, x_3\}$$

$$A_{([\gamma_1,\gamma_2])} = \{x_1, x_2\}$$

Definition 3.3 Let A_{IVNS} be any interval valued neutrosophic set on X. We define the (α, β, γ) strong cut of interval valued neutrosophic set A, denoted by $A_{([\alpha_1,\alpha_2],[\beta_1,\beta_2],[\gamma_1,\gamma_2])+}$, where $\alpha = [\alpha_1, \alpha_2]$, $\beta = [\beta_1, \beta_2]$ and $\gamma = [\gamma_1, \gamma_2] \in [0,1]$ such that $0 \leq \alpha + \beta + \gamma \leq 3$,

$$A_{([\alpha_1,\alpha_2],[\beta_1,\beta_2],[\gamma_1,\gamma_2])+} = \left\{ \left\langle \begin{array}{l} x\text{: } \inf T_A(x) > \alpha_1 = \alpha_1, \sup T_A(x) > \alpha_2 = \alpha_2, \\ \quad \inf I_A(x) < \beta_1 = \beta_1, \sup I_A(x) < \beta_2 = \beta_2, \\ \quad \inf F_A(x) < \gamma_1 = \gamma_1, \sup F_A(x) < \gamma_2 = \gamma_2 \end{array} \right\rangle ; x \in X \right\}$$

we define

1. $A_{([\alpha_1,\alpha_2])+} = \{\langle x\text{: } \inf T_A(x) > \alpha_1, \sup T_A(x) > \alpha_2 \rangle; x \in X\}$ the strong α-cut of the truth membership function generated by A,
2. $A_{([\beta_1,\beta_2])+} = \{\langle x\text{: } \inf I_A(x) < \beta_1, \sup I_A(x) < \beta_2 \rangle; x \in X\}$ the strong β-cut of the indeterminacy membership function generated by A and
3. $A_{([\gamma_1,\gamma_2])+} = \{\langle x\text{: } \inf F_A(x) < \gamma_1, \sup F_A(x) < \gamma_2 \rangle; x \in X\}$ the strong γ-cut of the falsity membership function generated by A.

Example 3.4 Let $X = \{x_1, x_2, x_3\}$. Consider an interval valued neutrosophic set:

$$\begin{aligned} A = \{ & x_1: ([0.5, 0.6];\ [0.3, 0.4];\ [0.2, 0.3]); \\ & x_2: ([0.5, 0.8];\ [0.4, 0.6];\ [0.4, 0.5]); \\ & x_3: ([0.1, 0.2];\ [0.4, 0.5];\ [0.7, 0.8])\} \end{aligned}$$

Let $\alpha_1 = 0.3; \beta_1 = 0.4; \gamma_1 = 0.5 \in [0,1]$,

$\alpha_2 = 0.5; \beta_2 = 0.6; \gamma_2 = 0.7 \in [0,1]$.

Then,

$$A_{([\alpha_1,\alpha_2],[\beta_1,\beta_2],[\gamma_1,\gamma_2])+} = \{x_1\}$$

$$A_{([\alpha_1,\alpha_2])+} = \{x_1, x_2\}$$

$A_{([\beta_1,\beta_2])+} = \{x_1\}$

$A_{([\gamma_1,\gamma_2])+} = \{x_1, x_2\}$

Moreover, we can define the cut sets where for any ordered $\alpha = [\alpha_1, \alpha_2]$, $\beta = [\beta_1, \beta_2]$ and $\gamma = [\gamma_1, \gamma_2] \in [0,1]$ such that $0 \le \alpha + \beta + \gamma \le 3$, as:

$A_{([\alpha_1,\alpha_2]+,[\beta_1,\beta_2],[\gamma_1,\gamma_2])} =$

$\{\langle x: inf\, T_A(x) > \alpha_1, sup\, T_A(x) > \alpha_2, \ inf\, I_A(x) \le \beta_1, \ sup\, I_A(x) \le \beta_2, inf\, F_A(x) \le \gamma_1, sup\, F_A(x) \le \gamma_2 \rangle; x \in X\}$ is $(\alpha+, \beta, \gamma)$,

$$A_{([\alpha_1,\alpha_2]+,[\beta_1,\beta_2],[\gamma_1,\gamma_2])} = \left\{ \langle \begin{matrix} x: inf\, T_A(x) > \alpha_1, sup\, T_A(x) > \alpha_2, \\ inf\, I_A(x) \le \beta_1, sup\, I_A(x) \le \beta_2, \\ inf\, F_A(x) \le \gamma_1, sup\, F_A(x) \le \gamma_2 \end{matrix} \rangle x \in X \right\}$$

is $(\alpha, \beta+, \gamma)$ and

$$A_{([\alpha_1,\alpha_2],[\beta_1,\beta_2],[\gamma_1,\gamma_2]+)} = \left\{ \langle \begin{matrix} x: inf\, T_A(x) \ge \alpha_1, sup\, T_A(x) \ge \alpha_2, \\ inf\, I_A(x) \le \beta_1, sup\, I_A(x) \le \beta_2, \\ inf\, F_A(x) < \gamma_1, sup\, F_A(x) < \gamma_2 \end{matrix} \rangle x \in X \right\}$$

is a $(\alpha, \beta, \gamma+)$-cut set of A, respectively. Similarly, we can define other cut sets of A.

Proposition 3.5 The cut sets of interval valued neutrosophic sets satisfy the following properties: for an interval valued neutrosophic set A on X, any ordered $\alpha = [\alpha_1, \alpha_2]$, $\beta = [\beta_1, \beta_2]$ and $= [\gamma_1, \gamma_2] \in [0,1]$, $\lambda = [\lambda_1, \lambda_2]$, $\mu = [\mu_1, \mu_2]$, $\delta = [\delta_1, \delta_2]$ such that $0 \le \alpha + \beta + \gamma \le 3$, and such that $0 \le \lambda + \mu + \delta \le 3$,

(1) $A_{([\alpha_1,\alpha_2],[\beta_1,\beta_2],[\gamma_1,\gamma_2])} = A_{([\alpha_1,\alpha_2])} \cap A_{([\beta_1,\beta_2])} \cap A_{([\gamma_1,\gamma_2])}$,

(2) If $[\alpha_1, \alpha_2] = [\gamma_1, \gamma_2]$, $A^c{}_{([\alpha_1,\alpha_2])} = A^c{}_{([\gamma_1,\gamma_2]+)}$,

(3) $A^c{}_{([\beta_1,\beta_2])} = 1 - A^c{}_{([\beta_1,\beta_2]+)}$ and $A^c{}_{([\gamma_1,\gamma_2]+)} = A^c{}_{([\alpha_1,\alpha_2])}$,

$$A_{([\alpha_1,\alpha_2],[\beta_1,\beta_2],[\gamma_1,\gamma_2])+} \subseteq A_{([\alpha_1,\alpha_2],[\beta_1,\beta_2],[\gamma_1,\gamma_2])},$$

(4) For $\alpha_1 \ge \lambda_1$ and $\alpha_2 \ge \lambda_2$, $\beta_1 \le \mu_1$ and $\beta_2 \le \mu_2$, $\gamma_1 \le \delta_1$ and $\gamma_2 \le \delta_2$, $A_{([\alpha_1,\alpha_2])} \subseteq A_{([\lambda_1,\lambda_2])}$, $A_{([\beta_1,\beta_2])} \subseteq A_{([\mu_1,\mu_2])}$, $A_{([\gamma_1,\gamma_2])} \subseteq A_{([\delta_1,\delta_2])}$ and $A_{([\alpha_1,\alpha_2],[\beta_1,\beta_2],[\gamma_1,\gamma_2])} \subseteq A_{([\lambda_1,\lambda_2],[\mu_1,\mu_2],[\delta_1,\delta_2])}$.

Proof. Let A_{IVNS} be any interval valued neutrosophic set on X for any ordered $\alpha = [\alpha_1, \alpha_2]$, $\beta = [\beta_1, \beta_2]$ and $\gamma = [\gamma_1, \gamma_2] \in [0,1]$, $= [\lambda_1, \lambda_2]$,

$\mu=[\mu_1,\mu_2]$, $\delta=[\delta_1,\delta_2]$ such that $0\le\alpha+\beta+\gamma\le 3$, and such that $0\le\lambda+\mu+\delta\le 3$,

(1) Follows directly from Definition 3.1.

(2) We know

$$A^c=\left\{\begin{matrix} x:[inf F_A(x),supF_A(x)],\\ \langle[1-supI_A(x),1-infI_A(x)],\rangle\, x\in X\\ [inf\,T_A(x),sup\,T_A(x)]\end{matrix}\right\}$$

so

$$A^c{}_{([\alpha_1,\alpha_2])}=\{\langle x:\ \inf F_A(x)\ge\gamma_1,\ \sup F_A(x)\ge\gamma_2\rangle;x\in X\}.$$

By definition

$$A_{([\gamma_1,\gamma_2])+}=\{\langle x:\inf F_A(x)<\gamma_1,\sup T_A(x)<\gamma_2\rangle;x\in X\},$$

$$A^c{}_{([\gamma_1,\gamma_2]+)}=\{\langle x:\ \inf F_A(x)\ge\gamma_1,\sup F_A(x)\ge\gamma_2\rangle;x\in X\}.$$

Therefore, $A^c{}_{([\alpha_1,\alpha_2])}=A^c{}_{([\gamma_1,\gamma_2]+)}$.
Similarly, $A^c{}_{([\beta_1,\beta_2])}=1-A^c{}_{([\beta_1,\beta_2]+)}$ and $A^c{}_{([\gamma_1,\gamma_2]+)}=A^c{}_{([\alpha_1,\alpha_2])}$ can be also proved.

(3) $A_{([\alpha_1,\alpha_2],[\beta_1,\beta_2],[\gamma_1,\gamma_2])+}$

$$=\left\{\begin{matrix} x:\ \inf T_A(x)>\alpha_1,\sup T_A(x)>\alpha_2,\\ \langle\ \ \inf I_A(x)<\beta_1,\sup I_A(x)<\beta_2,\ \ \rangle\, x\in X\\ \inf F_A(x)<\gamma_1,\sup F_A(x)<\gamma_2\end{matrix}\right\}$$

$$\subseteq\left\{\begin{matrix} x:inf\,T_A(x)\ge\alpha_1,sup\,T_A(x)\ge\alpha_2,\\ \langle\ \ inf\,I_A(x)\le\beta_1,sup\,I_A(x)\le\beta_2,\ \ \rangle\, x\in X\\ inf\,F_A(x)\le\gamma_1,sup\,F_A(x)\le\gamma_2\end{matrix}\right\}$$

$$=A_{([\alpha_1,\alpha_2],[\beta_1,\beta_2],[\gamma_1,\gamma_2])}$$

Hence, $A_{([\alpha_1,\alpha_2],[\beta_1,\beta_2],[\gamma_1,\gamma_2])+}\subseteq A_{([\alpha_1,\alpha_2],[\beta_1,\beta_2],[\gamma_1,\gamma_2])}$.

(4) By Definition 3.1, for any element x of $A_{([\alpha_1,\alpha_2])}$, $\alpha_1\ge\lambda_1$ and $\alpha_2\ge\lambda_2$ $\inf T_A(x)\ge\alpha_1\ge\lambda_1,\sup T_A(x)\ge\alpha_2\ge\lambda_2$ imply that $A_{([\alpha_1,\alpha_2])}\subseteq A_{([\lambda_1,\lambda_2])}$.

For any element x of $A_{([\beta_1,\beta_2])}$, $\beta_1\le\mu_1$ and $\beta_2\le\mu_2$, $\inf I_A(x)\le\beta_1\le\mu_1$, $\sup I_A(x)\le\beta_2\le\mu_2$ show that $A_{([\beta_1,\beta_2])}\subseteq A_{([\mu_1,\mu_2])}$.

Similarly, for $\gamma_1 \leq \delta_1$ and $\gamma_2 \leq \delta_2$, we obtain $A_{([\gamma_1,\gamma_2])} \subseteq A_{([\delta_1,\delta_2])}$.

Therefore, we can say that $A_{([\alpha_1,\alpha_2],[\beta_1,\beta_2],[\gamma_1,\gamma_2])} \subseteq A_{([\lambda_1,\lambda_2],[\mu_1,\mu_2],[\delta_1,\delta_2])}$.

Example 3.6: Let us consider set A defined in Example 3.2. Let $X = \{x_1, x_2, x_3\}$ be an interval valued neutrosophic set

$$A = \begin{Bmatrix} x_1: ([0.5,0.6],[0.3,0.4],[0.2,0.3]), \\ \langle x_2: ([0.5,0.8],[0.4,0.6],[0.4,0.5]),\rangle \\ x_3: ([0.1,0.2],[0.4,0.5],[0.7,0.8]) \end{Bmatrix}$$

Let $\alpha_1 = 0.3; \beta_1 = 0.4; \gamma_1 = 0.5 \in [0,1]$

$\alpha_2 = 0.5; \beta_2 = 0.6; \gamma_2 = 0.7 \in [0,1]$.

Then, $A_{([\alpha_1,\alpha_2],[\beta_1,\beta_2],[\gamma_1,\gamma_2])} = \{x_1, x_2\}$ and $A_{([\alpha_1,\alpha_2])} = \{x_1, x_2\}$, $A_{([\beta_1,\beta_2])} = \{x_1, x_2, x_3\}$, $A_{([\gamma_1,\gamma_2])} = \{x_1, x_2\}$.

We can realize that $A_{([\alpha_1,\alpha_2],[\beta_1,\beta_2],[\gamma_1,\gamma_2])} = \{x_1, x_2\}$ is actually the intersection of $A_{([\alpha_1,\alpha_2])}$, $A_{([\beta_1,\beta_2])}$ and $A_{([\gamma_1,\gamma_2])}$, i.e. $A_{([\alpha_1,\alpha_2],[\beta_1,\beta_2],[\gamma_1,\gamma_2])} = A_{([\alpha_1,\alpha_2])} \cap A_{([\beta_1,\beta_2])} \cap A_{([\gamma_1,\gamma_2])}$.

4 Algebraic Operations over Interval Valued Neutrosophic Sets

Let A and B be two interval valued neutrosophic sets on a universal set X. We define algebraic operations such as addition, subtraction, multiplication, division and scalar multiplication over the interval neutrosophic sets A and B as follows:

1. $A + B = \{z, \max[inf\, T_A(x) + \inf T_B(x) - \inf T_A(x). \inf T_B(x)$
   $\sup T_A(x) + \sup T_B(x) - \sup T_A(x). \sup T_B(x)]$
   $\min[(inf\, I_A(x). \inf I_B(x), (\sup I_A(x). \sup I_B(x)],$
   $\min[(inf\, F_A(x). \inf F_B(x), (\sup F_A(x). \sup F(x)]$
   $: z = x + y, (x, y) \in X \times Y\}.$
2. $A - B = \{z, \max[inf\, T_A(x) + inf\, T_B(x) - inf\, T_A(x). inf\, T_B(x),$
   $sup\, T_A(x) + sup\, T_B(x) - sup\, T_A(x). sup\, T_B(x)],$
   $\min[(inf\, I_A(x). inf\, I_B(x), (sup\, I_A(x). sup\, I_B(x)\ ],$
   $\min[(inf\, F_A(x). inf\, F_B(x), (sup\, F_A(x). sup\, F_B(x)\ ]$
   $: z = x - y, (x, y) \in X \times Y\}.$
3. $A.B = \{\langle z, \max[inf\, T_A(x). inf\, T_B(x), sup\, T_A(x). sup\, T_B(x)],$
   $min[inf\, I_A(x) + inf\, I_B(x) - inf\, I_A(x). inf\, I_B(x),$
   $sup\, I_A(x) + sup\, I_B(x) - sup\, I_A(x). sup\, I_B(x)],$
   $min[inf\, F_A(x) + inf\, F_B(x) - inf\, F_A(x). inf\, F_B(x),$

$sup\, F_A(x) + sup\, F_B(x) - \ sup\, F_A(x).\, sup\, F_B(x)]\rangle$
$: z = x.y, (x, y) \in X \times Y\}$.

4. $A/B \ = \{z, \max[inf\, T_A(x).\, inf\, T_B(x)\, , sup\, T_A(x).\, sup\, T_B(x)]\, ,$
$\min[inf\, I_A(x) + inf\, I_B(x) - inf\, I_A(x).\, inf\, I_B(x)\, ,$
$sup\, I_A(x)\, sup\, I_B(x) - \ sup\, I_A(x).\, sup\, I_B(x)],$
$\min[inf\, F_A(x) + inf\, F_B(x) - inf\, F_A(x).\, inf\, F_B(x),$
$sup\, F_A(x) + sup\, F_B(x) - \ sup\, F_A(x).\, sup\, F_B(x)]$
$: z = x/y, (x, y) \in X \times Y\},$

5. $\lambda B = \{\langle z, max\left[1 - (1 - infT_B(y))^{\lambda}, 1 - (1 - supT_B)^{\lambda}\right]$
$min\left[infI_B(y)^{\lambda}, supI_B(y)^{\lambda}\right],$
$min\left[infF_B(y)^{\lambda}, infF_B(y)^{\lambda}\right]\rangle\}$

where λ is any nonzero real number.

Example 4.1 Let

$A = \ \{3; ([0.2,0.3], [0.1,0.4], [0.2,0.5]), 2; ([0.3,0.5], [0.2,0.6], [0.3,0.4])\}$

and

$B = \{1; ([0.1,0.2], [0.2,0.3], [0.3,0.5]), 5; ([0.2,0.4], [0.1,0.3], [0.2,0.5])\}$
be two interval valued neutrosophic sets.

1. $A + B = \{4; ([0.28,0.44], [0.02,0.12], [0.06,0.25]),$
$8; ([0.36,0.58], [0.01,0.12], [0.04,0.25]),$
$3; ([0.37,0.6], [0.04,0.18], [0.09,0.2]),$
$7; ([0.44,0.7], [0.02,0.18], [0.06,0.2])\}.$
2. $A - B = \{2; ([0.28,0.44], [0.02,0.12], [0.06,0.25]),$
$-2; ([0.36,0.58], [0.01,0.12], [0.04,0.25]),$
$1; ([0.37,0.6], [0.04,0.18], [0.09,0.2]),$
$-3; ([0.44,0.7], [0.02,0.18], [0.06,0.2])\}.$
3. $A.B = \{3; ([0.02,0.06], [0.28,0.58], [0.44,0.7]),$
$15; ([0.06,0.2], [0.19,0.58], [0.36,0.75]),$
$2; ([0.03,0.1], [0.36,0.72], [0.51,0.58]),$
$10; ([0.06,0.2], [0.28,0.72], [0.51,0.7])\}.$
4. $A/B = \{3; ([0.02,0.06], [0.28,0.58], [0.44,0.7]),$
$3/5; ([0.06,0.2], [0.19,0.58], [0.36,0.75]),$
$2; ([0.03,0.1], [0.36,0.72], [0.51,0.58]),$
$2/5; ([0.06,0.2], [0.28,0.72], [0.51,0.7])\}.$
5. $B = \{3; ([0.001,0.027], [0.008,0.065], [0.097,0.025]),$
$15; ([0.008,0.016], [0.001,0.027], [0.008,0.025])\}.$

4.1 New Operations of Interval Valued Neutrosophic Sets

4.1.1 Disjunctive Sum

Let A and B be two interval valued neutrosophic sets on a universal set X. We define the disjunctive sum of two interval valued neutrosophic sets such that

$A \oplus B = (A \cap B^C) \cup (A^C \cap B)$ where
$A = \{\langle x, [inf\, T_A(x), sup\, T_A(x)], [inf\, I_A(x), sup\, I_A(x)],$
$[inf\, F_A(x), sup\, F_A(x)]\rangle : x \in X\},$

$B = \{\langle x, [inf\, T_B(x), sup\, T_B(x)], [inf\, I_B(x), sup\, I_B(x)],$
$[inf\, F_B(x), sup\, F_B(x)]\rangle : x \in X\},$

$A^c = \{\langle x: [inf F_A(x), sup F_A(x)], [1 - sup I_A(x), 1 - inf I_A(x)],$
$[inf\, T_A(x), sup\, T_A(x)]\rangle : x \in X\},$

$B^c = \{\langle x: [inf F_B(x), sup F_B(x)], [1 - sup I_B(x), 1 - inf I_B(x)],$
$[inf\, T_B(x), sup\, T_B(x)]\rangle : x \in X\},$

for all x in X. Then, we present the following on the disjunctive sum for the truth-membership function,

$T_{A\cap B^C} = \{ min[inf\, T_A(x), inf\, F_B(x)], min[sup\, T_A(x), sup\, F_B(x)]\},$
$T_{A^C\cap B} = \{ min[inf\, F_A(x), inf\, T_B(x)], min[sup\, F_A(x), sup\, T_B(x)]\},$
$T_{A\oplus B} = \{max(min[inf\, T_A(x), inf\, F_B(x)], min[inf\, F_A(x), inf\, T_B(x)],$
$max(min[sup\, T_A(x), sup\, F_B(x)], min[sup\, F_A(x), sup\, T_B(x)])\},$

Next, for the indeterminacy membership function,

$I_{A\cap B^c} = \{ max[inf\, I_A(x), 1 - sup\, I_B(x)], max[sup\, I_A(x), 1 - inf\, I_B(x)]\},$
$I_{A^c\cap B} = \{ max[1 - sup\, I_A(x), inf\, I_B(x)], max[1 - inf\, I_A(x), sup\, I_B(x)]\},$
$I_{A\oplus B} = \{\min(\max[inf\, I_A(x), 1 - sup\, I_B(x)], \max[1 - sup\, I_A(x), inf\, I_B(x)]),$
$min(max[sup\, I_A(x), 1 - inf\, I_B(x)], max[1 - inf\, I_A(x), sup\, I_B(x)]\},$

Finally, for the falsity-membership function,

$F_{A\cap B^c} = \{ max[inf\, F_A(x), inf\, T_B(x)], max[sup\, F_A(x), sup\, T_B(x)]\},$
$F_{A^c\cap B} = \{ max[inf\, T_A(x), inf\, F_B(x)], max[sup\, T_A(x), sup\, F_B(x)]\},$
$F_{A\oplus B} = \{min(max[inf\, F_A(x), inf\, T_B(x)], max[inf\, T_A(x), inf\, F_B(x)]),$
$min(max[sup\, F_A(x), sup\, T_B(x)], max[sup\, T_A(x), sup\, F_B(x)])\}.$

Therefore, we conclude that

$A \oplus B = \{x;\ [\max(\min[inf\, T_A(x), inf\, F_B(x)], \min[inf\, F_A(x), inf\, T_B(x)]),$
$max(min[sup\, T_A(x), sup\, F_B(x)], min[sup\, F_A(x), sup\, T_B(x)])],$
$[min(max[inf\, I_A(x), 1 - sup\, I_B(x)], max[1 - sup\, I_A(x), inf\, I_B(x)]),$
$\min(\max[sup\, I_A(x), 1 - inf\, I_B(x)], \max[1 - inf\, I_A(x), sup\, I_B(x)])],$

$[min(max[inf\, F_A(x), inf\, T_B(x)], max[inf\, T_A(x), inf\, F_B(x)]),$
$min(max[sup\, F_A(x), sup\, T_B(x)], max[sup\, T_A(x), sup\, F_B(x)])]\}.$

Example 4.2.2.: Let $A = \{x_1; ([0.2,0.3], [0.1,0.4], [0.2,0.5])\}$ and $B = \{x_2; ([0.1,0.2], [0.2,0.3], [0.3,0.5])\}$ be two interval valued neutrosophic sets. Then, the disjunctive sum is equal to

$$A \oplus B$$
$$= \{x; [max(min[0.2,0.3], min[0.2,0.1]), max(min[0.3,0.5], min[0.5,0.2])],$$
$$[min(max[0.1,0.7]\, max[0.6,0.2]), min(max[0.4,0.8], max[0.9,0.3])],$$
$$[min(max[0.2,0.1]\, max[0.2,0.3]), min(max[0.5,0.2], max[0.3,0.5])]\}.$$

Hence, we obtain from the above equality

$$A \oplus B = \{x; ([0.2,0.3], [0.6,0.8], [0.2,0.5])\}.$$

4.1.2 Difference of Interval Valued Neutrosophic Sets

Let A and B be two interval valued neutrosophic sets on a universal set X. We define the difference of two interval valued neutrosophic sets such that $A \ominus B = A \cap B^c$, where

$$A = \left\{ \begin{matrix} x, [inf\, T_A(x), sup\, T_A(x)], [inf\, I_A(x), sup\, I_A(x)], \\ [inf\, F_A(x), sup\, F_A(x)]: x \in X \end{matrix} \right\},$$

$$B = \left\{ \begin{matrix} x, [inf\, T_B(x), sup\, T_B(x)], [inf\, I_B(x), sup\, I_B(x)], \\ [inf\, F_B(x), sup\, F_B(x)]: x \in X \end{matrix} \right\},$$

$$B^c = \left\{ \left\langle \begin{matrix} x: [inf F_B(x), sup F_B(x)], [1 - sup I_B(x), 1 - inf I_B(x)], \\ [inf\, T_B(x), sup\, T_B(x)] \end{matrix} \right\rangle x \in X \right\}$$

for all x in X. Then, we present the following on the difference for the truth-membership function, the indeterminacy membership function and the falsity-membership function, respectively.

$$T_{A \cap B^c} = \{ min[inf\, T_A(x), inf\, F_B(x)], min[sup\, T_A(x), sup\, F_B(x)]\},$$

$$I_{A \cap B^c} = \{ max[inf\, I_A(x), 1 - sup\, I_B(x)], max[sup\, I_A(x), 1 - inf\, I_B(x)]\}$$

and

$$F_{A \cap B^c} = \{ max[inf\, F_A(x), inf\, T_B(x)], max[sup\, F_A(x), sup\, T_B(x)]\}.$$

Hence, we have that

$$A \ominus B = \{x; (min[inf\, T_A(x), inf\, F_B(x)], min[sup\, T_A(x), sup\, F_B(x)]),$$
$$(\max[inf\, I_A(x), 1 - sup\, I_B(x)], \max[sup\, I_A(x), 1 - inf\, I_B(x)]),$$
$$(max[inf\, F_A(x), inf\, T_B(x)], max[sup\, F_A(x), sup\, T_B(x)])\}.$$

Example 4.2.4: Using the values of example 4.2.2, we calculate the difference values and we obtain the following result:

$$A \ominus B = \left\{ \begin{matrix} x; (\min[0.2,0.3], \min[0.3,0.5]), \\ max[0.4,0.8]), (max[0.2,0.1], max[0.5,0.2]) \end{matrix} \right\}$$

Therefore,

$$A \ominus B = \{x; ([0.2,0.3], [0.7,0.8], [0.2,0.5])\}.$$

4.1.3 Cartesian Product of Interval Valued Neutrosophic Sets

Let A be an interval valued neutrosophic set on a universal set X. We define the power of the interval valued neutrosophic set A such that

$$A = \{x, [inf\ T_A(x), sup\ T_A(x)], [inf\ I_A(x), sup\ I_A(x)], [inf\ F_A(x), sup\ F_A(x)]: x \in X\},$$

$$A^2 = \{x, [inf\ T_A^2(x), sup\ T_A^2(x)], [inf\ I_A^2(x), sup\ I_A^2(x)], [inf\ F_A^2(x), sup\ F_A^2(x)]: x \in X\},$$

where $\inf T_A^2(x) = [\inf T_A(x)]^2$, $\sup T_A^2(x) = [\sup T_A(x)]^2$ and so are the other functions. Similarly, the m^{th} power of interval valued neutrosophic set A^m may be computed as

$$A^m = \{x, [inf\ T_A^m(x), sup\ T_A^m(x)], [inf\ I_A^m(x), sup\ I_A^m(x)], [\inf F_A^m(x), \sup F_A^m(x)]: x \in X\}.$$

Denoting $T_{A_1}(x), T_{A_2}(x), \dots, T_{A_n}(x)$, $I_{A_1}(x), I_{A_2}(x), \dots, I_{A_n}(x)$ and $F_{A_1}(x), F_{A_2}(x), \dots, F_{A_n}(x)$ as the truth-membership function, indeterminacy membership function and falsity membership function, respectively $A_1, A_2, \dots, A_n$ for all $x_1, x_2, \dots, x_n \in X$. Then, the probability for n-tuple $(x_1, x_2, \dots, x_n)$ to be involved in a Cartesian product neutrosophic set is

$$\begin{aligned} A_1 \times A_2 \times \dots \times A_n = \{x; [&min[inf\ T_{A_1}(x), inf\ T_{A_2}(x), \dots, inf\ T_{A_n}(x)], \\ &min[sup\ T_{A_1}(x), sup\ T_{A_2}(x), \dots, sup\ T_{A_n}(x)]\ ], \\ &[max[inf\ I_{A_1}(x), inf\ I_{A_2}(x), \dots, inf\ I_{A_n}(x)], \\ &max[sup\ I_{A_1}(x), sup\ I_{A_2}(x), \dots, sup\ I_{A_n}(x)]\ ], \\ &[\ max[inf\ F_{A_1}(x), inf\ F_{A_2}(x), \dots, inf\ F_{A_n}(x)], \\ &max[sup\ F_{A_1}(x), sup\ F_{A_2}(x), \dots, sup\ F_{A_n}(x)]\ ]\}. \end{aligned}$$

Example 4.2.6: Let $A = \{ x_1;\ ([0.2,0.3], [0.1,0.4], [0.2,0.5])\}$ be an interval valued neutrosophic set on X. Then, the triple Cartesian product for A is

$$A^3 = A \times A \times A = \{x;\ [min[0.2, 0.04, 0.008], min[0.3, 0.09, 0.027]], [max[0.1, 0.01, 0.001], max[0.4, 0.16, 0.064]], [max[0.2, 0.04, 0.008], max[0.5, 0.25, 0.125]]\}$$

Thus, we can easily see that

$$A^3 = \{x;\ [0.008, 0.027], [0.1, 0.4], [0.2, 0.5]\}.$$

What we conclude from the result is that the values of the indeterminacy membership function and falsity membership function in the Cartesian product are the same with A, whereas the value of the truth membership is decreased since the values are between 0 and 1 and the powers are less than themselves.

5 Conclusion

In this work, some new concepts have been put forward, such as the disjunctive sum, difference (α, β, γ)-cut interval valued neutrosophic sets and the (α, β, γ)-cut strong interval valued neutrosophic sets of interval valued neutrosophic sets. Some related properties have been established with examples and counter examples. It is hoped that our work will enhance the understanding of interval valued neutrosophic sets.

References

[1] Smarandache, F., *A Unifying Field in Logics. Neutrosophy: Neutrosophic Probability, Set and Logic*, American Research Press, Rehoboth, United States, 1998.

[2] Smarandache, F., *Neutrosophic Set - A Generalization of the Intuitionistic Fuzzy Sets*, International Journal of Pure and Applied Mathematics, **24**(3), pp. 287-297, 2005.

[3] Zadeh, L.A., *Fuzzy Sets*, Information and Control, **8**(3), pp. 338-353, 1965.

[4] Atanassov, K.T., *Intuitionistic Fuzzy Sets*, Fuzzy Sets Syst., **20**(1), pp. 87-96, 1986.

[5] Wang, H., Smarandache, F., Zhang, Y. & Sunderraman, R., *Single Valued Neutrosophic Sets*, Review of the Air Force Academy, **1**, pp. 10-14, 2010.

[6] Wang, H., Smarandache, F., Zhang, Y.Q. & Sunderraman, R., *Interval Neutrosophic Sets and Logic: Theory and Applications in Computing*, Hexis, Phoenix, Arizona, United States, 2005.

[7] Broumi, S., Talea, M., Bakali, A. & Smarandache, F., *Interval Valued Neutrosophic Graphs*, Publ. Soc. Math. Uncertain, **10**(5), pp. 5-33, 2016.

[8] Broumi, S., Bakali, A., Talea, M. & Smarandache, F. *Isolated Single Valued Neutrosophic Graphs,* Neutrosophic Sets Syst., **11**, pp. 74-78, 2016.

[9] Broumi, S., Bakali, A., Talea, M., Smarandache, F. & Ali, M., *Shortest Path Problem under Bipolar Neutrosphic Setting*, Applied Mechanics & Materials, **8**, pp. 59-66, 2017.

[10] Zhang, C., Li, D., Mu, Y. & Song, D., *An Interval-valued Hesitant Fuzzy Multigranulation Rough Set Over Two Universes Model for Steam Turbine Fault Diagnosis*, Applied Mathematical Modelling, **42**, pp. 693-704, 2017.

[11] Mukherjee, A., Datta, M. & Sarkar, S., *Restricted Interval Valued Neutrosophic Sets and Restricted Interval Valued Neutrosophic Topological Spaces*, Neutrosophic Sets and Systems, **12**, pp. 45-53, 2016.

[12] Zeng, W. & Shi, Y., *Note on Interval-valued Fuzzy Set*, in: Wang L. & Jin Y. (eds.), Fuzzy Systems and Knowledge Discovery, FSKD 2005, Lecture Notes in Computer Science, **3613**, Springer, Berlin, Heidelberg, 2005.

[13] Liu, P., Chu, Y., Li, Y. & Chen, Y., *Some Generalized Neutrosophic Number Hamacher Aggregation Operators and Their Application to Group Decision Making*, International Journal of Fuzzy Systems, **16**(2)**,** pp. 242-255, 2014.

[14] Liu, P. & Wang, Y., *Multiple Attribute Decision-making Method based on Single-valued Neutrosophic Normalized Weighted Bonferroni Mean*, Neural Computing and Applications, **25**(7-8), pp. 2001-2010, 2014.

[15] Liu, P. & Shi, L., *The Generalized Hybrid Weighted Average Operator based on Interval Neutrosophic Hesitant Set and Its Application to Multiple Attribute Decision Making*, *Neural Computing and Applications*, **26**(2), pp. 457-471, 2015.

[16] Liu, P. & Li, H., *Multiple Attribute Decision-making Method based on Some Normal Neutrosophic Bonferroni Mean Operators*, Neural Computing and Applications, **28**(1), pp. 179-194, 2014.

[17] Liu, P. & Shi, L., *Some Neutrosophic Uncertain Linguistic Number Heronian Mean Operators and their Application to Multi-attribute Group Decision Making*, Neural Computing and Applications, **28**(5), pp. 1079-1093, 2017.

[18] Liu, P. & Tang, G., *Multi-criteria Group Decision-making based on Interval Neutrosophic Uncertain Linguistic Variables and Choquet Integral*, Cognitive Computation, **8**(6), pp. 1036-1056, 2016.

[19] Liu, P., Zhang, L., Liu, X. & Wang, P., *Multi-valued Neutrosophic Number Bonferroni Mean Operators with their Applications in Multiple Attribute Group Decision Making*, International Journal of Information Technology & Decision Making, **15**(05), pp. 1181-1210, 2016.

[20] Liu, P., *The Aggregation Operators based on Archimedean t-Conorm and t-Norm for Single-valued Neutrosophic Numbers and their Application to Decision Making*, International Journal of Fuzzy Systems, **18**(5), pp. 849-863, 2016.

[21] Tahat, T., Bani Ahmad, F., Alhazaymeh, K. & Hassan, N., *Ordering on Vague Soft Set*, Global Journal of Pure and Applied Mathematics, **11**(5), pp. 3189-3193, 2015.

[22] Alhazaymeh, K. & Hassan, N., *Vague Soft Multiset Theory*, International Journal of Pure and Applied Mathematics, **93**(4), pp. 511-523, 2014.

[23] Alzu'bi, K.A., Alhazaymeh, K. & Hassan, N. *Vague Soft Set Relation Mapping*, Global Journal of Pure and Applied Mathematics, **11**(5), pp. 3195-3202, 2014.

[24] Smarandache, F., *Neutrosophic Modal Logic*, Preprint, 2017020017, 2017. DOI:10.20944/preprints201702.0017.v1

[25] Peng, J.J., Wang, J.Q. & Yang, W.E., *A Multi-valued Neutrosophic Qualitative Flexible Approach based on Likelihood for Multi-criteria Decision-making Problems*, International Journal of Systems Science, **48**(2), pp. 425-435, 2017.

[26] Sweety, C.A.C. & Arockiarani, I., *Rough Sets in Neutrosophic Approximation Space*, Annals of Fuzzy Mathematics and Informatics, **13**(4), pp. 449-463, 2017.

[27] Arockiarani, I. & Sumathi, I. R. *On (α,β)-Cut Fuzzy Neutrosophic Soft Sets*, International Journal of Mathematical Archive, **5**(1), pp. 263-272, 2014.

[28] Adak, A.K. & Bhowmik, M., *Interval Cut-set of Interval-valued Intuitionistic Fuzzy Sets*, African Journal of Mathematics and Computer Science Research, **4**(4), pp. 192-200, 2011.

[29] Şahin, R. & Liu, P., *Some Approaches to Multi Criteria Decision Making based on Exponential Operations of Simplified Neutrosophic Numbers*, Journal of Intelligent & Fuzzy Systems, **32**(3), pp. 2083-2099, 2017.

[30] Liu, C., *Interval Neutrosophic Fuzzy Stochastic Multi-criteria Decision-making Methods Based on MYCIN Certainty Factor and Prospect Theory*, Rev. Tec. Fac. Ing. Univ. Del Zulia, **39**, pp. 52-58, 2017.

[31] Liu, P. & Wang, Y., *Interval Neutrosophic Prioritized OWA Operator and Its Application to Multiple Attribute Decision Making*, Journal of Systems Science and Complexity, **29**(3), pp. 681-697, 2016.

[32] Liu, P. & Tang, G., *Some Power Generalized Aggregation Operators Based on the Interval Neutrosophic Sets and their Application to Decision Making*, Journal of Intelligent & Fuzzy Systems, **30**(5), pp. 2517-2528, 2016.

[33] Atanassov, K.T., *Intuitionistic Fuzzy Sets*, Physica-Verlag, Heidelberg, New York, 1999.

[34] Broumi, S., Deli, I. & Smarandache, F., *N-valued Interval Neutrosophic Sets and Their Application in Medical Diagnosis*, Critical Review, Center for Mathematics of Uncertainty, Creighton University, Omaha, NE, USA, **10**, pp. 45-69, 2015.