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Some new operations on single-valued neutrosophic matrices and their applications in multi-criteria group decision making

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Abstract

The single-valued neutrosophic set plays a crucial role to handle indeterminant and inconsistent information during decision making process. In recent research, a development in neutrosophic theory is emerged, called single-valued neutrosophic matrices, are used to address uncertainties. The beauty of single-valued neutrosophic matrices is that the utilizing of several fruitful operations in decision making. In this paper, some novel operations on neutrosophic matrices of are introduced, that is, type-1 product ($\tilde{\odot}$), type-2 product ($\tilde{\otimes}$) and minus ($\tilde{\ominus}$) between two single-valued neutrosophic matrices. Also, we introduced complement, transpose, upper and lower $\alpha-$level matrices of single-valued neutrosophic matrices and discussed related properties. Furthermore, we propose a multi-criteria group decision making method based on these new operations, and give an application of the proposed method in a real life problem. Finally, we compare proposed method in this paper with proposed methods previously.

Keywords Single-valued neutrosophic set · Single-valued neutrosophic matrix · Product operation · Complement matrix · $\alpha-$level of matrices · Decision making

1 Introduction

To describe situations mathematically which are indeterminant or inconsistent in nature, Smarandache introduced the theory of neutrosophic sets based on neutrosophy which is branch of philosopy. A neutrosophic set is characterized by three membership functions called truth- membership function (T), indeterminacy-membership function (I) and falsity-membership function (F). Range of all of the membership functions is real and non-real interval $]^{-}0, 1^{+}[$. In some areas such as engineering and real scientific fields, modeling of problems by using real standard or nonstandard subsets of $]^{-}0, 1^{+}[$ may not be easy sometimes. To cope with this issue concepts of single-valued neutrosophic set (SVN-set) and interval neutrosophic set (IN-set) were defined by Wang et al. in [23] and [24], respectively. Ye [41] defined concept of simplified neutrosophic sets (SNSs), and proposed a multi-criteria decision making method using aggregating operators of SNSs. Peng et al. [27] pointed out some problems in operations of SNSs defined by [41], and defined novel operations between simplified neutrosophic numbers(SNNs).

In many areas such as science, engineering, social sciences we encounter data, and need to stand for and computerize with matrix structures. From this aspect matrices have very important role. However, in the modeling of some problems involving uncertain data classical matrix theory may not be sufficient. Therefore, many researcher studied on matrix structures under fuzzy environment [16, 29, 32], intuitionistic fuzzy environment [13, 24, 25], soft environment [4], fuzzy soft environment [5] and intuitionistic fuzzy soft environment [9, 22]. As we know, fuzzy set and intuitionistic fuzzy sets are important tools for dealing with problems containing uncertainty and incomplete information. However, sometimes fuzzy sets and intuitionistic fuzzy sets may not suffice to model indeterminate and inconsistent information encountered in real world. Also mentioned failing are available fuzzy and intuitionistic fuzzy matrices. Therefore, concept of the neutrosophic matrix and square neutrosophic fuzzy matrices of which elements are belong to a neutrosophic field $K(I)$, was defined by

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Kandasamy and Smarandache [12], and they investigated some properties of them. They also for first time introduced notions of super neutrosophic matrices and quasi super neutrosophic matrices, and studied on their properties [11]. Dhar et al. [8] defined operations of addition and multiplication between two neutrosophic fuzzy matrices based on definitions given by Kandasamy and Smarandache, and obtained some properties of these operations. In 2014, Arockiarani and Sumathi [1] defined notion of fuzzy neutrosophic soft matrices. They also developed an decision making technique by defining a new score function which allows to evaluate proper of the alternatives. Deli and Broumi [6] introduced concept of neutrosophic soft matrices and operations between two neutrosophic soft matrices, and they also proposed a decision making method called NSM-decision making based on neutrosophic soft matrices. Determinants of neutrosophic matrices were studied in [14, 35]. Neutrosophic set is useful tool in multi-criteria decision making in terms of modeling of indeterminate and inconsistent information. Up to now, a lot of multi-criteria decision making (MCDM) and multi-criteria group decision making (MCGDM) method have been developed under neutrosophic environment. For example, Ye [39] introduced correlation and correlation coefficient of single-valued neutrosophic sets (SVNSs) and proposed a decision making method based on weighted correlation coefficient or the weighted cosine similarity measure of SVNSs. Ye [40] defined single valued neutrosophic cross entropy, and proposed a multi-criteria decision-making method based on the proposed single valued neutrosophic cross entropy. Ye [41] introduced concept of simplified neutrosophic set and proposed a simplified neutrosophic weighted arithmetic average operator and a simplified neutrosophic weighted geometric average operator, and then utilized two aggregation operators to develop a method for multi-criteria decision making problems under simplified neutrosophic environments. Peng et al. [26] developed some new operations of simplified neutrosophic set and proposed a novel outranking approach for multi-criteria decision-making (MCDM) problems. Peng et al. [27] pointed out certain problems regarding the existing operations of simplified neutrosophic numbers, their aggregation operators and the comparison method, and defined the new operations of simplified neutrosophic numbers. They also developed a multi-criteria decision making method and comparison method. Liu et al. [19] combined Hamacher operations and generalized aggregation operators to neutrosophic sets, and proposed some new operational rules for neutrosophic numbers (NNs) based on Hamacher operations. They also proposed some new operators such as generalized neutrosophic number Hamacher weighted averaging (GNNHWA) operator, generalized neutrosophic number Hamacher ordered weighted averaging (GNNHOWA) operator and generalized neutrosophic number Hamacher hybrid averaging (GNNHHA) operator, and they obtained some properties of them. Furthermore they proposed a multi attributive group decision making method based on these new operations. Other works on aggregating operators and multi-criteria (group) decision making under neutrosophic environment made by Liu et al. can be refer to [18, 20, 21]. Zhan and Wu [46] developed a new multi-criteria decision making with incomplete weight information under single-valued neutrosophic environment and applied a proposed method to a best global supplier selection problem. Tian et al. [34] proposed a MCDM method to solve problems including criteria that have different priority levels in form of simplified neutrosophic uncertain linguistic elements. In [33], in order to solve green product design selection problems using neutrosophic linguistic information they proposed a multi-criteria decision making method. Ji et al. [10] defined some new concepts such as Frank operations of SVNNs and Frank normalized prioritized Bonferroni mean (SVNFNPBM) operator for SVN-sets. Then developed a method for selecting TPL providers based on the proposed operator. Wu et al. [37] defined prioritized weighted average operator and prioritized weighted geometric operator for simplified neutrosophic numbers and proposed a new cross-entropy measures for simplified neutrosophic sets. Then they proposed a MCDM method based on these operators and cross-entropy measure. Biswas et al. [2] proposed a multi-attribute group decision-making by extending TOPSIS under single-valued neutrosophic environment. Peng et al. [28] suggested a multi-valued neutrosophic multi-criteria decision-making method based on extension of Elimination and Choice Translating Reality (ELECTRE) method, and supported this method by the comparison analysis with other existing methods. Some works related to multi-criteria decision making method under neutrosophic (single-valued neutrosophic, simplified neutrosophic etc.) in literature can be found in [17, 31, 38, 47].

Aim of this paper is to define some new operations between single-valued neutrosophic matrices, to investigate some of their properties and to propose a multi-criteria group decision making method based on these new matrix operations. The remaining part of this paper is organized as follows: In Section 2, some definitions and operations related to single-valued neutrosophic set, single-valued neutrosophic matrices are recalled. In Section 3, some new operations of single-valued neutrosophic matrices called type-1 product ($\tilde{\odot}$), type-2 product ($\tilde{\otimes}$), minus ($\tilde{\ominus}$), complement and transpose, based on single-valued neutrosophic number operations, and is obtained some results related to defined operations. In Section 4, a multi-criteria group decision making (MCGDM) method is proposed. In Section 5, an application of the proposed method is given to choose optimum alternative among firms considered to invest. In Section 6, it is given comparison proposed method with other methods developed previously. In Section 7, the concluding remarks are presented.

2 Preliminaries

In this section, we present the basic definitions of single-valued neutrosophic sets and single-valued neutrosophic matrices, which would be useful for subsequent discussions.

2.1 Single-valued neutrosophic sets

A neutrosophic set A on the universe of discourse X is defined as

$$A = \{\langle x, T_A(x), I_A(x), F_A(x)\rangle : x \in X\}$$

where $T_A, I_A, F_A : X \to]^{-}0, 1^{+}[$ and $^{-}0 \leq T_A(x) + I_A(x) + F_A(x) \leq 3^{+}$ [30]. From a philosophical point of view, the neutrosophic set takes the value from real standard or non-standard subsets of $]^{-}0, 1^{+}[$. But in real life application in scientific and engineering problems it is difficult to use neutrosophic set with value from real standard or non-standard subset of $]^{-}0, 1^{+}[$. Hence we consider the neutrosophic set which takes the value from the subset of [0, 1]. In some real life applications, modeling of problems by using real standard or nonstandard subsets of $]^{-}0, 1^{+}[$ may not be easy sometimes. Therefore concept of single valued neutrosophic set (SVN-set) was defined by Wang et al. [36].

Let $X \neq \emptyset$, with a generic element in X denoted by x. A single-valued neutrosophic set (SVN-set) A is characterized by three functions called truth membership function $T_A(x)$, indeterminacy membership function $I_A(x)$ and falsity membership function $F_A(x)$ such that $T_A(x)$, $I_A(x)$, $F_A(x) \in [0, 1]$ for all $x \in X$.

If X is continuous, a SVN-set A can be written as follows:

$$A = \int_X \langle T_A(x), I_A(x), F_A(x)\rangle / x, \quad \text{for all } x \in X.$$

If X is crisp set, a SVN-set A can be written as follows:

$$A = \sum_x \langle T_A(x), I_A(x), F_A(x)\rangle / x, \quad \text{for } all\ x \in X.$$

Here $0 \leq T_A(x) + I_A(x) + F_A(x) \leq 3$ for all $x \in X$. Also, we can denote SVN-set A over X by separating three parts as follows:

$$T_A = \{(x, T_A(x)) : x \in X\}, \quad I_A = \{(x, I_A(x)) : x \in X\},$$
$$F_A = \{(x, F_A(x)) : x \in X\}.$$

Operations and relations between two SVN-sets are defined in [36]. Let $A = \{\langle x, T_A(x), I_A(x), F_A(x)\rangle : x \in X\}$ and $B = \{\langle x, T_B(x), I_B(x), F_B(x)\rangle : x \in X\}$ be two SVN-sets. Then,

1. $A \subseteq B$ if and only if $T_A(x) \leq T_B(x)$, $I_A(x) \geq I_B(x)$, $F_A(x) \geq F_B(x)$ for all $x \in X$,
2. $A = B$ if and only if $A \subseteq B$ and $B \subseteq A$ for all $x \in X$,
3. $A^c = \{\langle x, F_A(x), 1 - I_A(x), T_A(x)\rangle : x \in X\}$,
4. $A \cup B = \{\langle x, (T_A(x) \vee T_B(x)), (I_A(x) \wedge I_B(x)), (F_A(x) \wedge F_B(x))\rangle : x \in X\}$,
5. $A \cap B = \{\langle x, (T_A(x) \wedge T_B(x)), (I_A(x) \vee I_B(x)), (F_A(x) \vee F_B(x))\rangle : x \in X\}$.

Operations between two SVN-numbers are defined by Liu and Wang [20]. Let $\mu(x) = \langle \mu_t(x), \mu_i(x), \mu_f(x)\rangle$ and $\mu(y) = \langle \mu_t(y), \mu_i(y), \mu_f(y)\rangle$ be two SVN-numbers. Then,

1. $\mu(x) \oplus \mu(y) = \langle \mu_t(x) + \mu_t(y) - \mu_t(x)\mu_t(y), \mu_i(x)\mu_i(y), \mu_f(x)\mu_f(y)\rangle$,
2. $\mu(x) \otimes \mu(y) = \langle \mu_t(x)\mu_t(y), \mu_i(x) + \mu_i(y) - \mu_i(x) + \mu_i(y), \mu_f(x) + \mu_f(y) - \mu_f(x) + \mu_f(y)\rangle$,
3. $\lambda\mu(x) = \langle 1 - (1 - \mu_t(x))^{\lambda}, \mu_i(x)^{\lambda}, \mu_f(x)^{\lambda}\rangle, \lambda > 0$,
4. $\mu(x)^{\lambda} = \langle (\mu_t(x))^{\lambda}, 1-(1-\mu_i(x))^{\lambda}, 1-(1-\mu_f(x))^{\lambda}\rangle$, $\lambda > 0$.

Let $\mu_j = \langle \mu^t_j, \mu^i_j, \mu^f_j\rangle$ $(j = 1, 2, ..., n)$ be a collection of SNNs, $SNNWA : SNN^n \to SNN$

$$SNNWA_w(\mu_1, \mu_2, ..., \mu_n) = \sum_{j=1}^{n} w_j\mu_j, \tag{1}$$

the SNNWA operator is called the simplified neutrosophic number weighted averaging operator of dimension n, where $w = (w_1, w_2, ..., w_n)$ is the weight vector of $\mu_j (j = 1, 2, ..., n)$, with $w_j \geq 0$ $(j = 1, 2, ..., n)$ and $\sum_{j=1}^{n} w_j = 1$ (see [27]). Here if it is taken all of w_j $(j = 1, 2, ..., n)$ as equal, then (1) can be written as follows:

$$SNNWA(\mu_1, \mu_2, ..., \mu_n) = \bigoplus_{j=1}^{n} \mu_j. \tag{2}$$

Similarly, neutrosophic number weighted geometric operator of dimension n can be defined as follows:

$$SNNWG(\mu_1, \mu_2, ..., \mu_n) = \bigotimes_{j=1}^{n} \mu_j. \tag{3}$$

[27]

Equations (2) and (3) can be written as more explicit, respectively, as follows:

$$\bigoplus_{j=1}^{n} \mu_j = \left\langle 1 - \prod_{j=1}^{n}(1 - \mu^t_j), \prod_{j=1}^{n} \mu^i_j, \prod_{j=1}^{n} \mu^f_j \right\rangle$$

and

$$\bigotimes_{j=1}^{n} \mu_j = \left\langle \prod_{j=1}^{n} \mu_j^t, 1 - \prod_{j=1}^{n}(1-\mu_j^i), 1 - \prod_{j=1}^{n}(1-\mu_j^f) \right\rangle.$$

Let $I^3 = [0,1] \times [0,1] \times [0,1]$ and $N(I^3) = \{(\alpha_1, \alpha_2, \alpha_3) : \alpha_1, \alpha_2, \alpha_3 \in [0,1]\}$. Then, $(N(I^3)$ is a lattice together with partial ordered relation $\preceq$, where order relation $\preceq$ on $N(I^3)$ can be defined by

$$\begin{aligned}(0,1,1) &\preceq (\alpha_1, \alpha_2, \alpha_3) \preceq (\beta_1, \beta_2, \beta_3) \\ &\preceq (1,0,0) \Leftrightarrow \alpha_1 \leq \beta_1, \alpha_2 \geq \beta_2, \alpha_3 \geq \beta_3\end{aligned}$$

for $(\alpha_1, \alpha_2, \alpha_3), (\beta_1, \beta_2, \beta_3) \in N(I^3)$ (see [15]).

Definition 1 Let $M_m(I^3) = \{\hat{A} = [(\mu^t_{pq}, \mu^i_{pq}, \mu^f_{pq})]_{m\times m} : (\mu^t_{pq}, \mu^i_{pq}, \mu^f_{pq}) \in I^3)\}$, where $I^3 = [0,1] \times [0,1] \times [0,1]$ is Cartesian product of unit intervals of the real line. Any matrix $\hat{A}$ in $M_m(I^3)$ is called a single-valued neutrosophic matrix (SVN-matrix).

3 Some new operations on SVN-matrices

In this section, we present some new operations on SVN-matrices based on the operations of SVNs. The type-1 product and type-2 product are introduced as in the following:

Definition 2 Let $\mu = [\langle \mu^t_{pq}, \mu^i_{pq}, \mu^f_{pq}\rangle]_{n\times r}$ and $\nu = [\langle \nu^t_{pq}, \nu^i_{pq}, \nu^f_{pq}\rangle]_{n\times r}$ be two SVN-matrices. Then, type-1 product operation between SVN-matrices μ and ν, denoted by $\mu\tilde{\odot}\nu$, is defined as follows:

$$\mu\tilde{\odot}\nu = \left[\langle \mu^t_{pq}\nu^t_{pq}, \mu^i_{pq}+\nu^i_{pq}-\mu^i_{pq}\nu^i_{pq}, \mu^f_{pq}+\nu^f_{pq}-\mu^f_{pq}\nu^f_{pq}\rangle\right]_{n\times r}.$$

Definition 3 Let $\mu = [\langle \mu^t_{pq}, \mu^i_{pq}, \mu^f_{pq}\rangle]_{n\times n}$, and $\nu = [\langle \nu^t_{pq}, \nu^i_{pq}, \nu^f_{pq}\rangle]_{n\times r}$. Then, type-2 product operation between SVN-matrices μ and ν, $\mu\tilde{\otimes}\nu$, is defined as follows:

$$\mu\tilde{\otimes}\nu = [\langle c^t_{pq}, c^i_{pq}, c^f_{pq}\rangle]_{n\times r},$$

where $c^t_{pq} = 1 - \prod_{s=1}^{n}(1-(\mu^t_{ps}\nu^t_{sq}))$, $c^i_{pq} = \prod_{s=1}^{n}(\mu^i_{ps} + \nu^i_{sq} - \mu^i_{ps}\nu^i_{sq})$ and $c^f_{pq} = \prod_{s=1}^{n}(\mu^f_{ps} + \nu^f_{sq} - \mu^f_{ps}\nu^f_{sq})$ for all $1 \leq p \leq n$ and $1 \leq q \leq r$.

Now the addition operation on SVN-matrices is presented as follow:

Definition 4 Let $\mu = [\langle \mu^t_{pq}, \mu^i_{pq}, \mu^f_{pq}\rangle]_{n\times r}$, and $\nu = [\langle \nu^t_{pq}, \nu^i_{pq}, \nu^f_{pq}\rangle]_{n\times r}$ be two SVN-matrices. Then, addition operation between SVN-matrices μ and ν, $\mu\tilde{\oplus}\nu$, is defined as follows:

$$\mu\tilde{\oplus}\nu = [\langle \mu^t_{pq} + \nu^t_{pq} - \mu^t_{pq}\nu^t_{pq}, \mu^i_{pq}\nu^i_{pq}, \mu^f_{pq}\nu^f_{pq}\rangle]_{n\times r}.$$

Transpose of SVN-matrix ν is defined as follows:

$$\nu' = [\langle \nu^t_{qp}, \nu^i_{qp}, \nu^f_{qp}\rangle]_{n\times m}.$$

Definition 5 Let $\mu = [\langle \mu^t_{pq}, \mu^i_{pq}, \mu^f_{pq}\rangle]_{n\times n}$ be a square SVN-matrix. If

$$\mu = \langle \mu^t_{pq}, \mu^i_{pq}, \mu^f_{pq}\rangle = \begin{cases}(1,0,0), & p = q \\ (0,1,1), & p \neq q\end{cases},$$

then μ is called SVN-unit matrix, and especially denoted by $\mathcal{I}_n = [\langle I^t_{pq}, I^i_{pq}, I^f_{pq}\rangle]_{n\times n}$.

Here

$$\mu^0 = \mathcal{I}_n = [\langle I^t_{pq}, I^i_{pq}, I^f_{pq}\rangle]_{n\times n}.$$

If $(\mu^t_{pq}, \mu^i_{pq}, \mu^f_{pq}) \preceq (\nu^t_{pq}, \nu^i_{pq}, \nu^f_{pq})$ for all $1 \leq p \leq n$,and $1 \leq q \leq r$, then SVN-matrix μ is smaller than SVN-matrix ν, and denoted by $\mu\tilde{\preceq}\nu$ (see [14, 35]).

Now we define some new operations related to SVN-matrices.

Definition 6 Let $\mu = [\langle \mu^t_{pq}, \mu^i_{pq}, \mu^f_{pq}\rangle]_{m\times m}$ and $\nu = [\langle \nu^t_{pq}, \nu^i_{pq}, \nu^f_{pq}\rangle]_{m\times m}$ be two SVN-matrices. Then, minus operation between SVN-matrices μ and ν, denoted by $\mu\tilde{\ominus}\nu$, is defined as follows:

$$\mu\tilde{\ominus}\nu = [\langle \mu^t_{pq} \ominus \nu^t_{pq}, \mu^i_{pq} \ominus \nu^i_{pq}, \mu^f_{pq} \ominus \nu^f_{pq}\rangle]_{m\times m},$$

where $\mu^t_{pq} \ominus \nu^t_{pq} = \begin{cases}\mu^t_{pq}, & \mu^t_{pq} > \nu^t_{pq} \\ 0, & \mu^t_{pq} \leq \nu^t_{pq}\end{cases}$, $\mu^i_{pq} \ominus \nu^i_{pq} = \begin{cases}1, & \mu^i_{pq} \geq \nu^i_{pq} \\ \mu^i_{pq}, & \mu^i_{pq} < \nu^i_{pq}\end{cases}$ and $\mu^f_{pq} \ominus \nu^f_{pq} = \begin{cases}1, & \mu^f_{pq} \geq \nu^f_{pq} \\ \mu^f_{pq}, & \mu^f_{pq} < \nu^f_{pq}\end{cases}$

Definition 7 Let $\mu = [\langle \mu^t_{pq}, \mu^i_{pq}, \mu^f_{pq}\rangle]_{m\times n}$ and $\alpha = (\alpha^t, \alpha^i, \alpha^f)$. Then upper $\alpha-$level matrix of SVN-matrix μ, denoted by $\mu^{(\alpha)}$, is defined as follows:

$$\mu^{(\alpha)} = [\langle (\mu^t_{pq})^{\alpha^t}, (\mu^i_{pq})^{\alpha^i}, (\mu^f_{pq})^{\alpha^f}\rangle]_{m\times n},$$

where $(\mu^t_{pq})^{\alpha^t} = \begin{cases}1, & \mu^t_{pq} \geq \alpha^t, \\ 0, & \mu^t_{pq} < \alpha^t,\end{cases}$, $(\mu^i_{pq})^{\alpha^i} = \begin{cases}1, & \mu^i_{pq} > \alpha^i, \\ 0, & \mu^i_{pq} \leq \alpha^i,\end{cases}$, $(\mu^f_{pq})^{\alpha^f} = \begin{cases}1, & \mu^f_{pq} > \alpha^f, \\ 0, & \mu^f_{pq} \leq \alpha^f.\end{cases}$

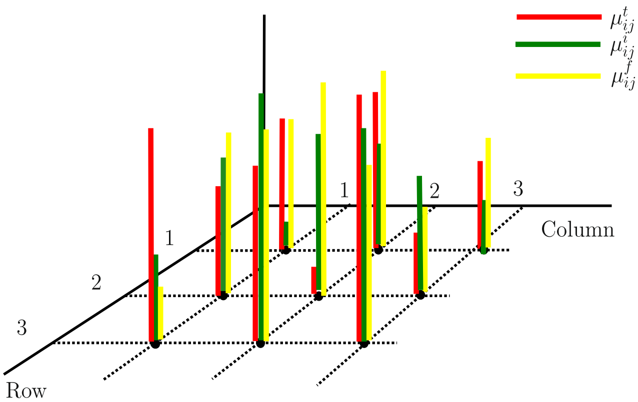

Fig. 1 Single-valued neutrosophic matrix μ

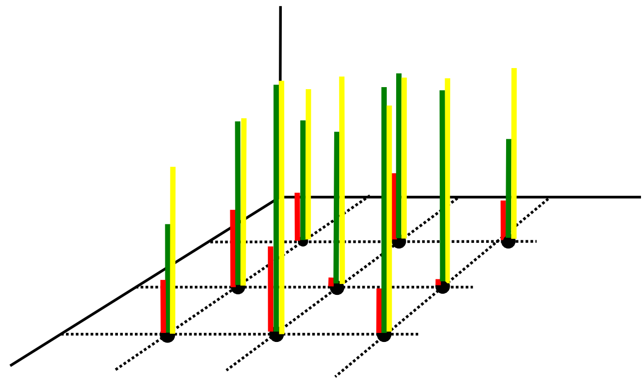

Fig. 3 Single-valued neutrosophic matrix $\mu\tilde{\odot}\nu$

Definition 8 Let $\mu = [\langle \mu^t_{pq}, \mu^i_{pq}, \mu^f_{pq}\rangle]_{m\times n}$ and $\alpha = (\alpha^t, \alpha^i, \alpha^f)$. Then lower $\alpha-$level matrix of SVN-matrix μ, denoted by $\mu_{(\alpha)}$, is defined as follows:

$$\mu_{(\alpha)} = [\langle (\mu^t_{pq})_{\alpha^t}, (\mu^i_{pq})_{\alpha^i}, (\mu^f_{pq})_{\alpha^f}\rangle]_{m\times n},$$

where

$$(\mu^t_{pq})_{\alpha^t} = \begin{cases} \mu^t_{pq}, & \mu^t_{pq} \geq \alpha^t, \\ 0, & \mu^t_{pq} < \alpha^t, \end{cases}, (\mu^i_{pq})_{\alpha^i} = \begin{cases} 1, & \mu^i_{pq} > \alpha^i, \\ \mu^i_{pq}, & \mu^i_{pq} \leq \alpha^i, \end{cases},$$

$$(\mu^f_{pq})_{\alpha^f} = \begin{cases} 1, & \mu^f_{pq} > \alpha^f, \\ \mu^f_{pq}, & \mu^f_{pq} \leq \alpha^f. \end{cases}$$

Definition 9 Let $\mu = [\langle \mu^t_{pq}, \mu^i_{pq}, \mu^f_{pq}\rangle]_{m\times n}$ be a SVN-matrix. Then, complement of SVN-matrices μ denoted by μ^c, is defined as follows:

$$\mu^c = [\langle \mu^f_{pq}, 1 - \mu^i_{pq}, \mu^t_{pq}\rangle]_{m\times n}.$$

In order to demonstrate these operation an example is presented and results are shown in images (Figs. 1, 2, 3, 4, 5, 6, 7, 8 and 9).

Example 1 Let us consider SVN-matrices μ and ν given as follows:

$$\mu = \begin{pmatrix} \langle 0.5, 0.1, 0.5\rangle & \langle 0.6, 0.4, 0.7\rangle & \langle 0.3, 0.2, 0.4\rangle \\ \langle 0.4, 0.5, 0.6\rangle & \langle 0.1, 0.6, 0.8\rangle & \langle 0.2, 0.4, 0.3\rangle \\ \langle 0.8, 0.3, 0.2\rangle & \langle 0.7, 0.9, 0.8\rangle & \langle 0.9, 0.8, 0.7\rangle \end{pmatrix},$$

$$\nu = \begin{pmatrix} \langle 0.4, 0.6, 0.5\rangle & \langle 0.5, 0.7, 0.4\rangle & \langle 0.6, 0.4, 0.8\rangle \\ \langle 0.8, 0.5, 0.4\rangle & \langle 0.9, 0.3, 0.5\rangle & \langle 0.1, 0.8, 0.9\rangle \\ \langle 0.3, 0.4, 0.7\rangle & \langle 0.5, 0.1, 0.6\rangle & \langle 0.2, 0.5, 0.5\rangle \end{pmatrix}.$$

Type-1 product on μ and ν:

$$\mu\tilde{\odot}\nu = \begin{pmatrix} \langle 0.2, 0.64, 0.75\rangle & \langle 0.3, 0.82, 0.82\rangle & \langle 0.18, 0.52, 0.88\rangle \\ \langle 0.32, 0.75, 0.76\rangle & \langle 0.09, 0.72, 0.9\rangle & \langle 0.02, 0.88, 0.93\rangle \\ \langle 0.24, 0.58, 0.76\rangle & \langle 0.35, 0.91, 0.92\rangle & \langle 0.18, 0.9, 0.85\rangle \end{pmatrix}$$

Type-2 product on μ and ν:

$$\mu\tilde{\otimes}\nu = \begin{pmatrix} \langle 0.62, 0.23, 0.50\rangle & \langle 0.71, 0.12, 0.45\rangle & \langle 0.38, 0.24, 0.54\rangle \\ \langle 0.27, 0.41, 0.56\rangle & \langle 0.34, 0.28, 0.49\rangle & \langle 0.28, 0.45, 0.54\rangle \\ \langle 0.78, 0.60, 0.48\rangle & \langle 0.88, 0.60, 0.41\rangle & \langle 0.60, 0.51, 0.57\rangle \end{pmatrix}$$

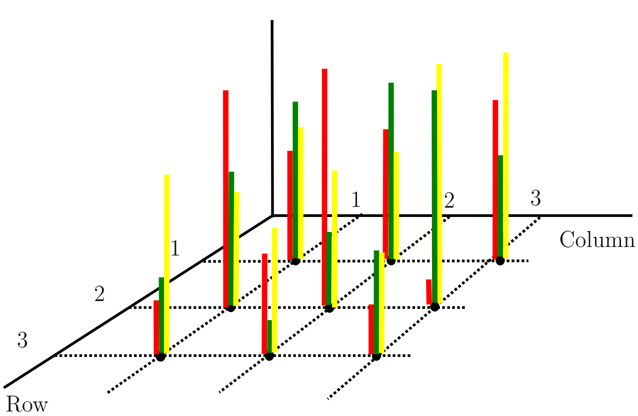

Fig. 2 Single-valued neutrosophic matrix ν

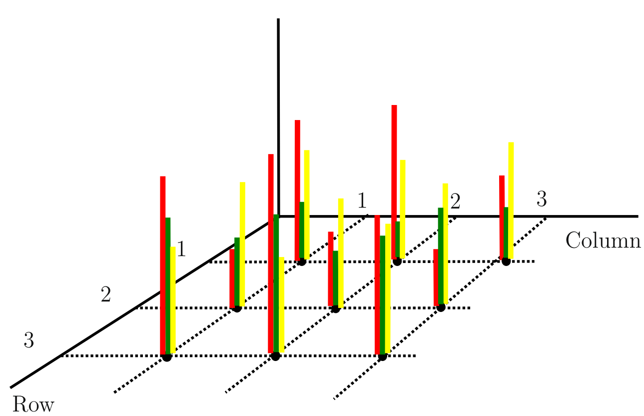

Fig. 4 Single-valued neutrosophic matrix $\mu\tilde{\otimes}\nu$

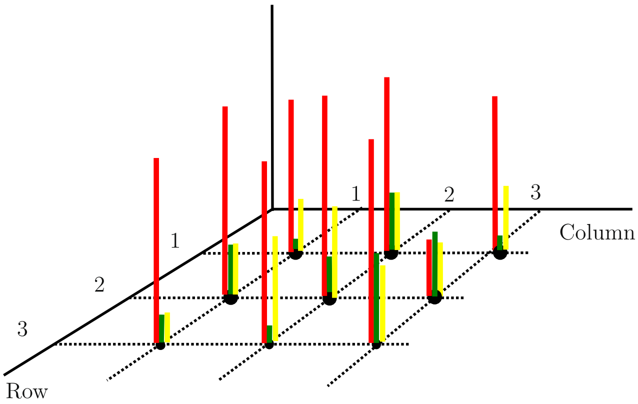

Fig. 5 Single-valued neutrosophic matrix $\mu\tilde{\oplus}\nu$

The addition operation on μ and ν:

$$\mu\tilde{\oplus}\nu = \begin{pmatrix} \langle 0.7, 0.06, 0.25\rangle & \langle 0.8, 0.28, 0.28\rangle & \langle 0.72, 0.08, 0.32\rangle \\ \langle 0.88, 0.25, 0.24\rangle & \langle 0.91, 0.18, 0.40\rangle & \langle 0.28, 0.32, 0.27\rangle \\ \langle 0.86, 0.12, 0.14\rangle & \langle 0.85, 0.09, 0.48\rangle & \langle 0.92, 0.40, 0.35\rangle \end{pmatrix}$$

The minus operation on μ and ν:

$$\mu\tilde{\ominus}\nu = \begin{pmatrix} \langle 0.5, 0.1, 1.0\rangle & \langle 0.6, 0.4, 1.0\rangle & \langle 0.0, 0.2, 0.4\rangle \\ \langle 0.0, 1.0, 1.0\rangle & \langle 0.0, 1.0, 1.0\rangle & \langle 0.2, 0.4, 0.3\rangle \\ \langle 0.8, 0.3, 0.2\rangle & \langle 0.7, 1.0, 1.0\rangle & \langle 0.9, 1.0, 1.0\rangle \end{pmatrix}$$

Now, suppose $\alpha = (0.4, 0.5, 0.3)$. Then **upper and lower** $\alpha - level$ of μ are calculated as in the following:

$$\mu^{(\alpha)} = \begin{pmatrix} \langle 1, 0, 1\rangle & \langle 1, 0, 1\rangle & \langle 0, 0, 1\rangle \\ \langle 1, 0, 1\rangle & \langle 0, 1, 1\rangle & \langle 0, 0, 0\rangle \\ \langle 1, 0, 0\rangle & \langle 1, 1, 1\rangle & \langle 1, 1, 1\rangle \end{pmatrix},$$

$$\mu_{(\alpha)} = \begin{pmatrix} \langle 0.5, 0.1, 1\rangle & \langle 0.6, 0.4, 1\rangle & \langle 0, 0.2, 1\rangle \\ \langle 0.4, 1, 1\rangle & \langle 0, 1, 1\rangle & \langle 0, 0.4, 0.3\rangle \\ \langle 0.8, 0.3, 0.2\rangle & \langle 0.7, 1, 1\rangle & \langle 0.9, 1, 1\rangle \end{pmatrix}.$$

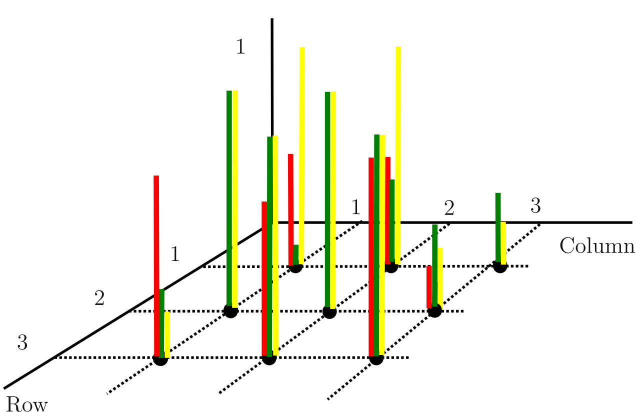

Fig. 6 Single-valued neutrosophic matrix $\mu\tilde{\ominus}\nu$

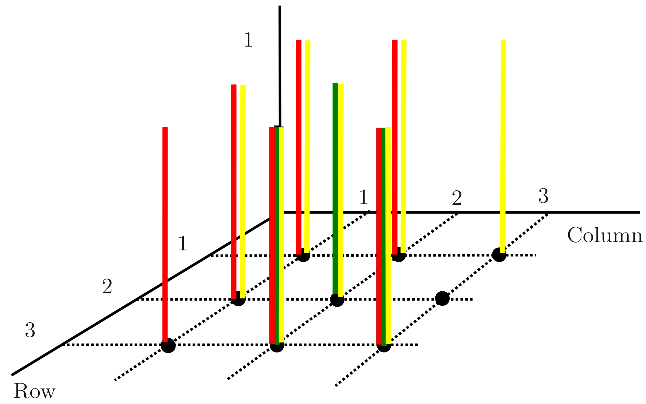

Fig. 7 Single-valued neutrosophic matrix $\mu^{(\alpha)}$

Next, the **complement** of μ is calculated as follows:

$$\mu^{c} = \begin{pmatrix} \langle 0.5, 0.9, 0.5\rangle & \langle 0.7, 0.6, 0.6\rangle & \langle 0.4, 0.8, 0.3\rangle \\ \langle 0.6, 0.5, 0.4\rangle & \langle 0.8, 0.4, 0.1\rangle & \langle 0.3, 0.6, 0.2\rangle \\ \langle 0.2, 0.7, 0.8\rangle & \langle 0.8, 0.1, 0.7\rangle & \langle 0.7, 0.2, 0.9\rangle \end{pmatrix},$$

Proposition 1 *Let μ, ν and γ be three $m \times n$ SVN-matrices. Then,*

1. $\mu\tilde{\oplus}\nu = \nu\tilde{\oplus}\mu$
2. $(\mu\tilde{\oplus}\nu)\tilde{\oplus}\gamma = \mu\tilde{\oplus}(\nu\tilde{\oplus}\gamma)$
3. $\mu\tilde{\odot}\nu = \nu\tilde{\odot}\mu$
4. $(\mu\tilde{\odot}\nu)\tilde{\odot}\gamma = \mu\tilde{\odot}(\nu\tilde{\odot}\gamma)$

Proof The proofs are obvious from definition of operations. □

Proposition 2 *Let μ and ν be two $m \times n$ and $n \times p$ SVN-matrices. Then,*

$(\mu\tilde{\otimes}\nu)\tilde{\otimes}\gamma = \mu\tilde{\otimes}(\nu\tilde{\otimes}\gamma)$

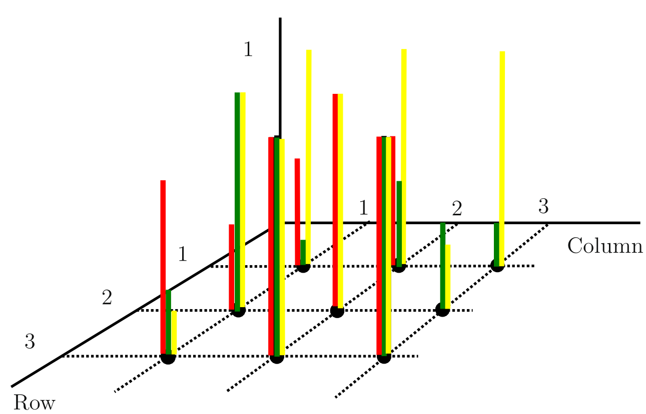

Fig. 8 Single-valued neutrosophic matrix $\mu_{(\alpha)}$

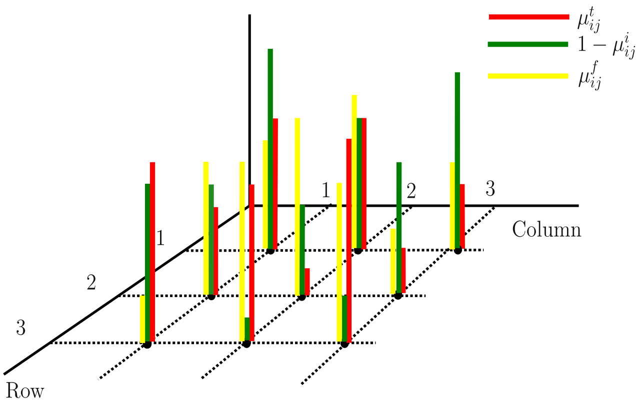

Fig. 9 Single-valued neutrosophic matrix μ^c

Note that $\mu\tilde{\otimes}\nu \neq \nu\tilde{\otimes}\mu$. Let us explain this situation with an example.

Example 2 Let us consider SVN-matrices μ and ν given in Example 1. We know that

$$\mu\tilde{\otimes}\nu = \begin{pmatrix} \langle 0.62, 0.23, 0.50\rangle & \langle 0.71, 0.12, 0.45\rangle & \langle 0.38, 0.24, 0.54\rangle \\ \langle 0.27, 0.41, 0.56\rangle & \langle 0.34, 0.28, 0.49\rangle & \langle 0.28, 0.45, 0.54\rangle \\ \langle 0.78, 0.60, 0.48\rangle & \langle 0.88, 0.60, 0.41\rangle & \langle 0.60, 0.51, 0.57\rangle \end{pmatrix}$$

and

$$\nu\tilde{\otimes}\mu = \begin{pmatrix} \langle 0.67, 0.32, 0.39\rangle & \langle 0.58, 0.63, 0.69\rangle & \langle 0.64, 0.49, 0.36\rangle \\ \langle 0.65, 0.31, 0.52\rangle & \langle 0.56, 0.49, 0.72\rangle & \langle 0.43, 0.33, 0.40\rangle \\ \langle 0.43, 0.16, 0.43\rangle & \langle 0.33, 0.39, 0.75\rangle & \langle 0.33, 0.22, 0.50\rangle \end{pmatrix},$$

$\mu\tilde{\otimes}\nu \neq \nu\tilde{\otimes}\mu$.

Proposition 3 *Let μ, ν and γ be three $n \times n$ SVN-matrices. Then,*

1. $\mu\tilde{\oplus}\nu\tilde{\succeq}\mu\tilde{\odot}\nu$,
2. $\mu\tilde{\oplus}\mu\tilde{\succeq}\mu$,
3. $\mu\tilde{\odot}\mu\tilde{\preceq}\mu$,
4. $\mu\tilde{\oplus}\nu\tilde{\succeq}\mu\tilde{\ominus}\nu$,
5. $\mu\tilde{\oplus}(\nu\tilde{\ominus}\gamma)\tilde{\succeq}(\mu\tilde{\oplus}\nu)\tilde{\ominus}(\mu\tilde{\oplus}\gamma)$,
6. $\mu\tilde{\ominus}(\nu\tilde{\oplus}\gamma)\tilde{\preceq}(\mu\tilde{\ominus}\nu)\tilde{\oplus}(\mu\tilde{\ominus}\gamma)$,
7. *If* $\mu\tilde{\preceq}\nu$, *then* $\mu\tilde{\oplus}\gamma\tilde{\preceq}\nu\tilde{\oplus}\gamma$, $\mu\tilde{\odot}\gamma\tilde{\preceq}\nu\tilde{\odot}\gamma$, $\mu\tilde{\otimes}\gamma\tilde{\preceq}\nu\tilde{\otimes}\gamma$ *and* $\mu\tilde{\ominus}\gamma\tilde{\preceq}\nu\tilde{\ominus}\gamma$.

Proof See Appendix A.1 □

Proposition 4 *Let μ, ν and γ be three $n \times n$ SVN-matrices. Then*

1. $\mu\tilde{\oplus}\nu = \nu\tilde{\oplus}\mu$
2. $(\mu\tilde{\oplus}\nu)\tilde{\oplus}\gamma = \mu\tilde{\oplus}(\nu\tilde{\oplus}\gamma)$
3. $\mu\tilde{\odot}\nu = \nu\tilde{\odot}\mu$
4. $(\mu\tilde{\odot}\nu)\tilde{\odot}\gamma = \mu\tilde{\odot}(\nu\tilde{\odot}\gamma)$
5. $\mu\tilde{\otimes}\mathcal{I} = \mu$
6. $(\mu\tilde{\otimes}\nu)\tilde{\otimes}\gamma = \mu\tilde{\otimes}(\nu\tilde{\otimes}\gamma)$

Proof The proofs are obvious from definitions of operations. □

Note that $\mu\tilde{\oplus}\mathcal{I} \neq \mu$ and $\mu\tilde{\odot}\mathcal{I} \neq \mu$. Let us consider SVN-matrices μ and ν given in Example 1. Then,

$$\begin{aligned} \mu\tilde{\oplus}\mathcal{I} &= \begin{pmatrix} \langle 0.5, 0.1, 0.5\rangle & \langle 0.6, 0.4, 0.7\rangle & \langle 0.3, 0.2, 0.4\rangle \\ \langle 0.4, 0.5, 0.6\rangle & \langle 0.1, 0.6, 0.8\rangle & \langle 0.2, 0.4, 0.3\rangle \\ \langle 0.8, 0.3, 0.2\rangle & \langle 0.7, 0.9, 0.8\rangle & \langle 0.9, 0.8, 0.7\rangle \end{pmatrix} \\ &\quad \tilde{\oplus} \begin{pmatrix} \langle 1, 0, 0\rangle & \langle 0, 1, 1\rangle & \langle 0, 1, 1\rangle \\ \langle 0, 1, 1\rangle & \langle 1, 0, 0\rangle & \langle 0, 1, 1\rangle \\ \langle 0, 1, 1\rangle & \langle 0, 1, 1\rangle & \langle 1, 0, 0\rangle \end{pmatrix} \\ &= \begin{pmatrix} \langle 1, 0, 0\rangle & \langle 0.6, 0.4, 0.7\rangle & \langle 0.3, 0.2, 0.4\rangle \\ \langle 0.4, 0.5, 0.6\rangle & \langle 1, 0, 0\rangle & \langle 0.2, 0.4, 0.3\rangle \\ \langle 0.8, 0.3, 0.2\rangle & \langle 0.7, 0.9, 1\rangle & \langle 1, 0.8, 0\rangle \end{pmatrix} \\ &\neq \mu \end{aligned}$$

$$\begin{aligned} \mu\tilde{\odot}\mathcal{I} &= \begin{pmatrix} \langle 0.5, 0.1, 0.5\rangle & \langle 0.6, 0.4, 0.7\rangle & \langle 0.3, 0.2, 0.4\rangle \\ \langle 0.4, 0.5, 0.6\rangle & \langle 0.1, 0.6, 0.8\rangle & \langle 0.2, 0.4, 0.3\rangle \\ \langle 0.8, 0.3, 0.2\rangle & \langle 0.7, 0.9, 0.8\rangle & \langle 0.9, 0.8, 0.7\rangle \end{pmatrix} \\ &\quad \tilde{\odot} \begin{pmatrix} \langle 1, 0, 0\rangle & \langle 0, 1, 1\rangle & \langle 0, 1, 1\rangle \\ \langle 0, 1, 1\rangle & \langle 1, 0, 0\rangle & \langle 0, 1, 1\rangle \\ \langle 0, 1, 1\rangle & \langle 0, 1, 1\rangle & \langle 1, 0, 0\rangle \end{pmatrix} \\ &= \begin{pmatrix} \langle 0.5, 0.1, 0.5\rangle & \langle 0, 1, 1\rangle & \langle 0, 1, 1\rangle \\ \langle 0, 1, 1\rangle & \langle 1, 0.6, 0.8\rangle & \langle 0, 1, 1\rangle \\ \langle 0, 1, 1\rangle & \langle 0, 1, 1\rangle & \langle 0.9, 0.8, 0.7\rangle \end{pmatrix} \\ &\neq \mu. \end{aligned}$$

Proposition 5 *Let μ and ν be two $n \times n$ SVN-matrices. Then*

1. $(\mu\tilde{\oplus}\nu)' = \mu'\tilde{\oplus}\nu'$
2. $(\mu\tilde{\odot}\nu)' = \mu'\tilde{\odot}\nu'$
3. $(\mu\tilde{\otimes}\nu)' = \nu'\tilde{\otimes}\mu'$
4. $(\mu\tilde{\ominus}\nu)' = \mu'\tilde{\ominus}\nu'$

Proof See Appendix A.2 □

Proposition 6 *Let μ and ν be two $n \times n$ SVN-matrices. Then*

1. $(\mu\tilde{\oplus}\nu)^c = \mu^c\tilde{\odot}\nu^c$
2. $(\mu\tilde{\odot}\nu)^c = \mu^c\tilde{\oplus}\nu^c$
3. $(\mu\tilde{\ominus}\nu)^c\tilde{\succeq}\mu^c\tilde{\ominus}\nu^c$.

Proof See Appendix A.3 □

Proposition 7 *Let μ be a $n \times n$ SVN-matrix and let α be a SVN-value. Then,*

$\mu_{(\alpha)}\tilde{\preceq}\mu^{(\alpha)}$.

Proof The proof is clear from definitions of μ_α and μ^α. □

Proposition 8 *Let μ and ν be two $n \times n$ SVN-matrices and let α be a SVN-value. If $\mu \tilde{\preceq} \nu$, then*

$\mu^{(\alpha)} \tilde{\preceq} \nu^{(\alpha)}$.

Proposition 9 *Let μ be a $n \times n$ SVN-matrix and let α, β be two SVN-values such that $\alpha \preceq \beta$. Then,*

1. $\mu^{(\beta)} \tilde{\preceq} \mu^{(\alpha)}$
2. $\mu_{(\beta)} \tilde{\preceq} \mu_{(\alpha)}$.

Proof See Appendix A.4 □

Proposition 10 *Let μ and ν be two $n \times n$ SVN-matrices and let α be a SVN-value. Then,*

1. $(\mu \tilde{\oplus} \nu)^{(\alpha)} \tilde{\succeq} \mu^{(\alpha)} \tilde{\oplus} \nu^{(\alpha)}$.
2. $(\mu \tilde{\oplus} \nu)_{(\alpha)} \tilde{\succeq} \mu_{(\alpha)} \tilde{\oplus} \nu_{(\alpha)}$.

Proof See Appendix A.5 □

Proposition 11 *Let μ and ν be two $n \times n$ SVN-matrices and let α be a SVN-value. Then,*

1. $(\mu \tilde{\ominus} \nu)^{(\alpha)} \tilde{\succeq} \mu^{(\alpha)} \tilde{\ominus} \nu^{(\alpha)}$.
2. $(\mu \tilde{\ominus} \nu)_{(\alpha)} \tilde{\succeq} \mu_{(\alpha)} \tilde{\ominus} \nu_{(\alpha)}$.

Proof See Appendix A.6 □

Proposition 12 *Let μ and ν be two $n \times n$ SVN-matrices and let α be a SVN-value. Then,*

1. $(\mu \tilde{\odot} \nu)^{(\alpha)} \tilde{\preceq} \mu^{(\alpha)} \tilde{\odot} \nu^{(\alpha)}$.
2. $(\mu \tilde{\odot} \nu)_{(\alpha)} \tilde{\preceq} \mu_{(\alpha)} \tilde{\odot} \nu_{(\alpha)}$.

Proof See Appendix A.7 □

4 Multi-criteria group decision making method

Let $D = \{D_1, D_2, ..., D_s\}$ be set of decision makers, $A_1, A_2, ..., A_p$ be possible alternatives among which decision makers have to choose an optimal alternative and $C_1, C_2, ..., C_r$ be criteria which alternatives performance are measured. The procedure of the proposed method as follows:

4.1 Constructing of decision matrices $D_k (i = 1, 2, ..., s)$

Suppose that $\tilde{d}_{ij}^k = \langle d_{ij(t)}^k, d_{ij(i)}^k, d_{ij(f)}^k \rangle$ is the evaluation value of the criteria C_j with respect to alternative A_i which is made by decision maker D_k. Then, decision matrix for decision maker D_k can be write as follows:

$$D_k = \begin{pmatrix} \tilde{d}_{11}^k & \tilde{d}_{12}^k & \cdots & \tilde{d}_{1r}^k \\ \tilde{d}_{21}^k & \tilde{d}_{22j}^k & \cdots & \tilde{d}_{2r}^k \\ \vdots & \vdots & \vdots & \vdots \\ \tilde{d}_{p1}^k & \tilde{d}_{p2}^k & \cdots & \tilde{d}_{pr}^k \end{pmatrix} \quad (4)$$

4.2 Constructing of reference matrix R

Let C^+ be a criteria set for positive ideal solution and C^- be a criteria set for negative ideal solution. Then, reference matrix R is constructed as follows:

$$R = \begin{pmatrix} \tilde{r}_{11} & \tilde{r}_{12} & \cdots & \tilde{r}_{1r} \\ \tilde{r}_{21} & \tilde{r}_{22} & \cdots & \tilde{r}_{2r} \\ \vdots & \vdots & \vdots & \vdots \\ \tilde{r}_{p1} & \tilde{r}_{p2} & \cdots & \tilde{r}_{pr} \end{pmatrix} \quad (5)$$

where

$$\tilde{r}_{ij} = \begin{cases} \langle \frac{1}{n}\sum_{k=1}^{n} \tilde{d}_{ij(t)}^k, \frac{1}{n}\sum_{k=1}^{n} \tilde{d}_{ij(i)}^k, \frac{1}{n}\sum_{k=1}^{n} \tilde{d}_{ij(f)}^k \rangle, \text{if } C_j \in C^+ \\ \langle \frac{1}{n}\sum_{k=1}^{n} \tilde{d}_{ij(f)}^k, \frac{1}{n}\sum_{k=1}^{n} (1-\tilde{d}_{ij(i)}^k), \frac{1}{n}\sum_{k=1}^{n} \tilde{d}_{ij(t)}^k \rangle, \text{if } C_j \in C^- \end{cases} \quad (6)$$

4.3 Construction of minus matrices $M_k (k = 1, 2, ..., s)$

Let $D_k = [\langle d_{ij(t)}^k, d_{ij(i)}^k, d_{ij(f)}^k \rangle]_{p \times r}$ be a decision maker matrix related to decision maker D_k. Then, minus matrix related to D_k, denoted by M_k is constructed as follows:

$$M_k = D_k \tilde{\ominus} R = \begin{pmatrix} \tilde{m}_{11}^k & \tilde{m}_{12}^k & \cdots & \tilde{m}_{1r}^k \\ \tilde{m}_{21}^k & \tilde{m}_{22}^k & \cdots & \tilde{m}_{2r}^k \\ \vdots & \vdots & \vdots & \vdots \\ \tilde{m}_{p1}^k & \tilde{m}_{p2}^k & \cdots & \tilde{m}_{pr}^k \end{pmatrix}, (k = 1, 2, ..., s) \quad (7)$$

where $\tilde{m}_{ij}^k = \langle d_{ij(t)}^k \ominus \tilde{r}_{ij(t)}, d_{ij(i)}^k \ominus \tilde{r}_{ij(i)}, d_{ij(f)}^k \ominus \tilde{r}_{ij(f)} \rangle$.

4.4 Finding of addition matrix A

Addition matrix, denoted by A, is calculated by using Definition 4 as follows:

$$A = \bigoplus_{k=1}^{s} M_k = \begin{pmatrix} \bigoplus_{k=1}^{s} \tilde{m}_{11}^k & \bigoplus_{k=1}^{s} \tilde{m}_{12}^k & \cdots & \bigoplus_{k=1}^{s} \tilde{m}_{1r}^k \\ \bigoplus_{k=1}^{s} \tilde{m}_{21}^k & \bigoplus_{k=1}^{s} \tilde{m}_{22}^k & \cdots & \bigoplus_{k=1}^{s} \tilde{m}_{2r}^k \\ \vdots & \vdots & \vdots & \vdots \\ \bigoplus_{k=1}^{s} \tilde{m}_{p1}^k & \bigoplus_{k=1}^{s} \tilde{m}_{p2}^k & \cdots & \bigoplus_{k=1}^{s} \tilde{m}_{pr}^k \end{pmatrix} \quad (8)$$

4.5 Constructing of score matrix based on addition matrix

We use following formula given in [27].

Definition 10 [27] Let $\mu = \langle \mu_t, \mu_i, \mu_f \rangle$ be a SVN-number. Then, the score function of SVN-number μ, denoted by $s(\mu)$, is defined as follows:

$$s(\mu) = \frac{\mu_t + 1 - \mu_i + 1 - \mu_f}{3} \tag{9}$$

Let us consider addition matrix A. Then, score matrix is a matrix constructed by calculating scores of elements of addition matrix A, and score matrix of A is denoted by $S(A) = [\tilde{s}_{ij}]_{p\times r}$, where $\tilde{s}_{ij} = s(\bigoplus_{k=1}^{s} \tilde{m}_{ij}^k)$.

4.6 Calculating of decision values of the alternatives

Let us consider score matrix $S(A)$. Then decision value of alternative A_i, denoted by $DV(A_i)$, is calculated by

$$DV(A_i) = \sum_{j=1}^{r} \tilde{s}_{ij}. \tag{10}$$

4.7 Determining of optimum alternative

We determine the addition matrix by summing the elements of minus matrix and we calculate decision values of alternatives according to scores of elements of addition matrix. Therefore, we rank the decision values of the alternatives and choose alternative which has minimum decision value as an optimum element.

The procedures of the proposed method is shown as an algorithm as follows:

5 Illustrative case study

In this section, a case study has been demonstrated to reveal the success of proposed method.

5.1 Firm selection problem

Let us consider numerical example in [18] where one investment company intends to select an firm from the following four alternatives to invest. The four firms are marked by $A_i (i = 1, 2, 3, 4)$, and they are measured by the three criteria: (1) C_1 (the risk index); (2) C_2(the growth index); (3) C_3(environmental impact index). Also there are three investment specialist as decision maker in the company. We denote these decision makers by D_1, D_2 and D_3.

The problem has been solved with proposed group decision making method as in the following steps:

Step 1: Construct decision matrix based on SVN-values of firms provided by DMs.

The SVN-values of the four firms provided by DMs as in Table 1, 2 and 3:

Step 2: Obtain reference matrix R based on decision matrix:

Let $C^+ = \{C_2, C_3\}$ and $C^- = \{C_1\}$. Then, reference matrix R is obtained by using (6) as follows:

$$R = \begin{pmatrix} \langle 0.467, 0.700, 0.600\rangle & \langle 0.400, 0.533, 0.400\rangle & \langle 0.367, 0.567, 0.333\rangle \\ \langle 0.767, 0.533, 0.367\rangle & \langle 0.633, 0.367, 0.467\rangle & \langle 0.533, 0.367, 0.533\rangle \\ \langle 0.300, 0.533, 0.633\rangle & \langle 0.567, 0.433, 0.500\rangle & \langle 0.567, 0.567, 0.500\rangle \\ \langle 0.400, 0.467, 0.533\rangle & \langle 0.667, 0.400, 0.333\rangle & \langle 0.800, 0.433, 0.500\rangle \end{pmatrix}.$$

Step 3: Find minus matrix: Minus matrices for each of decision makers are obtained by using Eq. 7 as follows:

$$M_1 = D_1 \tilde{\ominus} R = \begin{pmatrix} \langle 0.600, 0.700, 1.000\rangle & \langle 0.000, 1.000, 0.400\rangle & \langle 0.000, 1.000, 0.333\rangle \\ \langle 0.000, 0.533, 1.000\rangle & \langle 0.000, 0.367, 1.000\rangle & \langle 0.000, 1.000, 0.533\rangle \\ \langle 0.900, 0.533, 0.633\rangle & \langle 0.800, 1.000, 1.000\rangle & \langle 0.700, 0.567, 0.500\rangle \\ \langle 0.000, 1.000, 0.533\rangle & \langle 0.000, 1.000, 0.333\rangle & \langle 0.000, 1.000, 0.500\rangle \end{pmatrix},$$

$$M_2 = D_2 \tilde{\ominus} R = \begin{pmatrix} \langle 0.000, 0.700, 0.600\rangle & \langle 0.500, 1.000, 1.000\rangle & \langle 0.000, 1.000, 0.333\rangle \\ \langle 0.000, 0.533, 1.000\rangle & \langle 0.000, 1.000, 0.467\rangle & \langle 0.600, 0.367, 0.533\rangle \\ \langle 0.700, 1.000, 0.633.\rangle & \langle 0.000, 0.433, 0.500\rangle & \langle 0.900, 1.000, 0.500\rangle \\ \langle 0.000, 0.467, 0.533\rangle & \langle 0.800, 0.400, 1.000\rangle & \langle 0.000, 0.433, 1.000\rangle \end{pmatrix},$$

$$M_3 = D_3 \tilde{\ominus} R = \begin{pmatrix} \langle 0.700, 0.700, 0.600\rangle & \langle 0.000, 0.533, 1.000\rangle & \langle 0.500, 0.567, 1.000\rangle \\ \langle 0.000, 0.533, 1.000\rangle & \langle 0.900, 0.367, 1.000\rangle & \langle 0.600, 1.000, 1.000\rangle \\ \langle 0.000, 0.533, 0.633\rangle & \langle 0.000, 0.433, 0.500\rangle & \langle 0.000, 0.567, 1.000\rangle \\ \langle 0.800, 0.467, 0.533\rangle & \langle 0.700, 1.000, 0.333\rangle & \langle 0.900, 0.433, 1.000\rangle \end{pmatrix}.$$

Step 4: Calculate addition matrix: Addition matrix A is calculated by using (8) as follows:

$$A=\begin{pmatrix} \langle 0.880, 0.343, 0.360\rangle & \langle 0.500, 0.533, 0.400\rangle & \langle 0.500, 0.567, 0.111\rangle \\ \langle 0.000, 0.152, 1.000\rangle & \langle 0.900, 0.134, 0.467\rangle & \langle 0.840, 0.367, 0.284\rangle \\ \langle 0.970, 0.284, 0.254\rangle & \langle 0.800, 0.188, 0.250\rangle & \langle 0.970, 0.321, 0.250\rangle \\ \langle 0.800, 0.218, 0.152\rangle & \langle 0.940, 0.400, 0.111\rangle & \langle 0.900, 0.188, 0.500\rangle \end{pmatrix}.$$

Step 5: Construct score matrix: Score matrix $S(A)$ is constructed by using (9) as follows:

$$S(A)=\begin{pmatrix} 0.726 & 0.522 & 0.607 \\ 0.283 & 0.766 & 0.730 \\ 0.811 & 0.787 & 0.800 \\ 0.810 & 0.810 & 0.737 \end{pmatrix}.$$

Step 6: Find decision values of alternatives: Decision values of alternatives are found by using (10) as follows:

$DV(A_1) = 1.855$, $DV(A_2) = 1.779$ $DV(A_3) = 2.398$ $DV(A_4) = 2.357$.

Step 7: Rank decision values of the alternatives and select optimum element:

Ranking of decision values of the alternatives is $DV(A_2) < DV(A_1) < DV(A_4) < DV(A_3)$. Then, optimum element is A_2.

6 Comparison and discussion

In section, we compare our multi-criteria decision making procedure with the methods presented by Mukherjee et al. [23], Broumi et al. [7], Ye [42, 43, 45] and Ye and Fu [44]. Firstly, we modify similarity measure methods according to the proposed MCGDM method and then in Table 4 we give ranking orders of alternatives obtained by using existing methods and proposed method

1. Hamming Distance [23]:

$$L_H(D_k(A_i), R)=\frac{1}{6}\sum_{j=1}^{r}\Big\{|d^k_{ij(t)}-\tilde{r}_{ij(t)}|+|d^k_{ij(i)}-\tilde{r}_{ij(i)}|+|d^k_{ij(f)}-\tilde{r}_{ij(f)}|\Big\}. \quad (11)$$

2. Normalized Hamming distance [23]:

$$L_{NH}(D_k(A_i), R)=\frac{1}{6r}\sum_{j=1}^{r}\Big\{|d^k_{ij(t)}-\tilde{r}_{ij(t)}|+|d^k_{ij(i)}-\tilde{r}_{ij(i)}|+|d^k_{ij(f)}-\tilde{r}_{ij(f)}|\Big\}. \quad (12)$$

3. Euclidean distance [23]:

$$L_E(D_k(A_i), R)=\sqrt{\frac{1}{6}\sum_{j=1}^{r}\Big\{(d^k_{ij(t)}-\tilde{r}_{ij(t)})^2+(d^k_{ij(i)}-\tilde{r}_{ij(i)})^2+(d^k_{ij(f)}-\tilde{r}_{ij(f)})^2\Big\}}. \quad (13)$$

4. Normalized Euclidean distance [23]:

$$L_{NE}(D_k(A_i), R)=\sqrt{\frac{1}{6r}\sum_{j=1}^{r}\Big\{(d^k_{ij(t)}-\tilde{r}_{ij(t)})^2+(d^k_{ij(i)}-\tilde{r}_{ij(i)})^2+(d^k_{ij(f)}-\tilde{r}_{ij(f)})^2\Big\}}. \quad (14)$$

5. Extended(Normalized) Hausdorff Distance [7]:

$$L_{EH}(D_k(A_i), R)=\frac{1}{r}\sum_{j=1}^{r}\max\Big\{|d^k_{ij(t)}-\tilde{r}_{ij(t)}|, |d^k_{ij(i)}-\tilde{r}_{ij(i)}|, |d^k_{ij(f)}-\tilde{r}_{ij(f)}|\Big\}. \quad (15)$$

We also modified Extended (normalized) Euclidean Hausdorff distance given in [3].

6. Extended (normalized) Euclidean Hausdorff distance [3]:

$$L_{EEH}(D_k(A_i), R) = \sqrt{\frac{1}{r}\sum_{j=1}^{r} \max\left\{|d^k_{ij(t)} - \tilde{r}_{ij(t)}|^2, |d^k_{ij(i)} - \tilde{r}_{ij(i)}|^2, |d^k_{ij(f)} - \tilde{r}_{ij(f)}|^2\right\}}. \quad (16)$$

Similarity measure is defined by Mukherjee et al. [23] as follows:

$$S(D_k) = \frac{1}{1 + L(D_k, R)}$$

Based on this similarity measure formula, we define similarity score functions of alternatives, denoted by $S(A_i)$, and similarity measures, denoted by $SM(A_i, R)$, as follows:

$$S(A_i) = \frac{1}{s}\sum_{k=1}^{s} L(D_k(A_i), R) \quad (17)$$

and

$$SM(A_i, R) = \frac{1}{1 + S(A_i)} \quad (18)$$

7. Jaccard similarity measure [42]:

$$S_J(D_k(A_i), R) = \frac{1}{n}\sum_{j=1}^{n} = \frac{d^k_{ij(t)}\tilde{r}_{ij(t)} + d^k_{ij(i)}\tilde{r}_{ij(i)} + d^k_{ij(f)}\tilde{r}_{ij(f)}}{\begin{pmatrix} (d^2_{ij(t)} + d^2_{ij(i)} + d^2_{ij(f)} + \tilde{r}^2_{ij(t)} + \tilde{r}^2_{ij(t)} + \tilde{r}^2_{ij(t)}) \\ -(d^k_{ij(t)}\tilde{r}_{ij(t)} + d^k_{ij(i)}\tilde{r}_{ij(i)} + d^k_{ij(f)}\tilde{r}_{ij(f)}) \end{pmatrix}} \quad (19)$$

8. Dice similarity measure [42]:

$$S_D(D_k(A_i), R) = \frac{1}{n}\sum_{j=1}^{n} \frac{2(d^k_{ij(t)}\tilde{r}_{ij(t)} + d^k_{ij(i)}\tilde{r}_{ij(i)} + d^k_{ij(f)}\tilde{r}_{ij(f)})}{(d^2_{ij(t)} + d^2_{ij(i)} + d^2_{ij(f)} + \tilde{r}^2_{ij(t)} + \tilde{r}^2_{ij(t)} + \tilde{r}^2_{ij(t)})} \quad (20)$$

9. Cosine similarity measure [42]:

$$S_C(D_k(A_i), R) = \frac{1}{n}\sum_{j=1}^{n} \frac{(d^k_{ij(t)}\tilde{r}_{ij(t)} + d^k_{ij(i)}\tilde{r}_{ij(i)} + d^k_{ij(f)}\tilde{r}_{ij(f)})}{\sqrt{d^2_{ij(t)} + d^2_{ij(i)} + d^2_{ij(f)}}\sqrt{\tilde{r}^2_{ij(t)} + \tilde{r}^2_{ij(t)} + \tilde{r}^2_{ij(t)}}} \quad (21)$$

10. Improved cosine similarity measure [43]:

$$SC_1(D_k(A_i), R) = \frac{1}{n}\sum_{j=1}^{n} cos\left(\frac{\pi(|d^k_{ij(t)} - \tilde{r}_{ij(t)}| \vee |d^k_{ij(i)} - \tilde{r}_{ij(i)}| \vee |d^k_{ij(f)} - \tilde{r}_{ij(f)}|)}{2}\right) \quad (22)$$

$$SC_2(D_k(A_i), R) = \frac{1}{n}\sum_{j=1}^{n} cos\left(\frac{\pi(|d^k_{ij(t)} - \tilde{r}_{ij(t)}| + |d^k_{ij(i)} - \tilde{r}_{ij(i)}| + |d^k_{ij(f)} - \tilde{r}_{ij(f)}|)}{6}\right) \quad (23)$$

11. Tangent similarity measure [44]:

$$ST_1(D_k(A_i), R) = 1 - \frac{1}{n}\sum_{j=1}^{n} tan\left(\frac{\pi}{4} max(|d^k_{ij(t)} - \tilde{r}_{ij(t)}|, |d^k_{ij(i)} - \tilde{r}_{ij(i)}|, |d^k_{ij(f)} - \tilde{r}_{ij(f)}|)\right) \quad (24)$$

$$ST_2(D_k(A_i), R) = 1 - \frac{1}{n}\sum_{j=1}^{n} tan\left(\frac{\pi}{12}(|d^k_{ij(t)} - \tilde{r}_{ij(t)}| + |d^k_{ij(i)} - \tilde{r}_{ij(i)}| + |d^k_{ij(f)} - \tilde{r}_{ij(f)}|)\right) \quad (25)$$

12. Cotangent similarity measure [45]:

$$SCT_1(D_k(A_i), R) = \frac{1}{n}\sum_{j=1}^{n} cot\left(\frac{\pi}{4} + \frac{\pi}{4} max(|d^k_{ij(t)} - \tilde{r}_{ij(t)}|, |d^k_{ij(i)} - \tilde{r}_{ij(i)}|, |d^k_{ij(f)} - \tilde{r}_{ij(f)}|)\right) \quad (26)$$

$$SCT_2(D_k(A_i), R) = \frac{1}{n}\sum_{j=1}^{n} cot\left(\frac{\pi}{4} + \frac{\pi}{12}(|d^k_{ij(t)} - \tilde{r}_{ij(t)}| + |d^k_{ij(i)} - \tilde{r}_{ij(i)}| + |d^k_{ij(f)} - \tilde{r}_{ij(f)}|)\right) \quad (27)$$

Table 1 The evaluations of firms by decision maker 1 (D_1) for Firm Selection Problem

	C_1	C_2	C_3
A_1	$\langle 0.6, 0.5, 0.7\rangle$	$\langle 0.4, 0.7, 0.2\rangle$	$\langle 0.3, 0.6, 0.1\rangle$
A_2	$\langle 0.3, 0.5, 0.8\rangle$	$\langle 0.6, 0.2, 0.7\rangle$	$\langle 0.4, 0.4, 0.4\rangle$
A_3	$\langle 0.9, 0.2, 0.5\rangle$	$\langle 0.8, 0.6, 0.9\rangle$	$\langle 0.7, 0.5, 0.4\rangle$
A_4	$\langle 0.4, 0.7, 0.4\rangle$	$\langle 0.5, 0.4, 0.3\rangle$	$\langle 0.8, 0.9, 0.1\rangle$

Table 2 The evaluations of firms by decision maker 2 (D_2) for Firm Selection Problem

	C_1	C_2	C_3
A_1	$\langle 0.3, 0.3, 0.5\rangle$	$\langle 0.5, 0.6, 0.6\rangle$	$\langle 0.3, 0.6, 0.1\rangle$
A_2	$\langle 0.3, 0.5, 0.8\rangle$	$\langle 0.4, 0.7, 0.1\rangle$	$\langle 0.6, 0.3, 0.5\rangle$
A_3	$\langle 0.7, 0.7, 0.3\rangle$	$\langle 0.5, 0.3, 0.2\rangle$	$\langle 0.9, 0.7, 0.2\rangle$
A_4	$\langle 0.4, 0.1, 0.5\rangle$	$\langle 0.8, 0.3, 0.4\rangle$	$\langle 0.7, 0.2, 0.6\rangle$

Table 3 The evaluations of firms by decision maker 3 (D_3) for Firm Selection Problem

	C_1	C_2	C_3
A_1	$\langle 0.7, 0.1, 0.2\rangle$	$\langle 0.3, 0.3, 0.4\rangle$	$\langle 0.5, 0.5, 0.8\rangle$
A_2	$\langle 0.6, 0.4, 0.7\rangle$	$\langle 0.9, 0.2, 0.6\rangle$	$\langle 0.6, 0.4, 0.7\rangle$
A_3	$\langle 0.3, 0.5, 0.1\rangle$	$\langle 0.4, 0.4, 0.4\rangle$	$\langle 0.1, 0.5, 0.9\rangle$
A_4	$\langle 0.8, 0.2, 0.2\rangle$	$\langle 0.7, 0.5, 0.3\rangle$	$\langle 0.9, 0.2, 0.8\rangle$

Table 4 Ranking orders by our method and existing methods

Ranking order methods	A_1	A_2	A_3	A_4	Ranking order
Based on Hamming Distance [23]	0.745	0.844	0.776	0.812	$A_2 \succeq A_4 \succeq A_3 \succeq A_1$
Based on Normalized Hamming Distance [23]	0.950	0.956	0.933	0.945	$A_2 \succeq A_1 \succeq A_4 \succeq A_3$
Based on Euclidean Distance [23]	0.735	0.778	0.687	0.727	$A_2 \succeq A_1 \succeq A_4 \succeq A_3$
Based on Normalized Euclidean Distance [23]	0.847	0.875	0.814	0.842	$A_2 \succeq A_1 \succeq A_4 \succeq A_3$
Base on Hausdorff distance [7]	0.796	0.846	0.774	0.799	$A_2 \succeq A_4 \succeq A_1 \succeq A_3$
Extended Euclidean Hausdorff distance [3]	0.693	0.761	0.664	0.696	$A_2 \succeq A_4 \succeq A_1 \succeq A_3$
Jaccard similarity measure [42]	0.886	0.918	0.844	0.896	$A_2 \succeq A_4 \succeq A_1 \succeq A_3$
Dice similarity measure [42]	0.946	0.939	0.907	0.966	$A_4 \succeq A_1 \succeq A_2 \succeq A_3$
Cosine similarity measure [42]	0.9552	0.9617	0.9498	0.9615	$A_2 \succeq A_4 \succeq A_1 \succeq A_3$
Improved cosine similarity measure-1 [43]	0.914	0.950	0.885	0.909	$A_2 \succeq A_1 \succeq A_4 \succeq A_3$
Improved cosine similarity measure-2 [43]	0.9732	0.9739	0.9477	0.9650	$A_2 \succeq A_1 \succeq A_4 \succeq A_3$
Based on tangent function-1 [44]	0.796	0.856	0.765	0.798	$A_2 \succeq A_4 \succeq A_1 \succeq A_3$
Based on tangent function-2 [44]	0.889	0.902	0.847	0.878	$A_2 \succeq A_1 \succeq A_4 \succeq A_3$
Based on cotangent function-1 [45]	0.665	0.755	0.626	0.673	$A_2 \succeq A_4 \succeq A_1 \succeq A_3$
Based on cotangent function-2 [45]	0.802	0.828	0.740	0.787	$A_2 \succeq A_1 \succeq A_4 \succeq A_3$
Proposed group decision making method	1.855	1.779	2.398	2.357	$A_2 \succeq A_1 \succeq A_4 \succeq A_3$

Table 5 Obtained results by using (12)-(27) and proposed method

	Case 1	Case 2	Case 3	Case 4
X	(0, 0, 0)	(0.2, 0.3, 0.4)	(1,0,0)	(1,0,0)
Y	(0, 0, 0)	(0.2, 0.3, 0.4)	(0,0,0)	(0,1,1)
Based on Hamming Distance [23]	1	1	0.857	0.667
Based on Normalized Hamming Distance [23]	1	1	0.960	0.889
Based on Euclidean Distance [23]	1	1	0.710	0.586
Based on Normalized Euclidean Distance [23]	1	1	0.830	0.739
Base on Hausdorff distance [7]	1	1	0.750	0.750
Extended Euclidean Hausdorff distance [3]	1	1	0.634	0.634
Jaccard similarity measure [42]	null	1	0	0
Dice similarity measure [42]	null	1	0	0
Cosine similarity measure [42]	null	1	null	0
Improved cosine similarity measure-1 [43]	1	1	0	0
Improved cosine similarity measure-2 [43]	1	1	0.866	0
Based on tangent function-1 [44]	1	1	0	0
Based on tangent function-2 [44]	1	1	0.732	0
Based on cotangent function-1 [45]	1	1	0	0
Based on cotangent function-2 [45]	1	1	0.577	0
Proposed decision making method	1	1	0.667	0

We see that the proposed method in this paper is consistent with the other methods proposed previously.

For the comparison of the ranking order methods based on the proposed method with existing methods based on distance measures and similarity measures methods [7, 23, 42–45] in single-valued neutrosophic setting, a numerical example is presented to demonstrate the effectiveness and rationality of the proposed ranking order method of SVNSs. Let us consider two SVNSs X and Y in $A = \{a\}$ and compare the proposed method with the existing methods based on similarity measures and distance similarity measures in [7, 23, 42–45] for pattern recognitions. By applying (12)-(27), the similarity measure results for the pattern recognitions are indicated by the numerical example,as shown in Table 5.

By the Table 5:

1. In Case 1, values of Jaccard, Dice and Cosine similarity measures are undefined or unmeaningful.
2. In Case 3, similarity value based on improved cosine similarity-2 is equal 0, but similarity value based on improved cosine similarity-2 is equal 0.866. Hence the similarity measures are not suitable for some applications handling ranking order of alternatives and pattern recognition. Similar situations valid for similarity measures based on tangent function-1 and tangent function-2 and cotangent function-1 and cotangent function-2.
3. In case 4, since complement of (1,0,0) is (0,1,1), values of similarity measures based on Hamming distance, normalized Hamming distance, Euclidean distance, normalized Euclidean distance, Hausdorff distance and extended Euclidean Hausdorff distance are not meaningful.

As a result, we can say that proposed decision making method is more suitable for ranking order than other methods. Also, in the other methods many operations such as extraction, absolute value, e.i. are used. In our method, we use minus operation instead of distance measure, thus we obtain the results for ranking order easier.

7 Conclusion

In this paper, we defined some new operations on SVN-matrices, an investigated some properties of them. Especially, type-1 product operation of the SVN-matrices is similar Hadamard product of matrices. Therefore, type-1 product allows to make some applications related to Hadamard product under SVN environment. Also, in future it may be derived some properties of these operations under determinant function. Furthermore researchers may study on inverses of SVN-matrices and may propose a new decision making methods based on upper and lover $(\alpha)-$level SVN-matrices.

Compliance with Ethical Standards

Conflict of interests The authors declare that they have no conflict of interest.

Ethical approval This article does not contain any studies with human participants or animals performed by any of the authors.

Appendix A

A.1

1. Let $\alpha = \mu\tilde{\oplus}\nu$ and $\beta = \mu\tilde{\odot}\nu$. Then $\alpha^t_{pq} = \mu^t_{pq} + \nu^t_{pq} - \mu^t_{pq}\nu^t_{pq}$ and $\beta^t_{pq} = \mu^t_{pq}\nu^t_{pq}$. The proof will be made for membership, indeterminacy and non-membership degrees of elements of SVN-matrices.
   (a) Suppose that $\mu^t_{pq} + \nu^t_{pq} - \mu^t_{pq}\nu^t_{pq} \geq \mu^t_{pq}\nu^t_{pq}$. Since $0 \leq \mu^t_{pq} \leq 1$ and $0 \leq \nu^t_{pq} \leq 1$, $\mu^t_{pq}(1 - \nu^t_{pq}) + \nu^t_{pq}(1 - \mu^t_{pq}) \geq 0$. Hence $\alpha^t_{pq} \geq \beta^t_{pq}$.
   (b) Suppose that $\mu^i_{pq}\nu^i_{pq} \leq \mu^i_{pq} + \nu^i_{pq} - \mu^i_{pq}\nu^i_{pq}$. Since $0 \leq \mu^i_{pq} \leq 1$ and $0 \leq \nu^i_{pq} \leq 1$, $0 \leq \mu^i_{pq}(1 - \nu^i_{pq}) + \nu^i_{pq}(1 - \mu^i_{pq})$. Hence $\alpha^i_{pq} \leq \beta^i_{pq}$.
   (c) It can be shown that $\alpha^f_{pq} \leq \beta^f_{pq}$ with similar way to proof of $\alpha^i_{pq} \leq \beta^i_{pq}$.
2. Let $\alpha = \mu\tilde{\oplus}\mu$. Then $\alpha = (\mu^t_{pq} + \nu^t_{pq} - \mu^t_{pq}\nu^t_{pq}, \mu^i_{pq}\nu^i_{pq}, \mu^f_{pq}\nu^f_{pq})$. Since each of $\mu^t_{pq}, \mu^i_{pq}, \mu^f_{pq}, \nu^t_{pq}, \nu^t_{pq}$ and ν^f_{pq} are in interval [0,1], it is clear that $\mu^t_{pq} + \nu^t_{pq} - \mu^t_{pq}\nu^t_{pq} \geq \mu^t_{pq}, \mu^i_{pq}\nu^i_{pq} \leq \mu^i_{pq}$ and $\mu^f_{pq}\nu^f_{pq} \leq \mu^f_{pq}$. Therefore $\mu\tilde{\oplus}\mu \succeq \mu$.
3. Let $\alpha = \mu\tilde{\odot}\mu$. Then $\alpha = (\mu^t_{pq}\mu^t_{pq}, \mu^i_{pq} + \mu^i_{pq} - \mu^i_{pq}\mu^i_{pq}, \mu^f_{pq} + \mu^f_{pq} - \mu^f_{pq}\mu^f_{pq})$. Since $\mu^t_{pq}\nu^t_{pq} \leq \mu^t_{pq}$, $\mu^i_{pq} + \mu^i_{pq} - \mu^i_{pq}\mu^i_{pq} \geq \mu^i_{pq}$ and $\mu^f_{pq} + \mu^f_{pq} - \mu^f_{pq}\mu^f_{pq} \geq \mu^f_{pq}, \mu\tilde{\odot}\mu \preceq \mu$.
4. Let $\alpha_{pq} = \langle \mu^t_{pq} + \nu^t_{pq} - \mu^t_{pq}\nu^t_{pq}, \mu^i_{pq}\nu^i_{pq}, \mu^f_{pq}\nu^f_{pq} \rangle$ be an element of SVN-matrix $\mu\tilde{\oplus}\nu$. From definition of $\mu\tilde{\ominus}\nu$, we know that for truth-membership values of elements of SVN-matrix $\mu\tilde{\ominus}\nu$, if $\mu^t_{pq} \geq \nu^t_{pq}$, $\mu^t_{pq} \ominus \nu^t_{pq} = \mu^t_{pq}$ and if $\mu^t_{pq} < \nu^t_{pq}$, $\mu^t_{pq} \ominus \nu^t_{pq} = 0$. Since $\mu^t_{pq} + \nu^t_{pq} - \mu^t_{pq}\nu^t_{pq} \geq \mu^t_{pq}$ and $\mu^t_{pq} + \nu^t_{pq} - \mu^t_{pq}\nu^t_{pq} \geq 0$, truth-membership values of elements of SVN-matrix $\mu\tilde{\oplus}\nu$ are always greater than or equal to truth-membership values of elements of SVN-matrix $\mu\tilde{\ominus}\nu$. From definition of $\mu\tilde{\ominus}\nu$, if $\mu^i_{pq} \geq \nu^i_{pq}$, $\mu^i_{pq} \ominus \nu^i_{pq} = 1$, and if $\mu^i_{pq} < \nu^i_{pq}$, $\mu^i_{pq} \ominus \nu^i_{pq} = \mu_{pq}$. Since $1 \geq \mu^i_{pq}\nu^i_{pq}$ and $\mu^i_{pq} \geq \mu^i_{pq}\nu^i_{pq}$, indeterminacy-membership values of elements of SVN-matrix $\mu\tilde{\oplus}\nu$ are always smaller than or equal to indeterminacy-membership values of elements of SVN-matrix $\mu\tilde{\ominus}\nu$. Also, it can be shown that falsity-membership values of elements of SVN-matrix $\mu\tilde{\oplus}\nu$ are always smaller than or equal to falsity-membership values of elements of SVN-matrix $\mu\tilde{\ominus}\nu$. It is concluded that $\mu\tilde{\oplus}\nu\tilde{\succeq}\mu\tilde{\ominus}\nu$.
5. Let α_{pq} and β_{pq} be the pqth elements of the SVN-matrices $\mu\tilde{\oplus}(\nu\tilde{\ominus}\gamma)$ and $(\mu\tilde{\oplus}\nu)\tilde{\ominus}(\mu\tilde{\oplus}\gamma)$, respectively. The proof will be made for three cases.

   Case 1: For truth-membership values of the SVN-matrices. From Definition 6,

$$\alpha^t_{pq} = \begin{cases} \mu^t_{pq} + \nu^t_{pq} - \mu^t_{pq}\nu^t_{pq}, & \nu^t_{pq} > \gamma^t_{pq}, \\ \mu^t_{pq}, & \nu^t_{pq} \leq \gamma^t_{pq}. \end{cases}$$

   Since $\mu^t_{pq} + \nu^t_{pq} - \mu^t_{pq}\nu^t_{pq} > \mu^t_{pq} + \gamma^t_{pq} - \mu^t_{pq}\gamma^t_{pq}$ for $\nu^t_{pq} > \gamma^t_{pq}$, and $\mu^t_{pq} + \nu^t_{pq} - \mu^t_{pq}\nu^t_{pq} \leq \mu^t_{pq} + \gamma^t_{pq} - \mu^t_{pq}\gamma^t_{pq}$ for $\nu^t_{pq} \leq \gamma^t_{pq}$,

$$\beta^t_{pq} = \begin{cases} \mu^t_{pq} + \nu^t_{pq} - \mu^t_{pq}\nu^t_{pq}, & \nu^t_{pq} > \gamma^t_{pq}, \\ 0, & \nu^t_{pq} \leq \gamma^t_{pq}. \end{cases}$$

   Hence $\alpha^t_{pq} \geq \beta^t_{pq}$.
   Case 2: For indeterminacy-membership values of the SVN-matrices. From Definition 6,

$$\alpha^i_{pq} = \begin{cases} \mu^i_{pq}, & \nu^i_{pq} \geq \gamma^i_{pq}, \\ \mu^i_{pq}\nu^i_{pq}, & \nu^i_{pq} < \gamma^i_{pq}. \end{cases}$$

   Since $\mu^i_{pq}\nu^i_{pq} \geq \mu^i_{pq}\gamma^i_{pq}$ for $\nu^i_{pq} \geq \gamma^i_{pq}$, and $\mu^i_{pq}\nu^i_{pq} < \mu^i_{pq}\gamma^i_{pq}$ for $\nu^i_{pq} < \gamma^i_{pq}$,

$$\beta^i_{pq} = \begin{cases} 1, & \nu^i_{pq} \geq \gamma^i_{pq}, \\ \mu^i_{pq}\nu^i_{pq}, & \nu^i_{pq} < \gamma^i_{pq}. \end{cases}$$

   Thus $\alpha^i_{pq} \leq \beta^i_{pq}$
   Case 3: For falsity-membership values of the SVN-matrices. From Definition 6,

$$\alpha^f_{pq} = \begin{cases} \mu^f_{pq}, & \nu^f_{pq} \geq \gamma^f_{pq}, \\ \mu^f_{pq}\nu^f_{pq}, & \nu^f_{pq} < \gamma^f_{pq}. \end{cases}$$

   Since $\mu^f_{pq}\nu^f_{pq} \geq \mu^f_{pq}\gamma^f_{pq}$ for $\nu^f_{pq} \geq \gamma^f_{pq}$, and $\mu^f_{pq}\nu^f_{pq} < \mu^f_{pq}\gamma^f_{pq}$ for $\nu^f_{pq} < \gamma^f_{pq}$,

$$\beta^f_{pq} = \begin{cases} 1, & \nu^f_{pq} \geq \gamma^f_{pq}, \\ \mu^f_{pq}\nu^f_{pq}, & \nu^f_{pq} < \gamma^f_{pq}. \end{cases}$$

   Thus $\alpha^f_{pq} \leq \beta^f_{pq}$ and $\alpha_{pq} \succeq \beta_{pq}$.
   From Case 1,2 and 3, it is concluded that $\mu\tilde{\oplus}(\nu\tilde{\ominus}\gamma)\tilde{\succeq}(\mu\tilde{\oplus}\nu)\tilde{\ominus}(\mu\tilde{\oplus}\gamma)$.
6. The proof can be made with similar way to proof of statement 1.
7. Let $\alpha_{pq}, \beta_{pq}, \theta_{pq}, \sigma_{pq}, \delta_{pq}, \tau_{pq}, \rho_{pq}$ and ϵ_{pq} be pqth elements of $\mu\tilde{\oplus}\gamma, \nu\tilde{\oplus}\gamma, \mu\tilde{\odot}\gamma, \nu\tilde{\odot}\gamma, \mu\tilde{\otimes}\gamma, \nu\tilde{\otimes}\gamma$ and $\mu\tilde{\ominus}\gamma, \nu\tilde{\ominus}\gamma$, respectively. Since $\mu\tilde{\preceq}\nu$, $\mu^t_{pq} \leq \nu^t_{pq}, \mu^i_{pq} \geq \nu^i_{pq}$ and $\mu^f_{pq} \geq \nu^f_{pq}$. It is clear that $\alpha^t_{pq} = \mu^t_{pq} + \gamma^t_{pq} - \mu^t_{pq}\gamma^t_{pq} \leq \nu^t_{pq} + \gamma^t_{pq} - \nu^t_{pq}\gamma^t_{pq} = \beta^t_{pq}$, $\alpha^i_{pq} = \mu^i_{pq}\gamma^i_{pq} \geq \nu^i_{pq}\gamma^i_{pq} = \beta^t_{pq}$ and $\alpha^f_{pq} = \mu^f_{pq}\gamma^f_{pq} \geq \nu^f_{pq}\gamma^f_{pq} = \beta^f_{pq}$. Therefore $\alpha_{pq} \preceq \beta_{pq}$ and $\mu\tilde{\oplus}\gamma\tilde{\preceq}\nu\tilde{\oplus}\gamma$.

By given condition $\theta^t_{pq} = \mu^t_{pq}\gamma^t_{pq} \le \nu^t_{pq}\gamma^t_{pq} = \sigma^t_{pq}$, $\theta^i_{pq} = \mu^i_{pq} + \gamma^i_{pq} - \mu^i_{pq}\gamma^i_{pq} \ge \nu^i_{pq} + \gamma^i_{pq} - \nu^i_{pq}\gamma^i_{pq} = \sigma^t_{pq}$ and $\theta^f_{pq} = \mu^f_{pq} + \gamma^f_{pq} - \mu^f_{pq}\gamma^f_{pq} \ge \nu^f_{pq} + \gamma^f_{pq} - \nu^f_{pq}\gamma^f_{pq} = \sigma^f_{pq}, \theta_{pq} \preceq \sigma_{pq}$. Therefore $\mu\tilde{\odot}\gamma\tilde{\preceq}\nu\tilde{\odot}\gamma$.

By given condition $\delta^t_{pq} = 1 - \prod_{k=1}^n(1 - (\mu^t_{pk}\gamma^t_{kq})) \le 1 - \prod_{k=1}^n(1 - (\nu^t_{pk}\gamma^t_{kq})) = \tau^t_{pq}$, $\delta^i_{pq} = \prod_{k=1}^n(\mu^i_{pk}\gamma^i_{kq}) \ge \prod_{k=1}^n(\nu^i_{pk}\nu^i_{kq}) = \tau^i_{pq}$ and $\delta^f_{pq} = \prod_{k=1}^n(\mu^f_{pk}\gamma^f_{kq}) \ge \prod_{k=1}^n(\nu^f_{pk}\nu^f_{kq}) = \tau^f_{pq}$, $\delta_{pq} \preceq \tau_{pq}$. Therefore $\mu\tilde{\otimes}\gamma\tilde{\preceq}\nu\tilde{\otimes}\gamma$.

Since

$$\rho^t_{pq} = \begin{cases} \mu^t_{pq}, & \mu^t_{pq} \ge \gamma^t_{pq} \\ 0, & \mu^t_{pq} < \gamma^t_{pq} \end{cases}, \quad \epsilon^t_{pq} = \begin{cases} \nu^t_{pq}, & \nu^t_{pq} \ge \gamma^t_{pq} \\ 0, & \nu^t_{pq} < \gamma^t_{pq} \end{cases}$$

and $\mu^t_{pq} \le \nu^t_{pq}$ from hypothesis, $\rho^t_{pq} \le \epsilon^t_{pq}$. Also since

$$\rho^i_{pq} = \begin{cases} 1, & \mu^i_{pq} \ge \gamma^i_{pq} \\ \mu^i_{pq}, & \mu^i_{pq} < \gamma^i_{pq} \end{cases}, \quad \epsilon^i_{pq} = \begin{cases} 1, & \nu^i_{pq} \ge \gamma^i_{pq} \\ \nu^i_{pq}, & \nu^i_{pq} < \gamma^i_{pq} \end{cases}$$

and $\mu^i_{pq} \ge \nu^i_{pq}$ from hypothesis, $\rho^i_{pq} \ge \epsilon^i_{pq}$.

Similarly, $\rho^f_{pq} = \begin{cases} 1, & \mu^f_{pq} \ge \gamma^f_{pq} \\ \mu^f_{pq}, & \mu^f_{pq} < \gamma^f_{pq} \end{cases}$, $\epsilon^f_{pq} = \begin{cases} 1, & \nu^f_{pq} \ge \gamma^f_{pq} \\ \nu^f_{pq}, & \nu^f_{pq} < \gamma^f_{pq} \end{cases}$ and $\mu^f_{pq} \ge \nu^f_{pq}$ from hypothesis, $\rho^f_{pq} \ge \epsilon^f_{pq}$. Therefore $\rho_{pq} \preceq \epsilon_{pq}$ and so $\mu\tilde{\ominus}\gamma\tilde{\preceq}\nu\tilde{\ominus}\gamma$.

A.2

1. Suppose that $\alpha_{pq} = \langle\alpha^t_{pq}, \alpha^i_{pq}, \alpha^f_{pq}\rangle$ and $\beta_{pq} = \langle\beta^t_{pq}, \beta^i_{pq}, \beta^f_{pq}\rangle$ are ijth elements of $\mu\tilde{\oplus}\nu$ and $\mu'\tilde{\oplus}\nu'$, respectively. Then $\sigma_{pq} = \alpha_{qp}$ is the pqth element of $(\mu\tilde{\oplus}\nu)'$ and

$$\begin{aligned}\alpha_{pq} &= \langle\alpha^t_{pq}, \alpha^i_{pq}, \alpha^f_{pq}\rangle \\ &= \langle\mu^t_{pq} + \nu^t_{pq} - \mu^t_{pq}\nu^t_{pq}, \mu^i_{pq}\nu^i_{pq}, \mu^f_{pq}\nu^f_{pq}\rangle \\ \beta_{pq} &= \langle\beta^t_{pq}, \beta^i_{pq}, \beta^f_{pq}\rangle \\ &= \langle\mu^t_{qp} + \nu^t_{qp} - \mu^t_{qp}\nu^t_{qp}, \mu^i_{qp}\nu^i_{qp}, \mu^f_{qp}\nu^f_{qp}\rangle \\ \sigma_{pq} &= \langle\alpha^t_{qp}, \alpha^i_{qp}, \alpha^f_{qp}\rangle \\ &= \langle\mu^t_{qp} + \nu^t_{qp} - \mu^t_{qp}\nu^t_{qp}, \mu^i_{qp}\nu^i_{qp}, \mu^f_{qp}\nu^f_{qp}\rangle \\ &= \beta_{pq}.\end{aligned}$$

Thus, $\sigma_{pq} = \beta_{pq}$ for all p, q. Hence $(\mu\tilde{\oplus}\nu)' = \mu'\tilde{\oplus}\nu'$.

2. Suppose that $\alpha_{pq} = \langle\alpha^t_{pq}, \alpha^i_{pq}, \alpha^f_{pq}\rangle$ and $\beta_{pq} = \langle\beta^t_{pq}, \beta^i_{pq}, \beta^f_{pq}\rangle$ are pqth elements of $\mu\tilde{\odot}\nu$ and $\mu'\tilde{\odot}\nu'$, respectively. Suppose that $\alpha_{pq} = \langle\alpha^t_{pq}, \alpha^i_{pq}, \alpha^f_{pq}\rangle$ and $\beta_{pq} = \langle\beta^t_{pq}, \beta^i_{pq}, \beta^f_{pq}\rangle$ are pqth elements of $\mu\tilde{\odot}\nu$ and $\mu'\tilde{\odot}\nu'$, respectively. Then $\sigma_{pq} = \alpha_{qp}$ is the pqth element of $(\mu\tilde{\odot}\nu)'$ and

$$\begin{aligned}\alpha_{pq} &= \langle\alpha^t_{pq}, \alpha^i_{pq}, \alpha^f_{pq}\rangle \\ &= \langle\mu^t_{pq}\nu^t_{pq}, \mu^i_{pq} + \nu^i_{pq} - \mu^i_{pq}\nu^i_{pq}, \mu^f_{pq} + \nu^f_{pq} - \mu^f_{pq}\nu^f_{pq}\rangle \\ \beta_{pq} &= \langle\beta^t_{pq}, \beta^i_{pq}, \beta^f_{pq}\rangle \\ &= \langle\mu^t_{qp}\nu^t_{qp}, \mu^i_{qp} + \nu^i_{qp} - \mu^i_{qp}\nu^i_{qp}, \mu^f_{qp} + \nu^f_{qp} - \mu^f_{qp}\nu^f_{qp}\rangle \\ \sigma_{pq} &= \langle\sigma^t_{pq}, \sigma^i_{pq}, \sigma^f_{pq}\rangle \\ &= \langle\mu^t_{qp}\nu^t_{qp}, \mu^i_{qp} + \nu^i_{qp} - \mu^i_{qp}\nu^i_{qp}, \mu^f_{qp} + \nu^f_{qp} - \mu^f_{qp}\nu^f_{qp}\rangle\end{aligned}$$

Thus, $\sigma_{pq} = \beta_{pq}$ for all p, q. Hence $(\mu\tilde{\odot}\nu)' = \mu'\tilde{\odot}\nu'$.

3. Suppose that $\alpha_{pq} = \langle\alpha^t_{pq}, \alpha^i_{pq}, \alpha^f_{pq}\rangle$ and $\beta_{pq} = \langle\beta^t_{pq}, \beta^i_{pq}, \beta^f_{pq}\rangle$ are pqth elements of $\mu\tilde{\otimes}\nu$ and $\nu'\tilde{\otimes}\mu'$, respectively. Suppose that $\alpha_{pq} = \langle\alpha^t_{pq}, \alpha^i_{pq}, \alpha^f_{pq}\rangle$ and $\beta_{pq} = \langle\beta^t_{pq}, \beta^i_{pq}, \beta^f_{pq}\rangle$ are ijth elements of $\mu\tilde{\otimes}\nu$ and $\nu'\tilde{\otimes}\mu'$, respectively. Then $\sigma_{pq} = \alpha_{qp}$ is the pqth element of $(\mu\tilde{\otimes}\nu)'$ and

$$\begin{aligned}\alpha_{pq} &= \langle\alpha^t_{pq}, \alpha^i_{pq}, \alpha^f_{pq}\rangle = \langle 1 - \prod_{k=1}^n(1 - \mu^t_{pk}\nu^t_{kq}), \\ &\quad \prod_{k=1}^n(\mu^i_{pk} + \nu^i_{kq} - \mu^i_{pq}\nu^i_{pq}), \prod_{k=1}^n(\mu^f_{pq} + \nu^f_{pq} - \mu^f_{pq}\nu^f_{pq})\rangle \\ \beta_{pq} &= \langle\beta^t_{pq}, \beta^i_{pq}, \beta^f_{pq}\rangle = \langle 1 - \prod_{k=1}^n(1 - \nu^t_{qk}\mu^t_{kp}), \\ &\quad \prod_{k=1}^n(\nu^i_{qk} + \mu^i_{kp} - \nu^i_{qk}\mu^i_{kp}), \prod_{k=1}^n(\nu^f_{qk} + \mu^f_{kp} - \nu^f_{qk}\mu^f_{kp})\rangle \\ \sigma_{pq} &= \langle\sigma^t_{pq}, \sigma^i_{pq}, \sigma^f_{pq}\rangle = \langle 1 - \prod_{k=1}^n(1 - \mu^t_{qk}\nu^t_{kp}), \\ &\quad \prod_{k=1}^n(\mu^i_{qk} + \nu^i_{kp} - \mu^i_{qk}\nu^i_{kp}), \prod_{k=1}^n(\mu^f_{qk} + \nu^f_{kp} - \mu^f_{qk}\nu^f_{kp})\rangle \\ &= \beta_{pq}.\end{aligned}$$

Thus, $\sigma_{pq} = \beta_{pq}$ for all p, q. Hence $(\mu\tilde{\otimes}\nu)' = \nu'\tilde{\otimes}\mu'$.

4. Suppose that $\alpha_{pq} = \langle\alpha^t_{pq}, \alpha^i_{pq}, \alpha^f_{pq}\rangle$ and $\beta_{pq} = \langle\beta^t_{pq}, \beta^i_{pq}, \beta^f_{pq}\rangle$ are pqth elements of $\mu\tilde{\ominus}\nu$ and $\mu'\tilde{\ominus}\nu'$, respectively. Suppose that $\alpha_{pq} = \langle\alpha^t_{pq}, \alpha^i_{pq}, \alpha^f_{pq}\rangle$ and $\beta_{pq} = \langle\beta^t_{pq}, \beta^i_{pq}, \beta^f_{pq}\rangle$ are pqth elements of $\mu\tilde{\ominus}\nu$ and $\mu'\tilde{\ominus}\nu'$, respectively. Then $\sigma_{pq} = \alpha_{qp}$ is the pqth element of $(\mu\tilde{\ominus}\nu)'$ and

$$\begin{aligned}\alpha_{pq} &= \langle\alpha^t_{pq}, \alpha^i_{pq}, \alpha^f_{pq}\rangle = \langle\mu^t_{pq} \ominus \nu^t_{pq}, \mu^i_{pq} \ominus \nu^i_{pq}, \mu^f_{pq} \ominus \nu^f_{pq}\rangle, \\ \beta_{pq} &= \langle\beta^t_{pq}, \beta^i_{pq}, \beta^f_{pq}\rangle = \langle\mu^t_{qp} \ominus \nu^t_{qp}, \mu^i_{qp} \ominus \nu^i_{qp}, \mu^f_{qp} \ominus \nu^f_{qp}\rangle, \\ \sigma_{pq} &= \langle\sigma^t_{pq}, \sigma^i_{pq}, \sigma^f_{pq}\rangle = \langle\mu^t_{qp} \ominus \nu^t_{qp}, \mu^i_{qp} \ominus \nu^i_{qp}, \mu^f_{qp} \ominus \nu^f_{qp}\rangle, \\ &= \beta_{pq}.\end{aligned}$$

Thus, $\sigma_{pq} = \beta_{pq}$ for all p, q. Hence $(\mu\tilde{\ominus}\nu)' = \mu'\tilde{\ominus}\nu'$.

A.3

1. Let $\alpha_{pq} = \langle \alpha^t_{pq}, \alpha^i_{pq}, \alpha^f_{pq} \rangle$ and $\beta_{pq} = \langle \beta^t_{pq}, \beta^i_{pq}, \beta^f_{pq} \rangle$ be pqth elements of $\mu \tilde{\oplus} \nu$ and $\mu^c \tilde{\odot} \nu^c$, respectively, and let $\sigma_{pq} = \alpha^c_{pq}$. Then

$$\begin{aligned}\alpha_{pq} &= \left\langle \alpha^t_{pq}, \alpha^i_{pq}, \alpha^f_{pq} \right\rangle = \left\langle \mu^t_{pq} + \nu^t_{pq} - \mu^t_{pq}\nu^t_{pq}, \mu^i_{pq}\nu^i_{pq}, \mu^f_{pq}\nu^f_{pq} \right\rangle \\ \sigma_{pq} &= \alpha^c_{pq} = \left\langle \alpha^f_{pq}, 1-\alpha^i_{pq}, \alpha^t_{pq} \right\rangle = \left\langle \mu^f_{pq}\nu^f_{pq}, 1-\mu^i_{pq}\nu^i_{pq}, \mu^t_{pq} \right. \\ &\quad \left. + \nu^t_{pq} - \mu^t_{pq}\nu^t_{pq} \right\rangle\end{aligned}$$

$$\begin{aligned}\beta_{pq} &= \langle \beta^t_{pq}, \beta^i_{pq}, \beta^f_{pq} \rangle = \left\langle \mu^f_{pq}, 1-\mu^i_{pq}, \mu^t_{pq} \right\rangle \odot \left\langle \nu^f_{pq}, 1-\nu^i_{pq}, \nu^t_{pq} \right\rangle \\ &= \left\langle \mu^f_{pq}\nu^f_{pq}, 1-\mu^i_{pq}\nu^i_{pq}, \mu^t_{pq} + \nu^t_{pq} - \mu^t_{pq}\nu^t_{pq} \right\rangle.\end{aligned}$$

Thus, $\sigma_{pq} = \beta_{pq}$. Hence $(\mu \tilde{\oplus} \nu)^c = \mu^c \tilde{\odot} \nu^c$.

2. The proof can be proved by similar way to proof of 1.
3. Let $\alpha_{pq} = \langle \alpha^t_{pq}, \alpha^i_{pq}, \alpha^f_{pq} \rangle$ and $\beta_{pq} = \langle \beta^t_{pq}, \beta^i_{pq}, \beta^f_{pq} \rangle$ be pqth elements of $\mu \tilde{\ominus} \nu$ and $\mu^c \tilde{\ominus} \nu^c$, respectively, and let $\sigma_{pq} = \alpha^c_{pq}$. Then,

$$\begin{aligned}\alpha_{pq} &= \langle \alpha^t_{pq}, \alpha^i_{pq}, \alpha^f_{pq} \rangle \\ &= \left\langle \begin{cases} \mu^t_{pq}, & \mu^t_{pq} > \nu^t_{pq} \\ 0, & \mu^t_{pq} \le \nu^t_{pq} \end{cases}, \begin{cases} 1, & \mu^i_{pq} \ge \nu^i_{pq} \\ \mu^i_{pq}, & \mu^i_{pq} < \nu^i_{pq} \end{cases}, \begin{cases} 1, & \mu^f_{pq} \ge \nu^f_{pq} \\ \mu^t_{pq}, & \mu^f_{pq} < \nu^f_{pq} \end{cases} \right\rangle \\ \sigma_{pq} &= \alpha^c_{pq} = \left\langle \begin{cases} 1, & \mu^f_{pq} \ge \nu^f_{pq} \\ \mu^t_{pq}, & \mu^f_{pq} < \nu^f_{pq} \end{cases}, \begin{cases} 0, & \mu^i_{pq} \ge \nu^i_{pq} \\ 1-\mu^i_{pq}, & \mu^i_{pq} < \nu^i_{pq} \end{cases}, \begin{cases} \mu^t_{pq}, & \mu^t_{pq} > \nu^t_{pq} \\ 0, & \mu^t_{pq} \le \nu^t_{pq} \end{cases} \right\rangle \\ \beta_{pq} &= \left\langle \mu^f_{pq}, 1-\mu^i_{pq}, \mu^t_{pq} \right\rangle \tilde{\ominus} \left\langle \nu^f_{pq}, 1-\nu^i_{pq}, \nu^t_{pq} \right\rangle \\ &= \left\langle \begin{cases} \mu^f_{pq}, & \mu^f_{pq} \ge \nu^f_{pq} \\ 0, & \mu^f_{pq} < \nu^f_{pq} \end{cases}, \begin{cases} 1-\mu^i_{pq}, & \mu^i_{pq} > \nu^i_{pq} \\ 1 & \mu^i_{pq} \le \nu^i_{pq} \end{cases}, \begin{cases} 1, & \mu^t_{pq} \ge \nu^t_{pq} \\ \mu^t_{pq}, & \mu^t_{pq} < \nu^t_{pq} \end{cases} \right\rangle.\end{aligned}$$

For $\mu^f_{pq} \ge \nu^f_{pq}$, since $\mu^f_{pq} \le 1$, $1-\mu^i_{pq} \ge 0$, $1 \ge \mu^t_{pq}$ and $0 \le \mu^f_{pq}$, $1 \ge 1-\mu^i_{pq}$, $\mu^i_{pq} \ge 0$, $\sigma_{pq} \succeq \beta_{pq}$. Thus $(\mu \tilde{\ominus} \nu)^c \tilde{\succeq} \mu^c \tilde{\ominus} \nu^c$.

A.4

1. Let $\alpha \preceq \beta$. Then $\alpha^t \le \beta^t$, $\alpha^i \ge \beta^i$ and $\alpha^f \ge \beta^f$. The proof will be made each of μ^t_{ij}, μ^i_{pq} and μ^f_{pq}. Let $\mu^t_{pq} \ge \beta^t$. Then $(\mu^t_{pq})^{\beta^t} = 1$. Since $\beta^t \ge \alpha^t$, $\mu^t_{pq} \ge \alpha^t$, $(\mu^t_{pq})^{\alpha^t} = 1$. Therefore $(\mu^t_{pq})^{\alpha^t} = (\mu^t_{pq})^{\beta^t}$. If $\mu^t_{pq} < \beta^t$, there are two cases.

 Case 1: If $\mu^t_{pq} < \alpha^t \le \beta^t$, then $(\mu^t_{pq})^{\beta^t} = 0 = (\mu^t_{pq})^{\alpha^t}$.
 Case 2: If $\alpha^t \le \mu^t_{pq} < \beta^t$, then $(\mu^t_{pq})^{\beta^t} = 0 < 1 = (\mu^t_{pq})^{\alpha^t}$. Thus, $(\mu^t_{pq})^{\beta^t} \le (\mu^t_{pq})^{\alpha^t}$.

 Let $\mu^i_{pq} \ge \beta^i$. Then, there are two cases;

 Case 1: If $\mu^i_{pq} \ge \alpha^i \ge \beta^i$, then $(\mu^i_{pq})^{\alpha^i} = (\mu^i_{pq})^{\beta^i} = 1$.
 Case 2: If $\alpha^i \ge \mu^i_{pq} \ge \beta^i$, then $(\mu^i_{pq})^{\alpha^i} = 0 < (\mu^i_{pq})^{\beta^i} = 1$. Also if $\mu^i_{pq} < \beta^i$, then $(\mu^i_{pq})^{\beta^i} = 0$. Since $\beta^i \le \alpha^i$, $\mu^i_{pq} \le \alpha^i$, $(\mu^i_{pq})^{\alpha^i} = 0$. Thus, $(\mu^i_{pq})^{\beta^i} \ge (\mu^i_{pq})^{\alpha^i}$.

 Let $\mu^f_{pq} \ge \beta^f$. Then, there are two cases;

 Case 1: If $\mu^f_{pq} \ge \alpha^f \ge \beta^f$, then $(\mu^f_{pq})^{\alpha^f} = (\mu^f_{pq})^{\beta^f} = 1$.
 Case 2: If $\alpha^f \ge \mu^f_{pq} \ge \beta^f$, then $(\mu^f_{pq})^{\alpha^f} = 0 < (\mu^f_{pq})^{\beta^f} = 1$. Also if $\mu^f_{pq} < \beta^f$, then $(\mu^f_{pq})^{\beta^f} = 0$. Since $\beta^f \le \alpha^f$, $\mu^f_{pq} \le \alpha^f$, $(\mu^f_{pq})^{\alpha^f} = 0$. Thus, $(\mu^f_{pq})^{\beta^f} \ge (\mu^f_{pq})^{\alpha^f}$.

 It is concluded that $\mu^{(\beta)} \tilde{\preceq} \mu^{(\alpha)}$.
2. The proof can be made by similar way to proof of statement 1.

A.5

1. The proof will be made for truth, indeterminacy and falsity membership values of elements of SVN-matrices:

 Case 1: For truth-membership values:

 Subcase 1: Let $\mu^t_{pq}, \nu^t_{pq} \ge \alpha^t$. Then, $\mu^t_{pq} \oplus \nu^t_{pq} = \mu^t_{pq} + \nu^t_{pq} - \mu^t_{pq}\nu^t_{pq} \ge \mu^t_{pq}$ and ν^t_{pq}. Therefore $(\mu^t_{pq} \oplus \nu^t_{pq}) \ge \alpha^t$ and so $(\mu^t_{pq} \oplus \nu^t_{pq}) = 1$. Also, since $(\mu^t_{pq})^{\alpha} = 1$ and $(\nu^t_{pq})^{\alpha} = 1$, $(\mu^t_{pq})^{\alpha^t} \oplus (\nu^t_{pq})^{\alpha^t} = 1$. Thus, $(\mu^t_{pq} \oplus \nu^t_{pq})^{\alpha^t} = (\mu^t_{pq})^{\alpha^t} \oplus (\nu^t_{pq})^{\alpha^t}$.

Subcase 2: Let $\mu^t_{pq}, \nu^t_{pq} < \alpha^t$. Then, there are two cases: $\mu^t_{pq} \oplus \nu^t_{pq} < \alpha^t$ or $\mu^t_{pq} \oplus \nu^t_{pq} \geq \alpha^t$.

1. If $\mu^t_{pq} \oplus \nu^t_{pq} < \alpha^t$, $(\mu^t_{pq} \oplus \nu^t_{pq})^{\alpha^t} = 0$. Since $(\mu^t_{pq})^{\alpha^t} = 0$ and $(\nu^t_{pq})^{\alpha^t} = 0$, $(\mu^t_{pq})^{\alpha^t} \oplus (\nu^t_{pq})^{\alpha^t} = 0$. Hence $(\mu^t_{pq} \oplus \nu^t_{pq})^{\alpha^t} = (\mu^t_{pq})^{\alpha^t} \oplus (\nu^t_{pq})^{\alpha^t}$.
2. If $\mu^t_{pq} \oplus \nu^t_{pq} \geq \alpha^t$, $(\mu^t_{pq} \oplus \nu^t_{pq})^{\alpha^t} = 1$. Since $(\mu^t_{pq})^{\alpha^t} = 0$ and $(\nu^t_{pq})^{\alpha^t} = 0$, $(\mu^t_{pq})^{\alpha^t} \oplus (\nu^t_{pq})^{\alpha^t} = 0$. Thus $(\mu^t_{pq} \oplus \nu^t_{pq})^{\alpha^t} \geq (\mu^t_{pq})^{\alpha^t} \oplus (\nu^t_{pq})^{\alpha^t}$.

Subcase 3: Let $\mu^t_{pq} < \alpha^t$ and $\nu^t_{pq} \geq \alpha^t$. Then $\mu^t_{pq} \oplus \nu^t_{pq} = \mu^t_{pq} + \nu^t_{pq} - \mu^t_{pq}\nu^t_{pq} \geq \alpha^t$. Therefore $(\mu^t_{pq} \oplus \nu^t_{pq})^{\alpha^t} = 1$. Also, since $(\mu^t_{pq})^{\alpha^t} = 0$ and $(\nu^t_{pq})^{\alpha^t} = 1$, $(\mu^t_{pq})^{\alpha^t} \oplus (\nu^t_{pq})^{\alpha^t} = 1$. Thus, $(\mu^t_{pq} \oplus \nu^t_{pq})^{\alpha^t} = (\mu^t_{pq})^{\alpha^t} \oplus (\nu^t_{pq})^{\alpha^t}$.
Subcase 4: Let $\mu^t_{pq} \geq \alpha^t$ and $\nu^t_{pq} < \alpha^t$. Then $\mu^t_{pq} \oplus \nu^t_{pq} = \mu^t_{pq} + \nu^t_{pq} - \mu^t_{pq}\nu^t_{pq} \geq \alpha^t$. Therefore $(\mu^t_{pq} \oplus \nu^t_{pq})^{\alpha^t} = 1$. Also, since $(\mu^t_{pq})^{\alpha^t} = 1$ and $(\nu^t_{pq})^{\alpha^t} = 0$, $(\mu^t_{pq})^{\alpha^t} \oplus (\nu^t_{pq})^{\alpha^t} = 1$. Thus, $(\mu^t_{pq} \oplus \nu^t_{pq})^{\alpha^t} = (\mu^t_{pq})^{\alpha^t} \oplus (\nu^t_{pq})^{\alpha^t}$.

Case 2: For indeterminacy-membership values:

Subcase 1: Let $\mu^i_{pq}, \nu^i_{pq} > \alpha^i$. Then, there are two cases: $\mu^i_{pq} \oplus \nu^i_{pq} = \mu^i_{pq}\nu^i_{pq} \leq \alpha^i$ or $\mu^i_{pq} \oplus \nu^i_{pq} = \mu^i_{pq}\nu^i_{pq} > \alpha^i$.

1. If $\mu^i_{pq} \oplus \nu^i_{pq} = \mu^i_{pq}\nu^i_{pq} \leq \alpha^i$, then $(\mu^i_{pq} \oplus \nu^i_{pq})^{\alpha^i} = 0$. Since $(\mu^i_{pq})^{\alpha^i} = 1$ and $(\nu^i_{pq})^{\alpha^i} = 1$, $(\mu^i_{pq})^{\alpha^i} \oplus (\nu^i_{pq})^{\alpha^i} = 1$. Hence $(\mu^i_{pq} \oplus \nu^i_{pq})^{\alpha^i} \leq (\mu^i_{pq})^{\alpha^i} \oplus (\nu^i_{pq})^{\alpha^i}$.
2. If $\mu^i_{pq} \oplus \nu^i_{pq} = \mu^i_{pq}\nu^i_{pq} > \alpha^i$, $(\mu^i_{pq} \oplus \nu^i_{pq})^{\alpha^i} = 1$. Since $(\mu^i_{pq})^{\alpha^i} = 0$ and $(\nu^i_{pq})^{\alpha^i} = 0$, $(\mu^i_{pq})^{\alpha^i} \oplus (\nu^i_{pq})^{\alpha^i} = 1$. Thus $(\mu^i_{pq} \oplus \nu^i_{pq})^{\alpha^i} = (\mu^i_{pq})^{\alpha^i} \oplus (\nu^i_{pq})^{\alpha^i}$.

Subcase 2: Let $\mu^i_{pq}, \nu^i_{pq} \leq \alpha^i$. Then, $\mu^i_{pq} \oplus \nu^i_{pq} = \mu^i_{pq}\nu^i_{pq} \leq \alpha^i$ and so $(\mu^i_{pq} \oplus \nu^i_{pq})^{\alpha^i} = 0$. Since $(\mu^i_{pq})^{\alpha^i} = 0$ and $(\nu^i_{pq})^{\alpha^i} = 0$, $(\mu^i_{pq})^{\alpha^i} \oplus (\nu^i_{pq})^{\alpha^i} = 0$. Thus $(\mu^i_{pq} \oplus \nu^i_{pq})^{\alpha^i} = (\mu^i_{pq})^{\alpha^i} \oplus (\nu^i_{pq})^{\alpha^i}$.

Subcase 3: Let $\mu^i_{pq} \leq \alpha^i$ and $\nu^i_{pq} > \alpha^i$. If $\mu^i_{pq} = 0$, Then $\mu^i_{pq} \oplus \nu^i_{pq} = \mu^i_{pq}\nu^i_{pq} = 0 \leq \alpha^i$. Therefore $(\mu^i_{pq} \oplus \nu^i_{pq})^{\alpha^i} = 0$. Since $(\mu^i_{pq})^{\alpha^i} = 0$ and $(\nu^i_{pq})^{\alpha^i} = 1$, $(\mu^i_{pq})^{\alpha^i} \oplus (\nu^i_{pq})^{\alpha^i} = 0$. Thus, $(\mu^i_{pq} \oplus \nu^i_{pq})^{\alpha^i} = (\mu^i_{pq})^{\alpha^i} \oplus (\nu^i_{pq})^{\alpha^i}$. If $\mu^i_{pq} \neq 0$, then $\mu^i_{pq} \oplus \nu^i_{pq} = \mu^i_{pq}\nu^i_{pq} \leq \alpha^i$ and so $(\mu^i_{pq} \oplus \nu^i_{pq})^{\alpha^i} = 0$. Since $(\mu^i_{pq})^{\alpha^i} = 0$ and $(\nu^i_{pq})^{\alpha^i} = 1$, $(\mu^i_{pq})^{\alpha^i} \oplus (\nu^i_{pq})^{\alpha^i} = 0$. Thus, $(\mu^i_{pq} \oplus \nu^i_{pq})^{\alpha^i} = (\mu^i_{pq})^{\alpha^i} \oplus (\nu^i_{pq})^{\alpha^i}$.

Subcase 4: Let $\mu^i_{pq} > \alpha^i$ and $\nu^i_{pq} \leq \alpha^i$. If $\nu^i_{pq} = 0$, Then $\mu^i_{pq} \oplus \nu^i_{pq} = \mu^i_{pq}\nu^i_{pq} = 0 \leq \alpha^i$. Therefore $(\mu^i_{pq} \oplus \nu^i_{pq})^{\alpha^i} = 0$. Since $(\mu^i_{pq})^{\alpha^i} = 1$ and $(\nu^i_{pq})^{\alpha^i} = 0$, $(\mu^i_{pq})^{\alpha^i} \oplus (\nu^i_{pq})^{\alpha^i} = 0$. Thus, $(\mu^i_{pq} \oplus \nu^i_{pq})^{\alpha^i} = (\mu^i_{pq})^{\alpha^i} \oplus (\nu^i_{pq})^{\alpha^i}$. If $\nu^i_{pq} \neq 0$, then $\mu^i_{pq} \oplus \nu^i_{pq} = \mu^i_{pq}\nu^i_{pq} \leq \alpha^i$ and so $(\mu^i_{pq} \oplus \nu^i_{pq})^{\alpha^i} = 0$. Since $(\mu^i_{pq})^{\alpha^i} = 0$ and $(\nu^i_{pq})^{\alpha^i} = 1$, $(\mu^i_{pq})^{\alpha^i} \oplus (\nu^i_{pq})^{\alpha^i} = 0$. Thus, $(\mu^i_{pq} \oplus \nu^i_{pq})^{\alpha^i} = (\mu^i_{pq})^{\alpha^i} \oplus (\nu^i_{pq})^{\alpha^i}$.

Case 3: For indeterminacy-membership values: The proof can be made by similar way to proof of case 2.

When it is considered all of the cases, it is concluded that $(\mu \tilde{\oplus} \nu)^{(\alpha)} \tilde{\succeq} \mu^{(\alpha)} \tilde{\oplus} \nu^{(\alpha)}$.

2. The proof can be made by similar way to proof of statement 1.

A.6

1. The proof will be made for truth, indeterminacy and falsity-memberships values of elements of SVN-matrices.

1. For truth-membership values: There are four cases,
Case 1: $\mu^t_{pq}, \nu^t_{pq} \geq \alpha^t$.
Subcase 1: Let $\mu^t_{pq} > \nu^t_{pq}$. Then $\mu^t_{pq} \ominus \nu^t_{pq} = \mu^t_{pq}$. Since $\mu^t_{pq} \geq \alpha^t$, $(\mu^t_{pq} \ominus \nu^t_{pq})^{\alpha^t} = 1$. Also since $(\mu^t_{pq})^{\alpha^t} = 1$ and $(\nu^t_{pq})^{\alpha^t} = 1$, $(\mu^t_{pq})^{\alpha^t} \ominus (\nu^t_{pq})^{\alpha^t} = 0$. Therefore $(\mu^t_{pq} \ominus \nu^t_{pq})^{\alpha^t} \geq (\mu^t_{pq})^{\alpha^t} \ominus (\nu^t_{pq})^{\alpha^t}$.
Subcase 2: Let $\mu^t_{pq} \leq \nu^t_{pq}$. Then $\mu^t_{pq} \ominus \nu^t_{pq} = 0$. Since $0 \leq \alpha^t$, $(\mu^t_{pq} \ominus \nu^t_{pq})^{\alpha^t} = 0$ or 1. Also since $(\mu^t_{pq})^{\alpha^t} = 1$ and $(\nu^t_{pq})^{\alpha^t} = 1$, $(\mu^t_{pq})^{\alpha^t} \ominus (\nu^t_{pq})^{\alpha^t} = 0$. Therefore $(\mu^t_{pq} \ominus \nu^t_{pq})^{\alpha^t} \geq (\mu^t_{pq})^{\alpha^t} \ominus (\nu^t_{pq})^{\alpha^t}$.
Case 2: $\mu^t_{pq}, \nu^t_{pq} < \alpha^t$.
Subcase 1: Let $\mu^t_{pq} > \nu^t_{pq}$. Then $\mu^t_{pq} \ominus \nu^t_{pq} = \mu^t_{pq}$. Since $\mu^t_{pq} \geq \alpha^t$, $(\mu^t_{pq} \ominus \nu^t_{pq})^{\alpha^t} = 1$. Also since $(\mu^t_{pq})^{\alpha^t} = 0$ and $(\nu^t_{pq})^{\alpha^t} = 0$, $(\mu^t_{pq})^{\alpha^t} \ominus (\nu^t_{pq})^{\alpha^t} = 0$. Therefore $(\mu^t_{pq} \ominus \nu^t_{pq})^{\alpha^t} \geq (\mu^t_{pq})^{\alpha^t} \ominus (\nu^t_{pq})^{\alpha^t}$.
Subcase 2: Let $\mu^t_{pq} \leq \nu^t_{pq}$. Then $\mu^t_{pq} \ominus \nu^t_{pq} = 0$. Since $0 \leq \alpha^t$, $(\mu^t_{pq} \ominus \nu^t_{pq})^{\alpha^t} = 0$ or 1. Also since $(\mu^t_{pq})^{\alpha^t} = 0$ and $(\nu^t_{pq})^{\alpha^t} = 0$, $(\mu^t_{pq})^{\alpha^t} \ominus (\nu^t_{pq})^{\alpha^t} = 0$. Therefore, $(\mu^t_{pq} \ominus \nu^t_{pq})^{\alpha^t} \geq (\mu^t_{pq})^{\alpha^t} \ominus (\nu^t_{pq})^{\alpha^t}$.
Case 3: Let $\mu^t_{pq} < \alpha^t$ and $\nu^t_{pq} \geq \alpha^t$. Then, $\mu^t_{pq} < \nu^t_{pq}$ and $(\mu^t_{pq} \ominus \nu^t_{pq})^{\alpha^t} = 0$. Also, since $(\mu^t_{pq})^{\alpha^t} = 0$ and $(\nu^t_{pq})^{\alpha^t} = 1$, $(\mu^t_{pq})^{\alpha^t} \ominus (\nu^t_{pq})^{\alpha^t} = 0 \ominus 1 = 0$. Therefore $(\mu^t_{pq} \ominus \nu^t_{pq})^{\alpha^t} = (\mu^t_{pq})^{\alpha^t} \ominus (\nu^t_{pq})^{\alpha^t}$.
Case 4: Let $\mu^t_{pq} \geq \alpha^t$ and $\nu^t_{pq} < \alpha^t$. Then, $\mu^t_{pq} > \nu^t_{pq}$ and $(\mu^t_{pq} \ominus \nu^t_{pq}) = \mu^t_{pq}$. Since $\mu^t_{pq} \geq \alpha^t$, $(\mu^t_{pq} \ominus \nu^t_{pq})^{\alpha^t} = 1$. Also, since $(\mu^t_{pq})^{\alpha^t} = 1$ and

$(\nu^t_{pq})^{\alpha^t} = 0$, $(\mu^t_{pq})^{\alpha^t} \ominus (\nu^t_{pq})^{\alpha^t} = 1 \ominus 0 = 1$. Therefore $(\mu^t_{pq} \ominus \nu^t_{pq})^{\alpha^t} = (\mu^t_{pq})^{\alpha^t} \ominus (\nu^t_{pq})^{\alpha^t}$.
For indeterminacy-membership values: There are four cases,
Case 1: $\mu^i_{pq}, \nu^i_{pq} > \alpha^i$.
Subcase 1: Let $\mu^i_{pq} \geq \nu^i_{pq}$. Then $\mu^i_{pq} \ominus \nu^i_{pq} = 1$. Since $1 \geq \alpha^i$, $(\mu^i_{pq} \ominus \nu^i_{pq})^{\alpha^i} = 1 \; or \; 0$. Also since $(\mu^i_{pq})^{\alpha^i} = 1$ and $(\nu^i_{pq})^{\alpha^i} = 1$, $(\mu^i_{pq})^{\alpha^i} \ominus (\nu^i_{pq})^{\alpha^i} = 1$. Therefore $(\mu^i_{pq} \ominus \nu^i_{pq})^{\alpha^i} \leq (\mu^i_{pq})^{\alpha^i} \ominus (\nu^i_{pq})^{\alpha^i}$.
Subcase 2: Let $\mu^i_{pq} < \nu^i_{pq}$. Then $\mu^i_{pq} \ominus \nu^i_{pq} = \mu^i_{pq}$. Since $\mu^i_{pq} > \alpha^i$, $(\mu^i_{pq} \ominus \nu^i_{pq})^{\alpha^i} = 1$. Also since $(\mu^i_{pq})^{\alpha^i} = 1$ and $(\nu^i_{pq})^{\alpha^i} = 1$, $(\mu^i_{pq})^{\alpha^i} \ominus (\nu^i_{pq})^{\alpha^i} = 1$. Therefore $(\mu^i_{pq} \ominus \nu^i_{pq})^{\alpha^i} = (\mu^i_{pq})^{\alpha^i} \ominus (\nu^i_{pq})^{\alpha^i}$.
Case 2: $\mu^i_{pq}, \nu^i_{pq} \leq \alpha^i$.
Subcase 1: Let $\mu^i_{pq} \geq \nu^i_{pq}$. Then $\mu^i_{pq} \ominus \nu^i_{pq} = 1$. Since $\mu^i_{pq} \leq \alpha^i$, $(\mu^i_{pq} \ominus \nu^i_{pq})^{\alpha^i} = 0$. Also since $(\mu^i_{pq})^{\alpha^i} = 0$ and $(\nu^i_{pq})^{\alpha^i} = 0$, $(\mu^i_{pq})^{\alpha^i} \ominus (\nu^i_{pq})^{\alpha^i} = 1$. Hence, $(\mu^i_{pq} \ominus \nu^i_{pq})^{\alpha^i} \leq (\mu^i_{pq})^{\alpha^i} \ominus (\nu^i_{pq})^{\alpha^i}$.
Subcase 2: Let $\mu^i_{pq} < \nu^i_{pq}$. Then $\mu^i_{pq} \ominus \nu^i_{pq} = \mu^i_{pq}$. Since $\mu^i_{pq} \leq \alpha^i$, $(\mu^i_{pq} \ominus \nu^i_{pq})^{\alpha^i} = 0$. Also since $(\mu^i_{pq})^{\alpha^i} = 0$ and $(\nu^i_{pq})^{\alpha^i} = 0$, $(\mu^i_{pq})^{\alpha^i} \ominus (\nu^i_{pq})^{\alpha^i} = 1$. Therefore $(\mu^i_{pq} \ominus \nu^i_{pq})^{\alpha^i} \leq (\mu^i_{pq})^{\alpha^i} \ominus (\nu^i_{pq})^{\alpha^i}$.
Case 3: Let $\mu^i_{pq} \leq \alpha^i$ and $\nu^i_{pq} > \alpha^i$. Then, $\mu^i_{pq} < \nu^i_{pq}$ and $\mu^i_{pq} \ominus \nu^i_{pq} = \mu^i_{pq}$. Since $\mu^i_{pq} \leq \alpha^i$, $(\mu^i_{pq} \ominus \nu^i_{pq})^{\alpha^i} = 0$. Also, since $(\mu^i_{pq})^{\alpha^i} = 0$ and $(\nu^i_{pq})^{\alpha^i} = 1$, $(\mu^i_{pq})^{\alpha^i} \ominus (\nu^i_{pq})^{\alpha^i} = 0 \ominus 1 = 0$. Thus, $(\mu^t_{pq} \ominus \nu^t_{pq})^{\alpha^t} = (\mu^t_{pq})^{\alpha^t} \ominus (\nu^t_{pq})^{\alpha^t}$.
Case 4: Let $\mu^i_{pq} > \alpha^i$ and $\nu^i_{pq} \leq \alpha^i$. Then, $\mu^i_{pq} > \nu^i_{pq}$ and $(\mu^i_{pq} \ominus \nu^i_{pq}) = 1$. Since $1 \geq \alpha^t$, $(\mu^i_{pq} \ominus \nu^i_{pq})^{\alpha^i} = 1$ or 0. Also, since $(\mu^i_{pq})^{\alpha^i} = 1$ and $(\nu^i_{pq})^{\alpha^i} = 0$, $(\mu^i_{pq})^{\alpha^i} \ominus (\nu^i_{pq})^{\alpha^i} = 1 \ominus 0 = 1$. Therefore $(\mu^i_{pq} \ominus \nu^i_{pq})^{\alpha^i} \leq (\mu^i_{pq})^{\alpha^i} \ominus (\nu^i_{pq})^{\alpha^i}$.
For falsity-membership values: The proof can be made in similar way to the proof made for indeterminacy-membership values. When it is considered all of cases, it is concluded that $(\mu \tilde{\ominus} \nu)^{(\alpha)} \tilde{\succeq} (\mu)^{(\alpha)} \tilde{\ominus} (\nu)^{(\alpha)}$.
The proof will be made for truth, indeterminacy and falsity-memberships values of elements of SVN-matrices.
1. For truth-membership values: There are four cases,
Case 1: $\mu^t_{pq}, \nu^t_{pq} \geq \alpha^t$.
Subcase 1: Let $\mu^t_{pq} > \nu^t_{pq}$. Then $\mu^t_{pq} \ominus \nu^t_{pq} = \mu^t_{pq}$. Since $\mu^t_{pq} \geq \alpha^t$, $(\mu^t_{pq} \ominus \nu^t_{pq})_{\alpha^t} = \mu^t_{pq}$. Also since $(\mu^t_{pq})_{\alpha^t} = \mu^t_{pq}$ and $(\nu^t_{pq})_{\alpha^t} = \mu^t_{pq}$, $(\mu^t_{pq})_{\alpha^t} \ominus (\nu^t_{pq})_{\alpha^t} = 0$. Therefore, $(\mu^t_{pq} \ominus \nu^t_{pq})_{\alpha^t} \geq (\mu^t_{pq})_{\alpha^t} \ominus (\nu^t_{pq})_{\alpha^t}$.
Subcase 2: Let $\mu^t_{pq} \leq \nu^t_{pq}$. Then $\mu^t_{pq} \ominus \nu^t_{pq} = 0$. Since $0 \leq \alpha^t$, $(\mu^t_{pq} \ominus \nu^t_{pq})_{\alpha^t} = 0$ or μ^t_{pq}. Also since $(\mu^t_{pq})_{\alpha^t} = \mu^t_{pq}$ and $(\nu^t_{pq})_{\alpha^t} = \mu^t_{pq}$, $(\mu^t_{pq})_{\alpha^t} \ominus (\nu^t_{pq})_{\alpha^t} = 0$. Therefore, $(\mu^t_{pq} \ominus \nu^t_{pq})_{\alpha^t} \geq (\mu^t_{pq})_{\alpha^t} \ominus (\nu^t_{pq})_{\alpha^t}$.
Case 2: $\mu^t_{pq}, \nu^t_{pq} < \alpha^t$.
Subcase 1: Let $\mu^t_{pq} > \nu^t_{pq}$. Then, $\mu^t_{pq} \ominus \nu^t_{pq} = \mu^t_{pq}$. Since $\mu^t_{pq} \geq \alpha^t$, $(\mu^t_{pq} \ominus \nu^t_{pq})_{\alpha^t} = \mu^t_{pq}$. Also since $(\mu^t_{pq})_{\alpha^t} = 0$ and $(\nu^t_{pq})_{\alpha^t} = 0$, $(\mu^t_{pq})_{\alpha^t} \ominus (\nu^t_{pq})_{\alpha^t} = 0$. Therefore $(\mu^t_{pq} \ominus \nu^t_{pq})_{\alpha^t} \geq (\mu^t_{pq})_{\alpha^t} \ominus (\nu^t_{pq})_{\alpha^t}$.
Subcase 2: Let $\mu^t_{pq} \leq \nu^t_{pq}$. Then $\mu^t_{pq} \ominus \nu^t_{pq} = 0$. Since $0 \leq \alpha^t$, $(\mu^t_{pq} \ominus \nu^t_{pq})_{\alpha^t} = 0$ or μ^t_{pq}. Also since $(\mu^t_{pq})_{\alpha^t} = 0$ and $(\nu^t_{pq})_{\alpha^t} = 0$, $(\mu^t_{pq})_{\alpha^t} \ominus (\nu^t_{pq})_{\alpha^t} = 0$. Therefore $(\mu^t_{pq} \ominus \nu^t_{pq})_{\alpha^t} \geq (\mu^t_{pq})_{\alpha^t} \ominus (\nu^t_{pq})_{\alpha^t}$.
Case 3: Let $\mu^t_{pq} < \alpha^t$ and $\nu^t_{pq} \geq \alpha^t$. Then, $\mu^t_{pq} < \nu^t_{pq}$ and $(\mu^t_{pq} \ominus \nu^t_{pq}) = 0$. Since $0 \leq \alpha^t$, $(\mu^t_{pq} \ominus \nu^t_{pq})_{\alpha^t} = 0$ or μ^t_{pq}. Also, since $(\mu^t_{pq})_{\alpha^t} = 0$ and $(\nu^t_{pq})_{\alpha^t} = \nu^t_{pq}$, $(\mu^t_{pq})_{\alpha^t} \ominus (\nu^t_{pq})_{\alpha^t} = 0 \ominus \nu^t_{pq} = 0$. Therefore $(\mu^t_{pq} \ominus \nu^t_{pq})_{\alpha^t} \geq (\mu^t_{pq})_{\alpha^t} \ominus (\nu^t_{pq})_{\alpha^t}$.
Case 4: Let $\mu^t_{pq} \geq \alpha^t$ and $\nu^t_{pq} < \alpha^t$. Then, $\mu^t_{pq} > \nu^t_{pq}$ and $(\mu^t_{pq} \ominus \nu^t_{pq}) = \mu^t_{pq}$. Since $\mu^t_{pq} \geq \alpha^t$, $(\mu^t_{pq} \ominus \nu^t_{pq})_{\alpha^t} = \mu^t_{pq}$. Also, since $(\mu^t_{pq})_{\alpha^t} = \mu^t_{pq}$ and $(\nu^t_{pq})_{\alpha^t} = 0$, $(\mu^t_{pq})_{\alpha^t} \ominus (\nu^t_{pq})_{\alpha^t} = \mu^t_{pq} \ominus 0 = \mu^t_{pq}$. Therefore $(\mu^t_{pq} \ominus \nu^t_{pq})_{\alpha^t} = (\mu^t_{pq})_{\alpha^t} \ominus (\nu^t_{pq})_{\alpha^t}$.

2. For indeterminacy-membership values: There are four cases,

Case 1: $\mu^i_{pq}, \nu^i_{pq} > \alpha^i$.
Subcase 1: Let $\mu^i_{pq} \geq \nu^i_{pq}$. Then $\mu^i_{pq} \ominus \nu^i_{pq} = 1$. Since $1 \geq \alpha^i$, $(\mu^i_{pq} \ominus \nu^i_{pq})_{\alpha^i} = 1$ or μ^i_{pq}. Also since $(\mu^i_{pq})_{\alpha^i} = 1$ and $(\nu^i_{pq})_{\alpha^i} = 1$, $(\mu^i_{pq})_{\alpha^i} \ominus (\nu^i_{pq})_{\alpha^i} = 1$. Therefore $(\mu^i_{pq} \ominus \nu^i_{pq})_{\alpha^i} \leq (\mu^i_{pq})_{\alpha^i} \ominus (\nu^i_{pq})_{\alpha^i}$.
Subcase 2: Let $\mu^i_{pq} < \nu^i_{pq}$. Then $\mu^i_{pq} \ominus \nu^i_{pq} = \mu^i_{pq}$. Since $\mu^i_{pq} > \alpha^i$, $(\mu^i_{pq} \ominus \nu^i_{pq})_{\alpha^i} = 1$. Also since $(\mu^i_{pq})_{\alpha^i} = 1$ and $(\nu^i_{pq})_{\alpha^i} = 1$, $(\mu^i_{pq})_{\alpha^i} \ominus (\nu^i_{pq})_{\alpha^i} = 1$. Therefore $(\mu^i_{pq} \ominus \nu^i_{pq})_{\alpha^i} = (\mu^i_{pq})_{\alpha^i} \ominus (\nu^i_{pq})_{\alpha^i}$.
Case 2: $\mu^i_{pq}, \nu^i_{pq} \leq \alpha^i$.
Subcase 1: Let $\mu^i_{pq} \geq \nu^i_{pq}$. Then $\mu^i_{pq} \ominus \nu^i_{pq} = 1$. Since $1 \geq \alpha^i$, $(\mu^i_{pq} \ominus \nu^i_{pq})_{\alpha^i} = 1$ or μ^i_{pq}. Also since $(\mu^i_{pq})_{\alpha^i} = \mu^i_{pq}$ and $(\nu^i_{pq})_{\alpha^i} = \nu^i_{pq}$, $(\mu^i_{pq})_{\alpha^i} \ominus (\nu^i_{pq})_{\alpha^i} = \mu^i_{pq} \ominus \nu^i_{pq} = 1$. Therefore $(\mu^i_{pq} \ominus \nu^i_{pq})_{\alpha^i} \leq (\mu^i_{pq})_{\alpha^i} \ominus (\nu^i_{pq})_{\alpha^i}$.
Subcase 2: Let $\mu^i_{pq} < \nu^i_{pq}$. Then $\mu^i_{pq} \ominus \nu^i_{pq} = \mu^i_{pq}$. Since $\mu^i_{pq} \leq \alpha^i$, $(\mu^i_{pq} \ominus \nu^i_{pq})_{\alpha^i} = \mu^i_{pq}$. Also since $(\mu^i_{pq})_{\alpha^i} = \mu^i_{pq}$ and $(\nu^i_{pq})_{\alpha^i} = \nu^i_{pq}$, $(\mu^i_{pq})_{\alpha^i} \ominus (\nu^i_{pq})_{\alpha^i} = \mu^i_{pq} \ominus \nu^i_{pq} = \mu^i_{pq}$. Therefore $(\mu^i_{pq} \ominus \nu^i_{pq})_{\alpha^i} = (\mu^i_{pq})_{\alpha^i} \ominus (\nu^i_{pq})_{\alpha^i}$.
Case 3: Let $\mu^i_{pq} \leq \alpha^i$ and $\nu^i_{pq} > \alpha^i$. Then, $\mu^i_{pq} < \nu^i_{pq}$ and $\mu^i_{pq} \ominus \nu^i_{pq} = \mu^i_{pq}$. Since $\mu^i_{pq} \leq \alpha^i$, $(\mu^i_{pq} \ominus \nu^i_{pq})_{\alpha^i} = \mu^i_{pq}$. Also, since $(\mu^i_{pq})_{\alpha^i} = \mu^i_{pq}$

and $(\nu^i_{pq})_{\alpha^i} = 1$, $(\mu^i_{pq})_{\alpha^i}\ominus(\nu^i_{pq})_{\alpha^i} = \mu^i_{pq}\ominus 1 = 1$ or μ^i_{pq}. Therefore $(\mu^t_{pq}\ominus\nu^t_{pq})_{\alpha^t} \leq (\mu^t_{pq})_{\alpha^t}\ominus(\nu^t_{pq})_{\alpha^t}$.
Case 4: Let $\mu^i_{pq} > \alpha^i$ and $\nu^i_{pq} \leq \alpha^i$. Then, $\mu^i_{pq} > \nu^i_{pq}$ and $(\mu^i_{pq}\ominus\nu^i_{pq}) = 1$. Since $1 \geq \alpha^t$, $(\mu^i_{pq}\ominus\nu^i_{pq})_{\alpha^i} = 1$ or μ^i_{pq}. Also, since $(\mu^i_{pq})_{\alpha^i} = 1$ and $(\nu^i_{pq})_{\alpha^i} = \nu^i_{pq}$, $(\mu^i_{pq})_{\alpha^i}\ominus(\nu^i_{pq})_{\alpha^i} = 1\ominus\nu^i_{pq} = 1$. Therefore $(\mu^i_{pq}\ominus\nu^i_{pq})_{\alpha^i} = (\mu^i_{pq})_{\alpha^i}\ominus(\nu^i_{pq})_{\alpha^i}$.
3. For falsity-membership values: The proof can be made in similar way to the proof made for indeterminacy-membership values.
When it is considered all of cases, it is concluded that $(\mu\tilde{\ominus}\nu)_{(\alpha)}\tilde{\succeq}(\mu)_{(\alpha)}\tilde{\ominus}(\nu)_{(\alpha)}$.

A.7

1. The proof will be made for truth, indeterminacy and falsity membership values of elements of SVN-matrices:

Case 1: For truth-membership values:
Subcase 1: Let $\mu^t_{pq}, \nu^t_{pq} \geq \alpha^t$. Then, $\mu^t_{pq}\oplus\nu^t_{pq} = \mu^t_{pq}\nu^t_{pq} \geq \alpha^t$ or $\mu^t_{pq}\nu^t_{pq} > \alpha^t$.
If $\mu^t_{pq}\nu^t_{pq} \geq \alpha^t$, then $(\mu^t_{pq}\odot\nu^t_{pq})^{\alpha^t} = 1$. Since $(\mu^t_{pq})^{\alpha^t} = 1$ and $(\nu^t_{pq})^{\alpha^t} = 1$, $(\mu^t_{pq})^{\alpha^t}\odot(\nu^t_{pq})^{\alpha^t} = 1$. Thus $(\mu^t_{pq}\odot\nu^t_{pq})^{\alpha^t} = (\mu^t_{pq})^{\alpha^t}\odot(\nu^t_{pq})^{\alpha^t}$.
If $\mu^t_{pq}\nu^t_{pq} < \alpha^t$, then $(\mu^t_{pq}\odot\nu^t_{pq})^{\alpha^t} = 0$. Since $(\mu^t_{pq})^{\alpha^t} = 1$ and $(\nu^t_{pq})^{\alpha^t} = 1$, $(\mu^t_{pq})^{\alpha^t}\odot(\nu^t_{pq})^{\alpha^t} = 1$. Hence $(\mu^t_{pq}\odot\nu^t_{pq})^{\alpha^t} < (\mu^t_{pq})^{\alpha^t}\odot(\nu^t_{pq})^{\alpha^t}$.
Subcase 2: Let $\mu^t_{pq}, \nu^t_{pq} < \alpha^t$. Then, $\mu^t_{pq}\nu^t_{pq} = \mu^t_{pq}\odot\nu^t_{pq} < \alpha^t$ and so $(\mu^t_{pq}\odot\nu^t_{pq})^{\alpha^t} = 0$. Since $\mu^t_{pq} = 0$, $(\mu^t_{pq})^{\alpha^t}\odot(\mu^t_{pq})^{\alpha^t} = (\mu^t_{pq})^{\alpha^t}(\mu^t_{pq})^{\alpha^t} = 0$. Therefore $(\mu^t_{pq}\odot\nu^t_{pq})^{\alpha^t} = (\mu^t_{pq})^{\alpha^t}\odot(\mu^t_{pq})^{\alpha^t}$.
Subcase 3: Let $\mu^t_{pq} < \alpha^t$ and $\nu^t_{pq} \geq \alpha^t$. Then $\mu^t_{pq}\odot\nu^t_{pq} = \mu^t_{pq}\nu^t_{pq} \geq \alpha^t$. Therefore $(\mu^t_{pq}\odot\nu^t_{pq})^{\alpha^t} = 0$. Since $(\mu^t_{pq})^{\alpha^t} = 0$, $(\mu^t_{pq})^{\alpha^t}\odot(\nu^t_{pq})^{\alpha^t} = 0$. Hence, $(\mu^t_{pq}\odot\nu^t_{pq})^{\alpha^t} = (\mu^t_{pq})^{\alpha^t}\odot(\nu^t_{pq})^{\alpha^t}$.
Subcase 4: Let $\mu^t_{pq} \geq \alpha^t$ and $\nu^t_{pq} < \alpha^t$. Then $\mu^t_{pq}\odot\nu^t_{pq} = \mu^t_{pq}\nu^t_{pq} < \alpha^t$. Therefore $(\mu^t_{pq}\odot\nu^t_{pq})^{\alpha^t} = 0$. Since $(\nu^t_{pq})^{\alpha^t} = 0$, $(\mu^t_{pq})^{\alpha^t}\odot(\nu^t_{pq})^{\alpha^t} = 0$. Thus, $(\mu^t_{pq}\odot\nu^t_{pq})^{\alpha^t} = (\mu^t_{pq})^{\alpha^t}\odot(\nu^t_{pq})^{\alpha^t}$.
Case 2: For indeterminacy-membership values:
Subcase 1: Let $\mu^i_{pq}, \nu^i_{pq} > \alpha^i$. Then, $\mu^i_{pq}\odot\nu^i_{pq} = \mu^i_{pq}+\nu^i_{pq}-\mu^i_{pq}+\nu^i_{pq} > \alpha^i$ and so $(\mu^i_{pq}\odot\nu^i_{pq})^{\alpha^i} = 1$. Since $(\mu^i_{pq})^{\alpha^i} = 1$ and $(\nu^i_{pq})^{\alpha^i} = 1$, $(\mu^i_{pq})^{\alpha^i}\odot(\nu^i_{pq})^{\alpha^i} = 1+1-1 = 1$. Thus $(\mu^i_{pq}\odot\nu^i_{pq})^{\alpha^i} = (\mu^i_{pq})^{\alpha^i}\odot(\nu^i_{pq})^{\alpha^i}$.
Subcase 2: Let $\mu^i_{pq}, \nu^i_{pq} \leq \alpha^i$. Then, there are two cases:
1. If $\mu^i_{pq}\odot\nu^i_{pq} = \mu^i_{pq}+\nu^i_{pq}-\mu^i_{pq}\nu^i_{pq} \leq \alpha^i$, then $(\mu^i_{pq}\odot\nu^i_{pq})^{\alpha^i} = 0$. Since $(\mu^i_{pq})^{\alpha^i} = 0$ and $(\nu^i_{pq})^{\alpha^i} = 0$, $(\mu^i_{pq})^{\alpha^i}\odot(\nu^i_{pq})^{\alpha^i} = 0$. Therefore $(\mu^i_{pq}\odot\nu^i_{pq})^{\alpha^i} = (\mu^i_{pq})^{\alpha^i}\odot(\nu^i_{pq})^{\alpha^i}$.
2. If $\mu^i_{pq}\odot\nu^i_{pq} = \mu^i_{pq}+\nu^i_{pq}-\mu^i_{pq}\nu^i_{pq} > \alpha^i$, then $(\mu^i_{pq}\odot\nu^i_{pq})^{\alpha^i} = 1$. Since $(\mu^i_{pq})^{\alpha^i} = 0$ and $(\nu^i_{pq})^{\alpha^i} = 0$, $(\mu^i_{pq})^{\alpha^i}\odot(\nu^i_{pq})^{\alpha^i} = 0$. Hence $(\mu^i_{pq}\odot\nu^i_{pq})^{\alpha^i} > (\mu^i_{pq})^{\alpha^i}\odot(\nu^i_{pq})^{\alpha^i}$.
Subcase 3: Let $\mu^i_{pq} \leq \alpha^i$ and $\nu^i_{pq} > \alpha^i$. Then $\mu^i_{pq}\odot\nu^i_{pq} = \mu^i_{pq}+\nu^i_{pq}-\mu^i_{pq}\nu^i_{pq} > \alpha^i$ and so $(\mu^i_{pq}\odot\nu^i_{pq})^{\alpha^i} = 1$. Since $(\mu^i_{pq})^{\alpha^i} = 0$ and $(\nu^i_{pq})^{\alpha^i} = 1$, $(\mu^i_{pq})^{\alpha^i}\odot(\nu^i_{pq})^{\alpha^i} = 0+1-0 = 1$. Thus, $(\mu^i_{pq}\odot\nu^i_{pq})^{\alpha^i} = (\mu^i_{pq})^{\alpha^i}\odot(\nu^i_{pq})^{\alpha^i}$.
Subcase 4: Let $\mu^i_{pq} > \alpha^i$ and $\nu^i_{pq} \leq \alpha^i$. Then $\mu^i_{pq}\odot\nu^i_{pq} = \mu^i_{pq}+\nu^i_{pq}-\mu^i_{pq}\nu^i_{pq} > \alpha^i$ and so $(\mu^i_{pq}\odot\nu^i_{pq})^{\alpha^i} = 1$. Since $(\mu^i_{pq})^{\alpha^i} = 1$ and $(\nu^i_{pq})^{\alpha^i} = 0$, $(\mu^i_{pq})^{\alpha^i}\odot(\nu^i_{pq})^{\alpha^i} = 1+0-0$. Thus, $(\mu^i_{pq}\odot\nu^i_{pq})^{\alpha^i} = (\mu^i_{pq})^{\alpha^i}\odot(\nu^i_{pq})^{\alpha^i}$.
Case 3: For falsity-membership values: The proof can be made by similar way to proof of case 2.
When it is considered all of the cases, it is concluded that $(\mu\tilde{\odot}\nu)^{(\alpha)}\tilde{\preceq}\mu^{(\alpha)}\tilde{\odot}\nu^{(\alpha)}$.

2. The proof can be made by similar way to proof of statement 1.

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