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Two New Approaches for Multi-Attribute Group Decision-Making With Interval-Valued Neutrosophic Frank Aggregation Operators and Incomplete Weights

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ABSTRACT This paper investigates some Frank aggregation operators of interval-valued neutrosophic numbers (IVNNs) and applies to multi-attribute group decision-making (MAGDM) problems. First, the Frank t-conorm and t-norm are extended to interval-valued neutrosophic environment. Some new operational laws for IVNNs are defined and their related properties are investigated. Based on these new operational laws, some new aggregation operators for IVNNs are developed including the interval-valued neutrosophic Frank weighted averaging (IVNFWA) operator and the interval-valued neutrosophic Frank weighted geometric (IVNFWG) operator. Then some desirable properties and special cases of these new operators are further discussed. To solve the MAGDM with IVNNs, the weights of decision makers (DMs) are determined by using extended technique for order preference by similarity to ideal solution (TOPSIS) method based on cross-entropy. Additionally, attribute weights are determined based on the similarity degrees between alternatives and the absolute ideal solutions. Further, two new decision-making approaches for MAGDM with IVNNs are put forward by means of the IVNFWA and IVNFWG operators, respectively. Finally, a case study of selecting an agricultural socialization service provider is analyzed to illustrate the practicality and effectiveness of the developed two approaches.

INDEX TERMS Multi-attribute group decision-making, interval-valued neutrosophic set, interval-valued neutrosophic frank weighted averaging operator, interval-valued neutrosophic frank weighted geometric operator.

I. INTRODUCTION

Multi-attribute decision-making (MADM) and multi-attribute group decision-making (MAGDM) have been extensively used in real-life decision-making problems. The early studies of MADM are based on the information represented in the form of crisp values. However, due to uncertainty in the decision-making process, it is not suitable to use the crisp numbers to represent the decision information. To effectively handling fuzziness in actual decision-making problems, Zadeh [1] introduced the theory of fuzzy set (FS), which is an important tool to narrate fuzzy information. However, FS could not express the neutral state, that is, neither support nor objection. To overcome this defect, Atanassov [2], [3] introduced the concept of the intuitionistic fuzzy set (IFS). Compared with FS, IFS can simultaneously express three states of support, opposition and neutrality.

Although FSs and IFSs have been developed and generalized, they could not deal with the indeterminate and inconsistent information in real decision-making problems. To solve this issue, Smarandache [4] proposed the neutrosophic sets (NSs). Each element of the universe in NSs

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has degrees of truth, indeterminacy and falsity whose values lie in the non-standard unit interval of $]0^{-}, 1^{+}[$. Note that indeterminacy degree is independent of membership degree and non-membership degree which makes it hard to apply NSs to real scientific and engineering situations without specific description. Hence, Wang *et al.* [5] presented the notion of single-valued neutrosophic sets, which can be described by three real numbers in the real unit interval [0,1]. In some specified situations, it is difficult for decision maker (DM) to express the degrees of truth-membership, falsity-membership, and indeterminacy-membership by crisp values. Thus, Wang *et al.* [6] further proposed the concept of interval-valued neutrosophic (IVN) set (IVNS) which has significant power to express incomplete, indeterminate and inconsistent information.

Recently, IVNS has received extensive attention and has been widely applied to the fields of MADM [7]–[19] and MAGDM [16], [20], [21]. Numerous research results on MADM and MAGDM with IVN information have been reported. These results can be roughly divided into two categories: aggregation operators and decision-making methods.

The first category is the aggregation operators for IVNNs. To solve IVN MADM, Reddy *et al.* [8] defined an IVN arithmetic weighted average operator and an IVN geometric weighted average operator. Liu and Tang [22] proposed an IVN power generalized aggregation operator, an IVN power generalized weighted aggregation operator and an IVN power generalized ordered weighted aggregation operator. Zhao *et al.* [23] proposed an IVN generalized weighted aggregation operator for MADM. Liu and Wang [24] presented an IVN prioritized ordered weighted aggregation operator. Ye [25] developed an IVN ordered weighted averaging operator and an IVN ordered weighted geometric operator. Zhang *et al.* [26] developed an IVN weighted averaging operator and an IVN weighted geometric operator. Sun*et al.* [27] established an IVN Choquet integral operator. Ye [28] proposed a credibility-induced IVN weighted arithmetic averaging operator and a credibility-induced IVN weighted geometric averaging operator. Liu *et al.* [29] proposed an induced generalized IVN Shapley hybrid arithmetic averaging operator and an induced generalized IVN Shapley hybrid geometric mean operator. Li *et al.* [30] developed a generalized IVN Choquet ordered averaging operator and a generalized IVN Choquet ordered geometric operator. Liu and You [31] developed an IVN Muirhead mean operator, an IVN weighted Muirhead mean operator, an IVN weighted dual Muirhead mean operator and an IVN dual weighted Muirhead mean operator. Ye [32] proposed an IVN weighted exponential aggregation operator and a dual IVN weighted exponential aggregation operator. Garg and Nancy [33] developed a hybrid IVN weighted geometric operator and a hybrid ordered weighted geometric operator.

The second category is the decision-making methods for MADM and MAGDM with interval-valued neutrosophic numbers (IVNNs). Chi and Liu [34] proposed a Technique for Order Preference by Similarity to Ideal Solution (TOPSIS) method with IVNS. Dalapati *et al.* [16] developed a novel strategy for MAGDM based on the weighted cross-entropy. Pramanik and Mondal [35] studied the IVN MADM based on grey relational analysis. Bausys and Zavadskas [36] presented an IVN-VIKOR (from Serbian: VIseKriterijumska Optimizacija I Kompromisno Resenje) method to address MADM problem. Huang *et al.* [20] extended the VIKOR method to MAGDM with IVNNs. Karaşan and Kahraman [9] developed an IVN-Evaluation based on Distance from Average Solution (EDAS) method. Hu *et al.* [37] established a projection-based VIKOR method for MADM with IVNS. Zhang *et al.* [38] presented a new Elimination and Choice Translating Reality (ELECTRE) IV method with IVNNs. Bolturk and Kahraman [39] proposed an IVN-Analytic Hierarchy Process (AHP) method based on cosine similarity measure. Ye [12] established a MADM method with based on the similarity measure. Garg and Nancy [40] developed a new nonlinear programming method for solving MADM problem under IVNS. Şahin and Liu [21] constructed a maximizing deviation method to MAGDM with IVNS.

The literature review reveals that the studies on MADM and MAGDM with IVNNs have achieved fruitful results. However, there are some shortcomings which are summarized as follows:

(1) The current aggregation operators of IVNNs are mainly based on the algebraic operational laws of general t-norms and t-conorm, which are lack of flexibility and robustness. Besides the characteristics of general t-conorm and t-norm, Frank t-norm and t-conorm can make information fusion more flexible and robust with the aid of parameter. Therefore, it is necessary to extend Frank aggregation operators to IVN environment.
(2) Most of existing decision-making methods [12], [14], [15], [22]–[24], [28], [36], [40] are single decision rather than group decision. Only three references [16], [20], [21] studied the MAGDM with IVNNs. However, with increasing complexity of the socio-economic environment, many decision-making problems become more and more complex. It is difficult for single DM to fully consider all important aspects of a problem. To achieve accurate and reasonable decision result, a group of DMs from different fields should be invited to join the decision-making.
(3) The previous decision-making methods [10], [12], [14]–[17], [20], [23]–[26], [28], [38] assume that the DMs' weights or attribute weights are already known. Due to the lack of information, knowledge, and DM's limited expertise about the problem domain, it is very difficult or even impossible to give the weights of DMs and attributes precisely and objectively in advance. In actual situation, it is necessary to determine the attribute weights and DMs' weights based on the decision information, which can make decision result more objective and accurate.

To overcome the above limitations, the main motivations of this paper are outlined as follows:

(1) So far, different aggregation operators of IVNNs have been presented. Each operator has its distinctive characteristics and can work well for specific purpose. However, there is not yet an operator which can provide desirable generality and flexibility in aggregation of criterion values. This reality motivates us to extend the Frank t-conorm and t-norm to IVN environment due to the fact that Frank t-norm and t-conorm can make information fusion more flexible and robust with the aid of parameter.

(2) With the rapid development of modern technology and economy nowadays, group decision is an inevitable trend in most real-life decision-making problems. Thus, how to develop some new approaches for MAGDM with IVNNs is imperative. This is the second motivation of this paper.

(3) To integrate the information of attribute values, an effective tool is aggregation operator. Thus, it is necessary to define some new aggregation operators for IVNNs. This is the third motivation of this paper.

Therefore, this paper extends the Frank t-conorm and t-norm to IVN environment for the first time and then defines some new operational laws for IVNNs. Based on these new operational laws, some new aggregation operators for IVNNs are developed including the interval-valued neutrosophic Frank weighted averaging (IVNFWA) operator and the interval-valued neutrosophic Frank weighted geometric (IVNFWG) operator. For MAGDM with IVNNs, the weights of DMs are determined by using extended TOPSIS method based on cross-entropy. The attribute weights are determined based on the similarity degrees between alternatives and the absolute ideal solutions. Thereby, two new decision-making approaches for MAGDM with IVNNs are proposed by the IVNFWA and IVNFWG operators, respectively.

The main contributions of this paper are summarized as follows:

(1) This study firstly extends Frank operations to cope with the IVN information fusion. Some new operational laws for IVNNs are defined. The desirable properties are proven. Two new aggregation operators including the IVNFWA operator and the IVNFWG operator are developed. Some desirable properties and special cases of these new operators are discussed in details. Using the parameterized t-norms and t-conorms of Frank operations, DMs have more flexibility in choosing the parameters based on the degree of risk that one can bear.

(2) DMs' weights are determined by using extended TOPSIS method based on cross-entropy. Additionally, attribute weights are determined based on the similarity degree between alternatives and the absolute ideal solutions. In real-life MAGDM, neither attribute weights nor DM weights can be known in advance. This paper determines the attribute weights and DMs' weights based on the decision information, which is more in line with the actual situation of decision-making.

(3) Based on the IVNFWA operator and IVNFWG operator, two new approaches are presented for solving MAGDM under IVN environment, respectively. In the actual decision-making process, these two approaches enable DMs choose different parameters according to their preferences, which makes decision-making more flexible.

The remainder of this paper is organized as follows. Section 2 briefly reviews some basic concepts of IVNS and Frank operations. Section 3 introduces some Frank operations on IVNNs and investigates some desirable properties of the proposed operations. Based on Frank operations on IVNNs, we develop two Frank aggregation operators for IVN information in Section 4, including the IVNFWA operator and the IVNFWG operator. Section 5 proposes two approaches to MAGDM with IVN information based on the developed operators. In section 6, a case study of the selection of an agricultural socialization service provider is put forward to show the practicality and effectiveness of the proposed approaches. The conclusions are included in Section 7.

II. PRELIMINARIES

In this section, some basic notions are given, including IVNS and Frank operations.

A. DEFINITIONS AND OPERATIONS OF IVNS

Definition 1 [6]: Let X be a space of points (objects) with generic elements in X denoted by x. An IVNS A in X is characterized by a true-membership function $T_A(x)$, an indeterminacy-membership function $I_A(x)$, and a falsity-membership function $F_A(x)$. For each point x in X, there are $T_A(x) = [\inf T_A(x), \sup T_A(x)] \subseteq [0, 1]$, $I_A(x) = [\inf I_A(x), \sup I_A(x)] \subseteq [0, 1]$, $F_A(x) = [\inf F_A(x), \sup F_A(x)] \subseteq [0, 1]$ and $0 \leq \sup T_A(x) + \sup I_A(x) + \sup F_A(x) \leq 3$. Then, an IVNS A can be expressed as

$$A = \{\langle x, T_A(x), I_A(x), F_A(x) | x \in X \rangle\} \tag{1}$$

For convenience, we use $a = \langle [T^L, T^U], [I^L, I^U], [F^L, F^U] \rangle$ to represent an IVNN in an IVNS.

For two IVNSs $A = \langle [T_A^L, T_A^U], [I_A^L, I_A^U], [F_A^L, F_A^U] \rangle$ and $B = \langle [T_B^L, T_B^U], [I_B^L, I_B^U], [F_B^L, F_B^U] \rangle$, some relations for IVNSs A and B are defined as follows [6]:

(1) $A^c = \langle [F_A^L, F_A^U], [1 - I_A^U, 1 - I_A^L], [T_A^L, T_A^U] \rangle$ as the complement of an IVNS A;

(2) $A \subseteq B$ if and only if $T_A^L \leq T_B^L, T_A^U \leq T_B^U, I_A^L \geq I_B^L, I_A^U \geq I_B^U, F_A^L \geq F_B^L, F_A^U \geq F_B^U$;

(3) $A = B$ if and only if $A \subseteq B$ and $B \subseteq A$.

Definitions 2 [26]: Let two IVNSs be $A = \langle [T_A^L, T_A^U], [I_A^L, I_A^U], [F_A^L, F_A^U] \rangle$ and $B = \langle [T_B^L, T_B^U], [I_B^L, I_B^U], [F_B^L, F_B^U] \rangle$, and $\lambda > 0$. The operations for IVNSs are defined as:

(1) $A + B = \langle [T_A^L + T_B^L - T_A^L T_B^L, T_A^U + T_B^U - T_A^U T_B^U], [I_A^L I_B^L, I_A^U I_B^U], [F_A^L F_B^L, F_A^U F_B^U] \rangle$,

(2) $A \times B = \langle [T_A^L T_B^L, T_A^U T_B^U], [I_A^L + I_B^L - I_A^L I_B^L, I_A^U + I_B^U - I_A^U I_B^U], [F_A^L + F_B^L - F_A^L F_B^L, F_A^U + F_B^U - F_A^U F_B^U] \rangle$,

(3) $\lambda A = \langle [1 - (1 - T_A^L)^\lambda, 1 - (1 - T_A^U)^\lambda], [(I_A^L)^\lambda, (I_A^U)^\lambda], [(F_A^L)^\lambda, (F_A^U)^\lambda] \rangle$,

(4) $A^\lambda = \langle [(T_A^L)^\lambda, (T_A^U)^\lambda], [1 - (1 - I_A^L)^\lambda, 1 - (1 - I_A^U)^\lambda], [1 - (1 - F_A^L)^\lambda, 1 - (1 - F_A^U)^\lambda] \rangle$.

Definition 3 [26]: Let $A_j = \langle [T_{A_j}^L, T_{A_j}^U], [I_{A_j}^L, I_{A_j}^U], [F_{A_j}^L, F_{A_j}^U] \rangle$ $(j = 1, 2, \cdots, n)$ be a set of IVNNs. Based on definition 2, the IVN weighted averaging operator is shown as:

$$A(w) = \sum_{j=1}^n w_i A_j = \langle [1 - \prod_{j=1}^n (1 - T_{A_j}^L)^{w_j}, 1 - \prod_{j=1}^n (1 - T_{A_j}^U)^{w_j}], [\prod_{j=1}^n (I_{A_j}^L)^{w_j}, \prod_{j=1}^n (I_{A_j}^U)^{w_j}], [\prod_{j=1}^n (F_{A_j}^L)^{w_j}, \prod_{j=1}^n (F_{A_j}^U)^{w_j}] \rangle \tag{2}$$

It is worth noting that when $w = (1/n, 1/n, \cdots, 1/n)$, $A(W)$ is degenerated to arithmetic aggregation operator:

$$A(w) = \left\langle \left[1 - \sqrt[n]{\prod_{j=1}^n (1 - T_{A_j}^L)}, 1 - \sqrt[n]{\prod_{j=1}^n (1 - T_{A_j}^U)} \right], \left[\sqrt[n]{\prod_{j=1}^n (I_{A_j}^L)}, \sqrt[n]{\prod_{j=1}^n (I_{A_j}^U)} \right], \left[\sqrt[n]{\prod_{j=1}^n (F_{A_j}^L)}, \sqrt[n]{\prod_{j=1}^n (F_{A_j}^U)} \right] \right\rangle \tag{3}$$

For the purpose of compare the magnitudes of two IVNNs, Şahin and Liu [21] defined the score and accuracy functions for IVNNs and gave a simple comparison law as follows:

Definition 4 [21]: Let $a = \langle [T^L, T^U], [I^L, I^U], [F^L, F^U] \rangle$ be an IVNN. A score function $S(a)$ is defined as

$$S(a) = \frac{2 + T^L + T^U - 2I^L - 2I^U - F^L - F^U}{4} \tag{4}$$

where $S(a) \in [-1, 1]$.

Definition 5 [21]: Let $a = \langle [T^L, T^U], [I^L, I^U], [F^L, F^U] \rangle$ be an IVNN. An accuracy function $L(a)$ is defined as

$$L(a) = \frac{1}{2} \left(T^L + T^U - I^U \left(1 - T^U\right) - I^L \left(1 - T^L\right) - F^U \left(1 - I^U\right) - F^L \left(1 - I^L\right) \right) \tag{5}$$

where $L(a) \in [-1, 1]$.

Definition 6 [21]: Let $A = \langle [T_A^L, T_A^U], [I_A^L, I_A^U], [F_A^L, F_A^U] \rangle$ and $B = \langle [T_B^L, T_B^U], [I_B^L, I_B^U], [F_B^L, F_B^U] \rangle$ be two IVNNs. The comparative method of A and B can be defined as follows:

(1) If $S(A) > S(B)$, then A is bigger than B, denoted by $A \succ B$;

(2) If $S(A) = S(B)$, then $\begin{cases} L(A) > L(B) \Rightarrow A \succ B \\ L(A) = L(B) \Rightarrow A = B \end{cases}$

B. A CROSS-ENTROPY MEASURE OF IVNNS

Definition 7 [15]: For two IVNNs A and B in a universe of discourse $X = \{x_1, x_2, \cdots, x_n\}$, the cross- entropy between IVNNs A and B is defined as follows:

$$\begin{aligned} I_{NS}(A, B) = \frac{1}{2} \Bigg\{ \sum_{i=1}^n \Bigg[& \sqrt{\frac{(T_A^L(x_i))^2 + (T_B^L(x_i))^2}{2}} - \left(\frac{\sqrt{T_A^L(x_i)} + \sqrt{T_B^L(x_i)}}{2} \right)^2 \\ & + \sqrt{\frac{(I_A^L(x_i))^2 + (I_B^L(x_i))^2}{2}} - \left(\frac{\sqrt{I_A^L(x_i)} + \sqrt{I_B^L(x_i)}}{2} \right)^2 \\ & + \sqrt{\frac{(1 - I_A^L(x_i))^2 + (1 - I_B^L(x_i))^2}{2}} - \left(\frac{\sqrt{1 - I_A^L(x_i)} + \sqrt{1 - I_B^L(x_i)}}{2} \right)^2 \\ & + \sqrt{\frac{(F_A^L(x_i))^2 + (F_B^L(x_i))^2}{2}} - \left(\frac{\sqrt{F_A^L(x_i)} + \sqrt{F_B^L(x_i)}}{2} \right)^2 \Bigg] \\ + \sum_{i=1}^n \Bigg[& \sqrt{\frac{(T_A^U(x_i))^2 + (T_B^U(x_i))^2}{2}} - \left(\frac{\sqrt{T_A^U(x_i)} + \sqrt{T_B^U(x_i)}}{2} \right)^2 + \sqrt{\frac{(I_A^U(x_i))^2 + (I_B^U(x_i))^2}{2}} \\ & - \left(\frac{\sqrt{I_A^U(x_i)} + \sqrt{I_B^U(x_i)}}{2} \right)^2 + \sqrt{\frac{(1 - I_A^U(x_i))^2 + (1 - I_B^U(x_i))^2}{2}} \\ & - \left(\frac{\sqrt{1 - I_A^U(x_i)} + \sqrt{1 - I_B^U(x_i)}}{2} \right)^2 + \sqrt{\frac{(F_A^U(x_i))^2 + (F_B^U(x_i))^2}{2}} \\ & - \left(\frac{\sqrt{F_A^U(x_i)} + \sqrt{F_B^U(x_i)}}{2} \right)^2 \Bigg] \Bigg\} \end{aligned} \tag{6}$$

According to Definition 7, we define the cross-entropy between two IVNNs matrices.

Definition 8: Let $H^\Phi = \langle [T_{ij}^{L(\Phi)}, T_{ij}^{U(\Phi)}], [I_{ij}^{L(\Phi)}, I_{ij}^{U(\Phi)}], [F_{ij}^{L(\Phi)}, F_{ij}^{U(\Phi)}] \rangle_{m \times n}$ $(\Phi = 1, 2)$ be two IVNNs matrices.

The cross-entropy between H^1 and H^2 is defined as

$$
\begin{aligned}
&I_{NS}(H^1, H^2)\\
&= \frac{1}{2mn}\left\{\sum_{i=1}^{m}\sum_{j=1}^{n}\left[\sqrt{\frac{(T_{ij}^{L(1)})^2+(T_{ij}^{L(2)})^2}{2}}\right.\right.\\
&\quad -\left(\frac{\sqrt{T_{ij}^{L(1)}}+\sqrt{T_{ij}^{L(2)}}}{2}\right)^2+\sqrt{\frac{(I_{ij}^{L(1)})^2+(I_{ij}^{L(2)})^2}{2}}\\
&\quad -\left(\frac{\sqrt{I_{ij}^{L(1)}}+\sqrt{I_{ij}^{L(2)}}}{2}\right)^2+\sqrt{\frac{(1-I_{ij}^{L(1)})^2+(1-I_{ij}^{L(2)})^2}{2}}\\
&\quad -\left(\frac{\sqrt{1-I_{ij}^{L(1)}}+\sqrt{1-I_{ij}^{L(2)}}}{2}\right)^2\\
&\quad \left.+\sqrt{\frac{(F_{ij}^{L(1)})^2+(F_{ij}^{L(2)})^2}{2}}-\left(\frac{\sqrt{F_{ij}^{L(1)}}+\sqrt{F_{ij}^{L(2)}}}{2}\right)^2\right]\\
&\quad +\sum_{i=1}^{m}\sum_{j=1}^{n}\left[\sqrt{\frac{(T_{ij}^{U(1)})^2+(T_{ij}^{U(2)})^2}{2}}\right.\\
&\quad -\left(\frac{\sqrt{T_{ij}^{U(1)}}+\sqrt{T_{ij}^{U(2)}}}{2}\right)^2+\sqrt{\frac{(I_{ij}^{U(1)})^2+(I_{ij}^{U(2)})^2}{2}}\\
&\quad -\left(\frac{\sqrt{I_{ij}^{U(1)}}+\sqrt{I_{ij}^{U(2)}}}{2}\right)^2+\sqrt{\frac{(1-I_{ij}^{U(1)})^2+(1-I_{ij}^{U(2)})^2}{2}}\\
&\quad -\left(\frac{\sqrt{1-I_{ij}^{U(1)}}+\sqrt{1-I_{ij}^{U(2)}}}{2}\right)^2+\sqrt{\frac{(F_{ij}^{U(1)})^2+(F_{ij}^{U(2)})^2}{2}}\\
&\quad \left.\left.-\left(\frac{\sqrt{F_A^U(x_i)}+\sqrt{F_B^U(x_i)}}{2}\right)^2\right]\right\} \qquad (7)
\end{aligned}
$$

C. FRANK OPERATIONS

Definition 9 [41]: Frank operations consist of the Frank product $\otimes_F$ and Frank sum $\oplus_F$, which are t-norm and t-conorm, respectively. Let x and y be two real numbers satisfying $x, y \in [0, 1]$, and let $\lambda \in (1, +\infty)$. Then, the Frank product and Frank sum between x and y can be defined as follows:

$$x \oplus_F y = 1 - \log_\lambda(1 + \frac{(\lambda^{1-x}-1)(\lambda^{1-y}-1)}{\lambda - 1}) \qquad (8)$$

$$x \otimes_F y = \log_\lambda(1 + \frac{(\lambda^{x}-1)(\lambda^{y}-1)}{\lambda - 1}) \qquad (9)$$

In view of limit theory, it can be easily proven that when $\lambda \to 1$, $x \oplus_F y \to x + y - xy$ and $x \otimes_F y \to xy$, the Frank product and Frank sum are degenerated to the algebraic triangular norm and conorm, respectively. When $\lambda \to \infty$, $x \oplus_F y \to \min(x - y, 1)$ and $x \otimes_F y \to \max(0, x - y - 1)$, the Frank product and Frank sum are degenerated to the Lukasiewicz product and Lukasiewicz sum, respectively.

III. FRANK OPERATIONS OF IVNNS

In this section, the Frank operations for IVNSs are defined.

Definition 10: Let $h_1 = \langle[T_{h_1}^L, T_{h_1}^U], [I_{h_1}^L, I_{h_1}^U], [F_{h_1}^L, F_{h_1}^U]\rangle$ and $h_2 = \langle[T_{h_2}^L, T_{h_2}^U], [I_{h_2}^L, I_{h_2}^U], [F_{h_2}^L, F_{h_2}^U]\rangle$ be two IVNNs, $\lambda > 1$ and $\gamma > 0$. Frank operations of IVNNs are defined as follows:

(1)

$$
\begin{aligned}
h_1 \oplus_F h_2 = \langle[&1-\log_\lambda(1+\frac{(\lambda^{1-T_{h_1}^L}-1)(\lambda^{1-T_{h_2}^L}-1)}{\lambda-1}),\\
&1-\log_\lambda(1+\frac{(\lambda^{1-T_{h_1}^U}-1)(\lambda^{1-T_{h_2}^U}-1)}{\lambda-1})],\\
&[\log_\lambda(1+\frac{(\lambda^{I_{h_1}^L}-1)(\lambda^{I_{h_2}^L}-1)}{\lambda-1}),\\
&\log_\lambda(1+\frac{(\lambda^{I_{h_1}^U}-1)(\lambda^{I_{h_2}^U}-1)}{\lambda+1})],\\
&[\log_\lambda(1+\frac{(\lambda^{F_{h_1}^L}-1)(\lambda^{F_{h_2}^L}-1)}{\lambda-1}),\\
&\log_\lambda(1+\frac{(\lambda^{F_{h_1}^U}-1)(\lambda^{F_{h_2}^U}-1)}{\lambda-1})]\rangle \qquad (10)
\end{aligned}
$$

(2)

$$
\begin{aligned}
h_1 \otimes_F h_2 = \langle[&\log_\lambda(1+\frac{(\lambda^{T_{h_1}^L}-1)(\lambda^{T_{h_2}^L}-1)}{\lambda-1}),\\
&\log_\lambda(1+\frac{(\lambda^{T_{h_1}^U}-1)(\lambda^{T_{h_2}^U}-1)}{\lambda-1})],\\
&[1-\log_\lambda(1+\frac{(\lambda^{1-I_{h_1}^L}-1)(\lambda^{1-I_{h_2}^L}-1)}{\lambda-1}),\\
&1-\log_\lambda(1+\frac{(\lambda^{1-I_{h_1}^U}-1)(\lambda^{1-I_{h_2}^U}-1)}{\lambda-1})],\\
&[1-\log_\lambda(1+\frac{(\lambda^{1-F_{h_1}^L}-1)(\lambda^{1-F_{h_2}^L}-1)}{\lambda-1}),\\
&1-\log_\lambda(1+\frac{(\lambda^{1-F_{h_1}^U}-1)(\lambda^{1-F_{h_2}^U}-1)}{\lambda-1})]\rangle \qquad (11)
\end{aligned}
$$

(3)

$$
\begin{aligned}
\gamma \cdot_F h_1 = \langle[&1-\log_\lambda(1+\frac{(\lambda^{1-T_{h_1}^L}-1)^\gamma}{(\lambda-1)^{\gamma-1}}),\\
&1-\log_\lambda(1+\frac{(\lambda^{1-T_{h_1}^U}-1)}{(\lambda-1)^{\gamma-1}})],\\
&[\log_\lambda(1+\frac{(\lambda^{I_{h_1}^L}-1)^\gamma}{(\lambda-1)^{\gamma-1}}), \log_\lambda(1+\frac{(\lambda^{I_{h_2}^U}-1)^\gamma}{(\lambda-1)^{\gamma-1}})],\\
&[\log_\lambda(1+\frac{(\lambda^{F_{h_1}^L}-1)^\gamma}{(\lambda-1)^{\gamma-1}}), \log_\lambda(1+\frac{(\lambda^{F_{h_1}^U}-1)^\gamma}{(\lambda-1)^{\gamma-1}})]\rangle \qquad (12)
\end{aligned}
$$

(4)

$$h_1^{\wedge F\gamma} = \langle [\log_\lambda(1+\frac{(\lambda^{T_{h_1}^L}-1)^\gamma}{(\lambda-1)^{\gamma-1}}), \log_\lambda(1+\frac{(\lambda^{T_{h_1}^U}-1)}{(\lambda-1)^{\gamma-1}})], 1-\log_\lambda(1+\frac{(\lambda^{1-I_{h_1}^L}-1)^\gamma}{(\lambda-1)^{\gamma-1}})], 1-\log_\lambda(1+\frac{(\lambda^{1-I_{h_1}^U}-1)^\gamma}{(\lambda-1)^{\gamma-1}})], [1-\log_\lambda(1+\frac{(\lambda^{1-F_{h_1}^L}-1)^\gamma}{(\lambda-1)^{\gamma-1}}), 1-\log_\lambda(1+\frac{(\lambda^{1-F_{h_1}^U}-1)^\gamma}{(\lambda-1)^{\gamma-1}})]\rangle \quad (13)$$

Theorem 1: Let h_1, h_2 and h_3 be three IVNNs, and γ_1, $\gamma_2 > 0$. The following properties hold:

$(KP1) h_1 \oplus_F h_2 = h_2 \oplus_F h_1$;
$(KP2) h_1 \otimes_F h_2 = h_2 \otimes_F h_1$;
$(KP3) \gamma_1 \cdot_F (h_1 \oplus_F h_2) = \gamma_1 \cdot_F h_1 \oplus_F \gamma_1 \cdot_F h_2$;
$(KP4) (h_1 \otimes_F h_2)^{\wedge F\gamma_1} = h_1^{\wedge F\gamma_1} \otimes_F h_2^{\wedge F\gamma_1}$;
$(KP5) \gamma_1 \cdot_F h_1 \oplus_F \gamma_2 \cdot_F h_1 = (\gamma_1+\gamma_2) \cdot_F h_1$;
$(KP6) h_1^{\wedge F\gamma_1} \otimes_F h_1^{\wedge F\gamma_2} = h_1^{\wedge F(\gamma_1+\gamma_2)}$;
$(KP7) \gamma_1 \cdot_F (\gamma_2 \cdot_F h_1) = \gamma_2 \cdot_F (\gamma_1 \cdot_F h_1) = (\gamma_1\gamma_2) \cdot_F h_1$;
$(KP8) (h_1 \oplus_F h_2) \oplus_F h_3 = h_1 \oplus_F (h_2 \oplus_F h_3)$.

Proof: The proof of Theorem 1 can be easily derived from Frank operations of IVNNs. Therefore, it is omitted here.

IV. THE FRANK AGGREGATION OPERATORS FOR IVNNS

In this section, several Frank aggregation operators for IVN information are developed, including the IVNFWA operator and IVNFWG operator.

Definition 11: Let $x_i = \langle [T_{x_i}^L, T_{x_i}^U], [I_{x_i}^L, I_{x_i}^U], [F_{x_i}^L, F_{x_i}^U]\rangle$ $(i = 1, 2, \cdots, n)$ be a set of IVNNs, and let $w = (w_1, w_2 \cdots, w_n)^T$ be the weight vector of $x_i (i = 1, 2, \cdots, n)$ with $w_i \in [0, 1]$ and $\sum_{i=1}^n w_i$ =1. Then, an IVNFWA operator is a mapping $\Omega^n \to \Omega$, such that

$$IVNFWA(x_1, x_2 \cdots, x_n) = (w_1 \cdot_F x_1) \oplus_F (w_2 \cdot_F x_2) \cdots \oplus_F (w_n \cdot_F x_n) \quad (14)$$

where Ω is the set of all IVNNs.

In particular, if $w = (\frac{1}{n}, \frac{1}{n}, \cdots, \frac{1}{n})^T$, then the IVNFWA operator reduces to interval-valued neutrosophic Frank averaging operator as follows:

$$IVNFA(x_1, x_2, \cdots, x_n) = \frac{1}{n} \cdot_F (x_1 \oplus_F x_2 \oplus_F \cdots \oplus_F x_n) \quad (15)$$

Theorem 2: Let $x_i = \langle [T_{x_i}^L, T_{x_i}^U], [I_{x_i}^L, I_{x_i}^U], [F_{x_i}^L, F_{x_i}^U]\rangle$ $(i = 1, 2, \cdots, n)$ be a set of IVNNs, and let $w = (w_1, w_2 \cdots, w_n)^T$ be the weight vector of $x_i (i = 1, 2, \cdots, n)$, satisfying $w_i \in [0, 1]$ and $\sum_{i=1}^n w_i = 1$, then the aggregated value by using the IVNFWA operator is also an IVNNs, and

$$IVNFWA(x_1, x_2, \cdots, x_n) = \langle [1-\log_\lambda(1+\prod_{i=1}^n(\lambda^{1-T_{x_i}^L}-1)^{w_i}), 1-\log_\lambda(1+\prod_{i=1}^n(\lambda^{1-T_{x_i}^U}-1)^{w_i})], [\log_\lambda(1+\prod_{i=1}^n(\lambda^{I_{x_i}^L}-1)^{w_i}), \log_\lambda(1+\prod_{i=1}^n(\lambda^{I_{x_i}^U}-1)^{w_i})], [\log_\lambda(1+\prod_{i=1}^n(\lambda^{F_{x_i}^L}-1)^{w_i}), \log_\lambda(1+\prod_{i=1}^n(\lambda^{F_{x_i}^U}-1)^{w_i})]\rangle \quad (16)$$

Proof: See Appendix A.

In the following, we will investigate some desirable properties of the IVNFWA operator.

Theorem 3 (Idempotency): Let $x_i = \langle [T_{x_i}^L, T_{x_i}^U], [I_{x_i}^L, I_{x_i}^U], [F_{x_i}^L, F_{x_i}^U]\rangle (i = 1, 2, \cdots, n)$ be a set of INNs. If all $x_i (i = 1, 2, \cdots, n)$ are equal, i.e., $x_i = x = \langle [T^L, T^U], [I^L, I^U], [F^L, F^U]\rangle$, for all i, then

$$IVNFWA(x_1, x_2, \cdots x_n) = IVNFWA(x, x, \cdots, x) = x \quad (17)$$

Proof: See Appendix B.

Theorem 4 (Monotonicity): Let $x_i = \langle [T_{x_i}^L, T_{x_i}^U], [I_{x_i}^L, I_{x_i}^U], [F_{x_i}^L, F_{x_i}^U]\rangle$ and $x_i' = \langle [T_{x_i}^{L\prime}, T_{x_i}^{U\prime}], [I_{x_i}^{L\prime}, I_{x_i}^{U\prime}], [F_{x_i}^{L\prime}, F_{x_i}^{U\prime}]\rangle (i = 1, 2, \cdots, n)$ be two sets of IVNNs. If $T_{x_i}^L \le T_{x_i}^{L\prime}$, $T_{x_i}^U \le T_{x_i}^{U\prime}$, $I_{x_i}^L \ge I_{x_i}^{L\prime}$, $I_{x_i}^U \ge I_{x_i}^{U\prime}$, $F_{x_i}^L \ge F_{x_i}^{L\prime}$, $F_{x_i}^U \ge F_{x_i}^{U\prime}$, for all i, then

$$IVNFWA(x_1, x_2, \cdots, x_n) \le IVNFWA(x_1', x_2', \cdots, x_n') \quad (18)$$

Proof: See Appendix C.

Theorem 5 (Boundedness): Let $x_i = \langle [T_{x_i}^L, T_{x_i}^U], [I_{x_i}^L, I_{x_i}^U], [F_{x_i}^L, F_{x_i}^U]\rangle$ be a set of IVNNs, then

$$x^- \le IVNFWA(x_1, x_2, \cdots, x_n) \le x^+ \quad (19)$$

Proof: See Appendix D.

In the following, several special cases of the IVNFWA operator can be examined.

Let $x_i = \langle [T_{x_i}^L, T_{x_i}^U], [I_{x_i}^L, I_{x_i}^U], [F_{x_i}^L, F_{x_i}^U]\rangle (i = 1, 2, \cdots, n)$ be a set of IVNNs, and $w = (w_1, w_2 \cdots, w_n)^{\mathrm{T}}$ be the weight vector of $x_i (i = 1, 2, \cdots, n)$, satisfying $w_i \in [0, 1]$ and $\sum_{i=1}^n w_i = 1$. Then, we have some properties below.

(1) When $\lambda \to 1$, the IVNFWA operator reduces to the interval-valued neutrosophic weighted average (IVNWA) operator given by Zhang *et al.*, [26] based on the Algebraic t-conorm and t-norm:

$$\lim_{x\to 1} IVNFWA(x_1, x_2, \cdots, x_n) = IVNWA(x_1, x_2, \cdots, x_n) = \langle [1-\prod_{i=1}^n(1-T_{x_i}^L)^{w_i}, 1-\prod_{i=1}^n(1-T_{x_i}^U)^{w_i}], [\prod_{i=1}^n(I_{x_i}^L)^{w_i}, \prod_{i=1}^n(I_{x_i}^U)^{w_i}], [\prod_{i=1}^n(F_{x_i}^L)^{w_i}, \prod_{i=1}^n(F_{x_i}^U)^{w_i}]\rangle \quad (20)$$

Proof: In order to prove (20), we only need to prove that

$$\lim_{\lambda\to 1}(1-\log_\lambda(1+\prod_{i=1}^{n}(\lambda^{1-T_{x_i}^L}-1)^{w_i}))=1-\prod_{i=1}^{n}(1-T_{x_i}^L)^{w_i},$$

$$\lim_{\lambda\to 1}(1-\log_\lambda(1+\prod_{i=1}^{n}(\lambda^{1-T_{x_i}^U}-1)^{w_i}))=1-\prod_{i=1}^{n}(1-T_{x_i}^U)^{w_i},$$

$$\lim_{\lambda\to 1}(\log_\lambda(1+\prod_{i=1}^{n}(\lambda^{I_{x_i}^L}-1)^{w_i}))=\prod_{i=1}^{n}(I_{x_i}^L)^{w_i},$$

$$\lim_{\lambda\to 1}(\log_\lambda(1+\prod_{i=1}^{n}(\lambda^{I_{x_i}^U}-1)^{w_i}))=\prod_{i=1}^{n}(I_{x_i}^U)^{w_i},$$

$$\lim_{\lambda\to 1}(\log_\lambda(1+\prod_{i=1}^{n}(\lambda^{F_{x_i}^L}-1)^{w_i}))=\prod_{i=1}^{n}(F_{x_i}^L)^{w_i},$$

$$\lim_{\lambda\to 1}(\log_\lambda(1+\prod_{i=1}^{n}(\lambda^{F_{x_i}^U}-1)^{w_i}))=\prod_{i=1}^{n}(F_{x_i}^U)^{w_i}.$$

(a) We first prove that $\lim_{\lambda\to 1}(\log_\lambda(1+\prod_{i=1}^{n}(\lambda^{I_{x_i}^L}-1)^{w_i}))=\prod_{i=1}^{n}(I_{x_i}^L)^{w_i}$.

$$\lim_{\lambda\to 1}(\log_\lambda(1+\prod_{i=1}^{n}(\lambda^{I_{x_i}^L}-1)^{w_i}))$$
$$=\lim_{\lambda\to 1}\frac{\ln(1+\prod_{i=1}^{n}(\lambda^{I_{x_i}^L}-1)^{w_i})}{\ln\lambda}=\lim_{\lambda\to 1}\frac{\prod_{i=1}^{n}(\lambda^{I_{x_i}^L}-1)^{w_i}}{\ln\lambda}$$
$$=\lim_{\lambda\to 1}\frac{\prod_{i=1}^{n}(e^{I_{x_i}^L\ln\lambda}-1)^{w_i}}{\ln\lambda}=\lim_{\lambda\to 1}\frac{\prod_{i=1}^{n}(I_{x_i}^L\ln\lambda)^{w_i}}{\ln\lambda}$$
$$=\lim_{\lambda\to 1}\frac{\prod_{i=1}^{n}(\ln\lambda)^{w_i}\cdot\prod_{i=1}^{n}(I_{x_i}^L)^{w_i}}{\ln\lambda}=\prod_{i=1}^{n}(I_{x_i}^L)^{w_i}.$$

Similarly, we have

$$\lim_{\lambda\to 1}(\log_\lambda(1+\prod_{i=1}^{n}(\lambda^{I_{x_i}^U}-1)^{w_i}))=\prod_{i=1}^{n}(I_{x_i}^U)^{w_i},$$
$$\lim_{\lambda\to 1}(\log_\lambda(1+\prod_{i=1}^{n}(\lambda^{F_{x_i}^L}-1)^{w_i}))=\prod_{i=1}^{n}(F_{x_i}^L)^{w_i},$$
$$\lim_{\lambda\to 1}(\log_\lambda(1+\prod_{i=1}^{n}(\lambda^{F_{x_i}^U}-1)^{w_i}))=\prod_{i=1}^{n}(F_{x_i}^U)^{w_i}.$$

(b) Based on (a), it yields that $\lim_{\lambda\to 1}(1-\log_\lambda(1+\prod_{i=1}^{n}(\lambda^{1-T_{x_i}^L}-1)^{w_i}))=1-\log_\lambda(1+\prod_{i=1}^{n}(\lambda^{1-T_{x_i}^L}-1)^{w_i})=1-\prod_{i=1}^{n}(1-T_{x_i}^L)^{w_i}$. Similarly, one has $\lim_{\lambda\to 1}(1-\log_\lambda(1+\prod_{i=1}^{n}(\lambda^{1-T_{x_i}^U}-1)^{w_i}))=1-\prod_{i=1}^{n}(1-T_{x_i}^U)^{w_i}$.

(c) Based on (a) and (b), we can obtain

$$\lim_{x\to 1}IVNFWA(x_1,x_2,\cdots,x_n)$$
$$=IVNWA(x_1,x_2,\cdots,x_n)=\langle[1-\prod_{i=1}^{n}(1-T_{x_i}^L)^{w_i},\ 1-\prod_{i=1}^{n}(1-T_{x_i}^U)^{w_i}],[\prod_{i=1}^{n}(I_{x_i}^L)^{w_i},\prod_{i=1}^{n}(I_{x_i}^U)^{w_i}],[\prod_{i=1}^{n}(F_{x_i}^L)^{w_i},\prod_{i=1}^{n}(F_{x_i}^U)^{w_i}]\rangle.$$

This completes the Proof of (20).

(2) When $\lambda\to\infty$, the IVNFWA operator reduces to traditional arithmetic weighted average operator:

$$\lim_{\lambda\to\infty}IVNFWA(x_1,x_2,\cdots,x_n)$$
$$=\langle[\sum_{i=1}^{n}w_iT_{x_i}^L,\sum_{i=1}^{n}w_iT_{x_i}^U],[\sum_{i=1}^{n}w_iI_{x_i}^L,\sum_{i=1}^{n}w_iI_{x_i}^U],[\sum_{i=1}^{n}w_iF_{x_i}^L,\sum_{i=1}^{n}w_iF_{x_i}^U]\rangle \tag{21}$$

Proof: In order to prove (21), we only need to prove that

$$\lim_{\lambda\to\infty}(1-\log_\lambda(1+\prod_{i=1}^{n}(\lambda^{1-T_{x_i}^L}-1)^{w_i}))=\sum_{i=1}^{n}w_iT_{x_i}^L,$$
$$\lim_{\lambda\to\infty}(1-\log_\lambda(1+\prod_{i=1}^{n}(\lambda^{1-T_{x_i}^U}-1)^{w_i}))=\sum_{i=1}^{n}w_iT_{x_i}^U,$$
$$\lim_{\lambda\to\infty}(\log_\lambda(1+\prod_{i=1}^{n}(\lambda^{I_{x_i}^L}-1)^{w_i}))=\sum_{i=1}^{n}w_iI_{x_i}^L,$$
$$\lim_{\lambda\to\infty}(\log_\lambda(1+\prod_{i=1}^{n}(\lambda^{I_{x_i}^U}-1)^{w_i}))=\sum_{i=1}^{n}w_iI_{x_i}^U,$$
$$\lim_{\lambda\to\infty}(\log_\lambda(1+\prod_{i=1}^{n}(\lambda^{F_{x_i}^L}-1)^{w_i}))=\sum_{i=1}^{n}w_iF_{x_i}^L,$$
$$\lim_{\lambda\to\infty}(\log_\lambda(1+\prod_{i=1}^{n}(\lambda^{F_{x_i}^U}-1)^{w_i}))=\sum_{i=1}^{n}w_iF_{x_i}^U.$$

(a) We first prove that $\lim_{\lambda\to\infty}(1-\log_\lambda(1+\prod_{i=1}^{n}(\lambda^{1-T_{x_i}^L}-1)^{w_i}))=\sum_{i=1}^{n}w_iT_{x_i}^L$. According to the L'Hospital's rule in Calculus, it can be obtained that

$$\lim_{\lambda\to\infty}(1-\log_\lambda(1+\prod_{i=1}^{n}(\lambda^{1-T_{x_i}^L}-1)^{w_i}))$$
$$=1-\lim_{\lambda\to\infty}\frac{\ln(1+\prod_{i=1}^{n}(\lambda^{1-T_{x_i}^L})^{w_i})}{\ln\lambda}$$
$$=1-\lim_{x\to\infty}\frac{(\ln(1+\prod_{i=1}^{n}(\lambda^{1-T_{x_i}^L})^{w_i}))'}{(\ln\lambda)'}$$
$$=1-\frac{\dfrac{\prod_{i=1}^{n}(\lambda^{1-T_{x_i}^L}-1)^{w_i}\sum_{i=1}^{n}\frac{w_i(1-T_{x_i}^L)}{\lambda-\lambda^{T_{x_i}^L}}}{1+\prod_{i=1}^{n}(\lambda^{1-T_{x_i}^L}-1)^{w_i}}}{\frac{1}{\lambda}}$$
$$=1-\frac{\dfrac{\prod_{i=1}^{n}(\lambda^{1-T_{x_i}^L}-1)^{w_i}\sum_{i=1}^{n}\frac{w_i(1-T_{x_i}^L)}{\lambda-\lambda^{T_{x_i}^L}}}{1+\prod_{i=1}^{n}(\lambda^{1-T_{x_i}^L}-1)^{w_i}}}{1+\prod_{i=1}^{n}(\lambda^{1-T_{x_i}^L}-1)^{w_i}}$$
$$=1-\sum_{i=1}^{n}w_i(1-T_{x_i}^L)=\sum_{i=1}^{n}w_iT_{x_i}^L.$$

Similarly, we obtain $\lim_{\lambda\to\infty}(1-\log_\lambda(1+\prod_{i=1}^{n}(\lambda^{1-T_{x_i}^U}-1)^{w_i}))=\sum_{i=1}^{n}w_iT_{x_i}^U$.

(b) We then prove that $\lim_{\lambda\to\infty}(\log_\lambda(1+\prod_{i=1}^{n}(\lambda^{I_{x_i}^L}-1)^{w_i}))=\sum_{i=1}^{n}w_iI_{x_i}^L$. By means of the L'Hospital's rule, we can obtain that

$$\lim_{\lambda\to\infty}(\log_\lambda(1+\prod_{i=1}^{n}(\lambda^{I_{x_i}^L}-1)^{w_i}))$$
$$=\lim_{\lambda\to\infty}(\frac{\ln(1+\prod_{i=1}^{n}(\lambda^{I_{x_i}^L}-1)^{w_i})}{\ln\lambda})$$
$$=\lim_{\lambda\to\infty}(\frac{(\ln(1+\prod_{i=1}^{n}(\lambda^{I_{x_i}^L}-1)^{w_i}))'}{(\ln\lambda)'})$$
$$=\lim_{\lambda\to\infty}(\frac{\frac{\prod_{i=1}^{n}(\lambda^{I_{x_i}^L}-1)^{w_i}\sum_{i=1}^{n}\frac{w_iI_{x_i}^L\lambda^{I_{x_i}^L-1}}{\lambda^{I_{x_i}^L}-1}}{1+\prod_{i=1}^{n}(\lambda^{I_{x_i}^L}-1)^{w_i}}}{\frac{1}{\lambda}})$$
$$=\lim_{\lambda\to\infty}(\frac{\frac{\prod_{i=1}^{n}(\lambda^{I_{x_i}^L}-1)^{w_i}\sum_{i=1}^{n}\frac{w_iI_{x_i}^L\lambda^{I_{x_i}^L-1}}{\lambda^{I_{x_i}^L}-1}}{1+\prod_{i=1}^{n}(\lambda^{I_{x_i}^L}-1)^{w_i}}}{1+\prod_{i=1}^{n}(\lambda^{I_{x_i}^L}-1)^{w_i}})=\sum_{i=1}^{n}w_iI_{x_i}^L.$$

Similarly, we have

$$\lim_{\lambda\to\infty}(\log_\lambda(1+\prod_{i=1}^{n}(\lambda^{I_{x_i}^U}-1)^{w_i}))=\sum_{i=1}^{n}w_iI_{x_i}^U,$$
$$\lim_{\lambda\to\infty}(\log_\lambda(1+\prod_{i=1}^{n}(\lambda^{F_{x_i}^L}-1)^{w_i}))=\sum_{i=1}^{n}w_iF_{x_i}^L,$$
$$\lim_{\lambda\to\infty}(\log_\lambda(1+\prod_{i=1}^{n}(\lambda^{F_{x_i}^U}-1)^{w_i}))=\sum_{i=1}^{n}w_iF_{x_i}^U.$$

(c) Based on (a) and (b), it is easy to verify that

$$\lim_{\lambda\to\infty}IVNFWA(x_1,x_2,\cdots,x_n)$$
$$=\langle[\sum_{i=1}^{n}w_iT_{x_i}^L,\sum_{i=1}^{n}w_iT_{x_i}^U],$$
$$[\sum_{i=1}^{n}w_iI_{x_i}^L,\sum_{i=1}^{n}w_iI_{x_i}^U],[\sum_{i=1}^{n}w_iF_{x_i}^L,\sum_{i=1}^{n}w_iF_{x_i}^U]\rangle.$$

This completes the proof of (21).

Theorem 6: Let $x_i=\langle[T_{x_i}^L,T_{x_i}^U],[I_{x_i}^L,I_{x_i}^U],[F_{x_i}^L,F_{x_i}^U]\rangle(i=1,2,\cdots,n)$ be a set of IVNNs, and let $w=(w_1,w_2,\cdots,w_n)^{\mathrm{T}}$ be the weight vector of $x_i(i=1,2,\cdots,n)$, satisfying $w_i\in[0,1]$ and $\sum_{i=1}^{n}w_1=1$, then the aggregated value by using the IVNFWG operator is also an IVNNs, and

$$IVNFWG(x_1,x_2,\cdots,x_n)$$
$$=\langle[\log_\lambda(1+\prod_{i=1}^{n}(\lambda^{T_{x_i}^L}-1)^{w_i}),$$
$$\log_\lambda(1+\prod_{i=1}^{n}(\lambda^{T_{x_i}^U}-1)^{w_i})],$$
$$[1-\log_\lambda(1+\prod_{i=1}^{n}(\lambda^{1-I_{x_i}^L}-1)^{w_i}),$$
$$1-\log_\lambda(1+\prod_{i=1}^{n}(\lambda^{1-I_{x_i}^U}-1)^{w_i})],$$
$$[1-\log_\lambda(1+\prod_{i=1}^{n}(\lambda^{1-F_{x_i}^L}-1)^{w_i}),$$
$$1-\log_\lambda(1+\prod_{i=1}^{n}(\lambda^{1-F_{x_i}^U}-1)^{w_i})]\rangle \tag{22}$$

The IVNFWG operator has some desirable characteristics similar to the IVNFWA operator as follows. It should be noted that the proofs of these characteristics are also similar to those of the IVNFWA operator. Therefore, we just list out these properties without further proofs.

Theorem 7 (Idempotency): Let $x_i=\langle[T_{x_i}^L,T_{x_i}^U],[I_{x_i}^L,I_{x_i}^U],[F_{x_i}^L,F_{x_i}^U]\rangle(i=1,2,\cdots,n)$ be a set of IVNNs, if all $x_i(i=1,2,\cdots,n)$ are equal, i.e., $x_i=x=\langle[T^L,T^U],[I^L,I^U],[F^L,F^U]\rangle$, for all i, then

$$IVNFWG(x_1,x_2,\cdots,x_n)=IVNFWG(x,x,\cdots,x)=x \tag{23}$$

Theorem 8 (Monotonicity): Let $x_i=\langle[T_{x_i}^L,T_{x_i}^U],[I_{x_i}^L,I_{x_i}^U],[F_{x_i}^L,F_{x_i}^U]\rangle$ and $x_i'=\langle[T_{x_i}^{L\prime},T_{x_i}^{U\prime}],[I_{x_i}^{L\prime},I_{x_i}^{U\prime}],[F_{x_i}^{L\prime},F_{x_i}^{U\prime}]\rangle$ $(i=1,2,\cdots,n)$ be two sets of IVNNs, if $T_{x_i}^L\le T_{x_i}^{L\prime}$, $T_{x_i}^U\le T_{x_i}^{U\prime}$, $I_{x_i}^L\ge I_{x_i}^{L\prime}$, $I_{x_i}^U\ge I_{x_i}^{U\prime}$, $F_{x_i}^L\ge F_{x_i}^{L\prime}$, $F_{x_i}^U\ge F_{x_i}^{U\prime}$, for all i, then

$$IVNFWG(x_1,x_2,\cdots,x_n)\le IVNFWG(x_1',x_2',\cdots,x_n') \tag{24}$$

Theorem 9 (Boundedness): Let $x_i=\langle[T_{x_i}^L,T_{x_i}^U],[I_{x_i}^L,I_{x_i}^U],[F_{x_i}^L,F_{x_i}^U]\rangle(i=1,2,\cdots,n)$ be a set of IVNNs, then

$$x^-\le IVNFWG(x_1,x_2,\cdots,x_n)\le x^+ \tag{25}$$

where $x^-=\langle[\min_i\{T_{x_i}^L\},\min_i\{T_{x_i}^U\}],[\max_i\{I_{x_i}^L\},\max_i\{I_{x_i}^U\}],[\max_i\{F_{x_i}^L\},\max_i\{F_{x_i}^U\}]\rangle$, $x^+=\langle[\max_i\{T_{x_i}^L\},\max_i\{T_{x_i}^U\}],[\min_i\{I_{x_i}^L\},\min_i\{I_{x_i}^U\}],[\min_i\{F_{x_i}^L\},\min_i\{F_{x_i}^U\}]\rangle$.

The IVNFWG operator has also some special cases similar to the IVNFWA operator, which are given as follows: Let $x_i=\langle[T_{x_i}^L,T_{x_i}^U],[I_{x_i}^L,I_{x_i}^U],[F_{x_i}^L,F_{x_i}^U]\rangle$ $(i=1,2,\cdots,n)$ be a set of IVNNs, and $w=(w_1,w_2,\cdots,w_n)^{\mathrm{T}}$ be the weight vector of $x_i(i=1,2,\cdots n)$, satisfying $w_i\in[0,1]$ and $\sum_{i=1}^{n}w_i=1$. Then, we have

(1) When $\lambda\to1$, the IVNFWG operator reduces to the interval-valued neutrosophic weighted geometric (IVNWG) operator developed by Zhang *et al.*, [26] based on the Algebraic t-conorm and t-norm:

$$\lim_{\lambda\to1}IVNFWG(x_1,x_2,\cdots,x_n)$$
$$=IVNWG(x_1,x_2,\cdots,x_n)$$
$$=\langle[\prod_{i=1}^{n}(T_{x_i}^L)^{w_i},\prod_{i=1}^{n}(T_{x_i}^U)^{w_i}],$$
$$[1-\prod_{i=1}^{n}(1-I_{x_i}^L)^{w_i},1-\prod_{i=1}^{n}(1-I_{x_i}^U)^{w_i}],$$
$$[1-\prod_{i=1}^{n}(1-F_{x_i}^L)^{w_i},1-\prod_{i=1}^{n}(1-F_{x_i}^U)^{w_i}]\rangle \tag{26}$$

(2) When $\lambda \to \infty$, the IVNFWG operator reduces to traditional arithmetic weighted average operator:

$$\lim_{\lambda\to\infty} IVNFWG(x_1, x_2, \cdots, x_n) = \langle[\sum_{i=1}^{n} w_i T_{x_i}^{L}, \sum_{i=1}^{n} w_i T_{x_i}^{U}], [\sum_{i=1}^{n} w_i I_{x_i}^{L}, \sum_{i=1}^{n} w_i I_{x_i}^{U}], [\sum_{i=1}^{n} w_i F_{x_i}^{L}, \sum_{i=1}^{n} w_i F_{x_i}^{U}]\rangle \quad (27)$$

V. TWO NEW APPROACHES FOR MAGDM BASED ON THE FRANK AGGREGATION OPERATOR

This section develops two new approaches based on the proposed operators to settling the MAGDM problem with IVNNs.

A. PRESENTATION OF THE PROBLEM FOR MAGDM WITH IVNNS

For convenience, let $M = \{1, 2, \ldots, m\}$, $N = \{1, 2, \cdots, n\}$, $K = \{1, 2, \ldots, s\}$. The MAGDM problem involved in this paper is depicted as follows.

Let $C = \{C_1, C_2, \cdots C_m\}$ be the set of m feasible alternatives, $D = \{D_1, D_2, \cdots D_n\}$ be the set of attributes and $E = \{e_1, e_2 \cdots, e_s\}$ be the set of DMs. Suppose that $w = \{w_1, w_2, \cdots w_n\}^{\mathrm{T}}$ is the attributes weight vector, where $0 \le w_j \le 1 (j \in N)$ and $\sum_{j=1}^{n} w_j = 1$. $\varpi = \{\varpi_1, \varpi_2, \cdots, \varpi_s\}^{\mathrm{T}}$ is the DMs weight, where $0 \le \varpi_k \le 1$ and $\sum_{k=1}^{s} \varpi_k = 1 (k \in K)$. These two kinds of weights are unknown in this paper. Assume that $R^{(k)} = (a_{ij}^{(k)})_{m\times n}$ is an IVN decision matrix, given by the decision make e_k where $a_{ij}^{(k)} = \langle[T_{ij}^{L(k)}, T_{ij}^{U(k)}], [I_{ij}^{L(k)}, I_{ij}^{U(k)}], [F_{ij}^{L(k)}, F_{ij}^{U(k)}]\rangle$ expresses the rating of the alternative C_i with respect to the attribute D_j by the DM e_k.

Generally speaking, there are benefit attributes and cost attributes in MAGMD. It is necessary to convert cost attribute values into beneficial attribute values. Therefore, $R^{(k)} = (a_{ij}^{(k)})_{m\times n}$ can be transformed into $\hat{R}^{(k)} = (\hat{a}_{ij}^{(k)})_{m\times n}$, in which

$$\hat{a}_{ij}^{k} = \begin{cases} a_{ij}^{(k)}, & \text{for benefit attribute } D_j \\ (a_{ij}^{(k)})^c, & \text{for cost attribute } D_j \end{cases} \quad (28)$$

where $(a_{ij}^{(k)})^c$ is the complement of $a_{ij}^{(k)}$ such that $(a_{ij}^{(k)})^c = \langle[F_{ij}^{L(k)}, F_{ij}^{U(k)}], [1 - I_{ij}^{U(k)}, 1 - I_{ij}^{L(k)}], [T_{ij}^{L(k)}, T_{ij}^{U(k)}]\rangle$.

B. DETERMINE DMS' WEIGHTS USING EXTENDED TOPSIS METHOD

1) DETERMINE THE POSITIVE IDEAL DECISION MATRIX (PIDM)

Motived by the literature [42], a PIDM of all individual decision matrices is defined as the average matrix of all individual decision matrices $R^k (k \in K)$ as follows:

$$\hat{R}^{+} = (\hat{\alpha}_{ij}^{+})_{m\times n} = ([\hat{T}_{ij}^{L(+)}, \hat{T}_{ij}^{U(+)}], [\hat{I}_{ij}^{L(+)}, \hat{I}_{ij}^{U(+)}], [\hat{F}_{ij}^{L(+)}, \hat{F}_{ij}^{U(+)}])_{m\times n} \quad (29)$$

where $\hat{a}_{ij}^{+} = \frac{\sum_{k=1}^{s} \hat{a}_{ij}^{k}}{s}$ and calculated by (3).

The reason why PIDM is the arithmetic mean matrix of all individual decision matrices is that the PIDM is maximum compromise among all DMs. For example, in a sports competition, the final score of each player is the average of all referees' scores.

2) DETERMINE ALL THE NEGATIVE IDEAL DECISION MATRICES (NIDM)

The NIDM include the individual negative ideal decision matrix (INIDM), the left individual negative ideal decision matrix (LINIDM) and the right individual negative ideal decision matrix (RINIDM).

It is well known that the INIDM should be farthest from the PIDM. It is expected that the complement of the PIDM should be farthest from the PIDM. For this reason, the INIDM is displayed as.

$$\hat{R}_c^{+} = ((\hat{\alpha}_{ij}^{+}))_{m\times n}^{c} = ([T_{ij}^{L(*)}, T_{ij}^{U(*)}], [I_{ij}^{L(*)}, I_{ij}^{U(*)}], [F_{ij}^{L(*)}, F_{ij}^{U(*)}])_{m\times n} \quad (30)$$

where $T_{ij}^{L(*)} = \hat{F}_{ij}^{L(+)}, T_{ij}^{U(*)} = \hat{F}_{ij}^{U(+)}, I_{ij}^{L(*)} = 1 - \hat{I}_{ij}^{U(+)}, I_{ij}^{U(*)} = 1 - \hat{I}_{ij}^{L(+)}, F_{ij}^{L(*)} = \hat{T}_{ij}^{L(+)}, F_{ij}^{U(*)} = \hat{T}_{ij}^{U(+)}$.

Moreover, LINIDM which is the left farthest from the PIDM is represented as.

$$\hat{R}^{l-} = (\hat{\alpha}_{ij}^{l-})_{m\times n} = ([\hat{T}_{ij}^{L(l-)}, \hat{T}_{ij}^{U(l-)}], [\hat{I}_{ij}^{L(l-)}, \hat{I}_{ij}^{U(l-)}], [\hat{F}_{ij}^{L(l-)}, \hat{F}_{ij}^{U(l-)}])_{m\times n} \quad (31)$$

where $\hat{T}_{ij}^{L(l-)} = \min_{k\in K}\{\hat{T}_{ij}^{L(k)}\}$, $\hat{T}_{ij}^{U(l-)} = \min_{k\in K}\{\hat{T}_{ij}^{U(k)}\}$, $\hat{I}_{ij}^{L(l-)} = \max_{k\in K}\{\hat{I}_{ij}^{L(k)}\}$, $\hat{I}_{ij}^{U(l-)} = \max_{k\in K}\{\hat{I}_{ij}^{U(k)}\}$, $\hat{F}_{ij}^{L(l-)} = \max_{k\in K}\{\hat{F}_{ij}^{L(k)}\}$, $\hat{F}_{ij}^{U(l-)} = \max_{k\in K}\{\hat{F}_{ij}^{U(k)}\}(i \in C, j \in D)$.

Similarly, RINIDM which is the right farthest from the PIDM is expressed as.

$$\hat{R}^{r-} = ((\hat{\alpha}_{ij}^{+}))_{m\times n}^{r-} = ([\hat{T}_{ij}^{L(r-)}, \hat{T}_{ij}^{U(r-)}], [\hat{I}_{ij}^{L(r-)}, \hat{I}_{ij}^{U(r-)}], [\hat{F}_{ij}^{L(r-)}, \hat{F}_{ij}^{U(r-)}])_{m\times n} \quad (32)$$

where $\hat{T}_{ij}^{L(r-)} = \max_{k\in K}\{\hat{T}_{ij}^{L(k)}\}$, $\hat{T}_{ij}^{U(r-)} = \max_{k\in K}\{\hat{T}_{ij}^{U(k)}\}$ $\hat{I}_{ij}^{L(r-)} = \min_{k\in K}\{\hat{I}_{ij}^{L(k)}\}$, $\hat{I}_{ij}^{U(r-)} = \min_{k\in K}\{\hat{I}_{ij}^{U(k)}\}$, $\hat{F}_{ij}^{L(r-)} = \min_{k\in K}\{\hat{F}_{ij}^{L(k)}\}$, $\hat{F}_{ij}^{U(r-)} = \min_{k\in K}\{\hat{F}_{ij}^{U(k)}\}(i \in C, j \in D)$.

According to the above analysis, the LINIDM is the minimum matrix and RINIDM is the maximum matrix for the individual decision matrices provided by all DMs.

3) COMPUTE THE CROSS-ENTROPY $I_{NS}(\hat{R}^k, \hat{R}^+)$, $I_{NS}(\hat{R}^k, \hat{R}_c^+)$, $I_{NS}(\hat{R}^k, \hat{R}^{l-})$, $I_{NS}(\hat{R}^k, \hat{R}^{r-})$ BY (7)

4) COMPUTE AN EXTENDED RELATIVE CLOSENESS

An extended relative closeness of DM e_k with respect to ideal decisions including $\hat{R}^+$, $\hat{R}_c^+$, $\hat{R}^{l-}$, and $\hat{R}^{r-}$ is calculated as (33), shown at the top of the next page.

It is evident that if the individual decision matrix of DM e_k is closer the PIDM and farther from INIDM, LINIDM and RININM, then the decision of DM e_k is better.

$$G^k = \frac{I_{NS}(\hat{R}^k, \hat{R}_c^+) + I_{NS}(\hat{R}^k, \hat{R}^{l-}) + I_{NS}(\hat{R}^k, \hat{R}^{r-})}{I_{NS}(\hat{R}^k, \hat{R}^+) + I_{NS}(\hat{R}^k, \hat{R}_c^+) + I_{NS}(\hat{R}^k, \hat{R}^{l-}) + I_{NS}(\hat{R}^k, \hat{R}^{r-})}(k \in K) \tag{33}$$

5) DETERMINE DMS' WEIGHTS

Since G^k shows the degree of closeness between individual decision matrix $\hat{R}^k$ and PIDM, the larger the G^k, the greater the weight ϖ_k of DM e_k that should be assigned.

After normalizing G^k, the weight ϖ_k of DM e_k can be derived as follows:

$$\varpi_k = \frac{G^k}{\sum_{k=1}^{k} G^k}(k \in K) \tag{34}$$

C. DETERMINE THE ATTRIBUTE WEIGHTS BASED ON THE SIMILARITY DEGREE

After getting the DM's weight values based on (34), the evaluation values provided by all DMs can be aggregated into the collective decision matrix $\tilde{R}$ by (16) or by (22) as follows:

$$\tilde{R} = (\tilde{\alpha}_{ij})_{m\times n} = ([\tilde{T}_{ij}^L, \tilde{T}_{ij}^U], [\tilde{I}_{ij}^L, \tilde{I}_{ij}^U], [\tilde{F}_{ij}^L, \tilde{F}_{ij}^U])_{m\times n} \tag{35}$$

1) DETERMINE ABSOLUTE POSITIVE IDEAL SOLUTION

Let $\beta_i = \langle[1, 1], [0, 0], [0, 0]\rangle$ be the nth largest IVNN. $\tilde{R}^+ = (\beta_1, \beta_2, \cdots, \beta_n)$ is called an interval-valued neutrosophic absolute positive ideal solution (INAPIS).

2) COMPUTE CROSS-ENTROPY OF $\tilde{a}_{ij}$ FROM $I_{NS}(\tilde{\alpha}_{ij}, \beta_i)$ BY (7)

3) COMPUTE THE SIMILARITY DEGREE

Denote the similarity degree between $\tilde{a}_{ij}$ and β_i by $S(\tilde{\alpha}_{ij}, \beta_i)$. Then, the similarity degree $S(\tilde{\alpha}_{ij}, \beta_i)$ is calculated as

$$S(\tilde{\alpha}_{ij}, \beta_i) = 1 - \frac{I_{NS}(\tilde{\alpha}_{ij}, \beta_i)}{\sum_{j=1}^{n} I_{NS}(\tilde{\alpha}_{ij}, \beta_i)} \tag{36}$$

The similarity between $\tilde{R}_j = (\tilde{a}_{1j}, \tilde{a}_{2j}, \cdots, \tilde{a}_{mj})$ and INAPIS $\tilde{R}^+ = (\beta_1, \beta_2, \cdots, \beta_n)$ is computed as follows:

$$S_j = \sum_{i=1}^{m} S(\tilde{\alpha}_{ij}, \beta_i) \tag{37}$$

4) DETERMINE ATTRIBUTE WEIGHTS

In accordance with the information theory, the attribute which has a large similarity with the INAPIS should have a large weight. Hence, the weight of the $jth(j \in N)$ attribute can be obtained as follows:

$$w_j = S_j \Big/ \sum_{j=1}^{n} S_j \tag{38}$$

D. TWO NEW APPROACHES FOR MAGDM UNDER IVN ENVIRONMENT

In the light of above analysis, two new approaches are presented for MAGDM with IVNNs.

1) THE FIRST APPROACH BASED ON THE IVNFWA OPERATOR

The decision-making steps of this approach are shown as follows.

Step 1: Establish each DM's IVN decision matrix $R^k = (\alpha_{ij}^k)_{m\times n}$.

Step 2: Transform the IVN decision matrix $R^k = (\alpha_{ij}^k)_{m\times n}$ into a normalized IVN decision matrix $\hat{R}^k = (\hat{a}_{ij}^k)_{m\times n}$ using (28).

Step 3: Determine weight vector $\varpi = (\varpi_1, \varpi_2, \cdots, \varpi_s)^{\mathrm{T}}$ of DMs by (34).

Step 4: Aggregate all individual decision matrices $\hat{R}^k = (\hat{a}_{ij}^k)_{m\times n}$ into the collective decision matrix $\tilde{R}$ by (16).

Step 5: Calculate weight vector $w = (w_1, w_2, \cdots, w_n)^{\mathrm{T}}$ of attributes by (38).

Step 6: Calculate the comprehensive evaluation value for each alternative.

Using (16) and weights of attributes, the comprehensive evaluation value for alternative C_i is calculated, denoted by T_i.

Step 7: Calculate the score values and accuracy values of T_i.

Use (4) and (5) to calculate the score values $S(T_i)$ and accuracy values $L(T_i)$ of comprehensive values T_i of the alternatives C_i, respectively.

Step 8: Rank all the alternatives and select the best one(s).

Use Definition 5 to rank all the alternatives and select the best one(s) according to $S(T_i)$ and $L(T_i)$.

2) THE SECOND APPROACH BASED ON THE IVNFWG OPERATOR

The decision-making steps of this approach are shown as follows.

Steps 1-3: The same as steps 1-3 in the first approach.

Step 4: Aggregate all individual decision matrix $\hat{R}^k = (\hat{a}_{ij}^k)_{m\times n}$ into the collective decision matrix $\tilde{R}$ by (22).

Step 5: Calculate weight vector of attributes.

It is the same as step 5 in the first approach.

Step 6: Calculate the comprehensive evaluation value for each alternative.

By using (22) and weights of attributes, the comprehensive evaluation value for alternative C_i is calculated, denoted by T_i.

Steps 7-8: the same as steps 7-8 in the first approach.

VI. A CASE STUDY OF THE SELECTION OF AN AGRICULTURAL SOCIALIZATION SERVICE (ASS) PROVIDER

In this section, a case study of ASS provider selection in china agriculture is presented to illustrate the practicality and

TABLE 1. Decision Matrices R^k Given by DM e_k.

DM	Alternative	Attribute			
		D_1	D_2	D_3	D_4
e_1	C_1	<[0.3,0.4], [0.6,0.7], [0.3,0.5]>	<[0.4,0.5], [0.2,0.3], [0.1,0.2]>	<[0.1,0.2], [0.4,0.5], [0.1,0.2]>	<[0.3,0.4], [0.5,0.6], [0.2,0.3]>
	C_2	<[0.5,0.7], [0.6,0.8], [0.2,0.4]>	<[0.5,0.6], [0.3,0.5], [0.2,0.3]>	<[0.5,0.7], [0.4,0.6], [0.2,0.3]>	<[0.6,0.7], [0.3,0.4], [0.2,0.3]>
	C_3	<[0.4,0.5], [0.5,0.6], [0.2,0.3]>	<[0.3,0.4], [0.5,0.6], [0.1,0.2]>	<[0.3,0.4], [0.1,0.2], [0.2,0.3]>	<[0.4,0.5], [0.1,0.2], [0.3,0.4]>
	C_4	<[0.6,0.7], [0.2,0.3], [0.1,0.2]>	<[0.4,0.5], [0.1,0.2], [0.2,0.3]>	<[0.4,0.5], [0.2,0.3], [0.1,0.2]>	<[0.3,0.4], [0.4,0.5], [0.2,0.3]>
e_2	C_1	<[0.4,0.6], [0.5,0.7], [0.3,0.4]>	<[0.6,0.7], [0.5,0.6], [0.5,0.6]>	<[0.5,0.6], [0.4,0.5], [0.3,0.4]>	<[0.6,0.7], [0.4,0.5], [0.3,0.4]>
	C_2	<[0.6,0.9], [0.4,0.5], [0.3,0.4]>	<[0.7,0.8], [0.6,0.7], [0.4,0.5]>	<[0.7,0.8], [0.3,0.4], [0.3,0.4]>	<[0.8,0.9], [0.4,0.5], [0.3,0.4]>
	C_3	<[0.8,0.9], [0.8,0.9], [0.4,0.5]>	<[0.7,0.8], [0.5,0.6], [0.5,0.6]>	<[0.7,0.8], [0.1,0.2], [0.3,0.4]>	<[0.8,0.9], [0.5,0.6], [0.2,0.3]>
	C_4	<[0.6,0.7], [0.3,0.4], [0.5,0.6]>	<[0.8,0.9], [0.5,0.6], [0.6,0.7]>	<[0.5,0.6], [0.2,0.3], [0.4,0.5]>	<[0.5,0.6], [0.7,0.9], [0.3,0.4]>
e_3	C_1	<[0.7,0.8], [0.4,0.5], [0.4,0.5]>	<[0.7,0.8], [0.3,0.4], [0.6,0.7]>	<[0.6,0.7], [0.3,0.4], [0.4,0.5]>	<[0.5,0.6], [0.4,0.5], [0.4,0.5]>
	C_2	<[0.6,0.7], [0.5,0.6], [0.4,0.5]>	<[0.7,0.8], [0.6,0.7], [0.5,0.6]>	<[0.8,0.9], [0.2,0.3], [0.7,0.8]>	<[0.6,0.7], [0.3,0.4], [0.4,0.6]>
	C_3	<[0.7,0.8], [0.3,0.4], [0.5,0.6]>	<[0.8,0.9], [0.2,0.4], [0.6,0.7]>	<[0.8,0.9], [0.2,0.4], [0.4,0.5]>	<[0.9,1.0], [0.1,0.2], [0.5,0.6]>
	C_4	<[0.7,0.8], [0.4,0.5], [0.6,0.7]>	<[0.6,0.9], [0.1,0.2], [0.7,0.8]>	<[0.6,0.7], [0.1,0.2], [0.5,0.6]>	<[0.6,0.7], [0.3,0.4], [0.4,0.5]>

effectiveness of the proposed MAGDM approaches. Meanwhile, comparison analyses are conducted to show the advantages of the proposed approaches.

A. BACKGROUND DESCRIPTION OF AN ASS PROVIDER SELECTION

The selection of the most suitable ASS provider is critical for agricultural enterprise to survive from fierce competitive environment. The choice of ASS provider should depend on a wide range of attributes such as quality of the service, price, reliability of the service, timeliness of the service, service variety, technological ability, market reputation, practical experience, supply capability, etc. Therefore, selecting ASS provider may be regarded as a type of MAGDM problem.

Jiangxi Green Energy Agricultural Development Co., Ltd (Green Energy for short) is a leading industry of agricultural industrialization. It was established in 2010 and settled in Dinghu Town, Anyi County, Jiangxi Province. To reduce the cost of agricultural production and improve the yield and profitability of agricultural products, this company (i.e., Green Energy) is going to make decision about outsourcing of ASS and has declared its intent to select ASS provider. After preliminary screening of ASS providers, this company can select one of four potential providers which are providers C_1, C_2, C_3 and C_4, respectively.

ASS providers are assessed by DMs with respect to four attributes: (1) the cost of service (D_1); (2) the quality of service (D_2); (3) market reputation (D_3); (4) technological ability (D_4), where D_2, D_3 and D_4 are benefit attributes, while D_1 is a cost attribute. Due to uncertainty of decision process, it is better that the rating of alternative C_i with respect to attribute D_j is represented by INN. To find most suitable ASS provider, Green Energy invites three experts, denoted by e_1, e_2 and e_3 to evaluate each ASS provider against four attributes. Table 1 lists all ratings of ASS providers against attributes provided by e_1, e_2 and e_3.

B. RESOLUTION PROCESS OF THE PROPOSED APPROACHES OF THIS PAPER

Next, the proposed two approaches are carried out to derive the most desirable provider.

1) THE FIRST APPROACH

To get the best provider, the following steps are involved:

TABLE 2. Normalized Decision Matrices $\hat{R}^k$.

DM	Alternative	Attribute D_1	D_2	D_3	D_4
e_1	C_1	<[0.3,0.5], [0.3,0.4], [0.3,0.4]>	<[0.4,0.5], [0.2,0.3], [0.1,0.2]>	<[0.1,0.2], [0.4,0.5], [0.1,0.2]>	<[0.3,0.4], [0.5,0.6], [0.2,0.3]>
	C_2	<[0.2,0.4], [0.2,0.4], [0.5,0.7]>	<[0.5,0.6], [0.3,0.5], [0.2,0.3]>	<[0.5,0.7], [0.4,0.6], [0.2,0.3]>	<[0.6,0.7], [0.3,0.4], [0.2,0.3]>
	C_3	<[0.2,0.3], [0.4,0.5], [0.4,0.5]>	<[0.3,0.4], [0.5,0.6], [0.1,0.2]>	<[0.3,0.4], [0.1,0.2], [0.2,0.3]>	<[0.4,0.5], [0.1,0.2], [0.3,0.4]>
	C_4	<[0.1,0.2], [0.7,0.8], [0.6,0.7]>	<[0.4,0.5], [0.1,0.2], [0.2,0.3]>	<[0.4,0.5], [0.2,0.3], [0.1,0.2]>	<[0.3,0.4], [0.4,0.5], [0.2,0.3]>
e_2	C_1	<[0.3,0.4], [0.3,0.5], [0.4,0.6]>	<[0.6,0.7], [0.5,0.6], [0.5,0.6]>	<[0.5,0.6], [0.4,0.5], [0.3,0.4]>	<[0.6,0.7], [0.4,0.5], [0.3,0.4]>
	C_2	<[0.3,0.4], [0.5,0.6], [0.6,0.9]>	<[0.7,0.8], [0.6,0.7], [0.4,0.5]>	<[0.7,0.8], [0.3,0.4], [0.3,0.4]>	<[0.8,0.9], [0.4,0.5], [0.3,0.4]>
	C_3	<[0.4,0.5], [0.1,0.2], [0.8,0.9]>	<[0.7,0.8], [0.5,0.6], [0.5,0.6]>	<[0.7,0.8], [0.1,0.2], [0.3,0.4]>	<[0.8,0.9], [0.5,0.6], [0.2,0.3]>
	C_4	< [0.5,0.6], [0.6,0.7], [0.6,0.7]>	<[0.8,0.9], [0.5,0.6], [0.6,0.7]>	<[0.5,0.6], [0.2,0.3], [0.4,0.5]>	<[0.5,0.6], [0.7,0.9], [0.3,0.4]>
e_3	C_1	<[0.4,0.5], [0.5,0.6], [0.7,0.8]>	<[0.7,0.8], [0.3,0.4], [0.6,0.7]>	<[0.6,0.7], [0.3,0.4], [0.4,0.5]>	<[0.5,0.6], [0.4,0.5], [0.4,0.5]>
	C_2	<[0.4,0.5], [0.4,0.5], [0.6,0.7]>	<[0.7,0.8], [0.6,0.7], [0.5,0.6]>	<[0.8,0.9], [0.2,0.3], [0.7,0.8]>	<[0.6,0.7], [0.3,0.4], [0.4,0.6]>
	C_3	<[0.5,0.6], [0.6,0.7], [0.7,0.8]>	<[0.8,0.9], [0.2,0.4], [0.6,0.7]>	<[0.8,0.9], [0.2,0.4], [0.4,0.5]>	<[0.9,1.0], [0.1,0.2], [0.5,0.6]>
	C_4	<[0.6,0.7], [0.5,0.6], [0.7,0.8]>	<[0.6,0.9], [0.1,0.2], [0.7,0.8]>	<[0.6,0.7], [0.1,0.2], [0.5,0.6]>	<[0.6,0.7], [0.3,0.4], [0.4,0.5]>

TABLE 3. Cross-Entropy of Each DM.

$I_{NS}(R^1,R_c^+)$	$I_{NS}(R^2,R_c^+)$	$I_{NS}(R^3,R_c^+)$	$I_{NS}(R^1,R^{l-})$	$I_{NS}(R^2,R^{l-})$	$I_{NS}(R^3,R^{l-})$
0.1854	0.1368	0.1760	0.1259	0.0984	0.1232
$I_{NS}(R^1,R^{r-})$	$I_{NS}(R^2,R^{r-})$	$I_{NS}(R^3,R^{r-})$	$I_{NS}(R^1,R^+)$	$I_{NS}(R^2,R^+)$	$I_{NS}(R^3,R^+)$
0.0211	0.0218	0.0191	0.0723	0.0274	0.0258

Step 1: Using (28), we transform the IVN decision matrices $R^k = (\alpha_{ij}^k)_{4\times4}$ into the normalized IVN decision matrices $\hat{R}^k = (\hat{a}_{ij}^k)_{4\times4}$ (see Table 2).

Step 2: Determine the weights of DMs.

Using (29)-(32), PIDM, INIDM, LINIDM and RINIDM are easily identified.

Then by (7), we get $I_{NS}(R^k, R_c^+)$, $I_{NS}(R^k, R^{l-})$ and $I_{NS}(R^k, R^{r-})$ shown in Table 3.

Using (33), the extended relative closeness degrees of three DMs are calculated as $G^1 = 0.8214$, $G^2 = 0.9038$, $G^3 = 0.9251$.

By normalizing G^1, G^2 and G^3, the weights of three DMs are obtained as follows: $\varpi_1 = 0.310$, $\varpi_2 = 0.341$, $\varpi_3 = 0.349$.

Step 3: Obtain the collective decision matrix.

By (16) with $\lambda = 2$, the collective decision matrix is acquired as $\widetilde{R}$, shown at the top of the next page.

Step 4: Determine the attribute weights.

Combining (7) with (36)-(38), the weights of four attributes are calculated as follows:

$$w_1 = 0.22, w_2 = 0.25, w_3 = 0.27, w_4 = 0.26.$$

Step 5: Compute the comprehensive values of the alternatives C_i using (16) as follows:

$$C_1 = \langle[0.474, 0.585], [0.362, 0.476], [0.315, 0.435]\rangle,$$
$$C_2 = \langle[0.616, 0.739], [0.354, 0.486], [0.375, 0.506]\rangle,$$
$$C_3 = \langle[0.648, 1.000], [0.215, 0.357], [0.364, 0.477]\rangle,$$
$$C_4 = \langle[0.527, 0.669], [0.281, 0.410], [0.388, 0.502]\rangle.$$

$$\widetilde{R}=\begin{pmatrix}
\left\langle\begin{matrix}[0.336,0.468],\\ [0.360,0.498],\\ [0.449,0.589]\end{matrix}\right\rangle & \left\langle\begin{matrix}[0.587,0.693],\\ [0.317,0.423],\\ [0.333,0.459]\end{matrix}\right\rangle & \left\langle\begin{matrix}[0.439,0.545],\\ [0.351,0.463],\\ [0.239,0.353]\end{matrix}\right\rangle & \left\langle\begin{matrix}[0.483,0.587],\\ [0.429,0.529],\\ [0.294,0.397]\end{matrix}\right\rangle\\
\left\langle\begin{matrix}[0.308,0.437],\\ [0.351,0.498],\\ [0.567,0764]\end{matrix}\right\rangle & \left\langle\begin{matrix}[0.647,0.752],\\ [0.488,0.632],\\ [0.352,0.458]\end{matrix}\right\rangle & \left\langle\begin{matrix}[0.693,0.821],\\ [0.277,0.413],\\ [0.362,0.437]\end{matrix}\right\rangle & \left\langle\begin{matrix}[0.683,0.792],\\ [0.311,0.432],\\ [0.294,0.424]\end{matrix}\right\rangle\\
\left\langle\begin{matrix}[0.382,0.484],\\ [0.295,0.419],\\ [0.621,0.725]\end{matrix}\right\rangle & \left\langle\begin{matrix}[0.656,0.774],\\ [0.367,0.522],\\ [0.333,0.459]\end{matrix}\right\rangle & \left\langle\begin{matrix}[0.656,0.774],\\ [0.118,0.256],\\ [0.294,0.397]\end{matrix}\right\rangle & \left\langle\begin{matrix}[0.774,1.000],\\ [0.177,0.295],\\ [0.315,0.420]\end{matrix}\right\rangle\\
\left\langle\begin{matrix}[0.439,0.545],\\ [0.592,0.692],\\ [0.634,0.734]\end{matrix}\right\rangle & \left\langle\begin{matrix}[0.638,0.832],\\ [0.177,0.295],\\ [0.459,0.573]\end{matrix}\right\rangle & \left\langle\begin{matrix}[0.509,0.611],\\ [0.149,0.261],\\ [0.287,0.407]\end{matrix}\right\rangle & \left\langle\begin{matrix}[0.484,0.587],\\ [0.442,0.572],\\ [0.294,0.397]\end{matrix}\right\rangle
\end{pmatrix}$$

Step 6: Calculate the score values of the comprehensive values of the alternatives as follows:

$$S(C_1)=0.156,\quad S(C_2)=0.197,$$
$$S(C_3)=0.415,\quad S(C_4)=0.225.$$

Step 7: Rank the alternatives C_i and select the best one.

In terms of the score values, the ranking order of alternatives is $C_3 \succ C_4 \succ C_2 \succ C_1$, where the symbol "$\succ$" means "superior to". Thus, the most desirable provider is C_3.

2) THE SECOND APPROACH

To get the best provider by the second approach, the following steps are involved:

Steps 1-2 are the same as steps 1-2 in the first approach, so they are omitted here. The results are shown in Table 2 and Table 3.

Step 3: Obtain the collective decision matrix.

By (22) with $\lambda = 2$, the collective decision matrix is acquired as $\widetilde{R}$, shown at the bottom of the next page.

Step 4: Determine the attribute weights.

Combining (7) with (36)-(38), the weights of four attributes are calculated as follows:

$$w_1=0.22,\quad w_2=0.25,\quad w_3=0.27,\quad w_4=0.26.$$

Step 5: Compute the comprehensive values of the alternatives C_i using (22) as follows:

$$C_1=\langle[0.410,0.530],[0.384,0.491],[0.386,0.502]\rangle,$$
$$C_2=\langle[0.540,0.668],[0.390,0.512],[0.435,0.596]\rangle,$$
$$C_3=\langle[0.532,0.641],[0.304,0.424],[0.458,0.573]\rangle,$$
$$C_4=\langle[0.460,0.585],[0.402,0.536],[0.475,0.584]\rangle.$$

Step 6: Calculate the score values of the comprehensive values of the alternatives as follows:

$$S(C_1)=0.080,\quad S(C_2)=0.103,$$
$$S(C_3)=0.185,\quad S(C_4)=0.043.$$

Step 7: Rank the alternatives C_i and select the best one.

In terms of the score values, the ranking order of alternatives is $C_3 \succ C_2 \succ C_1 \succ C_4$ where the symbol "$\succ$" means "superior to". Thus, the most desirable provider is C_3.

C. INFLUENCE OF PARAMETER λ ON THE FINAL GROUP DECISION-MAKING RESULTS

It should be noted that we take the value of parameter $\lambda = 2$ in the above analysis. In fact, different DMs can set different values of parameter λ according to their different preferences, which will have different effects on the ranking of alternatives. To examine influence of the parameter λ on the ranking of four ASS providers, we assign λ some different values between 1 to 100 and computer the scores of these four ASS providers. The ranking results with respect to the IVNFWA and IVNFWG operators are shown in Table 4.

Table 4 shows that the scores of alternatives based on IVNFWA operator decrease with the increase of parameter λ, but the ordering of four alternatives is always $C_3 \succ C_4 \succ C_2 \succ C_1$ and the best choice is always C_3.

As we can see from Table 4, the scores of alternatives by using IVNFWG operator increase with the increase of parameter λ. Although there are slight changes in the rankings among the four alternatives, the best one still is C_3.

By further analyzing Fig. 1 and Fig. 2, we can intuitively find that the scores of alternatives based on IVNFWA operator are larger than those based on IVNFWG operator for the same value of the parameter λ. The IVNFWA operator can obtain more optimistic expectations and be regarded as an optimistic operator, while the IVNFWG operator can obtain more pessimistic expectations and can be regarded as a pessimistic operator. Thus, the parameter λ can be considered as the risk attitude of DMs. It is concluded that the pessimistic DMs could use the IVIFFWG operator and select larger values for the parameter λ, while the optimistic DMs could use the IVIFFWA operator and select smaller values for the parameter λ.

D. DISCUSSION ON THE RELIABILITY OF THE PROPOSED APPROACHES

We test the relative performance of the proposed approaches based on three test criteria established by Wang and Triantaphyllou [43]. These three criteria are as follows:

Test Criterion 1: An effective MADM method should not change the indication of the best alternative when a worse

TABLE 4. Ranking Orders of the Alternatives With Different Values of Parameter λ.

	IVNFWA operator		IVNFWG operator	
Parameter λ	Score value $S(C_i)$	Ranking order	Score value $S(C_i)$	Ranking order
$\lambda \to 1$	$S(C_1)$ =0.160 $S(C_2)$ =0.203 $S(C_3)$ =0.423 $S(C_4)$ =0.237	$C_3 \succ C_4 \succ C_2 \succ C_1$	$S(C_1)$ =0.058 $S(C_2)$ =0.078 $S(C_3)$ =0.150 $S(C_4)$ =0.003	$C_3 \succ C_2 \succ C_1 \succ C_4$
$\lambda = 2$	$S(C_1)$ =0.156 $S(C_2)$ =0.197 $S(C_3)$ =0.415 $S(C_4)$ =0.225	$C_3 \succ C_4 \succ C_2 \succ C_1$	$S(C_1)$ =0.080 $S(C_2)$ =0.103 $S(C_3)$ =0.185 $S(C_4)$ =0.043	$C_3 \succ C_2 \succ C_1 \succ C_4$
$\lambda = 5$	$S(C_1)$ =0.147 $S(C_2)$ =0.187 $S(C_3)$ =0.402 $S(C_4)$ =0.208	$C_3 \succ C_4 \succ C_2 \succ C_1$	$S(C_1)$ =0.090 $S(C_2)$ =0.115 $S(C_3)$ =0.203 $S(C_4)$ =0.056	$C_3 \succ C_2 \succ C_1 \succ C_4$
$\lambda = 10$	$S(C_1)$ =0.142 $S(C_2)$ =0.181 $S(C_3)$ =0.393 $S(C_4)$ =0.196	$C_3 \succ C_4 \succ C_2 \succ C_1$	$S(C_1)$ =0.095 $S(C_2)$ =0.121 $S(C_3)$ =0.214 $S(C_4)$ =0.076	$C_3 \succ C_2 \succ C_1 \succ C_4$
$\lambda = 50$	$S(C_1)$ =0.133 $S(C_2)$ =0.169 $S(C_3)$ =0.375 $S(C_4)$ =0.196	$C_3 \succ C_4 \succ C_2 \succ C_1$	$S(C_1)$ =0.104 $S(C_2)$ =0.132 $S(C_3)$ =0.231 $S(C_4)$ =0.096	$C_3 \succ C_2 \succ C_1 \succ C_4$
$\lambda = 100$	$S(C_1)$ =0.130 $S(C_2)$ =0.165 $S(C_3)$ =0.369 $S(C_4)$ =0.166	$C_3 \succ C_4 \succ C_2 \succ C_1$	$S(C_1)$ =0.107 $S(C_2)$ =0.136 $S(C_3)$ =0.236 $S(C_4)$ =0.102	$C_3 \succ C_2 \succ C_1 \succ C_4$

alternative is substituted for a non-optimal alternative without change in the relative importance of each decision criterion.

Test Criterion 2: An effective MADM method should follow the transitive property.

Test Criterion 3: For the same decision problem and based on the same MADM method, after a MADM problem is decomposed into a group of smaller problems, the new overall ranking of alternatives, combining the ranking of a group of smaller problems, should be identical to the original overall ranking of un-decomposed problem.

Next, these three criteria are used to evaluate the reliability of the alternatives' rankings obtained by the proposed approaches.

$$\widetilde{R} = \begin{pmatrix} \left\langle \begin{matrix}[0.332, 0.464],\\ [0.376, 0.509],\\ [0.501, 0.640]\end{matrix} \right\rangle & \left\langle \begin{matrix}[0.561, 0.663],\\ [0.347, 0.449],\\ [0.439, 0.545]\end{matrix} \right\rangle & \left\langle \begin{matrix}[0.333, 0.459],\\ [0.377, 0.467],\\ [0.281, 0.382]\end{matrix} \right\rangle & \left\langle \begin{matrix}[0.457, 0.560],\\ [0.433, 0.533],\\ [0.308, 0.408]\end{matrix} \right\rangle \\ \left\langle \begin{matrix}[0.294, 0.433],\\ [0.381, 0.509],\\ [0.571, 0792]\end{matrix} \right\rangle & \left\langle \begin{matrix}[0.632, 0.733],\\ [0.522, 0.647],\\ [0.382, 0.484]\end{matrix} \right\rangle & \left\langle \begin{matrix}[0.663, 0.801],\\ [0.311, 0.439],\\ [0.449, 0.562]\end{matrix} \right\rangle & \left\langle \begin{matrix}[0.663, 0.764],\\ [0.335, 0.436],\\ [0.308, 0.451]\end{matrix} \right\rangle \\ \left\langle \begin{matrix}[0.352, 0.458],\\ [0.395, 0.502],\\ [0.672, 0.787]\end{matrix} \right\rangle & \left\langle \begin{matrix}[0.573, 0.681],\\ [0.408, 0.538],\\ [0.439, 0.545]\end{matrix} \right\rangle & \left\langle \begin{matrix}[0.573, 0.681],\\ [0.139, 0.275],\\ [0.308, 0.408]\end{matrix} \right\rangle & \left\langle \begin{matrix}[0.681, 0.786],\\ [0.257, 0.362],\\ [0.346, 0.448]\end{matrix} \right\rangle \\ \left\langle \begin{matrix}[0.333, 0.459],\\ [0.603, 0.706],\\ [0.638, 0.739]\end{matrix} \right\rangle & \left\langle \begin{matrix}[0.588, 0.756],\\ [0.257, 0.362],\\ [0.545, 0.656]\end{matrix} \right\rangle & \left\langle \begin{matrix}[0.498, 0.599],\\ [0.172, 0.266],\\ [0.358, 0.461]\end{matrix} \right\rangle & \left\langle \begin{matrix}[0.458, 0.561],\\ [0.495, 0.683],\\ [0.308, 0.408]\end{matrix} \right\rangle \end{pmatrix}$$

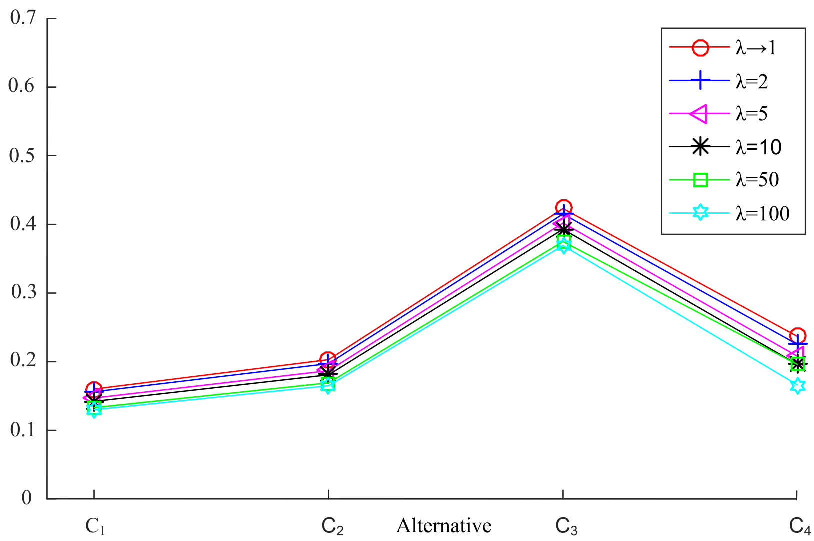

FIGURE 1. Score values for alternatives obtained by IVNFWA operator.

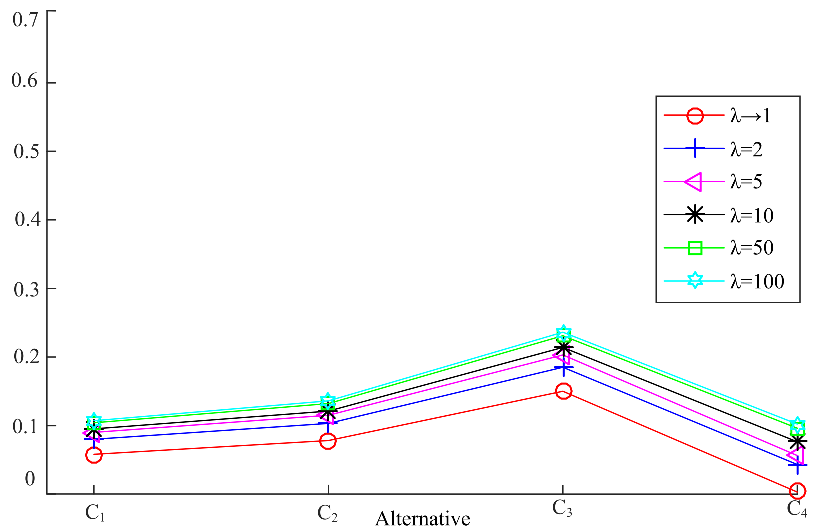

FIGURE 2. Score values for alternatives obtained by IVNFWA operator.

1) RELIABILITY TEST OF THE RESULTS OBTAINED BY THE PROPOSED APPROACHES UNDER CRITERION 1

To evaluate the reliability of the alternatives' rankings obtained by the proposed approaches in terms of Criterion 1, a non-optimum alternative C_1 was randomly replaced by the worse one C_1' shown in Table 5.

Other data remain the same as before. For the modified MAGDM problem, following the steps of the first approach, the score values of the comprehensive values of the alternatives are generated as: $S(C_1') = -0.053$, $S(C_2) = 0.196$, $S(C_3) = 0.413$, $S(C_4) = 0.226$. In terms of the score values, the ranking order of alternatives is $C_3 \succ C_4 \succ C_2 \succ C_1'$. Therefore, the most desirable provider is still C_3. Similarly, the score values of the comprehensive values of the alternatives for the second approach are generated as: $S(C_1') = 0.128$, $S(C_2) = 0.148$, $S(C_3) = 0.241$, $S(C_4) = -0.102$. The ranking order of alternatives is $C_3 \succ C_2 \succ C_1' \succ C_4$. Thus, the most desirable provider is also C_3.

TABLE 5. Evaluation Values of Alternative C_1' for Different DMs.

Alternative	DM	Attribute			
		D_1	D_2	D_3	D_4
C_1'	e_1	<[0.2,0.4], [0.4,0.5], [0.4,0.5]>	<[0.3,0.4], [0.3,0.4], [0.2,0.3]>	<[0.1,0.2], [0.5,0.6], [0.2,0.3]>	<[0.2,0.3], [0.6,0.7], [0.3,0.4]>
	e_2	<[0.2,0.3], [0.4,0.6], [0.5,0.7]>.	<[0.5,0.6], [0.6,0.7], [0.6,0.7]>	<[0.4,0.5], [0.5,0.6], [0.4,0.5]>	<[0.5,0.6], [0.5,0.6], [0.4,0.5]>
	e_3	<[0.3,0.4], [0.6,0.7], [0.8,0.9]>	<[0.6,0.7], [0.4,0.5], [0.7,0.8]>	<[0.5,0.6], [0.4,0.5], [0.5,0.6]>	<[0.4,0.5], [0.5,0.6], [0.5,0.6]>

TABLE 6. The Comparison With Different Methods.

Methods	The final ranking	The best alternative
Method 1 based on IVNN VIKOR in [20]	$C_3 \succ C_2 \succ C_1 \succ C_4$	C_3
Method 1 with IVNWA operator in [26]	$C_3 \succ C_4 \succ C_2 \succ C_1$	C_3
Method 2 with IVNWG operator in [26]	$C_3 \succ C_2 \succ C_1 \succ C_4$	C_3
Proposed approach with IVNFWA operator	$C_3 \succ C_4 \succ C_2 \succ C_1$	C_3
Proposed approach with IVNFWG operator	$C_3 \succ C_2 \succ C_1 \succ C_4$	C_3

It is obvious that the most desirable provider for modified MAGDM problem under the two proposed approaches is still C_3, which is the same as that for the original problem. As a result, the proposed approaches do not change the indication of the best alternative when a worse alternative is substituted for a non-optimal alternative. Thus, the proposed approaches are reliable under test Criterion 1.

2) RELIABILITY TEST OF THE RESULTS OBTAINED BY THE PROPOSED APPROACHES UNDER CRITERIA 2 AND 3

To evaluate the reliability of the alternatives' rankings obtained by the proposed approaches in terms of Criteria 2 and 3, original MAGDM problem is divided into six groups of smaller MAGDM problems $\{C_1, C_2\}$, $\{C_1, C_3\}$, $\{C_1, C_4\}$,$\{C_2, C_3\}$,$\{C_2, C_4\}$and $\{C_3, C_4\}$, respectively. Using the same steps of the first approach, corresponding rankings are respectively derived as: $C_2 \succ C_1$, $C_3 \succ C_1$, $C_4 \succ C_1$, $C_3 \succ C_2$, $C_4 \succ C_2$ and $C_3 \succ C_4$. Combined the rankings of six sub-problems, the final overall ranking of alternatives for the first approach is consistent with the original overall ranking of un-decomposed problem. Similarly, using the same steps of the second approach, corresponding rankings are respectively derived as: $C_2 \succ C_1$, $C_3 \succ C_1$, $C_1 \succ C_4$, $C_3 \succ C_2$, $C_2 \succ C_4$ and $C_3 \succ C_4$. Combined the rankings of six sub-problems, the final overall ranking of alternatives for the second approach is consistent with the original overall ranking of un-decomposed problem. Thus, the proposed approaches are reliable under test Criteria 2 and 3.

E. COMPARATIVE ANALYSES WITH THE EXISTING METHODS

In this subsection, some comparative analyses are conducted to illustrate the effectiveness and advantages of the proposed approaches of this paper.

For MADM in the IVN environment, Zhang et al. [26] used IVNWA and IVNWG operators to aggregate the performance of each criterion for alternatives and then calculated the degrees of possibility of alternatives. Huang et al. [20] proposed the VIKOR method for MAGDM with IVNSs. Since Zhang et al. [26] only considered the single decision-making problem, we use the IVNGA operator to integrate the individual decision matrices of the above ASS provider selection example to obtain the collective decision matrix. Then, for this collective decision matrix, the ranking order of all providers is derived by method in [20]. At the same time, we directly employ method in [20] to solve the above ASS provider selection example. The decision results obtained by different methods are given in Table 6.

It is easily seen from Table 6 that the ranking orders obtained by method in [26] are the same as those by our proposed two approaches. The ranking orders obtained by method in [20] are the same as the ones obtained by our proposed approach with IVNFWA operator. These further verify the effectiveness and stability of the proposed approaches of this paper.

By comparison with methods in [20], [26], three main advantages of our approaches are shown as follows:

Firstly, method [20] cannot deal with MAGDM with unknown weights of attributes and DMs, whereas our approaches are capable of solving MAGDM with unknown attribute weights and DM weights. In real-life MAGDM, both attribute weights and DM weights are usually unknown. This paper sufficiently takes this situation into consideration and can make the result of decision more consistent with actual situations.

Secondly, the IVNWA and IVNWG operators, proposed by Zhang et al. [26], are the aggregation operators based on Algebraic product and Algebraic sum. They are only the special cases of the IVNFWA and IVNFWG operators in (20) and (26) with parameter $\lambda \rightarrow 1$. The arithmetic and geometric aggregated assessments based on IVNFWA and IVNFWG operators are monotonically decreasing and increasing respectively relating to parameter λ, which enables DMs to choose appropriate value based on their risk attitudes. In addition, the arithmetic aggregated assessment is larger than the geometric aggregated assessment for the same parameter λ. Therefore, the proposed IVNFWA and IVNFWG operators can provide more choices for MDs with the help of parameter λ, in other word, they are more flexible for decision-making.

Finally, method [26] is only applicable for MADM problems and cannot be used to solve MAGDM problems. The proposed approaches of this paper can not only solve MAGDM but also solve MADM. Therefore, the proposed approaches of this paper have wide scope of applications.

To measure the ranking differences between approaches [20], [26] and the proposed approaches, Spearman's rank-correlation test, a technique allowing for ascertaining whether there is statistically significant rank-correlation between two sets of values, is applied to determine the ranking differences.

In this test, ranking values of alternatives are calculated by their corresponding ranking orders. For example, given a ranking order $C_4 \succ C_2 \succ C_1 \succ C_3$, ranking values of alternatives C_1, C_2, C_3 and C_4 are 3, 2, 4 and 1, respectively. In order to measure the ranking differences between two ranking orders, two statistics r_s and Z are computed as follows:

$$r_s = 1 - \frac{6}{n(n^2-1)} \sum\nolimits_{i=1}^{n} (d_i)^2 \tag{39}$$

$$Z = r_s\sqrt{n-1} \tag{40}$$

where d_i is the ranking difference of alternative C_i between two ranking orders, n is the number of alternatives. In the light of [44], r_s is Spearman's rank-correlation coefficient which ranks from -1 to 1, where -1 shows a perfect negative relationship between the two ranking orders, while 1 shows a perfect positive relationship between the two ranking orders. In addition, the closer the value of r_s is to 1 or -1, the stronger the correlation between two ranking orders. Z is a test statistic. If $Z \geq 1.645$, thereby it can be pointed out that there is evidence of a positive relation between two ranking orders. Otherwise, it is believed that the two ranking orders are different. Using (39) and (40), differences between ranking orders of alternatives obtained by methods [20], [26] and those generated by the proposed approaches are given in Table 7.

TABLE 7. Ranking by two Methods and Their Differences.

Providers	Ranking			Ranking differences	
	The proposed second approach (A)	Method in [20] (B)	Method in [26] (C)	A-B	A-C
C_1	3	3	3	0	0
C_2	2	2	2	0	0
C_3	1	1	1	0	0
C_4	4	4	4	0	0
Spearman's test result			r_s	1	1
			Z	1.732	1.732

It can been seen from Table 7 that the values of r_s obtained by methods [20], [26] and the proposed second approach are all equal to 1 and the values of Z are all equal to 1.732, which exceeds the critical 1.645. Therefore, it is concluded that the rankings obtained by methods [20], [26] are perfectly positively correlated with the proposed second approach. For the proposed first approach, the same conclusion holds. Thus, Spearman's rank-correlation test fully testifies the validity of the proposed approaches.

VII. CONCLUSION

This paper extends the Frank t-conorm and t-norm to IVN environment. Then some new operational laws for IVNNs are defined and their related properties are investigated. Based on these new operational laws, several new aggregation operators for IVNNs are developed including the IVNFWA operator and the IVNFWG operator. Various desirable properties and some special cases of these operators are discussed in detail. Two new decision-making approaches are proposed to solve MAGDM problem with IVN information. A case study of the selection of an ASS provider is analyzed to illustrate the practicality and feasibility of the proposed approaches. The reliability and effectiveness of the proposed approaches are further demonstrated through the comparative analyses with other methods.

However, this paper fails to consider the IVN Frank ordered weighted averaging (IVNFOWA) operator and the IVN Frank ordered weighted geometric (IVNFOWG) operator. In fact, the IVNFOWA and IVNFOWG operators are useful for integrating IVN information. Therefore, we will further study the IVNFOWA and IVNFOWG operators and apply them to MAGDM in near future.

APPENDIX A
PROOF OF THEOREM 2

Proof: In order to prove Theorem 2, we firstly prove that when $w = (w_1, w_2 \cdots, w_n)^T$ is any vector, i.e., without any constraint for w, the following equation is right:

$$\begin{aligned}
&IVNFWA(x_1, x_2, \cdots, x_n) \\
&= \langle[1 - \log_\lambda(1 + \frac{\prod_{i=1}^{n}(\lambda^{1-T_{x_i}^L}-1)^{w_i}}{(\lambda-1)^{\sum_{i=1}^{n} w_i - 1}}), \\
&\quad 1 - \log_\lambda(1 + \frac{\prod_{i=1}^{n}(\lambda^{1-T_{x_i}^U}-1)^{w_i}}{(\lambda-1)^{\sum_{i=1}^{n} w_i - 1}})], \\
&\quad [\log_\lambda(1 + \frac{\prod_{i=1}^{n}(\lambda^{I_{x_i}^L}-1)^{w_i}}{(\lambda-1)^{\sum_{i=1}^{n} w_i - 1}}), \\
&\quad \log_\lambda(1 + \frac{\prod_{i=1}^{n}(\lambda^{I_{x_i}^L}-1)^{w_i}}{(\lambda-1)^{\sum_{i=1}^{n} w_i - 1}})], \\
&\quad [\log_\lambda(1 + \frac{\prod_{i=1}^{n}(\lambda^{F_{x_i}^L}-1)^{w_i}}{(\lambda-1)^{\sum_{i=1}^{n} w_i - 1}}), \\
&\quad \log_\lambda(1 + \frac{\prod_{i=1}^{n}(\lambda^{F_{x_i}^U}-1)^{w_i}}{(\lambda-1)^{\sum_{i=1}^{n} w_i - 1}})]\rangle
\end{aligned} \tag{41}$$

We prove (41) by using mathematical induction method below.

When $n = 2$, we have $w_1 \cdot_F x_1 = \langle[1 - \log_\lambda(1 + \frac{(\lambda^{1-T_{x_1}^L}-1)^{w_1}}{(\lambda-1)^{w_1-1}}), 1 - \log_\lambda(1 + \frac{(\lambda^{1-T_{x_1}^U}-1)^{w_1}}{(\lambda-1)^{w_1-1}})], [\log_\lambda(1 + \frac{(\lambda^{I_{x_1}^L}-1)^{w_1}}{(\lambda-1)^{w_1-1}}), \log_\lambda(1 + \frac{(\lambda^{I_{x_1}^U}-1)^{w_1}}{(\lambda-1)^{w_1-1}})], [\log_\lambda(1 + \frac{(\lambda^{F_{x_1}^L}-1)^{w_1}}{(\lambda-1)^{w_1-1}}), \log_\lambda(1 + \frac{(\lambda^{F_{x_1}^U}-1)^{w_1}}{(\lambda-1)^{w_1-1}})]\rangle$, and $w_2 \cdot_F x_2 = \langle[1 - \log_\lambda(1 + \frac{(\lambda^{1-T_{x_2}^L}-1)^{w_2}}{(\lambda-1)^{w_2-1}}), 1 - \log_\lambda(1 + \frac{(\lambda^{1-T_{x_2}^U}-1)^{w_2}}{(\lambda-1)^{w_2-1}})],$

$[\log_\lambda(1+\frac{(\lambda^{I_{x_2}^L}-1)^{w_2}}{(\lambda-1)^{w_2-1}}), \log_\lambda(1+\frac{(\lambda^{I_{x_2}^U}-1)^{w_2}}{(\lambda-1)^{w_2-1}})]$, $[\log_\lambda(1+\frac{(\lambda^{F_{x_2}^L}-1)^{w_2}}{(\lambda-1)^{w_2-1}}), \log_\lambda(1+\frac{(\lambda^{F_{x_2}^U}-1)^{w_2}}{(\lambda-1)^{w_2-1}})]\rangle$.

Then, $IVNFWA(x_1 x_2) = w_1 \cdot_F x_1 \oplus_F w_2 \cdot_F x_2 =$

$$\langle[1-\log_\lambda(1+\frac{(\lambda^{\log_\lambda(1+\frac{(\lambda^{1-T_{x_1}^L}-1)^{w_1}}{(\lambda-1)^{w_1-1}})}-1)(\lambda^{\log_\lambda(1+\frac{(\lambda^{1-T_{x_2}^L}-1)^{w_2}}{(\lambda-1)^{w_2-1}})}-1)}{\lambda-1}),$$

$$1-\log_\lambda(1+\frac{(\lambda^{\log_\lambda(1+\frac{(\lambda^{1-T_{x_1}^U}-1)^{w_1}}{(\lambda-1)^{w_1-1}})}-1)(\lambda^{\log_\lambda(1+\frac{(\lambda^{1-T_{x_2}^U}-1)^{w_2}}{(\lambda-1)^{w_2-1}})}-1)}{\lambda-1})],$$

$$[\log_\lambda(1+\frac{(\lambda^{\log_\lambda(1+\frac{(\lambda^{I_{x_1}^L}-1)^{w_1}}{(\lambda-1)^{w_1-1}})}-1)^{w_1}(\lambda^{\log_\lambda(1+\frac{(\lambda^{I_{x_2}^L}-1)^{w_2}}{(\lambda-1)^{w_2-1}})}-1)^{w_2}}{\lambda-1}),$$

$$\log_\lambda(1+\frac{(\lambda^{\log_\lambda(1+\frac{(\lambda^{I_{x_1}^U}-1)^{w_1}}{(\lambda-1)^{w_1-1}})}-1)^{w_1}(\lambda^{\log_\lambda(1+\frac{(\lambda^{I_{x_2}^U}-1)^{w_2}}{(\lambda-1)^{w_2-1}})}-1)^{w_2}}{\lambda-1})],$$

$$[\log_\lambda(1+\frac{(\lambda^{\log_\lambda(1+\frac{(\lambda^{F_{x_1}^L}-1)^{w_1}}{(\lambda-1)^{w_1-1}})}-1)^{w_1}(\lambda^{\log_\lambda(1+\frac{(\lambda^{F_{x_2}^L}-1)^{w_2}}{(\lambda-1)^{w_2-1}})}-1)^{w_2}}{\lambda-1}),$$

$$\log_\lambda(1+\frac{(\lambda^{\log_\lambda(1+\frac{(\lambda^{F_{x_1}^U}-1)^{w_1}}{(\lambda-1)^{w_1-1}})}-1)^{w_1}(\lambda^{\log_\lambda(1+\frac{(\lambda^{F_{x_2}^U}-1)^{w_2}}{(\lambda-1)^{w_2-1}})}-1)^{w_2}}{\lambda-1})]\rangle =$$

$$\langle[1-\log_\lambda(1+\frac{\prod_{i=1}^{2}(\lambda^{1-T_{x_i}^L}-1)^{w_i}}{(\lambda-1)^{w_1+w_2-1}}), 1-\log_\lambda(1+\frac{\prod_{i=1}^{2}(\lambda^{1-T_{x_i}^U}-1)^{w_i}}{(\lambda-1)^{w_1+w_2-1}})],$$

$$[\log_\lambda(1+\frac{\prod_{i=1}^{2}(\lambda^{I_{x_i}^L}-1)^{w_i}}{(\lambda-1)^{w_1+w_2-1}}), \log_\lambda(1+\frac{\prod_{i=1}^{2}(\lambda^{I_{x_i}^U}-1)^{w_i}}{(\lambda-1)^{w_1+w_2-1}})], [\log_\lambda(1+\frac{\prod_{i=1}^{2}(\lambda^{F_{x_i}^L}-1)^{w_i}}{(\lambda-1)^{w_1+w_2-1}}), \log_\lambda(1+\frac{\prod_{i=1}^{2}(\lambda^{F_{x_i}^U}-1)^{w_i}}{(\lambda-1)^{w_1+w_2-1}})]\rangle.$$

That is, (41) holds for $n = 2$. Suppose (41) holds for $n = k$, i.e.,

$$
\begin{aligned}
&IVNFWA(x_1, x_2, \cdots x_k)\\
&= \langle[1-\log_\lambda(1+\frac{\prod_{i=1}^{k}(\lambda^{1-T_{x_i}^L}-1)^{w_i}}{(\lambda-1)^{\sum_{i=1}^{k} w_i-1}}),\\
&\quad 1-\log_\lambda(1+\frac{\prod_{i=1}^{k}(\lambda^{1-T_{x_i}^U}-1)^{w_i}}{(\lambda-1)^{\sum_{i=1}^{k} w_i-1}})],\\
&\quad [\log_\lambda(1+\frac{\prod_{i=1}^{k}(\lambda^{I_{x_i}^L}-1)^{w_i}}{(\lambda-1)^{\sum_{i=1}^{k} w_i-1}}),\\
&\quad \log_\lambda(1+\frac{\prod_{i=1}^{k}(\lambda^{I_{x_i}^U}-1)^{w_i}}{(\lambda-1)^{\sum_{i=1}^{k} w_i-1}})],\\
&\quad [\log_\lambda(1+\frac{\prod_{i=1}^{k}(\lambda^{F_{x_i}^L}-1)^{w_i}}{(\lambda-1)^{\sum_{i=1}^{k} w_i-1}}),\\
&\quad \log_\lambda(1+\frac{\prod_{i=1}^{k}(\lambda^{F_{x_i}^U}-1)^{w_i}}{(\lambda-1)^{\sum_{i=1}^{k} w_i-1}})]\rangle.
\end{aligned}
$$

Then, when $n = k+1$, by Definition 10, we have, $w_{k+1} \cdot_F x_{k+1}$ and $IVNFWA(x_1, x_2, \cdots x_k, x_{k+1})$, as shown at the top of the next page.

In other words, (41) holds for $n = k+1$. Thus, (41) holds for all n.

Because (41) is right without any constraint for w.

Therefore, when $w_i \in [0, 1]$ and $\sum_{i=1}^{n} w_i = 1$, (41) reduces to (16).

Moreover, since $[T_{x_i}^L, T_{x_i}^U] \in [0, 1]$, $[I_{x_i}^L, I_{x_i}^U] \in [0, 1]$, $[F_{x_i}^L, F_{x_i}^U] \in [0, 1]$ and $0 \leq T_{x_i}^U + I_{x_i}^U + F_{x_i}^U \leq 3 (i = 1, 2, \cdots, n)$, we have

$$
\begin{aligned}
0 &= 1-\log_\lambda(1+\prod_{i=1}^{n}(\lambda^{1-0}-1)^{w_i})\\
&\leq 1-\log_\lambda(1+\prod_{i=1}^{n}(\lambda^{1-T_{x_i}^L}-1)^{w_i})\\
&\leq 1-\log_\lambda(1+\prod_{i=1}^{n}(\lambda^{1-T_{x_i}^U}-1)^{w_i})\\
&\leq 1-\log_\lambda(1+\prod_{i=1}^{n}(\lambda^{1-1}-1)^{w_i}) = 1,\\
0 &= \log_\lambda(1+\prod_{i=1}^{n}(\lambda^{0}-1)^{w_i}) \leq \log_\lambda(1+\prod_{i=1}^{n}(\lambda^{I_{x_i}^L}-1)^{w_i})\\
&\leq \log_\lambda(1+\prod_{i=1}^{n}(\lambda^{I_{x_i}^U}-1)^{w_i})\\
&\leq \log_\lambda(1+\prod_{i=1}^{n}(\lambda^{1}-1)^{w_i}) = 1,\\
0 &= \log_\lambda(1+\prod_{i=1}^{n}(\lambda^{0}-1)^{w_i}) \leq \log_\lambda(1+\prod_{i=1}^{n}(\lambda^{F_{x_i}^L}-1)^{w_i})\\
&\leq \log_\lambda(1+\prod_{i=1}^{n}(\lambda^{F_{x_i}^U}-1)^{w_i})\\
&\leq \log_\lambda(1+\prod_{i=1}^{n}(\lambda^{1}-1)^{w_i}) = 1.
\end{aligned}
$$

Then it has

$$
\begin{aligned}
0 &= 1-\log_\lambda(1+\prod_{i=1}^{n}(\lambda^{1-0}-1)^{w_i})\\
&\quad + \log_\lambda(1+\prod_{i=1}^{n}(\lambda^{0}-1)^{w_i}) + \log_\lambda(1+\prod_{i=1}^{n}(\lambda^{0}-1)^{w_i})\\
&\leq 1-\log_\lambda(1+\prod_{i=1}^{n}(\lambda^{1-T_{x_i}^U}-1)^{w_i})\\
&\quad + \log_\lambda(1+\prod_{i=1}^{n}(\lambda^{I_{x_i}^U}-1)^{w_i}) + \log_\lambda(1+\prod_{i=1}^{n}(\lambda^{F_{x_i}^U}-1)^{w_i})\\
&\leq 1-\log_\lambda(1+\prod_{i=1}^{n}(\lambda^{1-1}-1)^{w_i}) + \log_\lambda(1+\prod_{i=1}^{n}(\lambda^{1}-1)^{w_i})\\
&\quad + \log_\lambda(1+\prod_{i=1}^{n}(\lambda^{1}-1)^{w_i}) = 3,
\end{aligned}
$$

$$
\begin{aligned}
w_{k+1} \cdot_F x_{k+1} &= \langle [1-\log_\lambda(1+\frac{(\lambda^{1-T^L_{x_{k+1}}}-1)^{w_{k+1}}}{(\lambda-1)^{w_{k+1}-1}}), 1-\log_\lambda(1+\frac{(\lambda^{1-T^U_{x_{k+1}}}-1)^{w_{k+1}}}{(\lambda-1)^{w_{k+1}-1}})],\\
&\quad [\log_\lambda(1+\frac{(\lambda^{I^L_{x_{k+1}}}-1)^{w_{k+1}}}{(\lambda-1)^{w_{k+1}-1}}), \log_\lambda(1+\frac{(\lambda^{I^U_{x_{k+1}}}-1)^{w_{k+1}}}{(\lambda-1)^{w_{k+1}-1}})],\\
&\quad [\log_\lambda(1+\frac{(\lambda^{F^L_{x_{k+1}}}-1)^{w_{k+1}}}{(\lambda-1)^{w_{k+1}-1}}), \log_\lambda(1+\frac{(\lambda^{F^U_{x_{k+1}}}-1)^{w_{k+1}}}{(\lambda-1)^{w_{k+1}-1}})]\rangle.\\
IVNFWA(x_1, x_2, \cdots x_k, x_{k+1}) &= IVNFWA(x_1, x_2, \cdots x_k) \oplus_F w_{k+1} \cdot_F x_{k+1}\\
&= \langle [1-\log_\lambda(1+\frac{(\lambda^{\log_\lambda(1+\frac{\prod_{i=1}^k(\lambda^{1-T^L_{x_i}}-1)^{w_i}}{(\lambda-1)^{\sum_{i=1}^k w_i-1}})}-1)(\lambda^{\log_\lambda(1+\frac{(\lambda^{1-T^L_{x_{k+1}}}-1)^{w_{k+1}}}{(\lambda-1)^{w_{k+1}-1}})}-1)}{\lambda-1}),\\
&\quad 1-\log_\lambda(1+\frac{(\lambda^{\log_\lambda(1+\frac{\prod_{i=1}^k(\lambda^{1-T^U_{x_i}}-1)^{w_i}}{(\lambda-1)^{\sum_{i=1}^k w_i-1}})}-1)(\lambda^{\log_\lambda(1+\frac{(\lambda^{1-T^U_{x_{k+1}}}-1)^{w_{k+1}}}{(\lambda-1)^{w_{k+1}-1}})}-1)}{\lambda-1})],\\
&\quad [\log_\lambda(1+\frac{(\lambda^{\log_\lambda(1+\frac{\prod_{i=1}^k(\lambda^{I^L_{x_i}}-1)^{w_i}}{(\lambda-1)^{\sum_{i=1}^k w_i-1}})}-1)(\lambda^{\log_\lambda(1+\frac{(\lambda^{I^L_{x_{k+1}}}-1)^{w_{k+1}}}{(\lambda-1)^{w_{k+1}-1}})}-1)}{\lambda-1}),\\
&\quad \log_\lambda(1+\frac{(\lambda^{\log_\lambda(1+\frac{\prod_{i=1}^k(\lambda^{I^U_{x_i}}-1)^{w_i}}{(\lambda-1)^{\sum_{i=1}^k w_i-1}})}-1)(\lambda^{\log_\lambda(1+\frac{(\lambda^{I^U_{x_{k+1}}}-1)^{w_{k+1}}}{(\lambda-1)^{w_{k+1}-1}})}-1)}{\lambda-1})],\\
&\quad [\log_\lambda(1+\frac{(\lambda^{\log_\lambda(1+\frac{\prod_{i=1}^k(\lambda^{F^L_{x_i}}-1)^{w_i}}{(\lambda-1)^{\sum_{i=1}^k w_i-1}})}-1)(\lambda^{\log_\lambda(1+\frac{(\lambda^{F^L_{x_{k+1}}}-1)^{w_{k+1}}}{(\lambda-1)^{w_{k+1}-1}})}-1)}{\lambda-1}),\\
&\quad \log_\lambda(1+\frac{(\lambda^{\log_\lambda(1+\frac{\prod_{i=1}^k(\lambda^{F^U_{x_i}}-1)^{w_i}}{(\lambda-1)^{\sum_{i=1}^k w_i-1}})}-1)(\lambda^{\log_\lambda(1+\frac{(\lambda^{F^U_{x_{k+1}}}-1)^{w_{k+1}}}{(\lambda-1)^{w_{k+1}-1}})}-1)}{\lambda-1})]\rangle\\
&= \langle (1-\log_\lambda(1+\frac{\prod_{i=1}^{k+1}(\lambda^{1-T^L_{x_i}}-1)^{w_i}}{(\lambda-1)^{\sum_{i=1}^{k+1} w_i-1}}), 1-\log_\lambda(1+\frac{\prod_{i=1}^{k+1}(\lambda^{1-T^U_{x_i}}-1)^{w_i}}{(\lambda-1)^{\sum_{i=1}^{k+1} w_i-1}})],\\
&\quad [\log_\lambda(1+\frac{\prod_{i=1}^{k+1}(\lambda^{I^L_{x_i}}-1)^{w_i}}{(\lambda-1)^{\sum_{i=1}^{k+1} w_i-1}}), \log_\lambda(1+\frac{\prod_{i=1}^{k+1}(\lambda^{I^U_{x_i}}-1)^{w_i}}{(\lambda-1)^{\sum_{i=1}^{k+1} w_i-1}})],\\
&\quad [\log_\lambda(1+\frac{\prod_{i=1}^{k+1}(\lambda^{F^L_{x_i}}-1)^{w_i}}{(\lambda-1)^{\sum_{i=1}^{k+1} w_i-1}}), \log_\lambda(1+\frac{\prod_{i=1}^{k+1}(\lambda^{F^U_{x_i}}-1)^{w_i}}{(\lambda-1)^{\sum_{i=1}^{k+1} w_i-1}})]\rangle.
\end{aligned}
$$

which indicates that $IVNFWA(x_1, x_2, \cdots x_n)$ is an IVNN. This completes the proof of Theorem 2.

APPENDIX B
PROOF OF THEOREM 3

Proof:

$$
\begin{aligned}
&IVNFWA(x_1, x_2, \cdots, x_n)\\
&= \langle [1-\log_\lambda(1+\frac{\prod_{i=1}^{n}(\lambda^{1-T^L_{x_i}}-1)^{w_i}}{(\lambda-1)^{\sum_{i=1}^n w_i-1}}),\\
&\quad 1-\log_\lambda(1+\frac{\prod_{i=1}^{n}(\lambda^{1-T^U_{x_i}}-1)^{w_i}}{(\lambda-1)^{\sum_{i=1}^n w_i-1}})],\\
&\quad [\log_\lambda(1+\frac{\prod_{i=1}^{n}(\lambda^{I^L_{x_i}}-1)^{w_i}}{(\lambda-1)^{\sum_{i=1}^n w_i-1}}),\\
&\quad \log_\lambda(1+\frac{\prod_{i=1}^{n}(\lambda^{I^L_{x_i}}-1)^{w_i}}{(\lambda-1)^{\sum_{i=1}^n w_i-1}})],\\
&\quad [\log_\lambda(1+\frac{\prod_{i=1}^{n}(\lambda^{F^L_{x_i}}-1)^{w_i}}{(\lambda-1)^{\sum_{i=1}^n w_i-1}}),\\
&\quad \log_\lambda(1+\frac{\prod_{i=1}^{n}(\lambda^{F^U_{x_i}}-1)^{w_i}}{(\lambda-1)^{\sum_{i=1}^n w_i-1}})]\rangle
\end{aligned}
$$

$$
\begin{aligned}
&= \langle[1-\log_\lambda(1+\prod_{i=1}^{n}(\lambda^{1-T^L}-1)^{w_i}),\\
&\quad 1-\log_\lambda(1+\prod_{i=1}^{n}(\lambda^{1-T^U}-1)^{w_i})],\\
&\quad [\log_\lambda(1+\prod_{i=1}^{n}(\lambda^{I^L}-1)^{w_i}),\\
&\quad \log_\lambda(1+\prod_{i=1}^{n}(\lambda^{I^U}-1)^{w_i})], [\log_\lambda(1+\prod_{i=1}^{n}(\lambda^{F^L}-1)^{w_i}),\\
&\quad \log_\lambda(1+\prod_{i=1}^{n}(\lambda^{F^U}-1)^{w_i})]\rangle\\
&= \langle[1-\log_\lambda(1+(\lambda^{1-T^L}-1)^{\sum_{i=1}^{n}w_i}),\\
&\quad 1-\log_\lambda(1+(\lambda^{1-T^U}-1)^{\sum_{i=1}^{n}w_i})],\\
&\quad [\log_\lambda(1+(\lambda^{I^L}-1)^{\sum_{i=1}^{n}w_i}),\\
&\quad \log_\lambda(1+(\lambda^{I^U}-1)^{\sum_{i=1}^{n}w_i})], [\log_\lambda(1+(\lambda^{F^L}-1)^{\sum_{i=1}^{n}w_i}),\\
&\quad \log_\lambda(1+(\lambda^{F^U}-1)^{\sum_{i=1}^{n}w_i})]\rangle\\
&= \langle[1-\log_\lambda(\lambda^{1-T^L}), 1-\log_\lambda(\lambda^{1-T^U})],\\
&\quad [\log_\lambda(\lambda^{I^L}), \log_\lambda(\lambda^{I^U})],\\
&\quad [\log_\lambda(\lambda^{F^L}), \log_\lambda(\lambda^{F^U})]\rangle\\
&= \langle[T^L, T^U], [I^L, I^U], [F^L, F^U]\rangle.
\end{aligned}
$$

This completes the proof of Theorem 3.

APPENDIX C
PROOF OF THEOREM 4

Proof: For two sets of IVNNs $x_i = \langle[T_{x_i}^L, T_{x_i}^U], [I_{x_i}^L, I_{x_i}^U], [F_{x_i}^L, F_{x_i}^U]\rangle$ and $x_i' = \langle[T_{x_i}^{L\prime}, T_{x_i}^{U\prime}], [I_{x_i}^{L\prime}, I_{x_i}^{U\prime}], [F_{x_i}^{L\prime}, F_{x_i}^{U\prime}]\rangle$ $(i = 1, 2, \cdots, n)$, if $T_{x_i}^L \leq T_{x_i}^{L\prime}$, $T_{x_i}^U \leq T_{x_i}^{U\prime}$, $I_{x_i}^L \geq I_{x_i}^{L\prime}$, $I_{x_i}^U \geq I_{x_i}^{U\prime}$, $F_{x_i}^L \geq F_{x_i}^{L\prime}$, $F_{x_i}^U \geq F_{x_i}^{U\prime}$, for all i, then

$$
1-\log_\lambda(1+\prod_{i=1}^{n}(\lambda^{1-T_{x_i}^L}-1)^{w_i}) \leq 1-\log_\lambda(1+\prod_{i=1}^{n}(\lambda^{1-T_{x_i}^{L\prime}}-1)^{w_i}) \tag{42}
$$

$$
1-\log_\lambda(1+\prod_{i=1}^{n}(\lambda^{1-T_{x_i}^U}-1)^{w_i}) \leq 1-\log_\lambda(1+\prod_{i=1}^{n}(\lambda^{1-T_{x_i}^{U\prime}}-1)^{w_i}) \tag{43}
$$

$$
\log_\lambda(1+\prod_{i=1}^{n}(\lambda^{I_{x_i}^L}-1)^{w_i}) \geq \log_\lambda(1+\prod_{i=1}^{n}(\lambda^{I_{x_i}^{L\prime}}-1)^{w_i}) \tag{44}
$$

$$
\log_\lambda(1+\prod_{i=1}^{n}(\lambda^{I_{x_i}^U}-1)^{w_i}) \geq \log_\lambda(1+\prod_{i=1}^{n}(\lambda^{I_{x_i}^{U\prime}}-1)^{w_i}) \tag{45}
$$

$$
\log_\lambda(1+\prod_{i=1}^{n}(\lambda^{F_{x_i}^L}-1)^{w_i}) \geq \log_\lambda(1+\prod_{i=1}^{n}(\lambda^{F_{x_i}^{L\prime}}-1)^{w_i}) \tag{46}
$$

$$
\log_\lambda(1+\prod_{i=1}^{n}(\lambda^{F_{x_i}^U}-1)^{w_i}) \geq \log_\lambda(1+\prod_{i=1}^{n}(\lambda^{F_{x_i}^{U\prime}}-1)^{w_i}) \tag{47}
$$

By Definition 3, it yields that

$$
\begin{aligned}
&S(IVNFWA(x_1, x_2, \cdots, x_n))\\
&= \frac{2}{4} + \frac{1-\log_\lambda(1+\prod_{i=1}^{n}(\lambda^{1-T_{x_i}^L}-1)^{w_i})}{4}
+ \frac{1-\log_\lambda(1+\prod_{i=1}^{n}(\lambda^{1-T_{x_i}^U}-1)^{w_i})}{4}\\
&\quad - \frac{2\log_\lambda(1+\prod_{i=1}^{n}(\lambda^{I_{x_i}^L}-1)^{w_i})}{4}
- \frac{2\log_\lambda(1+\prod_{i=1}^{n}(\lambda^{I_{x_i}^U}-1)^{w_i})}{4}\\
&\quad - \frac{\log_\lambda(1+\prod_{i=1}^{n}(\lambda^{F_{x_i}^L}-1)^{w_i})}{4}
- \frac{\log_\lambda(1+\prod_{i=1}^{n}(\lambda^{F_{x_i}^U}-1)^{w_i})}{4}\\
&\leq \frac{2}{4} + \frac{1-\log_\lambda(1+\prod_{i=1}^{n}(\lambda^{1-T_{x_i}^{L\prime}}-1)^{w_i})}{4}
+ \frac{1-\log_\lambda(1+\prod_{i=1}^{n}(\lambda^{1-T_{x_i}^{U\prime}}-1)^{w_i})}{4}\\
&\quad - \frac{2\log_\lambda(1+\prod_{i=1}^{n}(\lambda^{I_{x_i}^{L\prime}}-1)^{w_i})}{4}
- \frac{2\log_\lambda(1+\prod_{i=1}^{n}(\lambda^{I_{x_i}^{U\prime}}-1)^{w_i})}{4}\\
&\quad - \frac{\log_\lambda(1+\prod_{i=1}^{n}(\lambda^{F_{x_i}^{L\prime}}-1)^{w_i})}{4}
- \frac{\log_\lambda(1+\prod_{i=1}^{n}(\lambda^{F_{x_i}^{U\prime}}-1)^{w_i})}{4}\\
&= S(IVNFWA(x_1', x_2', \cdots, x_n')).
\end{aligned}
$$

If $S(IVNFWA(x_1, x_2, \cdots, x_n)) < (IVNFWA(x_1', x_2', \cdots, x_n'))$, then by Definition 3, we have $IVNFWA(x_1, x_2, \cdots, x_n) < IVNFWA(x_1', x_2', \cdots, x_n')$.

If $S(IVNFWA(x_1, x_2, \cdots, x_n)) < S(IVNFWA(x_1', x_2', \cdots, x_n'))$, i.e.,

$$
\begin{aligned}
&S(IVNFWA(x_1, x_2, \cdots, x_n))\\
&= \frac{2}{4} + \frac{1-\log_\lambda(1+\prod_{i=1}^{n}(\lambda^{1-T_{x_i}^L}-1)^{w_i})}{4}
+ \frac{1-\log_\lambda(1+\prod_{i=1}^{n}(\lambda^{1-T_{x_i}^U}-1)^{w_i})}{4}
\end{aligned}
$$

$$-\frac{2\log_\lambda(1+\prod_{i=1}^{n}(\lambda^{I_{x_i}^{L}}-1)^{w_i})}{4}$$
$$-\frac{2\log_\lambda(1+\prod_{i=1}^{n}(\lambda^{I_{x_i}^{U}}-1)^{w_i})}{4}$$
$$-\frac{\log_\lambda(1+\prod_{i=1}^{n}(\lambda^{F_{x_i}^{L}}-1)^{w_i})}{4}$$
$$-\frac{\log_\lambda(1+\prod_{i=1}^{n}(\lambda^{F_{x_i}^{U}}-1)^{w_i})}{4} < \frac{2}{4}$$
$$+\frac{1-\log_\lambda(1+\prod_{i=1}^{n}(\lambda^{1-T_{x_i}^{L\prime}}-1)^{w_i})}{4}$$
$$+\frac{1-\log_\lambda(1+\prod_{i=1}^{n}(\lambda^{1-T_{x_i}^{U\prime}}-1)^{w_i})}{4}$$
$$-\frac{2\log_\lambda(1+\prod_{i=1}^{n}(\lambda^{I_{x_i}^{L\prime}}-1)^{w_i})}{4}$$
$$-\frac{2\log_\lambda(1+\prod_{i=1}^{n}(\lambda^{I_{x_i}^{U\prime}}-1)^{w_i})}{4}$$
$$-\frac{\log_\lambda(1+\prod_{i=1}^{n}(\lambda^{F_{x_i}^{L\prime}}-1)^{w_i})}{4}$$
$$-\frac{\log_\lambda(1+\prod_{i=1}^{n}(\lambda^{F_{x_i}^{U\prime}}-1)^{w_i})}{4}$$
$$= S(IVNFWA(x_1', x_2', \cdots, x_n')),$$

then by (42), (43), (44), (45), (46), and (47), we have

$$1-\log_\lambda(1+\prod_{i=1}^{n}(\lambda^{1-T_{x_i}^{L}}-1)^{w_i}) = 1-\log_\lambda(1+\prod_{i=1}^{n}(\lambda^{1-T_{x_i}^{L\prime}}-1)^{w_i}),$$
$$1-\log_\lambda(1+\prod_{i=1}^{n}(\lambda^{1-T_{x_i}^{U}}-1)^{w_i}) = 1-\log_\lambda(1+\prod_{i=1}^{n}(\lambda^{1-T_{x_i}^{U\prime}}-1)^{w_i}),$$
$$\log_\lambda(1+\prod_{i=1}^{n}(\lambda^{I_{x_i}^{L}}-1)^{w_i}) = \log_\lambda(1+\prod_{i=1}^{n}(\lambda^{I_{x_i}^{L\prime}}-1)^{w_i}),$$
$$\log_\lambda(1+\prod_{i=1}^{n}(\lambda^{I_{x_i}^{U}}-1)^{w_i}) = \log_\lambda(1+\prod_{i=1}^{n}(\lambda^{I_{x_i}^{U\prime}}-1)^{w_i}),$$
$$\log_\lambda(1+\prod_{i=1}^{n}(\lambda^{F_{x_i}^{L}}-1)^{w_i}) = \log_\lambda(1+\prod_{i=1}^{n}(\lambda^{F_{x_i}^{L\prime}}-1)^{w_i}),$$
$$\log_\lambda(1+\prod_{i=1}^{n}(\lambda^{F_{x_i}^{U}}-1)^{w_i}) = \log_\lambda(1+\prod_{i=1}^{n}(\lambda^{F_{x_i}^{U\prime}}-1)^{w_i}).$$

Thus,

$$L(IVNFWA(x_1, x_2, \cdots, x_n))$$
$$= \frac{1}{2}\{1-\log_\lambda(1+\prod_{i=1}^{n}(\lambda^{1-T_{x_i}^{L}}-1)^{w_i})$$
$$+1-\log_\lambda(1+\prod_{i=1}^{n}(\lambda^{1-T_{x_i}^{U}}-1)^{w_i})$$
$$-\log_\lambda(1+\prod_{i=1}^{n}(\lambda^{I_{x_i}^{U}}-1)^{w_i})\log_\lambda(1+\prod_{i=1}^{n}(\lambda^{1-T_{x_i}^{U}}-1)^{w_i})$$
$$-\log_\lambda(1+\prod_{i=1}^{n}(\lambda^{I_{x_i}^{L}}-1)^{w_i})\log_\lambda(1+\prod_{i=1}^{n}(\lambda^{1-T_{x_i}^{L}}-1)^{w_i})$$
$$-\log_\lambda(1+\prod_{i=1}^{n}(\lambda^{F_{x_i}^{U}}-1)^{w_i})(1-\log_\lambda(1+\prod_{i=1}^{n}(\lambda^{I_{x_i}^{U}}-1)^{w_i}))$$
$$-\log_\lambda(1+\prod_{i=1}^{n}(\lambda^{F_{x_i}^{L}}-1)^{w_i})(1-\log_\lambda(1+\prod_{i=1}^{n}(\lambda^{I_{x_i}^{L}}-1)^{w_i}))\}$$
$$= \frac{1}{2}\{1-\log_\lambda(1+\prod_{i=1}^{n}(\lambda^{1-T_{x_i}^{L\prime}}-1)^{w_i})$$
$$+1-\log_\lambda(1+\prod_{i=1}^{n}(\lambda^{1-T_{x_i}^{U\prime}}-1)^{w_i})$$
$$-\log_\lambda(1+\prod_{i=1}^{n}(\lambda^{I_{x_i}^{U\prime}}-1)^{w_i})\log_\lambda(1+\prod_{i=1}^{n}(\lambda^{1-T_{x_i}^{U\prime}}-1)^{w_i})$$
$$-\log_\lambda(1+\prod_{i=1}^{n}(\lambda^{I_{x_i}^{L\prime}}-1)^{w_i})\log_\lambda(1+\prod_{i=1}^{n}(\lambda^{1-T_{x_i}^{L\prime}}-1)^{w_i})$$
$$-\log_\lambda(1+\prod_{i=1}^{n}(\lambda^{F_{x_i}^{U\prime}}-1)^{w_i})(1-\log_\lambda(1+\prod_{i=1}^{n}(\lambda^{I_{x_i}^{U\prime}}-1)^{w_i})$$
$$-\log_\lambda(1+\prod_{i=1}^{n}(\lambda^{F_{x_i}^{L\prime}}-1)^{w_i})(1-\log_\lambda(1+\prod_{i=1}^{n}(\lambda^{I_{x_i}^{L\prime}}-1)^{w_i})\}$$
$$= L(IVNFWA(x_1', x_2', \cdots, x_n')),$$

which implies that

$$IVNFWA((x_1, x_2, \cdots, x_n)) = IVNFWA(x_1', x_2', \cdots, x_n').$$

Hence, Theorem 4 always holds.

APPENDIX D
PROOF OF THEOREM 5

Proof: Let

$$IVNFWA(x_1, x_2, \cdots, x_n)$$
$$= \langle[1-\log_\lambda(1+\prod_{i=1}^{n}(\lambda^{1-T_{x_i}^{L}}-1)^{w_i}),$$
$$1-\log_\lambda(1+\prod_{i=1}^{n}(\lambda^{1-T_{x_i}^{U}}-1)^{w_i})],$$
$$[\log_\lambda(1+\prod_{i=1}^{n}(\lambda^{I_{x_i}^{L}}-1)^{w_i}),$$
$$\log_\lambda(1+\prod_{i=1}^{n}(\lambda^{I_{x_i}^{U}}-1)^{w_i})], [\log_\lambda(1+\prod_{i=1}^{n}(\lambda^{F_{x_i}^{L}}-1)^{w_i}),$$
$$\log_\lambda(1+\prod_{i=1}^{n}(\lambda^{F_{x_i}^{U}}-1)^{w_i})]\rangle$$
$$= x = \langle[T^L, T^U], [I^L, I^U], [F^L, F^U]\rangle.$$

Because

$$\min_i\{T^L_{x_i}\} \le T^L_{x_i} \le \max_i\{T^L_{x_i}\},\quad \min_i\{T^U_{x_i}\} \le T^U_{x_i} \le \max_i\{T^U_{x_i}\},$$
$$\min_i\{I^L_{x_i}\} \le I^L_{x_i} \le \max_i\{I^L_{x_i}\},\quad \min_i\{I^U_{x_i}\} \le I^U_{x_i} \le \max_i\{I^U_{x_i}\},$$
$$\min_i\{F^L_{x_i}\} \le F^L_{x_i} \le \max_i\{F^L_{x_i}\},$$
$$\min_i\{F^U_{x_i}\} \le F^U_{x_i} \le \max_i\{F^U_{x_i}\},$$

we have

$$\begin{aligned}\min_i\{T^L_{x_i}\} &= 1-\log_\lambda(1+\prod_{i=1}^n(\lambda^{1-\min_i\{T^L_{x_i}\}}-1)^{w_i})\\ &\le 1-\log_\lambda(1+\prod_{i=1}^n(\lambda^{1-T^L_{x_i}}-1)^{w_i}) = T^L\\ &\le 1-\log_\lambda(1+\prod_{i=1}^n(\lambda^{1-\max_i\{T^L_{x_i}\}}-1)^{w_i})\\ &= \max_i T^L \end{aligned}\tag{48}$$

$$\begin{aligned}\min_i\{T^U_{x_i}\} &= 1-\log_\lambda(1+\prod_{i=1}^n(\lambda^{1-\min_i\{T^U_{x_i}\}}-1)^{w_i})\\ &\le 1-\log_\lambda(1+\prod_{i=1}^n(\lambda^{1-T^U_{x_i}}-1)^{w_i}) = T^U\\ &\le 1-\log_\lambda(1+\prod_{i=1}^n(\lambda^{1-\max_i\{T^U_{x_i}\}}-1)^{w_i})\\ &= \max_i T^U \end{aligned}\tag{49}$$

$$\begin{aligned}\min_i\{I^L_{x_i}\} &= \log_\lambda(1+\prod_{i=1}^n(\lambda^{\min_i\{I^L_{x_i}\}}-1)^{w_i})\\ &\le \log_\lambda(1+\prod_{i=1}^n(\lambda^{I^L_{x_i}}-1)^{w_i}) = I^L\\ &\le \log_\lambda(1+\prod_{i=1}^n(\lambda^{\max_i\{I^L_{x_i}\}}-1)^{w_i})\\ &= \max_i\{I^L_{x_i}\} \end{aligned}\tag{50}$$

$$\begin{aligned}\min_i\{I^U_{x_i}\} &= \log_\lambda(1+\prod_{i=1}^n(\lambda^{\min_i\{I^U_{x_i}\}}-1)^{w_i})\\ &\le \log_\lambda(1+\prod_{i=1}^n(\lambda^{I^U_{x_i}}-1)^{w_i}) = I^U\\ &\le \log_\lambda(1+\prod_{i=1}^n(\lambda^{\max_i\{I^U_{x_i}\}}-1)^{w_i})\\ &= \max_i\{I^U_{x_i}\} \end{aligned}\tag{51}$$

$$\begin{aligned}\min_i\{F^L_{x_i}\} &= \log_\lambda(1+\prod_{i=1}^n(\lambda^{\min_i\{F^L_{x_i}\}}-1)^{w_i})\\ &\le \log_\lambda(1+\prod_{i=1}^n(\lambda^{F^L_{x_i}}-1)^{w_i}) = F^L\\ &\le \log_\lambda(1+\prod_{i=1}^n(\lambda^{\max_i\{F^L_{x_i}\}}-1)^{w_i})\\ &= \max_i\{F^L_{x_i}\} \end{aligned}\tag{52}$$

$$\begin{aligned}\min_i\{F^U_{x_i}\} &= \log_\lambda(1+\prod_{i=1}^n(\lambda^{\min_i\{F^U_{x_i}\}}-1)^{w_i})\\ &\le \log_\lambda(1+\prod_{i=1}^n(\lambda^{F^U_{x_i}}-1)^{w_i}) = F^U\\ &\le \log_\lambda(1+\prod_{i=1}^n(\lambda^{\max_i\{I^U_{x_i}\}}-1)^{w_i})\\ &= \max_i\{I^U_{x_i}\} \end{aligned}\tag{53}$$

Then, we can obtain, $S(x)$, as shown at the top of the next page.

If $S(x) < S(x^+)$ and $S(x) > S(x^-)$, then by Definition 3,

$$x^- \le IVNFWA(x_1, x_2, \cdots, x_n) \le x^+ \tag{54}$$

If $S(x) = S(x^+)$, i.e., $\frac{2+T^L+T^U-2I^L-2I^U-F^L-F^U}{4} \le \frac{2+\max_i\{T^L_{x_i}\}+\max_i\{T^U_{x_i}\}-2\min_i\{I^L_{x_i}\}-2\min_i\{I^U_{x_i}\}-\min_i\{F^L_{x_i}\}-\min_i\{F^U_{x_i}\}}{4}$, then, by (48)-(53), it follows that $T^L = \max_i\{T^L_{x_i}\}$, $T^U = \max_i\{T^U_{x_i}\}$, $I^L = \min_i\{I^L_{x_i}\}$, $I^U = \max_i\{I^U_{x_i}\}$, $F^L = \min_i\{F^L_{x_i}\}$, $F^U = \min_i\{F^U_{x_i}\}$ and that $L(x) = \frac{1}{2}\{T^L + T^U - I^U(1-T^U) - I^L(1-T^L) - F^U(1-I^U) - F^L(1-I^L)\} = \frac{1}{2}\{\max_i\{T^L_{x_i}\} + \max_i\{T^U_{x_i}\} - \min_i\{I^U_{x_i}\}(1-\max_i\{T^U_{x_i}\}) - \min_i\{I^L_{x_i}\}(1-\max_i\{T^L_{x_i}\}) - \min_i\{F^U_{x_i}\}(1-\min_i\{I^U_{x_i}\}) - \min_i\{F^L_{x_i}\}(1-\min_i\{I^L_{x_i}\})\} = L(x^+)$, which implies that

$$IVNFWA(x_1, x_2, \cdots, x_n) = x^+ \tag{55}$$

If $S(x) = S(x^+)$, i.e., $\frac{2+T^L+T^U-2I^L-2I^U-F^L-F^U}{4} \ge \frac{2+\min_i\{T^L_{x_i}\}+\min_i\{T^U_{x_i}\}-2\max_i\{I^L_{x_i}\}-2\max_i\{I^U_{x_i}\}-\max_i\{F^L_{x_i}\}-\max_i\{F^U_{x_i}\}}{4}$, then, by (48)-(53), we have $T^L = \min_i\{T^L_{x_i}\}$, $T^U = \min_i\{T^U_{x_i}\}$, $I^L = \max_i\{I^L_{x_i}\}$, $I^U = \max_i\{I^U_{x_i}\}$, $F^L = \max_i\{F^L_{x_i}\}$, $F^U = \max_i\{F^U_{x_i}\}$.

Thus

$$\begin{aligned}L(x) &= \frac{1}{2}\{T^L + T^U - I^U(1-T^U) - I^L(1-T^L)\\ &\quad - F^U(1-I^U) - F^L(1-I^L)\}\\ &= \frac{1}{2}\{\min_i\{T^L_{x_i}\} + \min_i\{T^U_{x_i}\} - \max_i\{I^L_{x_i}\}(1-\min_i\{T^U_{x_i}\})\\ &\quad - \max_i\{I^L_{x_i}\}(1-\min_i\{T^L_{x_i}\}) - \max_i\{F^U_{x_i}\}(1-\max_i\{I^U_{x_i}\})\\ &\quad - \max_i\{F^L_{x_i}\}(1-\max_i\{I^L_{x_i}\})\} = L(x^-).\end{aligned}$$

$$S(x) = \frac{2 + T^L + T^U - 2I^L - 2I^U - F^L - F^U}{4}$$
$$\leq \frac{2 + \max_i\{T^L_{x_i}\} + \max_i\{T^U_{x_i}\} - 2\min_i\{I^L_{x_i}\} - 2\min_i\{I^U_{x_i}\} - \min_i\{F^L_{x_i}\} - \min_i\{F^U_{x_i}\}}{4} = S(x^+),$$
$$S(x) = \frac{2 + T^L + T^U - 2I^L - 2I^U - F^L - F^U}{4}$$
$$\geq \frac{2 + \min_i\{T^L_{x_i}\} + \min_i\{T^U_{x_i}\} - 2\max_i\{I^L_{x_i}\} - 2\max_i\{I^U_{x_i}\} - \max_i\{F^L_{x_i}\} - \max_i\{F^U_{x_i}\}}{4} = S(x^-).$$

It implies that

$$IVNFWA(x_1, x_2, \cdots, x_n) = x^- \tag{56}$$

By combining (54) and (55) with (56), we can conclude that Theorem 5 always holds.

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